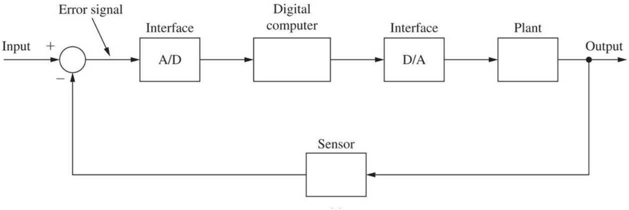
[2024-1 Digital Control]
Chapter 3. Sampling and Reconstruction

Gyunghoon Park

School of ECE, University of Seoul

Sampled-data control systems and questions of interest

Sampled-data system = Continuous-time plant + Discrete-time controller + Sampler and (data) hold (that we will discuss here)



- Q: How can we analyze the stability/performance of this system?

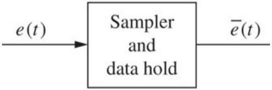
  A: z-transform (and state-space approach)
- Q: How can we represent the system in z-transform?

  A: We have to study more in the following two chapters.

Sampling and reconstruction

3.2. Sampled-data Control Systems

Sampler and data hold are two key components of sampled-data systems.



- Sampler: From analog to digital; that is, $e(t) \to e^{\mathsf{d}}(k)$
- (Data) hold: From digital to analog; that is, $e^{\mathsf{d}}(k) \to \bar{e}(t)$

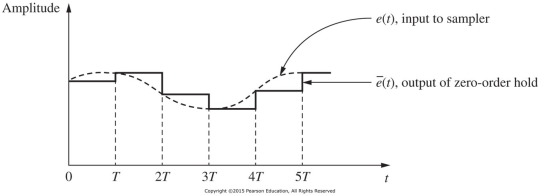
Example:

- Ideal sampler: $e^{\mathsf{d}}(k) = e(kT)$, $\forall k = 0, 1, 2, \ldots$.
- Zero-order hold: $\bar{e}(t) = e^{\mathsf{d}}(k)$, $\forall kT \leq t < (k+1)T$.

This process can be regarded as sampling and reconstruction of $e(t)$.

Sample and reconstruction of $e(t)$

3.2. Sampled-data Control Systems



- $e(t)$: The input of sampler (dashed).
- $\bar{e}(t)$: The output of sampling and reconstruction process (solid).
- With the unit step function $\mathbf{1}(t) := \begin{cases} 1, & t \geq 0, \\ 0, & \text{otherwise} \end{cases}$, we have

$$\begin{aligned} \bar{e}(t) &= e(0)[\mathbf{1}(t) - \mathbf{1}(t-T)] \\ &\quad + e(T)[\mathbf{1}(t-T) - \mathbf{1}(t-2T)] \\ &\quad + e(2T)[\mathbf{1}(t-2T) - \mathbf{1}(t-3T)] + \cdots \end{aligned}$$

Laplace transform of $\bar{e}(t)$

3.2. Sampled-data Control Systems

The Laplace transform of $\overline{e}(t)$ is computed by

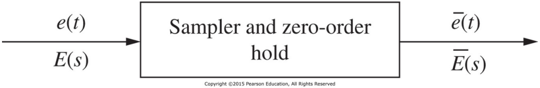
$$
\begin{aligned}
&\mathcal{L}[\overline{e}(t)] \\
&= \overline{E}(s) \\
&= e(0)\left[\frac{1}{s} - \frac{\mathrm{e}^{-Ts}}{s}\right] + e(T)\left[\frac{\mathrm{e}^{-Ts}}{s} - \frac{\mathrm{e}^{-2Ts}}{s}\right] + e(2T)\left[\frac{\mathrm{e}^{-2Ts}}{s} - \frac{\mathrm{e}^{-3Ts}}{s}\right] + \cdots \\
&= \left[e(0) + e(T)\mathrm{e}^{-Ts} + e(2T)\mathrm{e}^{-2Ts} + \cdots\right]\left[\frac{1 - \mathrm{e}^{-Ts}}{s}\right] \\
&= E^*(s)\left[\frac{1 - \mathrm{e}^{-Ts}}{s}\right]
\end{aligned}
$$

where $E^*(s)$ is the starred transform of $e(t)$ defined by

$$E^*(s) := \sum_{n=0}^{\infty} e(nT)\mathrm{e}^{-nTs}.$$

Key messages behind $\overline{E}(s) = E^*(s)\left[\frac{1-e^{-Ts}}{s}\right]$

- The equation represents the entire process of sampling and reconstruction.

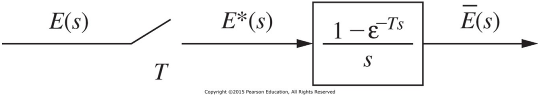


- We will see below that

$$E^*(s) = \text{The Laplace transform of sampled signal } e^\star(t) \text{ in a pulse form,}$$

$$\frac{1-e^{-Ts}}{s} = \text{The transfer function of the zero-order hold}$$

- The figure above can be re-expressed as a cascade connection:



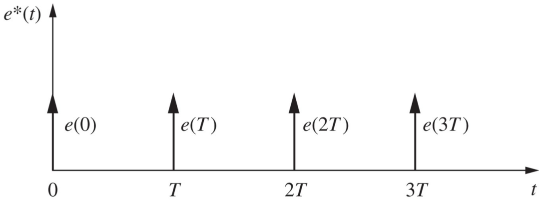
Sampled signal in a pulse form

3.3. The Ideal Sampler

One can compute the inverse Laplace transform of $E^*(s)$ as

$$e^*(t) := \mathcal{L}^{-1}[E^*(s)] = e(0)\delta(t) + e(T)\delta(t-T) + e(2T)\delta(t-2T) + \cdots$$

where $\delta(t)$ is the impulse function.



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Note:

- Roughly speaking, $e^*(t)$ is a continuous-time counterpart of $e^{\mathsf{d}}(k) = e(kT)$.
- $e^*(t)$ does not exist in a physical system. (Why?)
- It seems that $\exists$ no transfer function between $E(s)$ and $E^*(s)$.

Impulse modulator

3.3. The Ideal Sampler

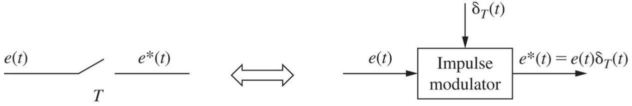
For a simpler expression of $e^*(t)$, we define the impulse modulator

$$\delta_T(t) = \delta(t) + \delta(t-T) + \cdots = \sum_{n=0}^{\infty} \delta(t-nT).$$

Then the sampled signal $e^*(t)$ in a pulse form can be rewritten as

$$\begin{aligned} e^*(t) &= e(0)\delta(t) + e(T)\delta(t-T) + e(2T)\delta(t-T) + \cdots \\ &= e(t)\delta(t) + e(t)\delta(t-T) + e(t)\delta(t-2T) + \cdots \\ &= e(t)\delta_T(t). \end{aligned}$$

$\therefore$ the impulse modulator represents the ideal sampler.



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Remarks & Examples

Remark:

- $e^*(t)$ may not be defined when $e(t)$ has the discontinuity at $t = kT$.
- But for sampled-data control systems, this is not problematic! (Why?)

Examples: Compute $E^*(s)$ for

- Example 3.1: $e(nT) = 1$, $n = 0, 1, \ldots$.
- Example 3.2: $e(t) = \mathrm{e}^{-t}$

Representation of $E^*(s)$ in terms of $E(s)$

3.4. Evaluation of $E^*(s)$

Suppose that $e(t)$ is continuous at all sampling instants $t = kT$.

Then, by definition,

$$e^*(t) = e(t)\delta_T(t) \quad \xrightarrow{\mathcal{L}(\cdot)} \quad E^*(s) = E(s) * \Delta_T(s)$$

where

- $*$ denotes the convolution, and
- $\Delta_T(s) = \mathcal{L}(\delta_T(t)) = 1 + \mathrm{e}^{-Ts} + \mathrm{e}^{-2Ts} + \cdots = \dfrac{1}{1 - \mathrm{e}^{-Ts}}$.

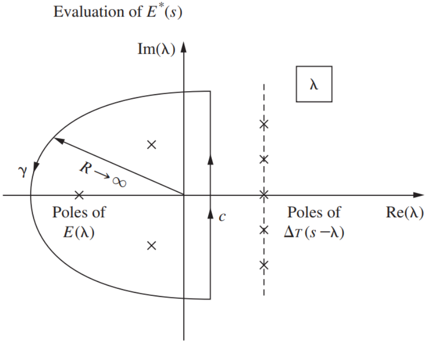
$\therefore$ With proper selection of c and s, we have

$$E^*(s) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} E(\lambda)\Delta_T(s-\lambda)\mathrm{d}\lambda = \frac{1}{2\pi j} \oint_{\gamma} E(\lambda)\Delta_T(s-\lambda)\mathrm{d}\lambda$$

where a closed path γ can be chosen in various ways.

Expression of $E^*(s)$ (1/3)

3.4. Evaluation of $E^*(s)$



By the residue theorem, we have another expression of $E^*(s)$:

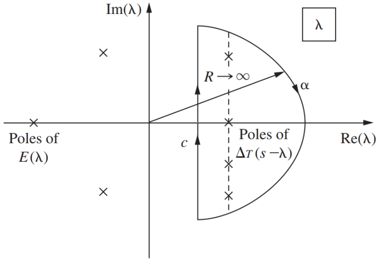
$$E^*(s) = \sum_{\text{at poles of } E(\lambda)} \left[\text{residues of } E(\lambda) \frac{1}{1 - \mathrm{e}^{-T(s-\lambda)}} \right]$$

Note: It is enough to deal with the poles of $E(\lambda)$ only. (Why?)

Expression of $E^*(s)$ (2/3)

3.4. Evaluation of $E^*(s)$

Suppose that we choose the path γ in the opposite way as follows:



We then have

$$E^*(s) = -\sum_{\text{at poles of } \Delta_T(s-\lambda)} \left[\text{residues of } E(\lambda)\frac{1}{1-\mathrm{e}^{-T(s-\lambda)}}\right]$$

$$= \cdots = \frac{1}{T}\sum_{n=-\infty}^{\infty} E(s + jn\omega_s), \quad \text{where } \omega_s = \frac{2\pi}{T} \text{ is the sampling frequency}$$

Expression of $E^*(s)$ (3/3)

3.4. Evaluation of $E^*(s)$

In summary, we have so far observed that there are 3 ways of expressing $E^*(s)$:

$$\begin{aligned} E^*(s) &= \sum_{k=0}^{\infty} e(kT)\mathrm{e}^{-kTs} \\ &= \sum_{\text{at poles of } E(\lambda)} \left[\text{residues of } E(\lambda)\frac{1}{1-\mathrm{e}^{-T(s-\lambda)}}\right] \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} E(s+jn\omega_s). \end{aligned}$$

Note:

- The above equations are valid when $e(t)$ is continuous (at least at $t = kT$).
- For example, if $e(t)$ is discontinuous at $t = 0$, then

$$E^*(s) = \cdots = \frac{1}{T}\sum_{n=-\infty}^{\infty} E(s+jn\omega_s) + \frac{e(0)}{2}$$

Examples

3.4. Evaluation of $E^*(s)$

Find $E^*(s)$ for

- Example 3.3: $E(s) = \dfrac{1}{(s+1)(s+2)}$
- Example 3.4: $E(s)\dfrac{\beta}{(s-j\beta)(s+j\beta)}$
- Example 3.5: $E(s) = \dfrac{1}{s(s+1)}$

Properties of $E^*(s)$

- Property 1: $E^*(s)$ is periodic in s with the period $j\omega_s = j\dfrac{2\pi}{T}$: that is,

$$E^*(s + jm\omega_s) = E^*(s), \quad \forall m = \ldots, -1, 0, 1, \ldots$$

Proof:

$$\begin{aligned} E^*(s + jm\omega_s) &= E^*(s)|_{s \to s + jm\omega_s} \\ &= \sum_{n=0}^{\infty} e(nT)\mathrm{e}^{-nT(s + jm\omega_s)} \\ &= \sum_{n=0}^{\infty} e(nT)\mathrm{e}^{-nTs} \quad (\because \omega_s T = 2\pi,\ \mathrm{e}^{j\theta} = \cos\theta + j\sin\theta) \\ &= E^*(s) \end{aligned}$$

- Property 2: If $E(s)$ has a pole at $s = s_1$,
  then $E^*(s)$ must have poles at $s = s_1 + jm\omega_s$, $m = 0, \pm 1, \pm 2, \cdots$.

  Proof: By the 3rd expression of $E^*(s)$,

  $$E^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} E(s + jn\omega_s)$$
  $$= \frac{1}{T} \left[\cdots + E(s - j\omega_s) + E(s) + E(s + j\omega_s) + \cdots \right]$$

  $\Rightarrow$ For any $s = s_1 + jm\omega_s$, $E(s - jm\omega_s) = \infty$
  $\Rightarrow$ $s = s_1 + jm\omega_s$ is a pole of $E^*(s)$.

Lesson 1: Periodicity of pole-zero location of $E^*(s)$

3.6. Properties of $E^*(s)$

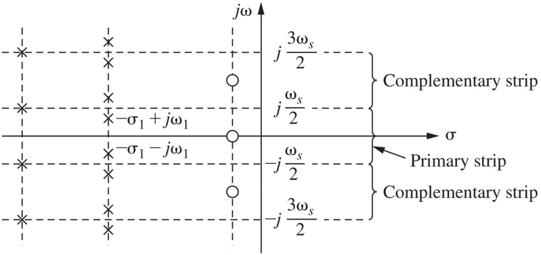
By Property 1, poles and zeros of $E^*(s)$ are periodic in s with period $j\omega_s$.

$\therefore$ IF a pole ($\times$) or zero ($\circ$) of $E^*(s)$ is located in the

$$\text{primary strip} \quad -\frac{j\omega_s}{2} \leq \omega \leq \frac{j\omega_s}{2},$$

THEN we have another pole or zero in

$$\text{complementary strip} \quad -\frac{j\omega_s}{2} + m\omega_s \leq \omega \leq \frac{j\omega_s}{2} + m\omega_s, \quad m = \cdots, -1, 1, \cdots$$



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Lesson 2: Two distinct signals may have the same $E^*(s)$.

Consider two sinusoidal signals with $\omega_1 := \dfrac{\omega_s}{4}$

$$E_1(s) = \mathcal{L}[\cos \omega_1 t] = \frac{s}{s^2 + \omega_1^2} = \frac{s}{(s + j\omega_1)(s - j\omega_1)},$$
$$E_2(s) = \mathcal{L}[\cos 3\omega_1 t] = \frac{s}{(s + j3\omega_1)(s - j3\omega_1)}.$$

After some computations, we have (WHY?????)

$$E_1^*(s) = E_2^*(s).$$

Intuition: Shannon's sampling theorem!

Note also that, by the previous slide,

- Poles of $E_1^*(s)$: $s = \pm j\omega_1 + jm4\omega_1$
- Poles of $E_2^*(s)$: $s = \pm j3\omega_1 + jm4\omega_1$ (= the above terms!)

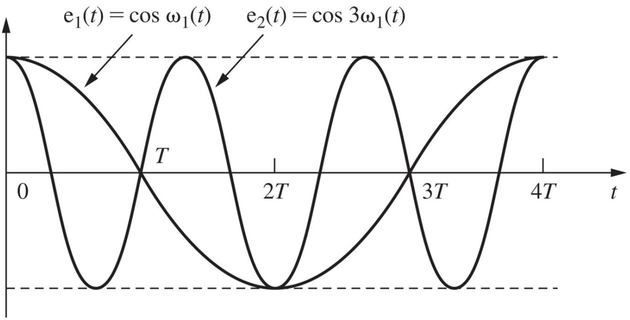
Shannon's sampling theorem

3.6. Properties of $E^*(s)$

The statement of Shannon:

> *"IF $e(t)$ contains no frequency components greater than f_s,*
> *it is uniquely determined by the values of $e(t)$*
> *at any set of sampling points $e(nT)$ spaced $1/(2f_s)$ seconds apart."*

In our case, $e_2(t)$ cannot be fully recovered from $e_2(nT)$ or $e_2^*(t)$.



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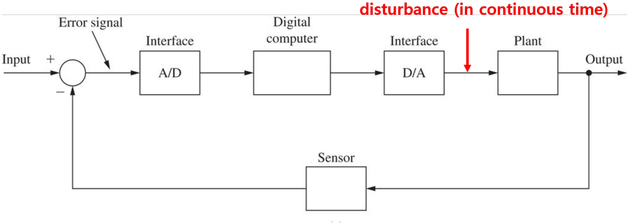
Fundamental limitation in digital control

3.6. Properties of $E^*(s)$



- Suppose that a disturbance with high-frequency components enters the system.
- The controller should generate fast control input to compensate the disturbance.
- Yet by the Shannon's sampling theorem,
  the high-frequency signals cannot be captured after passing the A/D device.
- $\therefore$ The controller may fail to generate the compensating input.

Data reconstruction

3.7. Data Reconstruction

Our goal: Recover $e(t)$ from $e(kT)$ or $e^*(t)$ (= Data reconstruction).

Key ingredient: Taylor's expansion of $e(t)$ about $t = nT$:

$$e(t) = e(nT) + e'(nT)(t - nT) + \frac{e''(nT)}{2!}(t - nT)^2 + \cdots$$

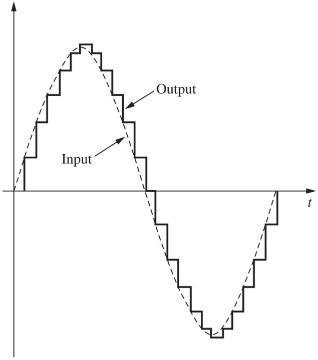
where $'$ means the derivative of a signal.

- Zero-order hold (that is mostly used):
  Approximation of $e(t)$ with up to the 0th-order term of T.E.
- First-order hold:
  Approximation of $e(t)$ with up to the 1st-order term of T.E.
- Generalized hold:
  Any extension of the concepts of holds above.

Graphical examples

3.7. Data Reconstruction

- Example of zero-order hold (left)
- Example of first-order hold (right)



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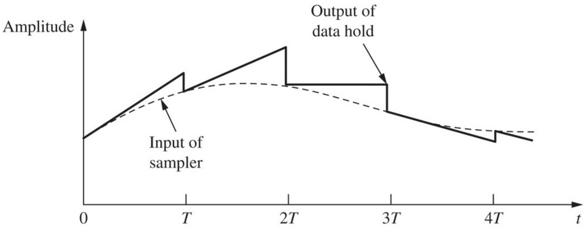


Zero-order hold

3.7. Data Reconstruction

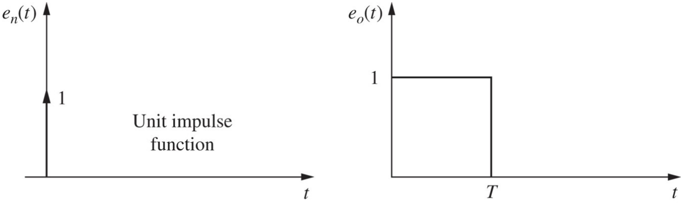
The output $\bar{e}_o(t)$ of the zero-order hold is computed by

$$\text{(Zero-order hold)} \quad \bar{e}(t) = e(nT), \quad nT \leq t < (n+1)T.$$

$\therefore$ IF $e(t)$ is the impulse signal,

THEN the output $\bar{e}(t)$ of the zero-order hold is of the pulse form:

$$\bar{e}(t) = \mathbf{1}(t) - \mathbf{1}(t-T) \quad \text{where } u(t) \text{ is the unit step function}$$



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(Cont'd)

Note:

Transfer function of a system

$=$ Laplace transform of the impulse response of the system

$\therefore$ the transfer function of the zero-order hold:

$$G_{h0}(s) = \mathcal{L}[\mathbf{1}(t) - \mathbf{1}(t-T)] = \frac{1 - \mathrm{e}^{-Ts}}{s}.$$

Note: $G_{h0}(s)$ describes the relation $\overline{E}(s) = G_{h0}(s)E^*(s)$ in the previous slide.

First-order hold

3.7. Data Reconstruction

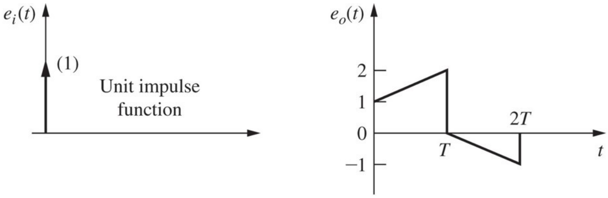
Similarly, we have the output $\overline{e}(t)$ of the first-order hold

$$\text{(First-order hold)} \quad \overline{e}(t) = e(nT) + e'(nT)(t - nT), \quad nT \leq t < (n+1)T$$

where $e'(nT)$ is an numerical derivative of $e(t)$ at $t = nT$:

$$e'(nT) = \frac{e(nT) - e((n-1)T)}{T}$$

$\Rightarrow$ The figure below depicts the impulse response of the first-order hold:



(a)

(Cont'd)

This $\overline{e}(t)$ can be written as follows:

$$\begin{aligned}\overline{e}(t) &= \mathbf{1}(t) + \frac{1}{T} t \cdot \mathbf{1}(t) - 2 \cdot \mathbf{1}(t-T) - \frac{2}{T}(t-T) \cdot \mathbf{1}(t-T) \\ &\quad + \mathbf{1}(t-2T) + \frac{1}{T}(t-2T) \cdot \mathbf{1}(t-2T)\end{aligned}$$

The Laplace transform of the impulse response gives

$$\begin{aligned}G_{h1}(s) &= \frac{1}{s}(1 - 2\mathrm{e}^{-Ts} + \mathrm{e}^{-2Ts}) + \frac{1}{Ts^2}(1 - 2\mathrm{e}^{-Ts} + \mathrm{e}^{-2Ts}) \\ &= \frac{1+Ts}{T}\left[\frac{1-\mathrm{e}^{-Ts}}{s}\right]^2\end{aligned}$$

which is the transfer function of the first-order hold.

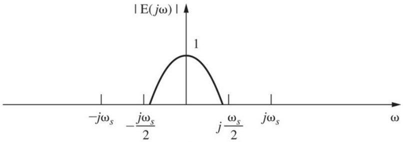
Understanding ideal sampler in frequency domain

3.6. Properties of $E^*(s)$

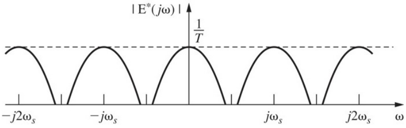
By the 3rd expression of $E^*(s)$, we have

$$E^*(j\omega) = E^*(s)|_{s=j\omega} = \frac{1}{T}\left[\cdots + E(j\omega - j\omega_s) + E(j\omega) + E(j\omega + j\omega_s) + \cdots\right]$$

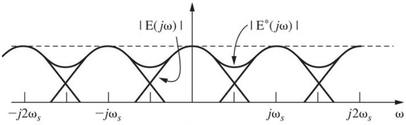
- IF $E(j\omega)$ is dominant only in the primary strip,
  THEN there is no loss of information in $|E^*(j\omega)|$ (left).
- IF NOT, we may encounter the aliasing effect (right).



(a)



(b)



(c)

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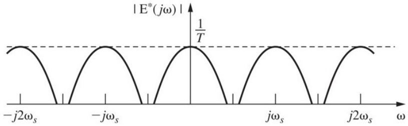
Understanding zero-order hold in frequency domain

Recall the frequency spectra of $E^*(j\omega)$ (without aliasing)



With the transfer function $G_h(s)$ of a hold, we have

$$\overline{E}(j\omega) = G_h(j\omega)E^*(j\omega).$$

Lesson: The hold can serve as a filter in the reconstruction process.

Candidates for $G_h(j\omega)$?

- Ideal low-pass filter $\times T$: Exact reconstruction, but not realizable
- Zero-order/First-order holds: Implementable, but is it a filter?

Frequency response of the zero-order hold

Remind that the transfer function $G_{h0}(s)$ of the zero-order hold is given by

$$G_{h0}(s) = \frac{1 - \mathrm{e}^{-Ts}}{s}.$$

Then, one has the frequency response of $G_{h0}(s)$ as follows:

$$\begin{aligned} G_{h0}(j\omega) &= \frac{1 - \mathrm{e}^{-j\omega T}}{j\omega} \mathrm{e}^{j(\omega T)/2} \mathrm{e}^{-j(\omega T)/2} \\ &= \frac{2\mathrm{e}^{-j(\omega T/2)}}{\omega} \left[\frac{\mathrm{e}^{j(\omega T/2)} - \mathrm{e}^{-j(\omega T/2)}}{2j} \right] \\ &= T \frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s} \mathrm{e}^{-j(\pi\omega/\omega_s)} \quad (\because \omega_s T = 2\pi) \end{aligned}$$

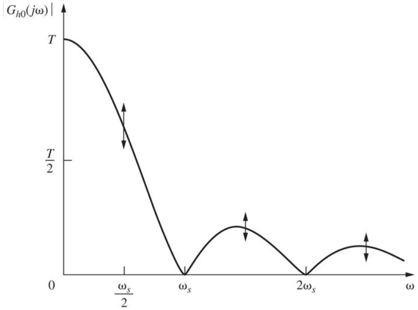
(Cont'd)

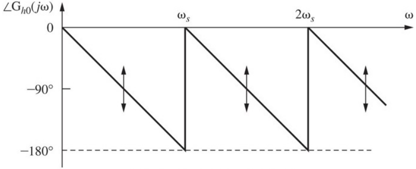
$\therefore$ The magnitude and phase shift of G_{h0}:

$$|G_{h0}(j\omega)| = T\left|\frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s}\right|,$$

$$\angle G_{h0}(j\omega) = -\frac{\pi\omega}{\omega_s} + \theta, \quad \theta = \begin{cases} 0, & \sin(\pi\omega/\omega_s) > 0, \\ \pi, & \sin(\pi\omega/\omega_s) < 0 \end{cases}$$

This shows that the zero-order hold works as a low-pass filter.





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Reconstruction of $e(t) = 2\cos(\omega_1 t)$ via zero-order hold

(a) $|E(j\omega)|$, (b) $|E^*(j\omega)|$, (c) $|E_n(j\omega)| = |G_{h0}(j\omega)E^*(j\omega)|$.