[2024-1 Digital Control]
Chapter 7. Stability Analysis Techniques

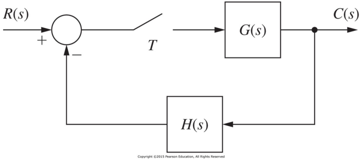
Gyunghoon Park

School of ECE, University of Seoul

Closed-loop system in discrete time

7.2. Stability

We consider a closed-loop system



whose transfer function is well-defined as

$$\frac{C^{\mathsf{d}}(z)}{R^{\mathsf{d}}(z)} = \frac{G^{\mathsf{d}}(z)}{1 + \overline{GH}^{\mathsf{d}}(z)}$$

In this case,

- $1 + \overline{GH}^{\mathsf{d}}(z) = 0$ is called the system characteristic equation.
- We denote the roots of the characteristic equation as p_i^{d}.
- $L^{\mathsf{d}}(z) := \overline{GH}^{\mathsf{d}}(z)$ is termed the open-loop transfer function.

Definition of stability (안정성): Frequency-domain approach

The closed-loop system is said to be

- stable IF p_i^{d} are located inside the open unit circle (or equivalently, $1 + \overline{GH}^{\mathsf{d}}(z) = 0$ is Schur stable).
- marginally stable IF p_i^{d} are located inside the closed unit circle, and each p_i^{d} on the unit circle is distinct (if exists).
- unstable IF the system is NOT (marginally) stable.

Important lessons from the last chapter:

- If the system is stable, the natural response $\to 0$ as $k \to \infty$.
  $\because$ $C^{\mathsf{d}}(z)$ can be decomposed as (for case of the simple poles)

$$C^{\mathsf{d}}(z) = \frac{k_1 z}{z - p_1^{\mathsf{d}}} + \cdots + \frac{k_n z}{z - p_n^{\mathsf{d}}} + C_R^{\mathsf{d}}(z).$$

- If the system is marginally stable, the natural response remains bounded as $k \to \infty$.

Definition of stability (안정성): State-variable approach

One also can represent the closed-loop system in the state space

$$\begin{bmatrix} \mathbf{x}^{\mathsf{d}}(k+1) \\ \mathbf{v}^{\mathsf{d}}(k+1) \end{bmatrix} = \mathbf{A}^{\mathsf{d}}_{\mathrm{cl}} \begin{bmatrix} \mathbf{x}^{\mathsf{d}}(k) \\ \mathbf{v}^{\mathsf{d}}(k) \end{bmatrix} + \mathbf{B}^{\mathsf{d}}_{\mathrm{cl}} r^{\mathsf{d}}(k),$$

$$y^{\mathsf{d}}(k) = \mathbf{C}^{\mathsf{d}}_{\mathrm{cl}} \begin{bmatrix} \mathbf{x}^{\mathsf{d}}(k) \\ \mathbf{v}^{\mathsf{d}}(k) \end{bmatrix} + D^{\mathsf{d}}_{\mathrm{cl}} r^{\mathsf{d}}(k)$$

- $\mathbf{x}^{\mathsf{d}}$: State variable for the plant (in discrete time)
- $\mathbf{v}^{\mathsf{d}}$: State variable for a digital controller (if exists)

Note: the solution of the state equation is given by

$$\begin{aligned} \begin{bmatrix} \mathbf{x}^{\mathsf{d}}(k) \\ \mathbf{v}^{\mathsf{d}}(k) \end{bmatrix} &= (\mathbf{A}^{\mathsf{d}}_{\mathrm{cl}})^k \begin{bmatrix} \mathbf{x}^{\mathsf{d}}(0) \\ \mathbf{v}^{\mathsf{d}}(0) \end{bmatrix} + \sum_{i=0}^{k-1} (\mathbf{A}^{\mathsf{d}}_{\mathrm{cl}})^{k-i} \mathbf{B}^{\mathsf{d}}_{\mathrm{cl}} r^{\mathsf{d}}(i) \\ &= \text{zero-input response} + \text{zero-(initial) state response} \end{aligned}$$

(Cont'd)

7.2. Stability

The system in state space is said to be

- asymptotically stable (with $r^{\mathsf{d}}(k) \equiv 0$) IF $(\mathbf{x}^{\mathsf{d}}, \mathbf{v}^{\mathsf{d}})(k) \to 0$ as $k \to \infty$.
- bounded-input-bounded-output (BIBO) stable IF $\mathbf{x}^{\mathsf{d}}(k)$ remains bounded for any bounded $r^{\mathsf{d}}(\cdot)$ and for all $k \geq 0$.

Note

- A sufficient condition for the stability:
  the characteristic polynomial $\det(z\mathbf{I} - \mathbf{A}_{\mathrm{cl}}^{\mathsf{d}}) = 0$ is Schur stable.
  (which implies that $(\mathbf{A}_{\mathrm{cl}}^{\mathsf{d}})^k \to 0$ as $k \to \infty$.)
- Two definitions of stability are related as

$$\{\text{The roots of } 1 + \overline{GH}^{\mathsf{d}}(z) = 0\} \subset \{\text{The roots of } \det(z\mathbf{I} - \mathbf{A}_{\mathrm{cl}}^{\mathsf{d}}) = 0\}$$

Q. Why these two sets of roots are NOT the same?

Note 1: Sample-and-hold process does not preserve stability

Consider two types of closed-loop systems:

- a continuous-time closed-loop system with the plant $G_p(s)$ and the controller $D(s)$:

$$G_p(s) = \frac{1}{s}, \quad D(s) = K.$$

- a sampled-data closed-loop system with the pulse transfer function $P^{\mathsf{d}}(z)$ and a discrete-time controller $D^{\mathsf{d}}(z)$:

$$G^{\mathsf{d}}(z) = \mathcal{Z}\left[\frac{1 - \mathrm{e}^{-Ts}}{s} G_p(s)\right], \quad D^{\mathsf{d}}(z) = K.$$

Questions

- Is the CT closed-loop system stable for all $K > 0$? Yes!
- Is the associated sampled-data system stable for all $K > 0$? No!

Note 2: Not all the discretization preserves stability.

We have learned some discretization methods, for example,

- Forward difference method

$$s = \frac{z-1}{T}, \quad z = 1 + Ts$$

- Backward difference method

$$s = \frac{1-z^{-1}}{T} = \frac{z-1}{Tz}, \quad z = \frac{1}{1-Ts}$$

In those methods,

$$\text{unit circle in } z\text{-domain} \quad \not\leftrightarrow \quad j\omega\text{-axis in } s\text{-domain}$$

Lesson? Discretization of a continuous-time system must be careful.

Bilinear transformation

7.3. Bilinear Transformation

The bilinear transformation (BT) is defined by

$$z = \frac{1 + (T/2)w}{1 - (T/2)w}, \quad \mathrm{w} = \frac{2}{T}\frac{z-1}{z+1}$$

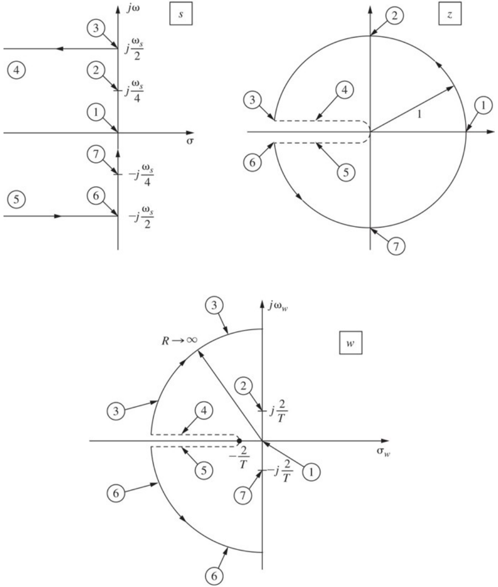
where w denotes a complex variable $\approx s$ in s-domain.

Note: For z being on the unit circle, one can represent the associated w as

$$\begin{aligned} z = \mathrm{e}^{j\omega T} \quad \xrightarrow{\text{BT}} \quad \mathrm{w} &= \frac{2}{T}\frac{\mathrm{e}^{j\omega T}-1}{\mathrm{e}^{j\omega T}+1} = \frac{2}{T}\frac{\mathrm{e}^{j\omega T/2}-\mathrm{e}^{-j\omega T/2}}{\mathrm{e}^{j\omega T/2}+\mathrm{e}^{-j\omega T/2}} \\ &= j\frac{2}{T}\tan\frac{\omega T}{2} \\ &=: j\omega_{\mathrm{w}} \\ &\approx j\omega \quad \text{where } \omega T \text{ is sufficiently small} \end{aligned}$$

Left-half plane in s-domain $\overset{\text{BT}}{\longleftrightarrow}$ Unit circle in z-domain

7.3. Bilinear Transformation



The Routh-Hurwitz criterion

- RH criterion provides a necessary and sufficient condition for stability of a continuous-time system.
- One can employ RH criterion for a discrete-time system, by
  1. transforming the DT system into a CT system via the bilinear transformation,
  2. applying the RH criterion to the transformed CT system.

TABLE 7-1 Basic Procedure for Applying the Routh–Hurwitz Criterion

1. Given a characteristic equation of the form

$$F(w) = b_n w^n + b_{n-1} w^{n-1} + \cdots + b_1 w + b_0 = 0$$

form the Routh array as

$$\begin{array}{c|cccc}
w^n & b_n & b_{n-2} & b_{n-4} & \cdots \\
w^{n-1} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\
w^{n-2} & c_1 & c_2 & c_3 & \cdots \\
\vdots & d_1 & d_2 & d_3 & \cdots \\
w^1 & j_1 & & & \\
w^0 & k_1 & & &
\end{array}$$

(Cont'd)

7.4. The Routh-Hurwitz Criterion

2. Only the first two rows of the array are obtained from the characteristic equation. The remaining rows are calculated as follows.

$$c_1 = \frac{b_{n-1}b_{n-2} - b_n b_{n-3}}{b_{n-1}} \qquad d_1 = \frac{c_1 b_{n-3} - b_{n-1}c_2}{c_1}$$

$$c_2 = \frac{b_{n-1}b_{n-4} - b_n b_{n-5}}{b_{n-1}} \qquad d_2 = \frac{c_1 b_{n-5} - b_{n-1}c_3}{c_1}$$

$$c_3 = \frac{b_{n-1}b_{n-6} - b_n b_{n-7}}{b_{n-1}} \qquad \vdots$$

3. Once the array has been formed, the Routh–Hurwitz criterion states that the number of roots of the characteristic equation with positive real parts is equal to the number of sign changes of the coefficients in the first column of the array.

4. Suppose that the w^{i-2} th row contains only zeros, and that the w^{i}th row above it has the coefficients $\alpha_1, \alpha_2, \ldots$. The auxiliary equation is then

$$\alpha_1 w^{i} + \alpha_2 w^{i-2} + \alpha_3 w^{i-4} + \cdots = 0$$

This equation is a factor of the characteristic equation.

Note:

- The CT-DT conversion via BT may be complicated...
- See Examples 7.2 – 7.3.

Jury's stability test

7.5. Jury's Stability Test

- is a discrete-time counterpart for the RH criterion.
- provides a necessary and sufficient condition under which all the roots are inside the unit circle.

Consider a polynomial of z

$$Q^{\mathsf{d}}(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

We compute the associated array for the Jury's stability test with

$$b_k := \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \quad c_k := \begin{vmatrix} b_0 & b_{n-1-k} \\ b_n & b_k \end{vmatrix},$$

as in the next slide.

(Cont'd)

7.5. Jury's Stability Test

TABLE 7-2 Array for Jury's Stability Test

z^0	z^1	z^2	$\dots$	z^{n-k}	$\dots$	z^{n-1}	z^n
a_0	a_1	a_2	$\dots$	a_{n-k}	$\dots$	a_{n-1}	a_n
a_n	a_{n-1}	a_{n-2}	$\dots$	a_k	$\dots$	a_1	a_0
b_0	b_1	b_2	$\dots$	b_{n-k}	$\dots$	b_{n-1}	
b_{n-1}	b_{n-2}	b_{n-3}	$\dots$	b_{k-1}	$\dots$	b_0	
c_0	c_1	c_2	$\dots$	c_{n-k}	$\dots$		
c_{n-2}	c_{n-3}	c_{n-4}	$\dots$	c_{k-2}	$\dots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
l_0	l_1	l_2	l_3				
l_3	l_2	l_1	l_0				
m_0	m_1	m_2					

Jury's stability test: Every roots of $Q^{\mathsf{d}}(z)$ (with $a_n > 0$) are inside the unit circle IF and ONLY IF the following holds:

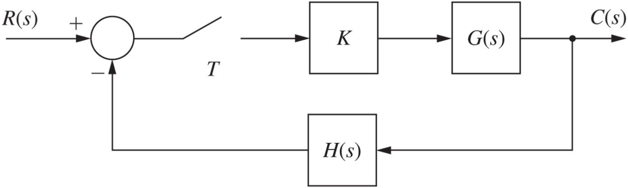
$$Q^{\mathsf{d}}(1) > 0, \quad (-1)^n Q^{\mathsf{d}}(-1) > 0,$$

$$|a_0| < a_n, \quad |b_0| > |b_{n-1}|, \quad |m_0| > |m_2|$$

See Examples 7.4–7.6.

Root locus technique is also available in discrete time.

Consider a sampled-data system



whose transfer function is computed by

$$\frac{C^{\mathsf{d}}(z)}{R^{\mathsf{d}}(z)} = \frac{KG^{\mathsf{d}}(z)}{1 + K\overline{GH}^{\mathsf{d}}(z)}.$$

The root locus technique aims to figure out
the root contour of $1 + K\overline{GH}^{\mathsf{d}}(z) = 0$ as K goes from 0 to $+\infty$.

Procedure for root locus (same as continuous-time version)

7.6. Root Locus

TABLE 7-3 Rules for Root-Locus Construction

For the characteristic equation

$$1 + K\overline{GH}(z) = 0$$

1. Loci originate on poles of $\overline{GH}(z)$ and terminate on the zeros of $\overline{GH}(z)$.
2. The root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros on the real axis.
3. The root locus is symmetrical with respect to the real axis.
4. The number of asymptotes is equal to the number of poles of $\overline{GH}(z)$, n_p, minus the number of zeros of $\overline{GH}(z)$, n_z, with angles given by $(2k + 1)\pi/(n_p - n_z)$.
5. The asymptotes intersect the real axis at σ, where

$$\sigma = \frac{\sum \textit{poles of } \overline{GH}(z) - \sum \textit{zeros of } \overline{GH}(z)}{n_p - n_z}$$

6. The breakaway points are given by the roots of

$$\frac{d[\overline{GH}(z)]}{dz} = 0$$

or, equivalently,

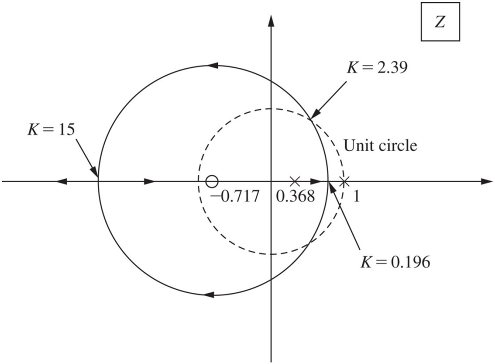
$$\mathrm{den}(z)\frac{d(\mathrm{num}(z))}{dz} - \mathrm{num}(z)\frac{d(\mathrm{den}(z))}{dz} = 0, \qquad \overline{GH}(z) = \frac{\mathrm{num}(z)}{\mathrm{den}(z)}$$

Example 7.7

7.6. Root Locus

Draw a root contour of the closed-loop system with

$$KG^{\mathsf{d}}(z) = \frac{0.368K(z+0.717)}{(z-1)(z-0.368)}.$$



See also Example 7.8.

The Cauchy's principle of argument

7.7. The Nyquist Criterion

Theorem: For a ratio $f(z)$ of two polynomials and the closed curve C in the z-plane, the following always holds:

$$N = Z - P$$

where

- N: The number of the clockwise encirclement of the origin
- Z: The number of the zeros of $f(z)$
- P: The number of the poles of $f(z)$

Q. What happens if $f(z)$ has the form $f(z) = 1 + g(z)$ for some $g(z)$?

The Nyquist criterion in s-plane

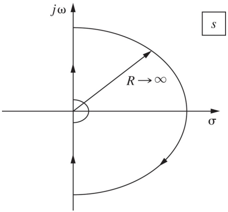
7.7. The Nyquist Criterion

For a continuous-time system whose characteristic equation is given by $1 + G_p(s)H(s) = 0$,

$$N = Z - P$$

- N: The number of the clockwise encirclement of $s = -1$
- Z: The number of the unstable zeros of $1 + G_p(s)H(s)$
- P: The number of the unstable poles of $1 + G_p(s)H(s)$

(if we take the Nyquist path C to enclose the right half-plane)



The Nyquist criterion in z-plane

7.7. The Nyquist Criterion

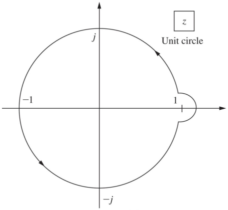
Let

- Z_i and P_i: The numbers of zeros/poles of $1 + \overline{GH}^{\mathsf{d}}(z)$ INSIDE the circle.
- Z_o and P_o: The numbers of zeros/poles of $1 + \overline{GH}^{\mathsf{d}}(z)$ OUTSIDE the circle.

Then, due to the Cauchy's principle of argument, we have

$$N = -(Z_i - P_i) = Z_o - P_o, \quad (\because \quad Z_i + Z_o = P_i + P_o)$$

where N: The number of clockwise encirclement of the $z = -1$



Bode diagram of a system in ω_{w}-domain

For a transfer function converted via the bilinear transformation:

$$G^{\mathsf{w}}(\mathrm{w}) = G^{\mathsf{d}}(z)\Big|_{z=\frac{2}{T}\frac{1-T\mathrm{w}}{1+T\mathrm{w}}}$$

one can draw its Bode diagram as in the following:

TABLE 7-5 Summary of Terms Employed in a Bode Diagram

1. A *constant term* K. When this term is present, the log magnitude plot is shifted up or down by the amount $20\log_{10}K$.
2. *The term* $j\omega_w$ or $1/j\omega_w$. If the term $j\omega_w$ is present, the log magnitude is $20\log_{10}\omega_w$, which is a straight line with a slope of +20 dB/decade, and the phase is constant at +90°. If the term $1/j\omega_w$ is present, the log magnitude is $-20\log\omega_w$, which is a straight line with a slope of −20 dB/decade, and the phase is constant at −90°. The Bode plots for these terms are shown in Fig. 7-20.
3. *The term* $(1 + j\omega_w\tau)$ or $[1/(1 + j\omega_w\tau)]$. The term $(1 + j\omega_w\tau)$ has a log magnitude of $20\log_{10}\sqrt{1+\omega_w^2\tau^2}$ which can be approximated as $20\log_{10}1 = 0$ when $\omega_w\tau \ll 1$ and as $20\log_{10}\omega_w\tau$, when $\omega_w\tau \gg 1$. The corner or "break" frequency is $\omega_w = 1/\tau$. The phase is given by the expression $tan^{-1}\omega_w\tau$. The term $[1/(1 + j\omega_w\tau)]$ is handled in a similar manner. The Bode plots for these functions are shown in Fig. 7-20.

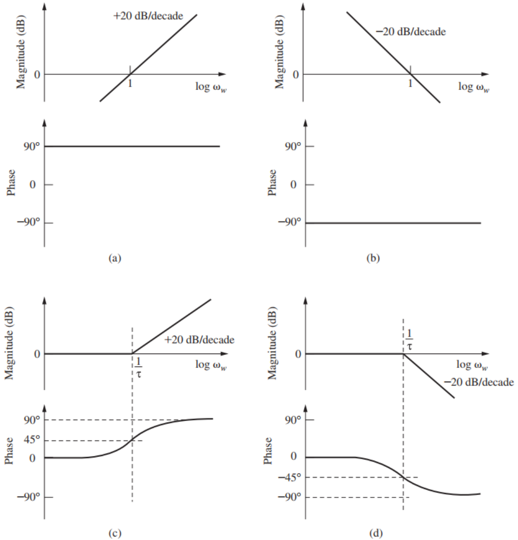


FIGURE 7-20 Short summary of terms employed in Bode diagrams: (a) Bode plot for $j\omega_w$; (b) Bode plot for $1/j\omega_w$; (c) Bode plot for $1 + j\omega_w\tau$; (d) Bode plot for $1/(1 + j\omega_w\tau)$.

Frequency response of a discrete-time system

For stable linear and time-invariant systems,
the steady-state response of the system associated with a sinusoidal input is also sinusoidal: that is,

$$e^{\mathsf{d}}(k) = A\sin(\omega T k) \quad \rightarrow \quad c_{\mathrm{ss}}^{\mathsf{d}}(k) = B\sin(\omega T k + \theta)$$

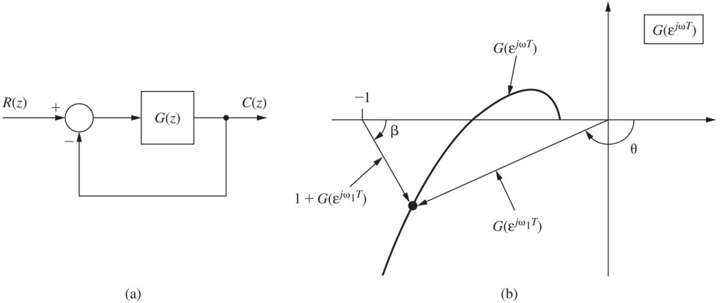
where B and θ are dependent of the frequency ω.

We thus define its frequency response $G^{\mathsf{d}}(\mathrm{e}^{j\omega T})$ as follows:

$$\begin{aligned} (\text{magnitude (dB)}) \quad & 20\log(B/A) = 20\log|G^{\mathsf{d}}(\mathrm{e}^{j\omega T})| \\ (\text{phase}) \quad & \theta = \angle G^{\mathsf{d}}(\mathrm{e}^{j\omega T}) \end{aligned}$$

The closed-loop frequency response

7.9. Interpretation of the Frequency Response



$$G^{\mathsf{d}}(\mathsf{e}^{j\omega T}) = |G^{\mathsf{d}}(\mathsf{e}^{j\omega T})|\mathsf{e}^{j\theta}, \quad 1 + G^{\mathsf{d}}(\mathsf{e}^{j\omega T}) = |1 + G^{\mathsf{d}}(\mathsf{e}^{j\omega T})|\mathsf{e}^{j\beta}$$

$\therefore$ The frequency response of the closed-loop system:

$$T^{\mathsf{d}}(\mathsf{e}^{j\omega T}) = \frac{G^{\mathsf{d}}(\mathsf{e}^{j\omega T})}{1 + G^{\mathsf{d}}(\mathsf{e}^{j\omega T})} = \frac{|G^{\mathsf{d}}(\mathsf{e}^{j\omega T})|}{|1 + G^{\mathsf{d}}(\mathsf{e}^{j\omega T})|}\mathsf{e}^{j(\theta-\beta)}$$

Constant magnitude circles

= The locus of points in the $G^{\mathsf{d}}(\mathsf{e}^{j\omega T})$ plane for which the magnitude of the closed-loop frequency response is constant.

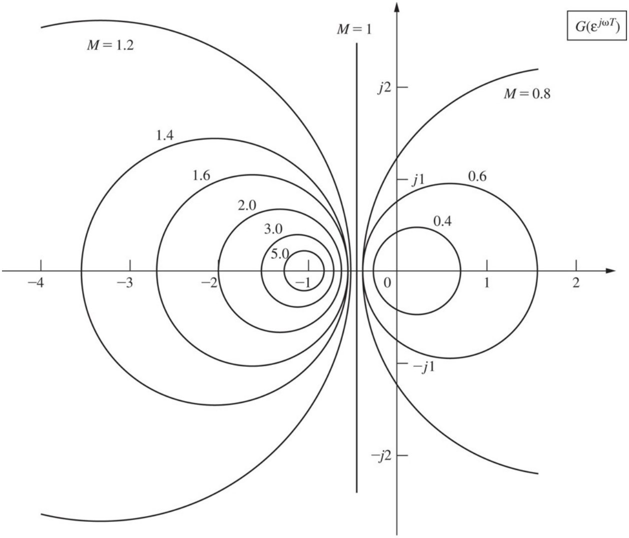
Denote $G^{\mathsf{d}}(\mathsf{e}^{j\omega T}) = X + jY$ and $|G^{\mathsf{d}}(\mathsf{e}^{j\omega T})|/|1 + G^{\mathsf{d}}(\mathsf{e}^{j\omega T})| = M$.

$$
\begin{aligned}
& M^2 = \frac{X^2 + Y^2}{(1+X)^2 + Y^2} \\
\Rightarrow\ & X^2(1 - M^2) - 2M^2 X - M^2 + (1 - M^2)Y^2 = 0 \\
\Rightarrow\ & \left[X + \frac{M^2}{M^2 - 1}\right]^2 + Y^2 = \frac{M^2}{(M^2 - 1)^2}
\end{aligned}
$$

$\therefore$ The constant magnitude circle is a circle with radius $|M/(M^2 - 1)|$ and the center $(-M^2/(M^2 - 1), 0)$.

(Cont'd)

7.10. Closed-loop Frequency Response



Constant phase circles

= The locus of points in the $G^{\mathrm{d}}(\mathrm{e}^{j\omega T})$ plane for which the phase of the closed-loop frequency response is constant.

$$\phi = \theta - \beta = \tan^{-1}\left(\frac{Y}{X}\right) - \tan^{-1}\left(\frac{Y}{1+X}\right).$$

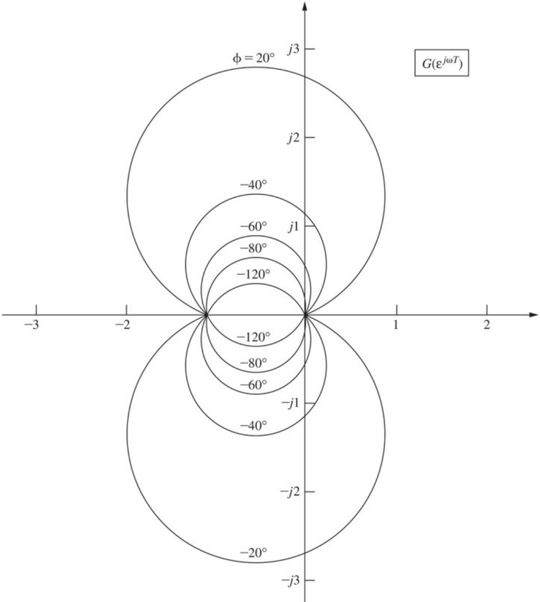
Letting $N = \tan(\theta - \beta)$, we have

$$\begin{aligned} N &= \frac{\tan\theta - \tan\beta}{1 + \tan\theta\tan\beta} = \frac{\frac{Y}{X} - \frac{Y}{1+X}}{1 + \frac{Y}{X}\left[\frac{Y}{1+X}\right]} \\ \Rightarrow\ & X^2 + X + Y^2 - \frac{1}{N}Y = 0. \\ \Rightarrow\ & \left(X + \frac{1}{2}\right)^2 + \left(Y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \left(\frac{1}{2N}\right)^2 \end{aligned}$$

$\therefore$ The constant phase circle is a circle with radius $\sqrt{1/4 + (1/2N)^2}$ and the center $(-1/2, 1/2N)$.

(Cont'd)

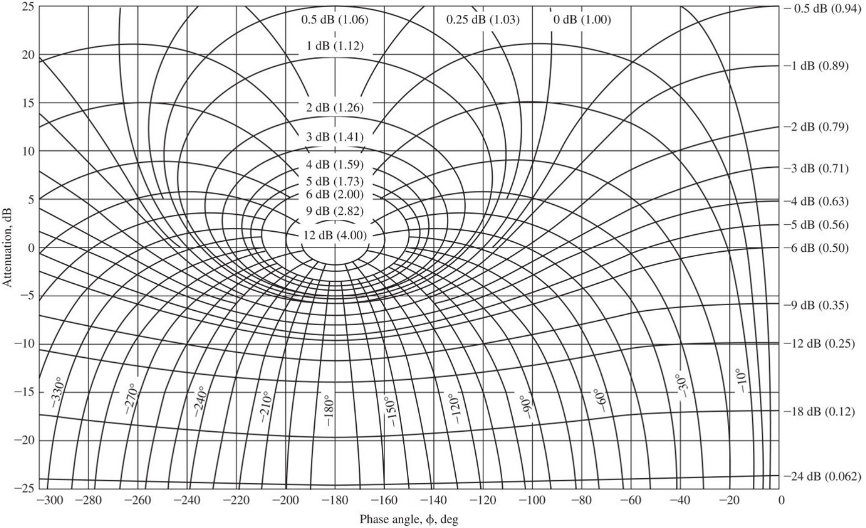
7.10. Closed-loop Frequency Response



Nichols chart

7.10. Closed-loop Frequency Response

= The constant magnitude/phase circles plotted in the gain-phase plane.



Model uncertainty?

= Mismatch between the actual system and a mathematical model of the system.

Example: Consider a mass-spring-damper system whose mathematical model is derived by

$$\frac{Y(s)}{U(s)} = G_p(s) = \frac{1}{Ms^2 + Bs + K}$$

- Parametric uncertainty: M, B, K are uncertain
- Unstructured uncertainty: $G_{\text{actual}}(s) = G_p(s) + \Delta(s)$ or $G_{\text{actual}}(s) = G_p(s)(1 + \Delta(s))$ with $\Delta(s)$ is an unmodelled part.

IF a system is stable for all admissible model uncertainty,
THEN the system is "robustly" stable.

Q. How to represent the model uncertainty in state space?

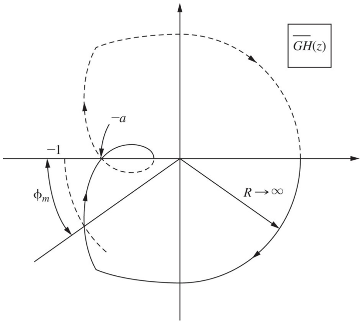
Relative stability & Gain/phase margin

7.8. The Bode Diagram

Relative stability = A "quantification" of robust stability.

- Phase-crossover point = The point of $\angle L^{\mathsf{d}}(\mathrm{e}^{j\omega T}) = \pi$.
- Gain-crossover point = The point of $|L^{\mathsf{d}}(\mathrm{e}^{j\omega T})| = 1$
- Gain margin = $-20\log|L^{\mathsf{d}}(\mathrm{e}^{j\omega T})|$ dB at the phase-crossover point.
- Phase margin = $\angle L^{\mathsf{d}}(\mathrm{e}^{j\omega T}) - (-\pi)$ at the gain-crossover point.

where $L^{\mathsf{d}}(z)$: open-loop transfer function.



Gain/phase margin in ω_{w}-domain

7.8. The Bode Diagram