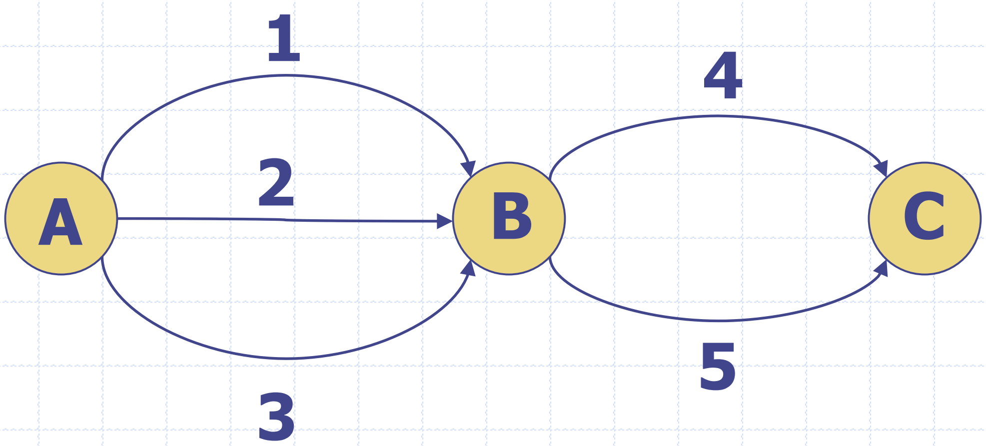
# Discrete Structures

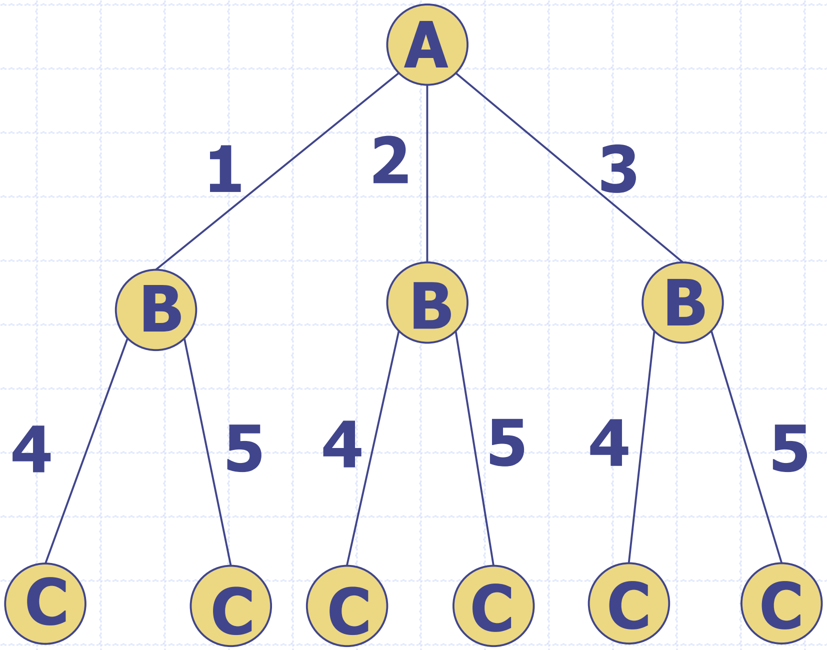
Counting

# Product Rule

Suppose a procedure can be broken into a sequence of two tasks. If there are $n_1$ ways to do the first task, and for each of these ways of doing the first task, there are $n_2$ ways to do the second task, there are $n_1 \times n_2$ ways to do the procedure.

Example: Going from *A* to *C*

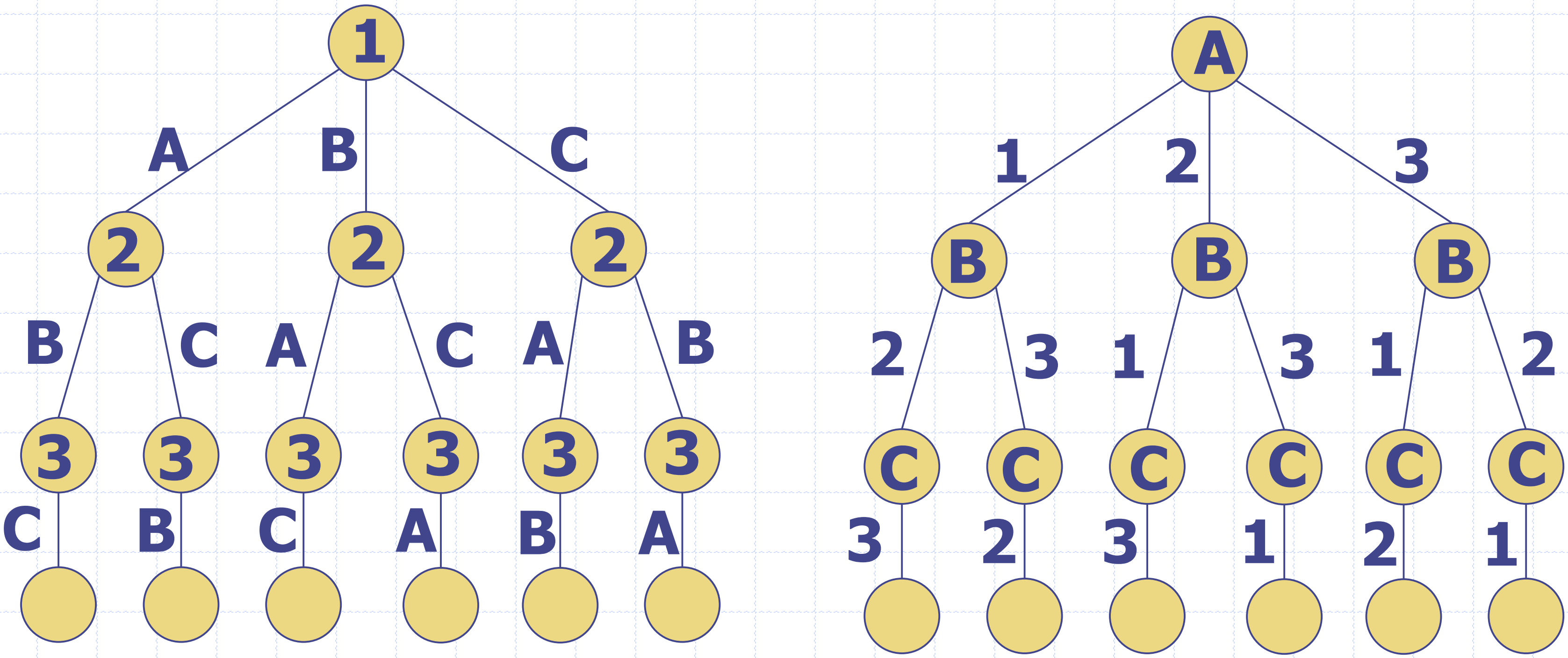


# Tree Diagram

Problem: #placing *A*, *B* and *C* in a row

Positons (from left to right): 1, 2, 3



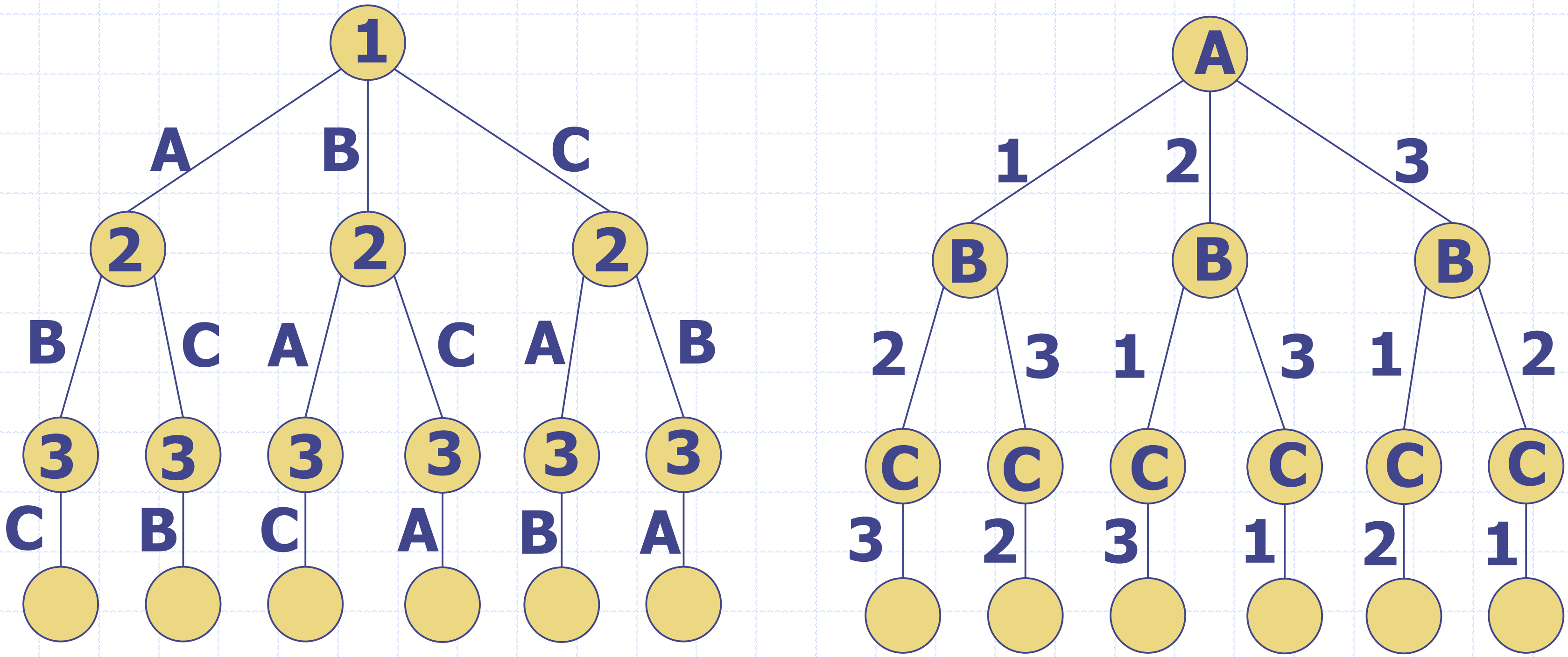

First position 1,
Then position 2,
Then position 3

First place *A*,
Then place B,
Then place *C*

# Tree Diagram

**Problem:** #placing A, B and C in a row

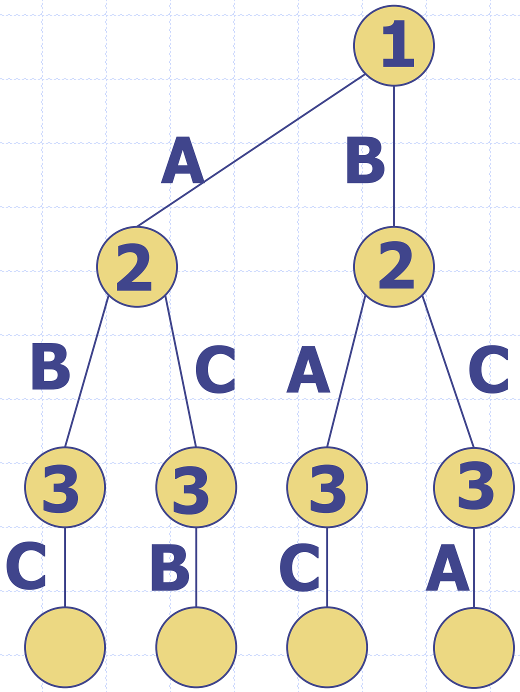
**Positons (from left to right):** 1, 2, 3

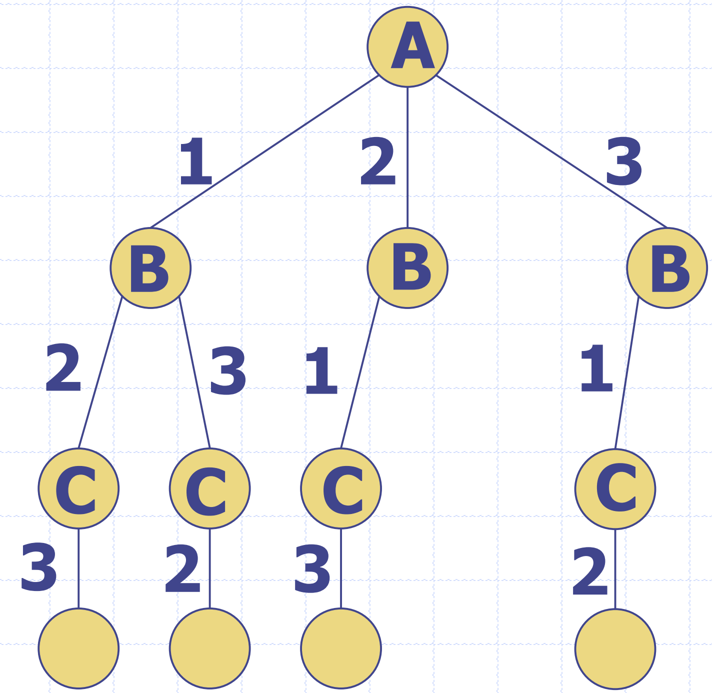



#ways = #leaves=3 × 2 × 1

# Tree Diagram

- Problem: #placing *A*, *B*, *C* in a row where *C* in not first
- Positons (from left to right): 1, 2, 3



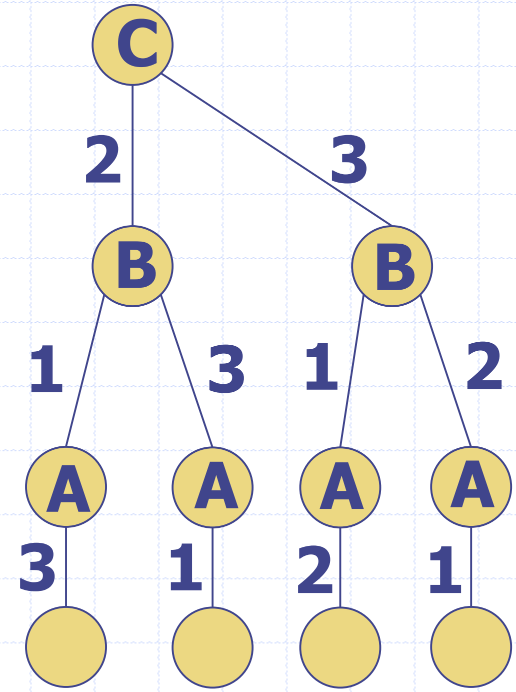

#ways = #leaves=$2 \times 2 \times 1$,

#ways = #leaves=???

# Tree Diagram

Problem: #placing $A$, $B$, $C$ in a row where $C$ is not first

Positons (from left to right): 1, 2, 3




#ways = #leaves=$2 \times 2 \times 1$

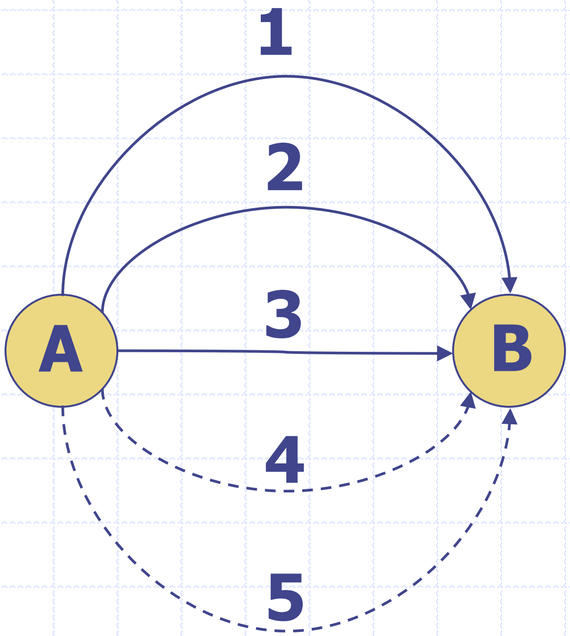
# Tree Diagram and Product Rule

#leaves: If a rooted tree has $k+1$ levels (assume root is at level 1) and each node at level $i$ has $n_i$ children, then the number of leaves of the tree is $n_1 \times n_2 \times \cdots \times n_k$.

Product Rule: Suppose that a procedure is carried out by performing the tasks $T_1, \dots, T_k$ in sequence. If task $T_i$ can be done in $n_i$ ways, regardless of how the previous tasks were done, then there are $n_1 \times n_2 \times \cdots \times n_k$ ways to carry out the procedure.

# Sum Rule

**Sum Rule:** if a task can be done either in one $n_1$ ways or in one of $n_2$ ways, where none of the set of $n_1$ ways is the same as any of the set of $n_2$ ways, then there are $n_1 + n_2$ ways to do the task.



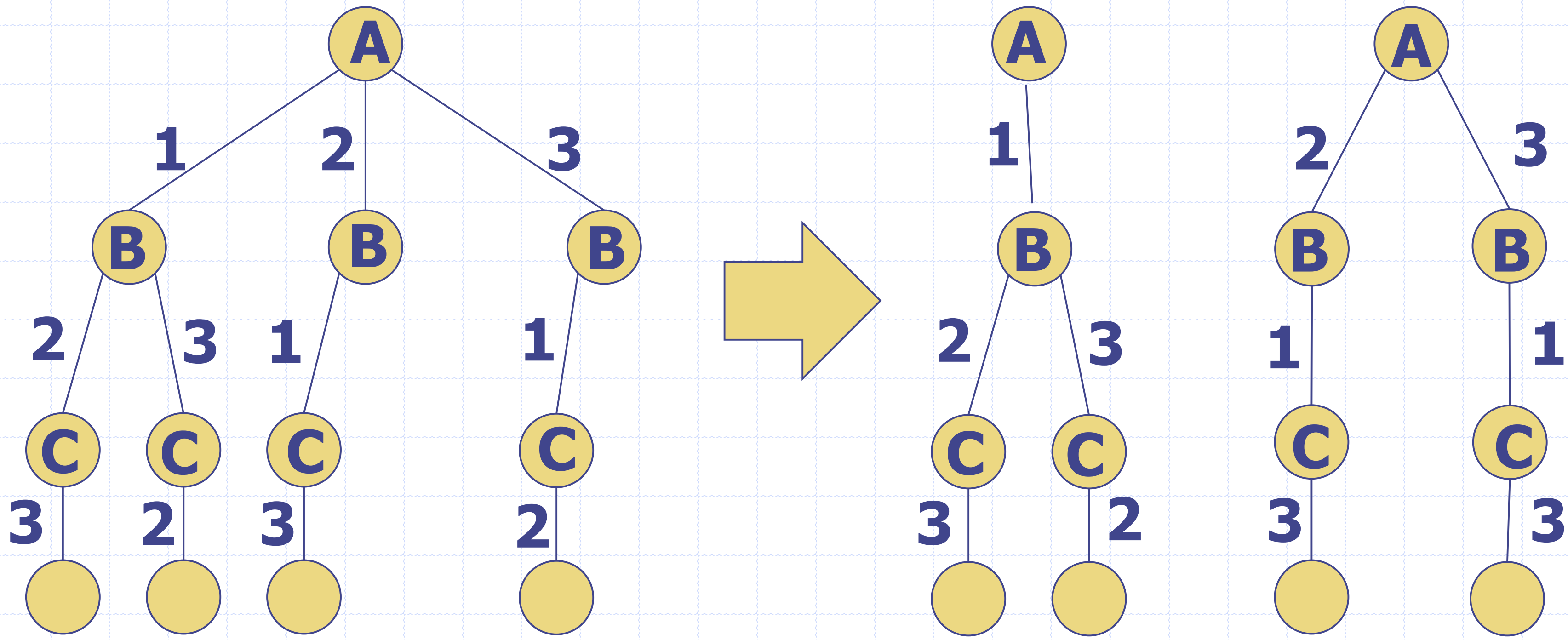

#ways=3+2=5

$$A \cap B = \emptyset \rightarrow |A \cup B| = |A| + |B|$$

# Tree Diagram

Problem: #placing A, B, C in a row where C in not first

Positons (from left to right): 1, 2, 3




#ways=$1 \times 2 \times 1 + 2 \times 1 \times 1 = 4$

# Some Problems

Problem: #bit stings of length $n$

Solution:

Constructing a string $s$ of length $n$

Task $T_i$: specifying the $i$-th bit of $s$

Constructing $s$ : doing the sequence of $T_i$ $(i = 1, .., n)$

#ways doing $T_i$ regardless of how $T_1, \ldots, T_{i-1}$ were done $= 2$

Then, #bit stings of length $n = 2^n$

Problem: #subsets of $X = \{x_1, \ldots, x_n\}$

Solution:

Constructing a subset $A$

Task $T_i$: determining whether $x_i \in A$

#ways doing $T_i$ regardless of how $T_1, \ldots, T_{i-1}$ were done $= 2$

Then, #subsets of $X = 2^n$

# Some Problems

Problem:

#one-to-one functions from $[n]$ to $[m]$ where $[n] = \{1, \dots, n\}$

Solution:

Constructing a one-to-one function $f: [n] \rightarrow [m]$

It is clear $n$ must be at most $m$

Task $T_i$: specifying the value of $f(i)$

Constructing $f$ : doing the sequence of $T_i$ $(i = 1, .., n)$

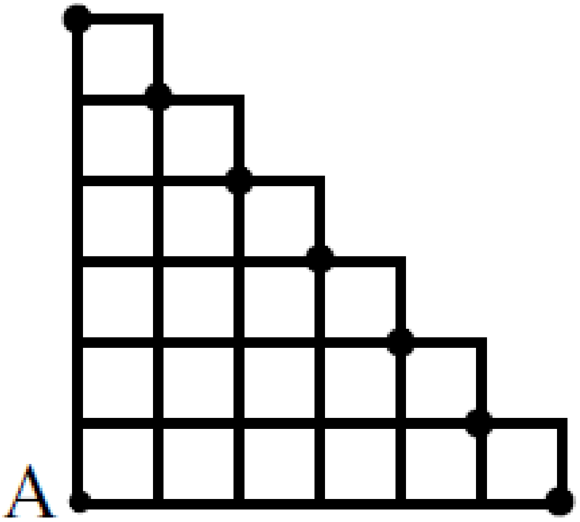
#ways of doing $T_i$ regardless of how $T_1, \dots, T_{i-1}$ were done= $\big(m - (i-1)\big) = m - i + 1$

#constructing $f = m \times (m-1) \times \cdots \times (m-n+1)$

# Some Problems

Problem:

#paths from $A$ to black disks when we are allowed to go right or up.




Solution:

Task $T_i$: specifying whether going right or up at the $i$-the step

A path: doing the sequence of $T_i$ $(i = 1, .., 6)$

#ways doing $T_i$ regardless of how $T_1, \dots, T_{i-1}$were done$= 2$

#constructing a path $=\ 2^6$

# Permutation

Definition:

A permutation of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of $r$ objects of a set is called $r$-permutation

Problem: #permutations of length $n$

Constructing a permutation $\pi$

Task $T_i$: specifying whom is placed at the i-th place, $\pi[i]$

Constructing $\pi$: doing the sequence of $T_i$ $(i = 1, .., n)$

#ways of doing $T_i$ regardless of how $T_1, \ldots, T_{i-1}$ were done= $\big(n - (i-1)\big) = n - i + 1$

#permutations= $n \times (n-1) \times \cdots \times 1 = n!$

#r-permutations $= \mathrm{P(n,r)} = n \times (n-1) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}$ (note $0!$ is defined 1)

# Cyclic Permutation

Definition:

A cyclic permutation of a set of distinct objects is an ordered arrangement of these objects around a circle.

Problem: #cyclic permutations of length $n$

Solution:

Constructing a cyclic permutation $\pi$

Task $T_i$: specifying the position of the i-th object

Constructing $\pi$: doing the sequence of $T_i$ $(i = 1, .., n)$

#ways of doing $T_1 = 1$

#ways of doing $T_i = i - 1$

#permutations$= 1 \times 1 \times 2 \times \cdots \times (n-1) = (n-1)!$

Other solution is to put $n$ identical chairs around a circle. The first person has one option, the second person has $n - 1$ options, third person has n-2 options, ....

# Some problems

Problem: #permutation of A,B,C and D where A and B are not adjacent.

Solution:

Answer is #permutation minus #permutation of A,B,C and D where A and B are adjacent.

Now we have to consider the cases where A,B are adjacent. Define a new character X which is the concatenation of A and B. Now we have to compute #permutation of C, D and X which 3!. There are two possibilities for X. So the answer is $4! - 2 \times 3!$

# Combination

Definition:

A $r$-combination of a set of $n$ distinct objects is unordered selection of $r$ objects from the set.

Problem:

#$r$-combinations of a set of size $n = \mathrm{C}(\mathrm{n}, \mathrm{r}) = \binom{n}{r}$

From each $r$-combination we can produce $r!$ $r$-permutations. We have $P(n, r)$ $r$-permutations in total.

Then $C(n, r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$

# Binomial Theorem

**Theorem:**

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

**Proof:**

- $(x+y)^n = (x+y)(x+y)\ldots(x+y)$
- Term $x^i y^{n-i}$ is produced when we choose $i$ $x$s from $n$ sums (so that the other $n-i$ terms in the product are $y$s)
- Therefore, the coefficient of $x^i y^{n-i}$ is $\binom{n}{i}$

$\binom{n}{i}$ is called a binomial coefficient

# Properties of Binomial Coefficient

- $x = y = 1 \rightarrow \sum_{i=0}^{n} \binom{n}{i} = 2^n$
- $x = -1, y = 1 \rightarrow \sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0 \rightarrow \sum \binom{n}{2\mathrm{i}} = \sum \binom{n}{2\mathrm{i}+1}$
- $x = 2,\ y = 1 \rightarrow \sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$
- $(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \rightarrow n(x+1)^{n-1} = \sum_{i=1}^{n} i \binom{n}{i} x^{\mathrm{i}-1} \rightarrow n2^{n-1} = \sum_{i=1}^{n} i \binom{n}{i} = \sum_{i=0}^{n} i \binom{n}{i}$

# Permutation with Repetition

Problem:

$n_i$ occurrences of object $i$ $(i = 1, \dots, k)$ and $\sum_{i=1}^{k} n_i = n$

Compute #permutations

For example, compute #permutations of $A, A, B, B, B, C, C$

Draw the tree diagram for this example when places from left to right are filled and see why you can not use the product rule

Solution:

Task $T_i$: Place all occurrences of object $i$

#ways of doing $T_i$: $\binom{n-(r_1+\cdots+r_{i-1})}{r_i}$

Indeed, selecting $r_i$ places of the remaining places.

#permutations= $\prod_{i=1}^{k} \binom{n-(r_1+\cdots+r_{i-1})}{r_i} = \frac{n!}{r_1!r_2!\dots r_k!}$

# One-to-One Correspondence

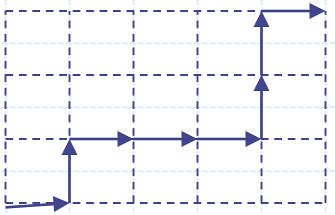
**Theorem:** if $X$ and $Y$ are two sets and there is one-to-one correspondence between $X$ and $Y$, then $|X| = |Y|$

**Problem:**#paths from $(0,0)$ to $(n, m)$ s.t. each path is made up of a series of steps, where step is a move one unit to the right or a move one unit upward.

**Solution:**

- There is one-to-one correspondence between such paths and permutations of $n$ $\rightarrow$s and $m$ $\uparrow$s.
- #permutations of $n$ $\rightarrow$s and $m$ $\uparrow$s$= \frac{(n+m)!}{n!m!} = \binom{n+m}{n}$

$\rightarrow\uparrow\rightarrow\rightarrow\rightarrow\uparrow\uparrow\rightarrow$ $\Longleftrightarrow$



# Some Problems

Problem:#rectangles constructed by $n$ vertical lines and $m$ horizontal lines.

Solution:

- There is one-to-one correspondence between such rectangles and pairs of $(\{h_l, h_r\}, \{v_b, v_t\})$ where $\{h_l, h_r\}$ is a subset of horizontal lines, and $\{v_b, v_t\}$ is a subset of vertical lines.
- #pairs of $(\{h_l, h_r\}, \{v_b, v_t\}) = \binom{n}{2}\binom{m}{2}$

# Some Problems

Problem:#non-decreasing sequences of length at most $n$ where each item is between $1$ and $n$ (inclusive)

For $n = 3$ these are few sequences $< 1,3,3 >, < 2,2,2 >,$ $< 1,2,3 >, < 2,2 >, < 1 >$

Solution:

Each such a sequence is correspondent to a permutation of $n$ 0s and $n$ 1s. To this end, ignore all 1 before the leftmost 0. For other 1, count the number of 0 before that. This corresponds to an item in the sequence. For instance,

100110 $\leftrightarrow < 2,2 >$, 010101 $\leftrightarrow < 1,2,3 >$, 000111 $\leftrightarrow < 3,3,3 >$

The answer is $\frac{(2n)!}{n!n!} = \binom{2n}{n}$

# Double Counting

Double counting is a proof technique for showing that two expressions are equal by showing that they are two ways of counting the size of one set.

Problem: $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$

Solution:

Count #subsets of size $r$ in two ways

One simple way: $\binom{n}{r}$

The other way: consider an element $a$.

Either $a$ is the subset or not.

#subsets containing a is $\binom{n-1}{r-1}$

#subsets not containing $a$ is $\binom{n-1}{r}$

# Some Problems

Problem: $\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$

Solution 1: Consider establishing a team of n persons from n boys and n girls. To count, sum over the number of boys in team. If the number of boys is $k$, the number of girls is $n-k$. The number of such teams in $\binom{n}{k}\binom{n}{n-k}=\binom{n}{k}^2$

Solution 2: Count the number of paths from $(0,0)$ to $(n,n)$ when we are allowed to go right or up. On one hand, it is $\binom{2n}{n}$. On the other hand, these paths can be decomposed into n disjoint sets, namely paths passing through points $(k,n-k)$ for $k=0,\ldots,n$.

#paths passing through $(k,n-k)$ is $\binom{n}{k}\binom{n}{k}=\binom{n}{k}^2$

# Inclusion-Exclusion Principle

Theorem:

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|\cup A_i| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \cdots$

Proof:

Consider an element $x$ and assume $m$ sets contain $x$ and $m \geq 1$.

See how many times we count $x$ in both sides (double counting) .

In the left side: answer is $1$

In the right side: answer is $m - \binom{m}{2} + \binom{m}{3} - \binom{m}{4} + \cdots$

We know $1 = \binom{m}{0}, m = \binom{m}{1},$ then we must show

$\sum \binom{m}{2i} = \sum \binom{m}{2i+1}$ (we proved it before)

# Balls into Bins

Problem: #ways putting n balls into $m$ bins

For different cases:

1. Balls are distinct and Bins are distinct
2. Balls are identical and Bins are distinct
3. Balls are distinct and Bins are identical
4. Balls are identical and Bins are identical

Example: 3 balls into 2 bins

# Balls into Bins

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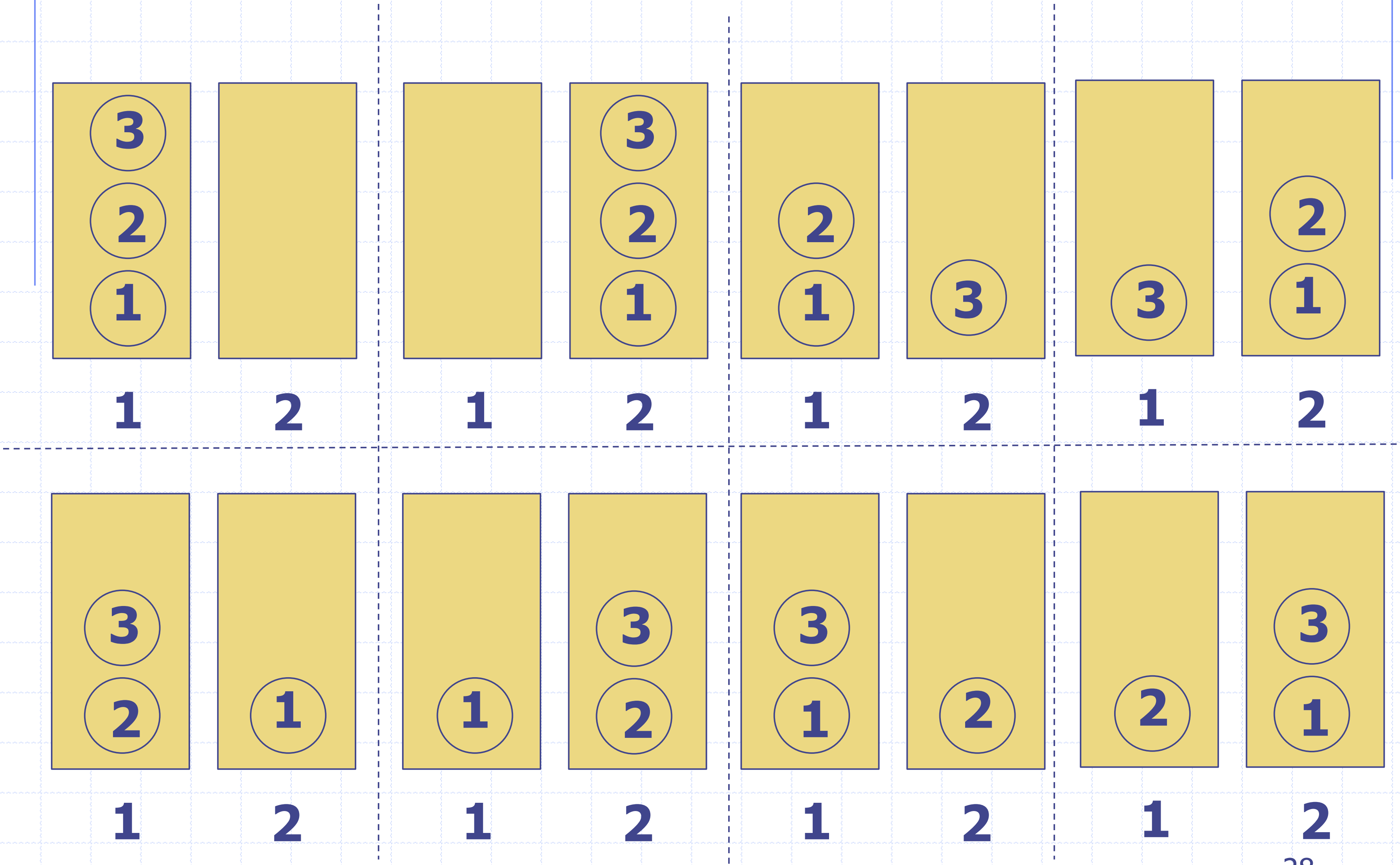
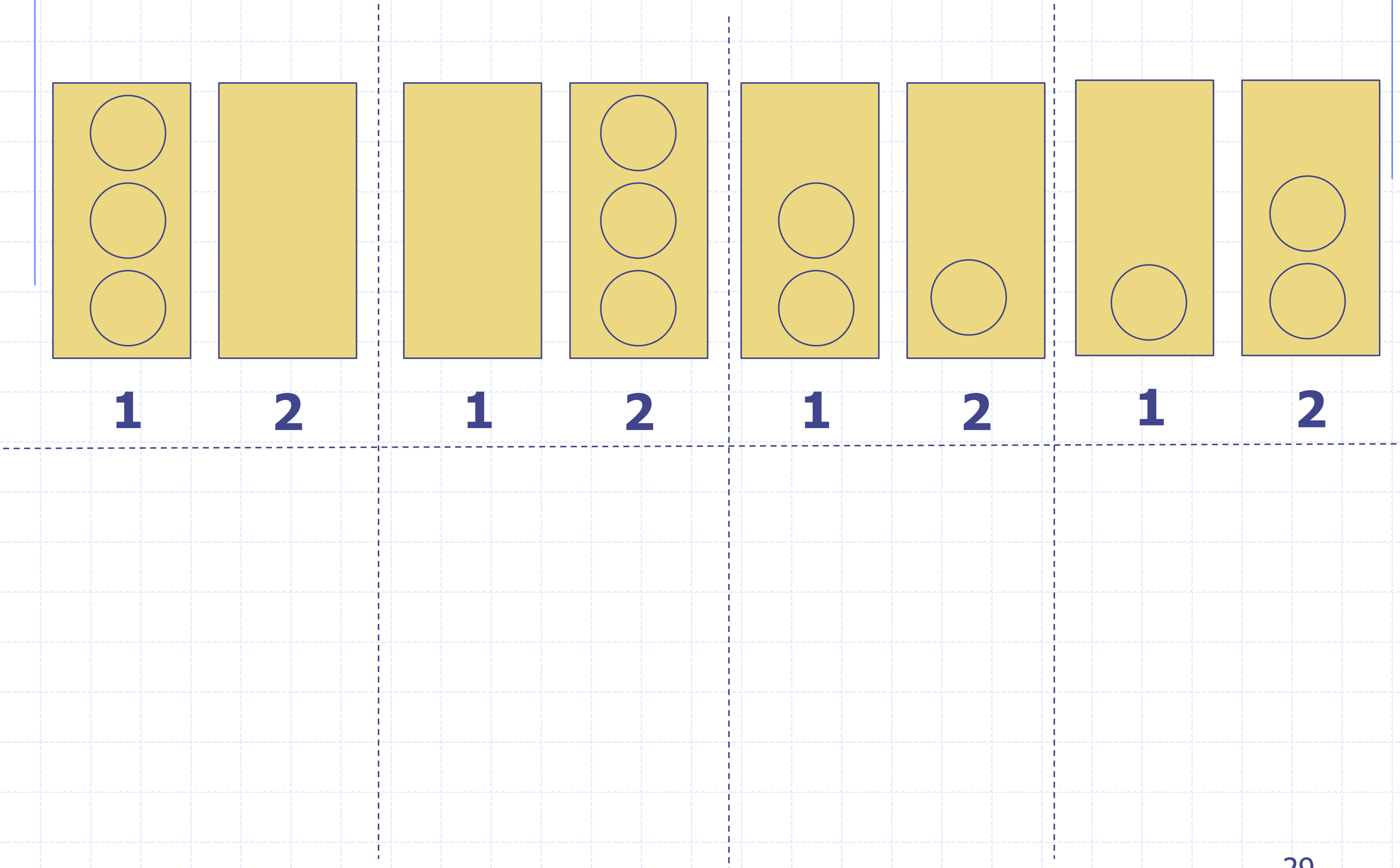
# Balls into Bins

Case 1: Distinct Balls and Distinct Bins



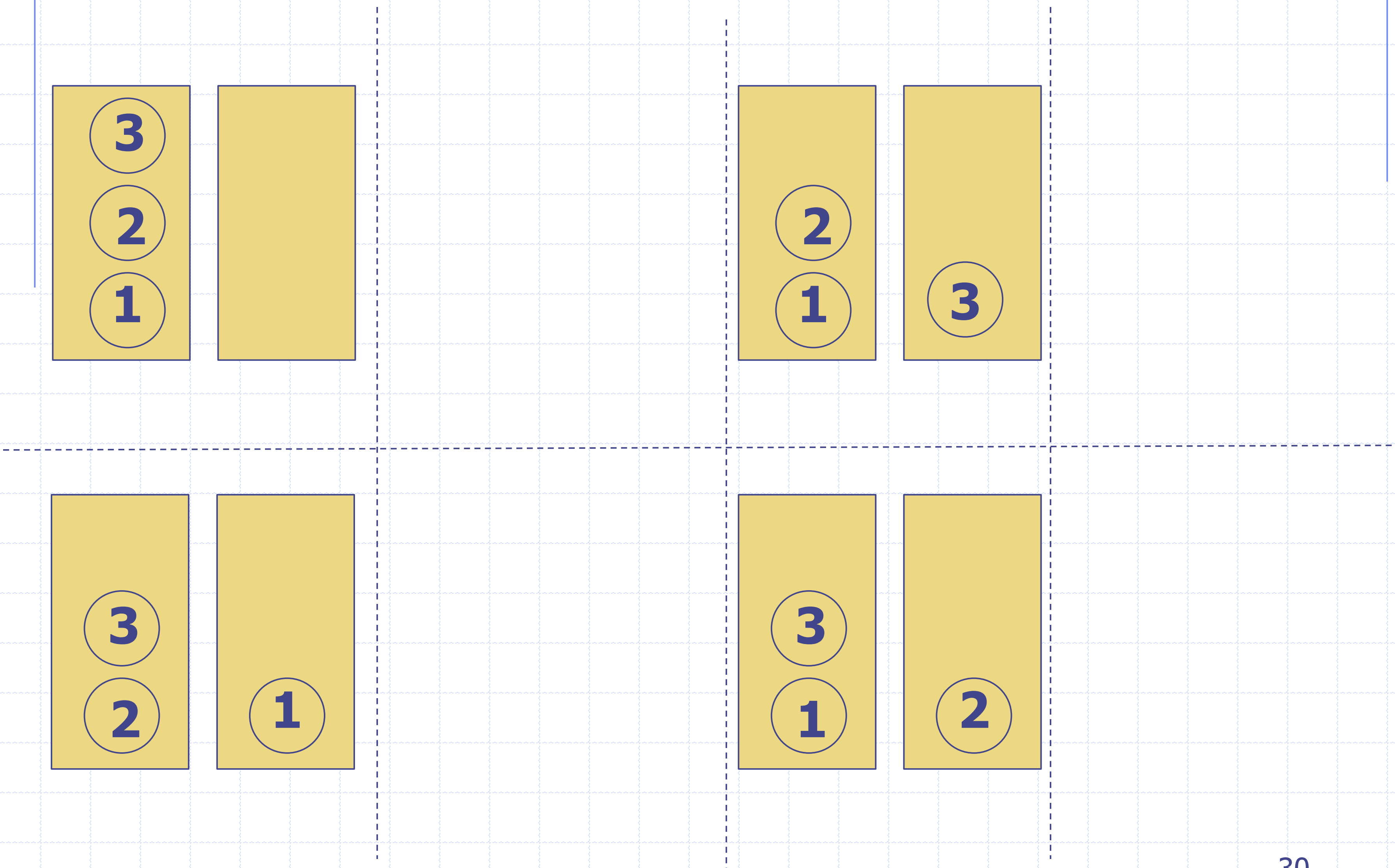
# Balls into Bins

## Case 2: Identical Balls and Distinct Bins



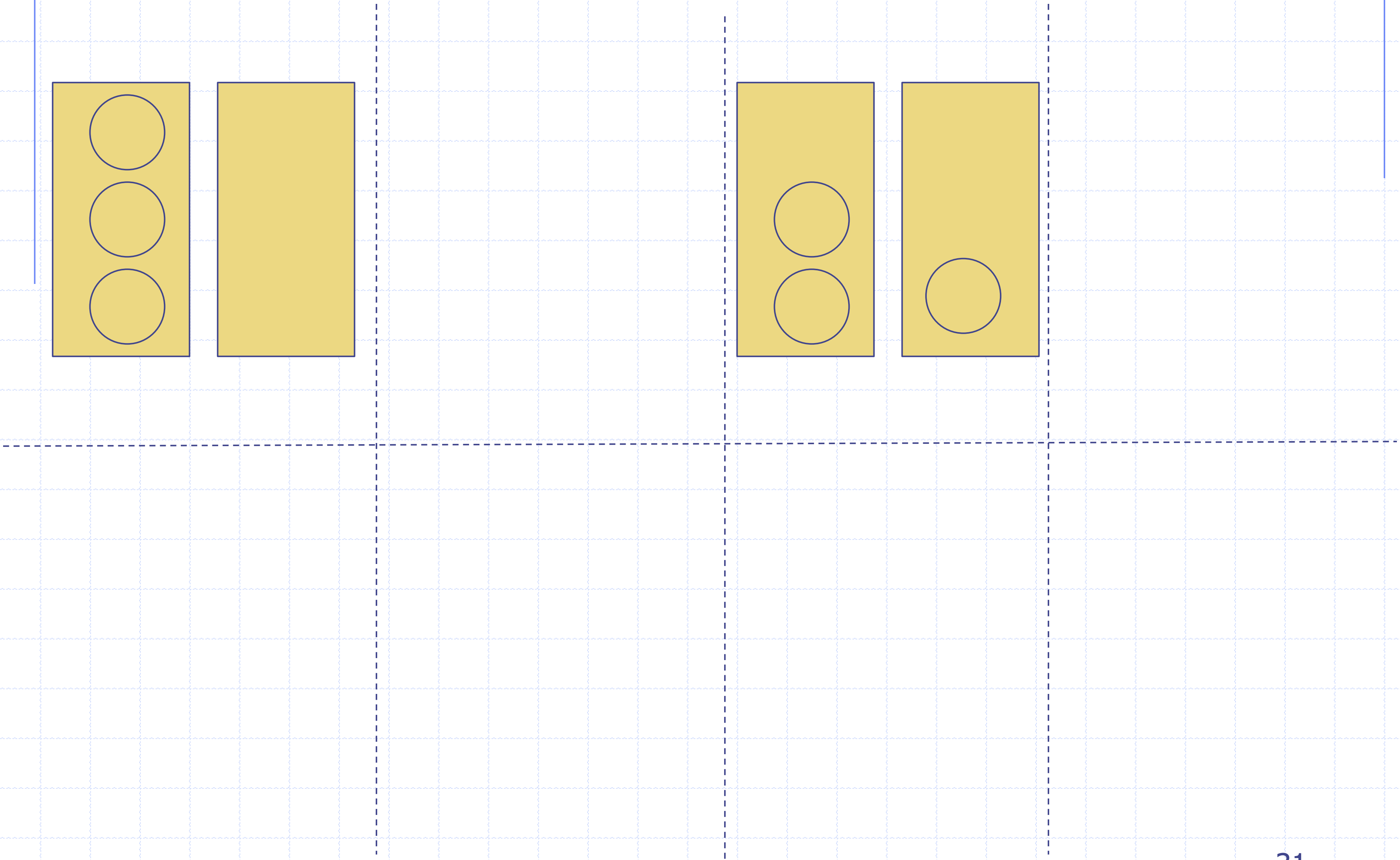
# Balls into Bins

## Case 3: Distinct Ball and identical Bins



# Balls into Bins

Case 4: Identical Balls and identical Bins



# Balls into Bins

Case 1: Distinct Balls and Distinct Bins

Solution:

Task $T_i$: Putting the $i$-th ball into bins

#ways of doing $T_i$: $m$

#ways of doing all $T_i = m^n$

Draw the tree diagram for the other cases for three balls and two bins to see why you can not use the product rule

# Balls into Bins

Case 2: Identical Balls and Distinct Bins

Solution:

- Let $x_i$ be the number of balls in the i-th bin.
- We have to count the number of solutions of $\sum_{i=1}^{m} x_i = n, 0 \le x_i$.
- Each solution is equivalent to an arrangement of n $0$s and $(m-1)$ $1$s (you can imagine $x_1$ is the number of $0$ before the leftmost $1$, $x_2$ is the number of $0$ between the two leftmost $1$, … )
- #permutations of n $0$s and $(m-1)$ $1$s $= \binom{n+m-1}{n}$

# Some Problems

**Problem:** Find #solutions of $x+y+z=30, 1 \leq x \leq 10, 3 \leq y \leq 15, 7 \leq z \leq 12$

**Solution:**

$x = x' + 1, \mathrm{x}' \geq 0$ and $y = y' + 3, \mathrm{y}' \geq 0$ and $z = z' + 7, \mathrm{z}' \geq 0$

We have $x' + y' + z' = 19, 0 \leq x' \leq 9, 0 \leq y' \leq 12, 0 \leq z' \leq 8$

#solutions of $x' + y' + z' = 19, 0 \leq x', \mathrm{y}', \mathrm{z}'$ is $\binom{21}{19}$

Let $A_{x' \geq 10}$ be the set of solutions of $x' + y' + z' = 19, 8 \leq x', 0 \leq \mathrm{y}', \mathrm{z}'$. Similarly define $A_{y' \geq 13}$ and $A_{z' \geq 9}$

answer to the problem is $\binom{21}{19} - |A_{x' \geq 10} \cup A_{y' \geq 13} \cup A_{z' \geq 9}|$

Then we have to compute

$$|A_{x'\geq 10}|, |A_{y'\geq 13}|, |A_{z'\geq 9}|, |A_{x'\geq 10} \cap A_{y'\geq 13}|, |A_{x'\geq 10} \cap A_{z'\geq 9}|,$$
$$|A_{y'\geq 13} \cap A_{z'\geq 9}|, |A_{x'\geq 10} \cap A_{y'\geq 13} \cap A_{z'\geq 9}|$$

# Balls into Bins

Case 3: Distinct Balls and Identical Bins

Solution:

- Let $S(n, k)$ be #putting $n$ distinct balls into $k$ identical bins s.t. no bin is empty (called Stirling number of the second kind).
- The answer to case (3) is $\sum_{k=1}^{m} S(n, k)$ .
- Let T$(n, k)$ be #putting $n$ distinct balls into $k$ distinct bins s.t. no bin is empty. We have $S(n, k) = T(n, k)/k!$
- T(n,k) is total ($k^n$) - #at least a bin is empty
- Let $A_i$ be set of cases of putting $n$ distinct balls into $k$ distinct bins s.t. the $i$-bin is empty
- $|\cup A_i| = \sum |A_i| - \sum |A_i \cap A_j| + \cdots = \binom{k}{1} (k-1)^n - \binom{k}{2} (k-2)^n + \cdots$
- $S(n, k) = (\frac{1}{k!}) \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n$

# Balls into Bins

Case 3: Identical Balls and Identical Bins

This is equivalent to #partitioning of $n$ into at most $m$ natural numbers denoted by $p_m(n)$

For instance $n = 6$ and $m = 4$

5,1

4,2

4,1,1

3,3

3,2,1

3,1,1,1

2,2,2

2,2,1,1

No simple closed formula exist for $p_m(n)$