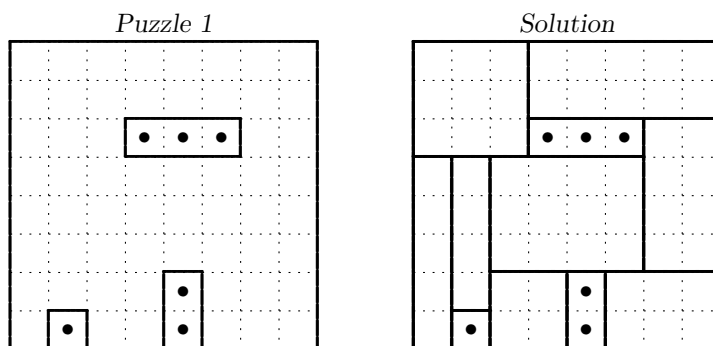


Rubik's Gridlock Puzzle via Metropolis

Rubik's Gridlock is a puzzle played on an 8×8 grid. The puzzle consists of three white rectangular pieces: 1×1 , 1×2 , and 1×3 — we'll call these the *puzzle rectangles*. These three pieces are placed in specified locations on the grid, as shown below (left). Eight additional rectangular pieces (of various unimportant colors) are of dimensions: (1) 1×4 ; (2) 1×5 ; (3) 2×2 ; (4) 2×3 ; (5) 2×4 ; (6) 2×5 ; (7) 3×3 ; and (8) 3×4 — we'll call these the *solution rectangles*. The collective area of the eleven rectangles ($1 + 2 + 3 + 4 + 5 + 4 + 6 + 8 + 10 + 9 + 12$) is 64, and your mission is to place the solution rectangles on the grid, without moving the puzzle rectangles, in such a manner that each of the 64 grid sites is covered by one rectangle. The Gridlock game comprises 88 different puzzles; puzzle number 1 is shown below along with its solution.



Our **state space** will consist of various ways to situate the eight solution rectangles on the grid. Preliminary, with $I = \{1, 2, \dots, 8\}$ and $G = I \times I$, consider the collection of configurations

$$C = \{((x, y, z)_i : i \in I) : \text{each } (x, y) \in G; \text{ each } z \in \{h, v\}\}.$$

Each configuration is an eight component vector where each component is a triplet. The index i identifies the solution rectangle in question with numbering as in the first paragraph above. The coordinates x and y indicate which grid site contains solution rectangle i 's lower-left corner. The value of z indicates if the rectangle is oriented horizontally ($z = h$) or vertically ($z = v$). For example, *Solution* above graphically illustrates the configuration

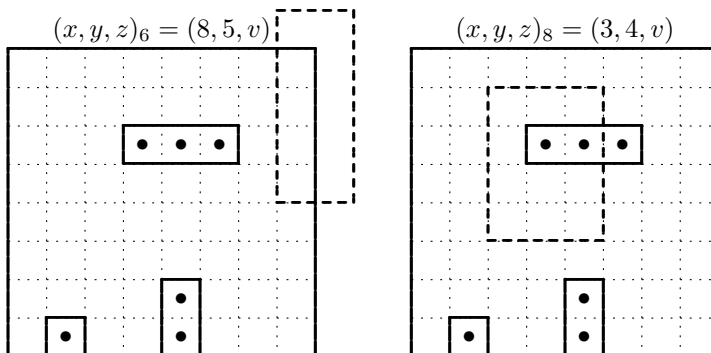
$$((2, 2, v), (1, 1, v), (3, 1, h), (6, 1, h), (7, 3, v), (4, 7, h), (1, 6, h), (3, 3, h)).$$

For the square rectangles, where $i = 3$ or 7 , the value of z is irrelevant: $z = h$ and $z = v$ produce the same graphical illustration. Since each of the eight triplet components can assume $8 \times 8 \times 2 = 128$ values, C contains $128^8 \approx 7.21 \times 10^{16}$ configurations.

Note that many configurations will have solution rectangles that do not lie entirely on the 8×8 grid. For example, if $(x, y, z)_6 = (8, 5, v)$ the 2×5 solution rectangle will have only four squares on the grid (see below, left). Additionally, a configuration may place a solution rectangle in such a way that it covers one or more puzzle rectangle grid sites. For example, if $(x, y, z)_8 = (3, 4, v)$ in *Puzzle 1*, two of the squares of the 1×3 puzzle rectangle are covered by solution rectangle

number 8 (below, right). Such rectangle positions cannot possibly be present in the solution and we will refer to configurations containing them as *defective*.

Let $D \subset C$ denote all such defective configurations. In the interest of algorithmic efficiency our state space will consist only of non-defective configurations: $S = C \setminus D$. Note that solution rectangles may overlap in non-defective states, but states where this happens will not be ground states per the energy function defined below. We will call two states **neighbors** if one can be obtained from the other by changing exactly one of the eight triplets involved. The number of configurations in D , and hence states in S , will depend on the location of the puzzle rectangles as will the number of neighbors each state possesses.



As for the **energy function**, in the solution each of the 64 grid sites will be covered by exactly one rectangle — either a puzzle or a solution rectangle. Letting $n_w(x, y)$ denote the number of rectangles that cover the grid site (x, y) in the state $w \in S$, we put

$$E(w) = \sum_{(x,y) \in G} |n_w(x, y) - 1|.$$

The Markov chain starts with a randomly generated state in S (which is therefore non-defective) and runs until a ground state with zero energy is found. (Due to the two square solution rectangles, there are four ground states all yielding the same graphical illustration.)

This is implemented in `RubiksGridlock.cpp`, which reads in a three line data file of the following format.

```
2 1 h
5 1 v
4 6 h
```

This is the data for *Puzzle 1* above and is found in the file `01.txt`. The first line indicates that the 1×1 puzzle rectangle is situated at grid site $(2, 1)$. The **h** indicates that it is oriented horizontally — not important as 1×1 is square. The second line indicates that the 1×2 rectangle's lower-left square is located at grid site $(5, 1)$ and is oriented vertically. Line three indicates that the 1×3 rectangle's lower left square is located at grid site $(4, 6)$ and is oriented horizontally. The solution may be viewed by Plain TeXing the file `RG.tex`. The code also reports the puzzle-dependent size of S as well as the number of neighbors each state has. For example, *Puzzle 1* above has approximately 1.99×10^{12} states each with 301 neighbors.

Data for each puzzle is found in the files `01.txt` through `88.txt`. The file `SpoilerAlert.pdf` contains solutions to the 88 puzzles as found by `RubiksGridlock.cpp`.