Discrete Structures

Graphs

Graph Terminology

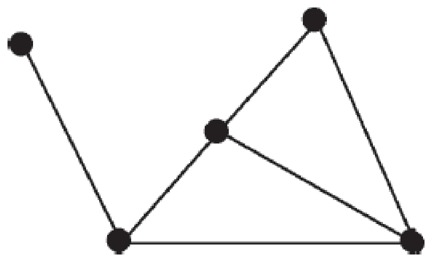
Definition: A graph $G = (V, E)$ consists of two sets, V a set of vertices, and E a set of edges.

- A edge joins two vertices called it its endpoints.
- Two vertices are adjacent if there is an edge joining them.

Graph Terminology

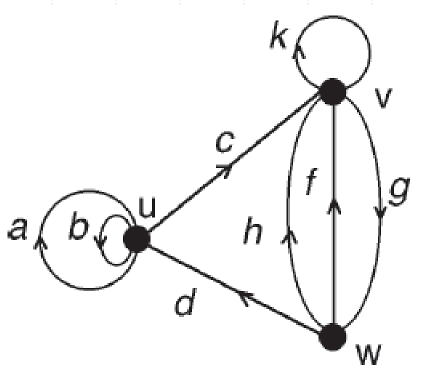
Definition: A graph $G = (V, E)$ is simple if:

- No self-loops
- There is at most one edge between any pair of vertices.



Definitions:

- A edge has direction if an arrow is added to designate what is forward.
- An arc is an edge with a direction
- A digraph (directed graph) is a collection of vertices and arcs
- When (u, v) is a directed edge, u is called the initial vertex and v is called the terminal vertex.



Graph Terminology

Definition: The degree of a vertex v, denoted by $\deg(v)$, is the number of edges incident to v (self-loops contributes 2)

Definition: The in-degree of a vertex v of a digraph $G = (V, E)$, denoted by $deg^{-}(v)$, is the number of edges with v as their terminal vertex. The out-degree, denoted by $deg^{+}(v)$, is the number of edges with v as their initial vertex.

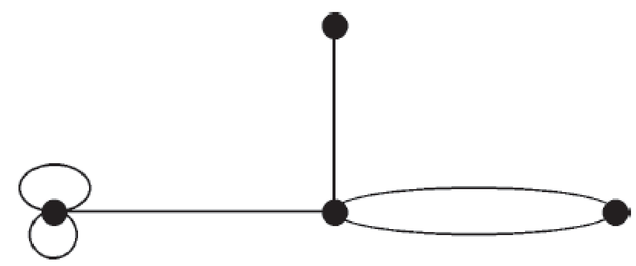
Theorem: Let $G = (V, E)$ be a digraph. Then

$$\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$$

Graph Terminology

Definition: The degree sequence of graph $G = (V, E)$ is a list of all degrees in ascending order

The degree sequence: 1, 2, 4, 5



Problem: Can ``1, 1, 2, 4,4,4" be a degree sequence of a simple graph?

Solution: Let A be all vertices with degree 4 and B be the rest. Each vertex of A must be adjacent with 2 vertices of B. Then the total edges coming out from A is at least 6 but the total edges incident to vertices of B is $1 + 1 + 2 = 4$

Handshaking Theorem

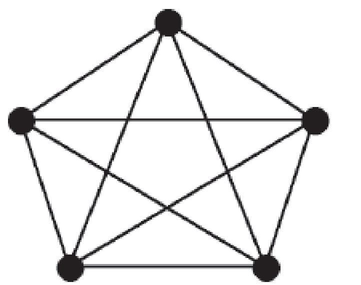
- **Theorem:** The sum of **degrees** is two time the number of edges.
- **Proof:** Each edge contributes two in the sum of degrees
- **Corollary:** A graph has even number of vertices with odd degree

Theorem: If G is a simple graph with at least two vertices, then G has two vertices with the same degree.

Proof: Let n be the number of vertices. The degree of a vertex is at least 0 and at most $n-1$ (n numbers). It is impossible two have two vertices, one with degree 0 and one with degree $n-1$. Then we have n numbers from a set of size at most $n-1$. Then based on the pigeonhole principle, two degrees should be the same.

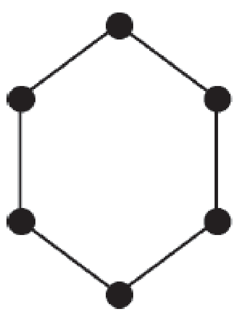
Special Simple Graphs

A Complete Graph is a simple graph with every pair of vertices joined by an edge. A complete graph with n vertices is denoted by K_n.



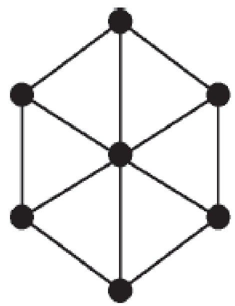
A Path Graph has vertices $\{v_1, v_2, \dots, v_n\}$ and edges $\{e_1, e_2, \dots, e_{n-1}\}$ s.t. e_k joins vertices $\{v_k, v_{k+1}\}$. This graph is denoted by P_n.

A Cycle Graph has vertices $\{v_1, v_2, \dots, v_n\}$ and edges $\{e_1, e_2, \dots, e_n\}$ s.t. e_k joins vertices $\{v_k, v_{k+1}\}$ where v_{k+1} is mod n. This graph is denoted by C_n.

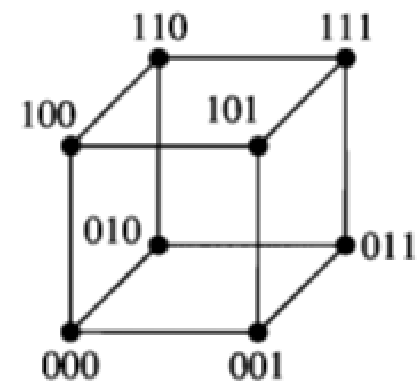


Special Simple Graphs

A Wheel Graph is a cycle graph with an additional hub vertex joined to every other vertex. A wheel graph with $n+1$ vertices is denoted by W_n.



The n-dimensional hypercube, or n-cube, denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent iff the bit strings that they represent differ in exactly one bit position.



A graph is regular if every vertex has the same degree.

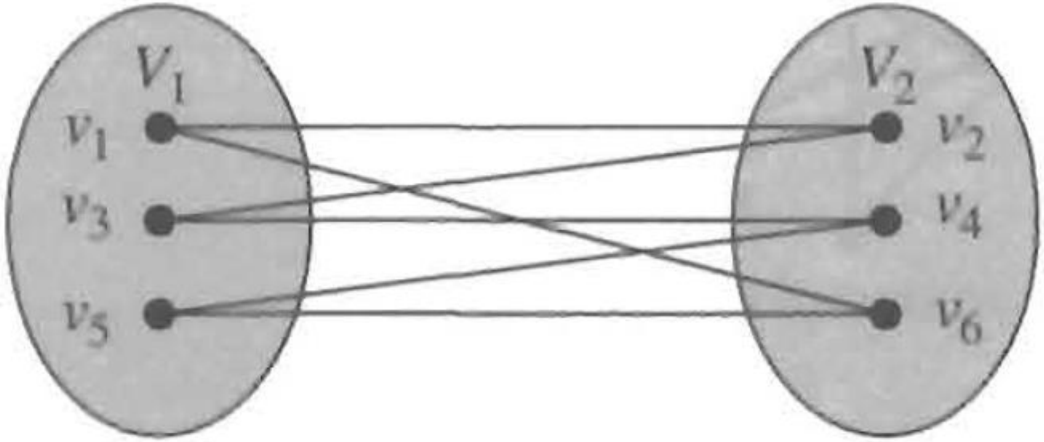
Examples: K_n, C_n, W_3

Bipartite Graphs

Definition: A simple graph G is called **bipartite** if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 to a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2).

$K_{n,m}$: A bipartite graph where $|V_1| = n, |V_2| = m$ and any vertex in V_1 is connected to any vertex in V_2.

Examples: C_6 is bipartite but K_3 is not.



Bipartite Graphs

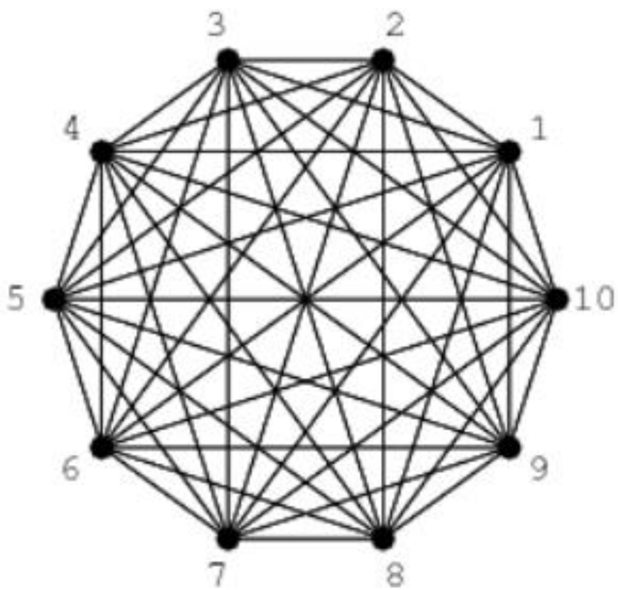
Theorem: A simple graph is bipartite simple iff it is possible to assign one of two colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

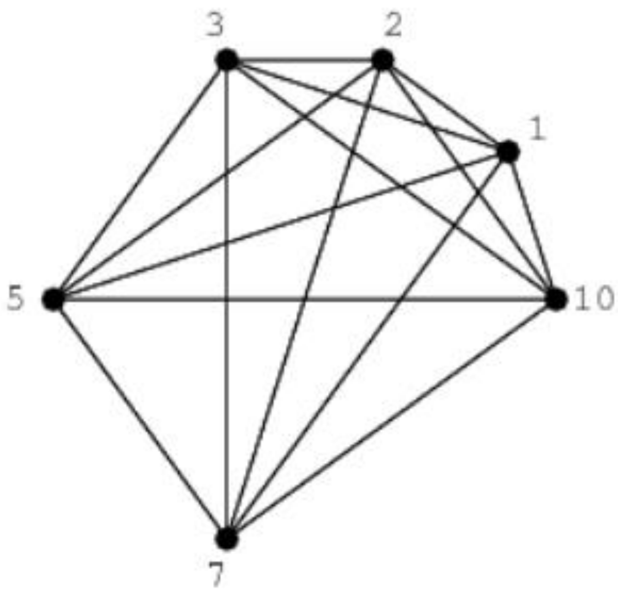
Proof: Suppose that it is possible to color the vertices with red and blue s.t. no two adjacent vertices are assigned the same color. Let V_1 be red vertices and V_2 be the blue vertices. Then we have a bipartite partition. The proof of the other way is also simple.

Subgraphs

Definition: A subgraph of a graph $G = (V, E)$ is a graph $H = (V', E')$, where $V' \subseteq V, E' \subseteq E$. If $H \neq G$, H is called a proper subgraph.

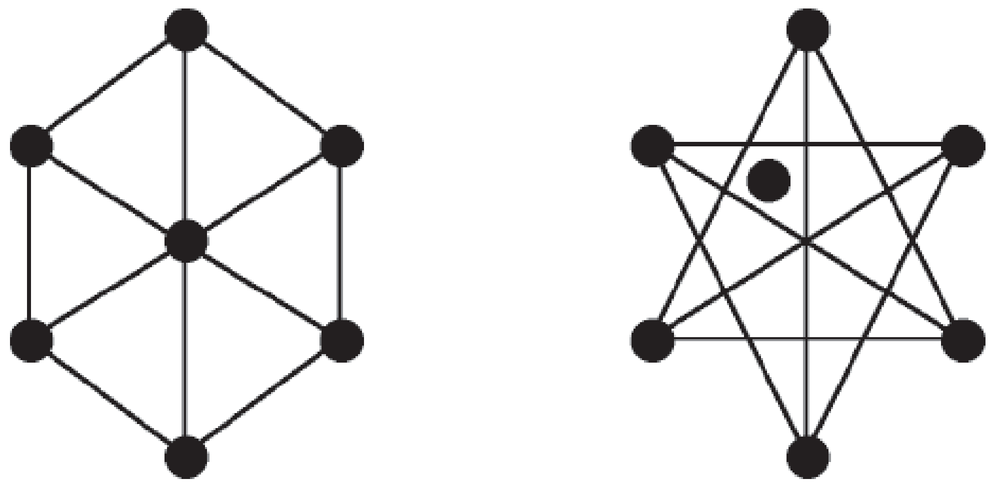
Definition: A induced subgraph of a graph $G = (V, E)$ is a graph formed from a subset V' of V and all edges of E connecting pairs of vertices in V'.



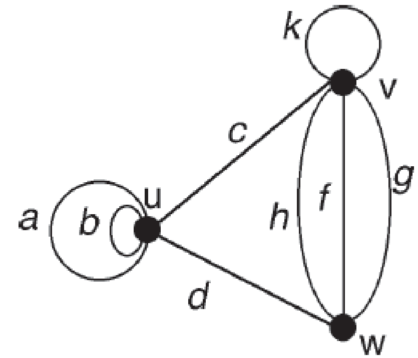


Edge-Complement Graph

Definition: The edge-complement of a simple graph $G = (V, E)$ is a graph $G^c = (V, E^c)$, where E^c is the complement of E.



Graph Representation



Representations of the given graph:

Incident Table:

edge	a	b	c	d	f	g	h	k
endpts	u	u	u	w	v	v	w	v
	u	u	v	u	w	w	v	v

Incident Matrix

	a	b	c	d	f	g	h	k
$u.$	2	2	1	1	0	0	0	0
$v.$	0	0	1	0	1	1	1	2
$w.$	0	0	0	1	1	1	1	0

Adjacency List

$u.$	u	u	v	w	
$v.$	u	v	w	w	w
$w.$	u	v	v	v	

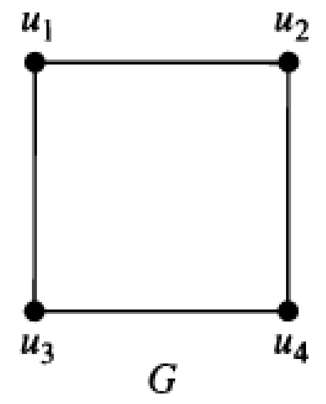
Adjacency Matrix

	u	v	w
$u.$	2	1	1
$v.$	1	1	3
$w.$	1	3	0

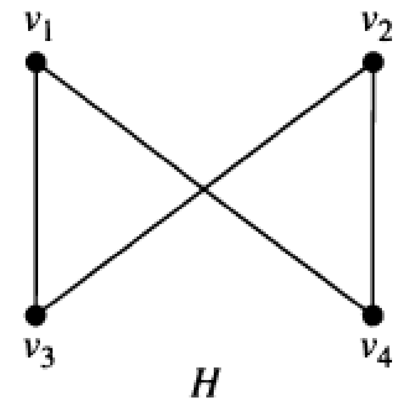
Isomorphism of Graphs

Definition: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is one-to-one and onto function $f: V_1 \rightarrow V_2$ with the property that v and u are adjacent in G_1 iff $\mathrm{f}(v)$ and $\mathrm{f}(u)$ are adjacent in G_2 for all v and u in V_1.

Example:



$f(u_1) = v_1, f(u_2) = v_4,$
$f(u_3) = v_3, f(u_4) = v_2$



Isomorphism of Graphs

- A property of a graph is said to be preserved under isomorphism if whenever G has that property, every graph isomorphic to G also has that property. A preserved property under isomorphism is called a graph invariant.

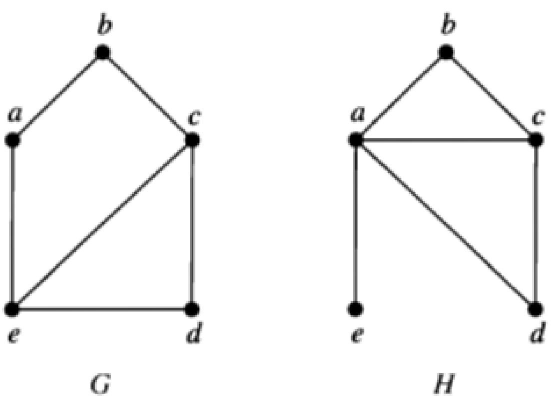
Graph invariants:

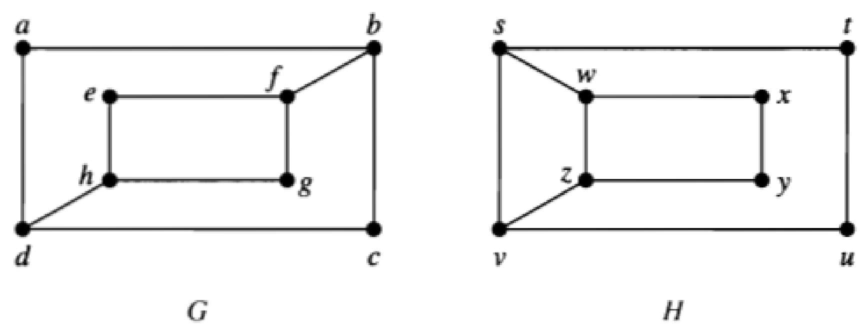
- Number of edges
- Number of vertices
- Degree sequence
- Connection properties

Isomorphism of Graphs

- It is often difficult to determine whether two graphs are isomorphic. There are $n!$ one-to-one correspondence between two sets of vertices with n vertices.
- Sometime it is not hard to show two graphs are not isomorphic. Indeed, if we can find a graph invariant only one of the two graphs has, then we are done.

Examples of non-isomorphic graphs





Path and Circuit

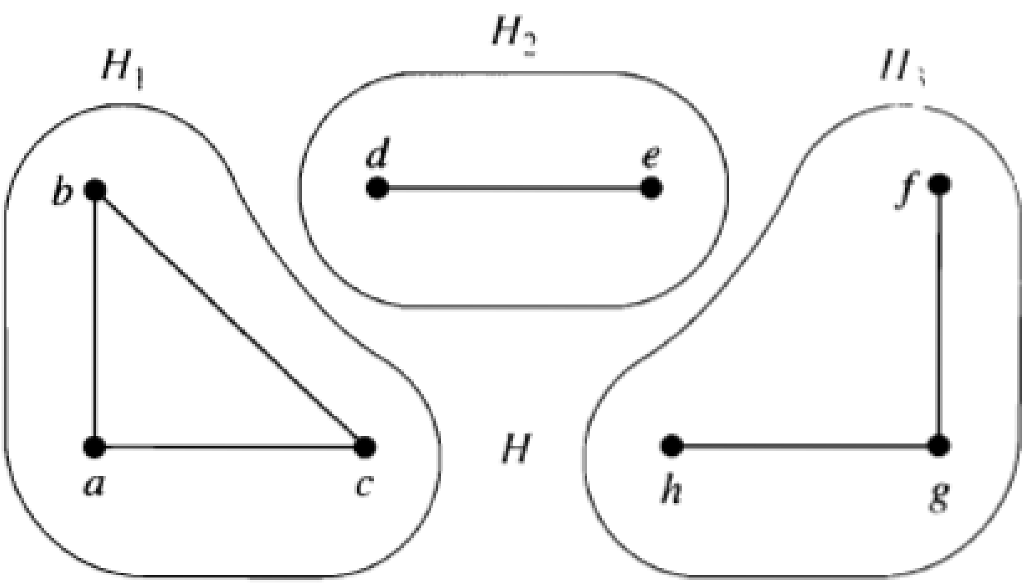
Definition: Let k be a non-negative integer and G an undirected edge. A path of length k from u to v in G is a sequence of k edges $e_1, e_2, \dots, e_k$ of G such that $e_i = \{v_i, v_{i+1}\}, v_1 = u, v_{k+1} = v$. When the graph is simple we can show the path by its vertex sequence $v_1, v_2, \dots, v_{k+1}$.

This path is a circuit if it begins and ends at the same vertex. A path or circuit is simple it does not contain the same edge more than once.

Connectedness

Definition: An undirected graph is called **connected** if there is a path between every pair of distinct vertices of the graph.

Definition: A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G.

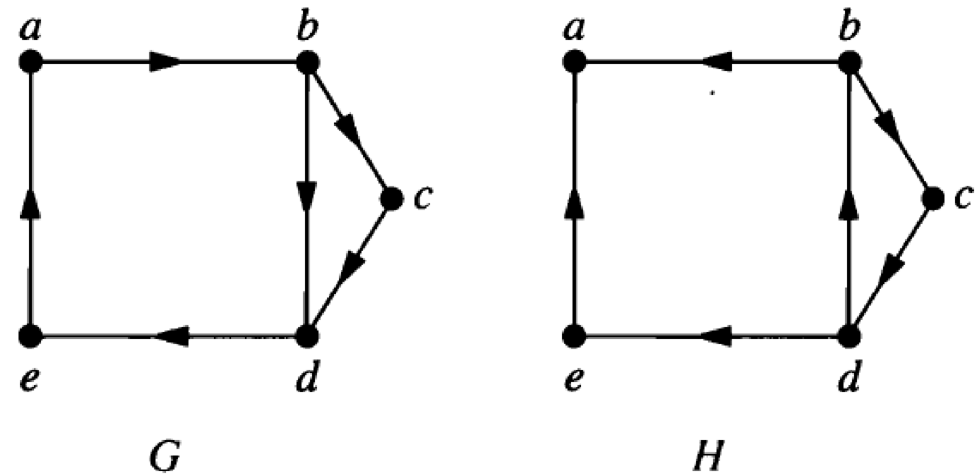


Connectedness in Digraphs

Definition: A digraph is **strongly connected** if there is a path from u to v and from v to u for any two vertices u and v.

Definition: A digraph is **weakly connected** if there is a path between any two vertices in the underlying undirected graph.

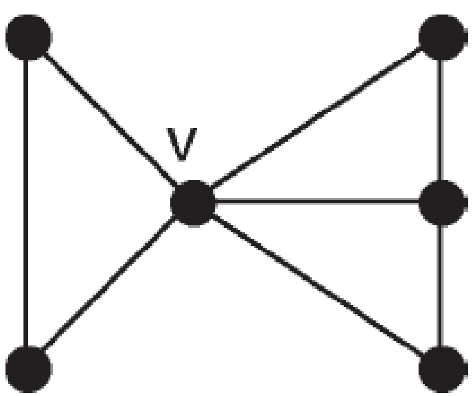
Definition: A subgraph of a digraph G is a strongly connected component iff it is strongly connected but not contained in larger strongly connected subgraphs.

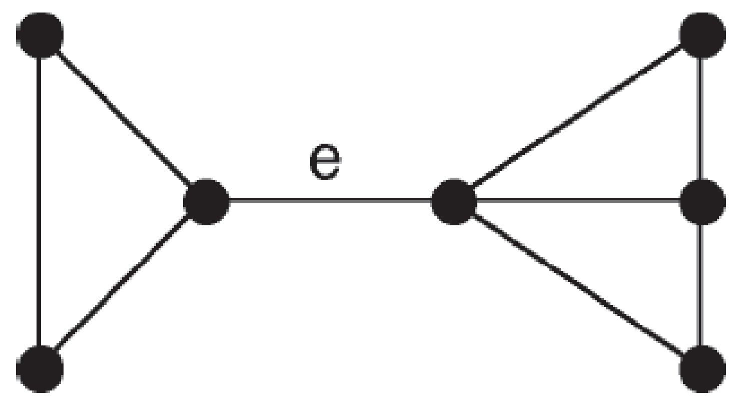


Cut Vertex and Cut Edge

Definition: a cut vertex of a undirected graph is a vertex whose removal increases the number of component.

Definition: a cut edge of a undirected graph is an edge whose removal increases the number of component.





Seven Bridges of Konigsberg

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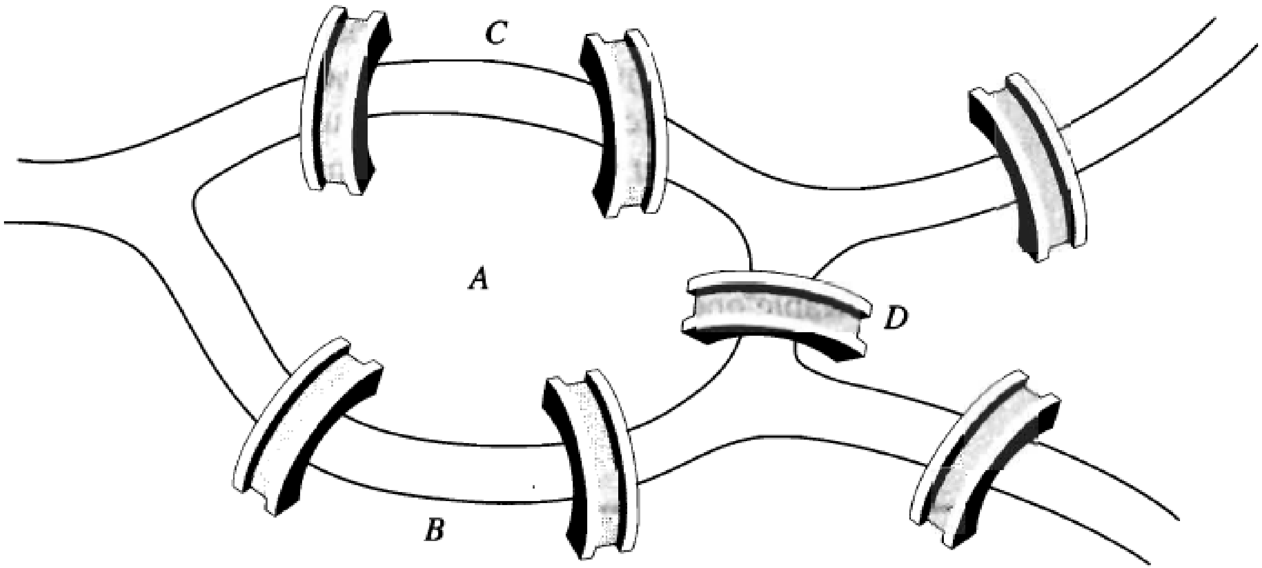
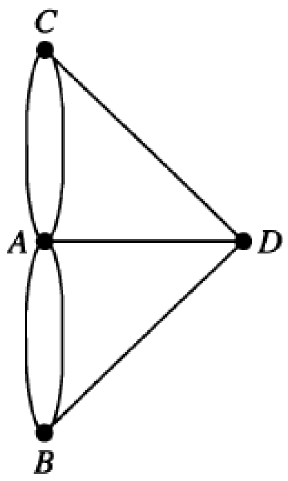




Euler Circuit and Path

Definition: An **Euler circuit** in a graph G is a simple circuit containing every edge of G. An **Euler path** in G is a simple path containing every edge of G.

Theorem 1: A connected multigraph with at least two vertices has an Euler circuit iff each vertices has even degree.

Theorem 2: A connected multigraph with at least two vertices has an Euler path not an Euler circuit iff it has exactly two vertices of odd degree.

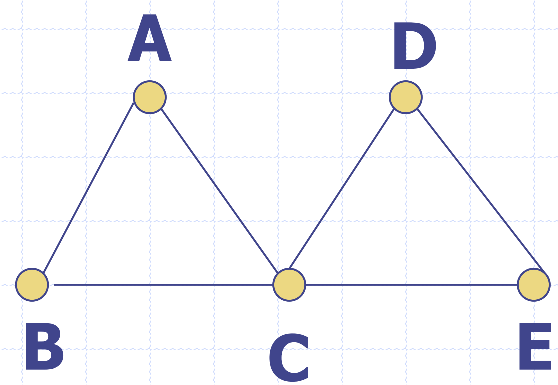
Theorem 2 can be simply proved using Theorem 1.

Euler Circuits and Paths

Proof of Theorem 1:

Let G has an Euler Circuit. Traverse this circuit staring from an arbitrary vertex A. When we enter a vertex other than A, we must exit it. Then for each entrance edge we have an exit edge. Since we pass all edges, then degree of all vertices other than A is even. The same argument is valid for A; only the first exit edge incident to A should be paired with the last entrance edge incident to A.

For the inverse, it seems walking arbitrarily gives us an Euler circuit. But as you see in the figure, if we follow ``A, B, C, A", then we get stuck.



Euler Circuits and Paths

Proof of Theorem 1:

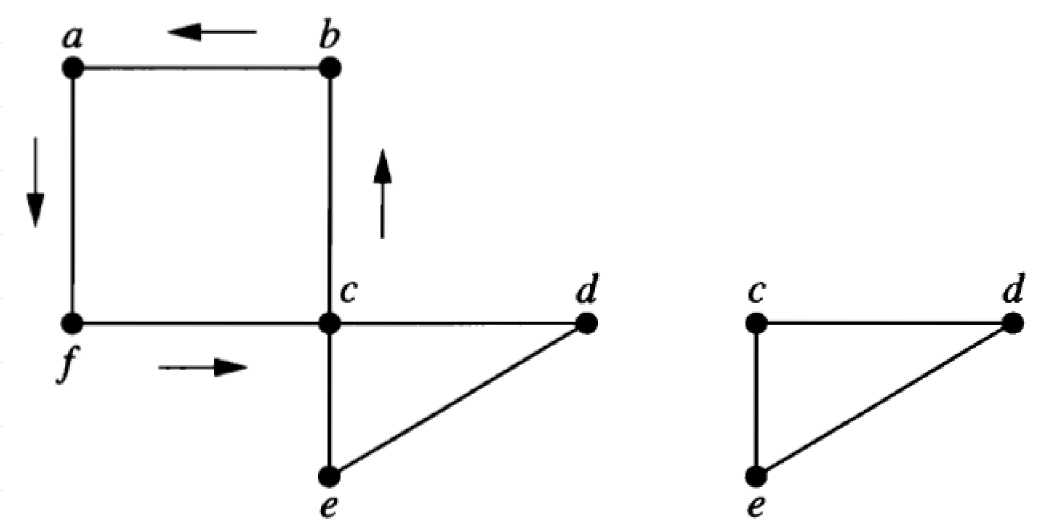
For the inverse we give an algorithm using induction to find an Euler circuit.

- Start walking from an arbitrary vertex A until you find a circuit C. We can always find such a circuit as when we enter a vertex, we can leave that vertex (note that the degree of each vertex is even).
- If the circuit C is an Euler circuit we are done.
- Otherwise, Consider the graph $G - C$ which is a proper subgraph of G and all degree in $G - C$ is even.
- Let $G - C$ has k connected components $G_1, \dots, G_k$.
- Based on the induction hypothesis, each G_i has an Euler Circuit.

Euler Circuits and Paths

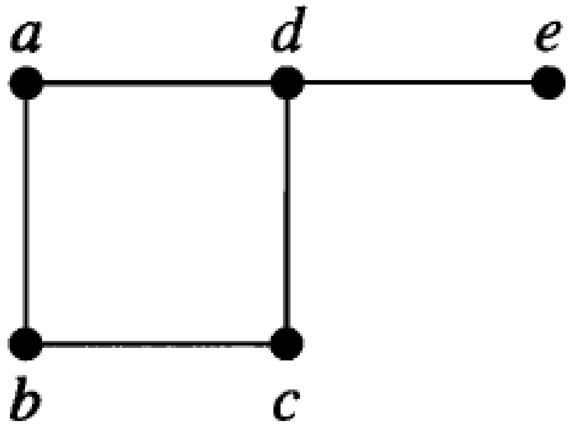
Proof of Theorem 1:

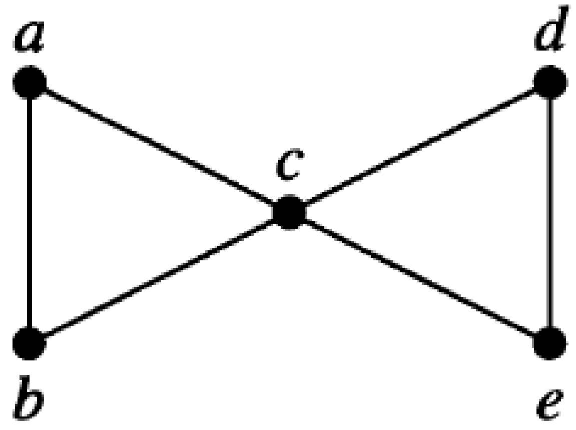
- Now the algorithm is as follows. Walk through C. When you reach a vertex v and v is a vertex of some G_i, use the Euler circuit of G_i to see all edges of G_i and get back to v. Just remember when you see another vertex of G_i while walking C, skip seeing all edges of G_i again.



Hamilton Circuits and Paths

Definition: An **Hamilton circuit** in a graph G is a simple circuit passing through every vertex of G exactly once. An **Hamilton path** in G is a simple path passing through every vertex of G exactly once.





Hamilton Circuits and Paths

Ore's Theorem: If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices u and v in G, then G has a Hamilton circuit.

Corollary: If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

Hamilton Circuits and Paths

Proof of Ore's Theorem: For the sake of contradiction, Assume G does not have a Hamilton circuit.

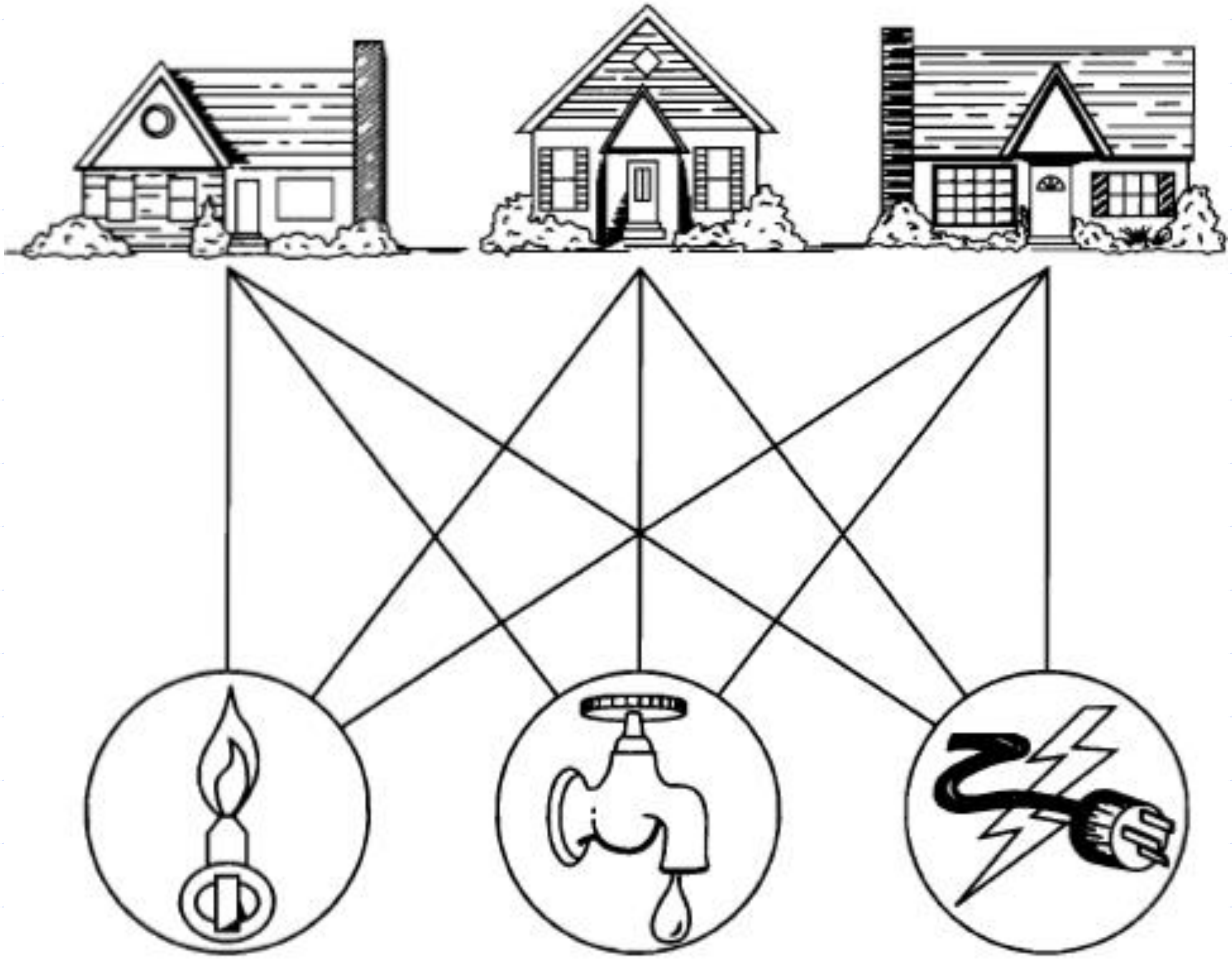
- Add as many edges as possible to G without producing a Hamilton circuit. Let 's call this graph H.
- Since G is a subgraph of H, we have $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices u and v in H.
- Note that adding any edge to H produces a Hamilton circuit.
- H is not a complete graph, since a complete graph does have a Hamilton circuit.
- Assume u and v are two non-adjacent vertices in H.
- Adding edge $\{u, v\}$ to H gives a Hamilton Path. So there is a Hamilton path from u to v.

Hamilton Circuits and Paths

Proof of Ore's Theorem:

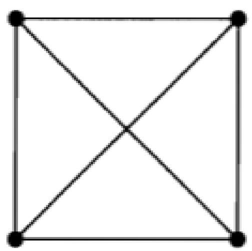
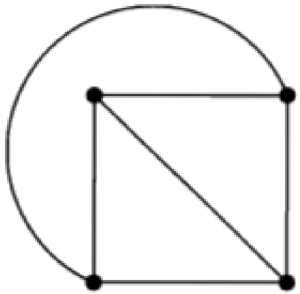
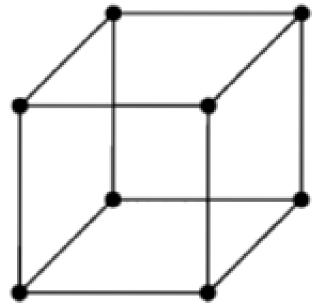
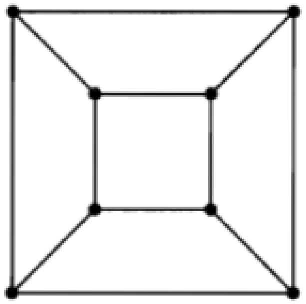
- Let $x_1, x_2, \dots, x_n$ be this Hamilton path where $x_1 = u, x_n = v$, and $\{x_1, x_2, \dots, x_n\}$ the set of all vertices of H.
- We claim that there exist some k such that edges $\{x_1, x_{k+1}\}, \{x_k, x_n\}$ exist in H. If so, The circuit $x_1, x_2, \dots, x_k, x_n, x_{n-1}, \dots, x_{k+1}, x_1$ is a Hamilton circuit which is a contradiction.
- To prove the claim, we can use the contradiction method as well.
- In one hand, we know $\deg(x_1) + \deg(x_n) \geq n$.
- Let $\{x_{i_1}, \dots, x_{i_t}\}$ be all adjacent vertices to x_1 where $t = \deg(x_1)$. If none of $x_{i_1 - 1}, \dots, x_{i_t - 1}$ are adjacent to x_n, then $\deg(x_n) \leq n - 1 - t < n - t = n - \deg(x_1)$. So we have $\deg(x_1) + \deg(x_n) < n$ which is a contradiction.

Planar Graphs



Planar Graphs

Definition: A graph G is called planar if it can be drawn in the plane without any edge crossing (where a crossing of edges is the intersection of the line or arcs representing them at a point other than their common endpoint). Such a drawing is called a planar representation of the graph or a plane graph of G.

Planar Graphs

Problem: Show that $K_{3,3}$ is not planar.

Proof: One elementary proof uses the following fact

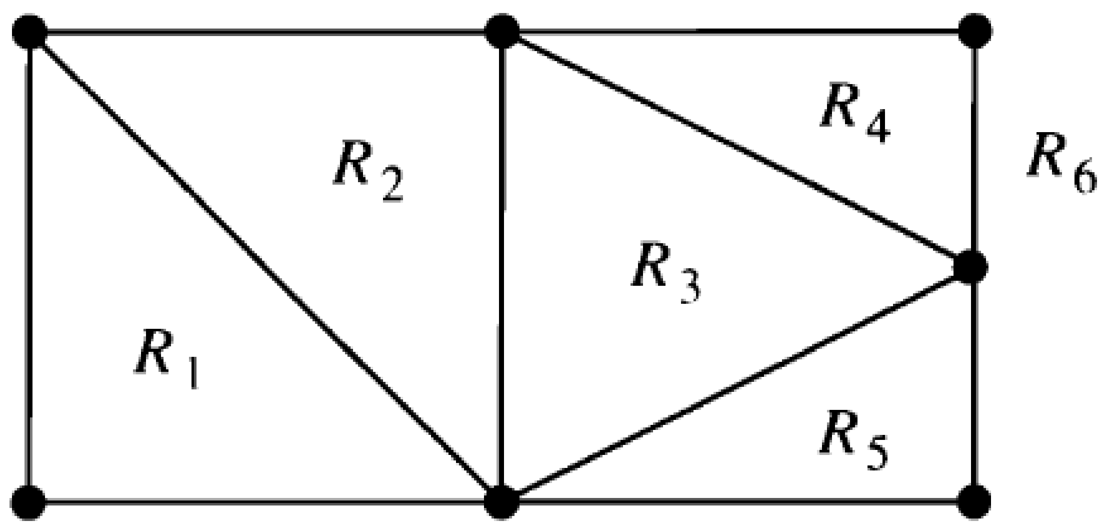
- If C be a closed curve and we want to connect a point inside C to a point outside C, we should intersect C.

Try yourself to use this fact and solve the problem. We later explain a combinatorial proof based on the Euler's formula for planar graphs.

Euler's Formula

A plane graph G splits the plane into regions (faces) including a unbounded region like the given example below. The number of regions (r) has been proved to be dependent only on the number of vertices (v) and the number of edges (e).

Theorem: $r = e - v + 2$



Euler's Formula

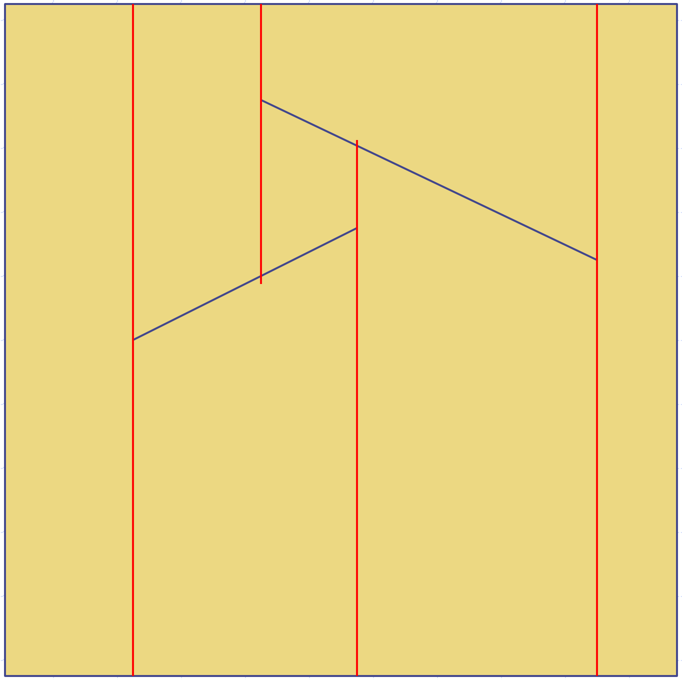
Proof:

We fix v and run induction on e. The smallest value for e to have a connected graph is $v-1$ when G is a tree (we prove this later). When G is a tree, the number of regions is 1 and then we have $1=(v-1)-v+2$ which of course is true.

Now let G have at least v edges (i.e. $e \geq v$). Then G has a circuit. Let $\{u,v\}$ be an edge of G in a circuit. Since $\{u,v\}$ is on a circuit, $\mathrm{G}'=G-\{u,v\}$ is connected. Let e', v', r' be the number of edges, vertices and regions of G'. It is clear that $e'=\mathrm{e}-1, \mathrm{v}'=\mathrm{v}, \mathrm{r}'=\mathrm{r}-1$. Since based on the induction hypothesis $\mathrm{r}'=e'-v'+2$, then we have $r=e-v+2$.

Euler's Formula

Problem: Given n disjoint segments inside a square. From each endpoint we draw a vertical line until it hits other segments or the edges of the square. Find the number of trapezoids. You can assume no two endpoint have the same x-coordinate. In the example given blow, the answer is 7 for two segments.



Euler's Formula

Solution: We can see the resulting drawing as a plane graph. We have to count the number of vertices and 'edges. Then by Euler's formula we have the number of regions which are trapezoids except the unbounded region. The number of vertices is $\mathrm{v} = 4 + 2n + 4n$ (four vertices of the square plus endpoints of n segments plus $4n$ vertices created by the $2n$ vertical lines). Before drawing the vertical segments, the number of edges is $n + 4$. Each vertical increase the number of edges by 4.

Then, $e = n + 4 + 4 \times 2n = 9n + 4$. So, $r = (9n + 4) - (6n + 4) + 2 = 3n + 2$. We should exclude the unbounded region. Then, the number of trapezoids is $3n + 1$.

The other solution is to compute the sum of the angles in two ways. Let x be the number of trapezoids. Then,

$$360x = 360 + 180 \times 6n \rightarrow x = 3n + 1$$

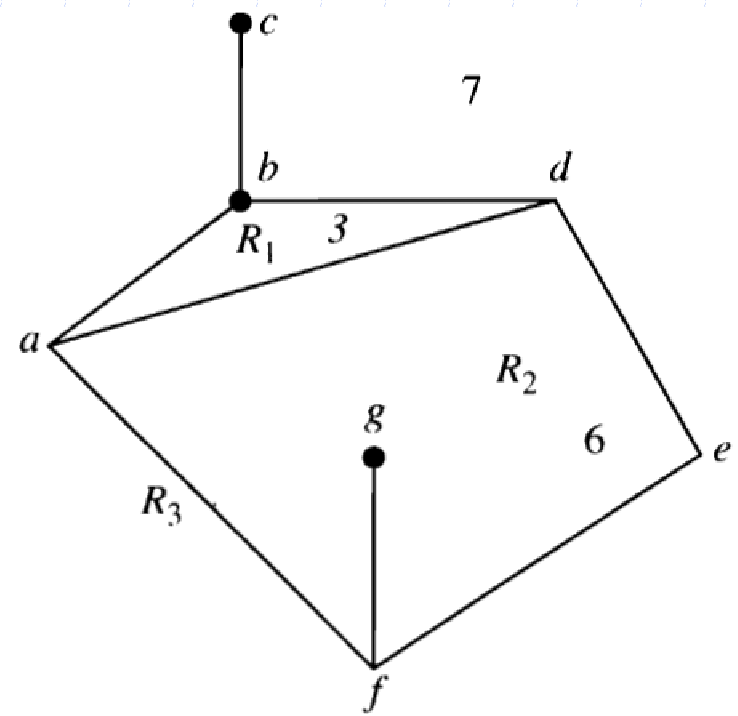
Bound on The Number of Edges

Theorem: if G is a connected planar simple graph with e edges and v vertices ($v \geq 3$), then $e \leq 3v - 6$.

Proof: It is easy to see $\sum_R |R| = 2e$, where R is a region and $|R|$ is the number of edges in the boundary of R. Since $|R| \geq 3$, we get $3r \leq 2e \rightarrow 3(e - v + 2) \leq 2e \rightarrow e \leq 3v - 6$.

Theorem: if G is a connected planar simple bipartite graph with e edges and v vertices ($v \geq 3$), then $e \leq 2v - 4$.

Proof: the proof is similar to the one given above. Just use the fact that $|R| \geq 4$.



Two Known Non-Planar Graphs

Theorem: K_5 is not planar.

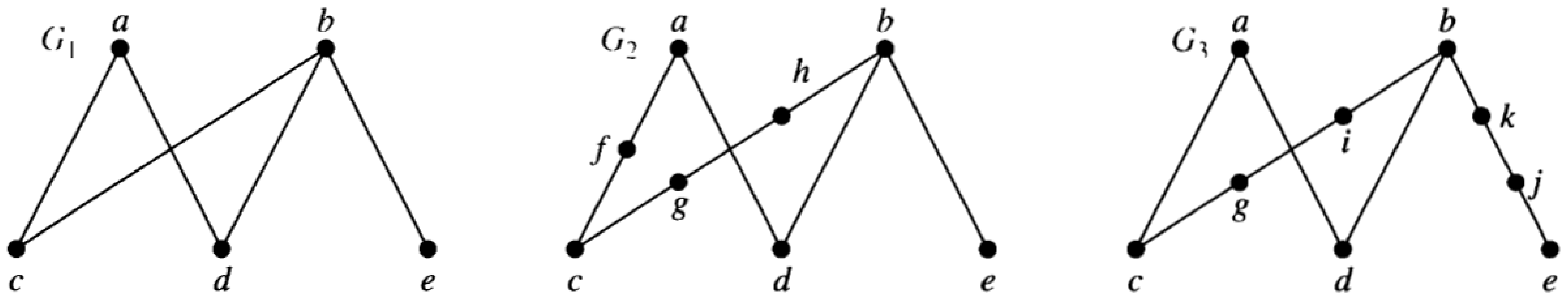
Proof: K_5 has 10 edges and does not satisfies $e \leq 3v - 6$ $(10 \leq 3 \times 5 - 6 = 9)$

Theorem: $K_{3,3}$ is not planar.

Proof: $K_{3,3}$ has 9 edges and does not satisfies $e \leq 2v - 4$ $(9 \leq 2 \times 6 - 4 = 8)$

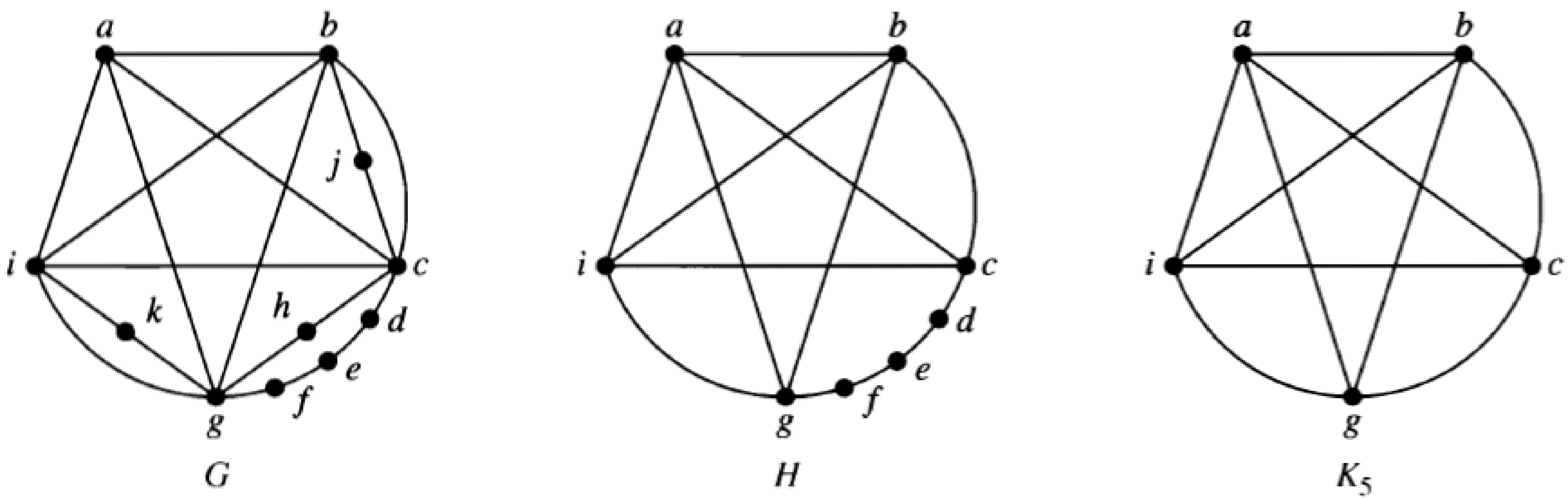
Homeomorphic Graphs

Definition: if a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$, and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an elementary subdivision. The graph G_1 and G_2 are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.



Kuratowski's Theorem

Theorem: *A graph is nonplanar iff it contains a subgraph homeomorphic to K_5 and $K_{3,3}$.*

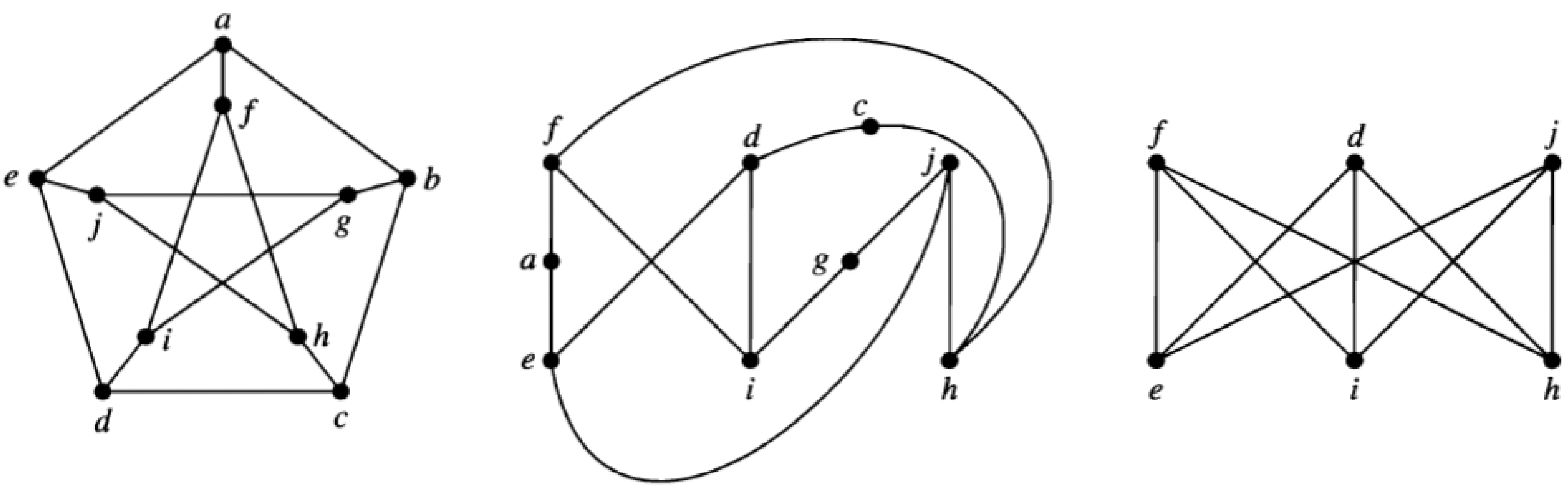


The undirected graph G, and its subgraph H homeomorphic to K_5.

Petersen Graph

Problem: Is the Petersen graph is planar?

Solution: The Petersen graph has a subgraph homeomorphic to $K_{3,3}$ as illustrated below.



Graph Coloring

Definition: A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Definition: The chromatic number of a graph is the least number of colors needed for a coloring of the graph. The chromatic number of a graph G is denoted by $\chi(G)$.

It is clear$\chi(K_n) = n, \chi\left(K_{n,m}\right) = 2.\chi(C_n) = 2 + n\ mod\ 2.$

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χ(Planar Graphs)

Theorem: If G is a connected planar graph, then $\chi(G) \leq 6$

Proof:

- The minimum degree of G is at most 5 as we know $e \leq 3v - 6$.
- Remove the vertex u with the min degree and use the induction hypothesis to color remaining vertices by 6 colors.
- Since u has 5 neighbors, then at most 5 colors are forbidden to be used for u. Therefore, there is one free color to color u.

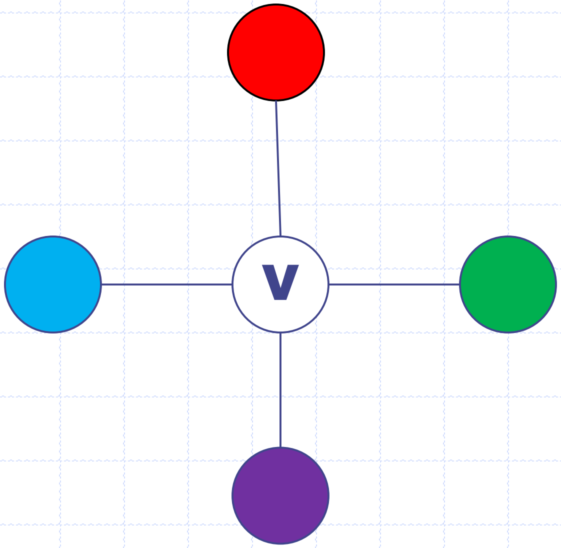
X(Planar Graphs)

Theorem: If G is a connected planar graph, then $\chi(G) \leq 5$

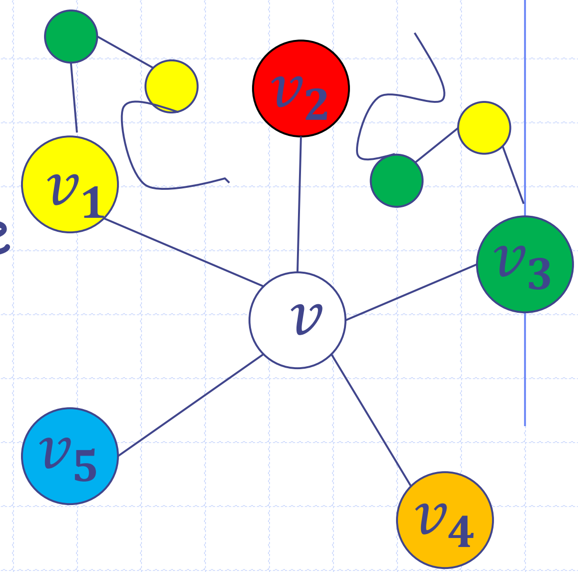
Proof:

Proof by contradiction

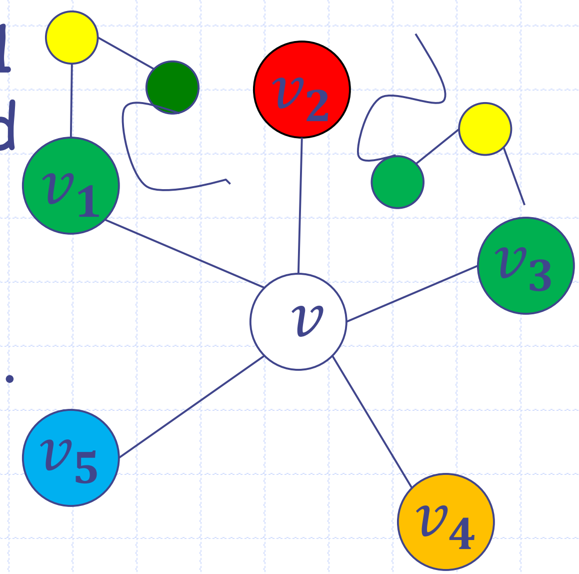
- Let G be the smallest planar graph (in term of the number of vertices) that can not be colored with five vertices.
- Let v be a vertex of G that has the minimum degree. We know $\deg(v) < 6$.
- Case 1: $\deg(v) < 6$. $G - \mathrm{v}$ can be colored with five colors. There are at most 4 colors that have been used on the neighbors of v. There is a free color for v.



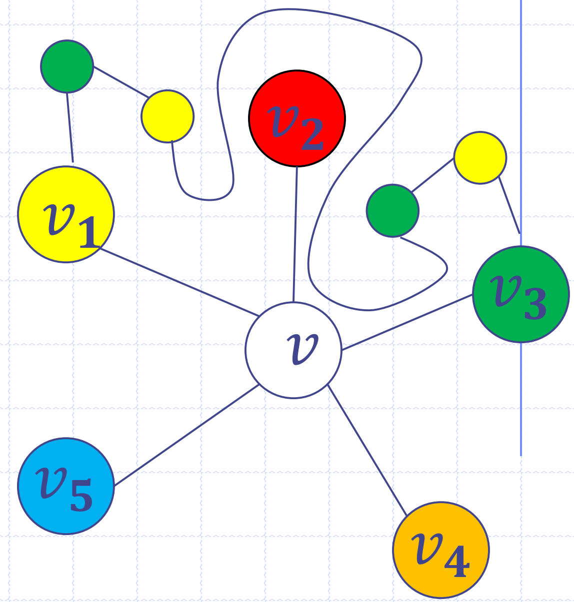
X(Planar Graphs)



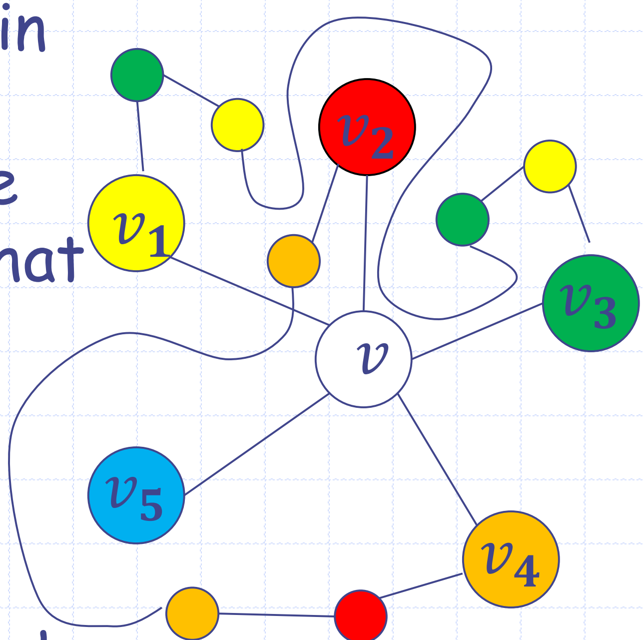
- Case 2: $\deg(v) = 5$. $G - \mathrm{v}$ can be colored with five colors. If two of the neighbors of v have the same color, then there is a free color for v.
- Assume vertices adjacent to v are colored with 1,2,3,4, and 5 in the clockwise order.
- Consider the induced graph on colors 1 and 3. if this subgraph is disconnected and v_1 and v_3 are in different components, then we can switch the color 1 and 3 in the component with v_1.
- This is still a 5-coloring of $G - v$. Moreover, v_1 is colored with 3. Now the color 1 is available for v.



X(Planar Graphs)



- Now let consider the case v_1 and v_3 are in the same component, i.e. there is path from v_1 to v_3 such that every vertex on this path is colored with either 1 or 3.
- Now, consider the induced subgraph on vertices with colors 2 and 4. Again if v_2 and v_4 are in different components, we are done. Otherwise there is a path from v_2 to v_4 such that every vertex on this path is colored with either 2 or 4.
- This means that there must be two edges cross each other. This contradicts the planarity of the graph



The Four Color Theorem

Theorem: If G is a connected planar graph, then $\chi(G) \leq 4$

Proof: It was open problem for more than 100 years. It was finally proved in 1976. The proof was relied on a careful case-by-case analysis carried out by computer. It was shown that if the four color theorem were false, there would have to be a counterexample of one of approximately 2000 different types, and it was shown that none of these types exists consuming about 1000 hours of computer time.

Applications of Coloring

Problem: how can the final exams at a university be scheduled so that no student has two exams at the same time?

Solution:

- This scheduling problem can be solved using a graph model, with vertices representing courses and with an edge between two vertices if there is a common student in the courses they represent.
- Each time slot for a final exam is represented by a different color. Therefore, a scheduling of the exams corresponds to a coloring of the associated graph.

Applications of Coloring

Problem: Television channels 1 through 17 are assigned to stations so that no two stations within 150km can operate on the same channel. How can the assignment of channels be modeled by graph coloring?

Solution:

- Construct a graph by assigning a vertex to each station.
- Two vertices are connected by an edge if they are located within 150km of each other.
- An assignment of channels corresponds to a coloring of the graph, where each color represent a different channel.

Applications of Coloring

When a map is colored, two regions with a common border are customarily assigned different colors to have better visualization. One way to ensure that no two adjacent regions never have the same color is to use a different color for each region. However, this is inefficient. We can model the map with a graph (called a dual graph) and apply the 4-color theorem to it.