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Stochastic distributions and the multiplicative model: relations, properties, estimators and applications to SAR image analysis

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ABSTRACT

The K distribution has been used as a flexible tool for the modelling of SAR data over non–homogeneous areas. It is characterized by three real–valued parameters; one of these parameters, the number of looks, is related to the kind of processing the raw data suffer in order to become an image. This distribution has been mostly used for one look data. In this paper the multilook case is considered for both quadratic and linear detections. A closed (recursive) computational form is provided for the K cumulative distribution function, as well as the estimators derived from the substitution method. Its limiting properties are proved using the multiplicative model representation. The sensitivity of the cumulative distribution function, with respect to possible discretizations of the parameters due to limitations imposed by the recursive form, is discussed. The recursive form of the cumulative distribution function of K multilook random variables is used to perform the Kolmogorov–Smirnov test of goodness of fit over SAREX data. It is shown that, mainly for forest data, the fit with the K multilook distribution is superior to some of other distributions that frequently appear in the literature. Specifically, the use of the normal distribution for this kind of data is discarded systematically.

Keywords: image analysis, K distributions, multiplicative model, statistical computing, synthetic aperture radar.

1. INTRODUCTION AND NOTATION

The precise knowledge of the statistical properties of SAR data plays a central role in image processing and understanding (7), (23). These properties can be used to discriminate different types of land use (see, for instance, (2), (3), (28), (29)).

Several studies have been conducted in order to relate physical features and statistical properties of SAR data ((19), (30)). In order to do this, some distributions are considered.

For 1–look data and homogeneous targets, a common hypothesis is the Exponential and Rayleigh distributions, for quadratic and linear detections, respectively. When the observed region cannot be assumed as homogeneous, other distributions are considered. Among these, the K distribution has received attention in the literature (see, for example, (11), (12), (13), (15)).

The statistical properties of multilook data are not completely known and, in some cases, their complexity offers theoretical and computational difficulties. In many cases, it is easier to derive the statistical properties of intensity data rather than amplitude data. For instance, the intensity speckle noise modelled as the sum of independent and exponentially distributed random variables has a well known distribution, the Gamma distribution, but this is not the case for amplitude speckle noise, since the convolution of Rayleigh distributions has no closed form.

Many authors (see (8), (22) and the references therein) have demonstrated the usefulness and the adequacy of multiplicative models for SAR data. In this work multiplicative models are used to carefully derive, in a unified notation, some suitable distributions for homogeneous and non–homogeneous regions with one and multiple looks.

The K distributions arise, within the framework of multiplicative models, in the modelling of non–homogeneous areas. Another way of deriving these distributions is through the use of *mixture models* (24); the equivalence between these two approaches is immediate from the notation used in this paper. Rayleigh and Exponential distributions are limit cases of amplitude and intensity K distributions, respectively; a proof of these limiting properties is provided, using the multiplicative model and a limiting property of Gamma distributions.

In order to use the K distributions for SAR data analysis, a closed form of their cumulative distribution functions is needed in many cases. A recursive form of these functions is presented. This form requires the discretization of one of the parameters of these distributions. The problems encountered in implementing the cumulative distribution functions of K distributions are commented upon, and the sensitivity of these functions with respect to the discretization of their parameters is studied.

The moment estimators of those parameters, as well as the scale properties of these distributions are provided. Finally, it is shown how these distributions are useful to the fitting of non–homogeneous areas of SAR images.

2. NOTATIONAL CONVENTIONS

The indicator function of the set A is defined, and denoted, as:

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else.} \end{cases}$$

The set of the real numbers is denoted $\mathbb{R}$, the set of positive real numbers $\mathbb{R}_+$, and the set of natural numbers $\mathbf{N}$.

Let $x \in \mathbb{R}$ be a number. The biggest integer smaller than or equal to x will be denoted as $\lfloor x \rfloor = \max\{k \in \mathbf{N} : k \leq x\}$. The closest integer to x will be defined, and denoted, as $[x] = \lfloor x + 1/2 \rfloor$.

The Gamma function, defined for every $\nu > -1$, is given by:

$$\Gamma(\nu + 1) = \int_{\mathbb{R}_+} t^{\nu} \exp(-t) dt.$$

It is immediate that if $\nu \in \mathbf{N}$, then $\Gamma(\nu + 1) = \nu!$. The non–italicized letter "Γ" shall be used to denote the Gamma distribution of random variables.

Random variables will be denoted with uppercase letters, and their outcomes with lowercase letters. A common underlying probability space will be assumed throughout this work: $(\Omega, \mathcal{A}, \Pr)$, where Ω denotes the sample set, $\mathcal{A}$ its σ–algebra, and "Pr" a probability. Therefore, real–valued random variables are measurable functions of the form $X : \Omega \to \mathbb{R}$.

The r–th order moment ($r \in \mathbb{R}$) of the real random variable X, if it exists, is defined and denoted as:

$$\mathrm{E}(X^r) = \int_{\mathbb{R}} x^r \, dF_X(x),$$

where, for every $x \in \mathbb{R}$, $F_X(x) = \Pr\{\omega \in \Omega : X(\omega) \leq x\}$ is the cumulative distribution function of the random variable X.

Other interesting quantities of the random variable X, if they exist, are its coefficients of skewness and kurtosis (denoted γ_1 and γ_2) given, respectively, by:

$$\begin{aligned} \gamma_1 &= \frac{\mathrm{E}(X - \mathrm{E}(X))^3}{(\mathrm{Var}(X))^{3/2}}, \quad \text{and} \\ \gamma_2 &= \frac{\mathrm{E}(X - \mathrm{E}(X))^4}{(\mathrm{Var}(X))^2} - 3, \end{aligned} \tag{1}$$

where $\mathrm{Var}(X) = \mathrm{E}(X - \mathrm{E}(X))^2$ is the variance of X.These values relate the symmetry and heavy–tailedness (how quickly the tails of the density fall off to zero) of its distribution (18). For normally distributed random variables, these quantities are identically zero.

The characteristic function of the random variable X is given, for every $t \in \mathbb{R}$, by $\phi(t) = \mathrm{E}(\exp(iXt))$, where $i = \sqrt{-1}$.

A useful estimation technique, the moment or substitution estimators, is based on the r–th order sample moments of a sample of size $n \geq r$, given by:

$$\hat{m}_r = \sum_{i=1}^{n} x_i^r.$$

3. SOME USEFUL DISTRIBUTIONS AND THEIR PROPERTIES

In this Section some distributions and their properties shall be either recalled or stated in the form of definitions, in order to maintain an uniform notation throughout this work. For a comprehensive account of random variables, their properties and their relations, the reader is referred to ((14), (17)).

Def. 1: The random variable U is said to be normally distributed with mean $\mu \in \mathbb{R}$ and variance σ^2, $\sigma \in \mathbb{R}_+$, if its density is given, for every $x \in \mathbb{R}$, by:

$$f_U(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right].$$

This case is denoted $U \sim \mathcal{N}(\mu,\sigma^2)$. It is immediate that if $U' \sim \mathcal{N}(\mu,\sigma^2)$ then $U = \sigma^{-1}(U' - \mu) \sim \mathcal{N}(0,1)$, which is called standard normal distribution.

Def. 2: The random variable U is said to be degenerated at $k \in \mathbb{R}$, or constantly equal to the real value k, if its cumulative distribution function is given, for every $x \in \mathbb{R}$, by $F_U(x) = 1_{[k,\infty)}(x)$. This case is denoted $U \sim \mathcal{C}(k)$. This somewhat awkward definition of a constant will be useful to keep an uniform notation along this work.

Def. 3: The random variable U is said to be (standard) exponentially distributed with unitary mean if its density, for every $x \in \mathbb{R}$, is given by:

$$f_U(x) = \exp(-x)1_{\mathbf{R}_+}(x).$$

This case is denoted $U \sim \mathcal{E}(1)$.

Def. 4: Let $U' \sim \mathcal{E}(1)$ and $\lambda \in \mathbb{R}_+$. The random variable $U = \lambda^{-1}U'$ is said to be exponentially distributed with parameter λ. Its density, for every $x \in \mathbb{R}$, is given by:

$$f_U(x;\lambda) = \big(\lambda\exp(-\lambda x)\big)1_{\mathbf{R}_+}(x),$$

therefore, its cumulative distribution function is given, for every $x \in \mathbb{R}$, by:

$$F_U(x;\lambda) = \big(1 - \exp(-\lambda x)\big)1_{\mathbf{R}_+}(x).$$

This case is denoted $U \sim \mathcal{E}(\lambda)$, and its r–th order moments are given by $\mathrm{E}(U^r) = \lambda^{-r}\Gamma(r+1)$. This distribution forms a scale family of distribution functions.

Def. 5: Let $U' \sim \mathcal{E}(1)$. The random variable $U = \sqrt{U'}$ is said to be standard Rayleigh distributed, and denoted $U \sim \mathcal{R}(1)$. Its density is given, for every $x \in \mathbb{R}$, by:

$$f_U(x;\lambda) = 2x\exp(-x^2)1_{\mathbf{R}_+}(x).$$

Def. 6: Let $U' \sim \mathcal{R}(1)$ and $\lambda \in \mathbb{R}_+$. The random variable $U = \lambda U'$ is said to be Rayleigh distributed with parameter $\theta = \lambda^{-2}$ and denoted $U \sim \mathcal{R}(\theta)$. Its density, for every $x \in \mathbb{R}$, is given by:

$$f_U(x;\theta) = 2\theta x \exp(-\theta x^2) 1_{\mathbb{R}_+}(x);$$

therefore, its cumulative distribution function is given, for every $x \in \mathbb{R}$, by:

$$F_U(x;\theta) = (1 - \exp(-\theta x^2) 1_{\mathbb{R}_+}(x).$$

Its r–th order moments are given by $\mathrm{E}(U^r) = \theta^{-r/2}\Gamma(r/2 + 1)$.

Def. 7: Let $n \in \mathbf{N}$ be a positive integer and let $(U_i)_{1 \le i \le n}$ be a collection of independent identically distributed random variables, such that $U_i \sim \mathcal{E}(1)$. Then, the random variable defined by the sum $U = \sum_{i=1}^{n} U_i$ is said to have a standard Gamma distribution with parameter n, and this case is denoted $U \sim \Gamma(n, 1)$. Its density is given, for every $x \in \mathbb{R}$, by:

$$f_U(x; n, 1) = \frac{x^{n-1}}{\Gamma(n)} \exp(-x) 1_{\mathbb{R}_+}(x).$$

Def. 8: Let $U' \sim \Gamma(n, 1)$ and $\lambda \in \mathbb{R}_+$. The random variable $U = \lambda^{-1}U'$ is said to be Gamma distributed with parameters n and λ, and its density, for every $x \in \mathbb{R}$, is given by:

$$f_U(x; n, \lambda) = \frac{\lambda^n}{\Gamma(n)} x^{n-1} \exp(-\lambda x) 1_{\mathbb{R}_+}(x).$$

This case is denoted $U \sim \Gamma(n,\lambda)$, and its r–th order moments are given by $\mathrm{E}(U^r) = \Gamma(n + r)/(\lambda^r\Gamma(n))$. In general, there is no closed form for its cumulative distribution function. More generally, the parameter n can be any positive number. A useful property of this distribution can be derived using its characteristic function, which is given by:

$$\phi(t) = \left(\frac{\lambda}{\lambda - it}\right)^n.$$

Consider the sequence of random variables $(U_n)_{n \ge 1}$, with $U_n \sim \Gamma(n,\lambda_n)$; it can be proved that if $\lambda_n \to \infty$ such that $n/\lambda_n \to \xi \in \mathbb{R}$ when $n \to \infty$, then:

$$\phi_n(t) = \left(\frac{\lambda_n}{\lambda_n - it}\right)^n \to \exp(it\xi).$$

Since $\exp(it\xi)$ is the characteristic function of a $\mathcal{C}(\xi)$ random variable, the aforementioned result proves that the sequence $(U_n)_{n \ge 1}$ converges in distribution to a $\mathcal{C}(\xi)$ distributed random variable. It can be seen in (14) that when a sequence of random variables converges in distribution to a constant, also convergence in probability holds.

An alternative way of proving this convergence, through the use of the Central Limit Theorem, is the following: assume $U_n \sim \Gamma(n,\lambda_n)$, with $\lambda_n \in \mathbb{R}_+$ for every $n \in \mathbf{N}$; therefore the random variable U_n can be written, by Def. 7 and Def. 8, as $U_n = \lambda_n^{-1}U_n'$, with $U_n' = \sum_{i=1}^{n} U_i$ and $U_i \sim \mathcal{E}(1)$ independent random variables. The mean $U_n'' = n^{-1}\sum_{i=1}^{n} U_i$ has unitary mean and variance equal to n, and it is immediate

that $\sqrt{n}(U_n'' - 1) \to Y$ in distribution with $Y \sim \mathcal{N}(0, 1)$, when $n \to \infty$. Then, also holds that $U_n'' \to 1$ in probability (14). Therefore, if $n\lambda_n^{-1} \to \xi$ as $n \to \infty$, then $U_n = n\lambda_n^{-1}U_n'' \to Y'$, with $Y' \sim \mathcal{C}(\xi)$, as $n \to \infty$. An intuitive interpretation of this result is the following (5): if a sufficiently large number of independent exponentially distributed random variables T_i, each with mean λ_n^{-1}, is added then the distribution of $n^{-1/2}\lambda_n\left(\sum_{i=1}^n T_i - n\lambda_n^{-1}\right)$ is approximately $\mathcal{N}(0, 1)$ or, equivalently, $U_n = \sum_{i=1}^n T_i$ has approximately a $\mathcal{N}(n\lambda_n^{-1}, n\lambda_n^{-2})$ distribution.

The coefficients of skewness and kurtosis (equations (1)) of $\Gamma(n,\lambda)$ distributed random variables are given by:

$$\gamma_1 = \frac{2}{\sqrt{n}}, \quad \text{and} \quad \gamma_2 = \frac{6}{n}.$$

These values, that do not depend on the parameter λ, show that, for sufficiently large values of n, $\Gamma(n, \cdot)$ distributed random variables are indistinguishable from normally distributed ones, from the skewness and kurtosis points of view.

Def. 9: Let $U' \sim \Gamma(n, 1)$, then, the random variable $U = \sqrt{U'}$ is said to have a standard Square Root of Gamma distribution. This case is denoted $U \sim \sqrt{\Gamma}(n, 1)$. Its density, for every $x \in \mathbb{R}$, is given by:

$$f_U(x; n, 1) = \frac{2}{\Gamma(n)}x^{2n-1}\exp(-x^2)1_{\mathbb{R}_+}(x).$$

Def. 10: Let $U' \sim \sqrt{\Gamma}(n, 1)$ and $\lambda \in \mathbb{R}_+$. The random variable $U = \lambda^{-1/2}U'$ is said to be Square Root of Gamma distributed with parameters n and λ, and its density is given, for every $x \in \mathbb{R}$, by:

$$f_U(x; n, \lambda) = \frac{2\lambda^n}{\Gamma(n)}x^{2n-1}\exp(-\lambda x^2)1_{\mathbb{R}_+}(x).$$

This case is denoted $U \sim \sqrt{\Gamma}(n,\lambda)$, and its r–th order moments are given by $\mathrm{E}(U^r) = \Gamma(n + r/2)/(\lambda^{r/2}\Gamma(n))$. In general, there is no closed form for its cumulative distribution function. Another way of defining a $U \sim \sqrt{\Gamma}(n,\lambda)$ random variable is the following: let $U' \sim \Gamma(n,\lambda)$, then $U = \sqrt{U'}$. Using Slutsky's theorem (14) on the convergence property mentioned in Def. 8 for a sequence of Gamma distributed random variables, it can be proved that if $\lambda_n \to \infty$ such that $n/\lambda_n \to \xi \in \mathbb{R}$ when $n \to \infty$, then the sequence $(U_n)_{n\geq 1}$ of $\sqrt{\Gamma}(n,\lambda_n)$ distributed random variables converges in probability to a $\mathcal{C}(\sqrt{\xi})$ distributed random variable. Also, as presented for $\Gamma(n,\lambda_n)$ distributed random variables, the intuitive counterpart of this result can be posed in the following manner: for sufficiently

large values of n, $U_n \sim \sqrt{\Gamma}(n, \lambda_n)$ has approximately a $\mathcal{N}(\mu_n, \sigma_n^2)$ distribution, with $\mu_n = \Gamma(n + 1/2)/(\lambda_n^{1/2}\Gamma(n))$ and $\sigma_n^2 = \lambda_n^{-1}(n - \Gamma^2(n + 1/2)/\Gamma^2(n))$.

The coefficients of skewness and kurtosis (equations (1)) of $\sqrt{\Gamma}(n, \lambda)$ distributed random variables are given by:

$$\gamma_1 = \frac{\Gamma(n + 1/2)\left(\Gamma^2(n) - 4n\Gamma^2(n) + 4\Gamma^2(n + 1/2)\right)}{2\Gamma^3(n)\left(n - \frac{\Gamma^2(n+1/2)}{\Gamma^2(n)}\right)^{3/2}}, \quad \text{and}$$

$$\gamma_2 = \frac{n - 2n^2 - \frac{2\Gamma^2(n+1/2)}{\Gamma^2(n)} + \frac{8n\Gamma^2(n+1/2)}{\Gamma^2(n)} - \frac{6\Gamma^4(n+1/2)}{\Gamma^4(n)}}{\left(n - \frac{\Gamma^2(n+1/2)}{\Gamma^2(n)}\right)^2}.$$

These values, that do not depend on the parameter λ, show that, for sufficiently large values of n, $\sqrt{\Gamma}(n, \cdot)$ distributed random variables are indistinguishable from normally distributed ones, from the skewness and kurtosis points of view, since it can be proved that $\lim_{n\to\infty} \gamma_1 = \lim_{n\to\infty} \gamma_2 = 0$.

Def. 11: The random variable U is said to have a χ distribution with parameter n and denoted $U \sim \chi(n)$ if its density, for every $x \in \mathbb{R}$, is given by:

$$f_U(x; n) = \frac{2^{1-n}}{\Gamma(n)} x^{2n-1} \exp\left(-\frac{x^2}{2}\right) 1_{\mathbb{R}_+}(x).$$

This distribution is often called χ with $2n$ degrees of freedom. Its r–th order moments are given by $\mathrm{E}(U^r) = 2^{r/2}\Gamma(n + r/2)/\Gamma(n)$. It can be shown that the $\sqrt{\Gamma}$ and χ distributions are related through a scale property: if $U' \sim \chi(n)$, then $U = (2\lambda)^{-1/2}U'$ has a $\sqrt{\Gamma}(n, \lambda)$ distribution.

4. SAR IMAGE FORMATION

A multiplicative model is commonly adopted for SAR image formation ((4), (9), (27)). This model assumes that the observed value in every pixel is the outcome of a random variable Z, defined as the product between the random variables X and Y, where X represents the random variable modelling the terrain backscatter and Y represents the random variable modelling the speckle noise, i.e. $Z = X \cdot Y$. The validity of this model has been analyzed by several authors (see (8), for example).

Different distributions for X and for Y yield different models for the observed data $Z(\omega) = z$. In this work, in particular, it is shown how the K distributions (amplitude and intensity, 1– and multilook) appear for SAR image modelling under the multiplicative model. More generally, the use of Rayleigh, Exponential, Gamma and Square Root of Gamma distributions is justified for that model. Most of the presented results of this Section can also be seen in (4) and in (26). In this paper, these results are posed in an unified manner.

Intensity variables shall be denoted with subscript "I", and amplitude variables with subscript "A", i.e. an intensity observation is modelled as $Z_I = X_I \cdot Y_I$, while an amplitude observation is modelled as $Z_A = X_A \cdot Y_A$.

This formulation, using the multiplicative model, allows an easy simulation of the models that appear in the literature for SAR data. This constructive formulation of the distributions helps the designing of simulation algorithms with little effort. For a detailed description of simulation techniques, the reader can see the book (6), among others.

4.1. Homogeneous Regions

The basic hypothesis that governs the modelling of homogeneous regions is that their backscatter is constant, though its value is unknown. In this manner, for this kind of regions, it is possible to assume that $X_I \sim \mathfrak{C}(k)$ if the image is intensity, and $X_A \sim \mathfrak{C}(\sqrt{k})$ if the image is amplitude.

4.1.1. One Look Case: In this case, the speckle noise is usually considered exponentially or Rayleigh distributed, depending on the kind of processing.

4.1.1.1. Intensity: When the image is in intensity, the backscatter is multiplied by the outcome of a standard Exponential random variable, i.e. $Y_I \sim \mathfrak{E}(1)$. In this manner, using Def. 4, it holds that:

$$Z_I = X_I \cdot Y_I \sim \mathfrak{E}(k^{-1}).$$

4.1.1.2. Amplitude: When the image is in amplitude, the backscatter is multiplied by the outcome of a standard Rayleigh random variable, i.e. $Y_A \sim \mathfrak{R}(1)$. In this manner, using Def. 6, it holds that:

$$Z_A = X_A \cdot Y_A \sim \mathfrak{R}(k^{-1}).$$

4.1.2. Multilook Case: In this case, the speckle noise has the distribution of the mean of n independent identically distributed random variables, namely $Y = n^{-1} \sum_{i=1}^{n} Y_i$. The number n is often called *number of looks*.

4.1.2.1. Intensity: Each random variable Y_i in the aforementioned sum has a standard exponential distribution. By Def. 7 it holds that $nY_I \sim \Gamma(n, 1)$, and by Def. 8 it holds that:

$$Z_I = X_I \cdot Y_I \sim \Gamma(n, nk^{-1}).$$

4.1.2.2. Amplitude: Each random variable Y_i in the aforementioned sum has a standard Rayleigh distribution. Though it is immediate that the distribution of such sum admits a density, there is no known explicit form for this density when the sum involves three or more terms (the case with two terms is considered at the end of this Section), therefore, it is customary to make some approximations.

Recalling Def. 5, a Rayleigh variable is the result of taking the square root of an Exponential random variable. Therefore, a sum of Rayleigh variables is a sum of square roots of Exponential random variables; a possible approximation consists of substituting such sum by the square root of a sum of Exponential random variables, i.e. instead of using $Y_A = n^{-1} \sum_{i=1}^{n} \sqrt{Y_i}$ with $Y_i \sim \mathfrak{E}(1)$, an alternative consists of using $Y_A = \sqrt{n^{-1} \sum_{i=1}^{n} Y_i}$ with $Y_i \sim \mathfrak{E}(1)$. Therefore, by Def. 7 and Def. 9, it holds that $n^{1/2} Y_A \sim \sqrt{\Gamma}(n, 1)$ and, by Def. 10, it holds that:

$$Z_A = X_A \cdot Y_A \sim \sqrt{\Gamma}(n, n/k).$$

Notice that, with the presented approximation, it holds that $Z_A = \sqrt{Z_I}$.

4.2. Non–homogeneous Regions

The basic hypothesis that governs the modelling of inhomogeneous regions is that their backscatter is not constant, though it can be modelled by a convenient distribution. In this work, following (26), for this kind of regions, it will be assumed that $X_I \sim \Gamma(\alpha, \lambda)$ if the processing is quadratic and that $X_A \sim \sqrt{\Gamma}(\alpha, \lambda)$, if it is linear. For details about the use of the Gamma and Square Root of Gamma distributions for backscatter modelling, and their relations to random walk statistics, the reader is referred to (4).

4.2.1. 1–Look Case: In this case, the speckle noise is usually considered exponentially or Rayleigh distributed, depending on the kind of processing.

4.2.1.1. Intensity: When the image is in intensity, the backscatter is multiplied by the outcome of a standard Exponential random variable, i.e. $Y_I \sim \mathcal{E}(1)$. In this manner, it holds that:

$$Z_I = X_I \cdot Y_I \sim \mathcal{KI}(\alpha, \beta, 1),$$

where $\beta = \alpha/\lambda = \mathrm{E}(Z_I)$, and $\mathcal{KI}(\alpha, \beta, 1)$ is called *K*–intensity one–look distribution with parameters α and β. A careful definition and the properties of this distribution shall be presented in Section 5.

4.2.1.2. Amplitude: When the image is in amplitude, the backscatter is multiplied by the outcome of a standard Rayleigh random variable, i.e. $Y_A \sim \mathcal{R}(1)$. In this manner it holds that:

$$Z_A = X_A \cdot Y_A \sim \mathcal{KA}(\alpha, \beta, 1),$$

where $\beta = \alpha/\lambda = \mathrm{E}(Z_A^2)$, and $\mathcal{KA}(\alpha, \beta, 1)$ is called *K*–amplitude one–look distribution with parameters α and β. A careful definition and the properties of this distribution shall be presented in Section 5.

4.2.2. Multilook Case: In this case, again, the speckle noise has the distribution of the mean of n independent identically distributed random variables, namely $Y = n^{-1} \sum_{i=1}^{n} Y_i$. The number n is often called *number of looks*.

4.2.2.1. Intensity: Each random variable Y_i in the aforementioned mean has a standard Exponential distribution. By Def. 7 and Def. 8, $Y_I \sim \Gamma(n, n)$. Then, it holds that:

$$Z_I = X_I \cdot Y_I \sim \mathcal{KI}(\alpha, \beta, n),$$

where $\beta = \alpha/\lambda = \mathrm{E}(Z_I)$, and $\mathcal{KI}(\alpha, \beta, n)$ is called *K*–intensity *n*–looks distribution with parameters α and β. A careful definition and the properties of this distribution shall be presented in Section 5.

4.2.2.2. Amplitude: Each random variable Y_i has a standard Rayleigh distribution. Since there is no closed form for the distribution of the mean of $n \geq 3$ such random variables, it is customary to make the same approximations mentioned in Section 4.1.2.2. Therefore, $n^{1/2} Y_A \sim \sqrt{\Gamma}(n, 1)$ and, by Def. 10 it holds that $Y_A \sim \sqrt{\Gamma}(n, n)$. It can be proved that:

$$Z_A = X_A \cdot Y_A \sim \mathcal{KA}(\alpha, \beta, n).$$

Notice that, with the presented approximation, it holds that $Z_A = \sqrt{Z_I}$. For the sake of completeness, if $Y_1 \sim \mathcal{R}(a)$ and $Y_2 \sim \mathcal{R}(b)$ are independent random variables, then it can be proved that the density of the random variable $Y' = Y_1 + Y_2$ is given, for every $x \in \mathbb{R}_+$, by:

$$f_{Y'}(x) = \frac{2abx}{(a+b)^2}\left(\frac{b}{\exp(ax^2)} + \frac{a}{\exp(bx^2)}\right) + \\ + \frac{ab\sqrt{\pi}\,(2abx^2 - a - b)}{(a+b)^{5/2}\exp(abx^2/(a+b))}\left[\Phi\left(\frac{ax}{\sqrt{a+b}}\right) + \Phi\left(\frac{bx}{\sqrt{a+b}}\right)\right],$$

zero elsewhere, where $\Phi(z) = 2\pi^{-1/2}\int_0^z \exp(-t^2)\,dt$ is called the normalized error function. If $a = b = 1$, i.e. standard Rayleighs are considered, this density simplifies yielding, for every $x \in \mathbb{R}$:

$$f_{Y'}(x) = \left(x\exp(-x^2) + \sqrt{\frac{\pi}{2}}(x^2 - 1)\Phi(x/\sqrt{2})\exp(-x^2/2)\right)1_{\mathbf{R}_+}(x).$$

5. THE *K* DISTRIBUTIONS, THEIR ESTIMATORS AND PROPERTIES

The aforementioned manner to derive $\mathcal{KA}(\alpha,\beta,n)$ distributed random variables (i.e. through the use of a multiplicative model) is not the only one. These random variables could also be defined as those constructed by the use of the *mixture model* presented in ((16), (24)) in the following manner for the intensity case: let the random variable modelling the returned signal, conditioned on the backscatter, be Gamma distributed: $(Z_I \mid X_I = x) \sim \Gamma(n, n/x)$. In this manner, $E(Z_I \mid X_I = x) = x$. If the backscatter has a Gamma distribution, i.e. if $X_I \sim \Gamma(\alpha,\lambda)$ then, by integration, it holds that $Z_I \sim \mathcal{KI}(\alpha,\beta,n)$. This definition of the K intensity distribution is equivalent to the previously presented since, by Def. 8, if $Y' = (Z_I \mid X_I = x) \sim \Gamma(n, n/x)$ then, defining $Y_I = xY'$, it holds that $Y_I \sim \Gamma(n,n)$; now, letting $Z_I = X_I \cdot Y_I$, with $X_I \sim \Gamma(\alpha,\lambda)$, it holds that $Z_I \sim \mathcal{KI}(\alpha,\beta,n)$ since this is the definition given in Section 4.2.2.1. As stated in 4.2.2.2., $Z_A = \sqrt{Z_I}$ and, therefore, the equivalence also holds for the amplitude distribution using Def. 9 and Def. 10.

A third way of deriving $\mathcal{K}$ random variables is through the use of the negative binomial distribution. Using this approach, it is possible to relate these kind of distributions to diffusion and birth–and–death stochastic models ((4), (11)).

5.1. Intensity Random Variables

The random variable Z_I is said to have K–intensity n–looks distribution with parameters $\alpha \in \mathbb{R}_+$, $\beta \in \mathbb{R}_+$ and $n \in \mathbb{R}_+$ (in symbols $Z \sim \mathcal{KI}(\alpha,\beta,n)$) if its density is given, for every $x \in \mathbb{R}$, by:

$$f_{Z_I}(x;\alpha,\beta,n) = \frac{2\alpha n}{\Gamma(\alpha)\beta\Gamma(n)}\left(\frac{\alpha n x}{\beta}\right)^{(\alpha+n)/2-1} K_{\alpha-n}\left[2\sqrt{\frac{\alpha n x}{\beta}}\right]1_{\mathbb{R}_+}(x), \quad (2)$$

where K_ν denotes the modified Bessel function of the third kind and order ν. It is possible to see that its r–th order moments are given by:

$$E(Z_I^r) = \left(\frac{\beta}{\alpha n}\right)^r \frac{\Gamma(r+\alpha)\Gamma(r+n)}{\Gamma(\alpha)\Gamma(n)}.$$

Using the fact that $K_\nu(x) = K_{-\nu}(x)$, a remarkable property of this distribution is the commutativity of the parameters α and n. This property, and a convenient discretization of one of these parameters, will allow the obtainment of a feasible computational –recursive– form of the cumulative distribution function of $\mathcal{KI}(\alpha,\beta,n)$ distributed random variables. In order to obtain this function, the following notation will be used: $\theta_1 = \alpha$ and $\theta_2 = n$, or $\theta_1 = n$ and $\theta_2 = \alpha$. The choice between these two reparametriza-

tions, and its implications in terms of CPU required time and precision, will be discussed later.

The cumulative distribution function of such random variable is given, for every $x \in \mathbb{R}$, by:

$$F_{Z_t}(x;\theta_1,\beta,\theta_2) = \frac{2^{2-\theta_1-\theta_2}}{\Gamma(\theta_1)\Gamma(\theta_2)} \int_0^{2\sqrt{\theta_1\theta_2 x/\beta}} t^{\theta_1+\theta_2-1} K_{\theta_1-\theta_2}(t)dt \; 1_{\mathbb{R}_+}(x).$$

The hypothesis of $\theta_2 \in \mathbb{N}$, instead of $\theta_2 \in \mathbb{R}_+$ is required in order to derive a closed form for the previous expression. Writing:

$$\int_0^{2\sqrt{\theta_1\theta_2 x/\beta}} t^{\theta_1+\theta_2-1} K_{\theta_1-\theta_2}(t)\,dt\, 1_{\mathbb{R}_+}(x) = f(\nu,k,z),$$

where $\nu = \theta_1 - \theta_2$, $k = 2\theta_2 - 1$ and $z = 2\sqrt{\frac{\theta_1\theta_2 x}{\beta}}$, and using the relations presented in (1), it holds that:

$$f(\nu,k,z) = \begin{cases} 2^{\nu}\Gamma(\nu+1) - z^{\nu+1} K_{\nu+1}(z) & \text{if } k = 1, \\ (k-1)(2\nu+k-1) f(\nu,k-2,z) + & \\ -z^{\nu+k} K_{\nu+1}(z) - (k-1)\, z^{\nu+k-1} K_{\nu}(z) & \text{else.} \end{cases}$$

Since, in the previous equation, the argument of the Γ functions may assume values in the set $\{0, -1, -2, \ldots\}$, the following form was adopted in order to avoid this problem:

$$F_{Z_t}(x;\theta_1,\beta,\theta_2) = 1 + \frac{2^{2-\theta_1-\theta_2}}{\Gamma(\theta_1)\Gamma(\theta_2)} g(\nu,k,z), \quad (3)$$

where:

$$g(\nu,k,z) = \begin{cases} -z^{\nu+1} K_{\nu+1}(z) & \text{if } k = 1, \\ (k-1)\,(2\nu+k-1)\, g(\nu,k-2,z) + & \\ -z^{\nu+k} K_{\nu+1}(z) - (k-1)\, z^{\nu+k-1} K_{\nu}(z) & \text{else.} \end{cases}$$

The number of required recursions in order to compute a single value $F_{Z_t}(x;\theta_1,\beta,\theta_2)$ is θ_2. Therefore, the best choice of reparametrization, under the computational speed criterion, is $\theta_1 = \max\{\alpha,n\}$ and $\theta_2 = [\min\{\alpha,n\}]$, however $F_{Z_t}(x;\max\{\alpha,n\},\beta,[\min\{\alpha,n\}])$ might be a coarser approximation of $F_{Z_t}(x;\alpha,\beta,n)$ than $F_{Z_t}(x;[\min\{\alpha,n\}],\beta,\max\{\alpha,n\})$ as will be presented in Section 6. through Figure 5.

An special case of $\mathcal{GI}(\alpha,\beta,n)$ distributed random variables is when $n = 1$ (see Section 4.2.1.1.). In this case the cumulative distribution function is given by a simpler equation, namely, for every $x \in \mathbb{R}$, is:

$$F_{Z_I}(x;\alpha,\beta,1) = 1 - \frac{2}{\Gamma(\alpha)}\left(\frac{\alpha x}{\beta}\right)^{\alpha/2} K_\alpha\left[2\sqrt{\frac{\alpha x}{\beta}}\right]1_{\mathbb{R}_+}(x).$$

5.1.1. Estimators for the Parameters of the Intensity K Distribution: The estimator of β based on the first sample moment is $\hat{\beta} = \hat{m}_1$, the sample mean. In order to obtain the moments estimators of α and n it is necessary to solve a system of equations; this system yields the solution $\hat{n} = (-B \pm \sqrt{C})/(2A)$, where:

$$A = 2\hat{m}_2^2 - \hat{m}_1^2\hat{m}_2 - \hat{m}_1\hat{m}_3$$

$$B = 4\hat{m}_2^2 - 3\hat{m}_1^2\hat{m}_2 - \hat{m}_1\hat{m}_3$$

$$C = \hat{m}_1^4\hat{m}_2^2 - 8\hat{m}_1\hat{m}_2^3 - 2\hat{m}_1^3\hat{m}_2\hat{m}_3 + 16\hat{m}_2^4 - 8\hat{m}_1\hat{m}_2^2\hat{m}_3 + \hat{m}_1^2\hat{m}_3^2.$$

One estimator of α, obtained by the moments method, is:

$$\hat{\alpha} = \frac{\hat{m}_1^2(\hat{n}+1)}{\hat{n}(\hat{m}_2 - \hat{m}_1^2) - \hat{m}_1^2}.$$

The problem of estimating the equivalent number of looks, the parameter n, has been treated extensively in (28). The maximum likelihood estimators of α and β, for $n = 1$ looks, are given in (4).

5.1.2. Scale Properties of the Intensity K Distribution: Let $U' \sim \mathcal{KI}(\alpha, 1, n)$, then the random variable defined as $U = \beta U'$, with $\beta \in \mathbb{R}_+$, has a $\mathcal{KI}(\alpha,\beta,n)$ distribution. Therefore, the parameter β could be called the *scale parameter* of this of distribution.

5.1.3. Limiting properties: Recall that if $X_I \sim \Gamma(\alpha,\lambda)$ and $Y_I \sim \Gamma(n,n)$ are independent then $Z_I = X_I \cdot Y_I \sim \mathcal{KI}(\alpha,\beta,n)$, where $\beta = \alpha/\lambda$. Using the limiting and scale properties mentioned in Def. 8, consider the sequence of random variables $Z_{I_m} = X_{I_m} \cdot Y_{I_m}$, with $X_{I_m} \sim \Gamma(\alpha_m,\lambda_m)$ and $Y_{I_m} \sim \Gamma(n_m, n_m)$, and the limits $Z_I = \lim_{m\to\infty} Z_{I_m}$, $X_I = \lim_{m\to\infty} X_{I_m}$ and $Y_I = \lim_{m\to\infty} Y_{I_m}$. If $\alpha_m \to \infty$, $\lambda_m \to \infty$ such that $\alpha_m/\lambda_m \to \beta \in \mathbb{R}$, and $n_m \to n$ when $m \to \infty$, then $X_I \sim \mathcal{C}(\beta)$ and $Z_I \sim \Gamma(n, n\beta^{-1})$. If the one look is considered, i.e. if $n = 1$, then $Z_I \sim \mathcal{E}(\beta^{-1})$ by Def. 4. Using the same properties, but now letting $n_m \to \infty$ for $\alpha_m \to \alpha$ and $\lambda_m \to \lambda$ when $m \to \infty$, it holds that $Y_I \sim \mathcal{C}(1)$; therefore $Z_I \sim \Gamma(\alpha,\lambda)$. If *all* the parameters go to infinity, i.e. if $\alpha_m \to \infty$, $\lambda_m \to \infty$ with $\alpha_m/\lambda_m \to \beta \in \mathbb{R}$ and $n_m \to \infty$ when $m \to \infty$, then $Z_I \sim \mathcal{C}(\beta)$.

Figure 1 illustrates the several distributions arising for different cases of homogeneity and of number of looks. Solid arrows indicate a high degree of heterogeneity that, thus, induces a $\mathcal{KI}(\alpha,\alpha/\lambda,n)$ distribution for the data if n is "small", and a $\Gamma(\alpha,\lambda)$ distribution for the data if $n \to \infty$. Dashed arrows indicate a high degree of homogeneity

that, thus, induces a $\Gamma(n,\lambda n/\alpha)$ distribution for the data if n is "small", a $\mathcal{C}(\alpha/\lambda)$ distribution for the data if $n \to \infty$, and a $\mathcal{N}(\alpha/\lambda, \alpha^2/(\lambda^2 n))$ distribution for intermediate values of n. The empty arrow indicates that the return from an intermediately homogeneous region, viewed with a "large" number of looks ($n \to \infty$) could be modelled as $\mathcal{N}(\alpha/\lambda, \alpha/\lambda^2)$ distributed data.

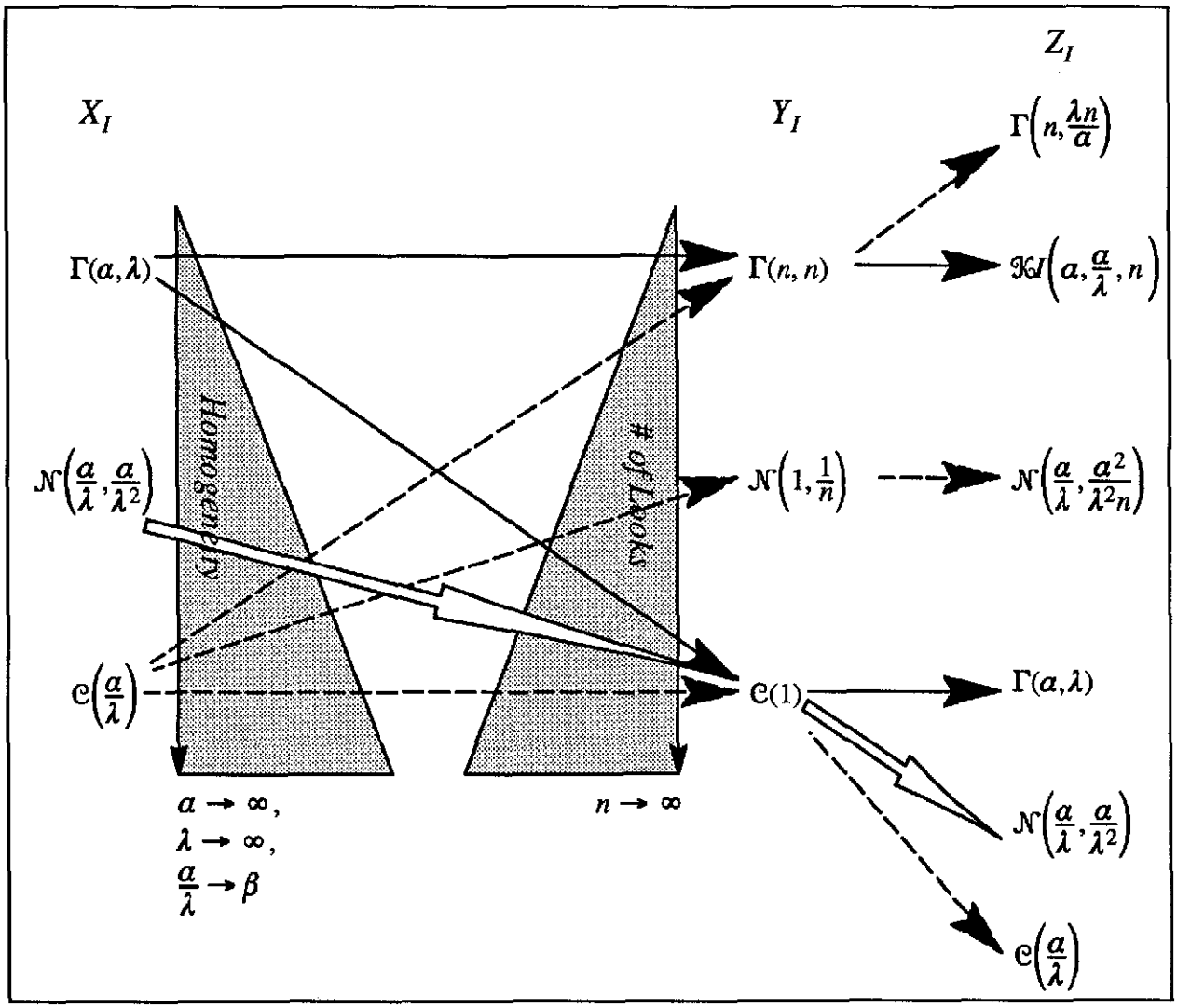

Figure 1 – Distributions for intensity data for several cases of homogeneity and of number of looks.

5.2. Amplitude Random Variables

The random variable Z_A is said to have K–amplitude n–looks distribution with parameters $\alpha \in \mathbb{R}_+$, $\beta \in \mathbb{R}_+$ and $n \in \mathbb{R}_+$ (in symbols $Z \sim \mathcal{KA}(\alpha,\beta,n)$) if its density is given, for every $x \in \mathbb{R}$, by:

$$f_{Z_A}(x;\alpha,\beta,n) = \frac{4x\alpha n}{\Gamma(\alpha)\beta\Gamma(n)}\left(\frac{\alpha n x^2}{\beta}\right)^{(\alpha+n-2)/2} K_{|\alpha-n|}\left[2x\sqrt{\frac{\alpha n}{\beta}}\right]1_{\mathbf{R}_+}(x). \quad (4)$$

It is possible to see that its r–th order moments are given by:

$$\mathrm{E}(Z_A^r) = \left(\frac{\beta}{\alpha n}\right)^{r/2} \frac{\Gamma(r/2+n)\Gamma(r/2+\alpha)}{\Gamma(\alpha)\Gamma(n)}.$$

Using the same reparametrization used in the derivation of equation (3), the cumulative distribution function of such random variable is given by $F_{Z_A}(x;\cdot) = F_{Z_I}(x^2;\cdot)$, with F_{Z_I} as defined for the $\mathcal{GI}(\theta_1,\beta,\theta_2)$ distributed random variables. In order to use the recursive form presented in equation (3), it is necessary to restrict the values of the parameter θ_2, i.e. to assume that $\theta_2 \in \mathbf{N}$.

5.2.1. Estimators for the Parameters of the Amplitude K Distribution: The estimators of α, β and n by the substitution method are given by $\hat{\beta} = \hat{m}_2$ and by the solution of the following system of equations:

$$f_1(\hat{\alpha},\hat{n}) = \sqrt{\frac{\hat{m}_2}{\hat{\alpha}\hat{n}}}\frac{\Gamma(\hat{\alpha}+\frac{1}{2})\Gamma(\hat{n}+\frac{1}{2})}{\Gamma(\hat{\alpha})\Gamma(\hat{n})} - \hat{m}_1 = 0$$

$$f_2(\hat{\alpha},\hat{n}) = \left(\frac{\hat{m}_2}{\hat{\alpha}\hat{n}}\right)^{3/2}\frac{\Gamma(\hat{\alpha}+\frac{3}{2})\Gamma(\hat{n}+\frac{3}{2})}{\Gamma(\hat{\alpha})\Gamma(\hat{n})} - \hat{m}_3 = 0.$$

The estimators $(\hat{\alpha},\hat{n})$ could be, alternatively, obtained by the minimization of the function $f_1^2 + f_2^2$. The behavior of this new function, with arguments values within a range that reflects real SAR data, is shown in Figure 2.

It is evident that the global minimization of this function is a hard task, due to the presence of multiple local minima. In practice, the value of the parameter n is either assumed known or estimated from homogeneous regions and other technique (28), and the parameter α is estimated using the first equation of the aforementioned system of equations.

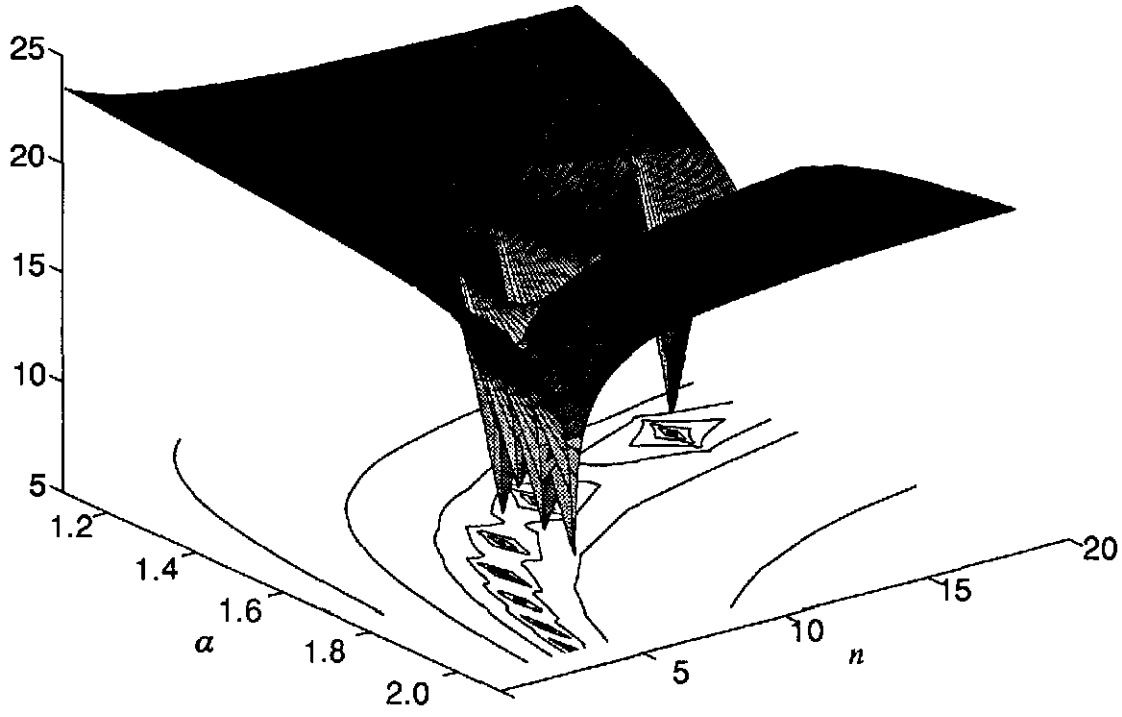

Figure 2 – A region of the parameter space where the value of the estimated parameter should be sought.

5.2.2. Scale Properties of the Amplitude K Distribution: Let $U' \sim \mathcal{KA}(\alpha, 1, n)$, then the random variable defined as $U = \sqrt{\beta}\,U'$, with $\beta \in \mathbb{R}_+$, has a $\mathcal{KA}(\alpha, \beta, n)$ distribution. Therefore, the parameter $\theta = \sqrt{\beta}$ could be called the *scale parameter* of this distribution.

5.2.3. Limiting properties: In analogy with the results presented in Section 5.1.3., recall that if $X_A \sim \sqrt{\Gamma}(\alpha,\lambda)$ and $Y_A \sim \sqrt{\Gamma}(n,n)$ are independent then $Z_A = X_A \cdot Y_A \sim \mathcal{KA}(\alpha,\beta,n)$, where $\beta = \alpha/\lambda$. Using the limiting and scale properties mentioned in Def. 10, consider the sequence of random variables $Z_{A_m} = X_{A_m} \cdot Y_{A_m}$, with $X_{A_m} \sim \sqrt{\Gamma}(\alpha_m,\lambda_m)$ and $Y_{A_m} \sim \sqrt{\Gamma}(n_m, n_m)$, and the limits $Z_A = \lim_{m\to\infty} Z_{A_m}$, $X_A = \lim_{m\to\infty} X_{A_m}$ and $Y_A = \lim_{m\to\infty} Y_{A_m}$. If $\alpha_m \to \infty$, $\lambda_m \to \infty$ such that $\alpha_m/\lambda_m \to \beta \in \mathbb{R}$, and $n_m \to n$ when $m \to \infty$, then $X_A \sim \mathcal{C}(\beta^{1/2})$ and $Z_A \sim \sqrt{\Gamma}(n, n\beta^{-1})$. If the one look is considered, i.e. if $n = 1$, then $Z_A \sim \mathcal{R}(\beta^{-1})$ by Def. 4. Using the same properties, but now letting $n_m \to \infty$, $\alpha_m \to \alpha$ and $\lambda_m \to \lambda$ when $m \to \infty$, it holds that $Y_A \sim \mathcal{C}(1)$; therefore $Z_A \sim \sqrt{\Gamma}(\alpha,\lambda)$. If *all* the parameters go to infinity, i.e. if $\alpha_m \to \infty$, $\lambda_m \to \infty$ with $\alpha_m/\lambda_m \to \beta \in \mathbb{R}$ and $n_m \to \infty$ when $m \to \infty$, then $Z_A \sim \mathcal{C}(\beta^{1/2})$.

Figure 1 can also be used to illustrate the distributions that appear for amplitude data, bearing in mind the proper parametrizations and reading $\sqrt{\Gamma}$ instead of Γ, $\mathcal{GA}$ instead of $\mathcal{GI}$. The constant and normal distributions are preserved (the latter by using the limiting properties presented in Def. 10).

6. IMPLEMENTATION PROBLEMS

In many situations like, for instance, to perform the Kolmogorov–Smirnov test, it is necessary to use the cumulative distribution function. Several problems appear in the implementation of the cumulative distribution function of K distributed random variables. Two sources of errors will be commented in this section, namely, the errors due to the implementations of Bessel functions K_ν, and the discretization required to use equation (3). The problems treated in this Section are related to the implementation of the cumulative distribution function of $\mathcal{KA}(\alpha, \beta, n)$ distributed random variables.

Three numerical tools were used, without achieving fully acceptable results, to obtain the K_ν Bessel functions in the required range: the Nag FORTRAN subroutines library ((20), (21)), the C built–in functions and the MatLab v. 3.5h system. To illustrate the limitations encountered, Figure 3 shows the numerical instabilities of the $F_{Z_A}(x; 10, 10000, 5)$ given in the use of equation (3) for $x \geq 120$, as obtained with the last mentioned system. Only values in the range $x \in [0; 140]$ are shown, since larger values produce gross instabilities.

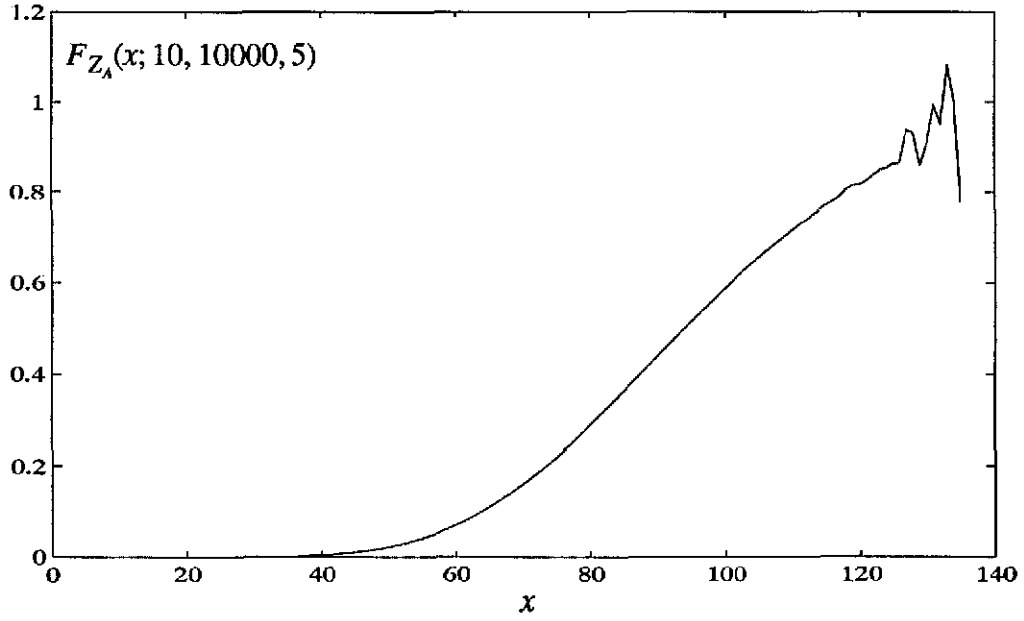

Figure 3 – Cumulative distribution function of a $\mathcal{KA}(10, 10000, 5)$ distributed random variable, using the MathLab V. 3.5h system.

The implementations of the Bessel functions provided with the the MatLab v. 4.0a system did not present those severe instabilities for large values of the argument, though minor instabilities are still present for small ones (see Figure 4). Notice that these instabilities are quite smaller than 10^{-10} and, for the considered parameters, are con-

fined to small values, approximately in the range $x \in (0; 3]$. This problem is less severe than the previously commented for the applications intended in this work since, for *byte* formatted images, the scope of variation of x is the interval [0; 255]. This source of errors is considered negligible in the rest of this Section.

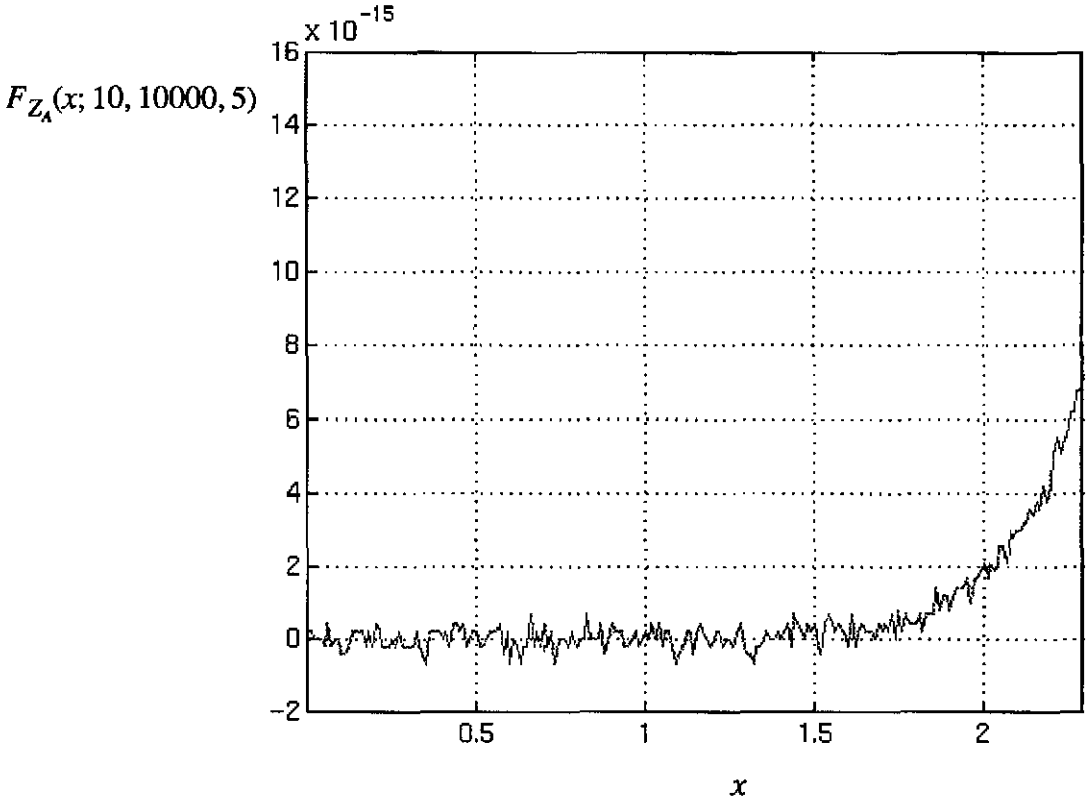

Figure 4 – Cumulative distribution function of a $\mathcal{KA}(10, 10000, 5)$ distributed random variable, using the MathLab V. 4.0a system.

An obvious alternative to obtain this cumulative distribution function is to perform a numerical integration of the density given in equation (4). This was performed, in order to compare CPU times and relative errors, using the adaptive recursive Newton–Cotes 8 panel rule method, implemented as the "`quad8()`" function in the MatLab v. 4.0a system (25). This alternative has the advantage of not requiring the discretization of one of the parameters.

As it was mentioned before equation (3), which is used to compute the F_{Z_A} function, was derived under the restriction of an integer parameter, i.e. assuming $\theta_2 \in \mathrm{N}$. In order to assess the influence of this discretization, a measure between distributions is used. Define the function $d : \mathfrak{D} \times \mathfrak{D} \rightarrow [0, 1]$, where $\mathfrak{D}$ is the set of all the cumulative distribution functions. The *distance in variation* between the distributions D_1 and D_2, both being elements of $\mathfrak{D}$, is defined as $d(D_1, D_2) = \sup_{x \in \mathrm{R}} |D_1(x) - D_2(x)|$. Using this

measure, the commutativy of the parameters a and n, and the scale property of K distributions, the influence of the discretization can be assessed through finding $d(D_1, D_2)$ with $D_1 = F_{Z_A}(\cdot;\theta_1, 1, \theta_2)$ and $D_2 = F_{Z_A}(\cdot;\theta_1, 1, \tau)$, with $\tau \in \mathbb{R}_+$ and $\theta_2 = [\tau]$. It can be shown that:

$$M(\theta_1, [\tau]) = \sup_{\varepsilon \in [-1/2, 1/2)} \left\{ d\Big(F_{Z_A}(\cdot;\theta_1, 1, [\tau] + \varepsilon), F_{Z_A}(\cdot;\theta_1, 1, [\tau])\Big)\right\} =$$

$$d\Big(F_{Z_A}(\cdot;\theta_1, 1, [\tau] - 1/2), F_{Z_A}(\cdot;\theta_1, 1, [\tau])\Big),$$

that is, the maximum absolute error due to the use of the discrete parameter $\theta_2 = [\tau]$ occurs when $\tau = [\tau] - 1/2$.

Figure 5 shows $M(\theta_1, \theta_2)$ for $\theta_1, \theta_2 \in [1(1)20]$. It is immediate that $M(\theta_1, \theta_2)$ increases with θ_1 for any value of the discretized parameter θ_2, though it is almost constant for small values of this last parameter. For any fixed value of θ_1, the error diminishes with θ_2. Notice that the maximum absolute error in the considered range is smaller than 10^{-1}.

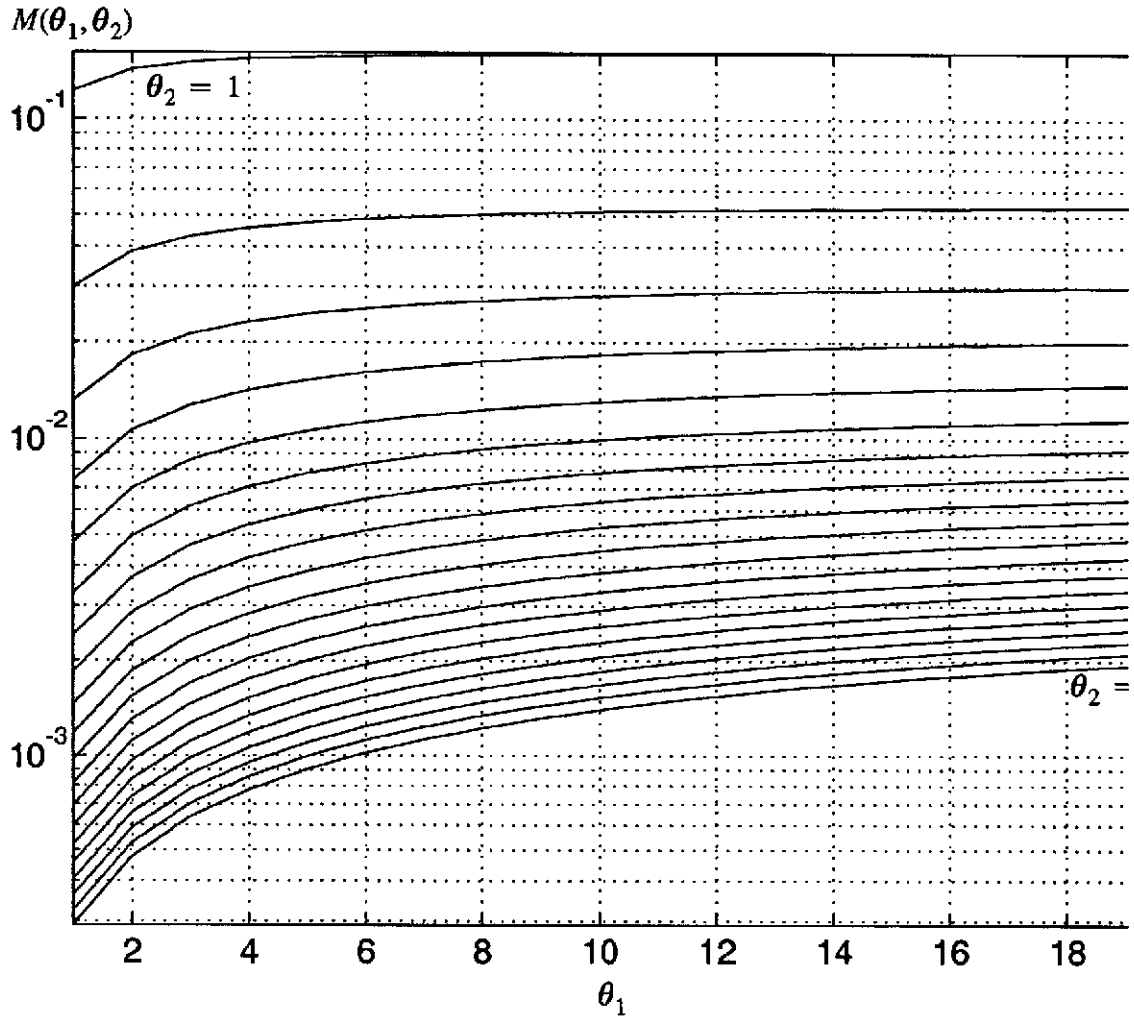

Figure 5 – Maximum distance in variation due to the discretization of θ_2, as a function of θ_1, for several values of the discretized parameter θ_2.

Figure 6 shows the difference of CPU times (in seconds) between the numerical integration and recursive methods, as functions of θ_1 for several values of the discretized parameter θ_2. It is immediate that the latter method is faster than the former, for almost every θ_1, and for $\theta_2 \lesssim 15$; after this value, the method based on numerical integration is usually faster than the recursive one. If the parameter θ_2 is associated to the number of looks, then the recursive method would almost always be the fastest, since SAR imagery products are distributed with processings that range from 1 look to, typically, less than 16 looks.

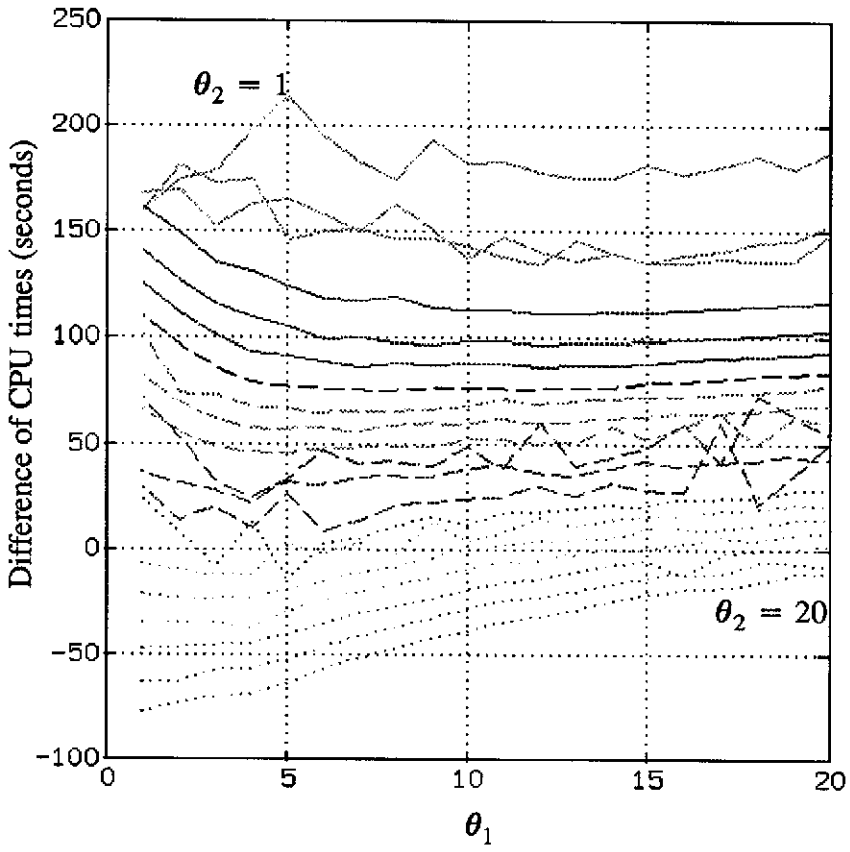

Figure 6 – Difference of CPU times (in seconds) between the numerical integration and recursive methods, as functions of θ_1 for several values of the discretized parameter θ_2.

7. APPLICATIONS OF THE K DISTRIBUTION TO SAR IMAGE ANALYSIS

As it was previously mentioned, the K distribution is useful for the modelling of non–homogeneous areas. In (28) the following study was performed: several distributions were fitted to homogeneous (non–forest: NF) and non–homogeneous (forest: F) data, obtained by the SAREX campaign over the Tapajós area, Brazil, and for bands HH and VV. After a visual segmentation of the image, several samples were taken from homogeneously occupied areas. These samples were fitted, after parameter estimation, by some of the aforementioned distributions. The Kolmogorov–Smirnov and χ^2 tests were applied, and those distributions for which the sample was fitted at at least $p = 0.01$ were recorded. Some distributions not mentioned in this paper were also fitted: the Weibull, Log–Normal, and Beta distributions. Though it is common to assume that multilook data can be well fitted by the Normal distribution, in that paper it was shown that this is not the case when the considered data are samples of forest areas.

In this paper, these data are first corrected to compensate the antenna pattern effect and, then, subsampled in order to obtain (approximately) uncorrelated samples. The subsampling factors are obtained from the estimation of the autocorrelation functions estimated in (28), and given in Table 1 below. The considered distributions are the Normal (N), Square Root of Gamma (SG) and the 6–looks amplitude K (KA6) distributions.

Table 2 presents the percentages of samples, from different bands and types of region (homogeneous and non–homogeneous), that do not reject the considered distribution for the Kolmogorov–Smirnov test at the 10% level. It is evident that the Normal distribution is quite inadequate for forest (non–homogeneous data) and both bands, whilst the Square Root of Gamma and amplitude K distributions fit these data well.

The results presented in this table are completely explained by the properties presented in Section 4. Square Root of Gamma and Normal distributions explain well non–forest data, since the backscatter could be considered homogeneous in these regions; the Normal distribution appears since the equivalent number of looks is, probably, large enough to hold the approximation to the Normal distribution. Also notice that the Square Root of Gamma fits the data well, since it arises as the product of a constant (the backscatter) and the Square Root of Gamma produced by the speckle noise (see Figure 1, where the intensity case is illustrated).

Square Root of Gamma and K distributions explain well forest data. The fit to the former distribution could be explained by the fact that the equivalent number of looks is large enough to admit the approximation of the speckle noise by a constant (see

Section 5.2.3.). Notice that the Normal distribution is clearly inadequate to describe this kind of data.

Though Figure 1 —properly converted to the amplitude case— suggests that samples from homogeneous areas should only be fitted by a Square Root of Gamma, a Normal or Constant distributions, Table 2 shows that the K distribution also provides a good fit for these areas, i.e. the acceptance or rejection of the K distribution is not an immediate criterion for the discrimination of forest and non–forest data. This departure from the theory could be explained by saying that this distribution is more general than the Square Root of Gamma in the sense that the latter is a limiting case of the former. It is important to notice, though, that their parameters may perform this discrimination (see Figures 7 and 8 for the estimated parameters of the Square Root of Gamma and K distributions, VV polarization, in semilogarithmic plot, respectively, and Table 3 for its means and standard deviations).

TABLE 1 – SUBSAMPLING FACTORS FOR FOREST AND NON–FOREST SAMPLES.

Sample \ Direction	Azimuth	Range
Forest	7	4
Non–forest	4	3

TABLE 2 – PERCENTAGES OF SAMPLES NOT REJECTED BY THE KOLMOGOROV–SMIRNOV TEST, AT THE 10% SIGNIFICANCE LEVEL.

Sample \ Distribution	SG	KA6	N
NF – VV	87	87	100
NF – HH	100	97	100
F – VV	88	95	39
F – HH	92	88	38

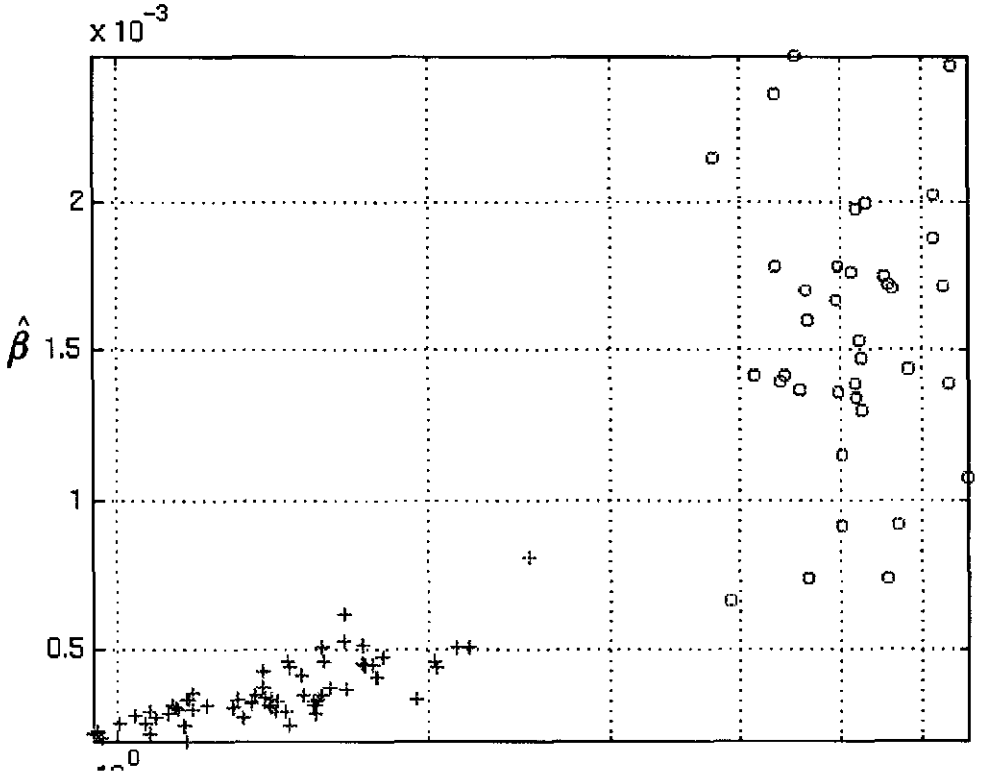

Figure 7 – Estimated parameters $\hat{\alpha}$ and $\hat{\beta}$ of the $\sqrt{\Gamma}$ distribution: "o" non forest, "+" forest.

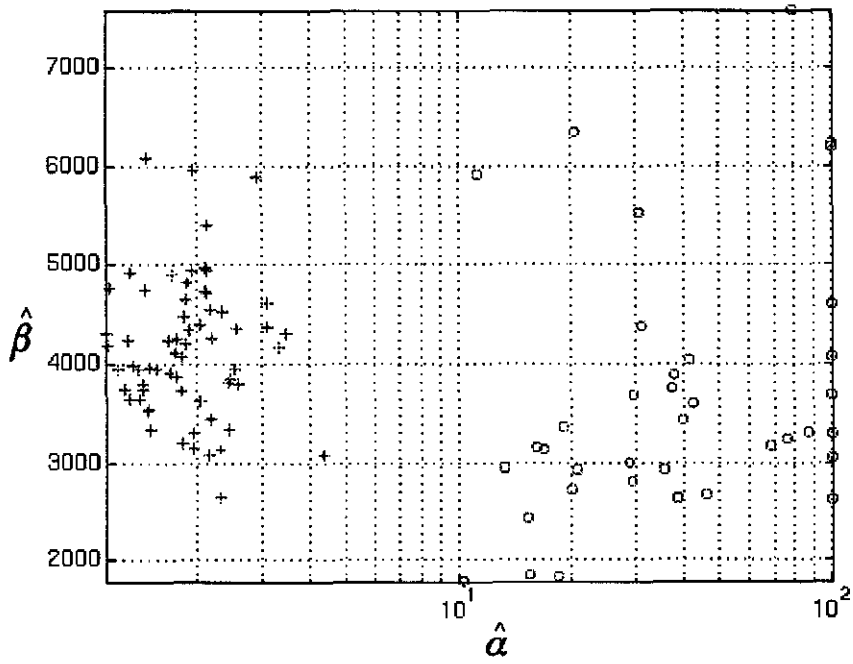

Figure 8 – Estimated parameters $\hat{\alpha}$ and $\hat{\beta}$ of the $\mathcal{GA}6$ distribution: "o" non forest, "+" forest.

8. CONCLUSIONS

A unified use of multiplicative models was presented to derive some useful distributions for SAR image modelling and analysis. The relations between these distributions was achieved using scale properties and a limiting property of Gamma distributed random variables. The cumulative distribution functions of multilook K distributed random variables was provided in a recursive form, that requires the discretization of one of its three parameters. The influence of this discretization on those functions was assessed, and the CPU times required for this method are compared to those needed to perform a numerical integration of the density. It was shown that for a wide range of parameters the recursive form is faster than the numerical integration technique, and that the error due to discretization is smaller than 10^{-1}. The moments estimators and scale properties of the distributions that arise with the use of the multiplicative model were provided. An example of the application of these distributions was given using SAREX data, where it was checked that the Normal and Square Root of Gamma distributions describe well non–forest samples, whilst the Square Root of Gamma and amplitude K distributions fit better forest samples. Though the Square Root of Gamma distribution fits well both types of samples, the estimated parameters allow a discrimination between forest and non–forest.

TABLE 3 – MEAN AND STANDARD DEVIATION OF ESTIMATED PARAMETERS OF SQUARE ROOT OF GAMMA (SG) AND 6–LOOKS AMPLITUDE K (KA6) DISTRIBUTIONS FOR SAREX DATA.

DISTRIBUTION	SG		KA6	
PARAMETER	EST. ALPHA	EST. BETA	EST. ALPHA	EST. BETA
NF–VV	5.16(0.72)	.00160(.0005)	48.45(33.49)	3665(1354)
NF–HH	5.42(0.75)	.00099(.0003)	63.18(36.22)	5736(1105)
F–VV	1.44(0.32)	.00035(.0001)	1.97(0.61)	4160(696)
F–HH	1.20(0.30)	.00017(.0001)	1.59(0.52)	7211(1447)

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