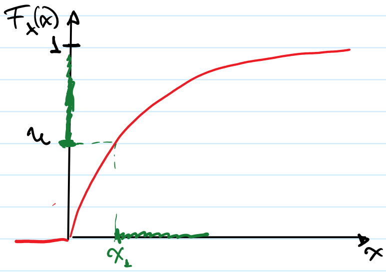
Método de la transformada inversa

Ejemplo: $X \sim Exp(\lambda)$, $F_X(x) = 1 - e^{-\lambda x}\,\mathbb{1}\{x>0\}$

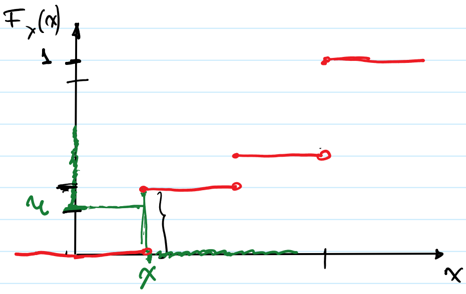



$u = F_X(x) = 1 - e^{-\lambda x}$, $U \sim \mathcal{U}(0;1)$

$x = \frac{\ln(1-u)}{-\lambda} = F_X^{-1}(u)$

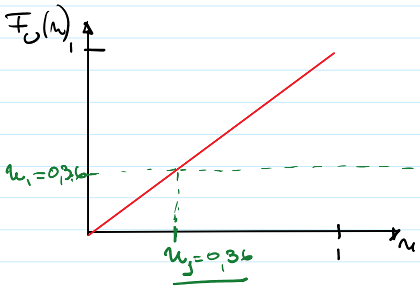
Definimos la inversa generalizada como:

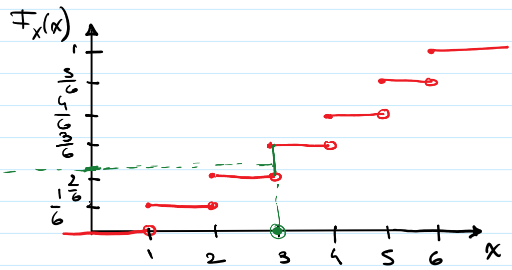
$$F_X^{-1}(u) = \min\{x \in \mathbb{R} : F_X(x) \geq u\},\ u \in (0,1)$$




Sea X el resultado de arrojar un dado equilibrado. A partir de 1000 realizaciones de una v.a. uniforme en el intervalo (0,1), simular 1000 realizaciones de X.

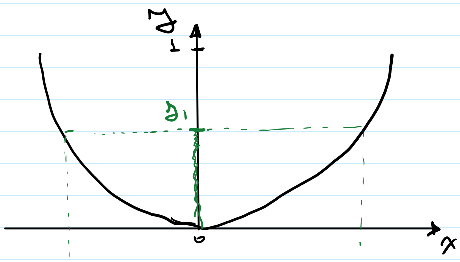
$U \sim \mathcal{U}(0;1)$




## Ejercicio 6

Sea X~U(-1,1), y sea Y=X$^2$. Hallar la función de densidad de Y



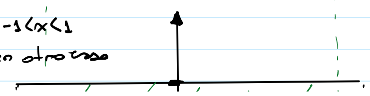

$F_Y(y) = P(Y \leq y) =$

$= P(X^2 \leq y) = P(|X| \leq \sqrt{y})$

$= P(-\sqrt{y} \leq X \leq \sqrt{y}) =$

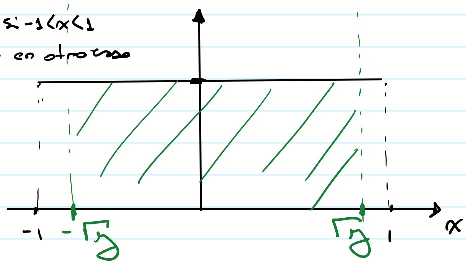
$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$.

$f_X(x) = \begin{cases} \frac{1}{2} & \text{si } -1<x<1 \\ 0 & \text{en otro caso} \end{cases}$



$F_X(x) = \begin{cases} 0 & \text{si } x < -1 \\ \frac{x+1}{2} & \text{si } -1 \leq x < 1 \end{cases}$

$f_X(x) = \begin{cases} \frac{1}{2} & \text{si } -1<x<1 \\ 0 & \text{en otro caso} \end{cases}$




$$F_X(x) = \begin{cases} 0 & \text{si } x<-1 \\ \frac{x+1}{2} & \text{si } -1 \leq x < 1 \\ 1 & \text{si } x \geq 1 \end{cases}$$

$$F_Y(y) = \frac{\sqrt{y}+1}{2} - \frac{-\sqrt{y}+1}{2} = \frac{2\sqrt{y}}{2} = \sqrt{y}$$

$$F_Y(y) = \begin{cases} 0 & \text{si } y<0 \\ \sqrt{y} & \text{si } 0 \leq y < 1 \\ 1 & \text{si } y \geq 1 \end{cases} \longrightarrow f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & \text{si } 0<y<1 \\ 0 & \text{en otro caso} \end{cases}$$

Otra forma: $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

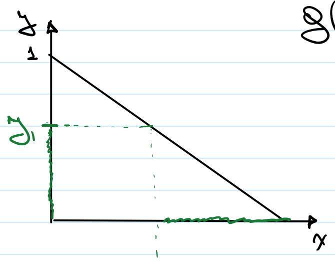
En gral.: $f_Y(y) = \dfrac{dF_Y(y)}{dy} \Rightarrow f_Y(y) = \dfrac{dF_X(g^{-1}(y))}{dx} \cdot \dfrac{d(g^{-1}(y))}{dy}$

$$Y = g(X) = X^2 \Rightarrow X = g^{-1}(Y) \Rightarrow X = \pm\sqrt{Y}$$

$$f_Y(y) = \frac{1}{2} \cdot \frac{1}{2\sqrt{y}} - \frac{1}{2} \cdot \left(-\frac{1}{2\sqrt{y}}\right) = \frac{1}{2\sqrt{y}} \mathbb{1}\{0<y<1\}$$

- Sea X una v.a.c. con función de densidad $f_X(x)$,
- Sea Y=g(X).
- g(x) es una función 1 a 1 (existe $g^{-1}(y)$)

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$



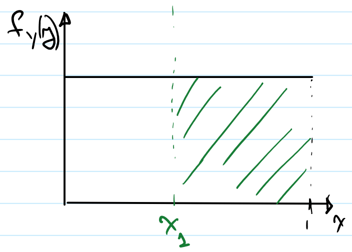

$g(X) = Y = 1-X \quad , \quad X \sim \mathcal{U}(0,1)$

$F_Y(y) = P(Y \leq y) = P(1-X \leq y) =$

$= P(X > 1-y)$

En gral.: $F_Y(y) = P(X > g^{-1}(y)) =$

$= 1 - F_X(g^{-1}(y))$




$$f_Y(y) = \frac{d[1-F_X(g^{-1}(y))]}{dx} \cdot \frac{d(g^{-1}(y))}{dy}$$

$$f_Y(y) = -f_X(g^{-1}(y)) \frac{d(g^{-1}(y))}{dy}$$

Va a resultar negativo

$$f_Y(y) = \underbrace{1}_{f_X(x)} \cdot \underbrace{|-1|}_{\frac{d(1-y)}{dy}} = \mathbb{1}\{0<y<1\}$$

## Ejercicio 7

Sean X e Y dos v.a. con distribución de Poisson de parámetros μ y λ respectivamente. Hallar la función de probabilidad de W = X + Y.

$$X \sim \mathcal{P}oi(\mu) \rightarrow p_X(x) = \frac{\mu^x}{x!}e^{-\mu},\ x \in \mathbb{N}_0$$

$$X \sim Po(\mu)\ ,\quad Y \sim Po(\lambda)\ ,\quad P_W(w) = ?$$

$$P_W(w) = P(W = w) = P(X+Y = w) = \sum_{x=0}^{w} P(X = x, Y = w-x) =$$

$$= \sum_{x=0}^{w} P(X=x) \cdot P(Y = w-x) = \sum_{x=0}^{w} \frac{\mu^x \cdot e^{-\mu}}{x!} \cdot \frac{\lambda^{(w-x)} \cdot e^{-\lambda}}{(w-x)!} =$$

$$= e^{-(\mu+\lambda)} \cdot \sum_{x=0}^{w} \frac{\mu^x \cdot \lambda^{(w-x)}}{x!\,(w-x)!} = \frac{e^{-(\mu+\lambda)}}{w!} \sum_{x=0}^{w} \frac{w!}{x!\,(w-x)!} \mu^x \cdot \lambda^{(w-x)} =$$

$$= \frac{e^{-(\mu+\lambda)}}{w!} \sum_{x=0}^{w} \frac{w!}{x!\,(w-x)!} \frac{\mu^x}{(\mu+\lambda)^x} \frac{\lambda^{(w-x)}}{(\mu+\lambda)^{(w-x)}} \cdot (\mu+\lambda)^x \cdot (\mu+\lambda)^{(w-x)} = \circledast$$

$$T \sim Bi(n, p)\ ,\quad P_T(t) = \binom{n}{t} \cdot p^t \cdot (1-p)^{n-t}$$

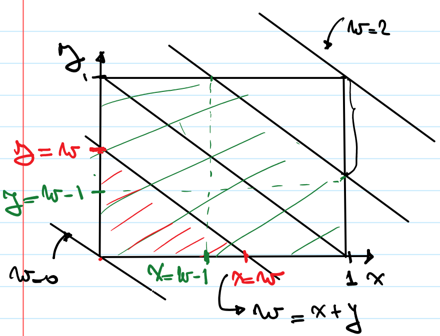
$$\circledast = \frac{e^{-(\mu+\lambda)}}{w!} \cdot (\mu+\lambda)^w \underbrace{\sum_{x=0}^{w} \frac{w!}{x!\,(w-x)!} \cdot \left(\frac{\mu}{\mu+\lambda}\right)^x \cdot \left(1 - \frac{\mu}{\mu+\lambda}\right)^{w-x}}_{=1} =$$

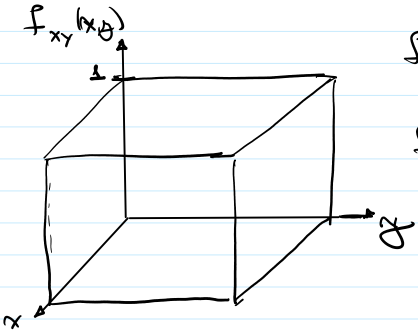
$$= \frac{e^{-(\mu+\lambda)}\,(\mu+\lambda)^w}{w!}$$

Entonces: $W \sim Po(\mu+\lambda)$

## Ejercicio 8

Sean X,Y ~ U(0,1) e independientes. Hallar la función de densidad de W = X+Y




$$f_{XY}(x;y) = f_X(x) \cdot f_Y(y)$$

$$f_{XY}(x,y) = \mathbb{I}\{0<x<1,\ 0<y<1\}$$

si $0 < w < 1$

$$F_W(w) = P(W \leq w) = \frac{w^2}{2}$$

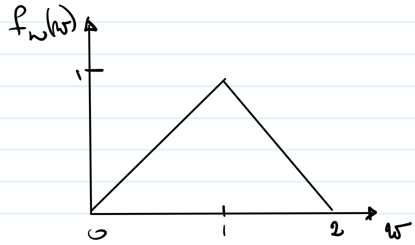
si $1 < w < 2$

$$F_W(w) = P(W \leq w) = 1 - \frac{[1-(w-1)]^2}{2} =$$

$$= 1 - \frac{(2-w)^2}{2}$$

$$F_W(w) = \begin{cases} 0 & \text{si } w < 0 \\ \frac{w^2}{2} & \text{si } 0 \leq w < 1 \\ 1 - \frac{(2-w)^2}{2} & \text{si } 1 \leq w < 2 \\ 1 & \text{si } w \geq 2 \end{cases} \Rightarrow f_W(w) = \begin{cases} w & \text{si } 0 < w < 1 \\ 2-w & \text{si } 1 \leq w < 2 \\ 0 & \text{en otro caso} \end{cases}$$

$$f_W(w) = w\,\mathbb{I}\{0 < w < 1\} + (2-w)\,\mathbb{I}\{1 \leq w < 2\}$$




Sean $X_1, X_2 \overset{i.i.d}{\sim} \mathcal{E}(\lambda)$ y sean $U = X_1 + X_2$ y $V = \frac{X_1}{X_1+X_2}$. Hallar $f_{U,V}(u,v)$ ¿Qué puede decir al respecto?

$X_1 \sim \mathcal{E}(\lambda)$ , $X_2 \sim \mathcal{E}(\lambda)$ , $\begin{cases} u = x_1 + x_2 > 0 \\ v = \dfrac{x_1}{x_1+x_2} > 0 \end{cases}$

$$f_{Y_1,Y_2} = f_{X_1,X_2}(x_1,x_2)\big|_{h_1^{-1}(u_1,u_2),h_2^{-1}(u_1,u_2)} \; |J|$$

$$f_{x_1x_2}(x_1,x_2) = f_{x_1}(x_1)\cdot f_{x_2}(x_2) = \lambda e^{-\lambda x_1}\cdot \lambda e^{-\lambda x_2}\, \mathbb{1}\{x_1>0, x_2>0\}$$

$$\begin{cases} u = x_1 + x_2 \Rightarrow x_2 = u - x_1 \Rightarrow x_2 = u - u\cdot v \\ v = \dfrac{x_1}{x_1+x_2} \Rightarrow v = \dfrac{x_1}{x_1 + u - x_1} = \dfrac{x_1}{u} \Rightarrow x_1 = u\cdot v \end{cases}$$

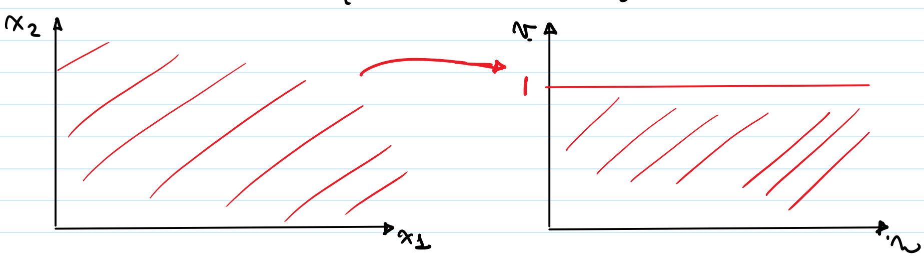
$$f_{U,V}(u;v) = f_{x_1x_2}(x_1;x_2)\Big|_{\substack{x_1 = h_1^{-1}(u;v) \\ x_2 = h_2^{-1}(u;v)}} \; |J|$$

$$|J| = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -uv - (1-v)u = -\cancel{uv} - u + \cancel{uv} = \\ = -u$$

$$f_{U,V}(u;v) = \lambda^2 \cdot e^{-\lambda(x_1+x_2)}\Big|_{\substack{x_1 = u\cdot v \\ x_2 = u - u\cdot v}} \; |u| \; \mathbb{1}\{u\cdot v>0;\ u - u\cdot v > 0\}$$

$$= u\cdot\lambda^2\cdot e^{-\lambda(u\cdot v + u - u\cdot v)} \; \mathbb{1}\{u>0;\ 0<v<1\} =$$

$$= u\cdot\lambda^2\cdot e^{-\lambda u} \; \mathbb{1}\{u>0;\ 0<v<1\} = \circledast$$




$$\circledast = \underbrace{\lambda^2 u e^{-\lambda u}\, \mathbb{1}\{u>0\}}\cdot \underbrace{\mathbb{1}\{0<v<1\}}$$

$$\circledast = \underbrace{\frac{\lambda^2}{1}\, u\, e^{-\lambda u}\, \mathbb{1}\{u>0\}}_{U \sim \Gamma(2;\lambda)} \cdot \underbrace{\mathbb{1}\{0<v<1\}}_{V \sim \mathcal{U}(0,1)}$$

Gamma

$$f_U(u) = \frac{\lambda^k}{\Gamma(k)} \cdot u^{k-1} \cdot e^{-\lambda u}\, \mathbb{1}\{u>0\} \Longrightarrow U \sim \Gamma(k, \lambda)$$