Discrete Structures

Countable and Uncountable Sets

Cardinality

Def: The cardinality of a set A is the number of elements in the set and denoted by $|A|$

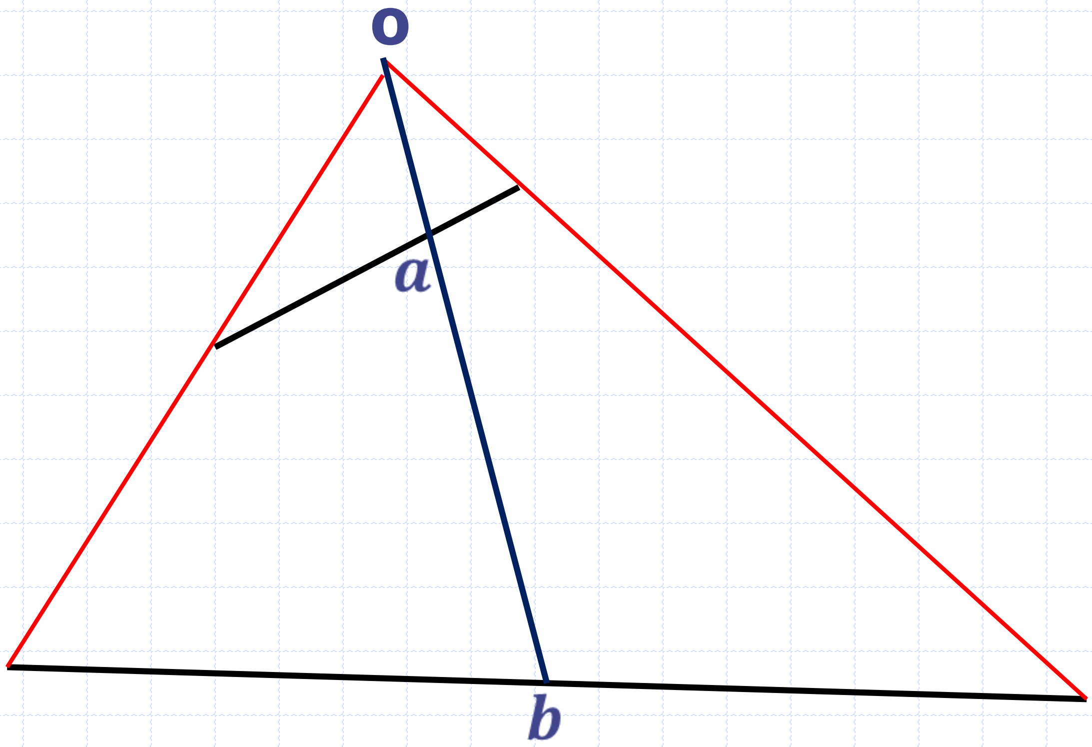
Def: Let A and B be two sets.

- $|A|=|B|$ iff there is a **one-to-one** and **onto** function (**bijection**) from A to B
- $|A| \leq |B|$ iff there is a **one-to-one** function from A to B
- If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$

  (Schröder-Bernstein theorem: No need to know the proof)

Cardinality

Cardinality



Countable Sets and Uncountable Sets

Def: Set A is **countable** if it is finite or if it has the same cardinality as the set of positive integers. Otherwise it is **uncountable**.

Examples:

- Infinite Countable Sets: N, Z^-, Z
- Infinite Uncountable Sets: R, R^+, R^-

Countable Sets and Uncountable Sets

How do you demonstrate that a set is countable ?

Suppose A is a set. If there is a one-to-one and onto function $f : A \to N$, then A is countable. Recall,

one-to-one means $\forall x \forall y (f(x) = f(y) \to x = y)$

onto means $\forall y \exists x (f(x) = y)$

Countable Sets and Uncountable Sets

Theorem:

Set **{ x | x is an odd postive integer}** is countable.

Proof: We need a one-to-one correspondence between this set and **N**

1, 3, 5, 7, 9, … corresponds to a_1, a_2, a_3, a_4, a_5 …

We could also consider f(2n-1) = n from the set of odd positive integers to **N**.

Countable Sets and Uncountable Sets

Theorem:

The set **Z** is countable.

Proof: List them like this: 0, -1, 1, -2, 2, -3, 3, -4, 4 …

Which corresponds to $a_1, a_2, a_3, a_4, a_5, a_6$ …

What we've actually done is given the one-to-one correspondence between all integers and the positive integers, i.e., the mapping from **Z** to **N**

What about an expression for this?

f(n) = 2n+1 when n is non-negative

f(n) = -2n when n is negative

Countability

Theorem: The set **P** of all ordered pairs of positive integers (n, m) is countable.

Proof: Can we find a one-to-one and onto function from P to **N**?

Countability

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	…
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	…
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	…
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	…
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	…
…	…	…	…	…	…

$$f(m,n) = ?$$

Countability

Easy one-to-one function but not onto from P to N

$$f(m,n) = 2^m 3^n$$

Countability

Easy one-to-one function but not onto from P to N

$$f(m,n) = 2^m 3^n$$

$$|P| \leq |N|$$

It is clear that $|N| \leq |P|$, then $|N| = |P|$

Countability

If A and B are countable sets, then the following are countable sets

- Any subset of A
- $A \cup B$
- $A \cap B$
- $A \times B$

Note that $P = N \times N$ and rational numbers can be seen as a subset of P

Uncountable sets

Theorem: The set of real numbers is uncountable.

If a subset of a set is uncountable, then the set is uncountable. The cardinality of a subset is at least as large as the cardinality of the entire set.

It is enough to prove that there is a subset of R that is uncountable

Theorem: The open interval of real numbers $[0,1) = \{r \in \mathbf{R} \mid 0 \leq r < 1\}$ is uncountable.

Proof by contradiction using the *Cantor diagonalization argument* (Cantor, 1879)

Uncountable Sets: R

Proof using *diagonalization*: Suppose **R** is countable (then any subset say [0,1) is also countable). So, we can list them: $r_1, r_2, r_3, \ldots$ where

$r_1 = 0.d_{11}d_{12}d_{13}d_{14}\ldots$ the d_{ij} are digits 0-9

$r_2 = 0.d_{21}d_{22}d_{23}d_{24}\ldots$

$r_3 = 0.d_{31}d_{32}d_{33}d_{34}\ldots$

$r_4 = 0.d_{41}d_{42}d_{43}d_{44}\ldots$

etc.

Now let $r = 0.d_1d_2d_3d_4\ldots$ where $d_i = 4$ if $d_{ii} \neq 4$

$d_i = 5$ if $d_{ii} = 4$

But r is not equal to any of the items in the list so it's missing from the list so we can't list them after all.

r differs from r_i in the i^{th} position, for all i. So, our assumption that we could list them all is incorrect.

References

- *Section 2.4 of the text book "Discrete Mathematics and its Applications" by Rosen, 6th edition.*