1. Sean X e Y dos variables aleatorias independientes con distribución exponencial de parámetro λ. Hallar la función de densidad conjunta de $V = X + Y$ y $W = X/(X + Y)$. ¿Qué puede decir al respecto?

$\mathbb{P}(V \leq v) = \mathbb{P}(X+Y \leq v)$

1) $\mathbb{P}(V \leq v, W \leq w) = \mathbb{P}\left(X+Y \leq v, \frac{X}{X+Y} \leq w\right)$

2) Método del jacobiano.

WACKERLY

WASSERMAN

(5) S = transmite canal cist

$S = \begin{cases} 1 \\ 0 \end{cases}$ $P(S=1) = 0.6$, $P(S=0) = 0.4$

S: 0,6 / 0,4

Ruido $N \sim N(0, 0.8)$

$X = S + N$. Si $S=1 \Rightarrow x \sim N(1, 0.8)$ ✓ ; si $S=0 \Rightarrow x \sim N(0, 0.8)$.

$N(0,0.8)(0.7532) = \frac{1}{\sqrt{2\pi\, 0.8}} \cdot e^{-\frac{x^2}{2(0.8)}} = 0,312880$ (b)

$N(1,0.8)(0.7532) = \frac{1}{\sqrt{2\pi\, 0.8}} \cdot e^{-\frac{(x-1)^2}{2(0.8)}} = 0,492370$ (a)

$P(S=1 \mid X=0,7532) = \frac{f_{X|S=1}(0,7532) \cdot 0,6}{(a) \cdot 0,6 + (b)\, 0,4} = 0,702426$

$\boxed{f_X(x) = 0.6 \cdot N(1, 0.8) + 0.4\, N(0, 0.8)}$

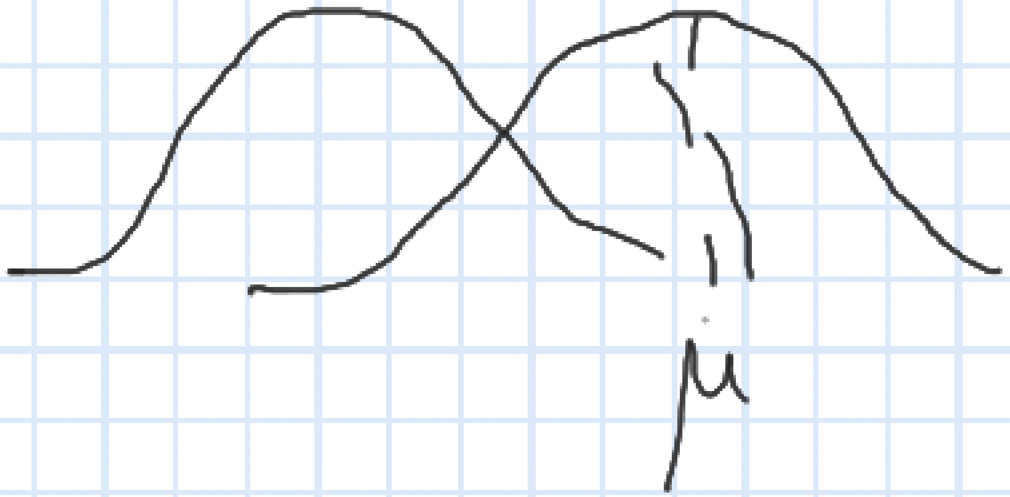
5. (Mezcla) En un sistema electrónico se debe determinar si se ha enviado señal o no. Se transmite señal ($S = 1$) con probabilidad 0.6. Además, por fabricación, el medio introduce un ruido (N) con distribución normal de media nula y varianza 0.8, independiente de lo que se envíe, de forma tal que se recibe $X = S + N$, con $S = \{0, 1\}$. Hallar la probabilidad de haber enviado una señal sabiendo que se recibió $X = 0.7532$.

$N \sim N(0, 0{,}8)$

$X = S + 1$ $S = \{0, 1\}$ $\mathbb{P}(S = 1) = 0{,}6$

$X|_{S=1} \sim N(1, 0{,}8)$

$X|_{S=0} \sim N(0, 0{,}8)$



4. La velocidad del viento (X) y el promedio de ozono en la atmósfera (Y), son dos variables aleatorias con función de densidad conjunta

$$f_{X,Y}(x,y) = \lambda e^{-\lambda x} \mu e^{-\mu x} y^{-(1+\mu)} I\{x > 0, y > e^{-x}\}$$

Motivación del ejercicio

Esta bien el ejercicio, sólo no llegabamos a ninguna distribución conocida :D (No hace falta hacer el truquito del logaritmo)

$Y|X = x$

$$\lambda e^{-\lambda x} \mu e^{-\mu x} y^{-1+\mu} I\{x>0, y > e^{-x}\}$$

$$\underbrace{\lambda e^{-\lambda x} I\{x>0\}}, \quad \mu e^{-\mu x} y^{-(1+\mu)} I\{y > e^{-x}\}.$$

$$y^{-(1+\mu)} = e^{\ln(y^{-(1+\mu)})} = e^{-(1+\mu)\ln(y)}$$

$$\mu e^{-\ln y - \mu(\ln(y) + x)}$$

$$I\{\ln(y) > \ln(e^{-x})\}$$

$$\ln(y) > -x$$

$$\mu e^{-\mu(\overset{w}{\ln(y)+x})} I\{\ln(y) > -x\}$$

$$\ln(Y) + x \sim \mathcal{E}(\mu)$$

$$\mathbb{E}\left[Y^2 \mid X\right] = \varphi(x)$$

$$\varphi(x) = \mathbb{E}\left[Y^2 \mid X = x\right]$$

$$\mathbb{V}\left(Y \mid X\right) = \mathbb{E}\left[Y^2 \mid X\right] - \mathbb{E}\left[Y \mid X\right]^2 \qquad \mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$= \sigma(X)$$

$Y \mid X = x \rightarrow$ dist conocida $\rightarrow \mathrm{Var}(Y \mid X = x) = \sigma(x)$

2. Sean $X \sim \mathcal{P}(\lambda)$ y $Y \sim \mathcal{P}(\mu)$ dos variables independientes. Hallar la distribución de $X|(X+Y)=m$. *Sugerencia: usar el resultado del ejercicio 5. de Transformaciones de variables.*

$$\mathbb{P}(X=x \mid X+Y=m) = \frac{\mathbb{P}(X=x,\ X+Y=m)}{\mathbb{P}(X+Y=m)} = \frac{\mathbb{P}(X=x,\ Y=m-x)}{\mathbb{P}(X+Y=m)}$$

$$= \frac{\mathbb{P}(X=x)\,\mathbb{P}(Y=m-x)}{\mathbb{P}(X+Y=m)}$$

$N(\mu, 9)$ $\qquad \hat{\mu} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

i.d

$$\mathbb{E}[\hat{\mu}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} \underbrace{\mathbb{E}[X_i]}_{\mu} = \frac{1}{n}\, n \cdot \mu = \mu$$

$\Rightarrow \hat{\mu}$ es INSESGADO ✓

$$B = 0 = \mathbb{E}[\hat{\mu} - \mu] = \mathbb{E}[\hat{\mu}] - \mu$$

$$\operatorname{var}(\hat{\mu}) = \operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\operatorname{var}\left(\sum_{i=1}^{n} X_i\right) \overset{\text{indep}}{=} \frac{1}{n^2}\sum_{i=1}^{n} \underbrace{\operatorname{var}(X_i)}_{9}$$

id

$$= \frac{1}{n^2}\, n \cdot 9 = \frac{9}{n} \xrightarrow{n \to \infty} 0$$

$B = 0$

$\operatorname{var}(\hat{\mu}) \xrightarrow{n\to\infty} 0$ $\Rightarrow \hat{\mu}$ es consistente

$ECM = \operatorname{var}(\hat{\mu}) + B(\mu)^2$ $\quad \hat{\mu}$ es " en media cuadrática

Ej. 1. $X|_{\Theta=\theta} \sim N(0, 1/\theta)$ $\qquad \pi_\Theta \sim \chi^2_8$ $\qquad \underline{X} = (X_1 \dots X_{10})$

$$f_{\Theta|\underline{X}=\underline{x}}(\theta) = \frac{f_{\underline{X}|\Theta=\theta}(\underline{x})\,\pi_\Theta(\theta)}{\int f_{\underline{X}|\Theta=\theta}(\underline{x})\,\pi_\Theta(\theta)\,d\theta} = (*)$$

$$T \sim \Gamma(\nu, \lambda)$$

$$\frac{1}{\Gamma(\nu)} \lambda^\nu t^{\nu-1} e^{-\lambda t} \mathbb{1}_{\{t>0\}}$$

$$\chi^2_8 = \Gamma(4, 1/2)$$

$$f_{\underline{X}|\Theta=\theta}(\underline{x}) = \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi \, 1/\theta}} e^{-\frac{1}{2}\frac{x_i^2}{1/\theta}} = \frac{\theta^{10}}{(2\pi)^5} e^{-\frac{\theta}{2}\sum_{i=1}^{10} x_i^2}$$

$$(*) \propto \frac{\theta^{10}}{\cancel{(2\pi)^5}} e^{-\frac{\theta}{2}\sum_{i=1}^{10} x_i^2} \frac{1}{\cancel{2^4\Gamma(4)}} \theta^3 e^{-\theta/2} \mathbb{1}_{\{\theta>0\}}$$

$$\propto \theta^{13} e^{-\frac{\theta}{2}\left(\sum_{i=1}^{10} x_i^2 + 1\right)} \mathbb{1}_{\{\theta>0\}}$$

$\Gamma(m) = (m-1)!$ $\quad m \in \mathbb{N}$

$$\lambda = \frac{1}{2}\left(\sum_{i=1}^{10} x_i^2 + 1\right)$$

$$\nu = 14$$

$$\Theta|\underline{X}=\underline{x} \sim \Gamma\left(14, \frac{1}{2}\left(\sum_{i=1}^{10} x_i^2 + 1\right)\right)$$

→ con la muestra $\Theta|\underline{X}=\underline{x} \sim \Gamma(14, 9)$

Ej 2 $\quad X_i|_{\mu=m} \overset{iid}{\sim} Po(m) \qquad \underline{X}=(X_1, \dots, X_{100}) \qquad \Pi_\mu(m) = \frac{1}{2} e^{-1/2\, m} \mathbb{1}\{m>0\}$

$$f_{\mu|\underline{X}=\underline{x}}(m) = \frac{P_{\underline{X}|\mu=m}(\underline{x})\, \Pi_\mu(m)}{\int P_{\underline{X}|\mu=m}(\underline{x})\, \Pi_\mu(m)\, dm} = \frac{\left[\prod_{i=1}^{100} \frac{1}{x_i!}\right] m^{\sum x_i} e^{-m100} \frac{1}{2} e^{-1/2\, m} \mathbb{1}\{m>0\}}{\int \text{- - - -}\, dm} \quad (*)$$

$\propto$

$$P_{\underline{X}|\mu=m}(\underline{x}) = \prod_{i=1}^{100} \frac{m^{x_i}}{x_i!} e^{-m} = \left[\prod_{i=1}^{100} \frac{1}{x_i!}\right] m^{\sum x_i} e^{-m \cdot 100}$$

$$\mathcal{E}(\lambda) = \Gamma(1, \lambda)$$

$$(*) \propto m^{\sum x_i} e^{-100.5\, m} \mathbb{1}\{m>0\}$$

$$\mu | \underline{X} = \underline{x} \sim \Gamma\left(\sum x_i + 1,\ 100,5\right) \rightsquigarrow$$

con los muestras

$\sum x_i = 212$

$$\mu | \underline{X} = \underline{x} \sim \Gamma(213, 100{,}5)$$

Error de Bayes p/ pérdida ℓ_2 del ej 1.

$$\mathbb{E}[\Theta \mid \underline{X}] = \varphi(\cancel{\underline{x}})$$

$$\varphi(x) = \mathbb{E}[\Theta \mid \underbrace{\underline{X} = \underline{x}}_{\Gamma(14,\,9)}] = \boxed{\frac{14}{9}}$$

evaluado en la muestra

$$\varphi(X) = \frac{14 \cdot 2}{\left(\sum X_i^2 + 1\right)} = \frac{14}{\left(\sum x_i^2 + 1\right)/2}$$

$$\mathbb{P}(X \in A) = \int_A \int_{\mathbb{R}} f_{X|\Theta=\theta}(x) \cdot f_{\Theta|\underline{X}=\underline{x}}(\theta)\, d\theta\, dx$$

En el contexto del Ej 2.

$$\mathbb{P}(X=0) = \int \mathbb{P}_{X|\mu=m}(0)\, f_{\mu|\underline{X}=\underline{x}}(m)\, dm.$$

$$= \int_0^\infty \frac{m^0}{0!} e^{-m} \cdot \frac{(100{,}5)^{213}}{\Gamma(213)}\, m^{212}\, e^{-m\,100{,}5}\, dm$$

$$e^{-m \cdot 101{,}5}$$

$$= \frac{(100{,}5)^{213}}{\cancel{(212)!}} \frac{\cancel{\Gamma(213)}}{101{,}5^{213}} \underbrace{\int_0^\infty \frac{101{,}5^{213}}{\Gamma(213)}\, m^{212}\, e^{-m\cdot 101{,}5}\, dm}_{1} = \left(\frac{100{,}5}{101{,}5}\right)^{213} = 0{,}1217$$

Es 5

1) $\underline{X} = (X_1, \dots, X_n) \qquad X_i \overset{iid}{\sim} \mathcal{E}(\lambda) \qquad \lambda > 0$

$$f_\lambda(\underline{x}) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} \mathbb{1}\{x_i > 0\} = \underbrace{\lambda^n e^{-\lambda \sum x_i}}_{g(T(\underline{x}), \lambda)} \underbrace{\prod_{i=1}^{n} \mathbb{1}\{x_i > 0\}}_{h(\underline{x})}$$

$$T(\underline{x}) = \sum_{i=1}^{n} x_i \Rightarrow T(\underline{X}) = \sum_{i=1}^{n} X_i$$

2) $$P_\theta(\underline{x}) = \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i} = \underbrace{\theta^{\sum x_i} (1-\theta)^{n - \sum x_i}}_{g(T(\underline{x}), \theta)} \cdot \underbrace{1}_{h(\underline{x})}$$

$$T(\underline{x}) = \sum_{i=1}^{n} x_i \Rightarrow T(\underline{X}) = \sum_{i=1}^{n} X_i$$

3) $$f_\theta(\underline{x}) = \prod_{i=1}^{n} \frac{1}{\theta} \mathbb{1}\{0 < x_i < \theta\} = \frac{1}{\theta^n} \underbrace{\prod_{i=1}^{n} \mathbb{1}\{0 < x_i < \theta\}}_{\mathbb{1}\{0 < \max(\underline{x}) < \theta\}} \cdot 1$$

$$\underbrace{\frac{1}{\theta^n}\, \mathbb{1}\{0 < \max(\underline{x}) < \theta\}}_{g(T(\underline{x}),\, \theta)} \cdot \underbrace{1}_{h(\underline{x}) = 1}$$

$$T(\underline{x}) = \max(\underline{x}) \Rightarrow T(\underline{X}) = \max(\underline{X})$$

Es 7 | $X_i \overset{iid}{\sim} Ber(p)$ $\underline{X} = (X_1 \dots, X_n)$ $p \in (0,1)$

$$L(p) = \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i} = p^{\sum x_i} (1-p)^{n-\sum x_i}$$

$$\frac{\partial \ln L(p)}{\partial p} = \frac{\partial}{\partial p}\left(\sum x_i \ln p + (n - \sum x_i) \ln(1-p) \right)$$

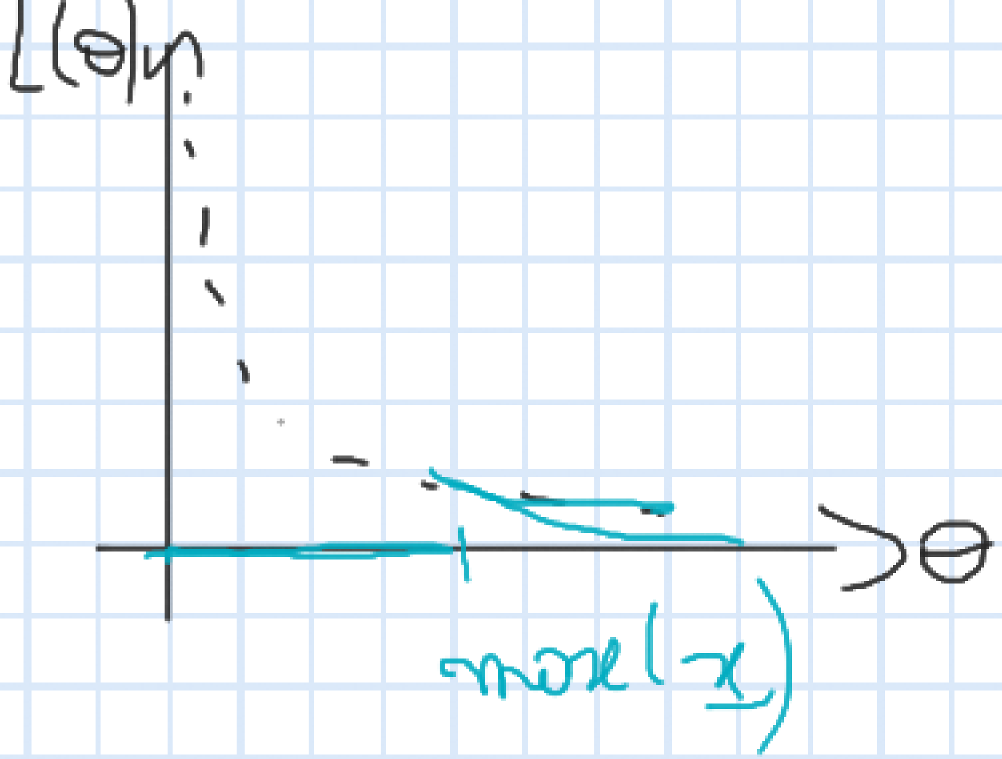
$$= \frac{\sum x_i}{p} - \frac{n - \sum x_i}{1-p} = 0$$

$$\frac{\sum x_i}{p} = \frac{n - \sum x_i}{1-p} \rightarrow$$

$$\sum x_i - p\cancel{\sum x_i} = np - p\cancel{\sum x_i}$$

$$\hat{p}(\underline{x}) = \frac{\sum x_i}{n} = \bar{x} \rightarrow \hat{p}(\underline{X}) = \bar{X}$$

Ej 8 | $L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} \mathbb{1}\{0 < x_i < \theta\} = \frac{1}{\theta^n} \mathbb{1}\{\max(\underline{x}) < \theta\} \cdot \mathbb{1}\{\min(\underline{x}) > 0\}$



$\hat{\theta}(\underline{x}) = \max(\underline{x})$

$\hat{\theta}(\underline{X}) = \max(\underline{X})$