Discrete Structures

Pigeon Hole

Pigeon Hole Principle

Simple Form: If we distribute $n+1$ balls into n bins, then there is at least a bin which contains at least two balls

General Form: If we distribute $kn+1$ balls into n bins, then there is at least a bin which contains at least $k+1$ balls.

Proof: For the sake of contradiction, assume each bin contains at most k ball. Then the number of balls is going to be at most nk which is a contradiction.

Some Problems

Problem: There are two persons out of 8 persons whose birthday happened in the same day of the week.

Solution:

Balls: 8 persons

Bins: 7 days of the week

Assigning balls into bins: If a person was born for instance on Sunday, it goes to the bin whose label is Sunday.

Some Problems

Problem: Assume $A \subseteq \{1, \ldots, 2n\}$, $|A| = n + 1$. Show that there are two numbers $a, b \in A$ s.t. $a|b$

Solution:

Each number n can be written as $2^{\alpha}b$ where b is an odd number (the odd part of n).

Balls: elements of A.

Bins: elements of $\{1,3, \ldots, 2n - 1\}$ (odd numbers)

Assigning balls into bins: an element of A goes to bin i iff its odd part is i.

#balls= $n + 1$, #bins= n, then two numbers have the same odd part, So, one is divisible by the other.

Some Problems

Problem: Assume $A \subseteq \{1, \ldots, 2n\}$, $|A| = n + 1$. Show that there are two numbers $a, b \in A$ s.t. $(a, b) = 1$

Solution:

Balls: elements of A.

Bins: the pairs $(2i - 1, 2i)$ $(i = 1, \ldots n)$

Assigning balls into bins: an element of A goes to bin $(2i - 1, 2i)$ iff it is equal to either $2i - 1$ or $2i$.

#balls= $n + 1$, #bins= n, then two numbers are in the same bin. These two number are relatively prime as they are consecutive.

Some Problems

Problem: Consider 6 distinct points and all segments whose endpoints are these 6 points. We color the segment with either red or blue. Prove that there is a triangle whose edges have the same color.

Solution:

Consider a point A. It has 5 segments incident to it. At least three of them have the same color, say red.

Call the other endpoints of these three red segments B, C and D. If one of segments BC, CD and DB is red, we have a red triangle. Otherwise all three must be blue. Then we have a blue triangle. In both cases, we have a triangle whose edges have the same color.

Some Problems

Problem: We have 9 numbers of form $2^\alpha 3^\beta 5^\gamma$. Show that there are two numbers whose product is a square number.

Solution:

Balls: 9 numbers

Bins: 8 triples (a, b, c) where $a, b, c \in \{0,1\}$

Assigning balls into bins: ball $2^\alpha 3^\beta 5^\gamma$ goes into bin (a, b, c) iff $\alpha \equiv a \ (mod\ 2), \beta \equiv b\ (mod\ 2), \gamma \equiv c\ (mod\ 2)$,

#balls$= 9$ and #bin$= 8$. Then, two numbers are in the same bin. So, the product of these two numbers is a square number.

Some Problems

Problem: Ali read his lecture notes 30 hours during the last 20 days to get prepared for the final exam. At the i-th day of his preparation, he spent a_i hours where a_i is a natural number. Prove that there were some consecutive days s.t. the total time he spent on these days is exactly 9 hours.

Solution:

Let $x_i = \sum_{j=1}^{i} a_j$. Clear that $x_i \in \{1,2,\dots,30\}$.

let $y_i = x_i + 9$. Clear that $y_i \in \{10,11,\dots,39\}$.

So, $x_i, y_i \in \{1,2,\dots,39\}$. Also clear that

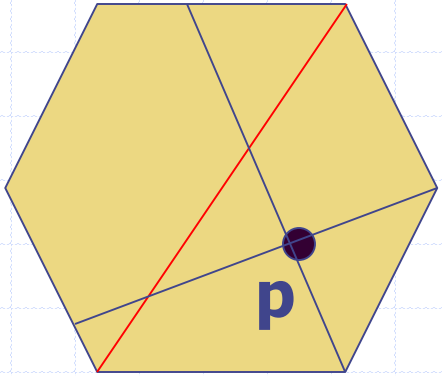
$x_1 < x_2 < \cdots < x_{20}$ and $y_1 < y_2 < \cdots < y_{20}$.

We have 40 numbers; each belonging to $\{1,2,\dots,39\}$. Then two of them must be equal. So, $x_i = y_j$ for some i and j. So $x_i - x_j = 9$.

Some Problems

Problem: Assume we have a convex polygon with $2n$ vertices and a point p inside it which is not on any diagonal of the polygon. For each vertex A, consider the line passing through A and p. This line intersects an edge of the polygon. Prove that there is an edge of the polygon not being intersected by these lines.

Solution:



- $A_1A_2 \dots A_{2n}$ are the vertices.
- p lies in one side of A_1A_{n+1}.
- Assume p is inside $A_1A_2 \dots A_nA_{n+1}$.
- Only lines $A_2p, A_3p, \dots, A_np$ (n-1 lines) may intersect edges $A_{n+1}A_{n+2}, A_{n+2}A_{n+3}, \dots, A_{2n}A_1$ (n edges).
- Then one edge must be free of intersection

Some Problems

Problem: Suppose we have a sequence of distinct numbers of length $n^2 + 1$. Prove that there is a subsequence of length $n + 1$ which is either decreasing or increasing.

Solution:

- $< a_1, \dots, a_{n^2+1} >$ is the sequence
- Assign a_k a pair (i_k, d_k) where i_k (d_k) is the length of the longest increasing (decreasing) subsequence starting (ending) at a_k
- If one of i_ks and d_ks is greater than n, we are done
- Otherwise, $n^2 + 1$ pairs (i_k, d_k) are of form (a, b) where $1 \leq a, b \leq n$. So, there are k and l s.t $(i_k, d_k) = (i_l, d_l)$
- Assume $k < l$. if $a_k < a_l$ $(a_k > a_l)$, then $i_k (d_l)$ must be at least $i_l + 1$ $(d_k + 1)$

$$a_k, \dots, a_l, \dots$$