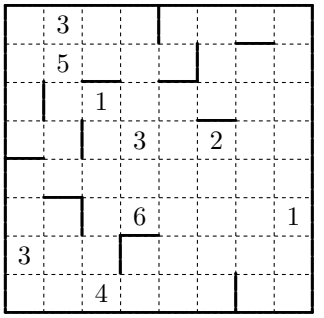
One Up

One Up is a nice puzzle currently (as of July 2024) appearing in the New York Times Magazine under Rodolfo Kurchan's byline. Here we apply the Metropolis algorithm to generate solutions to these puzzles. The puzzle consists of an 8×8 grid where each row and column is partitioned into (usually two) regions separated by "walls". You are to place the digits 1 through 8 throughout the grid so that if a region is of size n, the digits 1 through n all appear in the region — necessarily exactly once. Your solution must also agree with the pre-specified digits appearing at several sites. The puzzle appearing on 7/7/2024, together with its solution, is shown below.

Puzzle



Solution

4	3	1	2	2	3	1	4
3	5	2	4	1	1	2	3
1	4	1	5	3	2	6	7
2	1	5	3	1	2	4	6
4	2	6	1	5	3	7	8
1	2	3	6	2	4	5	1
3	1	2	2	4	1	3	5
2	3	4	1	6	5	1	2

Our **state space** S will consist of all possible allocations of the eight digits $D = \{1, 2, \ldots, 8\}$ to the 64 sites: $S = \{x : D \times D \to D\}$. There are $8^{64} \approx 6.277 \times 10^{57}$ such configurations in S. Two configurations x and y will be **neighbors** if y can be obtained from x by changing the digit at exactly one site. This is easily seen to adhere to the good neighbor rules of Chapter 3 of "The Metropolis Algorithm: Theory and Examples" (C Douglas Howard, FE Press, 2024). In particular, each configuration has $64 \times 7 = 448$ neighbors. One randomly chooses a configuration neighboring x by: (i) randomly choosing a row r; (ii) randomly choosing a column c; and (iii) randomly changing the digit $x(r, c)$. The **energy function** $E(x)$ is computed as follows. First, starting with $E(x) = 0$, $E(x)$ is incremented by 5 for each grid sited (r, c) with a pre-specified digit that disagrees with $x(r, c)$. The penalty of 5 is arbitrary but is intended to keep the Markov chain away from configurations

with such disagreements. $E(x)$ is then augmented for each row and column region's deviation from the "1 through n" rule. Specifically, if a region is of size n and $d \in D$ is present in the region m times then, in that region,

$$d\text{'s contribution to } E(x) \;=\; \begin{cases} |m-1| & \text{if } d \le n \\ 5m & \text{if } d > n. \end{cases}$$

(If $d > n$ then d cannot possibly be in the region.) For example, suppose a row region in configuration x is of size 6 and has the digits 4, 4, 7, 4, 7, 3. Then, summing over $d \in D$, that region's contribution to $E(x)$ is $1+1+0+2+1+1+10+0 = 16$. We perform this calculation for each row and column region and augment $E(x)$ accordingly. The solution has energy equal to 0.

This is implemented in `OneUp.cpp`, where the `.txt` data input file has the format:

```
11112222 Row regions
11111222
12222222
11222222
11111111
11222222
11122222
11111122

11111111 Column regions
11111121
11212121
11212221
21212221
22212221
22222221
22222221

9 Number of clues
1 2 3 Row 1, column 2's clue is 3
2 2 5
3 3 1
4 4 3
4 6 2
6 4 6
6 8 1
7 1 3
8 3 4
```

The above data corresponds to the 7/7/2024 puzzle shown above and can be found in `7-7-2024.txt`. The puzzle and solution may be viewed by TeXing `OneUp.tex` with Plain TeX (don't use Latex).