APPLIED ECONOMETRICS

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Handbook of Econometrics Vol 3

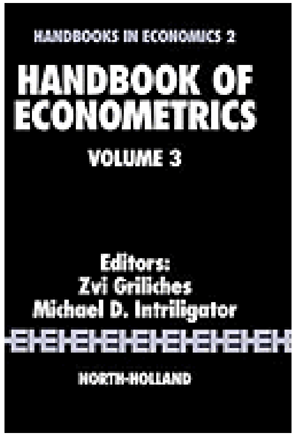
Chap 30 Demand Analysis

Angus Deaton

Chap 31 Econometric methods for modeling producer behavior

Dale W. Jorgenson

North Holland 1986

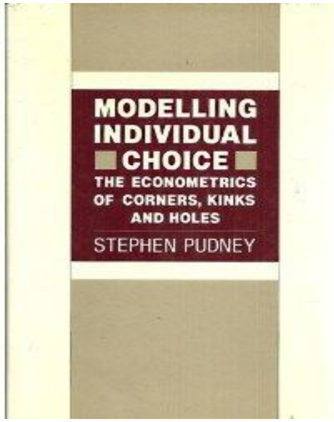


Modelling Individual Choice

The Econometrics of Corners, Kinks, and Holes

Stephen Pudney

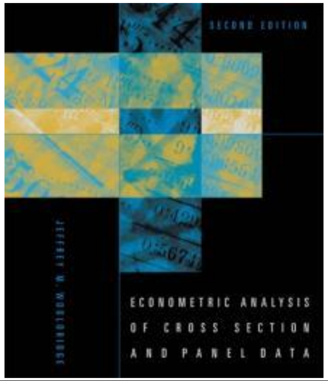
Blackwell 1989



Econometric Analysis of Cross Section and Panel Data

Jeffrey M. Wooldridge

MIT Press (second edition) 2010



APPLIED ECONOMETRICS

1. Production Function

   Cost Function and Factor demand – approaches to the estimation of productivity

2. Frontier Production functions

   analysis of efficiency

3. Consumer demand

   elasticity and welfare calculations

1

PRODUCTION FUNCTION

COST FUNCTION

FACTOR DEMAND

APPROACHES TO THE ESTIMATION OF PRODUCTIVITY

Contents

1. Introduction

- Different motivations:
  - To estimate the possibilities of substitution between the factors of production (*ex.:* Analysis of substitution possibilities between energy and capital after the first oil crisis, e.g. Berndt E.R and Wood D.O, *"Energy an econometric interpretations of energy capital complementarity,* AER, vol. 69, n°3, 1979)
  - To estimate input demand functions (used in macro econometric models, e.g. Edward A. Hudson & Dale W. Jorgenson, "U.S. energy policy and economic growth,1975-2000*",* Bell Journal of Economics, The RAND Corporation, vol. 5(2), pages 461-514, Autumn, 1974)
  - To estimate returns to scale (ex.: Are natural monopolies still existing in network industries? e.g. "Shin R.T. and Ying J.S., "Unnatural monopolies in local telephone*", Rand journal of Economics, vol23, n°2, 1992)*

2. The production function

2.1 Definition

Hypothesis: There exists a *technology* allowing to combine the different factors of production (or inputs) in order to produce one or more output(s),

Notations: One output, y; N inputs, $x_1, x_2, \dots, x_N$

Definition: a *production function* is a function that gives the greatest amount of output that can be obtained with a given selection of inputs, formally its is given by,

$$y = f\left(x_1, x_2, \dots, x_N\right) = f\left(x\right)$$

2.2 Two characteristics of the technology

2.2.1 Returns to scale

- Decreasing returns to scale:

$$\forall \lambda > 1, \quad f\left(\lambda x_1, \lambda x_2, \ldots, \lambda x_N\right) < \lambda f\left(x_1, x_2, \ldots, x_N\right)$$

- Constant returns to scale:

$$\forall \lambda \geq 0, \quad f\left(\lambda x_1, \lambda x_2, \ldots, \lambda x_N\right) = \lambda f\left(x_1, x_2, \ldots, x_N\right)$$

- Increasing returns to scale

$$\forall \lambda > 1, \quad f\left(\lambda x_1, \lambda x_2, \ldots, \lambda x_N\right) > \lambda f\left(x_1, x_2, \ldots, x_N\right)$$

- **Comment:** Increasing returns to scale are often the result of high fixed costs, and are a necessary condition for the natural monopoly theory to apply

2.2 Two characteristics of the technology

2.2.2 Substitution possibilities

- The issue is: by how much the firm is able to reduce the quantity of one input (let's say labor) by increasing the quantity of another input (let's say capital) while maintaining constant the level of production

- Measuring substitution possibilities: the concept of *elasticity of substitution*

2.2.2 Substitution possibilities (Continued)

- Elasticity of substitution (2 inputs case)
  - The production function is $y = f\left(x_1, x_2\right)$
  - The Marginal *Rate of Substitution* (MRS_{12}), between x_1 and x_2 is,

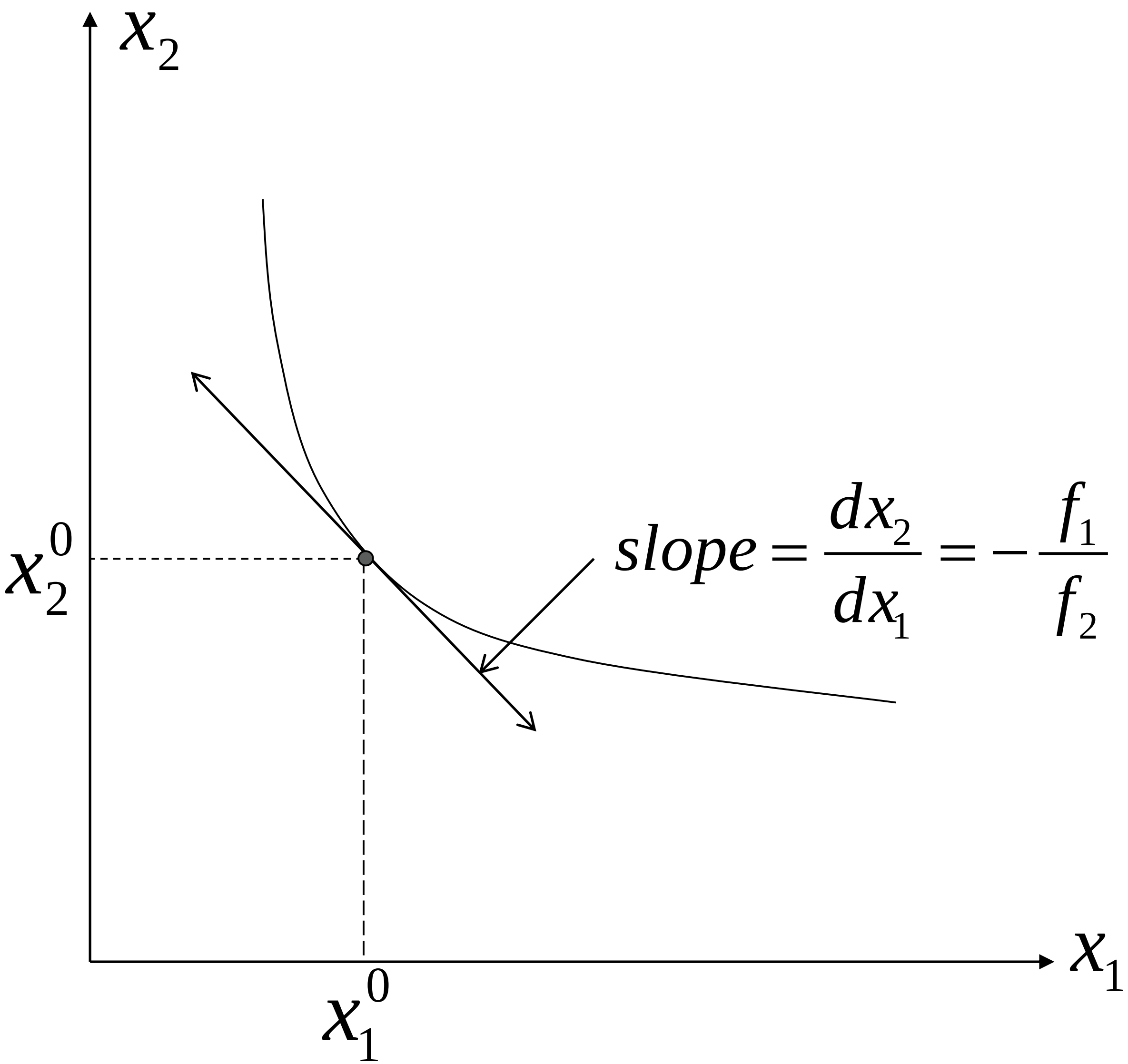
$$MRS_{12} = -\frac{\partial f / \partial x_1}{\partial f / \partial x_2} = -\frac{f_1}{f_2}$$

  - An *isoquant* is defined as the set of bundles $\left(x_1, x_2\right)$ that gives the same level of output (see fig. 1)
  - The slope of an isoquant at a given point is given by (see fig. 1),

$$dy = df\left(x_1, x_2\right) = 0 \Leftrightarrow f_1 dx_1 + f_2 dx_2 = 0 \Leftrightarrow \frac{dx_2}{dx_1} = -\frac{f_1}{f_2}$$

2.2.2 Substitution possibilities (Continued)

Figure 1: MRS between 2 inputs



2.2.2 Substitution possibilities (Continued)

The Allen Elasticity of Substitution

- MRS_{12} can be used as a measure of the possibilities of substitution between inputs 1 and 2

- A more convenient measure is the *Allen Elasticity of Substitution* (AES), σ, (Hicks (1932)) defined as,

$$\sigma = \frac{d\left(x_2/x_1\right)}{d\left(f_1/f_2\right)}\frac{f_1/f_2}{x_2/x_1} = \frac{d\ln\left(x_2/x_1\right)}{d\ln\left(f_1/f_2\right)}$$

- **Very Important remark:** in the definition of σ, the output, y, is held constant

2.2.2 Substitution possibilities

The Allen Elasticity of Substitution (continued)

- $$\sigma = \frac{\Delta_x}{\Delta_f} = \frac{d\ln(x_2/x_1)}{d\ln(f_1/f_2)} = \frac{d(x_2/x_1)/(x_2/x_1)}{d(f_2/f_1)/(f_2/f_1)}$$

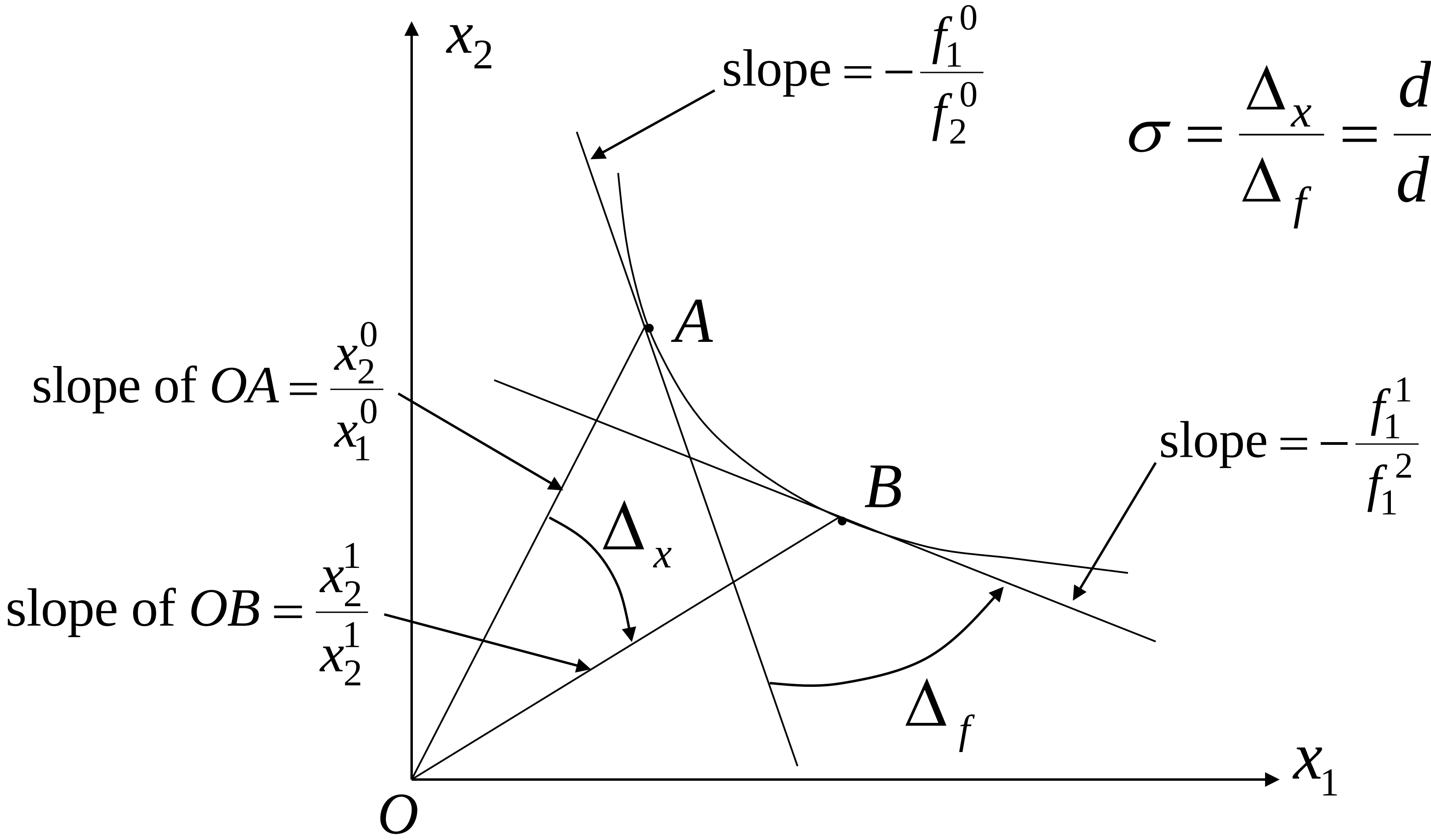
- When the isoquant is flat, $\Delta_f \to 0$ and, $\Delta_x/\Delta_f \to \infty$; this is the case of perfect substitutability between the 2 inputs

- The more the isoquant is convex, the more Δ_f is large and the more the elasticity is low (see fig. 2)

- **Remark:** With more than 2 inputs the above relation cannot be applied but it can be generalized (see hereafter)

2.2.2 Substitution possibilities (Continued)

Figure 2: The Allen Elasticity of Substitution



$$\sigma = \frac{\Delta_x}{\Delta_f} = \frac{d\ln\left(x_2/x_1\right)}{d\ln\left(f_1/f_2\right)}$$

2.2.2 Substitution possibilities (Continued)

General formula for the AES

- In the general case, the Allen elasticity is defined by,

$$\sigma_{ij} = \frac{\sum_i x_i f_i}{x_i x_j} \frac{|F_{ij}|}{|F|}$$

where, $|F|$, is the determinant of the bordered Hessian,

$$F = \begin{bmatrix} 0 & f_1(x) & f_2(x) & \cdots & f_N(x) \\ f_1(x) & f_{11}(x) & f_{12}(x) & \cdots & f_{1N}(x) \\ f_2(x) & f_{12}(x) & f_{22}(x) & \cdots & f_{2N}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_N(x) & f_{1N}(x) & f_{2N}(x) & \cdots & f_{NN}(x) \end{bmatrix}$$

and $|F_{ij}|$ the matrix cofactor of $\quad f_{ij} \equiv \partial f / \partial x_i \partial x_j$

2.2.2 Substitution possibilities (end)

Interpretation of elasticity

- $\sigma_{ij} > 0 \Leftrightarrow$ input i and input j are substitutable
- $\sigma_{ij} < 0 \Leftrightarrow$ input i and input j are complement
- **Remarks:**
  - When the technology includes only 2 inputs, they are necessarily substitutable (it is not possible to produce the same level of output with less of both inputs – see the move from A to B in fig. 2)
  - As $F_{ij} = F_{ji} \ \forall i \neq j$, the *AES* is symmetric: $\sigma_{ij} = \sigma_{ji} \ \forall i \neq j$

2.3 Parametric forms of the production function
2.3.1 The Cobb-Douglas (CD) form
Definition

- The CD production function (see R. Cobb and P. Douglas, "A theory of production", AER, 18:1, supplement, march 1928) is defined by,

$$y = A\prod_{k=1}^{N} x_k^{\alpha_k}$$

- The CD is homogeneous of degree $\mu = \sum \alpha_i$

- When $\sum \alpha_i = 1$, the CD has constant returns to scale

- **Remark:** As defined here the CD is a generalization to N inputs of the original CD function

2.3.1 The Cobb-Douglas (CD) form

Economic interpretation of the parameters

- **Hypothesis:** the producer maximizes its profit under the constraint that the technology is given

- Different cases can be considered:
  - *Monopolist case:* the producer is able to influence output price and/or input prices
  - *Perfect competition:* the producer considers prices as given (it is too small to influence prices)

- In the following we consider the second possibility

2.3.1 The Cobb-Douglas (CD) form

Economic interpretation of the parameters (continued)

- Mathematically the former hypothesis translates into,

$$\max_{y,x_1,\cdots,x_N} \left\{ p_y y - \sum_{k=1}^{N} p_k x_k : y \,,, \; f\left(x_1,\cdots,x_N\right) \right\}$$

Where $p_y, p_k \left(k=1,\cdots N\right)$ are output and input prices respectively

- The Lagrangian of this problem is,

$$L\left(y,x_1,\cdots,x_N,\lambda\right) = p_y y - \sum_{k=1}^{N} p_k x_k + \lambda\left[f\left(x_1,\cdots,x_N\right) - y \right]$$

- **Remark:** It is implicitly assumed that the production function is well behaved

2.3.1 The Cobb-Douglas (CD) form

Economic interpretation of the parameters (continued)

- The FOC of the former problem are:

$$p_y - \lambda = 0$$

$$p_k - \lambda f_k = 0 \quad k = 1, \ldots, N \quad (f_k \equiv \partial f / \partial x_k)$$

$$y = f(x_1, \cdots, x_N)$$

- When $f(x)$ is a CD, the FOC becomes (case of 2 inputs capital, K, and labor, L)

$$p_K - \lambda \alpha A K^{\alpha-1} L^{\beta} = 0$$

$$p_L - \lambda \beta A K^{\alpha} L^{\beta-1} = 0$$

$$y = A K^{\alpha} L^{\beta}$$

2.3.1 The Cobb-Douglas (CD) form

Economic interpretation of the parameters (continued)

- FOC can also be written,

$$\frac{p_K K}{p_y y} = \alpha \quad ; \quad \frac{p_L L}{p_y y} = \beta$$

- Optimum is reach when capital and labor value shares (wrt to production value) are set equal to the parameters α and β respectively

- In other words the value shares of inputs must be kept constant

- **Consequence:** the CD cannot be used by econometricians (it imposes, a priori, a very strong hypothesis on the producer behavior)

2.3.1 The Cobb-Douglas (CD) form

Economic interpretation of the parameters (continued)

- $\dfrac{p_K K}{p_y y} = \alpha$ and $\dfrac{p_L L}{p_y y} = \beta$ imply, $\dfrac{p_K K}{p_L L} = \dfrac{\alpha}{\beta}$
- Or, in logarithmic form,
- $$\ln \frac{K}{L} = \ln \frac{\alpha}{\beta} + \ln \frac{p_L}{p_K}$$
- This latter relation shows that,

$$\sigma_{KL} \equiv \frac{d \ln(K/L)}{d \ln(f_L/f_K)} = \frac{d \ln(K/L)}{d \ln(p_L/p_K)} = 1 \quad (\text{at optimum } \; f_L/f_K = p_L/p_K)$$

- **Conclusion:** Under a CD specification, σ_{KL} is equal to 1; the CD function is too much restrictive for being use in a econometric study

2.3.1 The Cobb-Douglas (CD) form

Economic interpretation of the parameters (end)

- Remarks:
  - With $N\ (N > 2)$ inputs, $\sigma_{ij} = 1\ \forall i, j; i \neq j$
  - It can be observed that the parameter of the CD can be estimated by calibration (i.e. using on year of observation only)
  - CD specification is often rejected on the basis of statistical tests:

Table 1: Estimating $\ln(K/L) = \alpha + \sigma \ln(p_L/p_K)$

(Data from Berndt and Wood, AER, vol.69 n°3, 1979)

	estimates	Standard error	Confidence. interval
α	-1.5059	0.0322	$]-1.5702, -1.4415[$
σ	0.6558	0.0779	$]0.5000, 0.8116[$

2.3.2 The CES form
Definition

- The CES (Constant Elasticity of Substitution production function (see Arrow, Chenery, Minhas and Solow, “Capital, labor substitution and economic efficiency”, RES, 43:5, August 1961) is defined by,

$$y = A\left[\sum_{i=1}^{N}\delta_i x_i^{-\rho}\right]^{-1/\rho} \qquad \text{where,} \sum_{i=1}^{N}\delta_i = 1$$

- The CES is clearly a constant returns to scale production function

2.3.2 The CES form

Economic interpretation of the parameters

- Let us consider the 2 inputs cases,

$$y = A\left[\delta K^{-\rho} + \left(1-\delta\right) L^{-\rho}\right]^{-1/\rho}$$

- From the producer's optimization program we derive,

$$\ln \frac{K}{L} = C^{te} + \frac{1}{1+\rho} \ln \frac{p_L}{p_K}$$

- The AES between K and L is then given by,

$$\sigma_{KL} \equiv \frac{d \ln\left(K/L\right)}{d \ln\left(p_L / p_K\right)} = \frac{1}{1+\rho}$$

Derivation of the preceding formula

- FOC are,

$$p_K = \lambda A\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]^{-1/\rho-1}\delta K^{-\rho-1}$$

$$p_L = \lambda A\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]^{-1/\rho-1}(1-\delta)L^{-\rho-1}$$

- So we have, $\dfrac{p_K}{p_L} = \dfrac{\delta}{1-\delta}\left(\dfrac{K}{L}\right)^{-\rho-1}$

- Or, $\dfrac{K}{L} = \left(\dfrac{\delta}{1-\delta}\right)^{\frac{1}{1+\rho}}\left(\dfrac{p_K}{p_L}\right)^{-\frac{1}{1+\rho}}$

- And finally, $\ln\dfrac{K}{L} = \dfrac{1}{1+\rho}\ln\dfrac{\delta}{1-\delta} + \dfrac{1}{1+\rho}\ln\dfrac{p_L}{p_K}$

2.3.2 The CES form

Economic interpretation of the parameters (continued)

- **Remarks about the CES:**
  - To estimate the parameters of the CES form we run an OLS estimation of the following equation (u is an unobserved residual),

$$\ln\frac{K}{L} = \alpha + \sigma \ln\frac{p_L}{p_K} + u \quad \text{where } \sigma = \frac{1}{1+\rho} \text{ and } \alpha = \sigma \ln\frac{\delta}{1-\delta}$$

  - The CES is reduced to a CD when $\rho \to 0 \quad \sigma_{KL} = 1/(1+\rho) \to 1$

  - With 3 inputs or more we have,

$$\sigma_{ij} = \frac{1}{1+\rho} \qquad \forall i, j = 1,\ldots,N \quad ; \quad i \neq j.$$

because FOC of the producer's program gives,

$$\ln\frac{x_i}{x_j} = C^{te} + \frac{1}{1+\rho}\ln\frac{p_j}{p_i} \qquad \forall i, j = 1,\ldots,N\,; i \neq j$$

2.3.2 The CES form

Economic interpretation of the parameters (end)

- **Conclusion about the CES:**
  - Any pair of inputs has the same AES, the econometric literature shows that this is far from being true (capital-labor AES is very different from capital-energy AES in all econometric studies)
  - The CES is a generalization of the CD but it imposes too restrictive a priori restrictions on the producer's behavior
  - A simple generalization of the CES is possible (see next subsection)

2.3.3 The nested CES form

- A nested CES is made of 2 CES that are nested; in the 3 inputs case (K, L, E), one possible way to define a nested CES is,

$$y = A\left[aQ^{-\rho} + (1-a)L^{-\rho}\right]^{-1/\rho} \quad \text{where,} \quad Q = \left[bK^{-\omega} + (1-b)E^{-\omega}\right]^{-1/\omega}$$

- It can be shown that, $\sigma_{KL} = \sigma_{LE} = \dfrac{1}{1+\rho}$, but the elasticity σ_{KE} is not constant

- **Remarks:**
  - The nested CES solves part of the problem only
  - Using a nested CES needs to implicitly assume a separability hypothesis that is far from being neutral as we will see later

2.3.4 The flexible forms

- Two definition have been used in the literature:
  - A flexible form is a functional form that does not impose any a priori restriction on the elasticities of substitution
  - a flexible form is a functional form that can be interpreted as a second order approximation of the true unknown production function at a given point
- The second definition is partially incorrect because an econometric estimation of a flexible form never gives a second order approximation of the true function (see H. White: "*Using Least Squares to Approximate Unknown Regression Functions*," *International Economic Review*, 21, 1980)

2.3.4 The flexible forms (continued)

- The most well-known flexible forms are
  - The ***Translog*** form introduced by Christensen, Jorgenson and Lau (see «*Conjugate duality and the transcendental logarithmic production function*» , Econometrica, 39, 1971)
  - The ***Generalized Leontief*** form introduced by, Diewert, «*An application of the Shephard duality theorem : a generalized Leontief production function*» , Journal of Political Economy, 79, 1971

2.3.4 The flexible forms (continued)

The Translog form (TL)

- The TL form is defined as,

$$\ln y = \alpha_0 + \sum_{k=1}^{N} \alpha_k \ln x_k + \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} \beta_{kl} \ln x_k \ln x_l \qquad \beta_{kl} = \beta_{lk} \, \forall l, k$$

- **Remarks:**
  - The TL form embodies the CD form since the logarithmic form of the CD is,

$$\ln y = \ln A + \sum_{k=1}^{N} \alpha_k \ln x_k$$

  - The constraints, $\beta_{kl} = \beta_{lk} \, \forall l, k$, are known as the *symmetry constraints* because the Hessian of the [illegible] is not symmetric when these constraints are not satisfied (This can be seen immediately by deriving the TL with respect to the logarithms of the input prices)

2.3.4 The flexible forms (continued)

The Generalized Leontief form (GL)

- The GL form is defined as,

$$y = \sum_{l=1}^{N}\sum_{k=1}^{N} a_{kl}\sqrt{x_l}\sqrt{x_k} \qquad a_{kl} = a_{lk}\ \forall l,k$$

- **Remarks:**
  - As defined above, the GL exhibits constant returns to scale, but it can be generalized to the case of non constant returns to scale
  - The GL has been less used than the TL in the literature; note that if the sample used for estimation includes some zero demands for some inputs, it is not possible to use the TL, the GL is then a possible candidate

2.3.4 The flexible forms (continued)

Estimating flexible forms of the production function

- The number of parameters to estimate is very large (with 4 inputs the TL form has 16 parameters)

- It is not possible to use the FOC of the producer's program to obtain the parametric form of the input demand functions

- **Consequences:**
  - OLS estimation of a flexible form often gives very bad results
  - Flexible forms of the production function are almost never estimated in the literature

3. The cost function
3.1 Definition

- The *cost function* gives the minimum cost associated with a given level of output and input prices, taking into account the available technology

- Formally we write,

$$C\left(y,p\right)=\min_{K,L,E,M}\left\{p_K K+p_L L+p_E E+p_M M : y\leq f\left(K,L,E,M\right)\right\}$$

3.2 The Shephard's lemma

- **Shephard's lemma:** The optimal demand of an input is given by the derivative of the cost function w.r.t. the price of that input formally we have,

$$x_i = \mathrm{x}_i\left(y, p\right) = \frac{\partial C\left(y, p\right)}{\partial p_i} \quad \forall i = 1, \cdots, N$$

- **Remark:** we implicitly consider here that the quantity of inputs that minimize the cost is unique, the conditions under which this is always true will be discussed later

3.2 The Shephard’s lemma (continued)

Comments

- When a functional form of the cost function has been chosen, the parametric form of the input demand functions are known (example: take a TL and derive)

- On the econometric point of view, when using a cost function we introduce a much more complete model with 5 equations (the cost function and 4 input demands) and not one only as when we use a production function, the model is then much more complete and includes much more information

3.2 The Shephard's lemma (continued)

Intuition of the proof in the 4 inputs case

- Note K, L, E, M the quantities of inputs used and p_K, p_L, p_E, p_M their respective prices

- Consider a production level, y, such that there exists a vector, (K,L,E,M) such that, $y \leq f(K,L,E,M)$

- Let us note (K^0,L^0,E^0,M^0) the (unique) input bundle minimizing the total cost of production when input prices are given by, $p_K^0, p_L^0, p_E^0, p_M^0$

- Consider a small change, h, of the price of capital such that the new price, $p_K^0 + h$, is positive

3.2 The Shephard's lemma

Intuition of the proof in the 4 inputs case (continued)

- From the definition of the cost function we necessarily have,

$$C\left(y, p_K^0 + h, p_L^0, p_E^0, p_M^0\right) \leq \left(p_K^0 + h\right)K^0 + p_L^0 L^0 + p_E^0 E^0 + p_M^0 M^0$$

- Or,

$$C\left(y, p_K^0 + h, p_L^0, p_E^0, p_M^0\right) - \left(p_K^0 K^0 + p_L^0 L^0 + p_E^0 E^0 + p_M^0 M^0\right) \leq hK^0$$

- Or,

$$C\left(y, p_K^0 + h, p_L^0, p_E^0, p_M^0\right) - C\left(y, p_K^0, p_L^0, p_E^0, p_M^0\right) \leq hK^0$$

- Since by definition of the cost function we have,

$$C\left(y, p_K^0, p_L^0, p_E^0, p_M^0\right) = p_K^0 K^0 + p_L^0 L^0 + p_E^0 E^0 + p_M^0 M^0$$

3.2 The Shephard's lemma

Intuition of the proof in the 4 inputs case (continued)

- Now consider the ratio,

$$g(h) \equiv \frac{C\left(y, p_K^0 + h, p_L^0, p_E^0, p_M^0\right) - C\left(y, p_K^0, p_L^0, p_E^0, p_M^0\right)}{h}$$

- We clearly have,

$$g(h) \geq K^0 \quad \text{if } h < 0$$

$$g(h) \leq K^0 \quad \text{if } h > 0$$

- And finally,

$$\lim_{h \to 0} g(h) = \frac{\partial C\left(y, p_K^0, p_L^0, p_E^0, p_M^0\right)}{\partial p_K} = K^0$$

- The same calculations could be done for labor, energy and materials

3.3 The Duality between cost and production function

- Previous results suggests that a relevant way to build an econometric model of the production behavior is to consider a parametric form of the cost function

- But one important question remains: are cost and production functions giving the same representation of the technology of the firm?

- Because the answer is yes (under certain conditions), almost all econometrics studies of the production sector use a cost function

3.3 The Duality between cost and production function (continued)

The duality theorem

- **Duality Theorem:** Consider a production function $y = f(x)$, continuously differentiable, increasing and quasi-concave,
  - There exists a cost function, $C = C(y, p)$, continuously differentiable, increasing in (y, p), linearly homogeneous and concave in p
  - The *implicit production function,*

$$f^*(x) = \max_y \left\{ y : \sum_{i=1}^{N} p_i x_i \geq C(y, p) \right\}$$

is such that, $f(x) = f^*(x)$

- **Remarks:** This theorem shows that there is a bijective correspondence between cost and production functions

3.3 The Duality between cost and production function (continued)

Some graphs to understand the duality theorem

- Consider a technology using only capital, K, and labor, L
- The curve C_1 on fig. 3 represents an *isoquant* ; by definition any point on C_1 is such that, $dy = df(K,L) = f_K dK + f_L dL = 0$, or such that, $dK/dL = -f_L / f_K$
- The set, $V(y) \equiv \{(K,L) : y \le f(K,L)\}$ is the *inputs requirement set* (hatched surface on fig. 3) ; this is the set of bundles (K,L) allowing to produce at least y
- The iso-cost line D_1 is the set of bundles (K,L) that gives the same cost at prices p_K and p_L ; the slope of the iso-cost line is $-p_L / p_K$ because,

$$C = p_K K + p_L L \Leftrightarrow K = \frac{C}{p_K} - \frac{p_L}{p_K} L$$

3.3 The Duality between cost and production function (continued)

Some graphs to understand the duality theorem

- Cost minimization is equivalent to profit maximization when the level of output is fixed

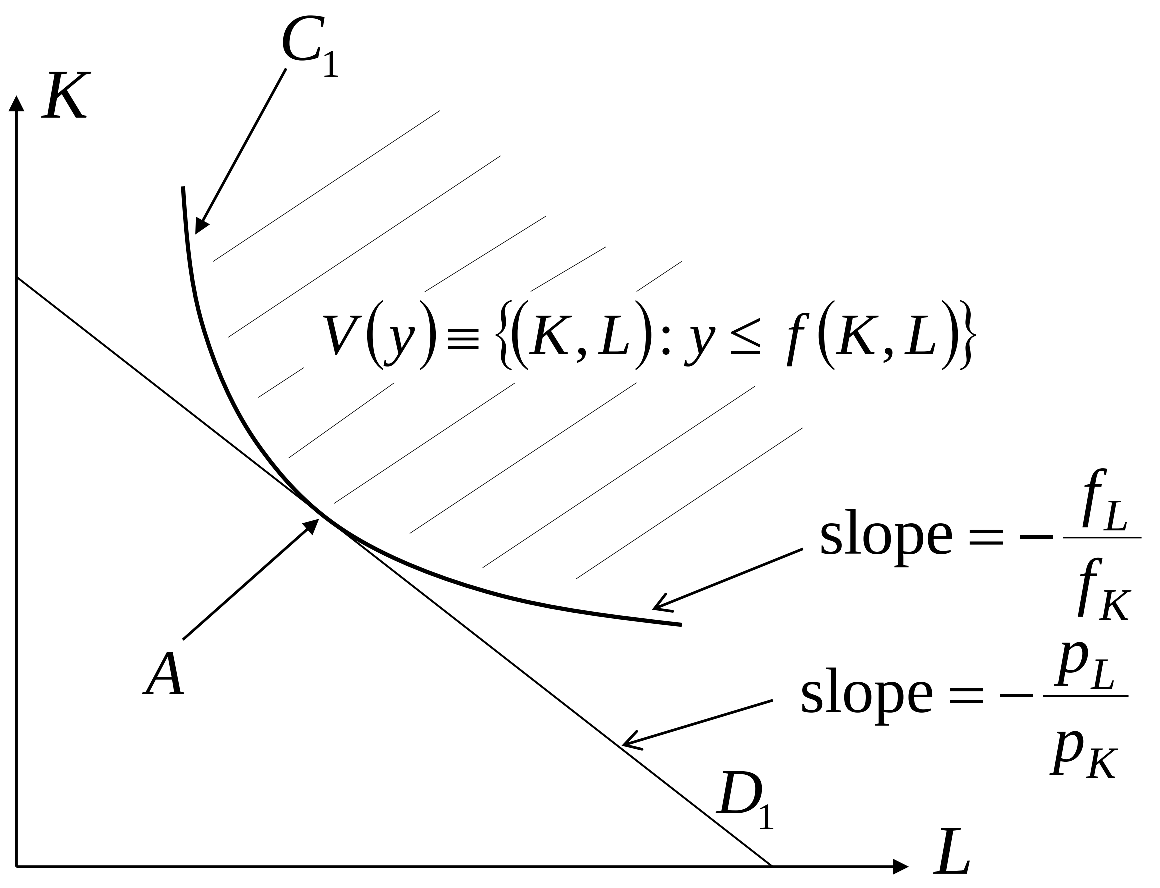
- So total cost is minimized when (see slide 18),

$$\begin{cases} p_K - \lambda f_K = 0 \\ p_L - \lambda f_L = 0 \end{cases} \Leftrightarrow \frac{f_L}{f_K} = \frac{p_L}{p_K}$$

- Thus, total cost is minimized at the point of tangency, A, between the iso-cost line and the isoquant (see fig. 3)

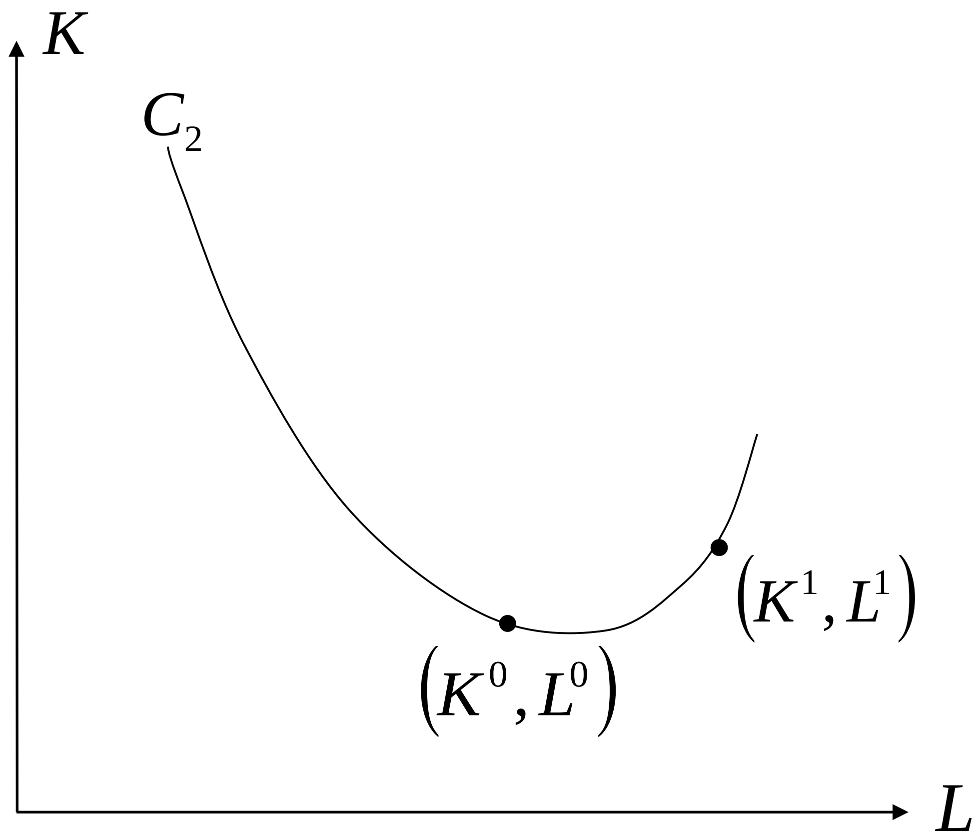
3.3 The Duality between cost and production function (continued)

Figure 3: Cost minimization



3.3 The Duality between cost and production function (continued)

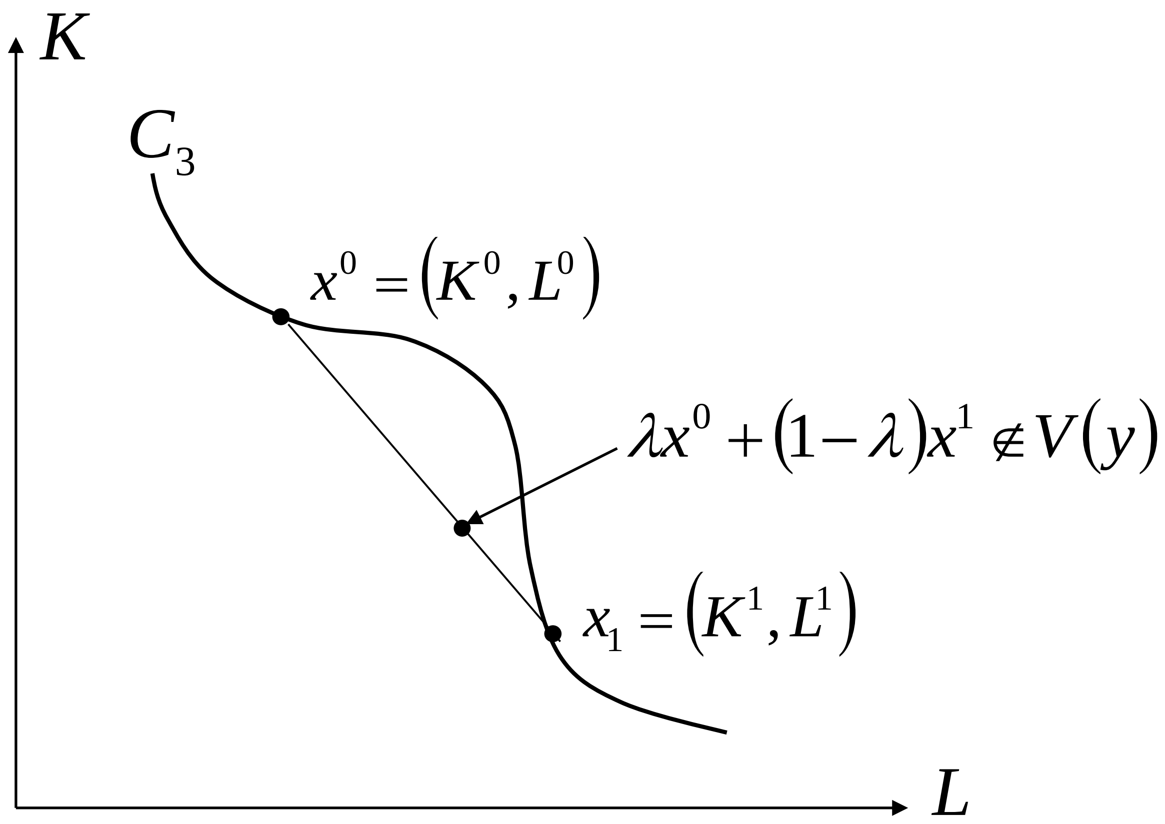
Figures 4: Isoquant when $y = f(K, L)$ is not an increasing function



- A non-decreasing production function is such that an isoquant of the type drawn on fig. 4 is not possible (the bundles (K^0, L^0) and (K^1, L^1) are on the same isoquant but $K^1 > K^0$ and $L^1 > L^0$)

3.3 The Duality between cost and production function (continued)

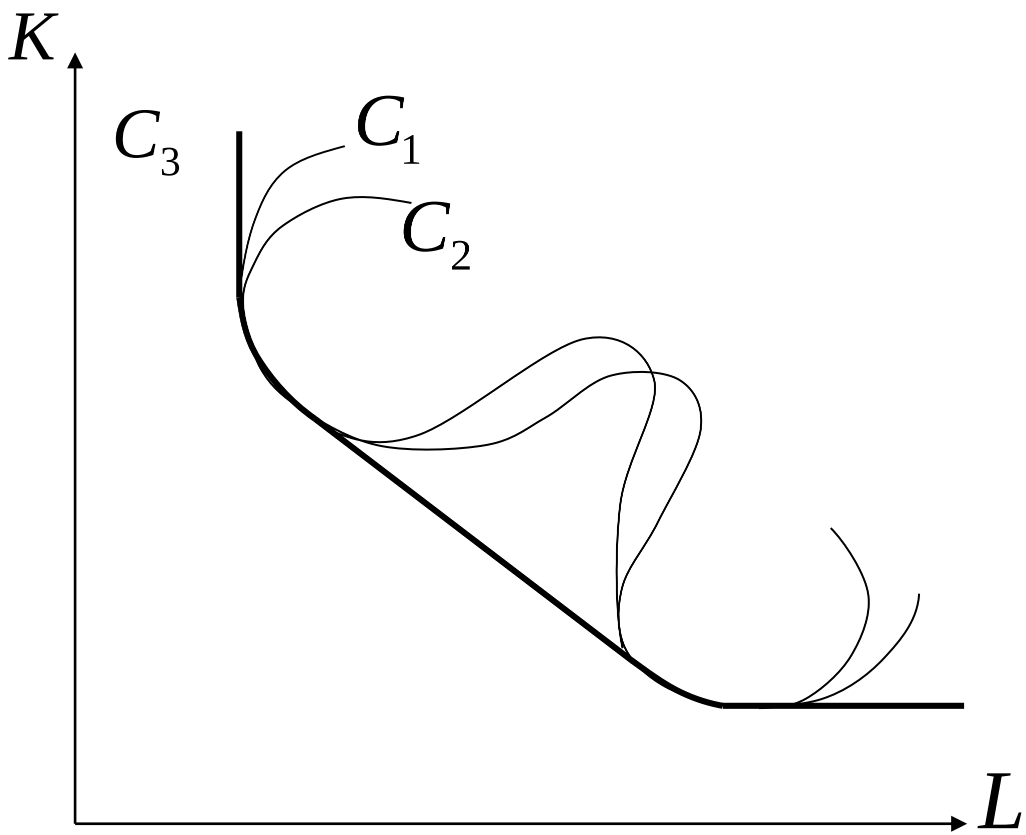
Figures 5: Isoquant when $y = f(K, L)$ is not a quasi-concave function



- A *quasi-concave production function* is such that an isoquant of the type drawn on fig. 5 is not possible (a linear combination of 2 input bundles included in the input requirement set should be included in the input requirement set)

3.3 The Duality between cost and production function (continued)

Figures 6: Duality between cost and production function



- Fig 6 shows an example in which 3 different technologies have the same dual cost function; this case is not possible for an increasing and quasi-concave production function because C_1 and C_2 are rejected

3.3 Cost function and returns to scale

- **Issue:** how to characterize returns to scale through the properties of a cost function?

- We consider fixed input prices in order to write the cost function as a function of y only, $C = C(y)$

- From the properties of the cost function (see the duality theorem) we know that $C(y)$ is increasing ($C'(y) \equiv \partial C / \partial y \geq 0$

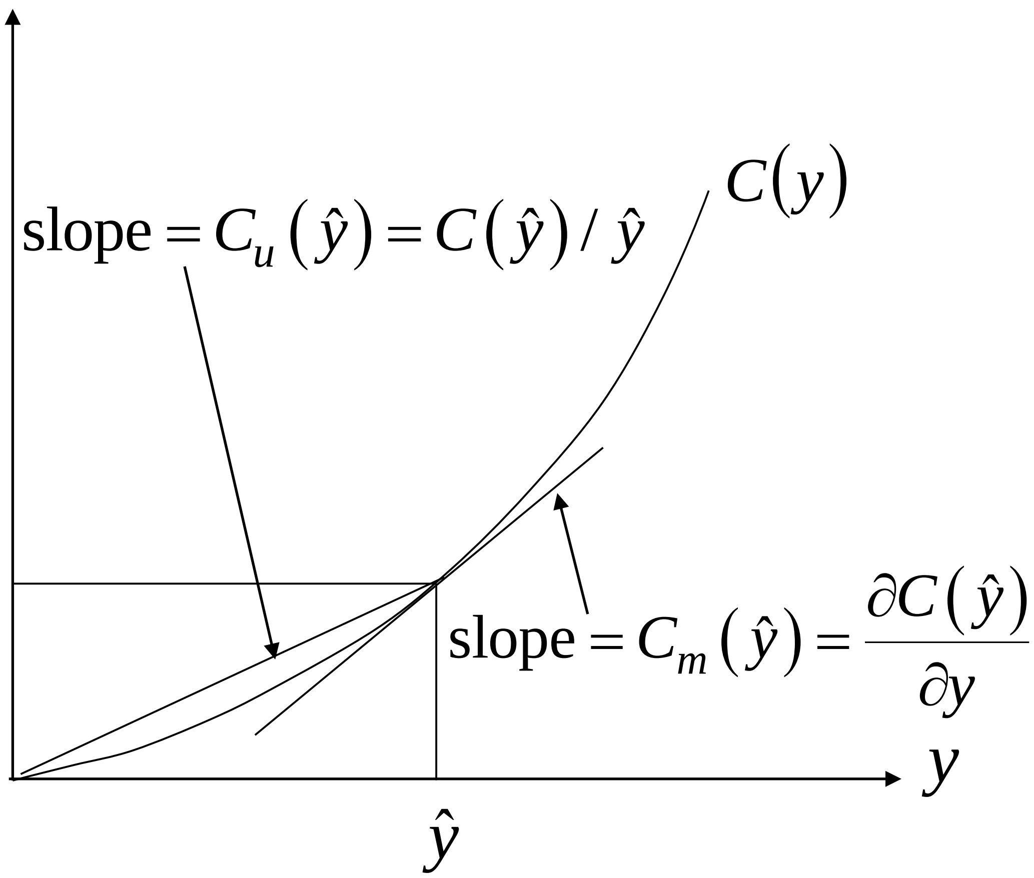
3.3 Cost function and returns to scale (continued)

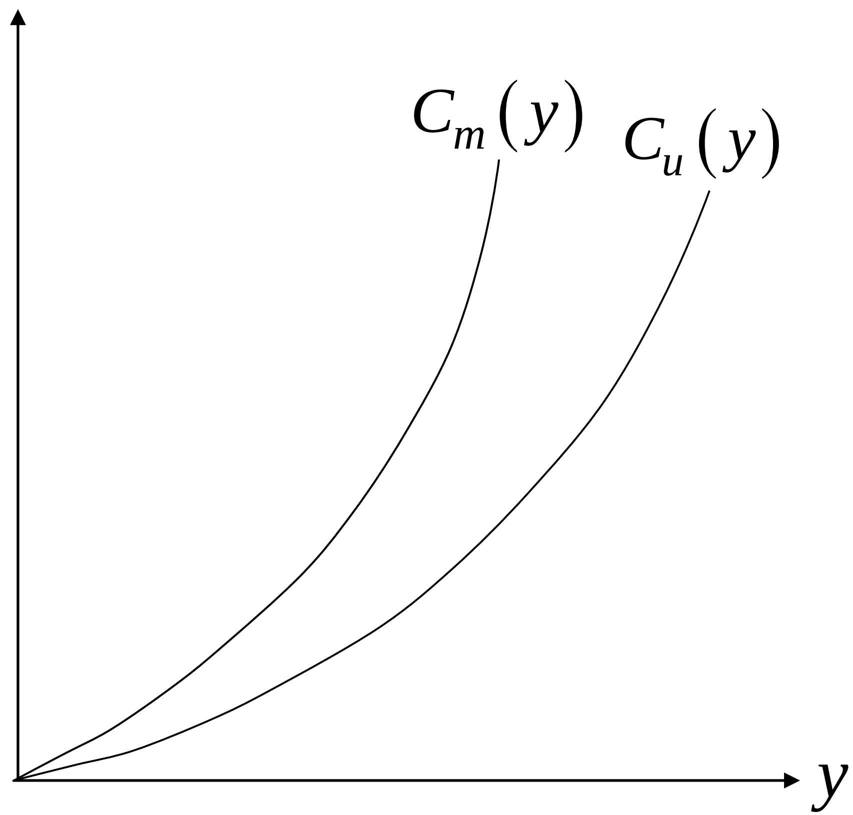
Decreasing returns to scale

- Assume the firm wants to multiply y by $\lambda > 1$
  - Under decreasing returns to scale we have $f(\lambda x) < \lambda f(x) = \lambda y$
  - Then to obtain λy the firm needs more than λx
  - **Consequences:** To produce λy the firm multiplies the total cost by more than λ; the unit cost function, $C_u(y) \equiv C(y)/y$ and the marginal cost function, $C_m(y) \equiv \partial C(y)/\partial C$ are both increasing with y (see fig. 7)
  - It is easily shown on figure 7 that under decreasing returns to scale, <u>the marginal cost is always larger than the unit cost</u>

3.3 Cost function and returns to scale (continued)

Figure 7: Decreasing returns to scale





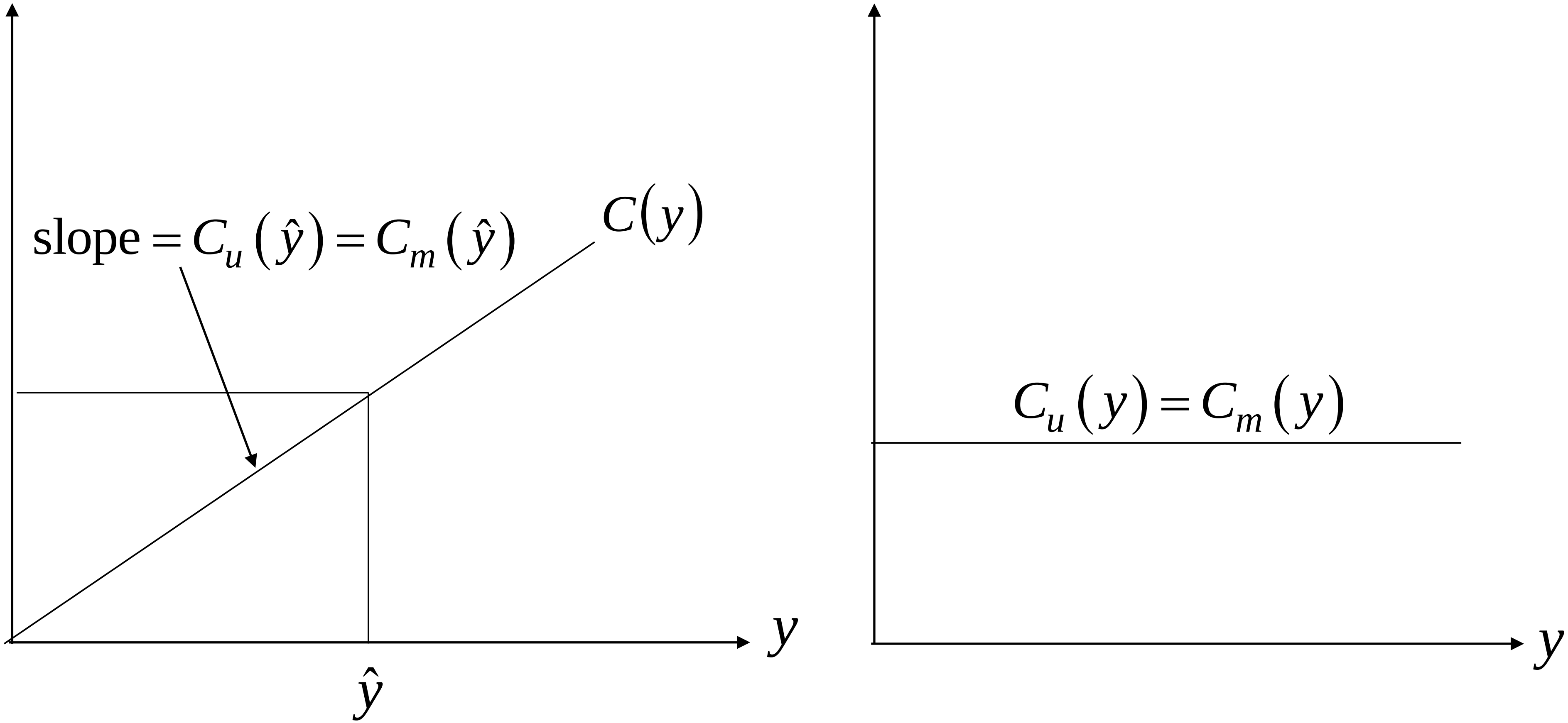
3.3 Cost function and returns to scale (continued)

Constant returns to scale

- Assume the firm wants to multiply y by $\lambda > 0$
  - Under constant returns to scale we have $f(\lambda x) = \lambda f(x) = \lambda y$
  - Then to obtain λy the firm needs λx
  - **Consequences:** Producing λy the firm multiplies the total cost by λ; the unit cost and the marginal cost are both constant (independent of y as shown on fig. 8)
  - It is easily shown on figure 8 that under constant returns to scale, <u>the marginal cost is always equal to the unit cost</u>

3.3 Cost function and returns to scale (continued)

Figure 8: Constant returns to scale



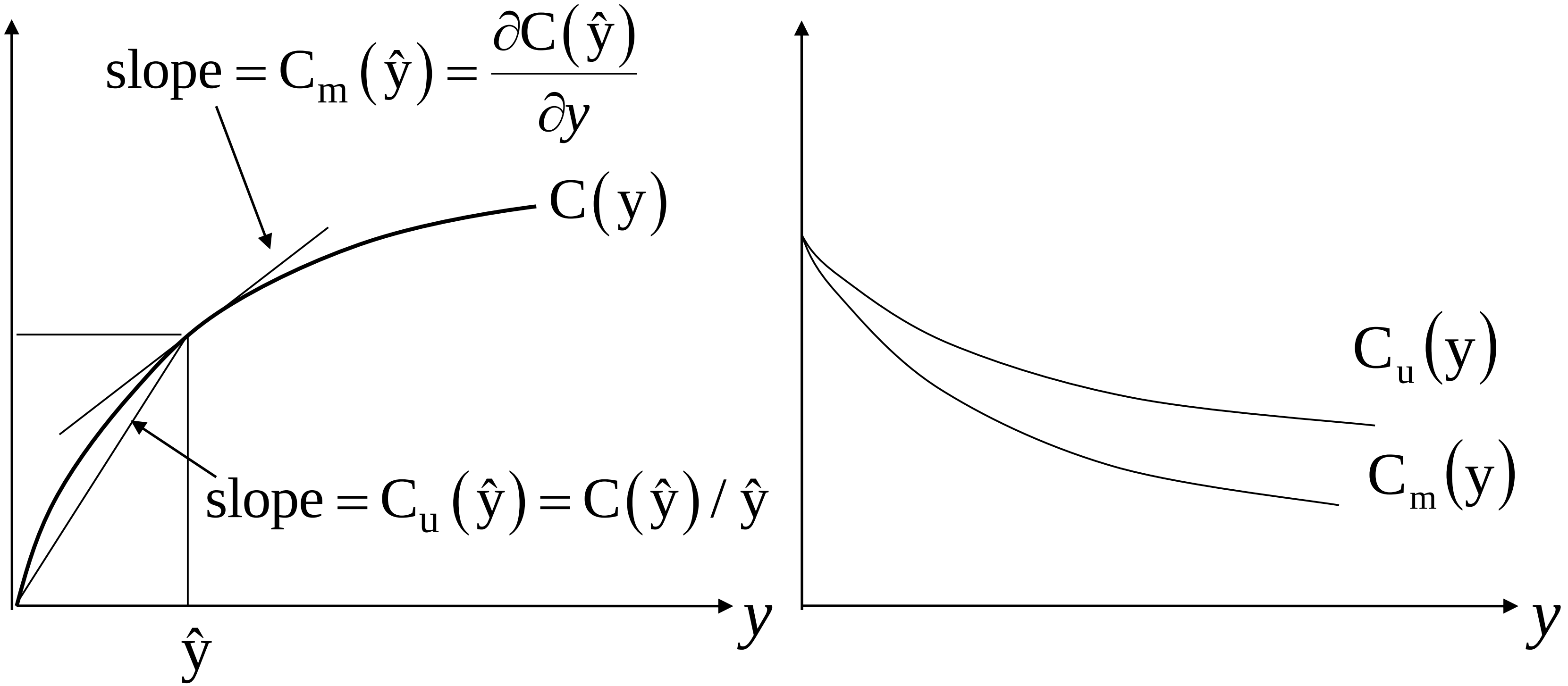
3.3 Cost function and returns to scale (continued)

Increasing returns to scale

- Assume the firm wants to multiply y by $\lambda > 1$
  - Under increasing returns to scale we have $f(\lambda x) > \lambda f(x) = \lambda y$
  - Then to obtain λy the firm needs less than λx
  - **Consequences:** Producing λy the firm multiplies the total cost by less than λ; the unit cost function, $C_u(y) \equiv C(y)/y$ and the marginal cost function, $C_m(y) \equiv \partial C(y)/\partial C$ are both decreasing with y (see fig. 9)
  - It is easily shown on figure 9 that under increasing returns to scale, <u>the marginal cost is always lower than the unit cost</u>

3.3 Cost function and returns to scale (continued)

Figure 9: Increasing returns to scale



3.3 Cost function and returns to scale (continued)

The case of constant returns to scale

- The production function, $y = f(x)$, is linearly homogeneous in x if and only if the dual cost function, $C(y,p)$, is linearly homogeneous of degree one in y

- **Consequence:** Constant returns to scale allows to simplify the expression of the cost function, we have,

$$C = C(y,p) = yC\left(\frac{1}{y}y, p\right) = yC(1,p) = yC_u(p)$$

- The function $C_u(p)$ is known as the *unit cost function*

3.3 Cost function and returns to scale

The case of constant returns to scale (continued)

- The proof of the preceding result is straightforward,
  - Assume $f(x)$ linearly homogeneous in x then we have ( ), $\lambda > 0$

$$C(\lambda y, p) = \min_x \{p'x : \lambda y \leq f(x)\} = \min_x \{p'x : y \leq \lambda^{-1} f(x)\}$$

  - But linear homogeneity of $f(x)$ implies,

$$\begin{aligned} C(y, p) &= \min_x \{\lambda p'(\lambda^{-1}x) : y \leq f(\lambda^{-1}x)\} \\ &= \min_z \{\lambda p'z : y \leq f(z)\} \quad \text{with } z \equiv \lambda^{-1}x \\ &= \lambda \min_z \{p'z : y \leq f(z)\} \\ &= \lambda C(y, p) \end{aligned}$$

3.3 Cost function and returns to scale

The case of constant returns to scale (continued)

- Assume $C(y,p)$ linearly homogeneous in y then ($\lambda > 0$),

$$f(\lambda x) = \max_y \left\{ y : p'(\lambda x) \leq C(y,p) \right\} = \max_y \left\{ y : p'x \leq \lambda^{-1} C(y,p) \right\}$$

- But linear homogeneity of $C(y,p)$ implies,

$$f(\lambda x) = \max_y \left\{ \lambda\left(\lambda^{-1} y\right) : p'x \leq C\left(\lambda^{-1} y, p\right) \right\}$$

$$= \max_z \left\{ \lambda z : p'x \leq C(z,p) \right\} \quad \text{with } z \equiv \lambda^{-1} y$$

$$= \lambda \max_z \left\{ z : p'x \leq C(z,p) \right\}$$

$$= \lambda f(x)$$

- QED

3.4 Elasticities of substitution and cost function

- From the FOC of the cost minimization problem we have,

$$p_j = \lambda f_j \qquad j = 1, \cdots, N$$

$$y = f\left(x_1, x_2, \cdots, x_N\right)$$

- Differentiating the FOC w.r.t. λ gives,

$$\begin{aligned} dp_j &= f_j d\lambda + \lambda df_j \\ &= f_j d\lambda + \lambda \sum_i f_{ij} dx_i \qquad j = 1, \cdots, N \end{aligned}$$

$$dy = \sum_{i=1}^{N} f_i dx_i$$

3.4 Elasticities of substitution and cost function (continued)

- In matrix form this gives,

$$\lambda \begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_N \\ f_1 & f_{11} & f_{12} & \cdots & f_{1N} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_N & f_{N1} & f_{N2} & \cdots & f_{NN} \end{bmatrix} \begin{bmatrix} d\lambda/\lambda \\ dx_1 \\ dx_2 \\ \vdots \\ dx_N \end{bmatrix} = \begin{bmatrix} \lambda dy \\ dp_1 \\ dp_2 \\ \vdots \\ dp_N \end{bmatrix}$$

3.4 Elasticities of substitution and cost function (continued)

- But as we are calculating an elasticity we have, $dy = 0$ and then, $dp_k = 0 \; \forall k \neq j$

$$\lambda \begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_N \\ f_1 & f_{11} & f_{12} & \cdots & f_{1N} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ f_j & f_{j1} & f_{j2} & \vdots & f_{jN} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ f_N & f_{N1} & f_{N2} & \cdots & f_{NN} \end{bmatrix} \begin{bmatrix} d\lambda/\lambda \\ dx_1 \\ \vdots \\ dx_j \\ \vdots \\ dx_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ dp_j \\ \vdots \\ 0 \end{bmatrix}$$

3.4 Elasticities of substitution and cost function (continued)

- We can also write,

$$\lambda\begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_N \\ f_1 & f_{11} & f_{12} & \cdots & f_{1N} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ f_j & f_{j1} & f_{j2} & \vdots & f_{jN} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ f_N & f_{N1} & f_{N2} & \cdots & f_{NN} \end{bmatrix}\begin{bmatrix} d\lambda / \lambda \\ dx_1 \\ \vdots \\ dx_j / dp_j \\ \vdots \\ dx_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

- Let us note that as $dp_k = 0 \; \forall k \neq j$ we have

$$dx_i / dp_j = \partial x_i / \partial p_j$$

3.4 Elasticities of substitution and cost function (continued)

- Using the Cramer's rule, we can then write,

$$\frac{\partial x_i}{\partial p_j} = \frac{|B_{ij}|}{|F|}$$

where F is the bordered Hessian matrix and B_{ij} is F with the ith column replaced by $[0, 0, \cdots, 0, 1, 0, \cdots, 0]'$

- Note that as the ith column of F has 0 everywhere except the jth component, $|B_{ij}|$ is also the cofactor of ijth element of F thus,

$$|B_{ij}| = (1/\lambda)|F_{ij}|$$

3.4 Elasticities of substitution and cost function (continued)

- Multiplying $\partial x_i / \partial p_j$ by p_j / x_i we have,

$$\varepsilon_{ij} \equiv \frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i} = \frac{\lambda f_j}{x_i}\frac{|F_{ij}|}{\lambda |F|}$$

- Then, dividing ε_{ij} by

$$S_j = p_j x_j / \sum_i p_i x_i = \lambda f_j x_j / \sum_i \lambda f_i x_i = f_j x_j / \sum_i f_i x_i$$

- We obtain,

$$\varepsilon_{ij} / S_j = \frac{\sum_i f_i x_i}{f_j x_j}\frac{\lambda f_j}{x_i}\frac{|F_{ij}|}{\lambda |F|} = \frac{\sum_i f_i x_i}{x_i x_j}\frac{|F_{ij}|}{|F|} = \sigma_{ij}$$

3.4 Elasticities of substitution and cost function (continued)

- ε_{ij} is known as the *cross-price elasticity*, note that we can write,

$$\varepsilon_{ij} \equiv \left.\frac{d \ln x_i}{d \ln p_j}\right|_{dy=0,\, dp_k=0\ \forall k \neq j}$$

- When $i = j$, ε_{ii} is known as the *own-price elasticity,* also named the *elasticity of input demand*

- **Remarks:**
  - The cross-price elasticities are not symmetric but have the same sign than the AES
  - The own-price elasticities are all negative when the production function is quasi-concave but the reverse is not true

3.4 Elasticities of substitution and cost function (continued)

Calculating elasticities in practice

- Assume that you have estimated a cost function and that you want to calculate the elasticities
- From the Shephard's lemma, we know that $x_i = \partial C / \partial p_i \equiv C_i$
- So we can write,

$$d\ln x_i = d\ln C_i = \frac{1}{C_i} dC_i = \frac{1}{C_i} \sum_{k=1}^{N} \frac{\partial C_i}{\partial p_k} dp_k = \frac{1}{C_i} \sum_{k=1}^{N} \frac{\partial^2 C}{\partial p_i \partial p_k} dp_k$$

- But, as all inputs prices are assumed constant except p_j, we have so we have ($dp_k = 0 \; \forall k \neq j$), $C_{ij} \equiv \partial^2 C / \partial p_i \partial p_j$

$$d\ln x_i = \frac{1}{C_i} \frac{\partial^2 C}{\partial p_i \partial p_j} dp_j = \frac{1}{C_i} \frac{\partial^2 C}{\partial p_i \partial p_j} p_j d\ln p_j = \frac{1}{C_i} C_{ij} p_j d\ln p_j$$

Calculating elasticities in practice (continued)

- Finally we can write,

$$\varepsilon_{ij} = \frac{p_j C_{ij}}{C_i} \quad \forall i, j; \quad \sigma_{ij} = \frac{C C_{ij}}{C_i C_j} \quad \forall i, j$$

- The cost function is linearly homogeneous in prices, so the input demand functions are homogeneous of degree 0 in prices; we have,

$$\sum_j \varepsilon_{ij} = \sum_j \frac{p_j C_{ij}}{C_i} = \frac{1}{C_i} \sum_j p_j \frac{\partial C_i}{\partial p_j} = \frac{1}{C_i} \sum_j p_j \frac{\partial x_i}{\partial p_j} = 0$$

- This last result is useful in practice, checking that your estimates of the elasticities verify the above relations allows to detect possible miscalculations

3.5 Two exercises

3.5.1 The dual cost function of a CD production function

- The CD production function is given by (2 inputs case),

$$y = AK^{\alpha}L^{\beta}$$

- The FOC of the cost minimization program are given by,

$$p_K - \lambda\alpha AK^{\alpha-1}L^{\beta} = 0$$

$$p_L - \lambda\beta AK^{\alpha}L^{\beta-1} = 0$$

$$y = AK^{\alpha}L^{\beta}$$

- The first and second FOC gives,

$$\frac{p_K}{p_L} = \frac{\alpha}{\beta}\frac{L}{K} \Leftrightarrow K = \frac{\alpha}{\beta}\frac{p_L}{p_K}L$$

- The last FOC gives,

$$L = A^{-\frac{1}{\beta}}K^{-\frac{\alpha}{\beta}}y^{\frac{1}{\beta}}$$

3.5.1 The dual cost function of a CD production function (continued)

- The demand function for capital can be written,

$$K=\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} A^{-\frac{1}{\alpha+\beta}}\left(\frac{p_L}{p_K}\right)^{\frac{\beta}{\alpha+\beta}} y^{\frac{1}{\alpha+\beta}}$$

- Symmetrically the labor demand function is given by,

$$L=\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} A^{-\frac{1}{\alpha+\beta}}\left(\frac{p_L}{p_K}\right)^{-\frac{\alpha}{\alpha+\beta}} y^{\frac{1}{\alpha+\beta}}$$

- Finally the dual cost function of the CD production function is,

$$C=p_K K+p_L L=A^{-\frac{1}{\alpha+\beta}} y^{\frac{1}{\alpha+\beta}} p_K^{\frac{\alpha}{\alpha+\beta}} p_L^{\frac{\beta}{\alpha+\beta}}\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}+\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right]=B y^{\frac{1}{\alpha+\beta}} p_K^{\frac{\alpha}{\alpha+\beta}} p_L^{\frac{\beta}{\alpha+\beta}}$$

3.5.1 The dual cost function of a CD production function (continued)

- First and second derivatives of the CD cost function are,

$$C_K = By^{\frac{1}{\alpha+\beta}} \frac{\alpha}{\alpha+\beta} p_K^{\frac{\alpha}{\alpha+\beta}-1} p_L^{\frac{\beta}{\alpha+\beta}}$$

$$C_L = By^{\frac{1}{\alpha+\beta}} \frac{\beta}{\alpha+\beta} p_K^{\frac{\alpha}{\alpha+\beta}} p_L^{\frac{\beta}{\alpha+\beta}-1}$$

$$C_{KL} = By^{\frac{1}{\alpha+\beta}} \frac{\alpha}{\alpha+\beta} \frac{\beta}{\alpha+\beta} p_K^{\frac{\alpha}{\alpha+\beta}-1} p_L^{\frac{\beta}{\alpha+\beta}-1}$$

- Thus it can be checked that,

$$\sigma_{KL} = \frac{CC_{KL}}{C_K C_L} = 1$$

3.5.1 The dual cost function of a CD production function (continued)

- **Comments:**
  - The dual cost function associated to a CD production function is itself a CD form, the CD form is said auto-dual
  - The previous slide illustrates the use of the Shephard's lemma to derive the input demand function and to use of the derivatives of the cost function to calculate the elasticity of substitution
  - We can easily see that when $\alpha + \beta = 1$, the CD cost function can be written as a unit cost function independent of the level of output (returns to scale are constant**)**

3.5.2 The dual cost function of a CES production function

Si la fonction de prod est CES => Alors la fonction de cost est aussi CES.

- The CES production function is given by (2 inputs case),

$$y = A\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]^{-\frac{1}{\rho}}$$

- The FOC of the cost minimization program are given by,

$$p_K - \lambda\delta A\left[\delta K^{-\rho} + (1-\delta)L^{1-\rho}\right]^{\frac{1}{\rho}-1} K^{-\rho-1} = 0$$

$$p_L - \lambda(1-\delta)A\left[\delta K^{-\rho} + (1-\delta)L^{1-\rho}\right]^{\frac{1}{\rho}-1} L^{-\rho-1} = 0$$

$$y = A\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]^{-\frac{1}{\rho}}$$

3.5.1 The dual cost function of a CES production function (continued)

- The first and second FOC gives,

$$\frac{K}{L}=\left(\frac{\delta}{1-\delta}\right)^{\frac{1}{1+\rho}}\left(\frac{p_L}{p_K}\right)^{\frac{1}{1+\rho}}$$

- Thus we can write,

$$\frac{p_K K}{p_L L}=\left(\frac{\delta}{1-\delta}\right)^{\frac{1}{1+\rho}} p_K^{\frac{\rho}{1+\rho}} p_L^{-\frac{\rho}{1+\rho}}=\frac{\delta^{\frac{1}{1+\rho}} p_K^{\frac{\rho}{1+\rho}}}{(1-\delta)^{\frac{1}{1+\rho}} p_L^{\frac{\rho}{1+\rho}}}$$

- Adding 1 on both sides of this last equation we then have,

$$\frac{p_K K}{p_L L}+1=\frac{C}{p_L L}=\frac{\delta^{\frac{1}{1+\rho}} p_K^{\frac{\rho}{1+\rho}}+(1-\delta)^{\frac{1}{1+\rho}} p_L^{\frac{\rho}{1+\rho}}}{(1-\delta)^{\frac{1}{1+\rho}} p_L^{\frac{\rho}{1+\rho}}}$$

3.5.1 The dual cost function of a CES production function (continued)

Ici on a les fonctions de demande de facteurs

- Finally we have, $$L=\frac{(1-\delta)^{\frac{1}{1+\rho}}\,p_L^{-\frac{1}{1+\rho}}C}{\delta^{\frac{1}{1+\rho}}\,p_K^{\frac{\rho}{1+\rho}}+(1-\delta)^{\frac{1}{1+\rho}}\,p_L^{\frac{\rho}{1+\rho}}}$$
- And symmetrically, $$K=\frac{\delta^{\frac{1}{1+\rho}}\,p_K^{-\frac{1}{1+\rho}}C}{\delta^{\frac{1}{1+\rho}}\,p_K^{\frac{\rho}{1+\rho}}+(1-\delta)^{\frac{1}{1+\rho}}\,p_L^{\frac{\rho}{1+\rho}}}$$
- From the last FOC we have, $$A^{\rho}y^{-\rho}=\delta K^{-\rho}+(1-\delta)L^{-\rho}$$

Sur nos données on peut estimer les fonctions de demande de facteurs (tout du moins essayer)

3.5.1 The dual cost function of a CES production function (continued)

- The 3 last equations put together gives,

$$A^{\rho} y^{-\rho} = \frac{\delta^{\frac{1}{1+\rho}} p_K^{\frac{\rho}{1+\rho}} C^{-\rho}}{\left[\delta^{\frac{1}{1+\rho}} p_K^{\frac{\rho}{1+\rho}} + (1-\delta)^{\frac{1}{1+\rho}} p_L^{\frac{\rho}{1+\rho}}\right]^{-\rho}} + \frac{(1-\delta)^{\frac{1}{1+\rho}} p_L^{\frac{\rho}{1+\rho}} C^{-\rho}}{\left[\delta^{\frac{1}{1+\rho}} p_K^{\frac{\rho}{1+\rho}} + (1-\delta)^{\frac{1}{1+\rho}} p_L^{\frac{\rho}{1+\rho}}\right]^{-\rho}}$$

$$= \left[\delta^{\frac{1}{1+\rho}} p_K^{\frac{\rho}{1+\rho}} + (1-\delta)^{\frac{1}{1+\rho}} p_L^{\frac{\rho}{1+\rho}}\right]^{1+\rho} C^{-\rho}$$

La forme de la fonction de production CES sera la même si c'est une fonction de cout (donc aussi une CES)

- The dual cost function of the CES production function is thus,

$$C = A^{-1}\left[\delta^{\frac{1}{1+\rho}} p_K^{\frac{\rho}{1+\rho}} + (1-\delta)^{\frac{1}{1+\rho}} p_L^{\frac{\rho}{1+\rho}}\right]^{\frac{1+\rho}{\rho}} y$$

Le cout total est proportionnel à la production => Si le cout total double la production double => Rendements d'échelle constant. Donc le cout unitaire ne dépend pas du niveau de production

3.5.1 The dual cost function of a CES production function (continued)

Réécrire cette équqtion dans notre sujet

- First and second derivatives of the CES cost function are,

$$C_K = A^{-1}\left[\delta^{\frac{1}{1+\rho}} p_K^{\frac{\rho}{1+\rho}} + (1-\delta)^{\frac{1}{1+\rho}} p_L^{\frac{\rho}{1+\rho}}\right]^{\frac{1}{\rho}} \delta^{\frac{1}{1+\rho}} p_K^{-\frac{1}{1+\rho}} y$$

$$C_L = A^{-1}\left[\delta^{\frac{1}{1+\rho}} p_K^{\frac{\rho}{1+\rho}} + (1-\delta)^{\frac{1}{1+\rho}} p_L^{\frac{\rho}{1+\rho}}\right]^{\frac{1}{\rho}} (1-\delta)^{\frac{1}{1+\rho}} p_L^{-\frac{1}{1+\rho}} y$$

$$C_{KL} = A^{-1}\frac{1}{1+\rho}\left[\delta^{\frac{1}{1+\rho}} p_K^{\frac{\rho}{1+\rho}} + (1-\delta)^{\frac{1}{1+\rho}} p_L^{\frac{\rho}{1+\rho}}\right]^{\frac{1}{\rho}-1} \delta^{\frac{1}{1+\rho}} (1-\delta)^{\frac{1}{1+\rho}} p_K^{-\frac{1}{1+\rho}} p_L^{-\frac{1}{1+\rho}} y$$

Comment le ratio d'inputs change en fonction des prix relatifs (indication synthéthique)

- It can then be checked that,

L'elasticité de substitution de Allen est définie plus haut

$$\sigma_{KL} \equiv \frac{CC_{KL}}{C_K C_L} = \frac{1}{1+\rho}$$

3.5.1 The dual cost function of a CES production function (continued)

- **Comments:**
  - The dual cost function associated to a CES production function is itself a CES form, the CES form is said auto-dual
  - The previous slide illustrates the use of the Shephard's lemma to derive the input demand function and the use the derivatives of the cost function to calculate the elasticity of substitution
  - We can easily see that the CES cost function can be written as a unit cost function independent of the level of output (returns to scale are constant)

4. Flexible forms of the cost functions

4.1 The Translog form (TL)

4.1.1 Definition

FONCTION DE COUT TRANSLOG

- The TL form of the cost function, $C = C(y, p)$ is defined as, (see Christensen, L.R., D.W. Jorgenson and L.J. Lau (1971), "*Conjugate Duality and the Transcendental Logarithmic Production Function*," *Econometrica* 39, 255-256)

C'est simplement une forme log linéaire avec effets d'interactions

$$\ln C = \alpha_0 + \sum_{i=1}^{N} \alpha_i \ln p_i + \alpha_y \ln y + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_{ij} \ln p_i \ln p_j$$

$$+ \sum_{i=1}^{N} \beta_{iy} \ln p_i \ln y + \frac{1}{2} \beta_{yy} (\ln y)^2$$

On estime l'équation et on a les paramètres

Where, $\beta_{ij} = \beta_{ji} \ \forall i, j \ (i \neq j) = K, L, E, M$

4.1 The Translog form (TL)

4.1.1 Definition (continued)

- **Comments:**
  - Remember that the conditions on the parameters β_{ij} are known as the *symmetry conditions*; they are necessary and sufficient conditions for the Hessian of the TL form to be symmetric (to see this calculate the hessian w.r.t. the logarithms of input prices)
  - As the log form of the CD form of the cost function is,

$$\ln C = \alpha_0 + \alpha_y \ln y + \sum_{i=1}^{N} \alpha_i \ln p_i$$

we can say that the TL form embodied the CD form

  - The following constraints on the parameters of the TL form reduce the TL form to a CD form

$$\beta_{ij} = \beta_{iy} = \beta_{yy} = 0 \; \forall i, j$$

pixi/C => part de dépense relative accordée à l'input i (S_i)

4.1 The Translog form (TL)

4.1.2 The value Shares of inputs

Le modèle translog repose sur une hypothèse plus réaliste que les parts de dépenses relatives ne sont pas constantes. La part de dépense relative peut etre facilement estimée (voir la fonction S_i en bas)

- As the TL depends on logarithms of the input prices it is better to use a logarithm form of the Shephard's lemma,

$$\frac{\partial \ln C}{\partial \ln p_i} = \frac{p_i}{C}\frac{\partial C}{\partial p_i} = \frac{p_i x_i}{C} \equiv S_i$$

- S_i is known as the value share of input i, we have,

$$S_i = \alpha_i + \sum_{j=1}^{n} \beta_{ij} \ln p_j + \beta_{iy} \ln y \quad (i = 1, \cdots, N)$$ il faut que ce soit = 1

si alpha_i = 1, les beta_ij = 0 et les beta_yy = 0, alors les params sont contraints

Tester un ensemble de contraintes sur les paramètres (avec test de Fisher - modèle contraints)

Modèle contraint ici => ln(C) = alpha_0 + \sum_i alpha_i * ln(pi) + ln(y) + 1/2 \sum_i\sum_j p_ij ln(pi) ln(pj)

4.1 The Translog form (TL)

4.1.3 Elasticities of substitution derived from a TL form

Test de restriction linéaire sur les paramètres à faire.

- To use the formula shown above, we calculate the first and second derivatives of the TL form

$$C_i \equiv \frac{\partial C}{\partial p_i} = \frac{C}{p_i}\frac{\partial \ln C}{\partial \ln p_i} = \frac{C}{p_i}S_i \qquad i = 1, \cdots, N$$

$$C_{ij} \equiv \frac{\partial^2 C}{\partial p_i \partial p_j} = \partial\left(\frac{C}{p_i}S_i\right) / \partial p_j$$

$$= \frac{C_j}{p_i}S_i + \frac{C}{p_i}\frac{\partial S_i}{\partial p_j} = \frac{p_j x_j}{p_i p_j}S_i + \frac{C}{p_i}\frac{1}{p_j}\frac{\partial S_i}{\partial \ln p_j}$$

$$= \frac{C}{p_i p_j}\left(\frac{p_j x_j}{C}S_i + \beta_{ij}\right) = \frac{C}{p_i p_j}\left(S_i S_j + \beta_{ij}\right) \qquad i, j = 1, \cdots, N; i \neq j$$

4.1.3 Elasticities of substitution derived from a TL form (Continued)

- The cross-price elasticity is then given by,

$$\varepsilon_{ij} = \frac{p_j C_{ij}}{C_i} = p_j \frac{C}{p_i p_j}\left(S_i S_j + \beta_{ij}\right)\frac{p_i}{C}\frac{1}{S_i} = \frac{\beta_{ij} + S_i S_j}{S_i} \quad i \neq j$$

- The AES is given by,

$$\sigma_{ij} \equiv \frac{1}{S_j}\varepsilon_{ij} = \frac{\beta_{ij} + S_i S_j}{S_i S_j} \quad i \neq j$$

4.1.3 Elasticities of substitution derived from a TL form (Continued)

- We also have,

$$C_{ii} \equiv \frac{\partial^2 C}{\partial p_i^2} = \partial\left(\frac{C}{p_i} S_i\right) / \partial p_i$$

$$= \frac{C_i}{p_i} S_i - \frac{C}{p_i^2} S_i + \frac{C}{p_i} \frac{\partial S_i}{\partial p_i} = \frac{p_i x_i}{p_i^2} S_i - \frac{C}{p_i^2} S_i + \frac{C}{p_i^2} \frac{\partial S_i}{\partial \ln p_i}$$

$$= \frac{C}{p_i^2}\left(\frac{p_i x_i}{C} S_i - S_i + \beta_{ii}\right) = \frac{C}{p_i^2}\left(S_i^2 - S_i + \beta_{ii}\right)$$

- So the own-price elasticity is given by,

$$\varepsilon_{ii} = \frac{p_i C_{ii}}{C_i} = \frac{p_i C}{p_i^2} \frac{p_i}{C S_i} \frac{\beta_{ii} + S_i^2 - S_i}{S_i} = \frac{\beta_{ii} + S_i\left(S_i - 1\right)}{S_i}.$$

4.1.3 Elasticities of substitution derived from a TL form (Continued)

- **Comments:**
  - The elasticities of substitution depend on the relative shares of inputs which are varying with input prices, these elasticities are thus non constant contrary to the CD or the CES cases
  - In general the elasticity between inputs i and j is different from the elasticity between k and l
  - From the above remarks it appears that the TL form does not put any a priori constraint on the elasticities of substitution, the TL form is clearly a *flexible form*
  - It is possible to have an estimated value of the parameter β_{ii} that is such that $\varepsilon_{ii} > 0$, in this case the estimated cost function is not concave, this problem is recurrent when estimating flexible forms

4.1 The Translog form (TL)

4.1.3 Returns to scale with a TL cost function

- The derivative of the TL cost function w.r.t. $\ln y$ is equal to the ratio of the marginal cost to the unit cost,

$$\frac{\partial \ln C}{\partial \ln y} = \frac{\partial C / \partial y}{C / y} = \frac{C_m}{C_u}$$

$$= \alpha_y + \beta_{Ky} \ln p_K + \beta_{Ly} \ln p_L + \beta_{Ey} \ln p_E + \beta_{My} \ln p_M + \beta_{yy} \ln y$$

- **Remarks:**
  - The returns to scale varies with input prices and output
  - To obtain an estimation of all the parameters included in the above equation, it is necessary to estimate, not only the relative shares of inputs, but also the cost function itself because C_m is not observed

4.1.3 Returns to scale with a TL cost function (continued)

- For the TL cost function to be linearly homogeneous in y, the following restrictions must be satisfied,

$$\alpha_y = 1$$

$$\beta_{iy} = \beta_{yy} = 0 \quad \forall i = 1, \ldots, N$$

- Under the above restriction the TL cost function can be written as the following unit cost function,

$$\ln \frac{C}{y} = \alpha_0 + \alpha \sum_{i=1}^{N} \alpha_i \ln p_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_{ij} \ln p_i \ln p_j$$

4.1 The Generalized Leontief form (GL)

4.1.1 Definition

- The GL form of the cost function, $C(y,p)$ is defined as (see DIEWERT, W.E. (1971), *"An Application of The Shephard Duality Theorem: A Generalized Leontief Production Function"*, *Journal of Political Economy* 79, 481-507.),

$$C = y\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}\left(p_i p_j\right)^{1/2} + \sum_{i=1}^{N} b_i p_i + b_{yy}\left(\sum_{i=1}^{N}\beta_i p_i\right) y^2$$

- Where $b_{ij} = b_{ji} \quad \forall i, j$

4.1 The Generalized Leontief form (GL)

4.1.1 Definition (continued)

- Comments:
  - As in the TL case, the symmetry conditions are necessarily imposed for the hessian of the GL form to be symmetric
  - It is not possible to identify all the parameters of GL form, the parameters β_i must be selected arbitrarily by the investigator (Diewert W.E. and T.J. Wales (1987), "*Flexible Functional Forms and Global Curvature Conditions*," *Econometrica* 55, 43-68 propose to use $\beta_i = \bar{x}_i$ where $\bar{x}_i$ is the sample mean of x_i)
  - For the TL cost function to be linearly homogeneous in y, the restrictions, , must be satisfied

$$b_i = b_{yy} = 0 \quad \forall i$$

4.1 The Generalized Leontief form (GL)

4.1.2 The input demand functions

- Under constant returns to scale, the GL form becomes,

$$C = y\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}\left(p_i p_j\right)^{1/2} \quad b_{ij} = b_{ji} \quad \forall i, j$$

- And the input demand functions (here written as input-output coefficients) are given by,

$$\frac{x_i}{y} = \sum_{j=1}^{N} b_{ij}\left(p_j / p_i\right)^{1/2} \quad i = 1, \ldots, N$$

- **Remark:** Under constant returns to scale all the parameters are identified

4.1 The Generalized Leontief form (GL)
4.1.3 Elasticities of substitution

- The elasticities of substitution derived from the GL form are given by,

$$\sigma_{ij} = \frac{1}{2} \frac{Cb_{ij} \left(p_i p_j \right)^{-1/2}}{y a_i a_j} \quad \forall i \neq j$$

$$\sigma_{ii} = \frac{-\frac{1}{2} C \sum_{\substack{i=1 \\ j \neq i}}^{N} b_{ij} p_j^{1/2} p_i^{-3/2}}{y a_i^2} \quad \forall i$$

5. Econometric estimation of the TL and GL forms
5.1 The TL form

- The TL function is linear in the parameter and could be easily estimated by OLS, unfortunately such an estimation would give a very bad result (too many parameters to estimate, colinearity of the independent variables)

- The Shephard's lemma allows to add N relative shares of inputs to the model and thus allows to increase the degrees of freedom of the initial model

- Hereafter we consider the case of a 4 inputs constant returns to scale technology because:
  - The expression of the equations to estimate is simpler
  - Generalization to non constant returns to scale technology is straightforward
  - There is a huge literature where 4 inputs unit cost functions are estimated (the so called $KLEM$ functions)

5.1 The TL form (continued)

The TL unit cost function

- K, L, E, M are respectively Capital, Labor, Energy and Materials, p_K, p_L, p_E, p_M are the corresponding prices
- The TL form of the unit cost function $C/y = C_u\left(p_K, p_L, p_E, p_M\right)$ is given by (with symmetry conditions included),

$$\ln\frac{C}{y} = \alpha_0 + \alpha_K \ln p_K + \alpha_L \ln p_L + \alpha_E \ln p_E + \alpha_M \ln p_M$$

$$+\frac{1}{2}\beta_{KK}\left(\ln p_K\right)^2 + \beta_{KL}\ln p_K \ln p_L + \beta_{KE}\ln p_K \ln p_E + \beta_{KM}\ln p_K \ln p_M$$

$$+\frac{1}{2}\beta_{LL}\left(\ln p_L\right)^2 + \beta_{LE}\ln p_L \ln p_E + \beta_{LM}\ln p_L \ln p_M$$

$$+\frac{1}{2}\beta_{EE}\left(\ln p_E\right)^2 + \beta_{EM}\ln p_E \ln p_M + \frac{1}{2}\beta_{MM}\left(\ln p_M\right)^2$$

5.1 The TL form (continued)

The relative shares of inputs

- The relative shares of inputs are given by,

$$S_K = \alpha_K + \beta_{KK} \ln p_K + \beta_{KL} \ln p_L + \beta_{KE} \ln p_E + \beta_{KM} \ln p_M$$
$$S_L = \alpha_L + \beta_{KL} \ln p_K + \beta_{LL} \ln p_L + \beta_{LE} \ln p_E + \beta_{LM} \ln p_M$$
$$S_E = \alpha_E + \beta_{KE} \ln p_K + \beta_{LE} \ln p_L + \beta_{EE} \ln p_E + \beta_{EM} \ln p_M$$
$$S_M = \alpha_M + \beta_{KM} \ln p_K + \beta_{LM} \ln p_L + \beta_{EM} \ln p_E + \beta_{MM} \ln p_M$$

- Note that all the parameters of the unit cost function (except α_0) are included in the above share equations so, it is possible to estimate the value shares of inputs only (remember that this is not possible when the cost function cannot be written as a unit cost function - case of non constant returns to scale)

5.1 The TL form (continued)

5.1.1 Data and Ordinary Least Squares (OLS) estimation

- Consider that, for a given period of time ($t = 1,\ldots,T$) we have observations on inputs quantities and prices we can write,

$$
\begin{aligned}
S_{Kt} &= \alpha_K + \beta_{KK} \ln p_{Kt} + \beta_{KL} \ln p_{Lt} + \beta_{KE} \ln p_{Et} + \beta_{KM} \ln p_{Mt} + u_{Kt} \\
S_{Lt} &= \alpha_L + \beta_{LK} \ln p_{Kt} + \beta_{LL} \ln p_{Lt} + \beta_{LE} \ln p_{Et} + \beta_{LM} \ln p_{Mt} + u_{Lt} \\
S_{Et} &= \alpha_E + \beta_{EK} \ln p_{Kt} + \beta_{EL} \ln p_{Lt} + \beta_{EE} \ln p_{Et} + \beta_{EM} \ln p_{Mt} + u_{Et} \\
S_{Mt} &= \alpha_M + \beta_{MK} \ln p_{Kt} + \beta_{ML} \ln p_{Lt} + \beta_{ME} \ln p_{Et} + \beta_{MM} \ln p_{Mt} + u_{Mt}
\end{aligned}
$$

- Where, $u_{Kt}, u_{Lt}, u_{Et}, u_{Mt}$, are unobserved residuals resulting, for instance, from producers' optimization errors

5.1 The TL form
5.1.1 Data and OLS estimation (continued)

- Hypothesis:
  - The input prices are considered as exogenous (and consequently non correlated with residuals)
  - In each equation, the residuals are supposed to be independently and identically distributed with mean 0 and variance $\sigma_i^2 \ (i=K,L,E,M)$; in other words we have

$$E\left(U_{it}\right)=0 \quad \forall i=K,L,E,M$$

$$E\left(U_{it}^2\right)=\sigma_i^2 \quad \forall t=1,\dots,T;\, i=K,L,E,M$$

$$E\left(U_{it}U_{jt'}\right)=0 \quad \forall i,j=K,L,E,M;\, \forall t,t'=1,\dots,T \quad \left(t\neq t'\right)$$

5.1.1 Data and OLS estimation (continued)

Some comments about OLS

- Under our hypothesis, Ordinary Least Square (OLS) estimator is BLUE (Best Linear Unbiased Estimator), but the chosen residuals specification can be considered as too restrictive, because there are possible correlations between contemporaneous residuals of different equations (all equations are obtained by deriving the same cost function)

- OLS is generally applied separately to each equation of the system and then does not allow to impose the symmetry constraints

- **Conclusion:** OLS is not necessarily the best estimator (see further developments)

5.1.1 Data and OLS estimation (continued)

A surprising result with OLS

- OLS estimation of the parameters of the 4 value share equations give parameters values such that the following relations are satisfied,

$$\alpha_K + \alpha_L + \alpha_E + \alpha_M = 1$$
$$\beta_{KK} + \beta_{LK} + \beta_{EK} + \beta_{MK} = 0$$
$$\beta_{KL} + \beta_{LL} + \beta_{EL} + \beta_{ML} = 0$$
$$\beta_{KE} + \beta_{LE} + \beta_{EE} + \beta_{ME} = 0$$
$$\beta_{KM} + \beta_{LM} + \beta_{EM} + \beta_{MM} = 0$$

- The above conditions of the parameters are known as the *additivity conditions*

5.1.1 Data and OLS estimation (continued)

Why are the additivity conditions applying?

- The additivity conditions apply because the TL satisfies the following conditions:
  - The value shares of inputs always sum to 1, i.e.

$$S_{Kt} + S_{Lt} + S_{Et} + S_{Mt} = 1 \quad \forall t = 1, \ldots, T$$

  - The same independent variables appear in all equations

Proof

- Let us note:

$$u_i = \begin{bmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iT} \end{bmatrix}; S_i = \begin{bmatrix} S_{i1} \\ S_{i2} \\ \vdots \\ S_{iT} \end{bmatrix}; \gamma_i = \begin{bmatrix} \alpha_i \\ \beta_{iK} \\ \beta_{iL} \\ \beta_{iE} \\ \beta_{iM} \end{bmatrix} \quad i = K, L, E, M$$

$$X = \begin{bmatrix} 1 & \ln p_{K1} & \ln p_{L1} & \ln p_{E1} & \ln p_{M1} \\ 1 & \ln p_{K2} & \ln p_{L2} & \ln p_{E2} & \ln p_{M2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \ln p_{KT} & \ln p_{LT} & \ln p_{ET} & \ln p_{MT} \end{bmatrix}$$

5.1.1 Data and OLS estimation

Proof (continued)

- OLS estimation of equation i, $\hat{\gamma}_i$, is given by,

$$\hat{\gamma}_i = (X'X)^{-1} X'S_i$$

- From this last relation it follows (e is a vector of 1),

$$\hat{\Gamma} = \sum_{i=K,L,E,M} \hat{\gamma}_i = \sum_{i=K,L,E,M} (X'X)^{-1} X'S_i$$

$$= (X'X)^{-1} X' \left(\sum_{i=K,L,E,M} S_i \right) = (X'X)^{-1} X'e$$

5.1.1 Data and OLS estimation

Proof (continued)

- $\left(X'X\right)^{-1}X'e$ is the OLS estimator of the vector of parameters Γ in the model, $e = X\Gamma + \varepsilon$, where ε is the vector of residuals
- The dependant variable of this model is always equal to one, there is no need of calculation to see that

$$\hat{\Gamma} = \left(1,0,0,0,0\right)'$$

- **Consequence:** the estimated residuals, $\hat{u}_i = S_i - X\hat{\gamma}_i$, are such that, $\sum_{i=1}^{N}\hat{u}_i = \vec{o}$, where $\vec{o}$ is the null vector

5.1 The TL form (continued)
5.1.1 Generalized Least Squares (GLS) estimation

- It is generally more appropriated to consider that contemporaneous residuals from different equations are correlated, because the different equations are derived from the same unit cost function

- Hereafter we will consider that residuals are uncorrelated with the dependent variable, have mean zero and satisfy,

$$E\left(U_{it}^2\right) = \sigma_i^2 \quad \forall i = K, L, E, M \quad \forall t = 1, \ldots, T$$

$$E\left(U_{it} U_{jt'}\right) = 0 \quad \forall i, j = K, L, E, M \quad \forall t, t' = 1, \ldots, T \quad t \neq t'$$

$$E\left(U_{it} U_{jt}\right) = \sigma_{ij} \quad \forall i, j = K, L, E, M \quad i \neq j \quad \forall t = 1, \ldots, T$$

5.1.1 GLS estimation (continued)

Some comments

- Now value share equations are related through the correlation of residuals of different equations, to capture this information when estimating the model we need to estimate simultaneously the 4 equations of the system of value share equations (this is necessary independently of the symmetry constraints)

- OLS estimator is not BLUE, the GLS estimator is more efficient, at least on sufficiently large samples

5.1.1 GLS estimation (continued)

Notations

- Let us note:

$$u = \begin{bmatrix} u_K \\ u_L \\ u_E \\ u_M \end{bmatrix}; S = \begin{bmatrix} S_K \\ S_L \\ S_E \\ S_M \end{bmatrix}; \gamma = \begin{bmatrix} \gamma_K \\ \gamma_L \\ \gamma_E \\ \gamma_M \end{bmatrix}; Z = \begin{bmatrix} X & O & O & O \\ O & X & O & O \\ O & O & X & O \\ O & O & O & X \end{bmatrix} = I_4 \otimes X,$$

where O is the null matrix of dimension $(T \times 4)$ and the I_4 identity matrix of order 4

5.1.1 GLS estimation (continued)

- The system of relative shares can be written in stacked form as (dimensions of the matrices are given in parenthesis),

$$\underset{(4T\times 1)}{S} = \underset{(4T\times 20)(20\times 1)}{Z\gamma} + \underset{(4T\times 1)}{u}$$

- Under the preceding hypothesis the covariance matrix of residuals, $\Omega \equiv E(uu')$, is given by

$$\underset{(4T\times 4T)}{\Omega} = \underset{(4\times 4)}{\Sigma} \otimes \underset{(T\times T)}{I_T}$$

- Where $\Sigma = [\sigma_{ij}]$ and $\otimes$ is the Kronecker product ($A\otimes B = [a_{ij}B]$ where $A = [a_{ij}]$)

5.1.1 GLS estimation (continued)

- OLS is not the most efficient estimator (the one with the smallest variance), because the covariance matrix of residuals is not diagonal

- Consider the following transformation of the initial model,

$$\Omega^{-\frac{1}{2}} S = \Omega^{-\frac{1}{2}} Z\gamma + \Omega^{-\frac{1}{2}} u$$

- Writing $\tilde{S} = \tilde{Z}\gamma + \tilde{u}$ (with evident notations); we have

$$E\left(\tilde{u}\tilde{u}'\right) = E\left(\Omega^{-\frac{1}{2}} uu'\Omega^{-\frac{1}{2}}\right) = \Omega^{-\frac{1}{2}} E\left(uu'\right)\Omega^{-\frac{1}{2}} = \Omega^{-\frac{1}{2}}\Omega\Omega^{-\frac{1}{2}} = I_{4\times T}$$

- Since the covariance matrix of residuals $\tilde{u}$ is diagonal, the OLS estimator is the most efficient one to estimate the parameters in the transformed model; we have,

$$\hat{\gamma} = \left(\tilde{Z}'\tilde{Z}\right)^{-1} \tilde{Z}'\tilde{S} = \left(\tilde{Z}'\Omega^{-\frac{1}{2}}\Omega^{-\frac{1}{2}}Z\right)^{-1} \tilde{Z}'\Omega^{-\frac{1}{2}}\Omega^{-\frac{1}{2}}S$$

$$= \left(\tilde{Z}'\Omega^{-1}Z\right)^{-1} \tilde{Z}'\Omega^{-1}S$$

- $\hat{\gamma} = \left(Z'\Omega^{-1}Z\right)^{-1} Z'\Omega^{-1}S$ is the usual expression of the GLS estimator

5.1.1 GLS estimation (continued)

The particular case in which all equations have the same exogenous variables

- When the exogenous variables are the same in all equations we can write $Z = I_4 \otimes X$ then,

$$\hat{\gamma} = \left[\left(I_4 \otimes X \right)' \left(\Sigma^{-1} \otimes I_T \right) \left(I_4 \otimes X \right) \right]^{-1} \left(I_4 \otimes X \right)' \left(\Sigma^{-1} \otimes I_T \right) S$$

$$= \left[\left(\Sigma^{-1} \otimes X' \right) \left(I_4 \otimes X \right) \right]^{-1} \left(\Sigma^{-1} \otimes X' \right) S$$

$$= \left(\Sigma^{-1} \otimes X'X \right)^{-1} \left(\Sigma^{-1} \otimes X' \right) S = \left[\Sigma \otimes \left(X'X \right)^{-1} \right] \left(\Sigma^{-1} \otimes X' \right) S$$

$$= \left[I \otimes \left(X'X \right)^{-1} X' \right] S$$

- **Conclusion:** When the same exogenous variables appear in all equations and when only contemporaneous residuals are correlated, OLS and GLS estimators are the same

5.1.1 GLS estimation (continued)

What about the TL value shares of inputs

- The TL value shares of inputs are such that OLS and GLS give the same results **but**, this is true only if we do not impose the symmetry restrictions

- When the symmetry restrictions are imposed, it is not possible to write $Z = I_4 \otimes X$ and the calculations of the preceding slide are not feasible

5.1.2 The SUR or Zellner's Estimator (ZEF)

- Zellner calls a regression of the type we are studying a Seemingly Unrelated Regression:
  - Unrelated because the system we are trying to estimate is not a simultaneous equations system (all endogenous variables are determined independently of the other endogenous variables)
  - Seemingly unrelated because the equation are independent only in appearance (the contemporaneous residuals of different equations are correlated)
- Zellner proposes to use a 2 steps method to estimate such a system (see "*An efficient method of estimating seemingly unrelated regressions and tests for aggregation bias*" , Journal of the American Statistical Association, 1962, pp 348-368)

5.1.2 The SUR or Zellner's Estimator (continued)

- We have to calculate $\hat{\gamma} = \left(Z'\Omega^{-1}Z\right)^{-1} Z'\Omega^{-1}S$ but Ω is unknown
- As $\Omega = \Sigma \otimes I_T$ we need in fact an estimation of Σ, Zellner proposes to use the following

$$\hat{\sigma}_{ij} = \frac{1}{T}\sum_{t=1}^{T}\hat{u}_{it}\hat{u}_{jt} = \frac{1}{T}\hat{u}_i'\hat{u}_j \quad i, j = K, L, E, M$$

- Where $\hat{u}_{it}\ \left(i, j = K, L, E, M\right)$ are the residuals of the model estimated by OLS

5.1.2 The SUR or Zellner's Estimator (continued)

Comments

- Denote $\hat{U} = \left[\hat{u}_K \ \hat{u}_L \ \hat{u}_E \ \hat{u}_M\right]'$ then $\hat{\Sigma} = \hat{U}'\hat{U}/T$

- Zellner shows that the SUR or Zellner's estimator is asymptotically more efficient than OLS, this means that on sufficiently large sample it has to be preferred to OLS

- The SUR estimator is a 2 steps procedure,
  - 1. Estimate the equations of the system by OLS and calculate $\hat{\Sigma}$
  - 2. Calculate $\hat{\gamma} = \left[Z'\left(\hat{\Sigma}^{-1} \otimes I_T\right)Z\right]^{-1} Z'\left(\hat{\Sigma}^{-1} \otimes I_T\right)S$

- but iterating on $\hat{\Sigma}$ is possible, this give the *Iterative Zellner's Estimator* (IZEF) **-see next slide-**

5.1.2 The SUR or Zellner's Estimator (continued)

The Iterative Zellner's Estimator (IZEF)

- From step 2 of the preceding procedure deduce a new estimation of $\hat{\Sigma}$ and Σ calculate again $\hat{\gamma} = \left[Z'\left(\hat{\Sigma}^{-1} \otimes I_T\right) Z \right]^{-1} Z'\left(\hat{\Sigma}^{-1} \otimes I_T\right) S$

- Repeat the preceding calculations until 2 consecutive calculations give (almost) the same value of the parameters

- It can be shown that the iterative Zellner's estimator is numerically equivalent to the Maximum Likelihood estimator (ML)

- **Remark:** The iterative process can be started from any value of Σ since it always converges towards the ML estimates, an OLS estimation at step 1 is therefore useless

5.1.2 The SUR or Zellner's Estimator (continued)

The particular case of the TL share equations

- An OLS estimation of the TL value shares of inputs is such that $\hat{u}_K+\hat{u}_L+\hat{u}_E+\hat{u}_M=\vec{O}$; consequently the columns of the matrix $\hat{U}$ are correlated and the matrix is singular: $\hat{\Sigma}=\hat{U}'\hat{U}/T$ cannot be $\hat{\gamma}$ calculated

- The solution is to estimate 3 of the 4 equations and to estimate the parameters of the equation that has been dropped by using the additivity constraints

- The results obtained that way are independent of the equation that has been dropped provided that the OLS estimation has been performed equation by equation (because the symmetry constraints are not imposed) or provided IZEF has been used

5.1.3 How to impose the symmetry constraints?

- We have to estimate 3 of the 4 equations of the TL value shares system, assume we have chosen to estimate equations K, L and E:

$$S_K = \alpha_K + \beta_{KK} \ln p_K + \beta_{KL} \ln p_L + \beta_{KE} \ln p_E + \beta_{KM} \ln p_M + u_K$$
$$S_L = \alpha_L + \beta_{KL} \ln p_K + \beta_{LL} \ln p_L + \beta_{LE} \ln p_E + \beta_{LM} \ln p_M + u_L$$
$$S_E = \alpha_E + \beta_{KE} \ln p_K + \beta_{LE} \ln p_L + \beta_{EE} \ln p_E + \beta_{EM} \ln p_M + u_E$$

- To estimate the parameters of the missing equation we use the additivity constraints:

$$\alpha_M = 1 - \alpha_K - \alpha_L - \alpha_E$$
$$\beta_{KM} = -\beta_{KK} - \beta_{KL} - \beta_{KE}$$
$$\beta_{LM} = -\beta_{KL} - \beta_{LL} - \beta_{LE}$$
$$\beta_{EM} = -\beta_{KE} - \beta_{LE} - \beta_{EE}$$
$$\beta_{MM} = -\beta_{KM} - \beta_{LM} - \beta_{EM}$$

5.1.3 How to impose the symmetry constraints? (Continued)

- **Consequence:** To have only one estimation of the parameters the system must be written $\beta_{KM}, \beta_{LM}, \beta_{EM}, \beta_{MM}$

$$S_K = \alpha_K + \beta_{KK} \ln \frac{p_K}{p_M} + \beta_{KL} \ln \frac{p_L}{p_M} + \beta_{KE} \ln \frac{p_E}{p_M} + u_K$$

$$S_L = \alpha_L + \beta_{KL} \ln \frac{p_K}{p_M} + \beta_{LL} \ln \frac{p_L}{p_M} + \beta_{LE} \ln \frac{p_E}{p_M} + u_L$$

$$S_E = \alpha_E + \beta_{KE} \ln \frac{p_K}{p_M} + \beta_{LE} \ln \frac{p_L}{p_M} + \beta_{EE} \ln \frac{p_E}{p_M} + u_E$$

5.1.3 How to impose the symmetry constraints? (Continued)

- When the symmetry constraints are imposed we have to estimate 9 parameters only, $\alpha_K, \beta_{KK}, \beta_{KL}, \beta_{KE}, \alpha_L, \beta_{LL}, \beta_{LE}, \alpha_E, \beta_{EE}$
- To impose the symmetry constraints, we write the system as, $S = W\delta + u$ and calculate $\hat{\delta} = \left[W'\left(\hat{\Sigma}^{-1} \otimes I_T\right)W\right]^{-1} W'\left(\hat{\Sigma}^{-1} \otimes I_T\right)S$ where,

$$W = \begin{bmatrix} e & x_K & x_L & x_E & o & o & o & o & o \\ o & o & x_K & o & e & x_L & x_E & o & o \\ o & o & o & x_K & o & o & x_L & e & x_E \end{bmatrix}; x_i \equiv \begin{bmatrix} \ln \frac{p_{i1}}{p_{M1}} \\ \ln \frac{p_{i2}}{p_{M2}} \\ \vdots \\ \ln \frac{p_{iT}}{p_{MT}} \end{bmatrix}; (i = K, L, E); \delta = \begin{bmatrix} \alpha_K \\ \beta_{KK} \\ \beta_{KL} \\ \beta_{KE} \\ \alpha_L \\ \beta_{LL} \\ \beta_{LE} \\ \alpha_E \\ \beta_{EE} \end{bmatrix}$$

5.1.3 How to impose the symmetry constraints? (Continued)

Remarks:

- An OLS estimation with symmetry constraints imposed can be easily obtained by replacing $\hat{\Sigma}$ by an identity matrix in the calculation of $\hat{\delta}$

- But remember that ZEF is a 2 steps procedure, and that for having invariance w.r.t. the equation dropped, the symmetry constraints must be imposed at the second step of the procedure only

5.1.4 The maximum Likelihood (ML) estimator

- We need to write the model in a different way; let us note,

$$s_t' = \begin{bmatrix} S_{Kt} \\ S_{Lt} \\ S_{Et} \end{bmatrix};\ x_t' = \begin{bmatrix} 1 \\ \ln\left(p_{Kt}/p_{Mt}\right) \\ \ln\left(p_{Lt}/p_{Mt}\right) \\ \ln\left(p_{Et}/p_{Mt}\right) \end{bmatrix};\ u_t' = \begin{bmatrix} u_{Kt} \\ u_{Lt} \\ u_{Et} \end{bmatrix};\ \theta = \begin{bmatrix} \alpha_K & \alpha_L & \alpha_E \\ \beta_{KK} & \beta_{KL} & \beta_{KE} \\ \beta_{KL} & \beta_{LL} & \beta_{LE} \\ \beta_{KE} & \beta_{LE} & \beta_{EE} \end{bmatrix}$$

- And write the model,

$$\underset{(1\times 3)}{s_t} = \underset{(1\times 4)}{x_t}\ \underset{(4\times 3)}{\theta} + \underset{(1\times 3)}{u_t} \qquad t = 1,\ldots,T$$

5.1.4 The ML estimator (continued)

- The model can also be written in stacked form,

$$s = x\theta + u$$

- With,

$$\underset{(T\times 3)}{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{bmatrix}; \underset{(T\times 3)}{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_T \end{bmatrix}; \underset{(T\times 4)}{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{bmatrix}.$$

5.1.4 The ML estimator (continued)

- At the contrary of GLS estimation, ML estimation needs to specify the distribution of the residuals of the model; we assume that the vectors u_t are identically normally distributed with mean 0 and covariance Σ; the density of u_t is,

$$f\left(u_t\right)=\frac{1}{\left(2\pi\right)^{3/2}}\frac{1}{\left|\Sigma\right|^{1/2}}\exp\left[-\frac{1}{2}u_t\Sigma^{-1}u_t'\right]$$

- The likelihood function is defined as $L\left(\theta,\Sigma\right)=\prod_{t=1}^{T} f\left(s_t\right)$ where $f\left(s_t\right)$ is the density of the random vector s_t

5.1.4 The ML estimator (continued)

- The density of the vector s_t is given by,

$$f\left(s_t\right)=\frac{1}{\left(2\pi\right)^{3/2}}\frac{\left|B\right|}{\left|\Sigma\right|^{1/2}}\exp\left[-\frac{1}{2}\sum_{T=1}^{T}u_t\Sigma^{-1}u_t'\right]$$

where $\left|B\right|$ is the absolute value of the Jacobian of the transformation from u_t to s_t

- Here, $B=\partial u_t/\partial s_t=I_3$ so the likelihood function is given by,

$$L\left(\theta,\Sigma\right)\equiv\prod_{i=1}^{T}f\left(s_t\right)=\frac{1}{\left(2\pi\right)^{3T/2}}\frac{1}{\left|\Sigma\right|^{T/2}}\exp\left[-\frac{1}{2}\sum_{T=1}^{T}\left(s_t-x_t\theta\right)\Sigma^{-1}\left(s_t-x_t\theta\right)'\right]$$

5.1.4 The ML estimator (continued)

- The ML estimator is obtained by maximizing the likelihood function or equivalently by maximizing the log-likelihood function,

$$\ln L(\theta,\Sigma) = -\frac{3T}{2}\ln 2\pi - \frac{T}{2}\ln|\Sigma| - \frac{1}{2}\sum_{T=1}^{T}(s_t - x_t\theta)\Sigma^{-1}(s_t - x_t\theta)'$$

- In practice the ML estimation of the parameters is obtained as the numerical solution of a non-linear optimization problem; to make this optimization easier, it is preferable to maximize the concentrated log-likelihood function because it is function of only

$$\theta$$

5.1.4 The ML estimator (continued)
The concentrated log-likelihood function

- As, $\sum_{T=1}^{T} u_t \Sigma^{-1} u_t' = \mathrm{tr}\left(u\Sigma^{-1}u'\right) = \mathrm{tr}\left(\Sigma^{-1}u'u\right)$ and $\left|\Sigma^{-1}\right| = 1/\left|\Sigma\right|$ the log-likelihood function writes,

$$\ln L(\theta,\Sigma) = -\frac{3T}{2}\ln 2\pi + \frac{T}{2}\ln\left|\Sigma^{-1}\right| - \frac{1}{2}\mathrm{tr}\Sigma^{-1}(s-x\theta)'(s-x\theta)$$

where “tr” is the trace of a matrix (when $A=\left[a_{ij}\right]$, $\mathrm{tr}A = \sum_i a_{ii}$)

- As $\ln\left|A^{-1}\right| = -\ln\left|A\right|$, $\frac{\partial \ln\left|A\right|}{\partial A} = \left(A'\right)^{-1}$ and $\frac{\partial}{\partial A}\mathrm{tr}AC = C'$, we have,

$$\frac{\partial \ln L}{\partial \Sigma^{-1}} = 0 \Leftrightarrow T\Sigma - (s-x\theta)'(s-x\theta) = 0$$

5.1.4 The ML estimator

The concentrated log-likelihood function (continued)

- The preceding relation shows that the value of Σ that maximizes the log-likelihood is given by,

$$\hat{\Sigma} = \frac{1}{T}(s - x\theta)'(s - x\theta)$$

- Substituting $\hat{\Sigma}$ for Σ in the former expression for $\ln L(\theta, \Sigma)$ we obtain the concentrated log-likelihood function,

$$\ln L^*(\theta) = const. - \frac{T}{2}\ln\left|(s - x\theta)'(s - x\theta)\right|$$

5.1.4 The ML estimator (continued)

- It can be shown that the ML estimator is consistent, asymptotically efficient and asymptotically normally distributed with mean θ and covariance $\left[I(\theta)\right]^{-1}$ where,

$$I(\theta) = -E\left(\frac{\partial^2 \ln L^*(\theta)}{\partial\theta\partial\theta'}\right)$$

- $I(\theta)$ can be estimated by calculating, $$\left[\hat{I}(\hat{\theta})\right]^{-1} = \left(-\frac{\partial^2 \ln L^*(\hat{\theta})}{\partial\hat{\theta}\partial\hat{\theta}'}\right)^{-1}$$

- But the second derivatives are sometimes complicated to derive, in this case $I(\theta)$ can be estimated as, $$\left[\hat{I}(\hat{\theta})\right]^{-1} = \left(\sum_{t=1}^{T}\hat{g}_t\hat{g}_t'\right)^{-1} = \left(\hat{G}'\hat{G}\right)^{-1}$$ where,

$$\hat{G} = \left[\hat{g}_1\hat{g}_2,\cdots,\hat{g}_T\right], \quad \hat{g}_t = \frac{\partial \ln f\left(x_t,\hat{\theta}\right)}{\partial\hat{\theta}}$$

5.2 The GL form

The relative shares of inputs

- The Shephard's lemma applied to the GL cost function gives the system of input-output coefficients,

$$
\begin{aligned}
a_K &= b_{KK} + b_{KL}\left(p_L / p_K\right)^{1/2} + b_{KE}\left(p_E / p_K\right)^{1/2} + b_{KM}\left(p_M / p_K\right)^{1/2} + u_K \\
a_L &= b_{LL} + b_{KL}\left(p_K / p_L\right)^{1/2} + b_{LE}\left(p_E / p_L\right)^{1/2} + b_{LM}\left(p_M / p_L\right)^{1/2} + u_L \\
a_E &= b_{EE} + b_{KE}\left(p_K / p_E\right)^{1/2} + b_{LE}\left(p_L / p_E\right)^{1/2} + b_{EM}\left(p_M / p_E\right)^{1/2} + u_E \\
a_M &= b_{MM} + b_{KM}\left(p_K / p_M\right)^{1/2} + b_{LM}\left(p_L / p_M\right)^{1/2} + b_{EM}\left(p_E / p_M\right)^{1/2} + u_M
\end{aligned}
$$

- Here we must estimate simultaneously the 4 equations of the system (and not only 3 as with the TL value shares)

6. Flexible forms and regularity conditions

- The theory tells us that the cost function is continuous and increasing in y and p, linearly homogeneous and concave in p
- **But**, clearly the flexible functional forms does not necessarily satisfy all the above conditions, a conflict between the theory and the econometric model is therefore possible
- The goal of this section is to see if there is a solution to this problem

6. Flexible forms and regularity conditions

6.1. The TL form

6.1.1. Symmetry, additivity and homogeneity constraints

- The TL form is homogeneous of degree 1 in prices if and only if its parameters satisfy the symmetry an the additivity conditions

- **Proof:** the condition is necessary
  - If the TL form is linearly homogeneous in prices, we can write,

$$C(y,p)=\sum_{i=1}^{N} p_i \frac{\partial C}{\partial p_i}=\sum_{i=1}^{N} p_i \frac{C}{p_i}\frac{\partial \ln C}{\partial \ln p_i}=C\sum_{i=1}^{N} S_i$$

  - But to have $C\sum_{i=1}^{N} S_i = C$ we need the symmetry and additivity restrictions

6.1. The TL form

6.1.1. Symmetry, additivity and homogeneity constraints (continued)

- **Proof:** The condition is sufficient
  - Using the additivity constraints,

$$\alpha_M = 1 - \alpha_K - \alpha_L - \alpha_E$$

$$\beta_{iM} = -\beta_{iK} - \beta_{iL} - \beta_{iE} \quad i = K, L, E, y$$

  - We can write the TL form as,

$$\ln \frac{C}{p_M} = \alpha_0 + \alpha_K \ln \frac{p_K}{p_M} + \alpha_L \ln \frac{p_L}{p_M} + \alpha_E \ln \frac{p_E}{p_M} + \frac{1}{2}\beta_{KK}\left(\ln \frac{p_K}{p_M}\right)^2 + \beta_{KL} \ln \frac{p_K}{p_M} \ln \frac{p_L}{p_M}$$

$$+\beta_{KE} \ln \frac{p_K}{p_M} \ln \frac{p_E}{p_M} + \frac{1}{2}\beta_{LL}\left(\ln \frac{p_L}{p_M}\right)^2 + \beta_{LE} \ln \frac{p_L}{p_M} \ln \frac{p_E}{p_M} + \frac{1}{2}\beta_{EE}\left(\ln \frac{p_E}{p_M}\right)^2$$

  - The above expression is clearly homogeneous of degree 1 in prices

6.1.1. Monotonicity and concavity constraints

Monotonicity

- $C(y,p)$ is increasing in p if and only if $\dfrac{\partial C}{\partial p_i} \geq 0 \quad \forall i,$
- As input prices are strictly positive the above conditions are equivalent to
$$S_i \geq 0 \quad \forall i$$
- Observed relative shares are always positive, therefore there is a high probability for the estimated relative shares to be positive when their estimation use observed values of input prices and output

- **Remarks:**
  - Using the econometric model to simulate producer' responses to huge increase of input prices (oil shock, energy taxation, …) can give negative estimated relative shares
  - The above conclusions also apply for monotonicity in y because we can suppose (though the marginal cost cannot be observed) that observed data are such that
$$\partial \ln C / \partial \ln y = C_m / C_u \geq 0$$

6.1.1. Monotonicity and concavity constraints

Concavity

- The TL form is not necessarily a globally concave in p function
- The second derivatives of the cost function are given by,

$$\frac{\partial^2 C}{\partial p_i \partial p_j} = \frac{C}{p_i p_j}\left(\beta_{ij} + S_i S_j\right) \forall i, j = K, L, E, M$$

$$\frac{\partial^2 C}{\partial p_i^2} = \frac{C}{p_i^2}\left(\beta_{ii} + S_i \left(S_i - 1\right)\right) \forall i = K, L, E, M$$

6.1.1. Monotonicity and concavity constraints

Concavity (continued)

- Let us note,

$$P = \begin{bmatrix} p_K & 0 & 0 & 0 \\ 0 & p_L & 0 & 0 \\ 0 & 0 & p_E & 0 \\ 0 & 0 & 0 & p_M \end{bmatrix}$$

$$H = \begin{bmatrix} \beta_{KK} + S_K\left(S_K - 1\right) & \beta_{KL} + S_K S_L & \beta_{KE} + S_K S_E & \beta_{KM} + S_K S_M \\ \beta_{KL} + S_K S_L & \beta_{LL} + S_L\left(S_L - 1\right) & \beta_{LE} + S_L S_E & \beta_{LM} + S_L S_M \\ \beta_{KE} + S_K S_E & \beta_{LE} + S_L S_E & \beta_{EE} + S_E\left(S_E - 1\right) & \beta_{EM} + S_E S_M \\ \beta_{KM} + S_K S_M & \beta_{LM} + S_L S_M & \beta_{EM} + S_E S_M & \beta_{MM} + S_M\left(S_M - 1\right) \end{bmatrix}$$

- The Hessian matrix of the TL cost function is then given by,

$$\nabla^2_{pp} C = C P^{-1} H P^{-1}$$

6.1.1. Monotonicity and concavity constraints

Concavity (continued)

- A necessary condition for concavity in p of the TL form is that its Hessian matrix be negative semi-definite

- But,

$$\nabla^2_{pp} C \text{ negative semi-definite} \Leftrightarrow y' C P^{-1} H P^{-1} y \leq 0 \ \forall y \neq 0$$

$$\Leftrightarrow C z' H z \leq 0 \ \forall z \equiv P^{-1} y \neq 0 \left(p_i > 0 \, \forall i, C > 0 \right)$$

$$\Leftrightarrow H \text{ negative semi-definite}$$

6.1.1. Monotonicity and concavity constraints

Concavity (continued)

- Let us note,

$$\beta = \left[\beta_{ij}\right]; S = \begin{bmatrix} S_K \\ S_L \\ S_E \\ S_M \end{bmatrix}; \tilde{S} = \begin{bmatrix} S_K & 0 & 0 & 0 \\ 0 & S_L & 0 & 0 \\ 0 & 0 & S_E & 0 \\ 0 & 0 & 0 & S_M \end{bmatrix}$$

- Then, $H = \beta + S'S - \tilde{S}$

- β negative semi-definite is necessary for H to be negative semi-definite ($S_K = 1, S_L = S_E = S_M = 0) \Rightarrow H = \left[\beta_{ij}\right]$

6.1.1. Monotonicity and concavity constraints

Concavity (continued)

- Assuming $S_i > 0 \quad \forall i = K, L, E, M; \sum_i S_i = 1$ (monotonicity condition), β negative semi-definite is also a necessary condition for H to be negative semi-definite indeed,
  - $S'S$ and $-\tilde{S}$ are both negative semi-definite (easy to check for a TL KLEM function)
  - The sum of negative semi-definite functions is negative semi-definite

- **Conclusion:** Assuming that the TL function is increasing in input prices, concavity in prices can be imposed by imposing the conditions on the parameters making β negative-semi definite

6.1.1. Monotonicity and concavity constraints

Concavity (continued)

- To impose concavity, Lau L. (1978, "*Testing an Imposing Monotonicity, Convexity and Quasi-Convexity Constraints*" , in M. Fuss and D. McFadden edors "*Production Economics : A Dual Approach to Theory and Applications*" , North Holland, Vol. 1), proposes a method based on the Cholesky decomposition of defined as, β

$$\beta = TDT'$$

- Where, $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t_{LK} & 1 & 0 & 0 \\ t_{EK} & t_{EL} & 1 & 0 \\ t_{MK} & t_{ML} & t_{ME} & 1 \end{bmatrix};\ D \begin{bmatrix} d_K & 0 & 0 & 0 \\ 0 & d_L & 0 & 0 \\ 0 & 0 & d_E & 0 \\ 0 & 0 & 0 & d_M \end{bmatrix}$

6.1.1. Monotonicity and concavity constraints

Concavity (continued)

- A Cholesky decomposition always exists for a negative semi-definite matrix

- We have,

$$\beta \text{ negative semi-definite} \Leftrightarrow x'\beta x \leq 0 \ \forall x \neq o$$

$$\Leftrightarrow x'TDT'x \leq 0 \ \forall x \neq o \Leftrightarrow y'Dy \leq 0 \ \forall y \neq o,\ y \equiv T'x$$

$$\Leftrightarrow \sum_{j=K,L,E,M} d_j y_j^2 \leq 0 \ \forall y \neq o,\ y \equiv T'x \Leftrightarrow d_j \leq 0 \ \forall j = K, L, E, M$$

- **Conclusion:** For a increasing TL function, the necessary and sufficient conditions for concavity are,

$$d_j \leq 0 \ \forall j = K, L, E, M$$

6.1.1. Monotonicity and concavity constraints

Concavity (continued)

- **Remark:** β negative semi-definite is equivalent to $-\beta$ positive semi-definite

- Since $d_i \geq 0 \quad \forall j$ we can write, $-\beta = TDT' = TD^{1/2}D^{1/2}T' = AA'$ with,

$$A = TD^{1/2} = \begin{bmatrix} a_{KK} & 0 & 0 & 0 \\ a_{LK} & a_{LL} & 0 & 0 \\ a_{EK} & a_{EL} & a_{EE} & 0 \\ a_{MK} & a_{ML} & a_{ME} & a_{MM} \end{bmatrix}$$

- β is negative semi-definite if and only if it can be written,

$$-\beta = AA'$$

6.1.1. Monotonicity and concavity constraints

How to impose the concavity constraints?

1. If you estimate the 3 first equations, replace the parameters β_{ij} in the value share equations by,

$$
\begin{aligned}
\beta_{KK} &= -a_{KK}^2 \\
\beta_{KL} &= -a_{KK}a_{LK} \\
\beta_{KE} &= -a_{KK}a_{EK} \\
\beta_{LL} &= -a_{LK}^2 - a_{LL}^2 \\
\beta_{LE} &= -a_{LK}a_{EK} - a_{LL}a_{EL} \\
\beta_{EE} &= -a_{EK}^2 - a_{EL}^2 - a_{EE}^2
\end{aligned}
$$

2. Estimate the parameters a_{ij} and use the above relations to obtain the estimates of the parameters β_{ij}

6.1.1. Monotonicity and concavity constraints

Concavity and flexibility

- The preceding relationships show that when concavity is imposed we necessarily have, $\beta_{ii} \leq 0 \quad \forall i$
- As $\varepsilon_{ii} = -1 + S_i + \frac{\beta_{ii}}{S_i}$, imposing concavity means that we impose a priori

$$\varepsilon_{ii} \leq -1 + S_i$$

- **Conclusion:** Imposing concavity destroys the flexibility of the TL form
- **Remark:** In practice the TL is often reduced to a CD when concavity is imposed

6.1.1. Monotonicity and concavity constraints

Is there a solution?

- The cost function is linearly homogeneous in prices so it depends only on relative prices; this means that prices can all be normalized at 1 for a given year of observation

- When prices are equal to 1, the matrix H defined on slide 131 reduces to,

$$H = \begin{bmatrix} \beta_{KK} + \alpha_K(\alpha_K - 1) & \beta_{KL} + \alpha_K\alpha_L & \beta_{KE} + \alpha_K\alpha_E & \beta_{KM} + \alpha_K\alpha_M \\ \beta_{KL} + \alpha_K\alpha_L & \beta_{LL} + \alpha_L(\alpha_L - 1) & \beta_{LE} + \alpha_L\alpha_E & \beta_{LM} + \alpha_L\alpha_M \\ \beta_{KE} + \alpha_K\alpha_E & \beta_{LE} + \alpha_L\alpha_E & \beta_{EE} + \alpha_E(\alpha_E - 1) & \beta_{EM} + \alpha_E\alpha_M \\ \beta_{KM} + \alpha_K\alpha_M & \beta_{LM} + \alpha_L\alpha_M & \beta_{EM} + \alpha_E\alpha_M & \beta_{MM} + \alpha_M(\alpha_M - 1) \end{bmatrix}$$

- The concavity in a neighborhood of $p_i = 1 \; \forall i$ can be imposed by imposing H negative semi-definite (the method is similar to the one developed in the previous slides)

6.2 The GL form

- The GL form is globally concave if and only if $b_{ij} \leq 0; \forall i \neq j$ these conditions rule out complementarity between 2 inputs, consequently, imposing concavity destroys the flexibility of the GL form

- An alternative solution is to impose concavity at a reference point as proposed by Ryan and Wales ("*Imposing local concavity in the Translog and Generalized Leontief cost functions*", Economics Letters, 67, 2000); this is what we did for the TL in the preceding slide

- About flexible forms and regularity conditions see also Diewert and Wales "*Flexible functional forms and global curvature conditions*", Econometrica, 55, 1987

7. Weak separability

- Some variable of the model for producer behavior are not always observed, for instance capital stocks is often difficult to estimate when data are given at the firm level (capital stock is the result of capital accumulation)
- It is not possible to use flexible forms when the number of inputs is very large (example: to study substitution between energy forms, gas, oil, coal and electricity, we have to consider a 7 inputs production function, using a flexible form you will have to estimate more 50 parameters)
- A solution in such a case is to introduce a *weak separability assumption*

7. Weak separability
7.1 Definition

- **Notations:** $x=\left(x^1,x^2,\cdots,x^r\right)$ and $p=\left(p^1,p^2,\cdots,p^r\right)$ are partitions into r separated vectors of $x=\left(x_1,x_2,\cdots,x_N\right)$ and $p=\left(p_1,p_2,\cdots,p_N\right)$ such that,

$$x_i\in x^j \Leftrightarrow x_i\notin x^k \;\; \forall j,k;\; j\neq k,$$

$$p_i\in p^j \Leftrightarrow p_i\notin p^k \;\; \forall j,k;\; j\neq k$$

- **Definition** (Leontief 1945, Sono 1961): The production function, $y=f\left(x\right)$ is weakly separable in r aggregates if and only if,

$$\forall i,j \quad \frac{\partial}{\partial x_k}\left(\frac{\partial f/\partial x_i}{\partial f/\partial x_j}\right)=0 \quad x_i,x_j\in x^i\,;x_k\notin x^i$$

7. Weak separability
7.1. Definition (continued)

- Introducing the weak separability assumption in an empirical analysis of the producer behaviour is far from being neutral

- **Example:** an empirical model to analyze substitutions between different energy forms
  - Inputs are, capital, labour, coal, gas, oil, electricity and materials
  - A weak separability assumption between energy and the other inputs is introduced
  - Suppose that to make substitutions between energy forms the firms need to invest
  - Our estimation of elasticities of substitution between energy forms through the model with weak separability will be biased (it is not possible to consider that those substitutions are independent of the price of capital since they need to modify the stock of capital)

7. Weak separability
7.2. Separability of production and cost function
Weakly separable production function

- **Theorem 1** (Blackorby C., Primont D. and Russel R. (BPR), *"Duality, separability and functional structure: theory and economic applications"*, Amsterdam, North Holland, 1978):

$$\left.\begin{array}{c} f(x) \text{ regular} \\ f \text{ separable in } r \text{ aggregates} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} f(x) = \hat{f}\left(f^{1}\left(x^{1}\right), \cdots, f^{r}\left(x^{r}\right)\right) \\ f(\cdot) \text{ and } f^{i}(\cdot)\ (i = 1, .. r) \text{ regulars} \end{array}\right.$$

- **Note:** For a production function, "*regular*" means, continue, increasing and quasi-concave, for a cost function "*regular*" means continue, increasing in p and y, linearly homogeneous and concave in p

7. Weak separability

7.2. Separability of production and cost function

Weakly separable cost function

- **Theorem 2** (BPR 1978):

$$\left.\begin{array}{c} C(y,p)\text{ regular} \\ C\text{ separable in } r+1\text{ aggregates}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} C(y,p)=\hat{C}\left(y,C^1\left(p^1\right),\cdots,C^r\left(p^r\right)\right) \\ \hat{C}(\cdot)\text{ and } C^i(\cdot)\ (i=1,..r)\text{ regulars}\end{array}\right.$$

- **Important note:** The cost function described in theorem 2 is not necessarily the dual form of the production function described in theorem 1; An additional assumption is needed to obtain the duality property of weakly separable production and cost functions (see theorem 3)

7. Weak separability

7.2. Separability of production and cost function

Dual Weakly separable production and cost functions

- **Theorem 3** (BPR 1978):

$$\left.\begin{array}{l} f(x) \text{ regular, } f(x) = \hat{f}\left(f^1\left(x^1\right),\cdots,f^r\left(x^r\right)\right) \\ \hat{f}(\cdot) \text{ and } f^r\left(X^r\right) \text{ regular and homothetic} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} C(y,p) = \hat{C}\left(y, C^1\left(p^1\right),\cdots,C^r\left(p^r\right)\right) \\ \hat{C}(\cdot) \text{ and } C^i(\cdot)\,(i=1,..r) \text{ regular} \end{array}\right.$$

- **Notes:**
  - The production function is said homothetically separable
  - $h(x)$ is said homothetic when it can be written $h(x) = g\left(f(x)\right)$ where $f(x)$ is a linearly homogeneous function and $g(\cdot)$ a continuous, positive and increasing function of f

7.3. An econometric model with weak separability assumption

- Consider a regular production function which is homothetically separable in r aggregates, from theorem 1 we have

$$f(x) = \hat{f}\left(f^1\left(x^1\right), \cdots, f^r\left(x^r\right)\right),$$

- From theorem 3, we know that there exists a dual cost function,

$$C(y, p) = \hat{C}\left(y, C^1\left(p^1\right), \cdots, C^r\left(p^r\right)\right).$$

7.3. An econometric model with weak separability assumption

- The functions $C^s\left(p^s\right)\left(s=1,\ldots,r\right)$ are linearly homogeneous and the functions $f^s\left(x^s\right)\left(s=1,\ldots,r\right)$ can be chosen linearly homogeneous (a particular case of homothetic functions); consequently those functions can be respectively considered as prices and quantities indices

- We note X_s and P_s the values taken by $f^s\left(p^s\right)$ and $C^s\left(p^s\right)$ respectively,

$$X_s = f^s\left(x^s\right) \; \left(s=1,\ldots,r\right)$$

$$P_s = C^s\left(p^s\right) \; \left(s=1,\ldots,r\right)$$

7.3. An econometric model with weak separability assumption (continued)

- The functions $f^s(x^s)$ and $C^s(p^s)$ are regular and can be interpreted as production and (unit) cost functions respectively

- The cost function $\hat{C}(\cdot)$ is defined as,

$$\hat{C}(y, P_1, \cdots, P_r) \equiv \min_{X_1, \ldots, X_s} \left\{ P_1 X_1 + \ldots + P_r X_r : \hat{f}(X_1, \cdots, X_r) \ldots y, P_s > 0 \ \forall s \right\}$$

- The unit cost functions $C^s(p^s)$ are defined as,

$$C^s(p^s) \equiv \frac{1}{X^s} \min_{X^s} \left\{ \left(p^{s\prime} x^s \right) : f^s(X^s) = X_s, \ p_i \in p^s > 0 \right\}, (s = 1, \ldots, r)$$

7.3. An econometric model with weak separability assumption (continued)

Decomposition of the producer’ program into 2 steps

- **First step:** The producer chooses the optimal expenditure devoted to each aggregate, $P_s X_s^* \left(s=1,\dots,r\right)$; applying the shephard’s lemma to the cost function $\hat{C}\left(y,P_1,\cdots,P_r\right)$ we have,

$$X_s^* = \frac{\partial \hat{C}\left(y,P_1,\cdots,P_r\right)}{\partial P_s} \quad \left(s=1,\dots,r\right)$$

- **Second step:** The producer chooses the optimal level of each input, $x_i^*\left(x_i \in x^s; s=1,\dots,r\right)$; applying the Shephard's lemma to the unit cost functions $C^s\left(p^s\right) \left(s=1,\dots,r\right)$ we have,

$$\frac{x_i^*}{X_s^*} = \frac{\partial C^s\left(p^s\right)}{\partial p_i} \quad p_i \in p^s \quad \left(s=1,\dots,r\right)$$

7.3. An econometric model with weak separability assumption (continued)

Comments

- As parametric forms of $\hat{C}(\,)$ and $C^s\left(p^s\right)$ can be estimated separately, it is possible to estimate substitutions between energy forms with data on energy prices and quantities only

- The weak homothetic separability assumption is far from being neutral since it supposes that the unit cost function of each aggregate depends only on prices of inputs making up this aggregate

- The R-square (generally noted R^2) is usually used in the context of OLS estimation; considering the classical linear model, $y = X\beta + \varepsilon$, it can be written as,

$$R^2 = 1 - \frac{\hat{\varepsilon}'\hat{\varepsilon}}{\left(y - e\overline{y}\right)'\left(y - e\overline{y}\right)} \quad \left(\hat{\varepsilon} = y - X\hat{\beta}\right)$$

- R-square has nice properties:
  - OLS estimator maximizes R^2 (because it minimizes $\hat{\varepsilon}'\hat{\varepsilon}$)
  - $0 \leq R^2 \leq 1$ because $0 \leq \hat{\varepsilon}'\hat{\varepsilon} \leq \left(y - e\overline{y}\right)'\left(y - e\overline{y}\right)$

8.1 Goodness of fit measurement (continued)

- The GLS estimator minimizes $q = \varepsilon'\Omega^{-\frac{1}{2}}\Omega^{-\frac{1}{2}}\varepsilon = \varepsilon'\Omega^{-1}\varepsilon$ therefore it does not maximizes R^2, moreover it can be shown that when using GLS estimator $R^2 < 0$ is possible

- An alternative to R^2 is, $\tilde{R}^2 = 1 - \dfrac{\tilde{\varepsilon}'\tilde{\varepsilon}}{\left(\tilde{y} - e\bar{\tilde{y}}\right)'\left(\tilde{y} - e\bar{\tilde{y}}\right)}$ where,

$$\tilde{\varepsilon} = \Omega^{-\frac{1}{2}}\varepsilon,\ \tilde{y} = \Omega^{-\frac{1}{2}}y,\ \bar{\tilde{y}} = \frac{1}{T}e'\tilde{y}$$

- Following the same idea, a measurement of goodness of fit in the case of iterative Zellner or ML estimators of TL input value shares could be,

$$\tilde{\tilde{R}}^2 = 1 - \frac{|u'u|}{\left|\left(s - e\bar{s}\right)'\left(s - e\bar{s}\right)\right|}$$

- Consider an estimator $\hat{\beta}$ of a parameter β such that $\hat{\beta}$ is normally distributed with mean β and variance $\sigma^2_{\hat{\beta}}$

- Denoting $\hat{\sigma}_{\hat{\beta}}$ an estimator of $\sigma_{\hat{\beta}}$, it can be shown that,

$$t = \frac{\hat{\beta} - \beta}{\hat{\sigma}_{\hat{\beta}}}$$

has a Student distribution with T-K degrees of freedom where T is the number of observations and K the number of parameters

8.2 Hypothesis testing
Confidence intervals (continued)

- A $(1-\alpha)\%$ confidence interval is such that,

$$\Pr\left[-t_{\alpha/2} \leq \frac{\hat{\beta}-\beta}{\hat{\sigma}_{\hat{\beta}}} \leq t_{\alpha/2}\right] = 1-\alpha$$

where $t_{\alpha/2}$ is such that, $\Pr\left[|T| \geq t_{\alpha/2}\right] = \alpha$

- The $(1-\alpha)\%$ confidence interval for β is therefore given by,

$$\left]\hat{\beta}-\hat{\sigma}_{\hat{\beta}}t_{\alpha/2} \leq \beta \leq \hat{\beta}+\hat{\sigma}_{\hat{\beta}}t_{\alpha/2}\right[$$

- We reject the null hypothesis $H_0 : \beta = \beta_0$ against the alternative hypothesis $H_1 : \beta \neq \beta_0$ when β_0 is outside the confidence interval

8.2 Hypothesis testing

The Wald test

- β is a $(K\times 1)$ vector of parameters, $\hat{\beta}$ is an estimator of β such that,

$$\hat{\beta}-\beta : \; N(0,\Omega)$$

- β satisfies the r linear constraints,

$$H_0 : R\beta = q; \; H_1 : R\beta \neq q$$

- We have

$$R\hat{\beta}-d : \; N(0, R\Omega R')$$

therefore,

$$(R\Omega R')^{-\frac{1}{2}}\left(R\hat{\beta}-d\right) : \; N(0, I)$$

- And finally the Wald test statistic is,

$$W = \left(R\hat{\beta}-d\right)'(R\Omega R')^{-\frac{1}{2}}(R\Omega R')^{-\frac{1}{2}}\left(R\hat{\beta}-d\right)$$

$$= \left(R\hat{\beta}-d\right)'(R\Omega R')^{-1}\left(R\hat{\beta}-d\right) : \; \chi^2(r)$$

8.2 Hypothesis testing

The Wald test (continued)

- The Wald test can also be used to test $H_0 : g(\beta) = 0$ against $H_1 : g(\beta) \neq 0$ where $g(\beta)$ is a vector of non-linear functions

- Using a first order Taylor series expansion, under the null hypothesis, of $g(\beta)$ we obtain,

$$g(\hat{\beta}) = g(\beta) + G(\beta)(\hat{\beta} - \beta) = G(\beta)(\hat{\beta} - \beta) \text{ where } G(\beta) \equiv \frac{\partial g(\beta)}{\partial \beta'}$$

- As, $g(\hat{\beta}) : \; N(0, G'\Omega G)$, the Wald test statistic is,

$$W = g(\hat{\beta})' (G\Omega G')^{-1} g(\hat{\beta}) : \; \chi^2_{\dim(g)}$$

8.2 Hypothesis testing

The Likelihood Ratio (LR) test

- Let $\hat{\beta}_u$ be the maximum likelihood estimate of β obtained without regard to the constraints $g(\beta)=0$ and let $\hat{\beta}_r$ be the constrained maximum likelihood estimator

- If $\hat{L}_u$ and $\hat{L}_r$ are the likelihood functions evaluated at these 2 estimates, the likelihood ratio is,

$$\lambda = \frac{\hat{L}_r}{\hat{L}_u}$$

- And the LR test statistic is,

$$LR = -2\ln\lambda \sim \chi^2_{\dim(g)}$$

8. An example from the literature

8.1 Berndt and Wood (AER 1979)

- Using time series data on K, L , E and M, econometricians often find E-K complementarity
- Engineering analysis show that energy efficiency increases with investment (new equipments use energy more efficiently than old ones), so they conclude to E-K substitutability
- Berndt & Wood consider that the source of this apparent conflict is that engineering and econometric analysis do not measure the same thing, the first consider that E-K substitutability can be analyzed independently of other inputs, the second considers that the producer's decisions about K, L , E and M are made simultaneously

8.1 Berndt and Wood (AER 1979) (continued)
The model

- The production function is, $Y = F\left(K, L, E, M\right)$

- $F(x)$ is supposed to be regular and weakly separable in such a way that it can be written,

$$Y = F^{*}\left(f_{K^{*}}\left(K, E\right), f_{L^{*}}\left(L, M\right)\right)$$

- $F^{*}(x)$, $f^{*}(x)$ are supposed to be regular;
  $f_{K}^{*}(x)$ and $f_{L}^{*}(x)$ linearly homogeneous

- We write, $K^{*} = f_{K^{*}}\left(K, E\right)$ and $L^{*} = f_{L^{*}}\left(L, M\right)$

8.1 Berndt and Wood (AER 1979)

The model (continued)

- The dual cost function is,

$$C = G\left(Y, p_K, p_L, p_E, p_M\right) = G^*\left(Y, g_{K^*}\left(p_K, p_E\right), g_{L^*}\left(p_L, p_M\right)\right)$$

- $G^*(x)$ is a regular cost function, $p_{K^*} = g_{K^*}\left(p_K, p_E\right)$ and $p_{L^*} = g_{L^*}\left(p_L, p_M\right)$ are regular unit cost functions

8.1 Berndt and Wood (AER 1979)

The cross-price elasticity between capital and energy

- $$\varepsilon_{EK} = \left.\frac{d\ln E}{d\ln p_K}\right|_{dY=0} = \left.\frac{\partial \ln E}{\partial \ln p_K}\right|_{dK^*=0} + \left.\frac{\partial \ln E}{\partial \ln K^*}\frac{\partial \ln K^*}{\partial \ln p_{K^*}}\frac{\partial \ln p_{K^*}}{\partial \ln p_K}\right|_{dY=0}$$

- Under the linear homogeneity assumption of $f_{K^*}(K,E)$ the demand for energy is $E = K^* \partial g_{K^*}(p_K, p_E)/\partial p_E$, it follows that, $\partial \ln E / \partial \ln K^* = 1$

- Denoting, $S_{KK^*} \equiv \dfrac{p_K K}{p_K K + p_L E} = \left.\dfrac{\partial \ln p_{K^*}}{\partial \ln p_K}\right|_{dY=0}$ and $\varepsilon_{K^*K^*} \equiv \left.\dfrac{\partial \ln K^*}{\partial \ln p_{K^*}}\right|_{dY=0}$ we have,

$$\varepsilon_{EK} = \varepsilon_{EK}^{*} + S_{KK^*}\varepsilon_{K^*K^*}$$

8.1 Berndt and Wood (AER 1979)

The cross-price elasticity between K and E (continued)

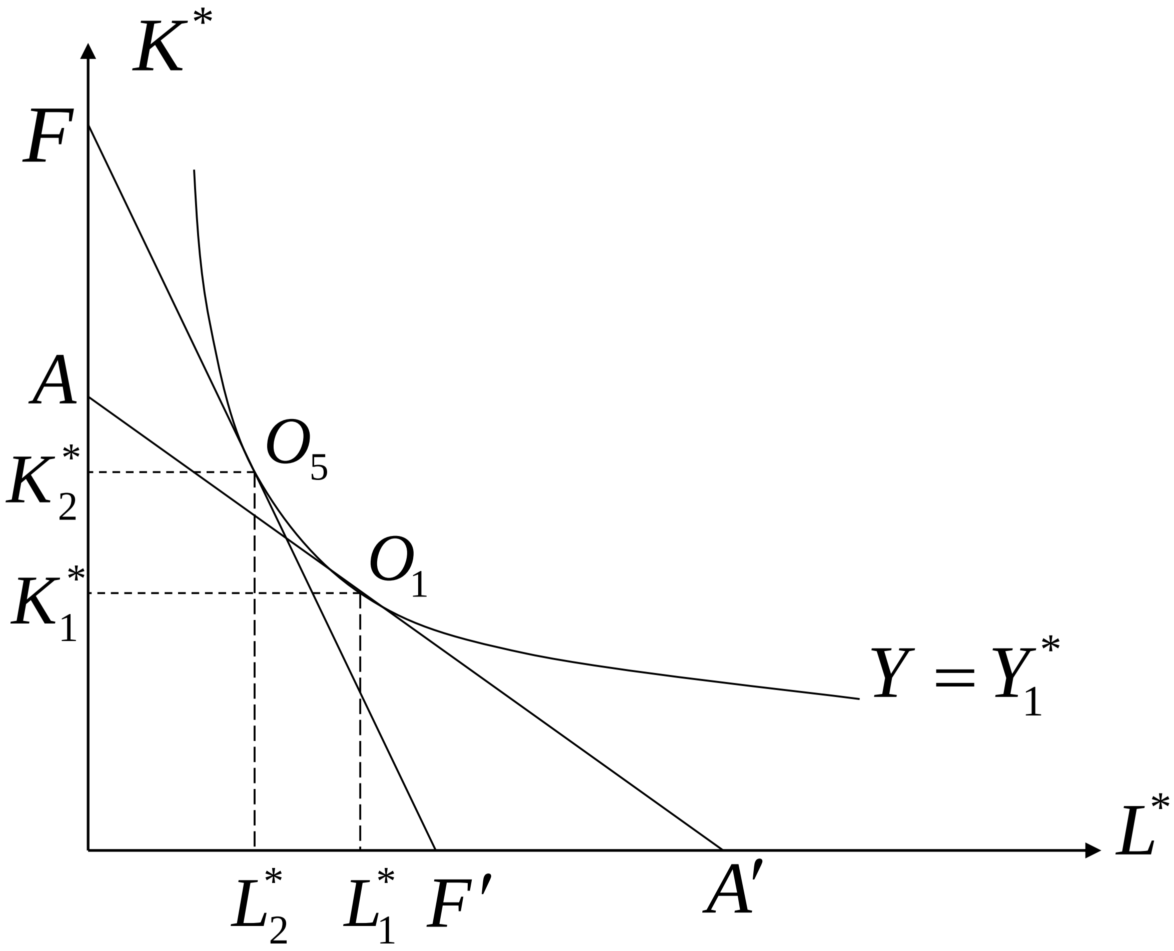
- In the expression, $\varepsilon_{EK} = \varepsilon^{*}_{EK} + S_{KK^*}\varepsilon_{K^*K^*}$ is the ε_{K^*E} gross-price elasticity because it is measured at constant when K^* any change of should also induce p_K a change of K^*

- ε_{KE} is the net-price elasticity, it takes into account any changes induced by a change of p_K

- $S_{KK^*}\varepsilon_{K^*K^*} < 0$ is the expansion elasticity because it measures the consequence for of a change of K^* p_K

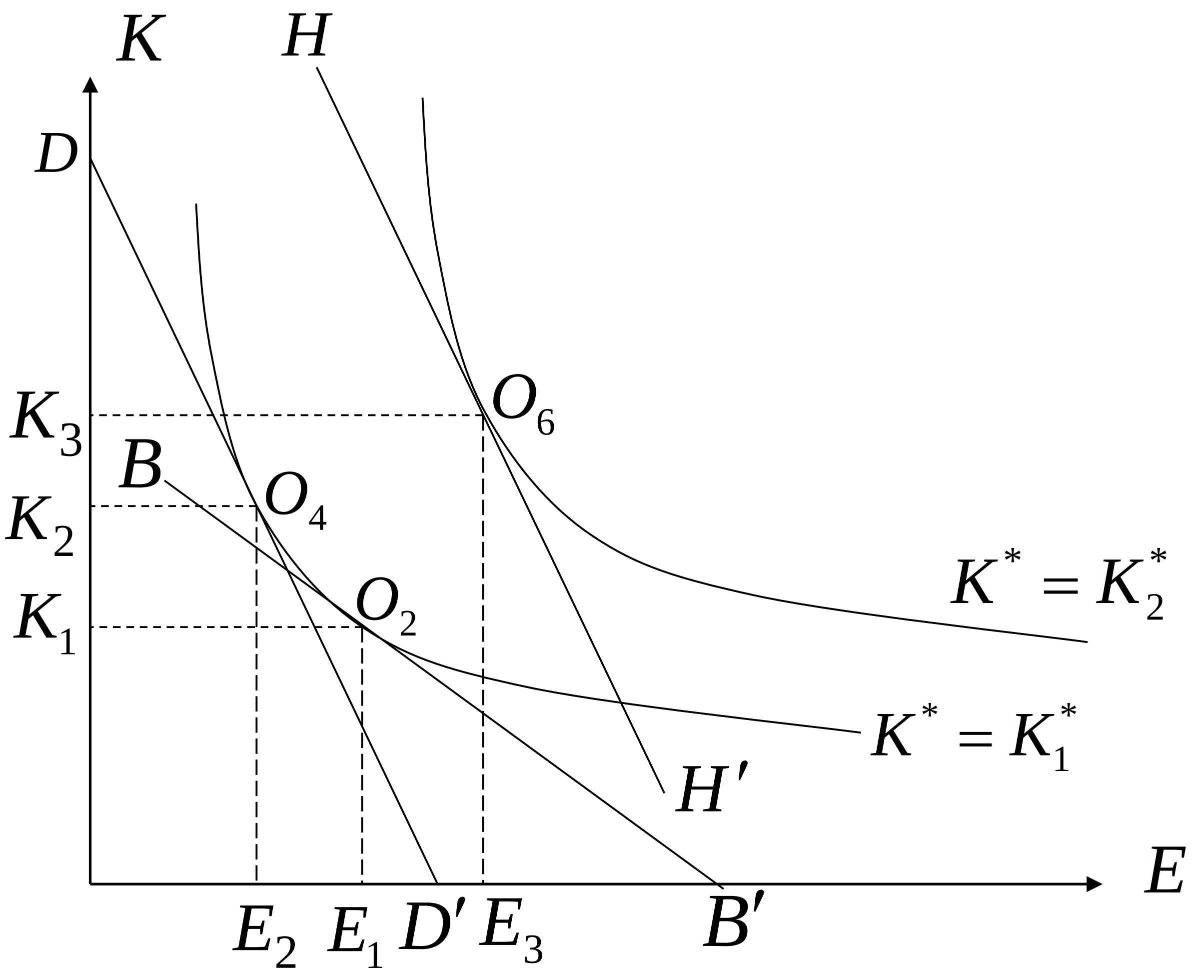
8.1 Berndt and Wood (AER 1979)

Figure 10: Master function



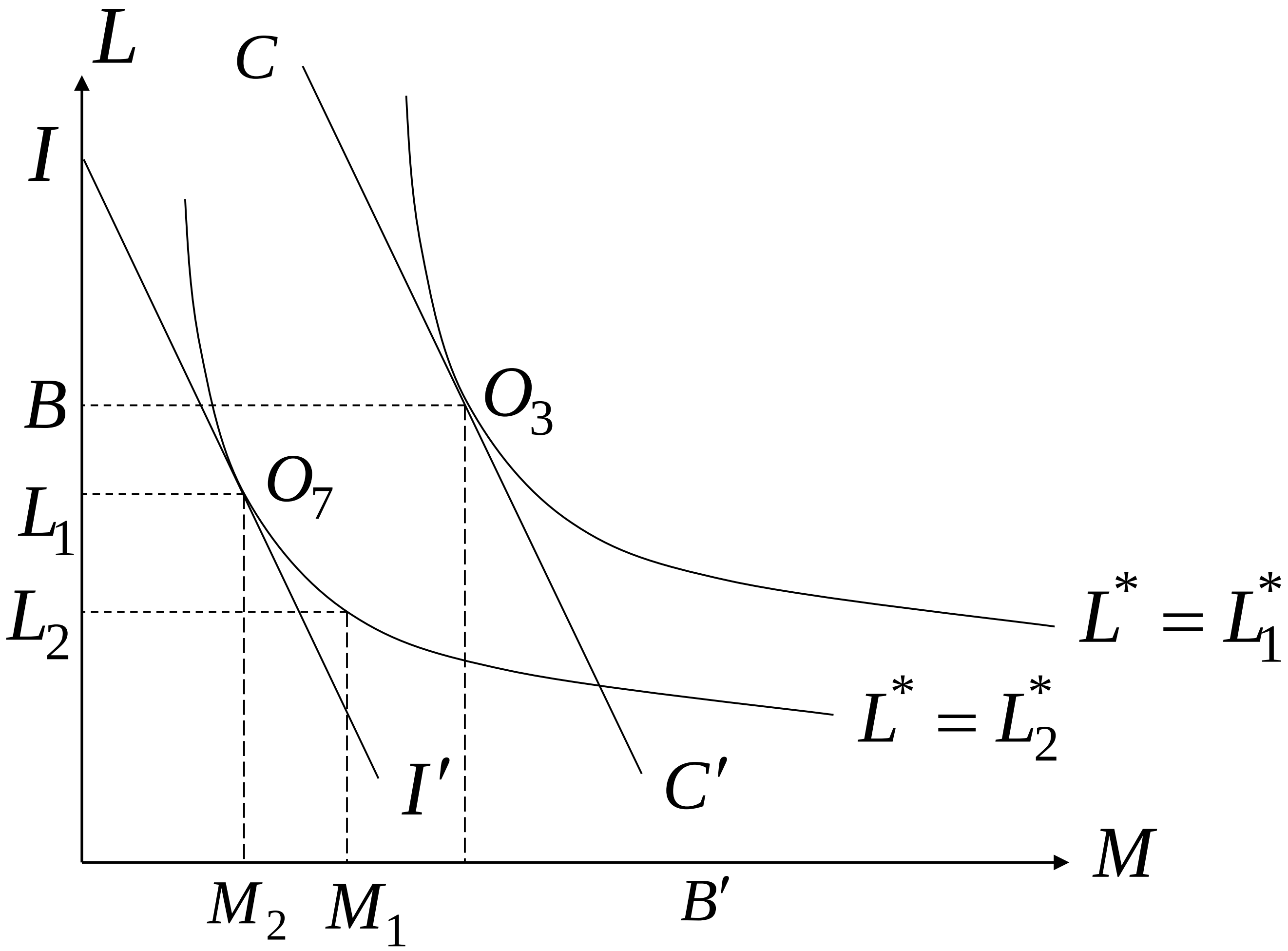
8.1 Berndt and Wood (AER 1979)

Figure 11: Capital-energy subfunction



8.1 Berndt and Wood (AER 1979)

Figure 12: Labor-materials subfunction



8.1 Berndt and Wood (AER 1979)

Reading the graphs

- O_1, O_2 and O_3 describe equilibrium when relative prices are given by the slopes of AA', BB' and CC' respectively
- Suppose p_K decreases:
  - The relative price p_E / p_K is given by the slope of DD' (see fig. 11)
  - Holding $K^* = K_1^*$ fixed, the new equilibrium is O_4 where K increases from K_1 to K_2 and E decreases from E_1 to E_2; this is the gross substitution effect measured by ε_{EK}^*
  - Decreasing p_K also decreases $p_{K^*} = g_{K^*}(p_K, p_E)$, the new relative price is given by p_{L^*} / p_{K^*} the slope of FF' and the new equilibrium is O_5 where K^* increases from K_1^* to K_2^* and decreases from L_1^* to L_2^* (see fig.10); this results in an outward shift of the K^* isoquant as shown on fig. 11

8.1 Berndt and Wood (AER 1979)
Reading the graphs (continued)

- At the new equilibrium, O_6 the expansion increases the demand for capital and energy from K_2 to K_3 and from E_2 to E_3
- For capital, the gross substitution effect (K_1 to K_2) and the expansion effect (K_2 to K_3) reinforce each other
- For energy, the gross substitution effect (E_1 to E_2) and the expansion effect (E_2 to E_3) works in opposite directions but
- in this particular example, the expansion effect ($E_3 > E_1$) dominates the gross substitution effect ($S_{KK^*}\varepsilon_{K^*K^*}$), and ε^*_{EK} E and K are gross substitutes but net complements

8.1 Berndt and Wood (AER 1979)
Empirical analysis

- We use TL forms for $g_{K^*}(p_K, p_E)$ and $g_{L^*}(p_L, p_M)$

$$\ln p_{K^*} = \gamma_{K^*} + \gamma_K \ln p_K + \gamma_E \ln p_E + \frac{1}{2}\gamma_{KK}(\ln p_K)^2$$

$$+\gamma_{KE} \ln p_K \ln p_E + \frac{1}{2}\gamma_{EE}(\ln p_E)^2$$

$$\ln p_{L^*} = \gamma_{L^*} + \gamma_L \ln p_L + \gamma_M \ln p_M + \frac{1}{2}\gamma_{LL}(\ln p_L)^2$$

$$+\gamma_{LM} \ln p_L \ln p_M + \frac{1}{2}\gamma_{MM}(\ln p_M)^2$$

- We use a CD form for $G^*(Y, p_{K^*}, p_{L^*})$

$$\ln(C/Y) = \beta + \beta_{K^*} \ln p_{K^*} + \beta_{L^*} \ln p_{L^*}$$

8.1 Berndt and Wood (AER 1979)
Empirical analysis (continued)

- The master cost function written in terms of the separable subfunction prices and parameters is given by,

$$\ln\left(C/Y\right) = \beta + \beta_{K^*}\gamma_{K^*} + \beta_{K^*}\gamma_{L^*} + \beta_{K^*}\gamma_K \ln p_K + \beta_{L^*}\gamma_L \ln p_L$$
$$+\beta_{K^*}\gamma_E \ln p_E + \beta_{L^*}\gamma_M \ln p_M + \frac{1}{2}\beta_{K^*}\gamma_{KK}\left(\ln p_K\right)^2$$
$$+\beta_{K^*}\gamma_{KE} \ln p_K \ln p_E + \frac{1}{2}\beta_{L^*}\gamma_{LL}\left(\ln p_L\right)^2 + \beta_{L^*}\gamma_{LM} \ln p_L \ln p_M$$
$$+\frac{1}{2}\beta_{K^*}\gamma_{EE}\left(\ln p_E\right)^2 + \frac{1}{2}\beta_{L^*}\gamma_{MM}\left(\ln p_M\right)^2$$

8.1 Berndt and Wood (AER 1979)

Empirical analysis (continued)

- The value shares of inputs are,

$$S_K = \beta_{K^*}\gamma_K + \beta_{K^*}\gamma_{KK} \ln p_K + \beta_{K^*}\gamma_{KE} \ln p_E$$
$$S_L = \beta_{L^*}\gamma_L + \beta_{L^*}\gamma_{LL} \ln p_L + \beta_{L^*}\gamma_{LM} \ln p_M$$
$$S_E = \beta_{K^*}\gamma_E + \beta_{K^*}\gamma_{KE} \ln p_K + \beta_{K^*}\gamma_{EE} \ln p_E$$
$$S_E = \beta_{L^*}\gamma_M + \beta_{L^*}\gamma_{LM} \ln p_L + \beta_{L^*}\gamma_{MM} \ln p_M$$

8.1 Berndt and Wood (AER 1979)

Empirical analysis (continued)

- Under symmetry and additivity restrictions the following equations are estimated

$$S_K = \beta_{K^*}\gamma_K + \beta_{K^*}\gamma_{KK} \ln \frac{p_K}{p_E} + u_K$$

$$S_L = \left(1-\beta_{K^*}\right)\gamma_L + \left(1-\beta_{K^*}\right)\gamma_{LL} \ln \frac{p_L}{p_M} + u_L$$

$$S_E = \beta_{K^*}\left(1-\gamma_K\right) - \beta_{K^*}\gamma_{KK} \ln \frac{p_K}{p_E} + u_E$$

- Based on annual time series data for US manufacturing, the ML estimator is used

8.1 Berndt and Wood (AER 1979)

Empirical analysis (continued)

- Using ML estimates of the parameters, it is possible to compute estimates of the net, expansion, and gross elasticities between and

K E

- The expression for the TL cross-price elasticity is of the form,

$$\frac{\beta_{EK} + S_K S_E}{S_E}$$

- Appling the above expression to the master cost function gives the net elasticity,

$$\varepsilon_{EK} = \frac{\beta_{K^*}\gamma_{KK} + S_K S_E}{S_E}$$

Empirical analysis (continued)

- The same expression applied to the TL form of the unit cost function $g_{K^*}(\cdot)$ gives the gross-elasticity,

$$\varepsilon^*_{EK} = \frac{-\gamma_{KK} + S_{KK^*} S_{EK^*}}{S_{EK^*}}$$

where,

$$S_{KK^*} = \frac{\partial \ln p_{K^*}}{\partial \ln p_K} = \gamma_K + \gamma_{KK} \ln \frac{p_K}{p_E}$$

$$S_{EK^*} = \frac{\partial \ln p_{K^*}}{\partial \ln p_E} = (1 - \gamma_K) - \gamma_{KK} \ln \frac{p_K}{p_E} = 1 - S_{KK^*}$$

8.1 Berndt and Wood (AER 1979)

Empirical analysis (continued)

- The expression for the TL own-price elasticity is of the form,

$$\frac{\beta_{KK} + S_K\left(S_K - 1\right)}{S_K}$$

- Appling the above expression to the master CD cost function gives the own-price elasticity,

$$\varepsilon_{K^*K^*} = \frac{0 + S_{K^*}\left(S_{K^*} - 1\right)}{S_{K^*}} = S_{K^*} - 1$$

- But, $S_{K^*} = \dfrac{\partial \ln\left(C/Y\right)}{\partial \ln p_{K^*}} = \beta_{K^*}$ therefore the expansion elasticity is given by

$$S_{KK^*}\varepsilon_{K^*K^*} = \left(\beta_{K^*} - 1\right)S_{KK^*}$$

8.1 Berndt and Wood (AER 1979)

Empirical analysis (continued)

	Gross SubstitutionElasticity	Expansion Elasticity	Net Elasticity
ε_{EK}	0.133 (0.026)	-0.462 (0.003)	-0.329 (0.026)

- The gross substitution effect (0.133) is dominated by the (always negative) expansion effect (-0.462) resulting in a value of the net elasticity of -0.329

- **Conclusion:** In the US manufacturing, energy and capital are gross substitutes but net complements