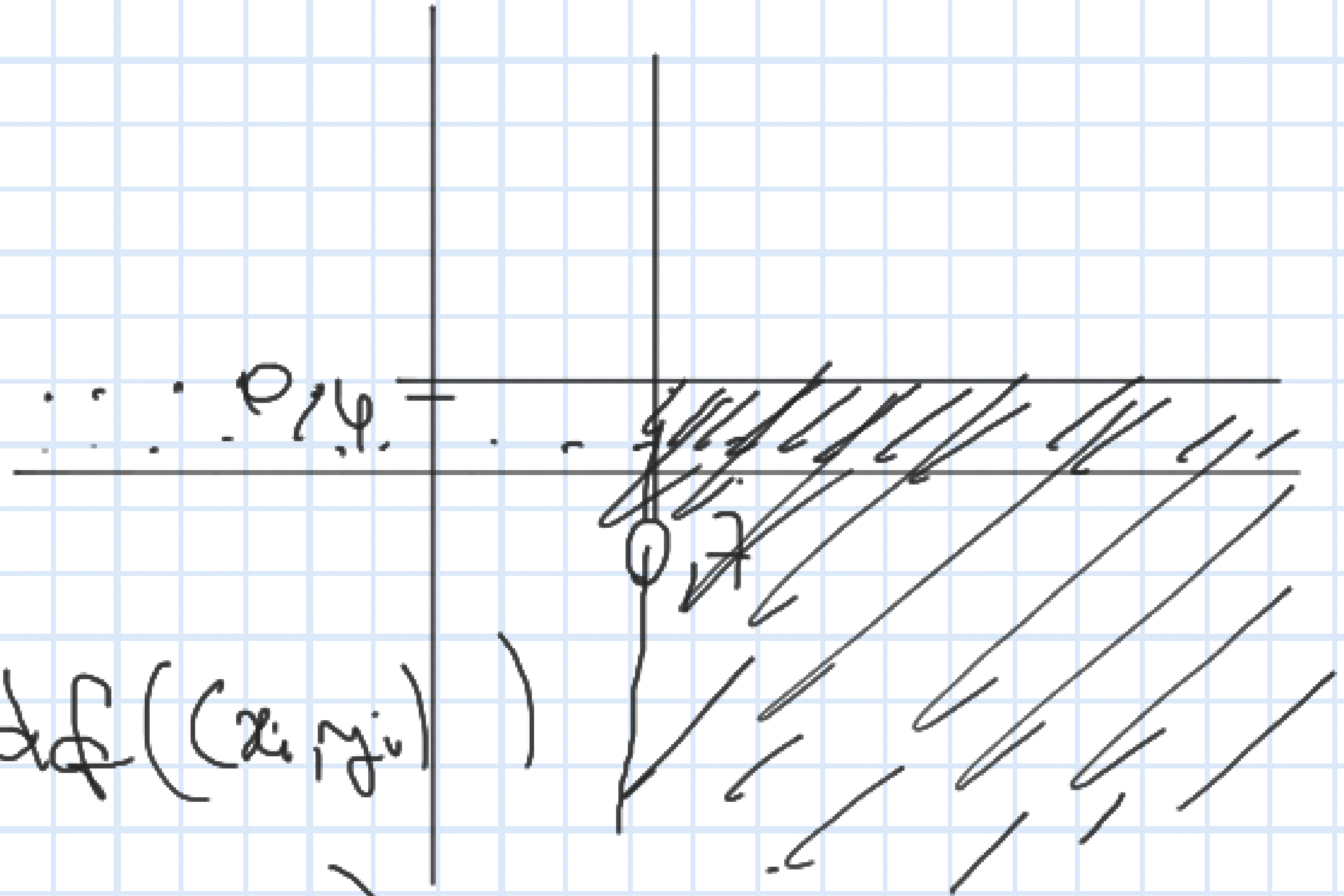
$X, Y \overset{iid}{\sim} \mathcal{N}(0,1)$



multivariate_normal.cdf$((x_i, y_i))$

$\mathbb{E}[\underbrace{(X,Y)}_{g(X,Y)=(X,Y)}] = (\mathbb{E}[X], \mathbb{E}[Y])$

$\iint x \, f_{X,Y}(x,y)\, dx\, dy.$

$X = g(X,Y)$

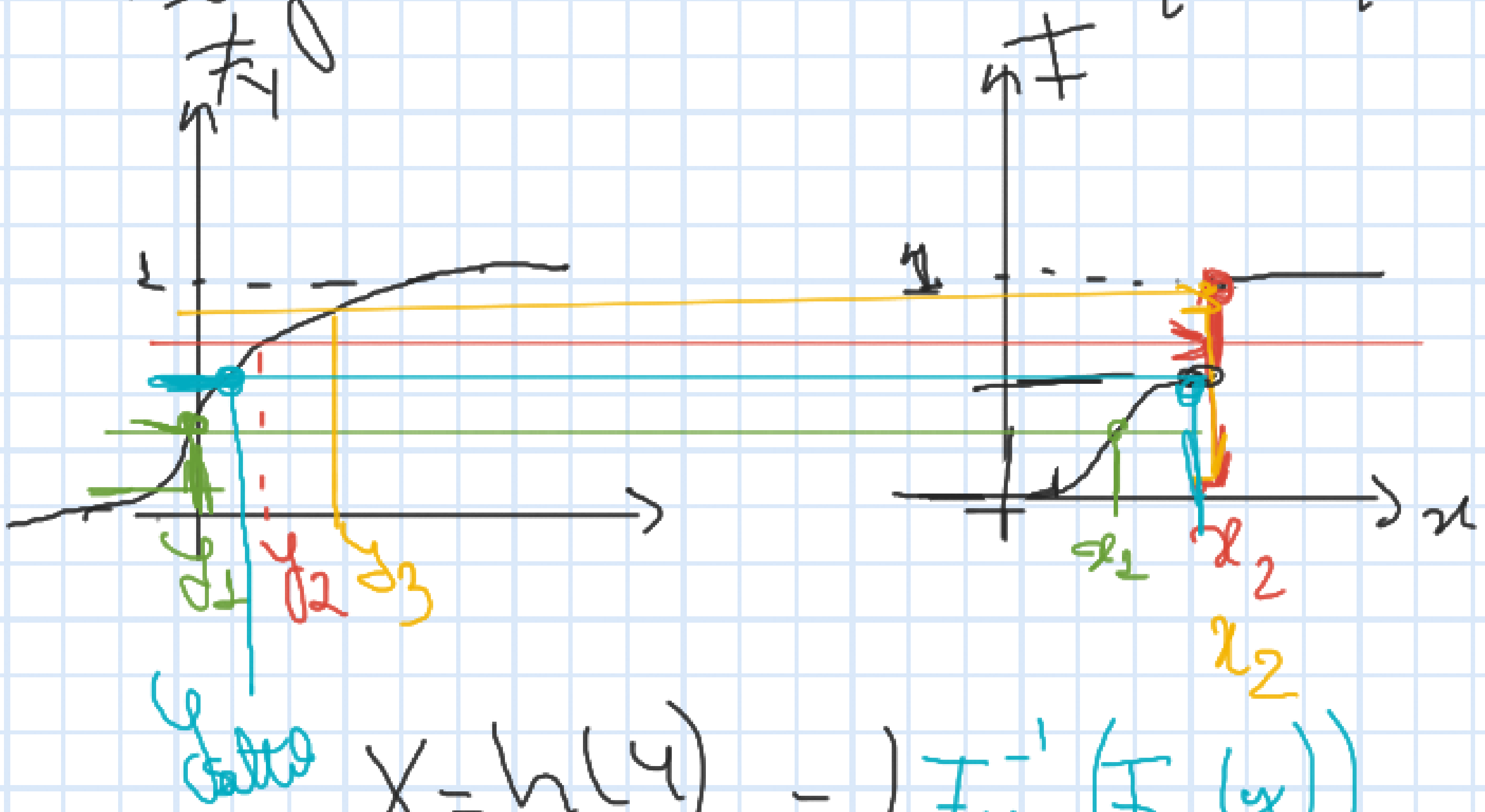
$\int x \underbrace{\left(\int f_{X,Y}(x,y)\, dy\right)}_{f_X(x)} dx$

$\mathbb{P}(X > 0.7, Y < 0.4) \underset{\text{indep}}{=} \mathbb{P}(X > 0.7)\,\mathbb{P}(Y < 0.4)$

Quiero $F(x)$

Tengo una v.a. Y c. $F_Y(y)$

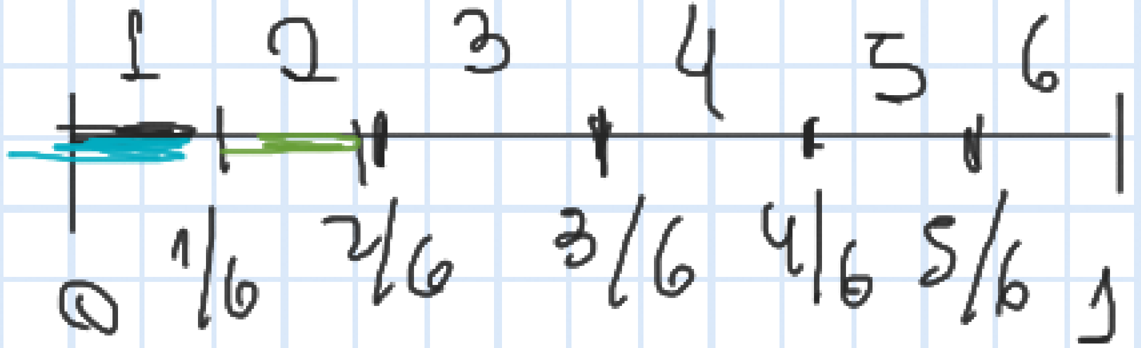
$F_Y(y) = F(x) \Rightarrow$ si no hay saltos puedo despejar sin problema



$$X = h(Y) = \begin{cases} F_X^{-1}(F_Y(y)) & y < y_{salto} \\ x_2 & y > y_{salto} \end{cases}$$

Ej 1

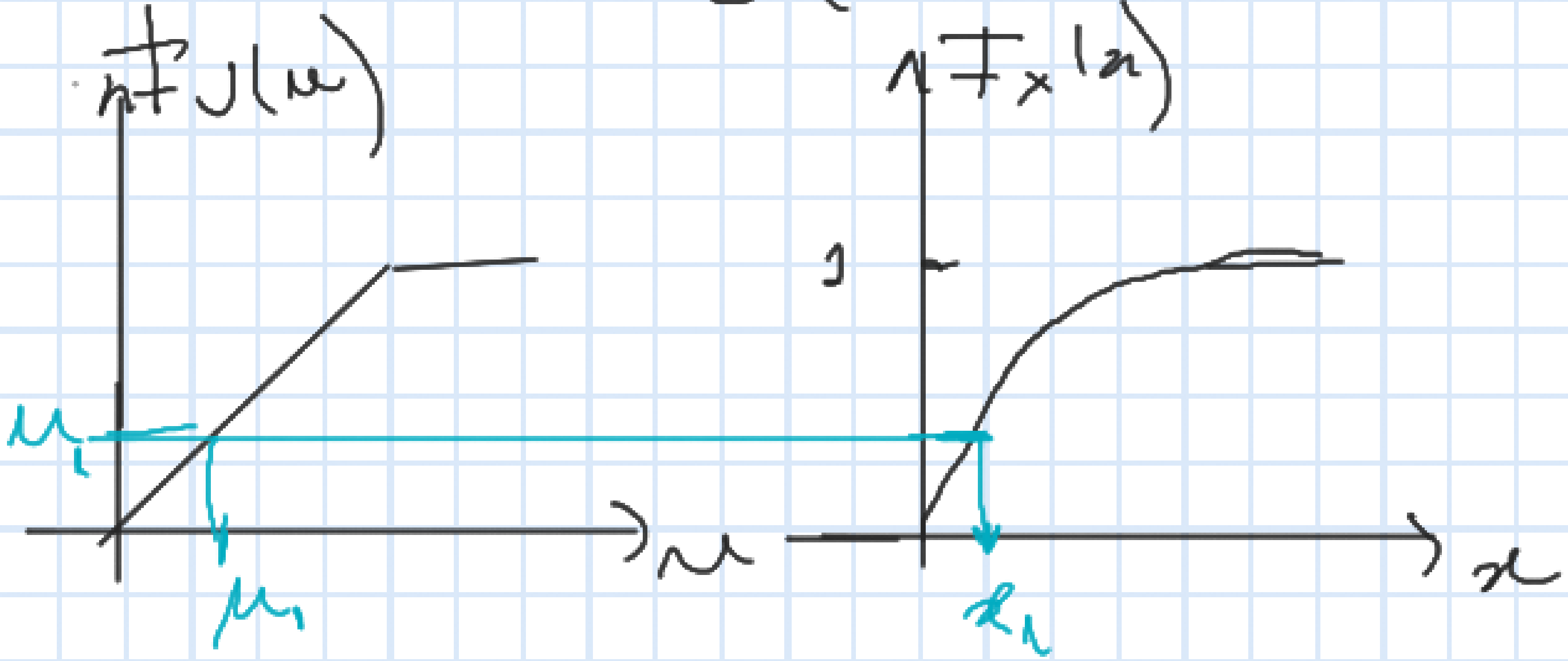
$U \sim U(0,1)$



$u = 0{,}37 \to x = 4$

$$X = h(U) = \begin{cases} 1 & \text{si } 0 \leq U < 1/6 \\ 2 & \text{si } 1/6 \leq U < 2/6 \\ \vdots & \end{cases}$$

Ej2. $X \sim \mathcal{E}(1/5)$



$$F_X(x) = \left(1 - e^{-x/5}\right) \mathbb{1}\{x > 0\}.$$

$$F_U(u) = u \, \mathbb{1}\{0 < u < 1\}$$

$$1 - e^{-x/5} = u$$

$$1 - u = e^{-x/5}$$

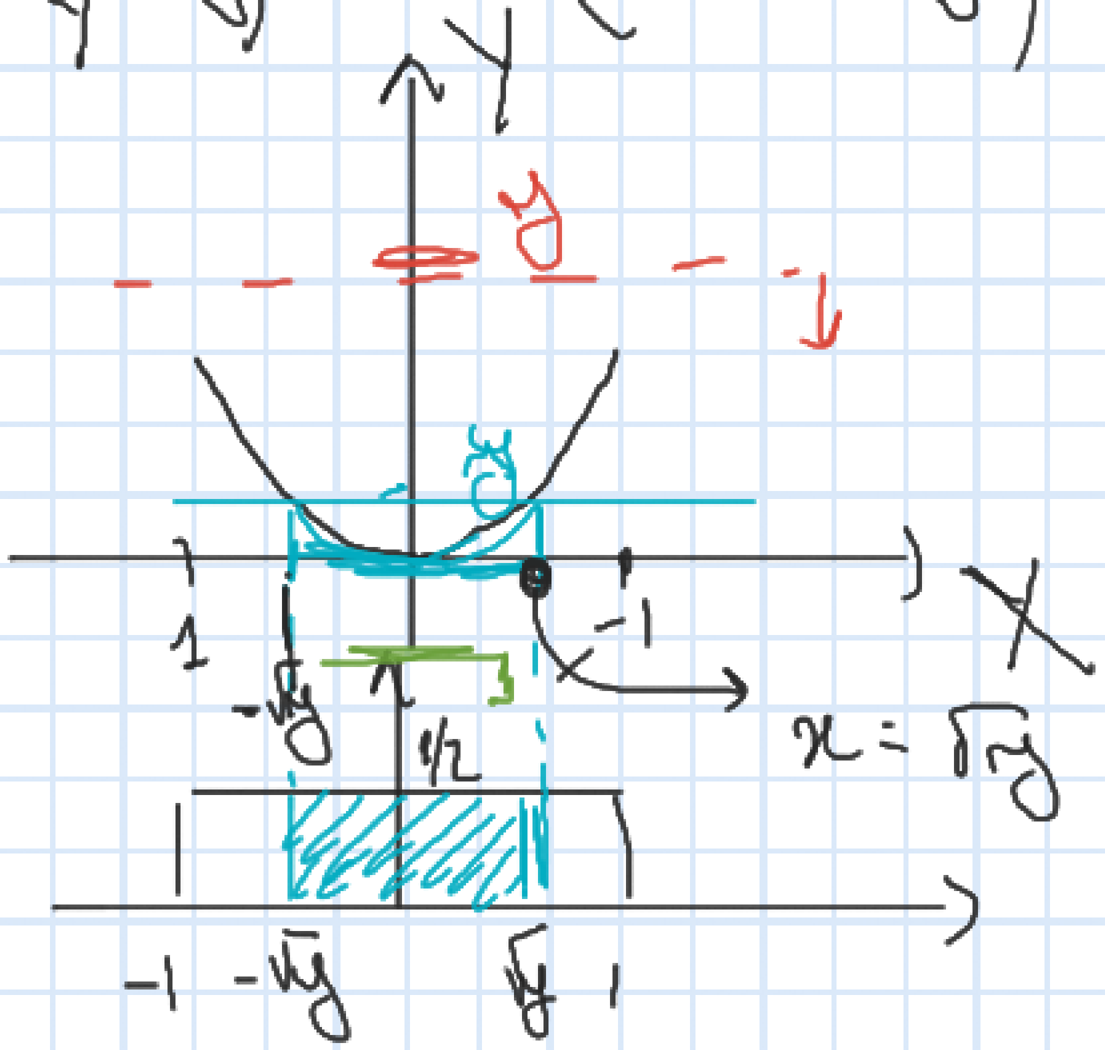
$$-\ln(1-u) = x/5$$

$$x = -5\ln(1-u)$$

$$\lambda e^{-\lambda x} \quad \mathbb{1}\{x > 0\}$$

Ej 3 $\quad X \sim U(-1,1) \qquad Y = X^2$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = \begin{cases} 0 & y < 0 \\ P(-\sqrt{y} < X < \sqrt{y}) & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases} = \begin{cases} 0 & y < 0 \\ \sqrt{y} & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases}$$



$X \sim Poi(\lambda)$
$Y \sim Poi(\mu)$

$W = X + Y$

$y \backslash x$	0	1	2	3 ...
0	0	1	2	
1	1	2		
2	2			
3				

ω	0	1	2 ...
$P_W(\omega)$	$P_X(0)P_Y(0)$	$P_X(0)P_Y(1) + P_X(1)P_Y(0)$	$P_X(0)P_Y(2) + P_X(1)P_Y(1) + P_X(2)P_Y(0)$

$$\frac{\lambda^x}{x!} e^{-\lambda}$$

$$P_W(\omega) = \sum_{x=0}^{\omega} P_X(x) P_Y(\omega - x) = \sum_{x=0}^{\omega} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\mu^{(\omega-x)}}{(\omega-x)!} e^{-\mu}$$

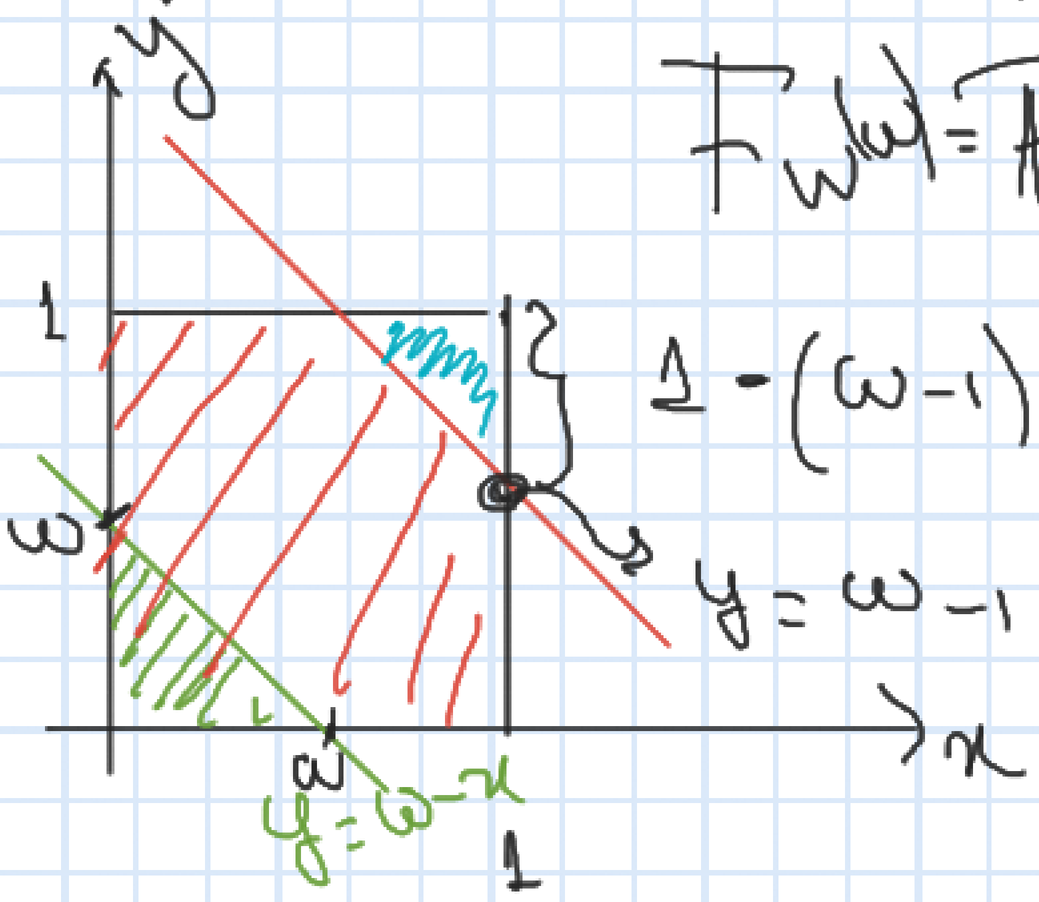
$$= \frac{(\lambda+\mu)^\omega}{\omega!} e^{-(\lambda+\mu)} \underbrace{\sum_{x=0}^{\omega} \frac{\omega!}{x!(\omega-x)!} \left(\frac{\lambda}{\lambda+\mu}\right)^x \left(\frac{\mu}{\lambda+\mu}\right)^{(\omega-x)}}_{1} = \frac{(\lambda+\mu)^\omega e^{-(\lambda+\mu)}}{\omega!}$$

$Bin(n, p)$ $\binom{n}{x} p^x (1-p)^{n-x}$

$W \sim Poi(\lambda + \mu)$

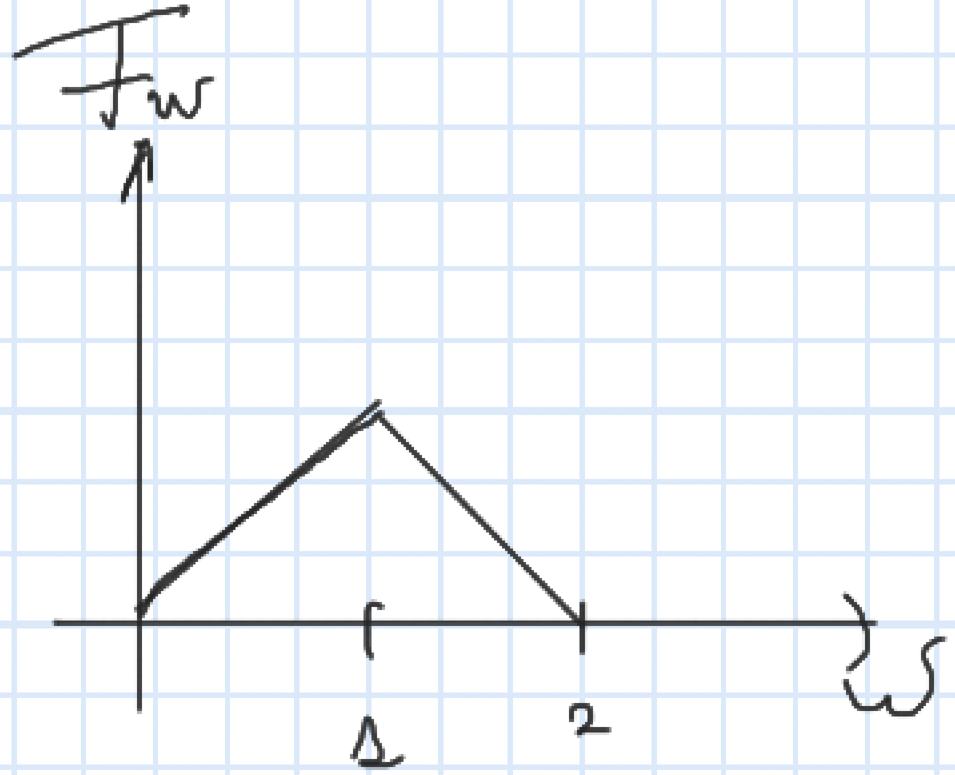
$X,Y \overset{iid}{\sim} \mathcal{U}(0,1) \qquad W = X+Y$

$$F_W(\omega) = P(W \le \omega) = P(X+Y \le \omega) = \begin{cases} 0 & \omega < 0 \\ (*) & 0 \le \omega < 2 \\ 1 & \omega \ge 2 \end{cases}$$



$1 - (\omega - 1) = 2 - \omega$

$$(*)\ P(\underbrace{X+Y \le \omega}_{Y \le \omega - X}) = \begin{cases} \omega^2/2 \cdot 1 & 0 \le \omega < 1 \\ 1 - \frac{(2-\omega)^2}{2} \cdot 1 & 1 \le \omega < 2 \end{cases}$$



$$f_W(\omega) = \omega\, \mathbb{1}\{0 \le \omega < 1\} + (2-\omega)\, \mathbb{1}\{1 \le \omega < 2\}$$

Clase 2.

Ej 1 $\quad X_1, X_2 \overset{iid}{\sim} N(0,1) \qquad U = X_1^2 + X_2^2 \qquad V = X_2/X_1$

$$u = x_1^2 + x_2^2 \rightarrow u = x_1^2 + (v x_1)^2 = (1+v^2)\, x_1^2$$

$$\hookrightarrow x_1 = \sqrt{\frac{u}{1+v^2}}$$

$$v = x_2/x_1 \rightarrow x_2 = v x_1$$

$$x_2 = v\sqrt{\frac{u}{1+v^2}}$$

$$\tilde{J} = \begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 & 2x_2 \\ -\frac{x_2}{x_1^2} & 1/x_1 \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{bmatrix}$$

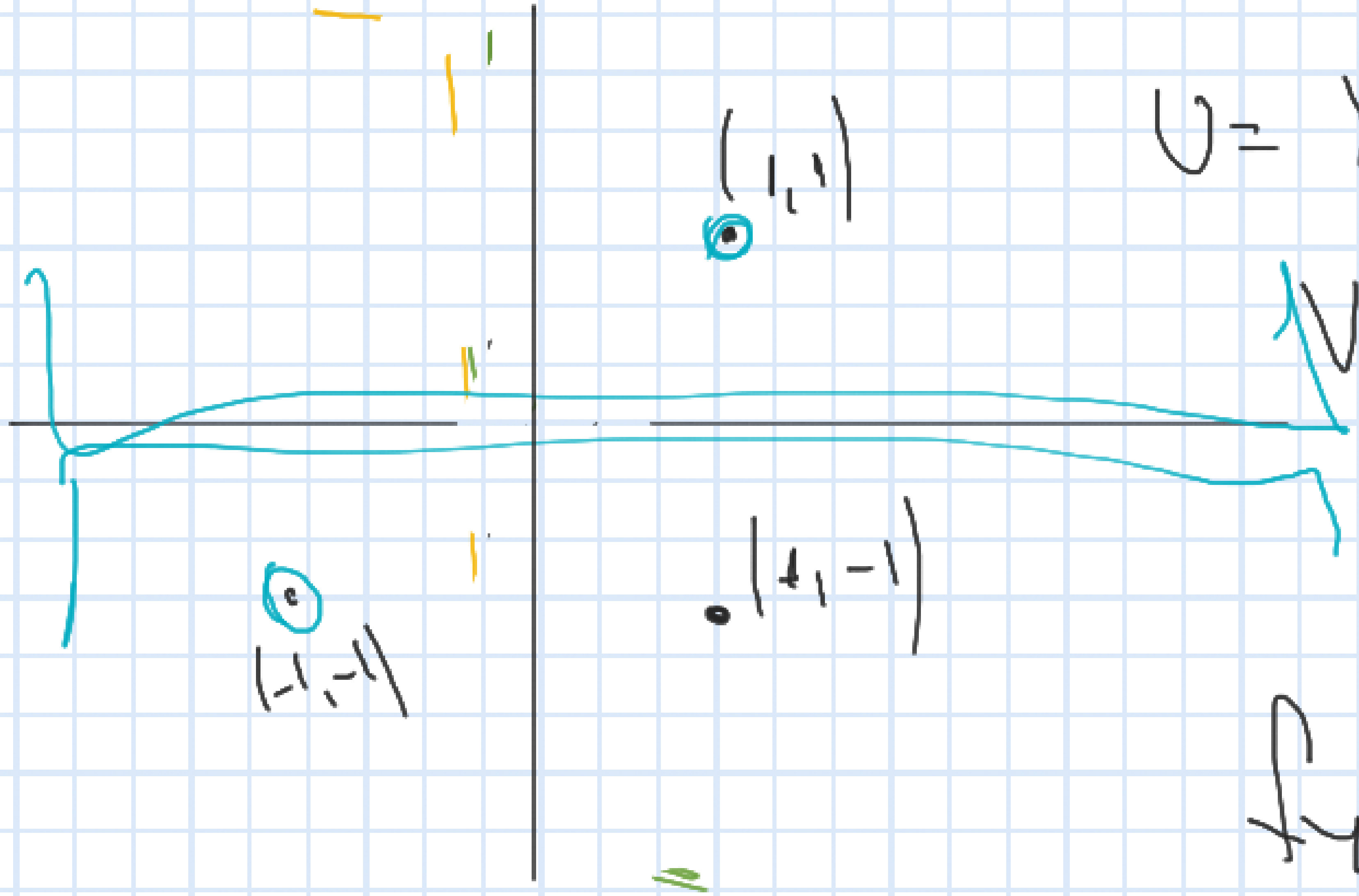
$$|J| = 2 + 2\frac{x_2^2}{x_1^2} = 2\left(1 + \frac{x_2^2}{x_1^2}\right)$$

$$f_{U,V}(u,v) = \frac{f_{X_1}\left(\sqrt{\frac{u}{1+v^2}}\right) f_{X_2}\left(v\sqrt{\frac{u}{1+v^2}}\right) \mathbb{1}\{\underbrace{v\sqrt{\tfrac{u}{1+v^2}} \geq 0}_{x_2 \geq 0}\}}{2\left(1+\frac{\left(v\sqrt{\cancel{\frac{u}{1+v^2}}}\right)^2}{\left(\sqrt{\cancel{\frac{u}{1+v^2}}}\right)^2}\right)} + \frac{f_{X_1}\left(\sqrt{\frac{u}{1+v^2}}\right) f_{X_2}\left(v\sqrt{\frac{u}{1+v^2}}\right)}{2(1+v^2)} \mathbb{1}\{v\sqrt{\tfrac{u}{1+v^2}} \dots$$

$$= \frac{2\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{u}{1+v^2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} v^2 \frac{u}{1+v^2}}}{2(1+v^2)} = \frac{\frac{2}{2\pi} e^{-\frac{1}{2}\cancel{\frac{u}{1+v^2}}\left(\cancel{1+v^2}\right)}}{2(1+v^2)}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}u} \frac{1}{2(1+v^2)} \mathbb{1}\{u \geq 0\} = \underbrace{\boxed{\frac{1}{2} e^{-1/2 u} \mathbb{1}\{u \geq 0\}}}_{U \sim \mathcal{E}(1/2)} \boxed{\frac{\cancel{2}}{\cancel{2}\pi(1+v^2)}}$$

U, V indep.



$U = X_1^2 + X_2^2 = 1$

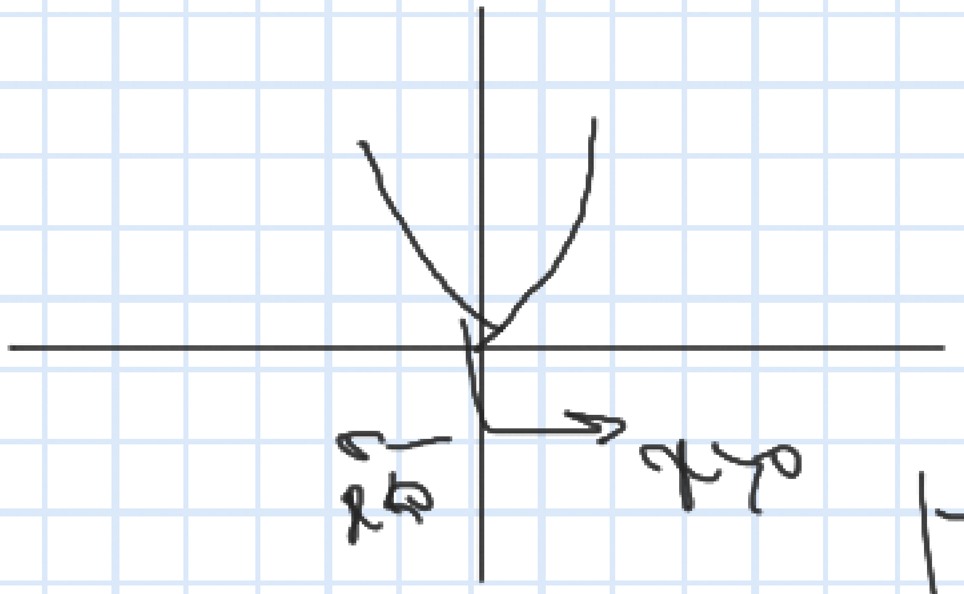
$V = X_2 \big|_{X_1} = -1$

$$f_Y(y) = \frac{f_X(\sqrt{y})\, \mathbb{1}\{\sqrt{y} > 0\}}{|2\sqrt{y}|} + \frac{f_X(-\sqrt{y})\, \mathbb{1}\{-\sqrt{y} < 0\}}{|-2\sqrt{y}|}$$

$1 \, \mathbb{1}\{0 < \sqrt{y} < 1\}$

1

$= \frac{\cancel{2}}{\cancel{2}\sqrt{y}} \mathbb{1}\{0 < y < 1\}$



$x > 0 \to x = \sqrt{y}$

$x < 0 \to x = -\sqrt{y}$

$|J| = \frac{\partial y}{\partial x} = |2x|$

$$f_{X,Y}(x,y) = f_{Y|X=x}(y)\, f_X(x)$$

$$f_{X|Y=y}(x)\, f_Y(y)$$

Si $f_X(x) \Rightarrow$ indep.

Si X, Y indep $\Rightarrow$

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

Ej 3

1. La probabilidad de acertar a un blanco es ⅓. Se realizan 10 tiros independientes y se cuenta la cantidad de aciertos. Sean X la cantidad de aciertos en los 10 tiros, e Y la cantidad de aciertos en el primer tiro. Hallar la distribución de X|Y=y y Y|X=x.

X = "# de aciertos en los tiros" $\quad X \sim Bin(10, 1/3)$

Y = "# de aciertos en el 1º tiro" $\quad Y \sim Ber(1/3)$ $\quad W \sim Bin(9, 1/3)$ indep

W = "# de aciertos en los tiros 2 al 10"

$$p_{X|Y=y}(x) = \begin{cases} \dfrac{P(X=x, Y=0)}{P(Y=0)} & y=0 \\ \dfrac{P(X=x, Y=1)}{P(Y=1)} & y=1 \end{cases}$$

$$P(X=x, Y=0) = P(W=x, Y=0) = p_W(x)\, p_Y(0)$$

$$P(X=x, Y=1) = P(W=x-1, Y=1) = p_W(x-1)\, p_Y(1)$$

$$= \begin{cases} p_W(x) & y=0 \quad 0 \leq x \leq 9 \\ p_W(x-1) & y=1 \quad 1 \leq x \leq 10 \end{cases}$$

$\mathbb{P}(Y \mid X = x) =$

$P_{Y|X=0}(y) = 1$ si $y = 0$

$P_{Y|X=1}(y) = \begin{cases} \frac{\mathbb{P}(X=1, Y=1)}{\mathbb{P}(X=1)} = 1/10 & y = 1 \\ \frac{\mathbb{P}(X=1, Y=0)}{\mathbb{P}(X=1)} = 9/10 & y = 0 \end{cases}$

$$\frac{\mathbb{P}(X=1, Y=1)}{\mathbb{P}(X=1)} = \frac{\mathbb{P}(W=0, Y=1)}{\mathbb{P}(X=1)} = \frac{\binom{9}{0} 1/3^0 \cancel{2/3^9} \quad 1/3}{\binom{10}{1} \cancel{1/3^1} \cancel{2/3^9}}$$

$P_{Y|X=2}(y) = \begin{cases} 8/10 & y = 0 \\ 2/10 & y = 1 \end{cases}$

$$\frac{\mathbb{P}(X=2, Y=0)}{\mathbb{P}(X=2)} = \frac{\mathbb{P}(W=2, Y=0)}{\mathbb{P}(X=2)} = 1/10$$

$$= \frac{\binom{9}{2} 1/3^2 \cancel{2/3^7} \quad 2/3}{\binom{10}{2} 1/3^2 \cancel{2/3^7}} = \frac{\frac{9!}{2!\,7!}}{\frac{10!}{2!\,8!}} = \frac{8}{10}$$

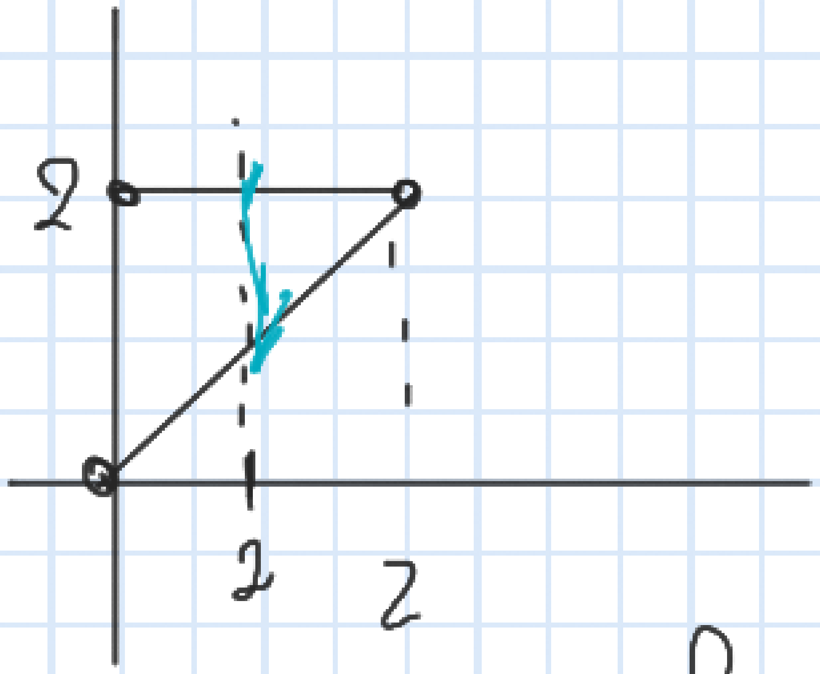
$Y \mid X = x \sim \mathrm{Ber}(x/10)$

2. Sea $Y=X^2$, con $X\sim U(-1,1)$. Hallar $f_{Y|X=x}(y)$

$$Y|X=x = x^2$$

$$\mathbb{P}(Y=y\,|\,X=x) = \mathbb{1}\{y=x^2\}$$

3. Sean X,Y dos v.a. conjuntamente uniformes en el triángulo de vértices (0,0), (2,2), (0,2). Hallar la función de densidad de Y|X=x.



$$f_{X,Y}(x,y) = \frac{1}{2}\mathbb{1}\{0<x<y<2\}$$

$$f_{Y|X=x_0}(y) = \frac{f_{X,Y}(x_0,y)}{f_X(x_0)} = \frac{\frac{1}{\cancel{2}}\mathbb{1}\{0<x_0<y<2\}}{\frac{2-x_0}{\cancel{2}}\mathbb{1}\{0<x_0<2\}} \quad (*)$$

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dy = \int_x^2 \frac{1}{2}\,dy\;\mathbb{1}\{0<x<2\}$$

$$= \frac{y}{2}\Big|_x^2 \mathbb{1}\{0<x<2\} = \left(1-\frac{x}{2}\right)\mathbb{1}\{0<x<2\}$$

$$= \frac{(2-x)}{2}\mathbb{1}\{0<x<2\}$$

$(*)$

$$f_{Y|X=x_0}(y) = \frac{1}{2-x_0}\mathbb{1}\{x_0<y<2\}$$

$$Y|X=x_0 \sim \mathcal{U}(x_0,2)$$