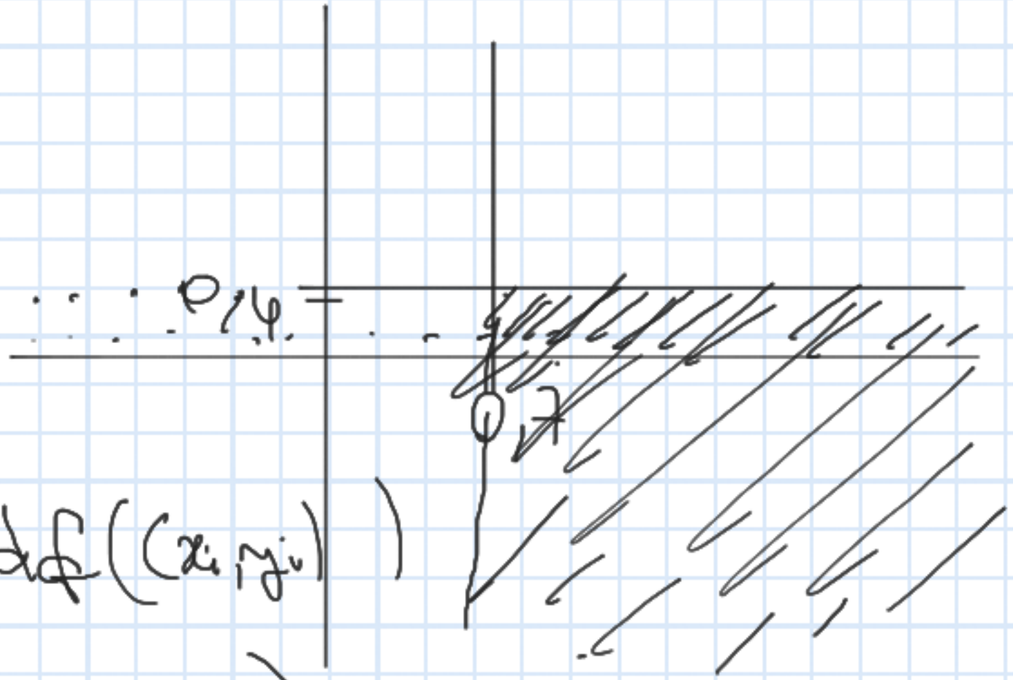
$X, Y \overset{iid}{\sim} \mathcal{N}(0,1)$



multivariate-normal. cdf$((x_i, y_i))$

$$\mathbb{E}[(X,Y)] = (\mathbb{E}[X], \mathbb{E}[Y])$$

$g(X,Y) = (X,Y)$

$$Y = g(X,Y)$$

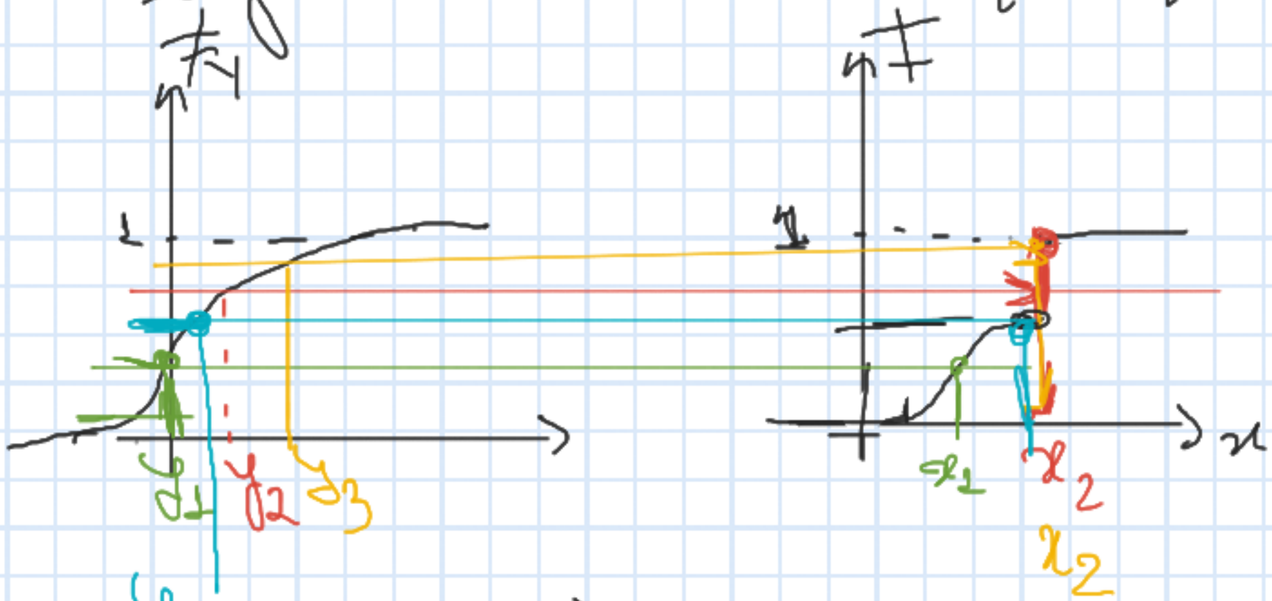
$$\int\int x f_{X,Y}(x,y)\,dx\,dy.$$

$$\int x \underbrace{\left(\int f_{X,Y}(x,y)\,dy \right)}_{f_X(x)} dx$$

$$P(X>0.7,\ Y<0.4) \overset{\text{indep}}{=} P(X>0.7)\,P(Y<0.4)$$

Quiero $F(x)$

Tengo una v.a. Y c. $F_Y(y)$



$F_Y(y) = F(x)$ → si no hay saltos puedo despejar sin problema

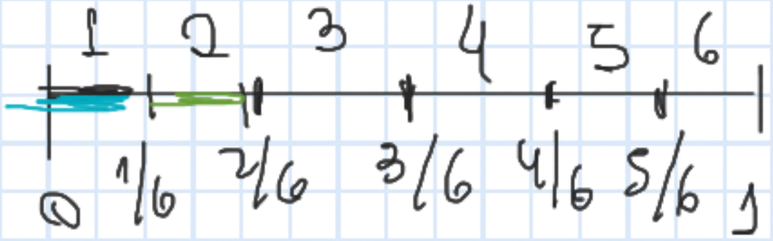
$X = h(Y) = F_X^{-1}(F_Y(y))$ $\quad x_2$

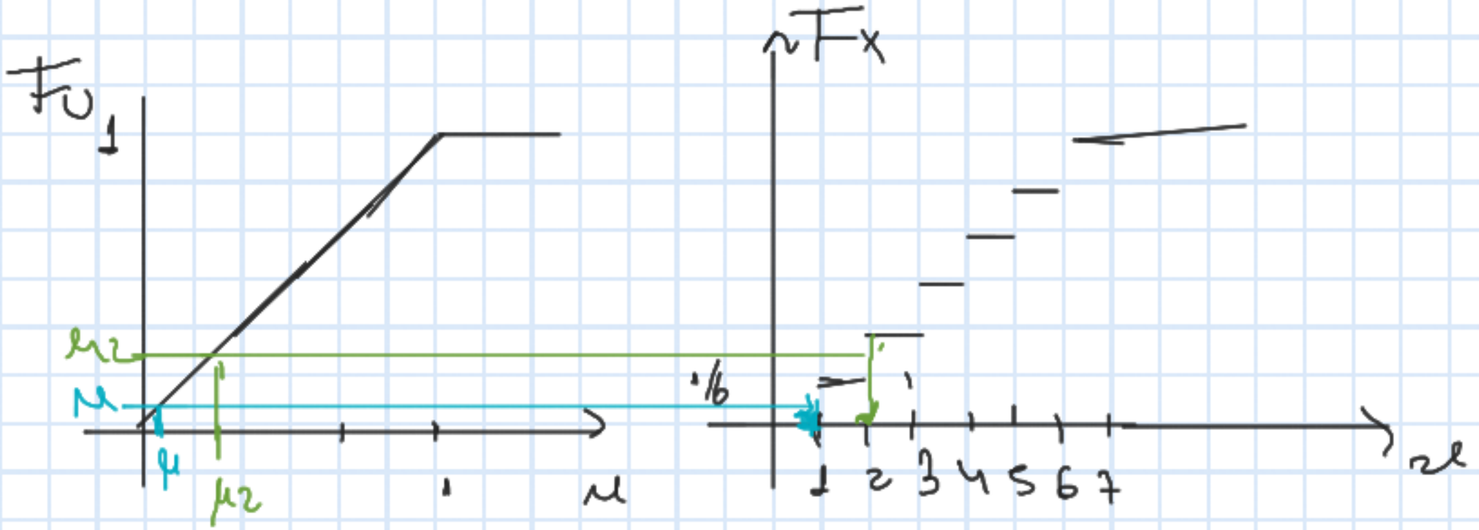
$y < y_{salto}$

$y > y_{salto}$

Ex 1

$U \sim \mathcal{U}(0,1)$

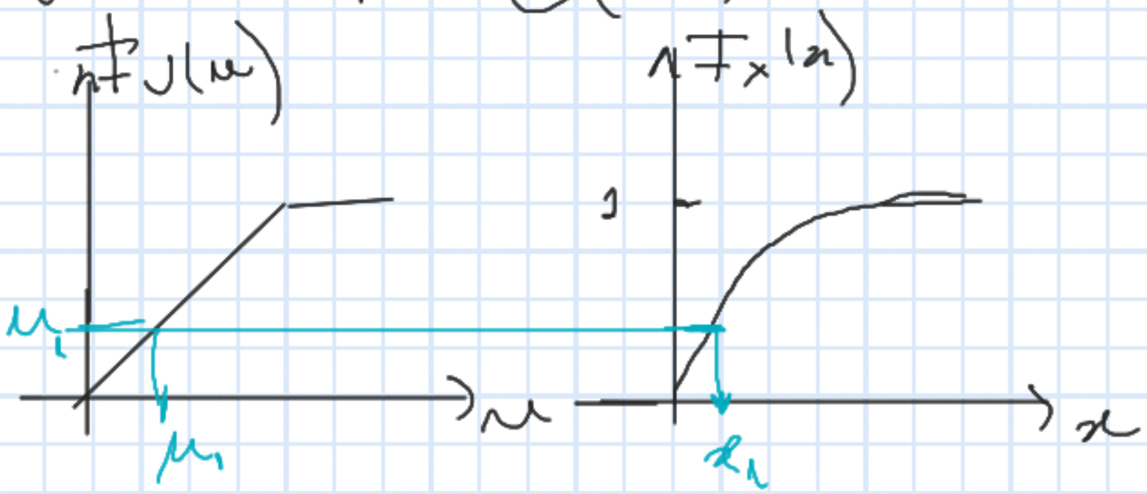




$u = 0,37 \rightarrow x = 4$

$$X = h(U) = \begin{cases} 1 & \text{si } 0 \leq U < 1/6 \\ 2 & \text{si } 1/6 \leq U < 2/6 \\ \vdots & \end{cases}$$

Ej 2. $X \sim \mathcal{E}(1/5)$



$$F_X(x) = \left(1 - e^{-x/5}\right) \mathbb{1}\{x > 0\}.$$

$$1 - e^{-x/5} = u$$

$$1 - u = e^{-x/5}$$

$$-\ln(1-u) = x/5$$

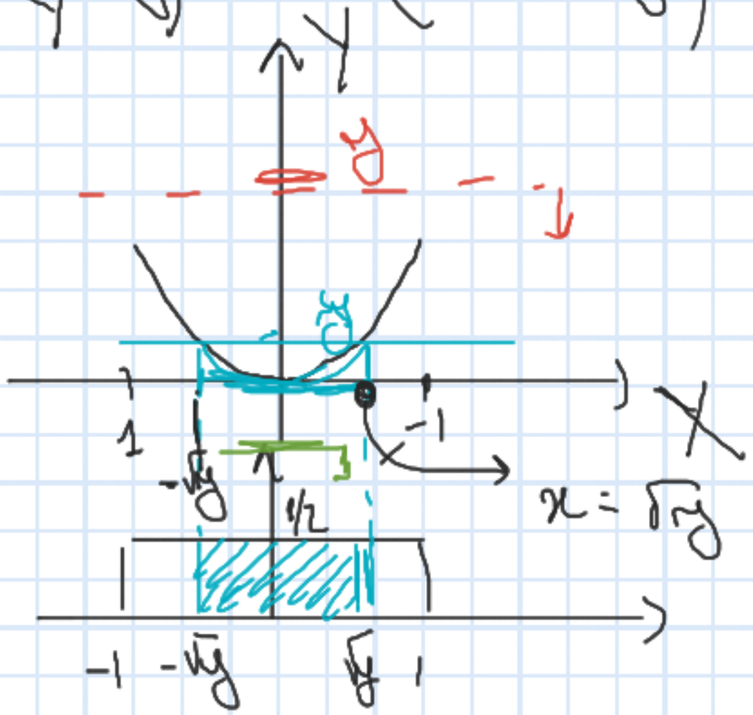
$$x = -5\ln(1-u)$$

$$F_U(u) = u\,\mathbb{1}\{0 < u < 1\}$$

$$\lambda e^{-\lambda x} \quad \mathbb{1}\{x > 0\}$$

Ej 3 $X \sim N(-1,1)$ $Y = X^2$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} < X < \sqrt{y}) \quad 0 \le y < 1.$$



$$\begin{cases} 0 & y < 0 \\ \sqrt{y} & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases}$$

$X \sim Poi(\lambda)$

$Y \sim Poi(\mu)$

$W = X + Y$

y \ x	0	1	2	3	...
0	0	1	2		
1	1	2			
2	2				
3					
...					

ω	0	1	2	...
$P_W(\omega)$	$P_X(0)P_Y(0)$	$P_X(0)P_Y(1) + P_X(1)P_Y(0)$	$P_X(0)P_Y(2) + P_X(1)P_Y(1) + P_X(2)P_Y(0)$	

$$P_W(\omega) = \sum_{x=0}^{\omega} P_X(x) P_Y(\omega - x) = \sum_{x=0}^{\omega} \frac{\lambda^x}{x!} e^{-\lambda} \frac{\mu^{(\omega-x)}}{(\omega-x)!} e^{-\mu}$$

$$= \frac{(\lambda+\mu)^{\omega}}{\omega!} e^{-(\lambda+\mu)} \sum_{x=0}^{\omega} \frac{\omega!}{x!(\omega-x)!} \left(\frac{\lambda}{\lambda+\mu}\right)^x \left(\frac{\mu}{\lambda+\mu}\right)^{(\omega-x)} = \frac{(\lambda+\mu)^{\omega}}{\omega!} e^{-(\lambda+\mu)}$$

$\sum = 1$

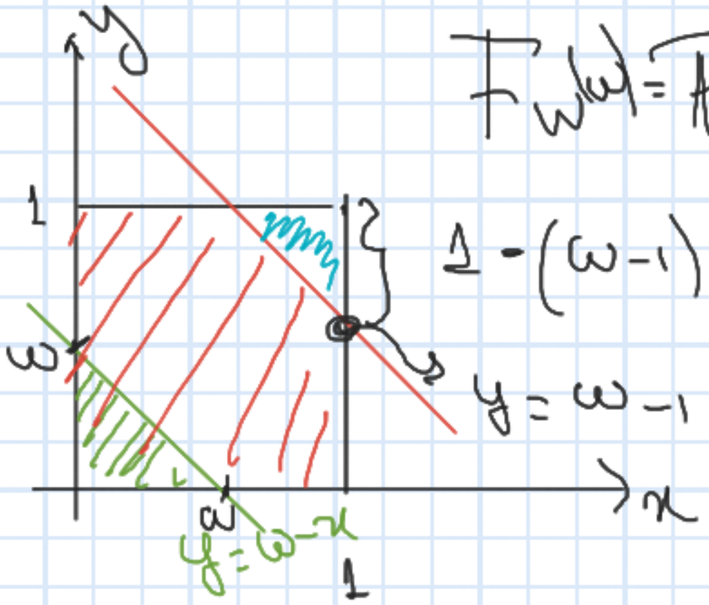
$\frac{\lambda^x}{x!} e^{-\lambda}$

$Bin(n,p)$ $\binom{n}{x} p^x (1-p)^{n-x}$

$W \sim Poi(\lambda + \mu)$ (1)

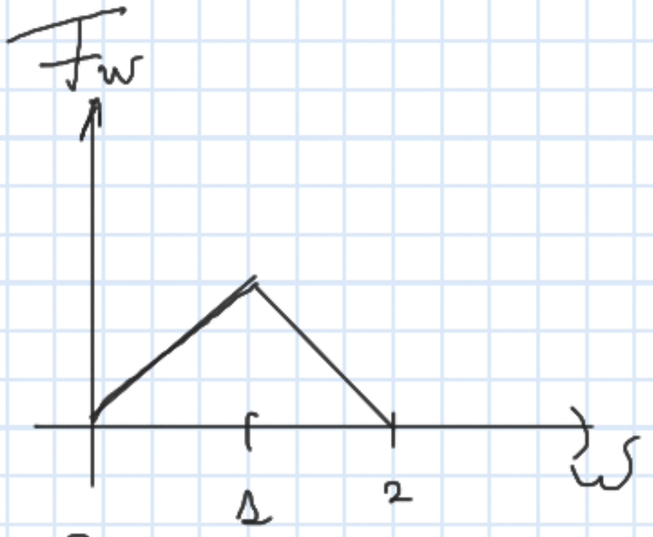
X, Y iid $U(0,1)$ $\quad W = X+Y$

$$F_W(\omega) = P(W \le \omega) = P(X+Y \le \omega) = \begin{cases} 0 & \omega < 0 \\ (*) & 0 \le \omega < 2 \\ 1 & \omega \ge 2 \end{cases}$$



$1 - (\omega - 1) = 2 - \omega$

$$(*)\ P(\underbrace{X+Y \le \omega}_{Y \le \omega - X}) = \begin{cases} \omega^2/2 \cdot 1 & 0 \le \omega < 1 \\ 1 - \dfrac{(2-\omega)^2}{2} \cdot 1 & 1 \le \omega < 2 \end{cases}$$



$$f_W(\omega) = \omega \mathbb{1}\{0 \le \omega < 1\} + (2-\omega)\mathbb{1}\{1 \le \omega < 2\}$$

Case 2.

Es 1 X_1, X_2 iid $N(0,1)$ $\quad U = X_1^2 + X_2^2 \quad V = X_2/X_1$

$$u = x_1^2 + x_2^2 \rightarrow u = x_1^2 + (v x_1)^2 = (1+v^2) x_1^2$$

$$\hookrightarrow x_1 = \sqrt{\frac{u}{1+v^2}}$$

$$v = x_2/x_1 \rightarrow x_2 = v x_1$$

$$x_2 = v \sqrt{\frac{u}{1+v^2}}$$

$$\underline{J} = \begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 & 2x_2 \\ -\frac{x_2}{x_1^2} & 1/x_1 \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{bmatrix}$$

$$|J| = 2 + 2\frac{x_2^2}{x_1^2} = 2\left(1 + \frac{x_2^2}{x_1^2}\right)$$

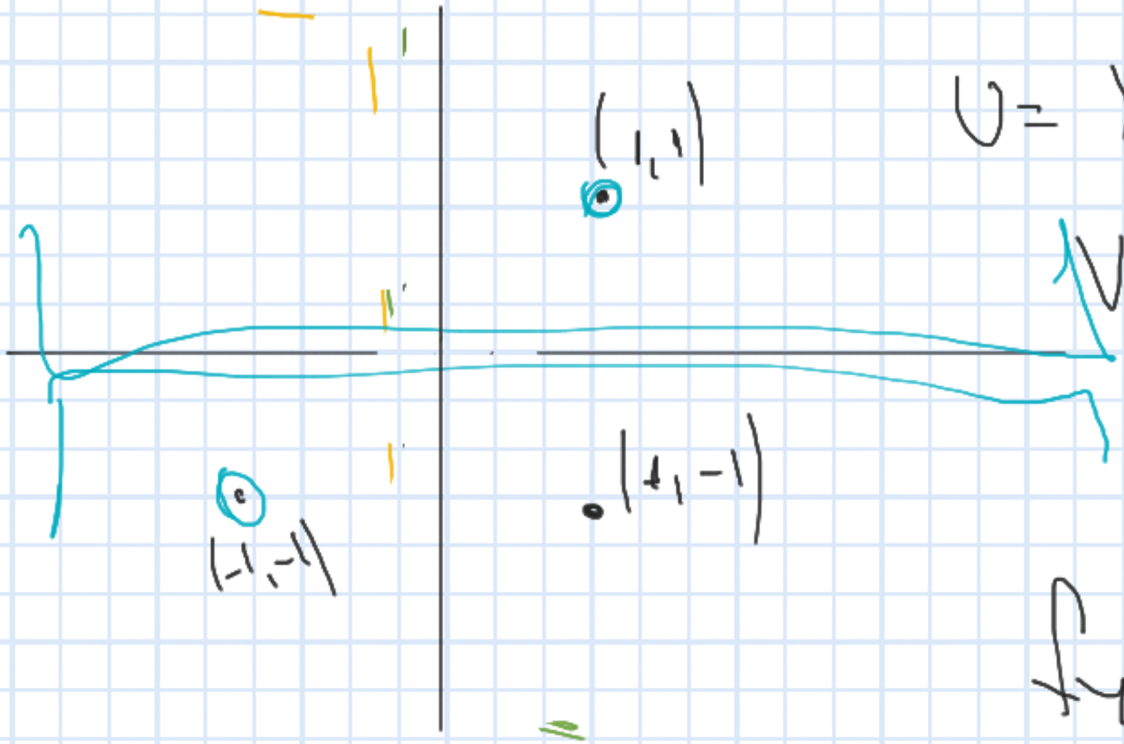
$$f_{U,V}(u,v) = \frac{f_{X_1}\left(\sqrt{\frac{u}{1+v^2}}\right) f_{X_2}\left(v\sqrt{\frac{u}{1+v^2}}\right) \mathbb{1}\{v\sqrt{\frac{u}{1+v^2}} > 0\} + f_{X_1}\left(-\sqrt{\frac{u}{1+v^2}}\right) f_{X_2}\left(-v\sqrt{\frac{u}{1+v^2}}\right) \mathbb{1}\{-v\sqrt{\frac{u}{1+v^2}} > 0\}}{2\left(1 + \frac{\left(v\sqrt{\frac{u}{1+v^2}}\right)^2}{\left(\sqrt{\frac{u}{1+v^2}}\right)^2}\right)} \cdot \left|\frac{u}{(1+v^2)^2}\right|$$

$$= \frac{2\,\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{u}{1+v^2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2\frac{u}{1+v^2}}}{2\left(1+v^2\right)} = \frac{2}{2\pi} e^{-\frac{1}{2}\frac{u}{1+v^2}\left(1+v^2\right)}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}u} \cdot \frac{1}{2\left(1+v^2\right)}$$

$$\mathbb{1}\{u>0\} = \underbrace{\frac{1}{2} e^{-\frac{1}{2}u} \mathbb{1}\{u>0\}}_{U \sim \mathcal{E}(1/2)} \cdot \frac{1}{2\pi\left(1+v^2\right)}$$

U, V indep.

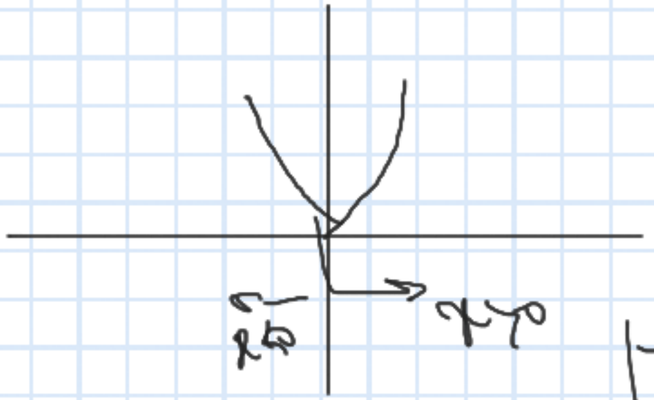


$$U = X_1^2 + X_2^2 = 1$$

$$V = X_2 \Big|_{X_1} = -1$$

$$f_Y(y) = \frac{f_X(\sqrt{y})\{0<\sqrt{y}<1\}}{|2\sqrt{y}|} + \frac{f_X(\sqrt{y})\{0<-\sqrt{y}<1\}}{|-2\sqrt{y}|}$$

$$= \frac{1}{2\sqrt{y}}\{0<y<1\}$$



$x > 0 \rightarrow x = \sqrt{y}$

$x < 0 \rightarrow x = -\sqrt{y}$

$$|J| = \frac{\partial y}{\partial x} = |2x|$$

$$f_{X,Y}(x,y) = f_{Y|X=x}(y)\, f_X(x)$$

$$f_{X|Y=y}(x)\, f_{Y|Y}(y)$$

Si $f_X(x) \Rightarrow$ indep

Si X, Y indep $\Rightarrow$

$$f_{X,Y}(x,y) = f_{X|X}(x) \cdot f_{Y|Y}(y)$$

Ej 3

1. La probabilidad de acertar a un blanco es ⅓. Se realizan 10 tiros independientes y se cuenta la cantidad de aciertos. Sean X la cantidad de aciertos en los 10 tiros, e Y la cantidad de aciertos en el primer tiro. Hallar la distribución de X|Y=y y Y|X=x.

X = "# de aciertos en 10 tiros"

Y = "# de aciertos en el 1º tiro"

W = "# de aciertos en los tiros 2 al 10"

$X \sim Bin(10, 1/3)$

$Y \sim Ber(1/3)$ $\quad W \sim Bin(9, 1/3)$ indep

$$p_{X|Y=y}(x) = \begin{cases} \dfrac{P(X=x, Y=0)}{P(Y=0)} & y=0 \\ \dfrac{P(X=x, Y=1)}{P(Y=1)} & y=1 \end{cases}$$

$$P(X=x, Y=0) = P(W=x, Y=0) = p_W(x)\, p_Y(0)$$

$$P(X=x, Y=1) = P(W=x-1, Y=1) = p_W(x-1)\, p_Y(1)$$

$$= \begin{cases} p_W(x) & y=0 \quad 0 < x < 9 \\ p_W(x-1) & y=1 \quad 1 < x < 10 \end{cases}$$

$$P(Y \mid X = x) =$$

$$P_{Y|X=0}(y) = 1 \text{ si } y = 0$$

$$P_{Y|X=1}(y) = \begin{cases} \frac{P(X=1, Y=1)}{P(X=1)} = 1/10 & y = 1 \\ \frac{P(X=1, Y=0)}{P(X=1)} = 9/10 & y = 0 \end{cases}$$

$$P_{Y|X=2}(y) = \begin{cases} 8/10 & y = 0 \\ 2/10 & y = 1 \end{cases}$$

$$Y \mid X = x \sim \text{Ber}\left(\frac{x}{10}\right)$$

$$\frac{P(X=1, Y=1)}{P(X=1)} = \frac{P(W=0, Y=1)}{P(X=1)}$$

$$= \frac{\binom{9}{0} 1/3^0 \, 2/3^9 \cdot 1/3}{\binom{10}{1} 1/3^1 \, 2/3^9} = 1/10$$

$$\frac{P(X=2, Y=0)}{P(X=2)} = \frac{P(W=2, Y=0)}{P(X=2)}$$

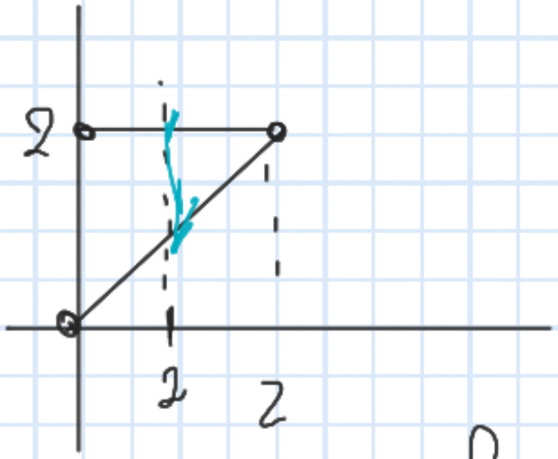
$$= \frac{\binom{9}{2} 1/3^2 \, 2/3^7 \cdot 8/3 \cdot \frac{9!}{2!7!}}{\binom{10}{2} 1/3^2 \, 2/3^7 \cdot \frac{10!}{2! \, 8!}} = \frac{8}{10}$$

2. Sea $Y=X^2$, con $X\sim U(-1,1)$. Hallar $f_{Y|X=x}(y)$

$Y|X=x \quad = x^2$

$P(Y=y \mid X=x) = 1\{y = x^2\}$

3. Sean X,Y dos v.a. conjuntamente uniformes en el triángulo de vértices (0,0), (2,2), (0,2). Hallar la función de densidad de Y|X=x.



$f_{X,Y}(x,y) = \frac{1}{2} 1\{0<x<y<2\}$

$f_{Y|X=x_0}(y) = \frac{f_{X,Y}(x_0,y)}{f_X(x_0)} = \frac{\frac{1}{2} 1\{0<x_0<y<2\}}{\frac{2-x_0}{2} 1\{0<x_0<2\}}$ (*)

$f_X(x) = \int_R f_{X,Y}(x,y)\,dy = \int_x^2 \frac{1}{2}\,dy\; 1\{0<x<2\}$

$= \frac{y}{2}\Big|_x^2 \; 1\{0<x<2\} = \left(1-\frac{x}{2}\right) 1\{0<x<2\}$

$= \frac{(2-x)}{2} 1\{0<x<2\}$

(*)

$f_{Y|X=x_0}(y) = \frac{1}{2-x_0} 1\{x_0<y<2\}$

$Y|X=x_0 \sim U(x_0,2)$