[2024-1 Digital Control]
Chapter 5. Closed-loop Systems

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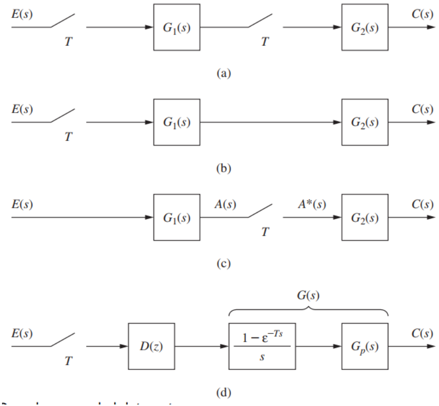
Cascaded open-loop system

5.1. Preliminary Concepts

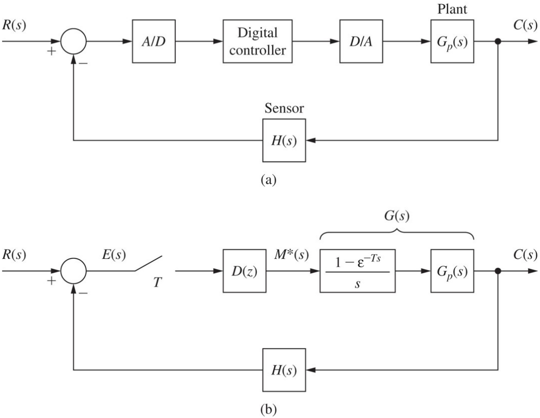
There are several types of open-loop sampled-data systems, such as



Note: NOT all the systems have a transfer function.

A closed-loop system in sampled-data framework

5.1. Preliminary Concepts



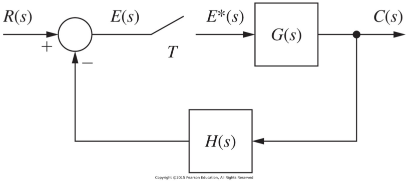
Why close the loop in control?

∵ The principle of feedback (i.e., we want to adjust uncertain environments)

A closed-loop system that has a transfer function

5.1. Preliminary Concepts

Consider a closed-loop system whose configuration is given by



(which is a usual form of the sampled-data system)

The output $C(s)$ can be represented as

$$C(s) = G(s)E^*(s)$$

and thus, one has

$$E(s) = R(s) - H(s)C(s) = R(s) - H(s)G(s)E^*(s)$$

(Cont'd)

5.1. Preliminary Concepts

Taking the starred transform $(\cdot)^*$ to both sides gives

$$E^*(s) = R^*(s) - \overline{GH}^*(s)E^*(s) \quad \Rightarrow \quad E^*(s) = \frac{1}{1+\overline{GH}^*(s)}R^*(s).$$

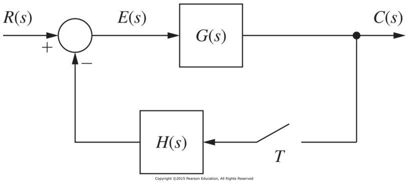
We finally have

$$\begin{aligned} C(s) &= G(s)E^*(s) = G(s)\frac{R^*(s)}{1+\overline{GH}^*(s)} \\ \Rightarrow\ C^*(s) &= \frac{G(s)}{1+\overline{GH}^*(s)}R^*(s) \\ \Rightarrow\ C^{\mathsf{d}}(z) &= \frac{G^{\mathsf{d}}(z)}{1+\overline{GH}^{\mathsf{d}}(z)}R^{\mathsf{d}}(z). \end{aligned}$$

$\therefore$ One can obtain a discrete-time transfer function btw. $R^{\mathsf{d}}(z)$ and $C^{\mathsf{d}}(z)$.

A closed-loop system that has no transfer function

Consider another example depicted below:



The error signal is computed by

$$E(s) = R(s) - H(s)C^*(s)$$

from which it follows that

$$\begin{aligned} & C(s) = G(s)E(s) = G(s)R(s) - G(s)H(s)C^*(s) \\ \Rightarrow \quad & C^*(s) = \overline{GR}^*(s) - \overline{GH}^*(s)C^*(s) \end{aligned}$$

(Cont'd)

5.2. Preliminary Concepts

In a simpler form,

$$C^*(s) = \frac{\overline{GR}^*(s)}{1 + \overline{GH}^*(s)},$$

$$\Rightarrow \; C^{\mathsf{d}}(z) = \frac{\overline{GR}^{\mathsf{d}}(z)}{1 + \overline{GH}^{\mathsf{d}}(z)}$$

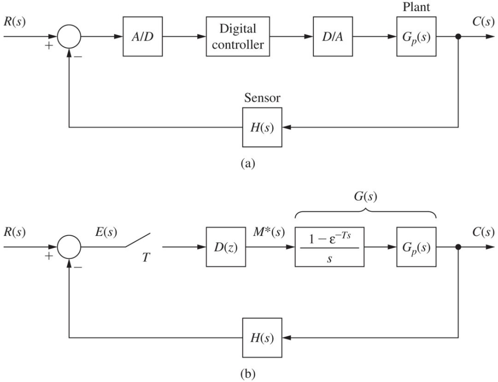
$\therefore$ there is NO specific transfer function btw. $R(s)$ and $C(s)$.

Why this happens? The input $R(s)$ is NOT sampled before reached at $C(s)$.

Example 5.1

5.3. Derivation Procedure

Show that the (discrete-time) transfer function of the closed-loop system

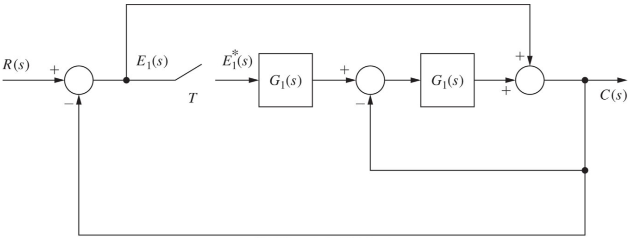


is computed by

$$C^{\mathsf{d}}(z) = \frac{D^{\mathsf{d}}(z)G^{\mathsf{d}}(z)}{1 + D^{\mathsf{d}}(z)\overline{GH}^{\mathsf{d}}(z)} R^{\mathsf{d}}(z)$$

Example 5.3

5.3. Derivation Procedure



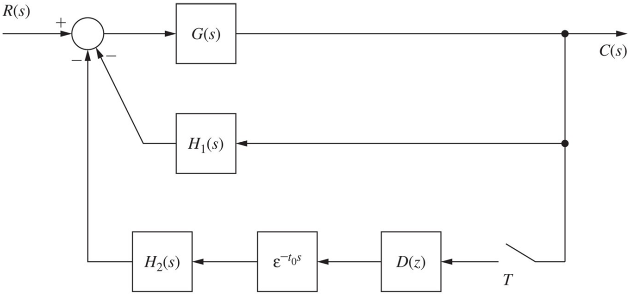
Note: No transfer function may be derived $\because R(s)$ is NOT sampled in the loop.

After some computations, we have

$$C^*(s) = \left[\frac{R}{2+G_2}\right]^*(s) + \frac{\left[\dfrac{G_1G_2}{2+G_2}\right]^*(s)}{1+\left[\dfrac{G_1G_2}{2+G_2}\right]^*(s)}\left[\frac{(1+G_2)R}{2+G_2}\right]^*(s)$$

Example 5.4

5.3. Derivation Procedure

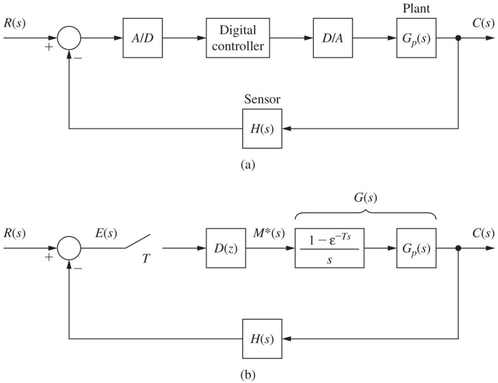


With help of the modified z-transform, one has (for $0 < t_0 < T$)

$$C^{\mathsf{d}}(z) = \frac{\left[\dfrac{GR}{1+GH_1}\right]^{\mathsf{d}}(z)}{1 + \left[\dfrac{GH_2}{1+GH_1}\right]^{\mathsf{d}}_{\mathrm{mod}}(z,m)D^{\mathsf{d}}(z)}$$

How about the closed-loop system in the state space?

5.4. State-variable Models

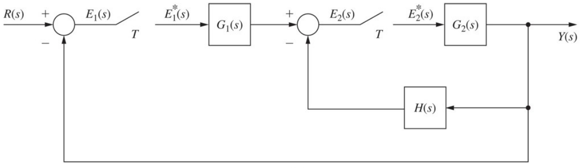


- Step 1: find the continuous-time state-space equation for $G_p(s)$ or $G_pH(s)$.
- Step 2: compute its discretization.
- Step 3: augment the discrete-time plant with the discrete-time controller.

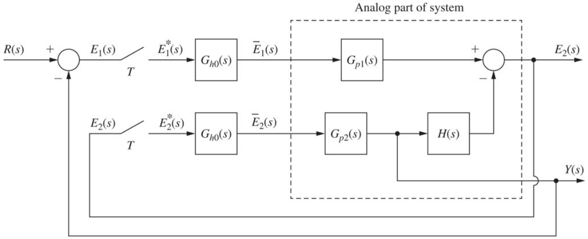
Example 5.5

5.4. State-variable Models

Consider the following closed-loop sampled-data system (Fig. (a)) that can be transformed into Fig. (b):



(a)



(b)

(Cont'd)

5.4. State-variable Models

Let the transfer functions in the blocks in Fig.(b) be

$$
\begin{aligned}
G_1(s) &= \frac{1-\mathsf{e}^{-Ts}}{s^2(s+1)} = \frac{1-\mathsf{e}^{-Ts}}{s} G_{p1}(s), \\
G_2(s) &= \frac{2(1-\mathsf{e}^{-Ts})}{s(s+2)} = \frac{1-\mathsf{e}^{-Ts}}{s} G_{p2}(s), \\
H(s) &= \frac{10}{s+10}
\end{aligned}
$$

Step 1: Obtain a continuous-time state-space equation for the analog part

$$
\dot{\mathbf{v}}^{\mathsf{c}}(t) = \mathbf{A}^{\mathsf{c}}\mathbf{v}^{\mathsf{c}}(t) + \mathbf{B}^{\mathsf{c}} \begin{bmatrix} \overline{e}_1^{\mathsf{c}}(t) \\ \overline{e}_2^{\mathsf{c}}(t) \end{bmatrix}, \quad \begin{bmatrix} y^{\mathsf{c}}(t) \\ e_2^{\mathsf{c}}(t) \end{bmatrix} = \mathbf{C}^{\mathsf{c}}\mathbf{v}^{\mathsf{c}}(t)
$$

Step 2: Together with the sampler and the zero-order hold,

the discretized model of the continuous-time system is given by

$$\mathbf{v}^{\mathsf{d}}(k+1) = \mathbf{A}^{\mathsf{d}}\mathbf{v}^{\mathsf{d}}(k) + \mathbf{B}^{\mathsf{d}}\begin{bmatrix}\overline{e}_1^{\mathsf{d}}(k)\\ \overline{e}_2^{\mathsf{d}}(k)\end{bmatrix}, \quad \begin{bmatrix}y^{\mathsf{d}}(k)\\ e_2^{\mathsf{d}}(k)\end{bmatrix} = \mathbf{C}^{\mathsf{d}}\mathbf{v}^{\mathsf{d}}(k)$$

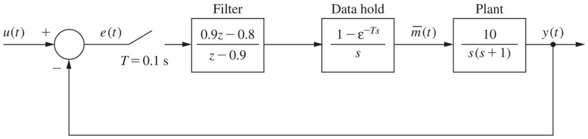
Step 3: Since

$$\begin{bmatrix}\overline{e}_1^{\mathsf{d}}(k)\\ \overline{e}_2^{\mathsf{d}}(k)\end{bmatrix} = \begin{bmatrix}r^{\mathsf{d}}(k)\\ 0\end{bmatrix} + \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}y^{\mathsf{d}}(k)\\ e_2^{\mathsf{d}}(k)\end{bmatrix} = \begin{bmatrix}r^{\mathsf{d}}(k)\\ 0\end{bmatrix} + \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\mathbf{C}^{\mathsf{d}}\mathbf{v}^{\mathsf{d}}(k),$$

the closed-loop system turns out to be

$$\mathbf{v}^{\mathsf{d}}(k+1) = \left(\mathbf{A}^{\mathsf{d}} + \mathbf{B}^{\mathsf{d}}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\mathbf{C}^{\mathsf{d}}\right)\mathbf{v}^{\mathsf{d}}(k) + \mathbf{B}^{\mathsf{d}}\begin{bmatrix}r^{\mathsf{d}}(k)\\ 0\end{bmatrix}$$

Example 5.7: When a digital filter is added in the loop



The system consists of the sampled-data system

$$\begin{bmatrix} v_1^{\mathsf{d}}(k+1) \\ v_2^{\mathsf{d}}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} \begin{bmatrix} v_1^{\mathsf{d}}(k) \\ v_2^{\mathsf{d}}(k) \end{bmatrix} + \begin{bmatrix} 0.0484 \\ 0.952 \end{bmatrix} m^{\mathsf{d}}(k),$$

$$y^{\mathsf{d}}(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_1^{\mathsf{d}}(k) \\ v_2^{\mathsf{d}}(k) \end{bmatrix}$$

and the digital filter or controller

$$v_3^{\mathsf{d}}(k+1) = 0.9 v_3^{\mathsf{d}}(k) + e^{\mathsf{d}}(k), \quad m^{\mathsf{d}}(k) = 0.01 v_3^{\mathsf{d}}(k) + 0.9 e^{\mathsf{d}}(k).$$

$\Rightarrow$ The state-space equation of the closed-loop system?