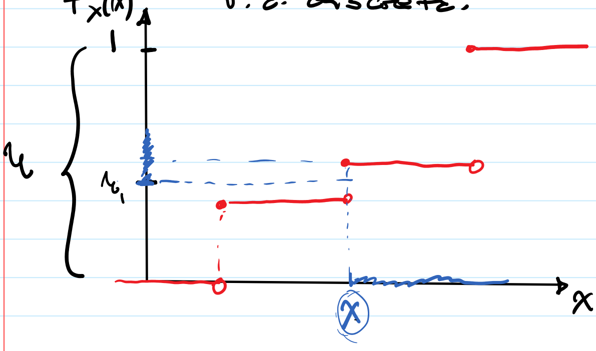
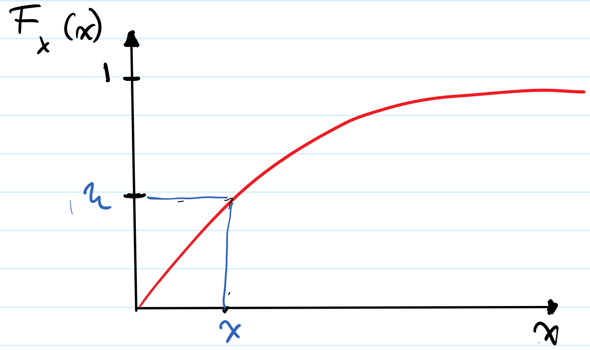
Definimos la inversa generalizada como:

$$F_X^{-1}(u) = \min\{x \in \mathbb{R} : F_X(x) \geq u\},\ u \in (0,1)$$

v. a. discreta.



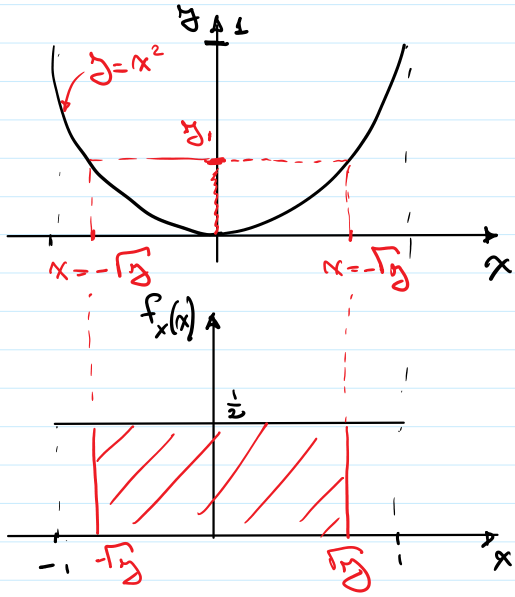


$$u = F_X(x) \Rightarrow u = 1 - e^{-\lambda x} \Rightarrow$$

$$x = \frac{\ln(1-u)}{-\lambda} = F^{-1}(u)$$

Ejercicio 6

Sea X~U(-1,1), y sea Y=X^2. Hallar la función de densidad de Y



$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) =$$
$$= P(-\sqrt{y} \leq X \leq \sqrt{y}) =$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}) =$$

$$F_X(x) = \begin{cases} 0 & \text{si } x < -1 \\ \frac{x+1}{2} & \text{si } -1 \leq x < 1 \\ 1 & \text{si } x \geq 1 \end{cases}$$

$$F_Y(y) = \frac{\sqrt{y}+1}{2} - \left(\frac{-\sqrt{y}+1}{2}\right) = \frac{2\sqrt{y}}{2} = \sqrt{y}$$

$$F_Y(y) = \begin{cases} 0 & \text{si } y < 0 \\ \sqrt{y} & \text{si } 0 \leq y < 1 \\ 1 & \text{si } y \geq 1 \end{cases} \longrightarrow \quad f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{2\sqrt{y}}$$

- $F_Y(y) = \sqrt{y}\,\mathbb{1}\{0 \leq y < 1\} + \mathbb{1}\{y \geq 1\}$

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & \text{si } 0 < y \leq 1 \\ 0 & \text{en otro caso.} \end{cases}$$

- $f_Y(y) = \frac{1}{2\sqrt{y}}\,\mathbb{1}\{0 < y < 1\}$

Ejercicio 7

Sean X e Y dos v.a. con distribución de Poisson de parámetros μ y λ respectivamente. Hallar la función de probabilidad de W = X + Y.

$$X \sim \mathcal{P}oi(\mu) \to p_X(x) = \frac{\mu^x}{x!}e^{-\mu},\ x \in \mathbb{N}_0$$

$\cdot X \sim \mathcal{P}o(\mu)$, $Y \sim \mathcal{P}o(\lambda)$ $\to$ X e Y son independientes

$$p_W(w) = P(W=w) = P(X+Y=w) = \sum_{x=0}^{w} P(X=x, Y=w-x) =$$

$$= \sum_{x=0}^{w} \frac{\mu^x \cdot e^{-\mu}}{x!} \cdot \frac{\lambda^{w-x} \cdot e^{-\lambda}}{(w-x)!} = e^{-(\mu+\lambda)} \sum_{x=0}^{w} \frac{\mu^x \cdot \lambda^{w-x}}{x! \, (w-x)!} =$$

$$= \frac{e^{-(\mu+\lambda)}}{w!} \sum_{x=0}^{w} \frac{w!}{x!\,(w-x)!} \cdot \mu^x \cdot \lambda^{w-x} =$$

$$= \frac{e^{-(\mu+\lambda)}}{w!} \sum_{x=0}^{w} \frac{w!}{x!\,(w-x)!} \cdot \left(\frac{\mu}{\mu+\lambda}\right)^x \cdot \left(\frac{\lambda}{\mu+\lambda}\right)^{w-x} \cdot (\mu+\lambda)^x \cdot (\mu+\lambda)^{w-x} = \circledast$$

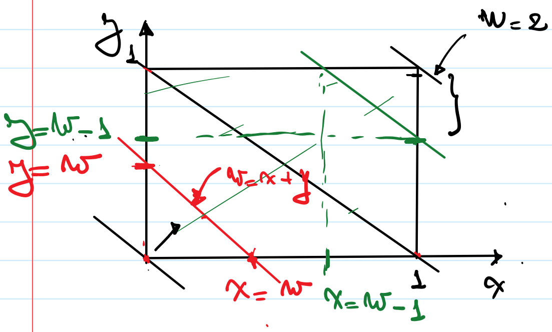
$$\frac{\mu}{\mu+\lambda} \qquad 1 - \frac{\mu}{\mu+\lambda} = \frac{\mu+\lambda-\mu}{\mu+\lambda} = \frac{\lambda}{\mu+\lambda}$$

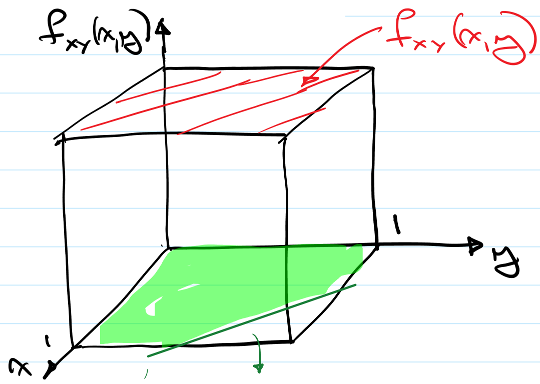
$$\circledast = \frac{e^{-(\mu+\lambda)}}{w!} \cdot (\mu+\lambda)^w \underbrace{\sum_{x=0}^{w} \frac{w!}{x!\,(w-x)!} \cdot \left(\overset{p}{\frac{\mu}{\mu+\lambda}}\right)^x \cdot \left(\frac{\lambda}{\mu+\lambda}\right)^{w-x}}_{=1\ ,\ X \sim Bi\left(w, \frac{\mu}{\mu+\lambda}\right)}$$

$$p_W(w) = \frac{(\mu+\lambda)^w}{w!} \cdot e^{-(\mu+\lambda)} \to W \sim \mathcal{P}o(\mu+\lambda)$$

Ejercicio 8

Sean X,Y ~ U(0,1) e independientes. Hallar la función de densidad de W = X+Y





$f_{XY}(x,y) = f_X(x) \cdot f_Y(y) =$

$= 1 \cdot 1 = 1$

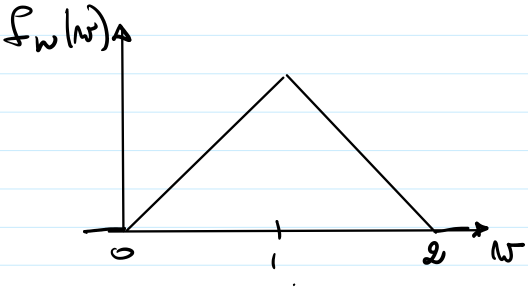
$$F_W(w) = \frac{w^2}{2} \longrightarrow \text{si } 0 < w < 1$$

$$F_W(w) = 1 - \frac{[1-(w-1)]^2}{2} \longrightarrow \text{si } 1 \leq w < 2$$

$$F_W(w) = 1 - \frac{[1-(w-1)]^2}{2} \longrightarrow \text{si } 1 \leq w < 2$$

$$F_W(w) = \frac{w^2}{2} \mathbb{1}\{0 < w < 1\} + 1 - \frac{(2-w)^2}{2} \mathbb{1}\{1 \leq w \leq 2\} + \mathbb{1}\{w \geq 2\}$$

$$f_W(w) = w \,\mathbb{1}\{0 < w < 1\} + (2-w)\,\mathbb{1}\{1 \leq w < 2\}$$

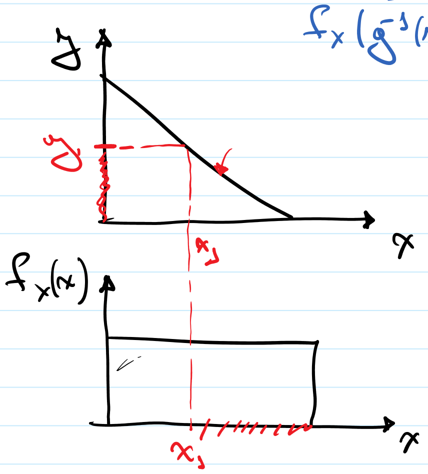


- Sea X una v.a.c. con función de densidad $f_X(x)$,
- Sea Y=g(X).
- g(x) es una función 1 a 1 (existe $g^{-1}(y)$)

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) =$$
$$= F_X(g^{-1}(y))$$

$$f_Y(y) = \underbrace{\frac{d F_X(\overbrace{g^{-1}(y)}^{x})}{dx}}_{f_X(g^{-1}(y))} \cdot \frac{d g^{-1}(y)}{dy}$$



$$F_Y(y) = P(Y \leq y) = 1 - P(X \leq g^{-1}(y)) =$$
$$= 1 - F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d[1 - F_X(g^{-1}(y))]}{dx} \cdot \frac{d \overbrace{g^{-1}(y)}^{x}}{dy}$$

$$f_Y(y) = -f_X(g^{-1}(y)) \cdot \frac{dg^{-1}(y)}{dy}$$

va a ser negativo porque X e Y tienen una relación inversa

Entonces en gral. : $f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{dg^{-1}(y)}{dy} \right|$

Método del Jacobiano

Ejercicio 1

Sean $X_1, X_2 \overset{i.i.d}{\sim} \mathcal{E}(\lambda)$ y sean $U = X_1 + X_2$ y $V = \frac{X_1}{X_1+X_2}$. Hallar $f_{U,V}(u,v)$ ¿Qué puede decir al respecto?

$u > 0$ y $v > 0$

$$X_1 \sim \mathcal{E}(\lambda), \quad X_2 \sim \mathcal{E}(\lambda) \to \text{ind.}, \quad f_{X_1X_2}(x_1,x_2) = \lambda e^{-\lambda x_1} \cdot \lambda e^{-\lambda x_2} = \lambda^2 \cdot e^{-\lambda(x_1+x_2)}$$

$$u = h_1(x_1, x_2), \quad v = h_2(x_1, x_2) \longrightarrow x_1 = h_1^{-1}(u,v), \quad x_2 = h_2^{-1}(u,v)$$

$$\begin{cases} u = x_1 + x_2 \Rightarrow x_1 = u - x_2 = u - u + uv = uv \\ v = \dfrac{x_1}{x_1 + x_2} \Rightarrow v = \dfrac{u - x_2}{u - x_2 + x_2} = \dfrac{u - x_2}{u} = 1 - \dfrac{x_2}{u} \Rightarrow \end{cases}$$

$$\Rightarrow x_2 = (1-v)\,u = u - u \cdot v$$

$$J = \det \begin{pmatrix} \overbrace{v}^{\frac{d h_1^{-1}(u,v)}{du}} & u \\ \underbrace{1-v}_{\frac{d h_2^{-1}(u,v)}{du}} & -u \end{pmatrix} = -uv - u(1-v) = -u$$

$$f_{UV}(u,v) = f_{X_1X_2}(x_1,x_2)\Big|_{\substack{x_1 = h_1^{-1}(u,v) \\ x_2 = h_2^{-1}(u,v)}} \cdot |J|$$

$$f_{UV}(u,v) = \lambda^2 \cdot e^{-\lambda(x_1+x_2)} \, \mathbb{1}\{x_1 > 0, x_2 > 0\} \Big|_{\substack{x_1 = uv \\ x_2 = u - uv}} |-u| \, .$$

$$f_{UV}(u,v) = \lambda^2 \cdot e^{-\lambda(uv + u - uv)} \cdot u \, \mathbb{1}\{u \cdot v > 0, \underbrace{u - uv > 0}\}$$

$$-v > -\frac{u}{u}$$

$$v < 1$$

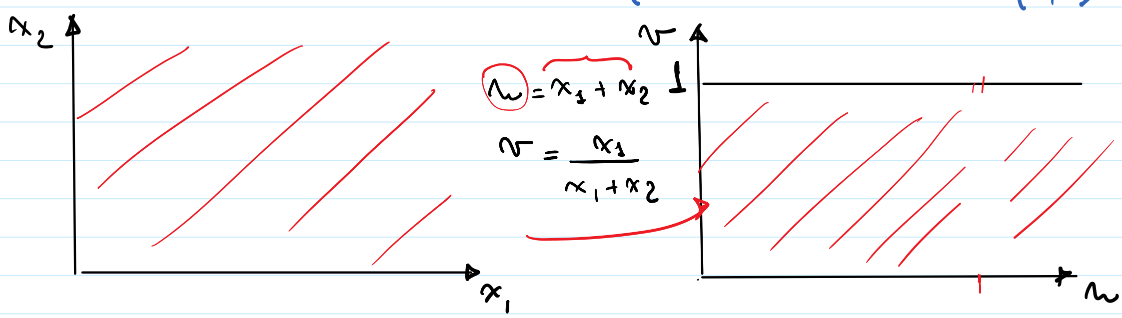
$$f_{UV}(u,v) = \underbrace{u \cdot \lambda^2 \cdot e^{-\lambda u}}_{\Gamma(2,\lambda)} \, \mathbb{1}\{u > 0, \; 0 < v < 1\}$$

$$Y \sim \Gamma(k, \lambda) \to f_Y(y) = \frac{\lambda^k}{\Gamma(k)} \cdot y^{k-1} \cdot e^{-\lambda y}$$

$$\Gamma(k) = (k-1)!$$

si es entero

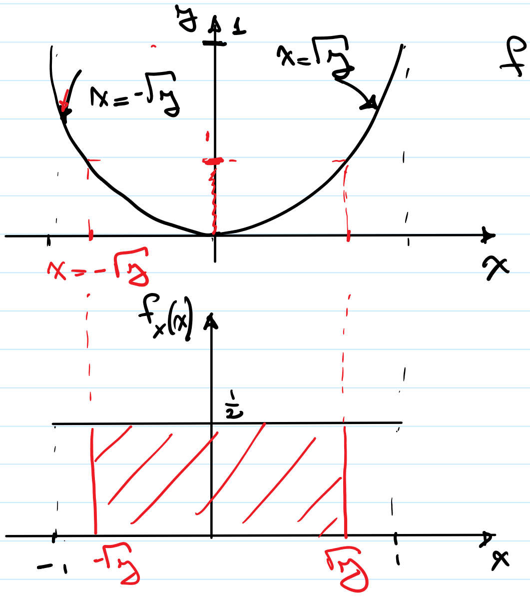
$$f_{UV}(u,v) = \underbrace{\lambda^2 \cdot u \cdot e^{-\lambda u} \, \mathbb{1}\{u > 0\}}_{U \sim \Gamma(2,\lambda)} \cdot \underbrace{1 \, \mathbb{1}\{0 < v < 1\}}_{V \sim \mathcal{U}(0,1)}$$



Ejercicio 2

Sea X~U(-1,1) e Y=X^2. Hallar por el método del Jacobiano generalizado la función de densidad de Y.

$Y = g(X) \longrightarrow$ no es biyectiva



$$f_Y(y) = \sum_{i=1}^{2} \frac{f_X(x)}{\left|\frac{dg(x)}{dx}\right|} \Bigg|_{x=g_i^{-1}(y)}$$

$$f_Y(y) = \frac{\frac{1}{2}}{|2x|}\Bigg|_{x=-\sqrt{y}} + \frac{\frac{1}{2}}{|2x|}\Bigg|_{x=\sqrt{y}} =$$

$$= \frac{1}{4\sqrt{y}} + \frac{1}{4\sqrt{y}} \, \mathbb{1}\{0<y<1\} =$$

$$= \frac{1}{2\sqrt{y}} \, \mathbb{1}\{0<y<1\}$$