[2024-1 Digital Control]
Chapter 6. System Time-response Characteristics

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A system response is determined by

Consider a continuous-time (open-loop) system

$$G(s) = K \frac{\prod(s - z_i)}{\prod(s - p_i)}$$

or a discrete-time (open-loop) system

$$G^{\mathsf{d}}(z) = K \frac{\prod(z - z_i^{\mathsf{d}})}{\prod(z - p_i^{\mathsf{d}})}.$$

The system response is determined by

- Poles p_i or p_i^{d}: Generate a certain "mode" of the output
- Zeros z_i or z_i^{d}: Block a certain "mode" of the input
- DC gain $\lim_{s \to 0} G(s)$ or $\lim_{z \to 1} G^{\mathsf{d}}(z)$
- Relative degree (상대차수) = # of poles − # of zeros

Some features of the sampled-data system

6.2. System Time Response

Consider a sampled-data system with the pulse transfer function

$$G^{\mathsf{d}}(z) = \mathcal{Z}\left[\frac{1-\mathsf{e}^{-Ts}}{s}G(s)\right].$$

- (Discrete-time) poles: Each pole has the exact form

$$p_i^{\mathsf{d}} = \mathsf{e}^{p_i T}$$

- (Discrete-time) zeros: Some are related to continuous-time zeros as

$$z_i^{\mathsf{d}} \approx \mathsf{e}^{z_i T} \quad \text{(for sufficiently small } T\text{)}$$

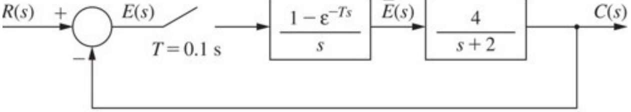
  while the others do not have a continuous-time counterpart.
- DC gain: $\lim_{z\to 1} G^{\mathsf{d}}(z) = \lim_{s\to 0} G(s)$
- Relative degree: For most cases, $\mathrm{rel.deg}(G^{\mathsf{d}}(z)) = 1$.

How to compute step response of closed-loop system

6.2. System Time Response

Example 6.1: We want to compute the inverse z-transform of

$$C^{\mathsf{d}}(z) = \frac{G^{\mathsf{d}}(z)}{1 + G^{\mathsf{d}}(z)} R^{\mathsf{d}}(z), \quad \text{with } R^{\mathsf{d}}(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}.$$



(a)

In the example above, the pulse transfer function is computed by

$$G^{\mathsf{d}}(z) = \mathcal{Z}\left[\frac{1 - \mathrm{e}^{-Ts}}{s} \frac{4}{s+2}\right] = \frac{z-1}{z} \mathcal{Z}\left[\frac{4}{s(s+2)}\right] = \frac{z-1}{z} \frac{2(1 - \mathrm{e}^{-2T})z}{(z-1)(z - \mathrm{e}^{-2T})}$$

$$= \frac{0.3625}{z - 0.8187}, \quad \text{if } T = 0.1 \text{ sec}$$

(Cont'd)

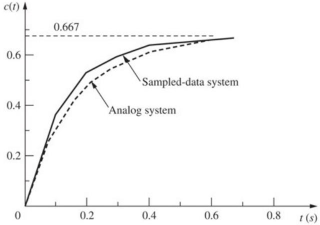
6.2. System Time Response

$\therefore$ We have the closed-loop transfer function

$$T^{\mathsf{d}}(z) = \frac{G^{\mathsf{d}}(z)}{1+G^{\mathsf{d}}(z)} = \frac{0.3625}{z-0.4562}.$$

$\Rightarrow$ the step response in discrete time has the form

$$\begin{aligned} c^{\mathsf{d}}(k) &= \mathcal{Z}^{-1}\left[\frac{G^{\mathsf{d}}(z)}{1+G^{\mathsf{d}}(z)}\frac{z}{z-1}\right] \\ &= \mathcal{Z}^{-1}\left[\frac{0.667z}{z-1} + \frac{-0.667z}{z-0.4562}\right] \\ &= 0.667 - 0.667(0.4562)^k \end{aligned}$$



(b)

Q. How can we derive the step response in continuous time?

(Cont'd)

The continuous-time closed-loop transfer function is given by

$$T_a(s) = \frac{G_p(s)}{1+G_p(s)} = \frac{4}{s+6}, \quad G_p(s) = \frac{4}{s+2}$$

$\Rightarrow$ We have the unit-step response in continuous time as:

$$C_a(s) = T_a(s)\frac{1}{s} = \frac{4}{s(s+6)} = \frac{0.667}{s} - \frac{0.667}{s+6},$$
$$\Rightarrow \quad c_a(t) = 0.667(1 - \mathrm{e}^{-6t})$$

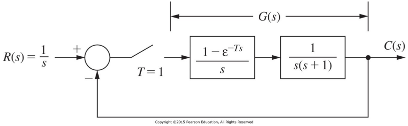
One important lesson: $c^{\mathsf{d}}(k)$ above is NOT the same as $c_a(kT)$: i.e.,

$$\begin{aligned} c^{\mathsf{d}}(k) = 0.667 - 0.667(0.4562)^k \neq c_a(kT) &= 0.667 - 0.667\mathrm{e}^{-0.6k} \\ &\approx 0.667 - 0.667(0.5488)^k \end{aligned}$$

Example 6.4

6.2. System Time Response

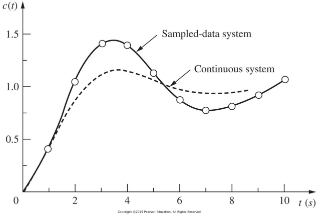
Consider the following sampled-data system with $T = 1$:



The pulse transfer function can be computed as

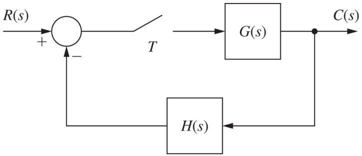
$$G^{\mathsf{d}}(z) = \frac{z-1}{z}\mathcal{Z}\left[\frac{1}{s^2(s+1)}\right]_{T=1} = \cdots = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

In this case, we have $c^{\mathsf{d}}(k)$ (solid) and $c_a(t)$ (dashed) as



System response of a closed-loop system

6.3. System Characteristic Equation



Then one can express the z transform of $c^{\mathsf{d}}(k) = c(kT)$ as

$$C^{\mathsf{d}}(z) = \frac{G^{\mathsf{d}}(z)}{1+\overline{GH}^{\mathsf{d}}(z)} R^{\mathsf{d}}(z) = \frac{K\prod^{m}(z-z_i^{\mathsf{d}})}{\prod^{n}(z-p_i^{\mathsf{d}})} R^{\mathsf{d}}(z)$$
$$= \frac{k_1 z}{z-p_1^{\mathsf{d}}} + \cdots + \frac{k_n z}{z-p_n^{\mathsf{d}}} + C_R(z)$$

where p_i^{d} is the root of $1+\overline{GH}^{\mathsf{d}}(z) = 0$ (assumed to be simple in this case).

Note:

- $\frac{k_1 z}{z-p_1^{\mathsf{d}}} + \cdots + \frac{k_n z}{z-p_n^{\mathsf{d}}}$ is called the natural response of $C^{\mathsf{d}}(z)$.
- $1+\overline{GH}^{\mathsf{d}}(z) = 0$ is termed the system characteristic equation.

Mapping from s-plane to z-plane: The case for $e(t) = \mathrm{e}^{-at}$

Main interest is to reveal the relation btw. continuous- and discrete-time responses.

For the simplest case $e(t) = \mathrm{e}^{-at}$, we have

$$E(s) = \frac{1}{s+a} \quad \to \quad E^*(s) = \frac{\mathrm{e}^{Ts}}{\mathrm{e}^{Ts} - \mathrm{e}^{-aT}} \quad \to \quad E^{\mathrm{d}}(z) = \frac{z}{z - \mathrm{e}^{-aT}}$$

which means that,

$$(\text{the pole } s = -a \text{ in } s\text{-plane}) \quad \leftrightarrow \quad (\text{the pole } z = \mathrm{e}^{-aT} \text{ in } z\text{-plane})$$

Note:

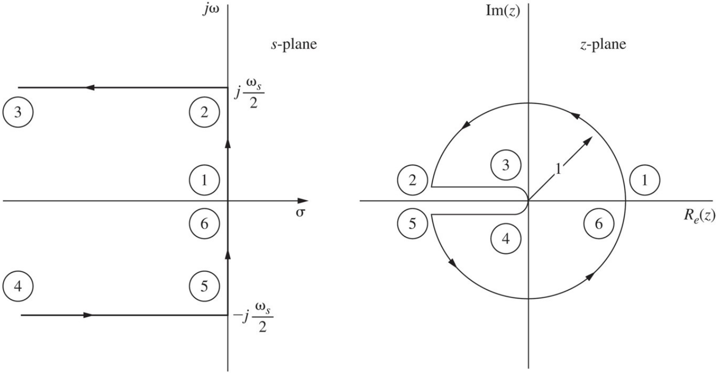
- This equivalence is satisfied for both $a \in \mathbb{R}$ and $a \in \mathbb{C}$.
- No such relation can be found in the zeros.

In the following slides, we examine the contour of $\mathrm{e}^{(\sigma + j\omega)T}$ in the complex plane.

Contour 1: Unit circle in z-plane

Case 1: For $z = \mathrm{e}^{j\omega T}$, we have

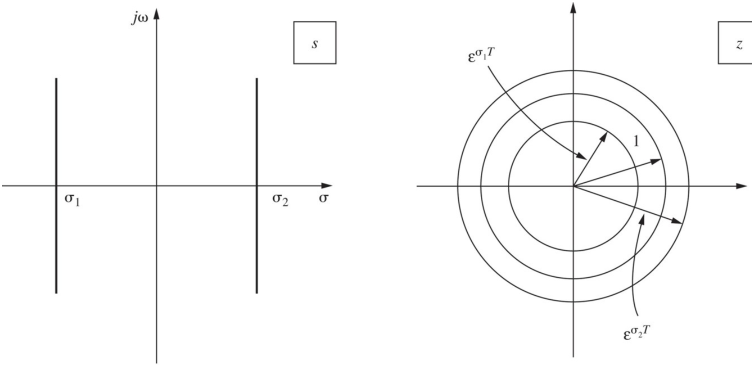
$$z = \mathrm{e}^{sT} = \mathrm{e}^{\sigma T}\mathrm{e}^{j\omega T} = \mathrm{e}^{j\omega T} = \cos(\omega T) + j\sin(\omega T) = 1\angle\omega T.$$



Contour 2: Circles with various radius

Case 2: For $s_1 = \sigma_1 + j\omega$,

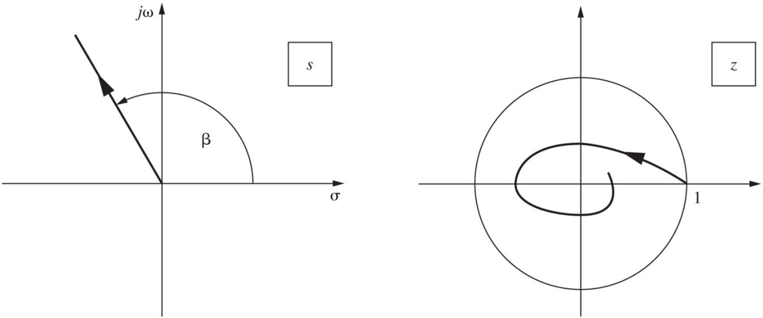
$$z = \mathrm{e}^{\sigma_1 T}\mathrm{e}^{j\omega T} = \mathrm{e}^{\sigma_1 T}\angle\omega T \quad \text{whose radius is } \mathrm{e}^{\sigma_1 T}$$



Contour 3: Constant damping ratio

Case 3: For $s_1 = \sigma + j\omega$ where $\omega/\sigma = \tan(\beta)$ with fixed β,

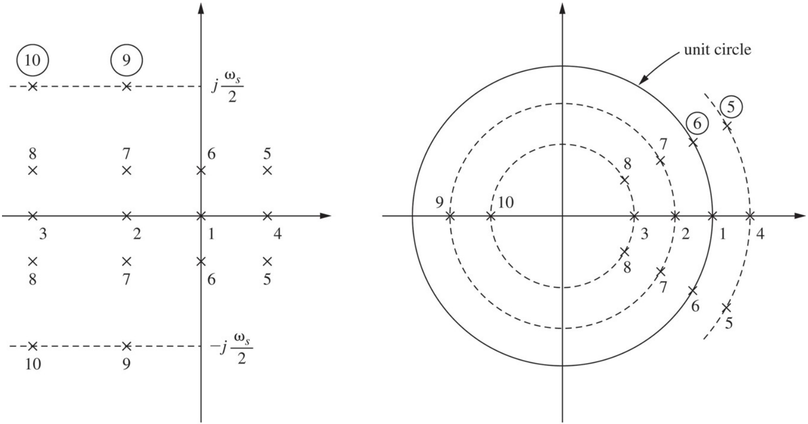
$$z = \mathrm{e}^{\sigma T}\mathrm{e}^{j\omega T} = \mathrm{e}^{\sigma T}\angle\omega T = \mathrm{e}^{\sigma T}\angle(\sigma T\tan\beta)$$



Summarizing so far,

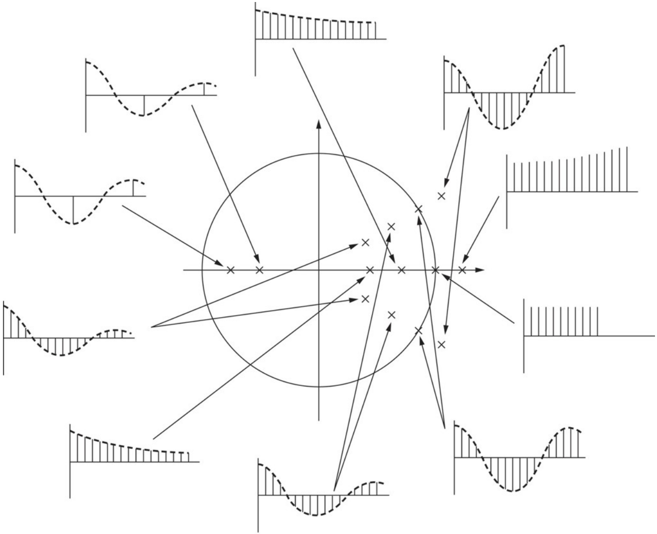
6.4. Mapping the s-plane into the z-plane

Each points in the z-plane are associated with points in the s-plane, with the equivalence $z = \mathrm{e}^{sT}$.



Transient response characteristics w.r.t. pole locations

6.4. Mapping the s-plane into the z-plane



Extension to 2nd-order system

Consider a continuous-time 2nd-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where the (continuous-time) system response of $G(s)$ is governed by

- ζ: damping ratio
- ω_n: natural frequency

$\Rightarrow$ The continuous-time poles of the system is computed by

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Equivalent discrete-time poles in z-plane?

6.4. Mapping the s-plane into the z-plane

Following the same procedure, we have the equivalent discrete-time poles in z-plane

$$p_{1,2}^{\mathsf{d}} = \mathsf{e}^{sT}|_{s=p_{1,2}} = \mathsf{e}^{-\zeta\omega_n T}\angle(\pm\omega_n T\sqrt{1-\zeta^2}) = \angle r(\pm\theta).$$

where the radius and the phase are computed by

$$\begin{aligned} r &= \mathsf{e}^{-\zeta\omega_n T} = \mathsf{e}^{-T/\tau}, \quad (\text{with } \tau := 1/(\zeta\omega_n)) \\ \theta &= \omega_n T\sqrt{1-\zeta^2} \end{aligned}$$

On the other hand, one can also derive ζ and ω_n as

$$\zeta = \frac{-\ln r}{\sqrt{(\ln r)^2 + \theta^2}}, \quad \omega_n = \frac{1}{T}\sqrt{(\ln r)^2 + \theta^2}.$$

Understanding the role of time constant τ

In summary, a part of the system characteristic equation in s-plane will be changed into the following in the z-plane:

- 1st-order equation:

$$s + 1/\tau \quad \rightarrow \quad z - \mathrm{e}^{-T/\tau}$$

- 2nd-order equation:

$$\begin{aligned}(s+1/\tau)^2 + \omega^2 \quad \rightarrow \quad & z^2 - 2z\mathrm{e}^{-T/\tau}\cos(\omega T) + \mathrm{e}^{-2T/\tau} \\ & = (z - z_1)(z - \overline{z}_1) \ \ \text{(where } z_1 = \mathrm{e}^{-T/\tau}\angle\omega T\text{)}\end{aligned}$$

Note: In both cases, τ and the radius r are related with each other as

$$\frac{\tau}{T} = -\frac{1}{\ln r}$$

which means that, the ratio btw. τ and T is important.

System type of a discrete-time system

Consider a discrete-time closed-loop system

$$\frac{C^{\mathsf{d}}(z)}{R^{\mathsf{d}}(z)} = \frac{G^{\mathsf{d}}(z)}{1 + G^{\mathsf{d}}(z)}$$

where the open-loop transfer function is expressed by

$$G^{\mathsf{d}}(z) = \frac{K \prod^{m} (z - z_i^{\mathsf{d}})}{(z-1)^{N} \prod^{n-N} (z - p_j^{\mathsf{d}})} \quad \text{with } p_j^{\mathsf{d}} \neq 1.$$

$\Rightarrow$ N is called the type of the system (in discrete time).

Note: For a system with $N \geq 1$, its DC gain = ∞.

Q. How about continuous-time case?

Steady-state step response and system type

For the unit-step input whose z-transform is computed by

$$R^{\mathsf{d}}(z) = \frac{z}{z-1}$$

one has the steady-state error of the closed-loop system as

$$e_{ss}^{\mathsf{d}} = \lim_{k\to\infty} e^{\mathsf{d}}(k) = \lim_{z\to 1}(z-1)E^{\mathsf{d}}(z) = \lim_{z\to 1}\frac{(z-1)R^{\mathsf{d}}(z)}{1+G^{\mathsf{d}}(z)} = \frac{1}{1+\lim_{z\to 1} G^{\mathsf{d}}(z)}$$
$$= \begin{cases} \text{constant} \neq 0, & \text{if } N = 0 \\ 0, & \text{if } N \geq 1 \end{cases}$$

provided that the closed-loop system is stable (as we will discuss later).

Lesson 1: A system with type $N \geq 1$ has zero-steady-state error in step response.

Steady-state ramp response and system type

6.5. Steady-state Accuracy

For the unit-ramp input whose z-transform is computed by

$$R^{\mathsf{d}}(z) = \frac{Tz}{(z-1)^2}$$

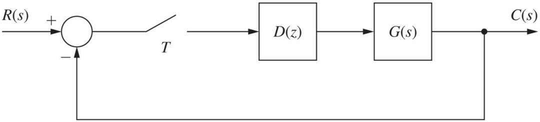
one has the steady-state error of the closed-loop system as

$$\begin{aligned} e_{ss}^{\mathsf{d}} = \lim_{k\to\infty} e^{\mathsf{d}}(k) = \lim_{z\to 1}(z-1)E^{\mathsf{d}}(z) = \lim_{z\to 1}\frac{(z-1)R^{\mathsf{d}}(z)}{1+G^{\mathsf{d}}(z)} &= \frac{T}{\lim_{z\to 1}(z-1)G^{\mathsf{d}}(z)} \\ &= \begin{cases} \infty, & \text{if } N = 0 \\ \text{constant} \neq 0, & \text{if } N = 1 \\ 0, & \text{if } N \geq 2 \end{cases} \end{aligned}$$

Lesson 2: A system with type $N \geq 2$ has zero-steady-state error in ramp response.

Example 6.8: Role of integrator in control

6.5. Steady-state Accuracy



$$G(s) = \frac{1}{s+1}\frac{1-\mathrm{e}^{-Ts}}{s}, \quad D^{\mathrm{d}}(z) = \frac{K_I z}{z-1} + K_P$$

Questions:

- Steady-state error for the step input?
- Steady-state error for the ramp input?

Note: A generalization of this concept = Internal model principle (내부모델이론)

That is, to regulate the reference or compensate the disturbance exactly, $D^{\mathrm{d}}(z)$ should have the generating model (such as $z/z-1$ for step reference)