Clase 2

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Sea $F:\mathbb{R}\to[0,1]$ una función de distribución, existe una variable aleatoria

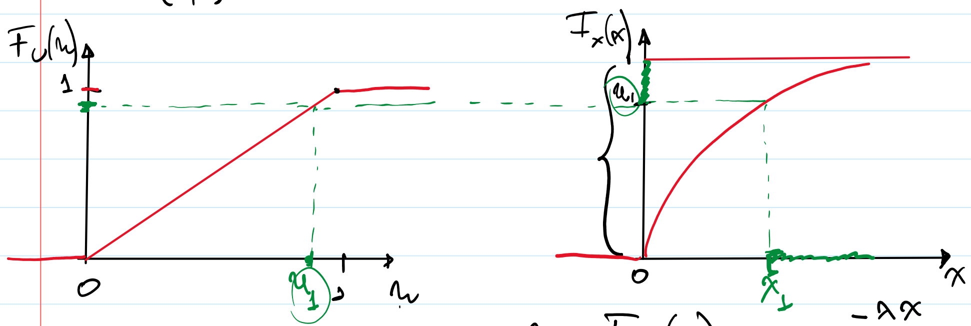
$$X/F(x)=\mathbb{P}(X\le x)$$

Definimos la inversa generalizada como:

$$F_X^{-1}(u)=\min\{x\in\mathbb{R}: F_X(x)\ge u\},\ u\in(0,1)$$

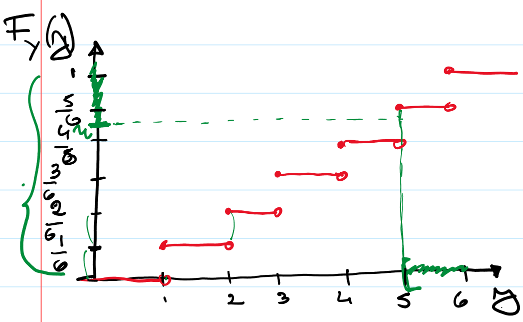
$U\sim \mathcal{U}(0,1)$

Ej: $X\sim \mathcal{E}xp(\lambda)$, $F_X(x)=1-e^{-\lambda}\mathbb{1}\{x>0\}$



$$u=F_X(x)=1-e^{-\lambda x}$$

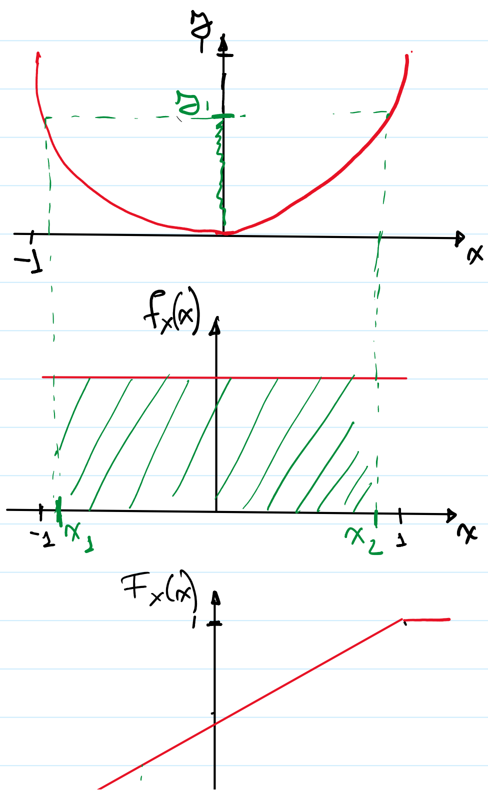
$$x=\frac{\ln(1-u)}{-\lambda}=F_X^{-1}(u)$$



$$F_X^{-1}(u)=\min\{x\in\mathbb{R}: F_X(x)\ge u\},\ u\in(0,1)$$

Ejercicio 6

Sea X~U(-1,1), y sea Y=X². Hallar la función de densidad de Y



$X\sim \mathcal{U}(-1,1)$, $Y=X^2=g(X)$

$F_Y(y)=P(Y\le y)=P(g(X)\le y)=$

$=P(X^2\le y)=P(|X|\le\sqrt{y})=$

$=P(-\sqrt{y}\le X\le\sqrt{y})=$

$=F_X(\sqrt{y})-F_X(-\sqrt{y})$ ⊗

$$F_X(x)=\begin{cases}0 & \text{si } x<-1\\ \frac{x+1}{2} & \text{si } -1\le x<1\\ 1 & \text{si } x\ge 1\end{cases}$$

$$f_X(x)=\begin{cases}\frac{1}{2} & \text{si } -1\le x<1\end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{si } -1 \leq x < 1 \\ 0 & \text{en otro caso.} \end{cases}$$

$$F_X(x) = \int_{-1}^{x} \frac{1}{2}\, dx = \frac{1}{2}(x+1)$$

$$\circledast \quad F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{\sqrt{y}+1}{2} - \frac{-\sqrt{y}+1}{2} =$$

$$= \sqrt{y}\ \mathbb{1}\{0 \leq y < 1\} + \mathbb{1}\{y \geq 1\}$$

$$f_Y(y) = \frac{1}{2\sqrt{y}}\ \mathbb{1}\{0 \leq y < 1\}$$

Otra forma: $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \underbrace{\frac{dF_X(\overset{x}{\sqrt{y}})}{dx}}_{f_X(\sqrt{y})} \cdot \frac{d(\sqrt{y})}{dy} - \underbrace{\frac{dF_X(\overset{x}{-\sqrt{y}})}{dx}}_{f_X(-\sqrt{y})} \cdot \frac{d(-\sqrt{y})}{dy} =$$

$$= \frac{1}{2}\ \frac{1}{2\sqrt{y}} - \frac{1}{2} \cdot \left(-\frac{1}{2\sqrt{y}}\right) = \frac{1}{2\sqrt{y}}\ \mathbb{1}\{0 \leq y \leq 1\}$$

Ejercicio 7

Sean X e Y dos v.a. con distribución de Poisson de parámetros μ y λ respectivamente. Hallar la función de probabilidad de W = X + Y.

$$X \sim \mathcal{P}oi(\mu) \rightarrow p_X(x) = \frac{\mu^x}{x!}e^{-\mu},\ x \in \mathbb{N}_0$$

$X \sim Poi(\mu)$, $Y \sim Poi(\lambda)$, $p_W(w) = ?$

$$p_W(w) = P(W = w) = P(X + Y = w) = \sum_{x=0}^{w} P(X = x; Y = w - x) =$$

$$= \sum_{x=0}^{w} P(X = x) \cdot P(Y = w - x) = \sum_{x=0}^{w} \frac{e^{-\mu} \cdot \mu^x}{x!} \cdot \frac{e^{-\lambda} \cdot \lambda^{w-x}}{(w-x)!} =$$

$X \sim Bi\left(w, \frac{\mu}{\mu+\lambda}\right)$

$$= \frac{e^{-(\mu+\lambda)}}{w!} \sum_{x=0}^{w} \frac{w!}{x!(w-x)!} \cdot \mu^x \cdot \lambda^{w-x} = \frac{e^{-(\mu+\lambda)}}{w!} \underbrace{\sum_{x=0}^{w} \frac{w!}{x!(w-x)!} \left(\frac{\mu}{\mu+\lambda}\right)^x \cdot \left(\frac{\lambda}{\mu+\lambda}\right)^{w-x}}_{=1} \cdot$$

$$\cdot (\mu+\lambda)^x \cdot (\mu+\lambda)^{w-x} =$$

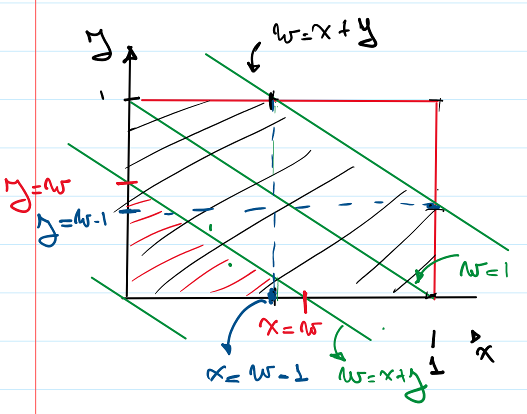
$T \sim Bi(n; p)$, $p_T(t) = \binom{n}{t} \cdot p^t \cdot (1-p)^{n-t}$

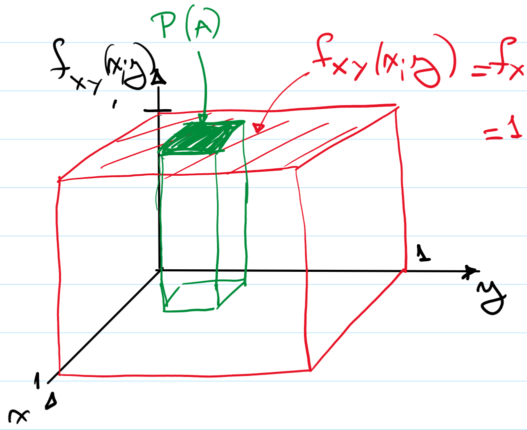
$$= \frac{e^{-(\mu+\lambda)}}{w!} \cdot (\mu+\lambda)^w = p_W(w)$$

Entonces: $W \sim Poi(\mu+\lambda)$.

Ejercicio 8

Sean X,Y ~ U(0,1) e independientes. Hallar la función de densidad de W = X+Y





$f_{XY}(x;y) = f_X(x) . f_Y(y) =$
$= 1 . 1 . \mathbb{1}\{0 \leq x < 1, 0 \leq y < 1\}$

$$F_W(w) = P(W \leq w)$$

- Si $0 \leq w \leq 1$: $F_W(w) = \frac{w^2}{2} . 1 \ \mathbb{1}\{0 \leq w \leq 1\}$
- Si $1 < w < 2$: $F_W(w) = 1 - \frac{[1-(w-1)]^2}{2} = 1 - \frac{(2-w)^2}{2} \mathbb{1}\{1 \leq w < 2\}$

$$F_W(w) = \frac{w^2}{2} \mathbb{1}\{0 \leq w < 1\} + \frac{2-(2-w)^2}{2} \mathbb{1}\{1 \leq w < 2\} + \mathbb{1}\{w \geq 2\}$$

$$F_W(w) = \begin{cases} 0 & \text{si } w < 0 \\ \frac{w^2}{2} & \text{si } 0 \leq w < 1 \\ \frac{2-(2-w)^2}{2} & \text{si } 1 \leq w < 2 \\ 1 & \text{si } w \geq 2 \end{cases} \qquad f_W(w) = \begin{cases} w & \text{si } 0 \leq w < 1 \\ 2-w & \text{si } 1 \leq w < 2 \\ 0 & \text{en otro} \end{cases}$$

Ejercicio 1

Sean $X_1, X_2 \overset{i.i.d}{\sim} \mathcal{E}(\lambda)$ y sean $U = X_1 + X_2$ y $V = \frac{X_1}{X_1+X_2}$. Hallar $f_{U,V}(u,v)$ ¿Qué puede decir al respecto?

`X_1, X_2 \overset{i.i.d}{\sim} \mathcal{E}(\lambda)`

Por X e Y ind.

$X_1 \sim \mathcal{E}xp(\lambda)$, $X_2 \sim \mathcal{E}xp(\lambda)$, $f_{X_1X_2}(x_1;x_2) \overset{\downarrow}{=} f_{X_1}(x_1) . f_{X_2}(x_2) =$

$u = h_1(x_1;x_2)$, $v = h_2(x_1;x_2)$

$= \lambda e^{-\lambda x_1} . \lambda e^{-\lambda x_2} \mathbb{1}\{x_1 > 0; x_2 > 0\} =$

$= \lambda^2 . e^{-\lambda(x_1+x_2)} \mathbb{1}\{x_1 > 0; x_2 > 0\}$

$u = h_1(x_1; x_2)$, $v = h_2(x_1; x_2)$ $\quad = \lambda e \cdot \lambda e \; \mathbb{1}\{x_1 > 0; x_2 > 0\} =$

$= \lambda^2 . e^{-\lambda(x_1+x_2)} \; \mathbb{1}\{x_1 > 0; x_2 > 0\}$

$$\begin{cases} u = x_1 + x_2 \Rightarrow x_2 = u - x_1 = u - u.v = h_2^{-1}(u; v) \\ v = \dfrac{x_1}{x_1 + x_2} \Rightarrow v = \dfrac{x_1}{x_1 + u - x_1} = \dfrac{x_1}{u} \Rightarrow x_1 = u.v = h_1^{-1}(u; v) \end{cases}$$

- $u \geq 0$ y $v \geq 0$

$$\det(J) = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -uv - u(1-v) = -u$$

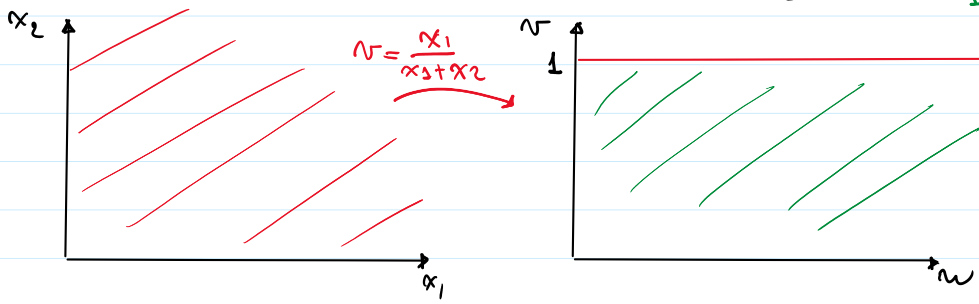
$$f_{U,V}(u; v) = \lambda^2 e^{-\lambda(x_1+x_2)} \mathbb{1}\{x_1 \geq 0; x_2 \geq 0\} \Big|_{\substack{x_1 = u.v \\ x_2 = u - u.v}} |-u| =$$

$$= \lambda^2 . e^{-\lambda(u.v + u - u.v)} \mathbb{1}\{u.v \geq 0; \underbrace{u - u.v}_{u.(1-v)} \geq 0\} . u =$$

$$= \lambda^2 . u . e^{-\lambda u} \mathbb{1}\{u \geq 0; 0 \leq v \leq 1\}$$

$\hookrightarrow 1 - v \geq \frac{0}{u}$

$1 \geq v$



$Y \sim \Gamma(\lambda, k)$, $f_Y(y) = \dfrac{\lambda^k}{\Gamma(k)} . y^{k-1} . e^{-\lambda y} \mathbb{1}\{y > 0\} \longrightarrow$ Distribución Gamma

$f_V(v) =$

$$f_{U,V}(u; v) = \underbrace{\frac{\lambda^2}{(2-1)!} u e^{-\lambda u} \mathbb{1}\{u \geq 0\}}_{U \sim \Gamma(\lambda; 2)} . \underbrace{1 . \mathbb{1}\{0 \leq v \leq 1\}}_{V \sim \mathcal{U}(0; 1)}$$

- Si k es un entero, entonces: $\Gamma(k) = (k-1)!$