1. Let (T, X) be a dynamical system.
   (a) Prove that if $x \in X$ is a fixed point of T, then the *basin of attraction* of x, A_x, is non-empty.
   (b) Fix $y \in X$. Prove that if $R \subseteq A_y$, then $T^{-1}(R) \subseteq A_y$. (Recall $T^{-1}(R) = \{w \in X : T(w) \in R\}$ is the *inverse image of R under T*. T is **not necessarily** invertible.)

2. You will prove that under certain conditions, Newton's method converges.
   Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function and suppose that $f(0) = 0$ and that f' and f'' are positive on the interval $(0, a]$. Let T_f be the function that applies one iteration of Newton's method to its input.
   (a) Find a function f satisfying the required properties and graph it. Does it look like Newton's method will converge on $(0, a]$?
   (b) Show that if $x \in (0, a)$, then $0 \le T_f(x) < a$.
   (c) The *Monotone Convergence Theorem* states that if (a_n) is a sequence of real numbers that is bounded below *and* $a_{n+1} \le a_n$, then $\lim\limits_{n\to\infty} a_n$ exists and is a real number.
   Use the Monotone Convergence Theorem to prove that $\lim\limits_{n\to\infty} T_f^n(x)$ converges for all $x \in (0, a]$.
   (d) A point y is called a *fixed point* of a function g if $g(y) = y$.
   Fix $x_0 \in (0, a]$, and define $\boldsymbol{x} = \lim\limits_{n\to\infty} T_f(x)$. Show that $\boldsymbol{x}$ is a fixed point of T_f.
   Hint: You may use the fact that if g is a continuous function, then $g(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} g(a_n)$ for any convergent sequence (a_n).
   (e) Prove that for any $x \in (0, a]$, $\lim\limits_{n\to\infty} T_f^n(x) = 0$.
   (f) Combine your results and explain why Newton's method will always converge for f if you pick an initial guess in $(0, a]$.

3. Let's prove more about Newton's method! Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $f(0) = 0$. Further, suppose f' and f'' are both positive on the interval $[-a, a]$ (for some $a > 0$).
   (a) Use a picture to argue that Newton's method might not converge for some $x \in [-a, a]$.
   (b) Show that there is some $b > 0$ so that for $x \in [-b, a]$, Newton's method always converges.
   (c) Let $g : \mathbb{R} \to \mathbb{R}$ and define $h : \mathbb{R} \to \mathbb{R}$ by $h(t) = g(-t)$. Prove that if Newton's method converges for g with a starting point of x_0, then Newton's method converges for h with a starting point of $-x_0$.
   (d) Prove that if $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 0$, and $f' < 0$ and $f'' > 0$ on the interval $[-a, a]$, then there exists a $b > 0$ so that Newton's method converges for f on $[-a, b]$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 1$ and let T_f be the function that applies Newton's method to its inputs.
   (a) Find $T_f^{-1}(\{1, 2, 3\})$ (recall that $T_f^{-1}(\{1, 2, 3\})$ is the *inverse image* of the set $\{1, 2, 3\}$ under T_f. It is not the same as applying the *inverse* of T_f, since T_f might not be invertible).
   (b) Is $T_f(x)$ defined for all $x \in \mathbb{R}$?
   (c) Find the largest possible domain, $X \subseteq \mathbb{R}$, so that $T_f : X \to X$ is a *dynamical system.*
   (d) Prove that $\lim_{n\to\infty} T_f^n(4)$ exists. What value is it?
   (e) Find all fixed points of T_f.
   (f) Fore each fixed point of T_f, find its *basin of attraction.*

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{2}{1+x^2} - \frac{5}{4}$ and let T_f be the function that applies Newton's method to its inputs.

   (a) Is 0 in the domain of T_f?

   (b) Find $T_f^{-1}(\{0\})$ (you can use a computer algebra system).

   (c) Find $T_f^{-2}(\{0\})$ (you can use a computer algebra system).

   (d) Describe largest possible domain, $X \subseteq \mathbb{R}$, so that $T_f : X \to X$ is a *dynamical system*. How many connected components does this domain consist of? (Use computers to help you!)