[2024-1 Digital Control]

Chapter 2. Discrete-Time Systems and the z-Transform

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Overview of Chapter 2

Key questions

- How can we represent a system in discrete time?
- What is the difference between continuous-time and discrete-time systems?

Part A: Transfer function approach

- Discrete-time systems
- Definition of z-transform
- Properties
- Relation with difference equation
- Inverse z-transform

Part B: State space approach

- Flow graphs
- State variables and state equations
- Relation with transfer function
- Solution of state equations

What we have studied on continuous-time system

2.1. Introduction

A continuous-time system

- has the continuous-time input $u(t)$ and output $y(t)$, and
- is governed by a differential equation (미분방정식)

$$y(t) = \beta_n \frac{\mathrm{d}^n u(t)}{\mathrm{d}t^n} + \cdots + \beta_1 \frac{\mathrm{d}u(t)}{\mathrm{d}t} + \beta_0 u(t) - \alpha_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} - \cdots - \alpha_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t}.$$

Example: Mass-spring-damper system $M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = u(t)$

- Output $y(t)$ = Position of a mass
- Input $u(t)$ = External force
- The defining equation can be rewritten as

$$y(t) = \frac{1}{K}u(t) - \frac{M}{K}\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - \frac{B}{K}\frac{\mathrm{d}y(t)}{\mathrm{d}t}.$$

Two ways of analyzing a continuous-time system

2.1. Introduction

For the case of the mass-spring-damper system $M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = u(t)$,

1. Laplace transform $\mathcal{L}(\cdot)$: In s-domain, we have

$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = u(t) \quad \xrightarrow{\mathcal{L}} \quad Ms^2Y(s) + BsY(s) + KY(s) = U(s)$$

   (with $Y(s) = \mathcal{L}(y(t))$, $U(s) = \mathcal{L}(u(t))$)

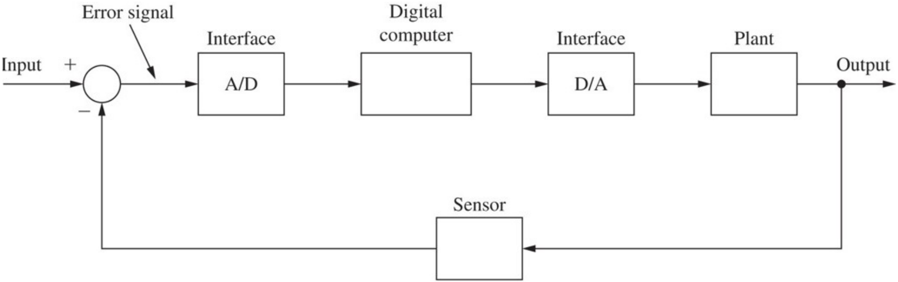
   $\therefore$ The transfer function of the system: $G(s) = \dfrac{Y(s)}{U(s)} = \dfrac{1}{Ms^2 + Bs + K}$

2. State-space model (상태 공간 모델):

$$x := \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u \quad && = Ax + Bu, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x && = Cx. \end{aligned}$$

Note: Similar approaches are possible in discrete time.

Overall diagram of “sampled-data” control system



Sampled-data control system

= Continuous-time plant + discrete-time controller + A/D and D/A converters

- Not all the components are in continuous time.
- The methods for continuous-time system are not enough.
- We need analysis tools for “discrete-time” system.

What is a discrete-time system?

A discrete-time system

- has the discrete-time input $u^{\mathsf{d}}(k)$ and output $y^{\mathsf{d}}(k)$, and
- is governed by a difference equation (차분방정식)

$$y^{\mathsf{d}}(k) = b_n u^{\mathsf{d}}(k) + \cdots + b_0 u^{\mathsf{d}}(k-n) - a_{n-1} y^{\mathsf{d}}(k-1) - \cdots - a_0 y^{\mathsf{d}}(k-n)$$

Example: Discretized mass-spring-damper system?

$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = u(t) \quad \xrightarrow{\text{discretization}} \quad (??)$$

On the discretization (이산화), we will learn

- discretized signal in Chapter 2, and
- discretized system in Chapter 3.

Discretization of a continuous-time signal

We here study discretizations of continuous-time signals

- a continuous-time signal: $x(t)$
- its derivative: $\dot{x}(t)$
- its integral: $\int_0^t x(\tau)\mathrm{d}\tau$
- $\cdots$

at $t = kT$ where $T > 0$ is a time period called sampling period (샘플링 주기).

Notes:

- In this course, the superscript "d" will be used for discrete-time signals. (e.g., $x(t)$ is a continuous-time signal, and $x^{\mathsf{d}}(k)$ is a discrete-time signal.)
- There are MANY candidates for (approximate) discretization.

Several ways of discretizing a continuous-time signal

- Continuous-time signal $x(t)$: $x^{\mathsf{d}}(k) = x(kT)$.
- Derivative $\dot{x}(t)$ of $x(t)$:

$$\dot{x}^{\mathsf{d}}_{\text{fwd}}(k) := \frac{x((k+1)T) - x(kT)}{T} \qquad \text{(forward difference)}$$

$$\text{or} \quad \dot{x}^{\mathsf{d}}_{\text{bwd}}(k) := \frac{x(kT) - x((k-1)T)}{T} \qquad \text{(backward difference)}$$

- Integral $c(t) := \int_0^t x(\tau)\mathrm{d}\tau$ of $x(t)$:

$$c(t)|_{t=kT} \approx c^{\mathsf{d}}_{\text{rec},1}(k) := \sum_{i=0}^{k-1} x\big(iT\big)T \qquad \text{(rectangular by left side)}$$

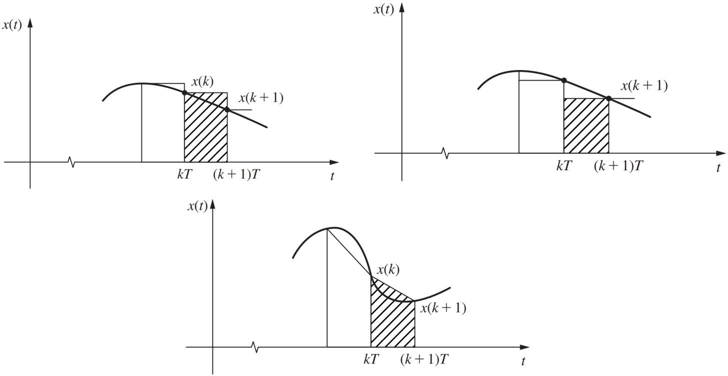
$$\text{or} \quad \approx c^{\mathsf{d}}_{\text{rec},2}(k) := \sum_{i=1}^{k} x\big(iT\big)T \qquad \text{(rectangular by right side)}$$

$$\text{or} \quad \approx c^{\mathsf{d}}_{\text{trap}}(k) := \sum_{i=1}^{k} \frac{x\big(iT\big) + x\big((i-1)T\big)}{2}T \qquad \text{(trapezoidal)}$$

(Cont'd)

2.2. Discrete-time Systems

- (Top-left) Rectangular rule by left side
- (Top-right) Rectangular rule by right side
- (Bottom) Trapezoidal rule



z-transform: Discrete-time analogue of Laplace transform

- (single-sided) z-transform $\mathcal{Z}[\{e^{\mathsf{d}}(k)\}]$ (or simply, $\mathcal{Z}[e^{\mathsf{d}}(k)]$):

$$E^{\mathsf{d}}(z) = \mathcal{Z}[\{e^{\mathsf{d}}(k)\}] := e^{\mathsf{d}}(0) + e^{\mathsf{d}}(1)z^{-1} + \cdots = \sum_{k=0}^{\infty} e^{\mathsf{d}}(k) z^{-k}.$$

- double-sided z-transform (not frequently used in this class)

$$E^{\mathsf{d}}(z) = \mathcal{Z}[\{e^{\mathsf{d}}(k)\}] = \sum_{k=-\infty}^{\infty} e^{\mathsf{d}}(k) z^{-k}.$$

- inverse z-transform of $E^{\mathsf{d}}(z)$:

$$e^{\mathsf{d}}(k) = \mathcal{Z}^{-1}[E^{\mathsf{d}}(z)] = \frac{1}{2\pi j} \oint_{\Gamma} E^{\mathsf{d}}(z) z^{k-1} \mathrm{d}z$$

where Γ is a closed path that encloses every poles of $E^{\mathsf{d}}(z)z^{k-1}$.

Examples

2.3. Transform Methods

Question: Find the z-transform of $\{e^{\mathsf{d}}(k)\}$ where

- $e_1^{\mathsf{d}}(k) = 1$, $k = 0, 1, \ldots,$
- $e_2^{\mathsf{d}}(k) = a^k$, $k = 0, 1, \ldots$

Answer:

$$E_1^{\mathsf{d}}(z) = \mathcal{Z}[e_1^{\mathsf{d}}(k)] = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}, \quad \text{where } |z^{-1}| < 1,$$

$$\begin{aligned} E_2^{\mathsf{d}}(z) = \mathcal{Z}[e_2^{\mathsf{d}}(k)] &= 1 + az^{-1} + a^2 z^{-2} + \cdots \\ &= 1 + (az^{-1}) + (az^{-1})^2 + \cdots \\ &= \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \qquad \text{where } |az^{-1}| < 1 \end{aligned}$$

Notice: The power series must converge only in a particular region of z.

Region of convergence (ROC)

= The region of z in which the power series of $e^{\mathsf{d}}(k)z^{-k}$ is well-defined: i.e.,

$$\text{ROC} = \left\{ z : \left| \sum_{k=0}^{\infty} e^{\mathsf{d}}(k) z^{-k} \right| < \infty \right\} \quad \text{(for single-sided } z\text{-transform)}$$

Note:

- ROC is important
  $\because$ without specifying ROC, the inverse z-transform MAY NOT be unique.

  Example: The following signals have the same double-sided z-transform:
  - $e_1^{\mathsf{d}}(k) = 0.5^k \mathbf{1}^{\mathsf{d}}(k)$ (with the Heaviside step function $\mathbf{1}^{\mathsf{d}}(k)$)
  - $e_2^{\mathsf{d}}(k) = -(0.5)^k \mathbf{1}^{\mathsf{d}}(-k-1)$.
- Nonetheless, it is not problematic (and will not be discussed further) if we pre-specify the 1-1 relation $E^{\mathsf{d}}(z) \leftrightarrow e^{\mathsf{d}}(k)$ (with a fixed ROC and use of single-sided z-transform).

Properties of the z-transform

- Addition and subtraction: $\mathcal{Z}[e_1^{\mathsf{d}}(k) \pm e_2^{\mathsf{d}}(k)] = \mathcal{Z}[e_1^{\mathsf{d}}(k)] \pm \mathcal{Z}[e_2^{\mathsf{d}}(k)]$.
- Multiplication by a constant: $\mathcal{Z}[ae^{\mathsf{d}}(k)] = a\mathcal{Z}[e^{\mathsf{d}}(k)]$

Note: The two properties above lead to the linearity property.

$$\mathcal{Z}[ae_1^{\mathsf{d}}(k) + be_2^{\mathsf{d}}(k)] = a\mathcal{Z}[e_1^{\mathsf{d}}(k)] + b\mathcal{Z}[e_2^{\mathsf{d}}(k)].$$

- Real translation 1: $\mathcal{Z}[e^{\mathsf{d}}(k-n)\mathbf{1}^{\mathsf{d}}(k-n)] = z^{-n}E^{\mathsf{d}}(z)$

  where $\mathbf{1}^{\mathsf{d}}(k) = \begin{cases} 1, & k = 0, 1, \ldots, \\ 0, & \text{otherwise} \end{cases}$
- Real translation 2: $\mathcal{Z}[e^{\mathsf{d}}(k+n)\mathbf{1}^{\mathsf{d}}(k)] = z^n \left[E^{\mathsf{d}}(z) - \sum_{k=0}^{n-1} e^{\mathsf{d}}(k)z^{-k}\right]$

(Cont'd)

- Complex translation:

$$\mathcal{Z}[\mathsf{e}^{akT}e^{\mathsf{d}}(k)] = E^{\mathsf{d}}(\mathsf{e}^{-aT}z), \quad (a\text{: real or complex})$$

where e in San-serif font means the natural constant (자연수).

- Initial value: $e^{\mathsf{d}}(0) = \lim_{z\to\infty} E^{\mathsf{d}}(z)$
- Final value: (If the limit of $e^{\mathsf{d}}(n)$ exists, then)

$$\lim_{n\to\infty} e^{\mathsf{d}}(n) = \lim_{z\to 1}(z-1)E^{\mathsf{d}}(z)$$

- Polynomial-in-time signal: $\mathcal{Z}[ke^{\mathsf{d}}(k)] = -z\dfrac{\mathrm{d}E^{\mathsf{d}}(z)}{\mathrm{d}z}$
- Summation over time: $\mathcal{Z}\left[\sum_{n=0}^{k} e^{\mathsf{d}}(n)\right] = \dfrac{z}{z-1}E^{\mathsf{d}}(z)$

(Cont'd)

2.5. Finding z-Transforms

TABLE 2-2 Properties of the z-Transform

Sequence	Transform
$e(k)$	$E(z) = \sum_{k=0}^{\infty} e(k)z^{-k}$
$a_1e_1(k) + a_2e_2(k)$	$a_1E_1(z) + a_2E_2(z)$
$e(k-n)u(k-n); \quad n \geq 0$	$z^{-n}E(z)$
$e(k+n)u(k); \quad n \geq 1$	$z^n\left[E(z) - \sum_{k=0}^{n-1} e(k)z^{-k}\right]$
$\varepsilon^{akT}e(k)$	$E(z\varepsilon^{-aT})$
$ke(k)$	$-z\dfrac{dE(z)}{dz}$
$e_1(k) * e_2(k)$	$E_1(z)E_2(z)$
$e_1(k) = \sum_{n=0}^{k} e(n)$	$E_1(z) = \dfrac{z}{z-1}E(z)$
Initial value: $e(0) = \lim_{z\to\infty} E(z)$	
Final value: $e(\infty) = \lim_{z\to 1}(z-1)E(z)$, if $e(\infty)$ exists	

Properties of the Laplace transform (for comparison)

2.5. Finding z-Transforms

TABLE A5-1 Laplace Transform Properties

Name	Theorem
Derivative	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0^+)$
nth-order derivative	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - s^{n-1}f(0^+) - \cdots - f^{(n-1)}(0^+)$
Integral	$\mathscr{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$
Shifting	$\mathscr{L}[f(t - t_0)\, u(t - t_0)] = \varepsilon^{-t_0 s}F(s)$
Initial value	$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$
Final value	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$
Frequency shift	$\mathscr{L}[\varepsilon^{-at}f(t)] = F(s + a)$
Convolution integral	$\mathscr{L}^{-1}[F_1(s)F_2(s)] = \int_0^t f_1(t - \tau)f_2(\tau)d\tau = \int_0^t f_1(\tau)f_2(t - \tau)d\tau$

Frequently-used results of the z-transform

2.5. Finding z-Transforms

TABLE 2-3 ***z*-Transforms**

Sequence	Transform
$\delta(k-n)$	z^{-n}
1	$\dfrac{z}{z-1}$
k	$\dfrac{z}{(z-1)^2}$
k^2	$\dfrac{z(z+1)}{(z-1)^3}$
a^k	$\dfrac{z}{z-a}$
ka^k	$\dfrac{az}{(z-a)^2}$
$\sin ak$	$\dfrac{z \sin a}{z^2 - 2z \cos a + 1}$
$\cos ak$	$\dfrac{z(z - \cos a)}{z^2 - 2z \cos a + 1}$
$a^k \sin bk$	$\dfrac{az \sin b}{z^2 - 2az \cos b + a^2}$
$a^k \cos bk$	$\dfrac{z^2 - az \cos b}{z^2 - 2az \cos b + a^2}$

For the proof of the last few results, one may have to use the Euler's relation

$$\cos(ak) = \frac{\mathrm{e}^{jak} + \mathrm{e}^{-jak}}{2}, \quad \sin(ak) = \frac{\mathrm{e}^{jak} - \mathrm{e}^{-jak}}{2}.$$

Then, we have the following:

$$\begin{aligned}
\mathcal{Z}[\cos(ak)] &= \sum_{k=0}^{\infty} \cos(ak) z^{-k} = \sum_{k=0}^{\infty} \left(\frac{\mathrm{e}^{jak} + \mathrm{e}^{-jak}}{2} \right) z^{-k} \\
&= \frac{1}{2} \left(\frac{z}{z-1} \right) \Big|_{z \to \mathrm{e}^{-ja} z} + \frac{1}{2} \left(\frac{z}{z-1} \right) \Big|_{z \to \mathrm{e}^{ja} z} \quad \text{(by complex trans.)} \\
&= \frac{1}{2} \frac{\mathrm{e}^{-ja} z}{\mathrm{e}^{-ja} z - 1} + \frac{1}{2} \frac{\mathrm{e}^{ja} z}{\mathrm{e}^{ja} z - 1} \\
&= \frac{1}{2} \frac{2z^2 - (\mathrm{e}^{-ja} + \mathrm{e}^{ja}) z}{(\mathrm{e}^{-ja} z - 1)(\mathrm{e}^{ja} z - 1)} = \frac{z^2 - (\mathrm{e}^{-ja} + \mathrm{e}^{ja}) z/2}{z^2 - (\mathrm{e}^{-ja} + \mathrm{e}^{ja}) z + 1} \\
&= \frac{z^2 - \cos a z}{z^2 - 2\cos a z + 1}.
\end{aligned}$$

(Cont'd)

2.5. Finding z-Transforms

- Prove the contents of Table 2-3 by yourself!
- A more comprehensive table is given in Appendix VI.

Note: The results in Appendix VI are derived from the following procedure:

$$E(s) \xrightarrow{\mathcal{L}^{-1}(\cdot)} e(t) \xrightarrow{\text{Discretization}} e^{\mathsf{d}}(k) \xrightarrow{\mathcal{Z}(\cdot)} E^{\mathsf{d}}(z) = \mathcal{Z}[e^{\mathsf{d}}(k)]$$

Laplace transform $E(s)$	**Time function** $e(t)$	**z-Transform** $E(z)$	**Modified z-transform** $E(z, m)$
$\frac{1}{s}$	$u(t)$	$\frac{z}{z-1}$	$\frac{1}{z-1}$
$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$	$\frac{mT}{z-1} + \frac{T}{(z-1)^2}$

Difference equation $\leftrightarrow$ Transfer function in z-domain

2.6. Solution of Difference Equations

Remind: A discrete-time system can be represented as a difference equation:

$$\begin{aligned} & m^{\mathsf{d}}(k) + a_{n-1} m^{\mathsf{d}}(k-1) + \cdots + a_0 m^{\mathsf{d}}(k-n) \\ & = b_n e^{\mathsf{d}}(k) + b_{n-1} e^{\mathsf{d}}(k-1) + \cdots + b_0 e^{\mathsf{d}}(k-n) \end{aligned}$$

where $m^{\mathsf{d}}(k)$: output, and $e^{\mathsf{d}}(k)$: input.

Now, taking the z-transform to the both sides, we have

$$\begin{aligned} & M^{\mathsf{d}}(z) + a_{n-1} z^{-1} M^{\mathsf{d}}(z) + \cdots + a_0 z^{-n} M^{\mathsf{d}}(z) \\ & = b_n E^{\mathsf{d}}(z) + b_{n-1} z^{-1} E^{\mathsf{d}}(z) + \cdots + b_0 z^{-n} E^{\mathsf{d}}(z) \end{aligned}$$

where $M^{\mathsf{d}}(z) = \mathcal{Z}[m^{\mathsf{d}}(k)]$, and $E^{\mathsf{d}}(z) = \mathcal{Z}[e^{\mathsf{d}}(k)]$.

(Cont'd)

$\therefore$ the input/output relation of the system can be expressed as a transfer function

$$\frac{M^{\mathsf{d}}(z)}{E^{\mathsf{d}}(z)} = \frac{b_n + b_{n-1}z^{-1} + \cdots + b_0 z^{-n}}{1 + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}}$$

$\Rightarrow$ the output $m^{\mathsf{d}}(k)$ is the result of the inverse z-transform

$$m^{\mathsf{d}}(k) = \mathcal{Z}^{-1}[M^{\mathsf{d}}(z)] = \mathcal{Z}^{-1}\left[\frac{b_n + b_{n-1}z^{-1} + \cdots + b_0 z^{-n}}{1 + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}} E^{\mathsf{d}}(z)\right].$$

We thus can compute the system response $m^{\mathsf{d}}(k)$, by

1. finding the difference equation (Done)
2. computing $E^{\mathsf{d}}(z)$ via the z-transform of $e^{\mathsf{d}}(k)$ (Done)
3. deriving the inverse z-transform of $M^{\mathsf{d}}(z)$ above. (How?)

Method 1: Power series method

If one can represent $E^{\mathsf{d}}(z)$ in the power series form

$$\begin{aligned} E^{\mathsf{d}}(z) &= (\star) + (\star)z^{-1} + (\star)z^{-2} + \cdots \quad \text{(with some } (\star)\text{)} \\ &= e^{\mathsf{d}}(0)z^{0} + e^{\mathsf{d}}(1)z^{-1} + e^{\mathsf{d}}(2)z^{-2} + \cdots \end{aligned}$$

the sequence $\{e^{\mathsf{d}}(k)\}$ can be directly computed by definition.

Example: Find $m^{\mathsf{d}}(k)$ generated by

$$m^{\mathsf{d}}(k) = e^{\mathsf{d}}(k) - e^{\mathsf{d}}(k-1) - m^{\mathsf{d}}(k-1), \quad e^{\mathsf{d}}(k) = \begin{cases} 1, & \text{for even } k, \\ 0, & \text{for odd } k. \end{cases}$$

via the power series method.

Method 2: Partial-fraction expansion method

If we decompose $E^{\mathsf{d}}(z)$ as a sum of partial fractions

$$E^{\mathsf{d}}(z) = E_1^{\mathsf{d}}(z) + E_2^{\mathsf{d}}(z) + \cdots + E_q^{\mathsf{d}}(z)$$

where $\mathcal{Z}^{-1}[E_i(z)]$, $i = 1, \ldots, q$, is already known,

then the inverse of $E^{\mathsf{d}}(z)$ is derived by

$$\mathcal{Z}^{-1}[E^{\mathsf{d}}(z)] = \mathcal{Z}^{-1}[E_1^{\mathsf{d}}(z)] + \mathcal{Z}^{-1}[E_2^{\mathsf{d}}(z)] + \cdots + \mathcal{Z}^{-1}[E_q^{\mathsf{d}}(z)].$$

Example: Compute the inverse z-transform of $E(z) = \dfrac{z}{(z-1)(z-2)}$ via the partial-fraction expansion method.

Method 3: Inversion-formula method

By the theorem of residues,

$$e^{\mathsf{d}}(k) = \frac{1}{2\pi j}\oint_{\Gamma} E^{\mathsf{d}}(z)z^{k-1}\mathrm{d}z = \sum_{\text{at poles of } E^{\mathsf{d}}(z)z^{k-1}} \left[\text{residue of } E^{\mathsf{d}}(z)z^{k-1}\right]$$

where a residue of $E(z)z^{k-1}$ is evaluated as

- for simple pole at $z = a$,

$$\left(\text{residue of } E^{\mathsf{d}}(z)z^{k-1}\right)_{z=a} = (z-a)E^{\mathsf{d}}(z)z^{k-1}\Big|_{z=a}$$

- for multiple poles $z = a$ of order m,

$$\left(\text{residue of } E^{\mathsf{d}}(z)z^{k-1}\right)_{z=a} = \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\left[(z-a)^m E^{\mathsf{d}}(z)z^{k-1}\right]\Big|_{z=a}$$

(Cont'd)

Example (revisited): Compute the inverse z-transform of $E(z) = \dfrac{z}{(z-1)(z-2)}$ via the inverse-formula method.

Example: Compute the inverse of $E(z) = \dfrac{z}{(z-1)^2}$ via the inverse-formula method.

Method 4: Discrete convolution

As similar to the Laplace transform case, the inverse of $E^{\mathsf{d}}(z)$ decomposed by

$$E^{\mathsf{d}}(z) = E_1^{\mathsf{d}}(z)E_2^{\mathsf{d}}(z)$$

is exactly the same as the convolution of two signals

$$e^{\mathsf{d}}(k) = e_1^{\mathsf{d}}(k) * e_2^{\mathsf{d}}(k) = \sum_{n=0}^{k} e_1^{\mathsf{d}}(n)e_2^{\mathsf{d}}(k-n) = \sum_{n=0}^{k} e_1^{\mathsf{d}}(k-n)e_2^{\mathsf{d}}(n).$$

Note: It is not easy to compute the convolution directly...

Overview of Chapter 2

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Part A: Transfer function approach

- Discrete-time systems
- Definition of z-transform
- Properties
- Relation with difference equation
- Inverse z-transform

Part B: State space approach

- Flow graphs
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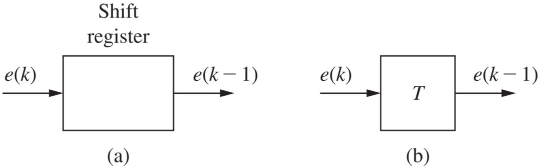
Time-delay element (or shift register)

- A linear difference equation can be represented by a simulation diagram.
- The time-delay element is a key of this conversion.



where the right version emphasizes the sampling period T.

$$e^{\mathsf{d}}(k) \xrightarrow{\text{Time-delay operator}} e^{\mathsf{d}}(k-1) \quad \xrightarrow{\mathcal{Z}} \quad E(z) \xrightarrow{\text{Time-delay operator}} z^{-1}E(z)$$

Note:

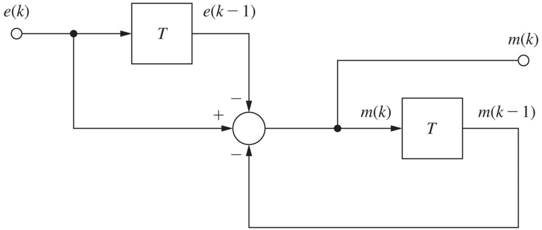
- A time-delay operation in discrete time $\leftrightarrow$ z^{-1} in z-domain
- Differentiation in continuous time $\leftrightarrow$ s in s-domain

Difference equation → Simulation diagram: Simple case

Consider a 1st-order difference equation

$$m^{\mathsf{d}}(k) = e^{\mathsf{d}}(k) - e^{\mathsf{d}}(k-1) - m^{\mathsf{d}}(k-1).$$

A simulation diagram of this system is given by



Note: One can rewrite the difference equation as

$$m^{\mathsf{d}}(k+1) = e^{\mathsf{d}}(k+1) - e^{\mathsf{d}}(k) - m^{\mathsf{d}}(k)$$

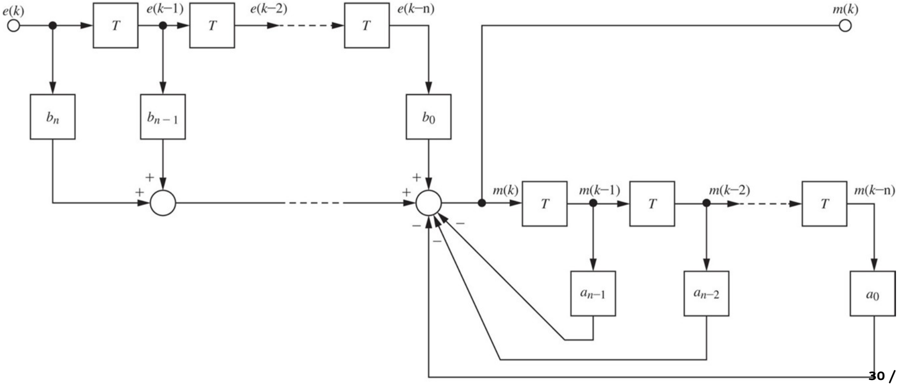
in order to include non-zero initial conditions (how?)

Difference equation → Simulation diagram: General case

For a general nth-order difference equation

$$
\begin{aligned}
m^{\mathsf{d}}(k) + a_{n-1}m^{\mathsf{d}}(k-1) + \cdots + a_0 m^{\mathsf{d}}(k-n) \\
= b_n e^{\mathsf{d}}(k) + b_{n-1}e^{\mathsf{d}}(k-1) + \cdots + b_0 e^{\mathsf{d}}(k-n)
\end{aligned}
$$

we have the simulation diagram as follows:



Difference equation → Transfer function: Simple case

Return to the 1st-order difference equation above:

$$
\begin{aligned}
m^{\mathsf{d}}(k) &= e^{\mathsf{d}}(k) - e^{\mathsf{d}}(k-1) - m^{\mathsf{d}}(k-1) \\
\xrightarrow{\mathcal{Z}} \quad M^{\mathsf{d}}(z) &= E^{\mathsf{d}}(z) - z^{-1}E^{\mathsf{d}}(z) - z^{-1}M^{\mathsf{d}}(z)
\end{aligned}
$$

Therefore, we have a transfer function

$$
G^{\mathsf{d}}(z) = \frac{M^{\mathsf{d}}(z)}{E^{\mathsf{d}}(z)} = \frac{1 - z^{-1}}{1 + z^{-1}} = \frac{z-1}{z+1}
$$

Note: You may reach the same conclusion with the modified difference equation

$$
m^{\mathsf{d}}(k+1) = e^{\mathsf{d}}(k+1) - e^{\mathsf{d}}(k) - m^{\mathsf{d}}(k)
$$

as long as the initial condition is set as zero.

Difference equation $\rightarrow$ Transfer function: General case

For a general nth-order difference equation

$$\begin{aligned} & m^{\mathsf{d}}(k) + a_{n-1} m^{\mathsf{d}}(k-1) + \cdots + a_0 m^{\mathsf{d}}(k-n) \\ & \quad = b_n e^{\mathsf{d}}(k) + b_{n-1} e^{\mathsf{d}}(k-1) + \cdots + b_0 e^{\mathsf{d}}(k-n) \end{aligned}$$

applying the z-transform to both sides leads to

$$\begin{aligned} & M^{\mathsf{d}}(z) + a_{n-1} z^{-1} M^{\mathsf{d}}(z) + \cdots + a_0 z^{-n} M^{\mathsf{d}}(z) \\ & \quad = b_n E^{\mathsf{d}}(z) + b_{n-1} z^{-1} E^{\mathsf{d}}(z) + \cdots + b_0 z^{-n} E^{\mathsf{d}}(z). \end{aligned}$$

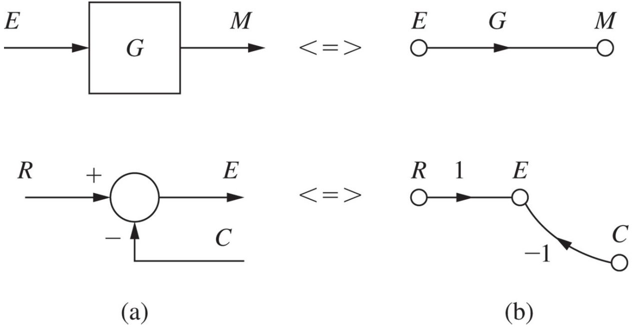
$\therefore$ one has the transfer function in the general case as

$$\begin{aligned} G^{\mathsf{d}}(z) := \frac{M^{\mathsf{d}}(z)}{E^{\mathsf{d}}(z)} &= \frac{b_n + b_{n-1} z^{-1} + \cdots + b_0 z^{-n}}{1 + a_{n-1} z^{-1} + \cdots + a_0 z^{-n}} \\ &= \frac{b_n z^n + b_{n-1} z^{n-1} + \cdots + \beta_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_0}. \end{aligned}$$

Block diagram $\leftrightarrow$ Flow graph

2.8. Simulation Diagrams and Flow Graphs

- Block diagram in (a) is a block-wise generalization of simulation diagram.
- Flow graph in (b) consists of node (= points) and branches (= lines)

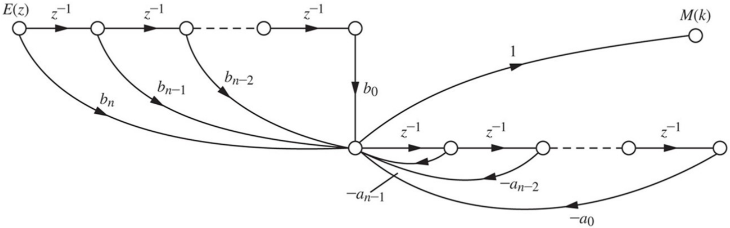


Upper represents $M^{\mathsf{d}}(z) = G^{\mathsf{d}}(z)E^{\mathsf{d}}(z)$, while lower $E^{\mathsf{d}}(z) = R^{\mathsf{d}}(z) - C^{\mathsf{d}}(z)$.

For a general nth-order difference equation

$$\begin{aligned} m^{\mathsf{d}}(k) + a_{n-1}m^{\mathsf{d}}(k-1) + \cdots + a_0 m^{\mathsf{d}}(k-n) \\ = b_n e^{\mathsf{d}}(k) + b_{n-1}e^{\mathsf{d}}(k-1) + \cdots + b_0 e^{\mathsf{d}}(k-n), \end{aligned}$$

the associated flow graph is given by



(b)

Note: In this class, we will not go further into the flow graph.

Example 2.16: Consider a system with the difference equation

$$y^{\mathsf{d}}(k+2) = u^{\mathsf{d}}(k) + 1.7y^{\mathsf{d}}(k+1) - 0.72y^{\mathsf{d}}(k), \quad (u^{\mathsf{d}}\text{: input, } y^{\mathsf{d}}\text{: output})$$

Our goal here is to find another expression of the same system (called state-variable model, 상태변수 모델), with

- a "1st-order" and "vectorized" difference equation (called state equation)

$$\mathbf{x}^{\mathsf{d}}(k+1) = \mathbf{A}^{\mathsf{d}}\mathbf{x}^{\mathsf{d}}(k) + \mathbf{B}^{\mathsf{d}}u^{\mathsf{d}}(k)$$

- and a linear equation (called output equation)

$$y^{\mathsf{d}}(k) = \mathbf{C}^{\mathsf{d}}\mathbf{x}^{\mathsf{d}}(k)$$

where

- $\mathbf{x}^{\mathsf{d}}$ is called a state variable (상태변수) that is a vector,
- $\mathbf{A}^{\mathsf{d}}$, $\mathbf{B}^{\mathsf{d}}$, $\mathbf{C}^{\mathsf{d}}$ are some matrices to be determined.

(Cont'd)

In our example, define the state variable $\mathbf{x}^{\mathsf{d}}$ as

$$\mathbf{x}^{\mathsf{d}}(k) := \begin{bmatrix} x_1^{\mathsf{d}}(k) \\ x_2^{\mathsf{d}}(k) \end{bmatrix} = \begin{bmatrix} y^{\mathsf{d}}(k) \\ y^{\mathsf{d}}(k+1) \end{bmatrix}.$$

Then, one can rewrite the difference equation as a state-variable model

$$\begin{aligned} \mathbf{x}^{\mathsf{d}}(k+1) &= \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix} \mathbf{x}^{\mathsf{d}}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u^{\mathsf{d}}(k) \qquad &&= \mathbf{A}^{\mathsf{d}}\mathbf{x}^{\mathsf{d}}(k) + \mathbf{B}^{\mathsf{d}}u^{\mathsf{d}}(k), \\ y^{\mathsf{d}}(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}^{\mathsf{d}}(k) &&= \mathbf{C}^{\mathsf{d}}\mathbf{x}^{\mathsf{d}}(k). \end{aligned}$$

Note:

- The state variable represents the the minimum amount of info. which is necessary to determine future behavior of the system.
- The state variable can be defined in different ways, which gives different $\mathbf{A}^{\mathsf{d}}$, $\mathbf{B}^{\mathsf{d}}$, $\mathbf{C}^{\mathsf{d}}$.

State-variable model: General form

Let us introduce the input, output, and state vectors

$$\mathbf{u}^{\mathsf{d}}(k) = \begin{bmatrix} u_1^{\mathsf{d}}(k) \\ \vdots \\ u_r^{\mathsf{d}}(k) \end{bmatrix}, \quad \mathbf{y}^{\mathsf{d}}(k) = \begin{bmatrix} y_1^{\mathsf{d}}(k) \\ \vdots \\ y_p^{\mathsf{d}}(k) \end{bmatrix}, \quad \mathbf{x}^{\mathsf{d}}(k) = \begin{bmatrix} x_1^{\mathsf{d}}(k) \\ \vdots \\ x_n^{\mathsf{d}}(k) \end{bmatrix}$$

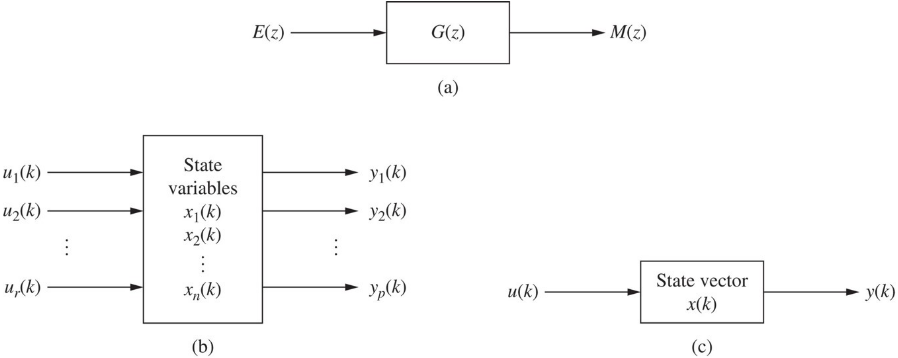
A general state-variable model with nonlinear functions $\mathbf{f}$ and $\mathbf{h}$:

$$\begin{aligned} \mathbf{x}^{\mathsf{d}}(k+1) &= \mathbf{f}(\mathbf{x}^{\mathsf{d}}(k), \mathbf{u}^{\mathsf{d}}(k)), && \text{(State equation)} \\ \mathbf{y}^{\mathsf{d}}(k) &= \mathbf{h}(\mathbf{x}^{\mathsf{d}}(k), \mathbf{u}^{\mathsf{d}}(k)). && \text{(Output equation)} \end{aligned}$$

IF $\mathbf{f}$ and $\mathbf{h}$ are linear functions on their arguments (so that the system is linear), THEN the state-variable model has the form

$$\begin{aligned} \mathbf{x}^{\mathsf{d}}(k+1) &= \mathbf{A}^{\mathsf{d}}\mathbf{x}^{\mathsf{d}}(k) + \mathbf{B}^{\mathsf{d}}\mathbf{u}^{\mathsf{d}}(k), && \text{(State equation)} \\ \mathbf{y}^{\mathsf{d}}(k) &= \mathbf{C}^{\mathsf{d}}\mathbf{x}^{\mathsf{d}}(k) + \mathbf{D}^{\mathsf{d}}\mathbf{u}^{\mathsf{d}}(k). && \text{(Output equation)} \end{aligned}$$

Several blocks that represents a state-variable model



- (a) Transfer function
- (b) State-variable model
- (c) A simplified version of the state-variable model

Transfer function vs. State-variable model

- Transfer function

$$G^{\mathsf{d}}(z) = \frac{b_n + b_{n-1}z^{-1} + \cdots + b_0 z^{-n}}{1 + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}}$$

  - Useful for the frequency-domain analysis
  - Not easy to understand the system response in time

- State-variable model

$$\begin{aligned} \mathbf{x}^{\mathsf{d}}(k+1) &= \mathbf{A}^{\mathsf{d}}\mathbf{x}^{\mathsf{d}}(k) + \mathbf{B}^{\mathsf{d}}\mathbf{u}^{\mathsf{d}}(k), && \text{(State equation)} \\ \mathbf{y}^{\mathsf{d}}(k) &= \mathbf{C}^{\mathsf{d}}\mathbf{x}^{\mathsf{d}}(k) + \mathbf{D}^{\mathsf{d}}\mathbf{u}^{\mathsf{d}}(k). && \text{(Output equation)} \end{aligned}$$

  - Helpful to understand time-domain response.
  - Available when the system is nonlinear.
  - Maybe we have to learn a lot of materials for that...

Transfer function $\rightarrow$ State-variable model: 2nd-order case

Consider a 2nd-order system whose transfer function is given by

$$G^{\mathsf{d}}(z) = \frac{Y^{\mathsf{d}}(z)}{U^{\mathsf{d}}(z)} = \frac{b_1 z + b_0}{z^2 + a_1 z + a_0}$$

Note that, for any $E^{\mathsf{d}}(z)$ (whose inverse z-transform is well-defined), the transfer function can also be represented as

$$G^{\mathsf{d}}(z) = \frac{Y^{\mathsf{d}}(z)}{U^{\mathsf{d}}(z)} = \frac{(b_1 z + b_0)E^{\mathsf{d}}(z)}{(z^2 + a_1 z + a_0)E^{\mathsf{d}}(z)}$$

$\therefore$ It is possible to express $Y^{\mathsf{d}}(z)$ and $U^{\mathsf{d}}(z)$ as follows:

$$\begin{aligned} Y^{\mathsf{d}}(z) &= (b_1 z + b_0)E^{\mathsf{d}}(z) && = b_1 z E^{\mathsf{d}}(z) + b_0 E^{\mathsf{d}}(z), \\ U^{\mathsf{d}}(z) &= (z^2 + a_1 z + a_0)E^{\mathsf{d}}(z) && = z^2 E^{\mathsf{d}}(z) + a_1 z E^{\mathsf{d}}(z) + a_0 E^{\mathsf{d}}(z) \end{aligned}$$

for a particular $E^{\mathsf{d}}(z)$.

(Cont'd)

2.9. State Variables

Taking the inverse z-transform gives a candidate for the state variable x_i^{d}:

$$\begin{aligned} E^{\mathsf{d}}(z) &\to e^{\mathsf{d}}(k) && =: x_1^{\mathsf{d}}(k) \\ zE^{\mathsf{d}}(z) &\to e^{\mathsf{d}}(k+1) && =: x_2^{\mathsf{d}}(k) \end{aligned}$$

Then we have the following relations
(whose form is exactly the state/output equations above):

$$\begin{aligned} (e^{\mathsf{d}}(k+1) =) \quad & x_1^{\mathsf{d}}(k+1) = x_2^{\mathsf{d}}(k) \\ (e^{\mathsf{d}}(k+2) =) \quad & x_2^{\mathsf{d}}(k+1) = -a_0 x_1^{\mathsf{d}}(k) - a_1 x_2^{\mathsf{d}}(k) + u^{\mathsf{d}}(k) \end{aligned}$$

and

$$y^{\mathsf{d}}(k) = b_0 x_1^{\mathsf{d}}(k) + b_1 x_2^{\mathsf{d}}(k).$$

Transfer function → State-variable model: General case

We will extend the idea in the previous slides to a general nth-order system, which is represented by a transfer function

$$G^{\mathsf{d}}(z) = \frac{Y^{\mathsf{d}}(z)}{U^{\mathsf{d}}(z)} = \frac{b_{n-1}z^{n-1} + \cdots + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_0}$$

or by a difference equation

$$\begin{aligned} y^{\mathsf{d}}(k) + a_{n-1}y^{\mathsf{d}}(k-1) + \cdots + a_0 y^{\mathsf{d}}(k-n) \\ = b_{n-1}u^{\mathsf{d}}(k-1) + \cdots + b_0 u^{\mathsf{d}}(k-n). \end{aligned}$$

(Cont'd)

2.9. State Variables

Following the same procedure,

we have the state-variable model for a general nth-order system:

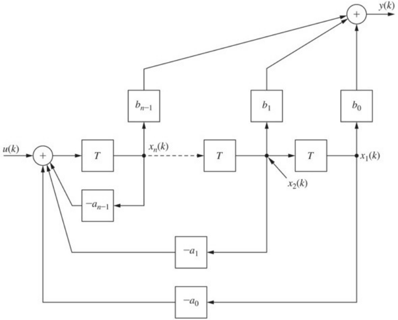
$$
\begin{aligned}
\begin{bmatrix} x_1^{\mathsf{d}}(k+1) \\ x_2^{\mathsf{d}}(k+1) \\ \vdots \\ x_n^{\mathsf{d}}(k+1) \end{bmatrix}
&= \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
 & & & \vdots & & & \\
-a_0 & -a_1 & -a_2 & -a_3 & -a_4 & \cdots & -a_{n-1}
\end{bmatrix}
\begin{bmatrix} x_1^{\mathsf{d}}(k) \\ x_2^{\mathsf{d}}(k) \\ \vdots \\ x_n^{\mathsf{d}}(k) \end{bmatrix}
+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u^{\mathsf{d}}(k) \\
&= \mathbf{A}^{\mathsf{d}} \mathbf{x}^{\mathsf{d}}(k) + \mathbf{B}^{\mathsf{d}} u^{\mathsf{d}}(k), \\
y^{\mathsf{d}}(k) &= \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix}
\begin{bmatrix} x_1^{\mathsf{d}}(k) \\ x_2^{\mathsf{d}}(k) \\ \vdots \\ x_n^{\mathsf{d}}(k) \end{bmatrix} \\
&= \mathbf{C}^{\mathsf{d}} \mathbf{x}^{\mathsf{d}}(k).
\end{aligned}
$$

State-variable model → Simulation diagram

The state equation above can be viewed as a chain of time-delay elements:

$$x_1^{\mathsf{d}}(k) \xleftarrow{\text{Delay}} x_1^{\mathsf{d}}(k+1) = x_2^{\mathsf{d}}(k) \xleftarrow{\text{Delay}} \cdots \xleftarrow{\text{Delay}} x_{n-1}^{\mathsf{d}}(k+1) = x_n^{\mathsf{d}}(k).$$

∴ One can represent the state-variable model as a simulation diagram



(b)

Example 2.19: Find a state-variable model of

$$G^{\mathsf{d}}(z) = \frac{Y^{\mathsf{d}}(z)}{U^{\mathsf{d}}(z)} = \frac{z^2 + 2z + 1}{z^3 + 2z^2 + z + 1/2}$$

Example 2.20: Find a state-variable model of

$$G^{\mathsf{d}}(z) = \frac{Y^{\mathsf{d}}(z)}{U^{\mathsf{d}}(z)} = \frac{b_2 z^2 + b_1 z + b_0}{z^2 + a_1 z + a_0}$$

Notice that the order of numerator and denominator are the same.

State-variable model $\rightarrow$ Transfer function

Consider a state-variable model

$$\mathbf{x}^{\mathsf{d}}(k+1) = \mathbf{A}^{\mathsf{d}}\mathbf{x}^{\mathsf{d}}(k) + \mathbf{B}^{\mathsf{d}}u^{\mathsf{d}}(k), \quad y^{\mathsf{d}}(k) = \mathbf{C}^{\mathsf{d}}\mathbf{x}^{\mathsf{d}}(k) + \mathbf{D}^{\mathsf{d}}u^{\mathsf{d}}(k).$$

When applying the z-transform to both sides of the state equation, we have

$$\begin{aligned} & z\mathbf{X}^{\mathsf{d}}(z) - z\mathbf{x}^{\mathsf{d}}(0) = \mathbf{A}^{\mathsf{d}}\mathbf{X}^{\mathsf{d}}(z) + \mathbf{B}^{\mathsf{d}}U^{\mathsf{d}}(z) \\ \Rightarrow \quad & [z\mathbf{I} - \mathbf{A}^{\mathsf{d}}]\mathbf{X}^{\mathsf{d}}(z) = \mathbf{B}^{\mathsf{d}}U^{\mathsf{d}}(z) \\ \Rightarrow \quad & \mathbf{X}^{\mathsf{d}}(z) = [z\mathbf{I} - \mathbf{A}^{\mathsf{d}}]^{-1}\mathbf{B}^{\mathsf{d}}U^{\mathsf{d}}(z). \end{aligned}$$

Similar computation gives

$$\begin{aligned} y^{\mathsf{d}}(k) = \mathbf{C}^{\mathsf{d}}\mathbf{x}^{\mathsf{d}}(k) + Du^{\mathsf{d}}(k) \xrightarrow{\mathcal{Z}} \quad Y^{\mathsf{d}}(z) &= \mathbf{C}^{\mathsf{d}}\mathbf{X}^{\mathsf{d}}(z) + D^{\mathsf{d}}U^{\mathsf{d}}(z) \\ &= (\mathbf{C}^{\mathsf{d}}[z\mathbf{I} - \mathbf{A}^{\mathsf{d}}]^{-1}\mathbf{B}^{\mathsf{d}} + D^{\mathsf{d}})U^{\mathsf{d}}(z) \end{aligned}$$

(Cont'd)

which implies that, the transfer function of the system is

$$G^{\mathsf{d}}(z) = \frac{Y^{\mathsf{d}}(z)}{U^{\mathsf{d}}(z)} = \mathbf{C}^{\mathsf{d}}[z\mathbf{I} - \mathbf{A}^{\mathsf{d}}]^{-1}\mathbf{B}^{\mathsf{d}} + D^{\mathsf{d}}.$$

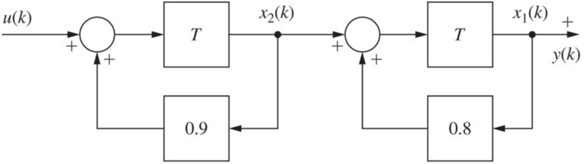
Note: For the same $G^{\mathsf{d}}(z)$, its state-variable model is NOT unique. (WHY???)

Example 2.22: Find a state-variable model of

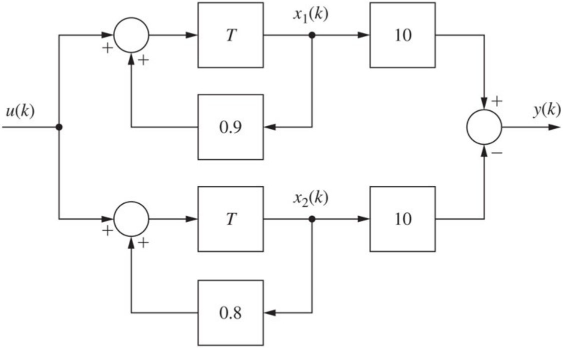
$$\begin{aligned} G^{\mathsf{d}}(z) = \frac{Y^{\mathsf{d}}(z)}{U^{\mathsf{d}}(z)} &= \frac{1}{z^2 - 1.7z + 0.72} \\ &= \left(\frac{1}{z-0.9}\right)\left(\frac{1}{z-0.8}\right) && \text{Series connection, Fig.(a)} \\ &= \frac{10}{z-0.9} + \frac{-10}{z-0.8}, && \text{Parallel connection, Fig.(b)} \end{aligned}$$

(Cont'd)

2.10. Other State-variable Formulations



(a)



Similar transformation (유사변환)

= A way of finding another state $\mathbf{w}^{\mathsf{d}}$ from the original $\mathbf{x}^{\mathsf{d}}$.

If we define another vector $\mathbf{w}^{\mathsf{d}}$ with nonsingular matrix $\mathbf{P}$,

$$\mathbf{x}^{\mathsf{d}} = \mathbf{P}\mathbf{w}^{\mathsf{d}}$$

(which is often called similar transformation (유사변환)) then,

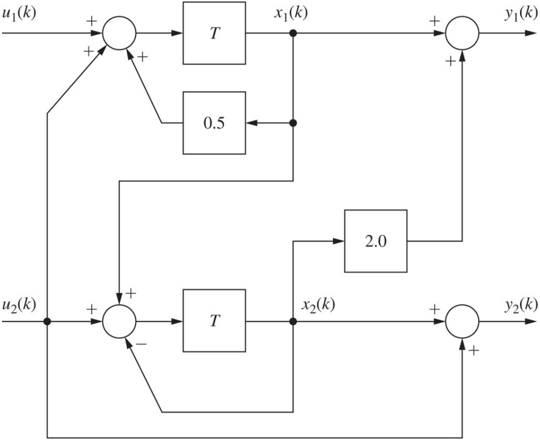
$$\begin{aligned}
\mathbf{w}^{\mathsf{d}}(k+1) &= (\mathbf{P}^{-1}\mathbf{A}^{\mathsf{d}}\mathbf{P})\mathbf{w}^{\mathsf{d}}(k) + (\mathbf{P}^{-1}\mathbf{B}^{\mathsf{d}})\mathbf{u}^{\mathsf{d}}(k) \\
&= \mathbf{A}^{\mathsf{d}}_{\mathsf{new}}\mathbf{w}^{\mathsf{d}}(k) + \mathbf{B}^{\mathsf{d}}_{\mathsf{new}}\mathbf{u}^{\mathsf{d}}(k), \\
\mathbf{y}^{\mathsf{d}}(k) &= (\mathbf{C}^{\mathsf{d}}\mathbf{P})\mathbf{w}^{\mathsf{d}}(k) + (\mathbf{D}^{\mathsf{d}})\mathbf{u}^{\mathsf{d}}(k) \\
&= \mathbf{C}^{\mathsf{d}}_{\mathsf{new}}\mathbf{w}^{\mathsf{d}}(k) + \mathbf{D}^{\mathsf{d}}_{\mathsf{new}}\mathbf{u}^{\mathsf{d}}(k)
\end{aligned}$$

But, these state-variable models represent the same system: that is,

$$\mathbf{C}^{\mathsf{d}}[z\mathbf{I} - \mathbf{A}^{\mathsf{d}}]^{-1}\mathbf{B}^{\mathsf{d}} + \mathbf{D}^{\mathsf{d}} = \mathbf{C}^{\mathsf{d}}\mathbf{P}^{\mathsf{d}}[z\mathbf{I} - {\mathbf{P}^{\mathsf{d}}}^{-1}\mathbf{A}^{\mathsf{d}}\mathbf{P}^{\mathsf{d}}]^{-1}\mathbf{B}^{\mathsf{d}}\mathbf{P}^{\mathsf{d}} + \mathbf{D}^{\mathsf{d}}$$

A multivariable case: Example

See Example 2.21 for example.



Solution of state equations

2.12. Solutions of the State Equations

When the initial condition $\mathbf{x}^{\mathsf{d}}(0)$ is given,
one can compute the solution $\mathbf{x}^{\mathsf{d}}(k)$ recursively by using the state equation.

$$
\begin{aligned}
\mathbf{x}^{\mathsf{d}}(1) &= \mathbf{A}^{\mathsf{d}}\mathbf{x}^{\mathsf{d}}(0) + \mathbf{B}^{\mathsf{d}}\mathbf{u}^{\mathsf{d}}(0), \\
\mathbf{x}^{\mathsf{d}}(2) &= \mathbf{A}^{\mathsf{d}}\mathbf{x}^{\mathsf{d}}(1) + \mathbf{B}^{\mathsf{d}}\mathbf{u}^{\mathsf{d}}(1) = \mathbf{A}^{2}\mathbf{x}^{\mathsf{d}}(0) + \mathbf{A}^{\mathsf{d}}\mathbf{B}^{\mathsf{d}}\mathbf{u}^{\mathsf{d}}(0) + \mathbf{B}^{\mathsf{d}}\mathbf{u}^{\mathsf{d}}(1) \\
&\vdots
\end{aligned}
$$

Thus, the solution $\mathbf{x}^{\mathsf{d}}(k)$ has the explicit form (w/ $\boldsymbol{\Phi}^{\mathsf{d}}(k) := (\mathbf{A}^{\mathsf{d}})^k$)

$$
\mathbf{x}^{\mathsf{d}}(k) = \boldsymbol{\Phi}^{\mathsf{d}}(k)\mathbf{x}^{\mathsf{d}}(0) + \sum_{j=0}^{k-1} \boldsymbol{\Phi}^{\mathsf{d}}(k-1-j)\mathbf{B}^{\mathsf{d}}\mathbf{u}^{\mathsf{d}}(j)
$$

This idea can be extended to time-varying case (omitted here).

Example

Example 2.29: Find the solution of the following system w/ $u^{\mathsf{d}}(k) = 1$.

$$\begin{aligned} \mathbf{x}^{\mathsf{d}}(k+1) &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x}^{\mathsf{d}}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u^{\mathsf{d}}(k), \\ y^{\mathsf{d}}(k) &= \begin{bmatrix} 3 & 1 \end{bmatrix} \mathbf{x}^{\mathsf{d}}(k) \end{aligned}$$

State transition matrix?

2.12. Solutions of the State Equations

For the linear time-invariant system, a state transition matrix $\mathbf{\Phi}^{\mathsf{d}}$ is defined by

$$\mathbf{\Phi}(k) := \left(\mathbf{A}^{\mathsf{d}}\right)^{k}$$

Why called "state transition"? Assuming $u^{\mathsf{d}} \equiv 0$,
$\mathbf{\Phi}^{\mathsf{d}}(k)$ maps the initial value $\mathbf{x}^{\mathsf{d}}(0)$ to the current state $\mathbf{x}^{\mathsf{d}}(k)$ as

$$\mathbf{x}^{\mathsf{d}}(k) = \mathbf{\Phi}^{\mathsf{d}}(k)\mathbf{x}^{\mathsf{d}}(0).$$

Properties:

- $\mathbf{\Phi}^{\mathsf{d}}(0) = \mathbf{I}$ (= identity matrix)
- $\mathbf{\Phi}^{\mathsf{d}}(k_1 + k_2) = \mathbf{\Phi}^{\mathsf{d}}(k_1)\mathbf{\Phi}^{\mathsf{d}}(k_2)$
- $\mathbf{\Phi}^{\mathsf{d}}(-k) = (\mathbf{\Phi}^{\mathsf{d}})^{-1}(k)$

Finding $\mathbf{\Phi}^{\mathsf{d}}(k)$ via z-transform

2.12. Solutions of the State Equations

Consider the case when there is no input $\mathbf{u}^{\mathsf{d}}(k)$, by which one has (why??)

$$\mathbf{X}^{\mathsf{d}}(z) = z[z\mathbf{I} - \mathbf{A}^{\mathsf{d}}]^{-1}\mathbf{x}^{\mathsf{d}}(0).$$

Take the inverse z-transform to both sides. Then we have

$$\mathbf{x}^{\mathsf{d}}(k) = \mathcal{Z}^{-1}[z[z\mathbf{I} - \mathbf{A}^{\mathsf{d}}]^{-1}]\mathbf{x}^{\mathsf{d}}(0) = (\mathbf{\Phi}^{\mathsf{d}})(k)\mathbf{x}^{\mathsf{d}}(0)$$

Example 2.30: Compute $\mathbf{\Phi}(k)$ for the system in Example 2.29.

Summary

Part A: Transfer function approach

- Discrete-time systems
- Definition of z-transform
- Properties
- Relation with difference equation
- Inverse z-transform

Part B: State space approach

- Flow graphs
- State variables and state equations
- Relation with transfer function
- Solution of state equations