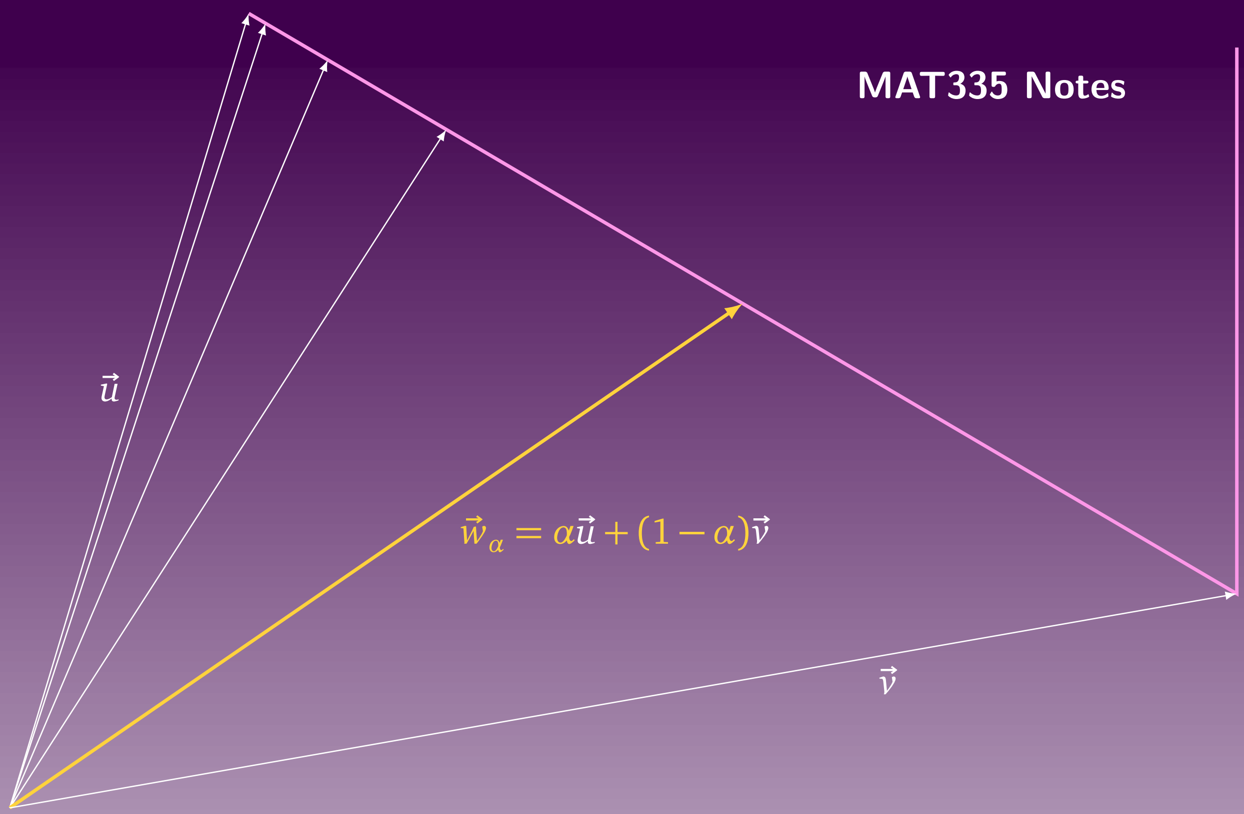
MAT335 Notes



Jason Siefken

Dynamical Systems

Dynamical System

DEF A pair (T,X) is called a ***dynamical system*** if X is a set and $T : X \to X$ is a function. In the context of a dynamical system, T is often called a ***transformation***.

This definition is very general, and most things you encounter could be considered a dynamical system. For example, if $X = \{\text{air molecules and their positions on earth}\}$ and $T : X \to X$ is the result of the wind blowing for one second, then (T,X) is a dynamical system. Alternatively, we could take the state of your computer's memory (RAM) to be a set and your processor executing a single instruction to be a transformation.

It's hard to say much about general dynamical systems. However, throughout the course, we will find ways to classify dynamical systems. Once we "narrow the field", we'll be able to say lot's of interesting things.

Newton's Method

Newton's method is a way of using tangent-line approximations to functions to estimate their roots. It is an iterative procedure.[1]

---

1 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $T_f : \{\text{guesses}\} \to \{\text{guesses}\}$ be a single application of Newton's method.

1.1 Find a general formula for T_f.

1.2 Let $f(x) = x(x-2)(x-3)$. Compute $T_f^n(4)$ for $n = 0, 1, 2, 3$.

1.3 Do you think

$$\lim_{n\to\infty} T_f^n(4)$$

converges? If so, what does it converge to? Can you prove your answer?

Fixed Point

DEF Let (T,X) be a dynamical system. A point $a \in X$ is called a ***fixed point*** if $T(a) = a$.

Basin of Attraction

DEF Let (T,X) be a dynamical system and let $x \in X$. The ***basin of attraction*** of x is the set

$$A_x = \{y \in X : \lim_{n\to\infty} T^n y = x\}.$$

Eventually, we will talk about more general *basins of attraction*, but for now we will limit ourselves to that of a single point.

---

2 Let $f(x) = x(x-2)(x-3)$ and let T_f be the function that applies a single iteration of Newton's method (as before).

2.1 Is $[3,4) \subseteq A_3$ for T_f? What about $[100, 1000]$? $(2,3)$?

2.2 Describe A_4.

2.3 Is A_4 connected?

---

[1] Whenever something is iterative, you should think dynamics!

Inverse Image

DEF

Let $f : A \to B$ be a function and let $X \subseteq B$. The ***inverse image*** of X under f, denoted $f^{-1}(X)$, is
$$f^{-1}(X) = \{x \in A : f(x) \in X\}.$$

Note: a function *need not* to have inverse images. In fact, the idea of inverse images applies to *every* function.

3

3.1 Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$. Find $g^{-1}(\{1\})$, $g^{-1}(\{4\})$, $g^{-1}(\{0\})$, $g^{-1}(\{-1\})$, and $g^{-1}([3,4])$.

3.2 Let f and T_f be as before. (Recall, $f(x) = x(x-2)(x-3)$). Find $T_f^{-1}([3,4))$.

3.3 Define
$$Q = \bigcup_{n \geq 0} T_f^{-n}([3,3.1))$$
where T_f^0 is the identity function.

Is $Q = A_3$? Why or why not?