Discrete Structures

Recurrence Relations

Some Problems

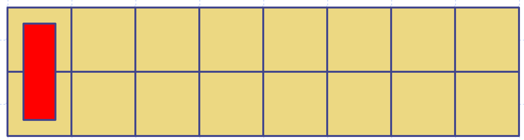
Problem: #ways of tiling a $2 \times n$ table with dominoes

Solution:

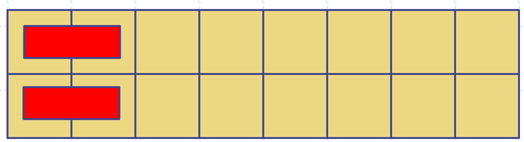
Let a_n be #ways of tiling a $2 \times n$ table with dominoes

First column can be tiled in either of two following ways

1. Case 1



2. Case 2



- Cases 1 and 2 are disjoint.
- #ways of doing case 1 $= a_{n-1}$
- #ways of doing case 2 $= a_{n-2}$

Then, $a_n = a_{n-1} + a_{n-2}$

We also know $a_1 = 1, a_2 = 2$

Some Problems

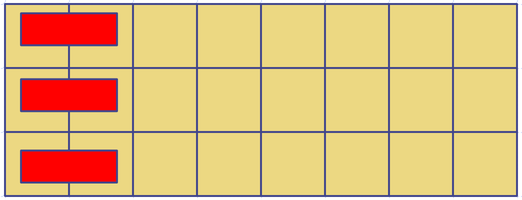
Problem: #ways of tiling a $3 \times n$ table with dominoes

Solution:

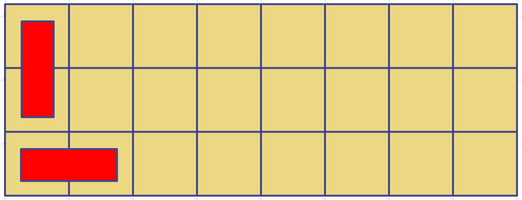
Let a_n be #ways of tiling a $3 \times n$ table with dominoes

First column can be tiled in one of three following ways

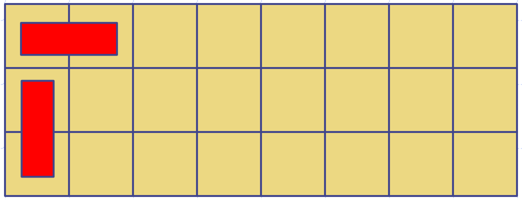
1. Case 1



2. Case 2



3. Case 3



Some Problems

- Cases 1, 2, and 3 are disjoint
- #ways of doing case 1 $= a_{n-2}$
- Cases 2 and 3: Tiling a $3 \times (n-1)$ table whose one of corners has already tiled

Let b_n be #ways of tiling a $3 \times n$ table whose one of corners has already tiled

- #ways of doing case 2$=$ #ways of doing case 3$= b_{n-1}$

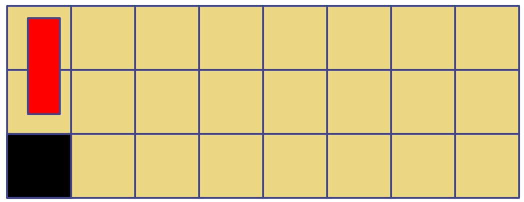
Therefore, $a_n = a_{n-2} + 2b_{n-1}$

Some Problems

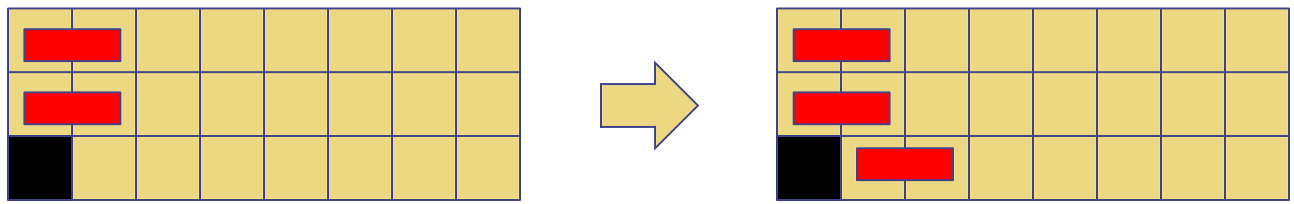
But how to obtain b_n: Of course recursively

First column can be tiled in either of two following ways

- Case 1



- Case 2



Then, $b_n = a_{n-1} + b_{n-2}$. So we have

$$\left\{\begin{array}{lcl} a_n = a_{n-2} + 2b_{n-1} & \Rightarrow & b_n = (a_{n+1} - a_{n-1})/2 \\ b_n = a_{n-1} + b_{n-2} & \Rightarrow & a_n = b_{n+1} - b_{n-1} \end{array}\right\} \Rightarrow$$

$$a_n = (a_{n+2} - a_n)/2 - (a_n - a_{n-2})/2 \Rightarrow a_{n+2} = 4a_n - a_{n-2}$$

We also know $a_0 = 1, a_1 = 0, a_2 = 3, a_3 = 0$

Linear Homogenous Recurrence

A linear homogenous recurrence of degree k

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$$

where $c_1, \ldots, c_k$ are real numbers and $c_k \neq 0$

It is called linear because the right-hand side is a sum of the previous terms of the sequence each multiplied by a real number.

It is called homogenous because no terms occur that are not multiple of a_js

Given the first k terms of the sequence (i.e. $a_0, \ldots, a_{k-1}$), the sequence is uniquely determined.

Solving L.H.R

We explain the solution for $k = 2$ and at the end we generalize it for any k

Observation: if we have two sequences $\{a_n\}, \{b_n\}$ s.t.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$
$$b_n = c_1 b_{n-1} + c_2 b_{n-2}$$

and their two first terms are equal (i.e. $a_0 = b_0, a_1 = b_1$)

Then, $a_n = b_n$ for all n

Solving L.H.R.

Problem: Compute a_n where $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and a_0, a_1 are given.

Solution:

Let r_1, r_2 be the roots of $x^2 = c_1 \mathrm{x} + c_2$ (characteristic equation). If we define $b_n = \alpha {r_1}^n$ and $d_n = \beta {r_2}^n$ for arbitrary and fix numbers α, β, it is easy (just replace) to show that regardless of what b_0, b_1, d_0, d_1 are, we have

$$b_n = c_1 b_{n-1} + c_2 b_{n-2}$$
$$d_n = c_1 d_{n-1} + c_2 d_{n-2}$$

Also if we define $h_n = b_n + d_n$, again we have

$$h_n = c_1 h_{n-1} + c_2 h_{n-2}$$

Sequences $\{a_n\}$ and $\{h_n = \alpha {r_1}^n + \beta {r_2}^n\}$ have the same recurrence formula. If $a_0 = h_0, a_1 = h_1$, then $\forall n: a_n = h_n$

So, find α, β s.t. $a_0 = h_0 = \alpha + \beta, a_1 = h_1 = \alpha r_1 + \beta r_2$

Some Problems

Problem: $f_n = f_{n-1} + f_{n-2}$ and $f_0 = 0, f_1 = 1.$

Solution:

- $x^2 = x + 1 \ \rightarrow r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$
- $f_n = \alpha(\frac{1+\sqrt{5}}{2})^n + \beta(\frac{1-\sqrt{5}}{2})^n$

$$\begin{cases} \alpha + \beta = 0 \\ \alpha\left(\frac{1+\sqrt{5}}{2}\right) + \beta\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{cases} \Rightarrow \begin{cases} \alpha = 1/\sqrt{5} \\ \beta = -1/\sqrt{5} \end{cases}$$

Some Problems

Problem: $a_n = 4a_{n-1} - 4a_{n-2}$ and $a_0 = 0, a_1 = 1.$

Solution:

- $x^2 = 4x - 4 \ \rightarrow r_1 = 2, r_2 = 2$
- $a_n = \alpha 2^n + \beta 2^n$

$$\begin{cases} \alpha + \beta = 0 \\ 2\alpha + 2\beta = 1 \end{cases} \Rightarrow \begin{cases} \alpha = ? \\ \beta = ? \end{cases}$$

In above equations, left sides are not independent (one is a multiplication of the other)

What we have to do when $r_1 = r_2$

Solving L.H.R.

Problem: Compute a_n where $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and a_0, a_1 are given where both roots of $x^2 = c_1 \mathrm{x} + c_2$ are equal

Solution:

Let r be the unique root of $x^2 = c_1 \mathrm{x} + c_2$. Define $b_n = \alpha r^n$ and $d_n = \beta n r^n$. it is easy to show that

$$b_n = c_1 b_{n-1} + c_2 b_{n-2}$$
$$d_n = c_1 d_{n-1} + c_2 d_{n-2}$$

Let's check the second one

We have to show $\beta n r^n = c_1 \beta (n-1) r^{n-1} + c_2 \beta (n-2) r^{n-2}$

We know r is the root of $x^n = c_1 x^{n-1} + c_2 x^{n-2}$ and its derivative $\mathrm{n}x^{n-1} = c_1 (n-1) x^{n-2} + c_2 (n-2) x^{n-3} \rightarrow \mathrm{n}x^n = c_1 (n-1) x^{n-1} + c_2 (n-2) x^{n-2}$. So we have $\beta n r^n = c_1 \beta (n-1) r^{n-1} + c_2 \beta (n-2) r^{n-2}$

Solving L.H.R.

Problem: Compute a_n where $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and a_0, a_1 are given where both roots of $x^2 = c_1\mathrm{x} + c_2$ are equal

Solution:

Let r be the unique root of $x^2 = c_1\mathrm{x} + c_2$. Define $b_n = \alpha r^n$ and $d_n = \beta n r^n$. it is easy to show that

$$b_n = c_1 b_{n-1} + c_2 b_{n-2}$$
$$d_n = c_1 d_{n-1} + c_2 d_{n-2}$$

Also if we define $h_n = b_n + d_n$, again we have

$$h_n = c_1 h_{n-1} + c_2 h_{n-2}$$

Sequences $\{a_n\}$ and $\{h_n = \alpha r^n + \beta n r^n\}$ have the same recurrence formula. If $a_0 = h_0, a_1 = h_1$, then $\forall n: a_n = h_n$

So, find α, β s.t. $a_0 = h_0 = \alpha, a_1 = h_1 = \alpha r + \beta r$

Some Problems

Problem: $a_n = 4a_{n-1} - 4a_{n-2}$ and $a_0 = 0, a_1 = 1.$

Solution:

- $x^2 = 4x - 4 \;\rightarrow r_1 = r_2 = 2$
- $a_n = \alpha 2^n + \beta n 2^n$

$$\begin{cases} \alpha = 0 \\ 2\alpha + 2\beta = 1 \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \beta = 1/2 \end{cases}$$

Some Problems

Problem: $a_n = 4a_{n-1} - 4a_{n-2}$ and $a_0 = 0, a_1 = 1.$

Solution:

- $x^2 = 4x - 4 \rightarrow r_1 = r_2 = 2$
- $a_n = \alpha 2^n + \beta n 2^n$

$$\begin{cases} \alpha = 0 \\ 2\alpha + 2\beta = 1 \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \beta = 1/2 \end{cases}$$

Solving L.H.R.

A linear homogenous recurrence of degree k

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$$

where $c_1, \ldots, c_k$ are real numbers and $c_k \neq 0$

Solution:

Let $r_1, \ldots, r_t$ be the roots of $x^k = c_1 x^{k-1} + \cdots + c_{k-1} x + c_k$ with multiplicities $m_1, \ldots, m_t$

$$a_n = \left(\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}\right) {r_1}^n + \cdots + \left(\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}\right) {r_t}^n$$

All k constants $\alpha_{i,j}$ can be computed by k first terms of the sequence.

L. Non-H.R.

A linear non-homogenous recurrence of degree k

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} + f(n)$$

where $c_1, \ldots, c_k$ are real numbers and $c_k \neq 0$

Solution:

Assume we find a sequence b_n satisfying the above formula regardless of what its k first terms are, then

$$d_n = b_n + \left(\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}\right) {r_1}^n + \cdots + \left(\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}\right) {r_t}^n$$

Satisfies the above formula where $r_1, \ldots, r_t$ are the roots of $x^k = c_1 x^{k-1} + \cdots + c_{k-1} x + c_k$ with multiplicities $m_1, \ldots, m_t$

L. Non-H.R.

A linear non-homogenous recurrence of degree k

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} + f(n)$$

where $c_1, \ldots, c_k$ are real numbers and $c_k \neq 0$

Solution:

Computing all k constants $\alpha_{i,j}$ by the k first terms of the sequence is straightforward. The main problem is to find b_n which is called a particular solution. There are only some hints how to find b_n for special functions $f(n)$.

L. Non-H.R.

A linear non-homogenous recurrence of degree k

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} + f(n)$$

where $c_1, \ldots, c_k$ are real numbers and $c_k \neq 0$

Solution:

If $f(n) = (b_t n^t + \cdots + b_1 n + b_0) s^n$, then a particular solution is of form $b_n = (\alpha_t n^t + \cdots + \alpha_1 n + \alpha_0) s^n$

If s is a root of the characteristic equation and its multiplicity is m, a particular solution is of form $b_n = n^m (\alpha_t n^t + \cdots + \alpha_1 n + \alpha_0) s^n$

Some Problems

Problem: $a_n = 6a_{n-1} - 9a_{n-2} + n3^n, a_0 = 0, a_1 = 1$

Characteristic equation: $x^2 = 6x - 9 \to r_1 = r_2 = 3$

Particular solution: $b_n = n^2(\alpha n + \beta)3^n$

$$n^2(\alpha n + \beta)3^n$$
$$= 6\left((n-1)^2(\alpha(n-1) + \beta)3^{n-1}\right)$$
$$- 9\left((n-2)^2(\alpha(n-2) + \beta)3^{n-2}\right) + n3^n \;\; \to \;\; \alpha n^3 + \beta n^2$$
$$= (2\alpha n^3 - 6\alpha n^2 + 6\alpha n - 2\alpha + 2\beta n^2 - 4\beta n + 2\beta)$$
$$- (\alpha n^3 - 6\alpha n^2 + 12\alpha n - 8\alpha + \beta n^2 - 4\beta n + 4\beta) + n \to$$
$$(-6\alpha + 1)n + 6\alpha - 2\beta = 0 \to -6\alpha + 1 = 0, 6\alpha - 2\beta = 0 \to$$
$$\alpha = \frac{1}{6}, \beta = \frac{1}{2} \to b_n = n^2\left(\left(\frac{1}{6}\right)n + \frac{1}{2}\right)3^n \to$$
$$a_n = (\alpha' n + \beta')3^n + n^2\left(\left(\frac{1}{6}\right)n + \frac{1}{2}\right)3^n \to \beta' = 0, \alpha' = -1$$
$$a_n = (1/2)(n^3 + 3n^2 - 6n)3^{n-1}$$