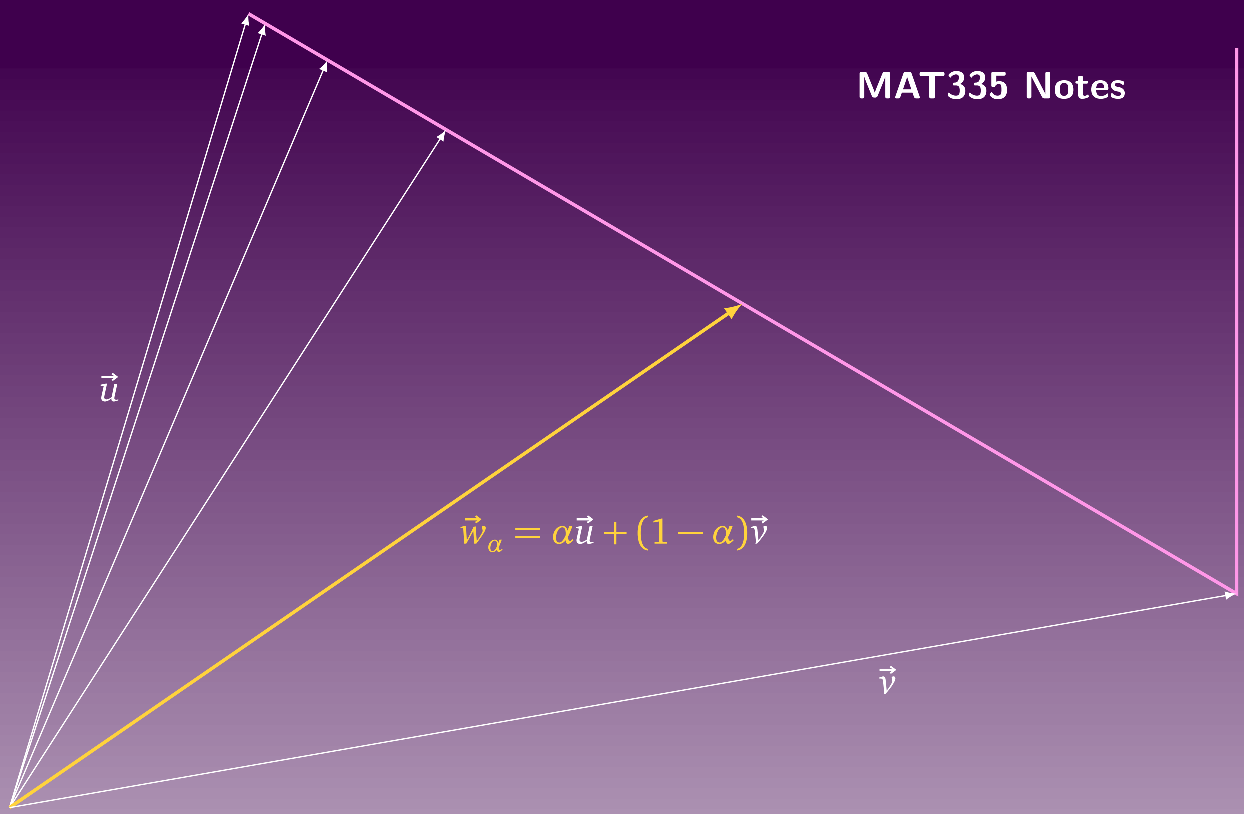
MAT335 Notes



Jason Siefken

Dynamical Systems

Dynamical System

DEF A pair (T,X) is called a ***dynamical system*** if X is a set and $T : X \to X$ is a function. In the context of a dynamical system, T is often called a ***transformation***.

This definition is very general, and most things you encounter could be considered a dynamical system. For example, if $X = \{\text{air molecules and their positions on earth}\}$ and $T : X \to X$ is the result of the wind blowing for one second, then (T,X) is a dynamical system. Alternatively, we could take the state of your computer's memory (RAM) to be a set and your processor executing a single instruction to be a transformation.

It's hard to say much about general dynamical systems. However, throughout the course, we will find ways to classify dynamical systems. Once we "narrow the field", we'll be able to say lot's of interesting things.

Newton's Method

Newton's method is a way of using tangent-line approximations to functions to estimate their roots. It is an iterative procedure.[1]

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1 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $T_f : \{\text{guesses}\} \to \{\text{guesses}\}$ be a single application of Newton's method.

1.1 Find a general formula for T_f.

1.2 Let $f(x) = x(x-2)(x-3)$. Compute $T_f^n(4)$ for $n = 0, 1, 2, 3$.

1.3 Do you think

$$\lim_{n\to\infty} T_f^n(4)$$

converges? If so, what does it converge to? Can you prove your answer?

Fixed Point

DEF Let (T,X) be a dynamical system. A point $a \in X$ is called a ***fixed point*** if $T(a) = a$.

Basin of Attraction

DEF Let (T,X) be a dynamical system and let $x \in X$. The ***basin of attraction*** of x is the set

$$A_x = \{y \in X : \lim_{n\to\infty} T^n y = x\}.$$

Eventually, we will talk about more general *basins of attraction*, but for now we will limit ourselves to that of a single point.

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2 Let $f(x) = x(x-2)(x-3)$ and let T_f be the function that applies a single iteration of Newton's method (as before).

2.1 Is $[3,4) \subseteq A_3$ for T_f? What about $[100, 1000]$? $(2,3)$?

2.2 Describe A_3.

2.3 Is A_3 connected?

[1]Whenever something is iterative, you should think dynamics!

Inverse Image

DEF Let $f : A \to B$ be a function and let $X \subseteq B$. The ***inverse image*** of X under f, denoted $f^{-1}(X)$, is

$$f^{-1}(X) = \{x \in A : f(x) \in X\}.$$

Note: a function *need not* be invertible to have inverse images. In fact, the idea of inverse images applies to *every* function.

3

3.1 Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$. Find $g^{-1}(\{1\})$, $g^{-1}(\{4\})$, $g^{-1}(\{0\})$, $g^{-1}(\{-1\})$, and $g^{-1}([3,4])$.

3.2 Let f and T_f be as before. (Recall, $f(x) = x(x-2)(x-3)$). Find $T_f^{-1}([3,4))$.

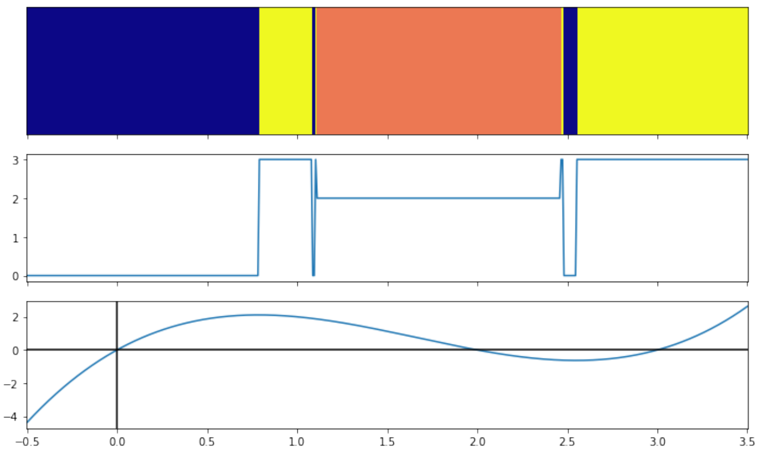
3.3 Define

$$Q = \bigcup_{n \geq 0} T_f^{-n}([3, 3.1))$$

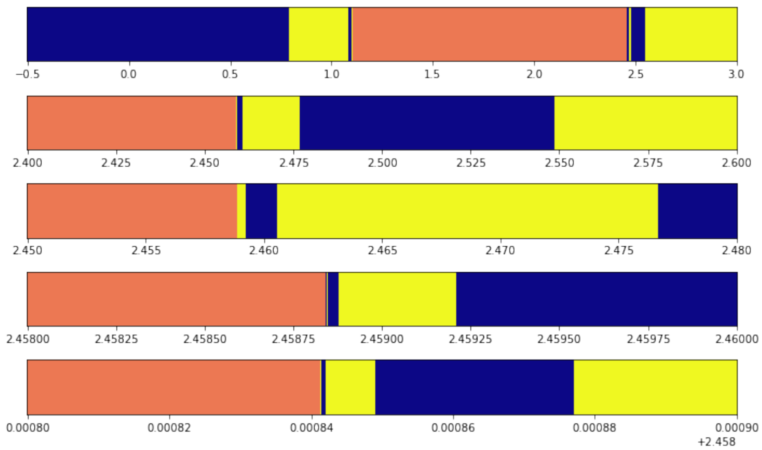
where T_f^0 is the identity function.

Is $Q = A_3$? Why or why not?

Using a computer, we can graph A_0, A_2, and A_3.



Zooming in around $x \approx 2.4588$:



We've just seen our first **fractal**! For now, we will define a *fractal* as a set with repeated patterns at all scales.

Fractals

Let's construct some famous fractals.

4 Let K_0 be an equilateral triangle with sides of length 1. Let K_1 be the result of applying $\rightarrow$ to each side of K_0. Repeat this process to get K_2 from K_1, etc. and define

$$K_\infty = \lim_{n \to \infty} K_n.$$

4.1 Draw K_0, K_1, and K_2.

4.2 Find the perimeter of K_0, K_1, and K_2. Find a general formula for the perimeter of K_n.

4.3 What is the perimeter of K_∞? What is the area enclosed by K_∞?

K_∞ is called the *Koch Snowflake*.

5 Let T_0 be a filled-in equilateral triangle. To get T_1, T_2, etc., apply the substitution rule  $\rightarrow$  to each (sub)triangle of T_0, T_1, etc.. Define T_∞ to be the limit of this process.

5.1 Draw T_0, T_1, and T_2.

5.2 Find a formula for the area of T_n.

5.3 Compute the area of T_∞.

5.4 Is T_∞ the empty set? Why or why not?

T_∞ is called *Sierpinski's Triangle*.

6 Let $C_0 = [0, 1]$ be the unit interval. Recursively define C_i by the substitution rule $\rightarrow$, which removes the middle $1/3$ of every interval. Define C_∞ to be the limit of this process.

6.1 Compute the length of C_n.

6.2 Compute the length of C_∞.

C_∞ is called the *Cantor set*.

Our normal sense of measurement fails when it comes to these fractals. We need a new idea: *similarity dimension*.

Dimension

Dimension can be thought of as a relationship between scale and content.[2]

7

7.1 Let $\ell = [0,1)$, $2\ell = [0,2)$, $3\ell = [0,3)$, etc.. How many disjoint copies of ℓ does it take to cover $n\ell$?

7.2 Let $S = [0,1)^2$, $2S = [0,2)^2$, etc.. How many disjoint copies of S does it take to cover nS?

7.3 Let $C = [0,1)^3$, $2C = [0,2)^3$, etc.. How many disjoint copies of C does it take to cover nC?

7.4 Based on the patterns you see, describe an algorithm that can be used to find the dimensions of ℓ, S, and C.

7.5 Let T be the filled-in equilateral triangle. Apply your algorithm to $2T$.

7.6 Let T_∞ be the Sierpinski triangle. Apply your algorithm to $2T_\infty$. Does the number you get make sense?

Similarity Dimension

DEFINITION

A set $Q \subseteq \mathbb{R}^n$ has ***similarity dimension*** d if there exists a $c \in \mathbb{Z}$ and $s \in \mathbb{R}^+$ satisfying

$$d = \log_s(c)$$

and sQ (Q scaled up by a factor of s) is covered by c copies of Q (at most overlapping on their boundaries).

8

8.1 Compute the similarity dimension of a (a) a line segment, (b) the Cantor set, and (c) Sierpinski's triangle.

8.2 Compute the similarity dimension of the Koch snowflake.

What about sets that aren't self-similar?

9

Let K'_∞ be the "Koch snowflake" obtained with the substitution rule $\to$.

9.1 Find the perimeter and dimension of K'_∞.

9.2 Let K_{strange} be the "Koch snowflake" obtained by the rule $\to$ or chosen randomly at each stage. What should the dimension of K_{strange} be? Can you compute it's similarity dimension?

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[2]Here "content" refers to the "stuff inside" of an object.

We need a way to define dimension for shapes that aren't self-similar. Let's again work from sets whose dimension we know: cubes.

Box Covering (DEFINITION)

A d-dimensional ***box covering*** of $X \subseteq \mathbb{R}^n$ is a collection $C = \{B_i\}$ of d-dimensional cubes which satisfy

1. if $i \neq j$, B_i and B_j intersect at most on their boundaries;
2. $B_i \cap X \neq \{\}$ for all i;
3. $X \subseteq \bigcup_i B_i$.

Outer Measure (DEFINITION)

The ***d-dimensional outer measure*** of $X \subseteq \mathbb{R}^n$ is

$$\lim_{n\to\infty} \text{volume}(C_n)$$

where C_n is a d-dimensional box covering of X with cubes of side-length $1/n$.

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10 Let $\ell \subseteq \mathbb{R}^3$ be the line segment from $\vec{0}$ to $(1,0,0)$.

10.1 Find the 1, 2, and 3-dimensional outer measures of ℓ.

10.2 Does ℓ have a 0-dimensional outer measure?

10.3 Let $T \subseteq \mathbb{R}^3$ be the filled in triangle with vertices $(0,0,0)$, $(1,0,0)$, and $(0,1,0)$. Find the 1, 2, and 3-dimensional outer measures of T.

10.4 Find the 1-dimensional outer measure of the Cantor set.

What would it mean to have a fractional-dimensional outer measure? Let B be a d-dimensional box with side lengths k. Its volume is k^d. Divide the box in half along every dimension and each sub-box has volume $(k/2)^d = (1/2)^d k^d$, and so there must be 2^d sub-boxes.

What if there were fewer "sub-boxes"?

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11 Let C_α be the Cantor-like set obtained by removing the middle α of each subinterval. (I.e., the standard Cantor set is $C_{1/3}$.)

11.1 Find the number of boxes in a 1-dimensional box-covering of C_0 (the interval) and $C_{1/3}$ where the width of each box is $1/3$, $1/9$, $1/27$, etc..

11.2 Based on what you know about how many width-k boxes it takes to fill d-dimensional space, find a formula relating d, the number of boxes, and the width of the boxes.

11.3 Use your formula to estimate d for $C_{1/3}$. How does this compare to the similarity-dimension of $C_{1/3}$?

Box-counting Dimension (DEFINITION)

Let $X \subseteq \mathbb{R}^n$ and let $B \subseteq \mathbb{R}^n$ and let B be a minimal-dimensional, minimally-sized box such that $X \subseteq B$. The ***box-counting dimension*** of X is

$$d = \lim_{n\to\infty} \frac{\log(\#\text{ of sub-boxes of } B_n \text{ that intersect } X)}{\log n},$$

where B_n is B "cut" along each axis into n equally-spaced slices.

11.4 Find the box-counting dimension of $C_{1/3}$.

11.5 Find the box-counting dimension of the unit simplex in $\mathbb{R}^2$. I.e. $\{\vec{v} \in \mathbb{R}^2 : \vec{v} = \alpha\vec{e}_1 + \beta\vec{e}_2 \text{ for some } \alpha, \beta \geq 0 \text{ satisfying } \alpha + \beta \leq 1\}$

11.6 Intuitively, what should $\lim_{\alpha\to 0} \dim(C_\alpha)$ be? Find the box-counting and similarity dimension of C_α and verify.