Discrete Structures

Generating Function

Generating Functions

Definition: the generating function for the sequence $a_0, a_1, a_2, \ldots$ of the real numbers in the infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Examples:

- $a_k = 3 \rightarrow G(x) = \sum_{k=0}^{\infty} 3x^k$
- $a_k = k + 1 \rightarrow G(x) = \sum_{k=0}^{\infty} (k+1)x^k$
- $a_k = 2^k \rightarrow G(x) = \sum_{k=0}^{\infty} 2^k x^k$
- For the sequence 1,2,3, 4, we have $G(x) = 1 + 2x + 3x^2 + 4x^3$.

Generating Functions

General Idea:

- Representing the terms of a sequence as coefficients of a polynomial
- Using the properties of polynomials to obtain the desired target

The main properties of polynomials

- Let $P(x) = \sum_{k=0}^{\infty} \mathrm{p}_k x^k$ and $Q(x) = \sum_{k=0}^{\infty} \mathrm{q}_k x^k$. If $P(x) = Q(x)$ for any $x \in [a, b]$ for any $a \neq b$, then $p_k = q_k$ for any integer $k \geq 0$.
- Also we know how to multiply or sum two polynomials

Generating Functions

Some known equalities:

- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x| < 1$
- $\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$ for $|x| < 1/|a|$ and $a \neq 0$
- For any real number u: $(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$ for $|x| < 1$

  Where $\binom{u}{k} = \frac{u(u-1)\ldots(u-k+1)}{k!}$

$$\binom{-2}{3} = \frac{(-2)(-3)\,(-4)}{3!} = -4,\ \binom{1/2}{3} = \frac{(1/2)\left(\frac{1}{2}-1\right)(1/2-2)}{3!} = 1/16$$

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$$

- $(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$

Counting Using Generating Functions

Problem: #solutions of $a+b+c=17, 2 \le a \le 5, 3 \le b \le 6, 4 \le c \le 7$

Solution:

The coefficient of x^{17} in the following polynomial

$$(x^2+x^3+x^4+x^5)(x^3+x^4+x^5+x^6)(x^4+x^5+x^6+x^7)$$

Problem: #ways of distributing 8 identical balls into 3 distinct bins; each bin has at least two and at most 4 balls

Solution:

The coefficient of x^8 in the following polynomial

$$(x^2+x^3+x^4)(x^2+x^3+x^4)(x^2+x^3+x^4) = (x^2+x^3+x^4)^3$$

Use a computer algebra system to the desired coefficient

Counting Using Generating Functions

Problem: #ways to insert 1\$, 2\$ and 5\$ into a vending machine to pay r\$ when order of coins does not matter

Solution:

The coefficient of x^r in the following polynomial

$$(1 + x^1 + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots)$$

Problem: #ways to insert 1\$, 2\$ and 5\$ into a vending machine to pay r\$ when the order of coins does matter

Solution:

If we use exactly k coins, the coefficient of x^r in the following polynomial is the answer

$$(x^1 + x^2 + x^5)^k$$

Since any number of coins may be inserted, the coefficient of x^r in $\sum_{k=0}^{\infty}(x^1 + x^2 + x^5)^k = \frac{1}{1-x^1-x^2-x^5}$ is the answer

Counting Using Generating Functions

Problem: #r-combinations of a set of n elements when repetition of elements is allowed

Solution:

The coefficient of x^r in the following polynomial

$$(1+x^1+x^2+x^3+\cdots)^n = (1-x)^{-n}$$

which is $\binom{n+r-1}{r}$

Counting Using Generating Functions

Problem: #ways to select r objects of n different kinds if we must select at least on object of each kind

Solution:

The coefficient of x^r in the following polynomial

$$(x^1 + x^2 + x^3 + \cdots)^n = x^n(1-x)^{-n}$$

$$x^n(1-x)^{-n} = x^n \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k$$

$$= \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^{n+k} = \sum_{t=n}^{\infty} \binom{t-1}{t-n} x^t$$

Counting Using Generating Functions

Problem: #ways to partition n

Solution:

The coefficient of x^n in the following polynomial

$(1 + x^1 + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)\ (1 + x^3 + x^6 + \cdots) \ldots = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$

Catalan Number

Problem: #expressions containing n pairs of parentheses which are correctly matched (Catalan number)

Solution:

Let c_n be the n-th Catalan number. It is easy to see

$$c_n = \sum_{k=0}^{n-1} c_k c_{n-k-1}, \qquad c_0 = c_1 = 1,$$

Let $G(x) = \sum_{k=0}^{\infty} c_k x^k$ be G.F. of the Catalan sequence.

$$G(x)^2 = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} c_i c_{k-i}\right) x^k = \sum_{k=0}^{\infty} c_{k+1} x^k = \frac{G(x) - 1}{x} \rightarrow$$

$$xG(x)^2 - G(x) + 1 = 0 \rightarrow G(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n$$

Solving Recurrence Relations

Problem: $a_n = 3a_{n-1} - 2a_{n-2} + 3^n, a_0 = 0, a_1 = 1$

Solution:

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. We have

$$a_n x^n = 3a_{n-1}x^n - 2a_{n-2}x^n + 3^n x^n$$

$$\sum_{n=2}^{\infty} a_n x^n = 3x \sum_{n=2}^{\infty} a_{n-1}x^{n-1} - 2x^2 \sum_{n=2}^{\infty} a_{n-2}x^{n-2} + \sum_{n=2}^{\infty} 3^n x^n \rightarrow$$

$$G(x) - a_1 x - a_0 = 3x(G(x) - a_0) - 2x^2 \mathrm{G(x)} + (\frac{1}{1-3x} - 3\mathrm{x} - 1) \rightarrow$$

$$(1 + 2x^2 - 3x)G(x) = \frac{1}{1-3x} - 2\mathrm{x} - 1 \rightarrow$$

$$G(x) = \frac{6x^2 + x}{(1-x)(1-2x)(1-3x)} = \frac{3.5}{1-x} - \frac{8}{1-2x} + \frac{4.5}{1-3x} \rightarrow$$

$$G(x) = \sum_{n=0}^{\infty} (3.5 - 8 \times 2^n + 4.5 \times 3^n) x^n$$