# Discrete Structures

## Number Theory

# Division

**Definition:**

- $a \mid b$ iff $\exists c : b = ac$
- $a$ divides $b$, $b$ is a multiple of $a$, $a$ is a factor of $b$
- If $a$ does not divides $b$, we write $a \nmid b$

**Properties:**

1. $a \mid b, b|c \rightarrow a|c$
2. $a|b, a|c \rightarrow a|bx + cy$
3. $a| \, b, b|a \leftrightarrow |a| = |b|$
4. $a \mid 1 \ \leftrightarrow |a| = 1$
5. $a \mid b \ \rightarrow a|bc$
6. $\forall n \geq 1 : a \mid b \ \leftrightarrow a^n|b^n$

# Some Proofs

Proof (2):

$a|b, a|c \rightarrow \exists b', c' : b = ab', c = ac' \rightarrow bx = ab'x, cy = ac'y$
$\rightarrow bx + cy = a(b'x + c'y) \rightarrow a|bx + cy$

Proof (6):

$a|b \rightarrow \exists c : b = ac \rightarrow b^n = a^n c^n \rightarrow a^n|b^n$

The reverse is not simple now; wait for GCD.

# Some Problems

Problem:

if $x_1x_2 + x_2x_3 + \cdots + x_nx_1 = 0$ and $x_i \in \{1, -1\}$, then $4|n$

Solution:

It is clear $2|n$. So, $n/2$ terms are $+1$ and $n/2$ terms $-1$

Multiply all terms. In one hand we have $(-1)^{n/2}$. In the other hand we have $(x_1x_2x_3 \ldots x_n)^2 = 1$. Then

$$(-1)^{n/2} = 1 \rightarrow 4|n$$

Problem:

$$a - c \mid ab + cd \ \rightarrow \ a - c | ad + bc$$

Solution:

$$a - c | (ad + bc) - (ab + cd) = (a - c)(d - b)$$

# Some Problems

**Problem:** $13|4^{2n+1}+3^{n+2}$

**Solution:**

- Basis Step: $13|4^1+3^2$
- Inductive Step:

$$13|4^{2n+1}+3^{n+2} \rightarrow 13|4^{2(n+1)+1}+3^{(n+1)+2}$$

$$(4^{2(n+1)+1}+3^{(n+1)+2})-(4^{2n+1}+3^{n+2})$$
$$=15\times 4^{2n+1}+2\times 3^{n+2}=13\times 4^{2n+1}+2(4^{2n+1}+3^{n+2})$$

**Problem:** $9|a^2+ab+b^2 \rightarrow 3|a, 3|b$

**Solution:**

$9|a^2+ab+b^2=(a-b)^2+3ab \ \rightarrow 3|a-b$

$\rightarrow 9|(a-b)^2 \rightarrow 9|3ab \rightarrow 3|a \vee 3|b$

$(3|a-b) \wedge (3|a \vee 3|b) \rightarrow 3|a \wedge 3|b$

# Greatest Common Divisor

**Definition:**

$GCD(a,b) = d$ iff

1. $d|a, d|b$
2. $\forall d'\colon d'|a, d'|b \rightarrow d' \leq d$

**Properties (Let 's denote $GCD(a,b)$ by $(a,b)$):**

1. $(a,b) = (b,a) = (-a,b) = (a,-b) = (-a,-b)$
2. $\forall k\colon (a,b) = (a, b+ak)$
3. $\exists x,y\colon ax + by = (a,b)$
4. $(a,b) = (a,c) = 1 \rightarrow (a,bc) = 1$
5. $(a^n, b^n) = (a,b)^n$
6. $a|bc, (a,b) = 1 \rightarrow a|c$
7. $(ka, kb) = |k|(a,b)$
8. $a|c, b|c, (a,b) = 1 \rightarrow ab|c$

# Some Proofs

Proof (2)

Let $(a, b) = d$, $(a, b + ak) = d'$

$(a, b) = d \rightarrow d|a, d|b \rightarrow d|a, d|b + ak \rightarrow d \leq d'$

Similarly $d' \leq d$, and then $d = d'$

Proof (6)

$(a, b) = 1 \rightarrow \exists x, y: ax + by = 1 \rightarrow acx + bcy = c$

$\rightarrow acx + aa'y = c \rightarrow a(cx + a'y) = c \rightarrow a|c$

Proof (8)

$(a, b) = 1 \rightarrow \exists x, y: ax + by = 1 \rightarrow acx + bcy = c$

$\rightarrow abb'x + baa'y = c \rightarrow ab(b'x + a'y) = c \rightarrow ab|c$

# Some Problems

Problem: show $(3n+2,\ 7n+5)=1$

Soluton 1:

$(3n+2,\ 7n+5)=d \to d|3n+2, d|7n+5 \to$
$d|7(3n+2)-3(7n+5)=-1 \to d=1$

Solution 2:

$(3n+2, 7n+5)=\big(3n+2, 7n+5-2(3n+2)\big)$
$=(3n+2, n+1)=(n+1, 3n+2)$
$=\big(n+1, 3n+2-3(n+1)\big)=(n+1,-1)=1$

Problem: $a \mid b \ \leftrightarrow a^n|b^n$

Solution: $a^n|b^n \to (a^n, b^n)=a^n \to (a,b)=a \to a|b$

# Some Problems

**Definition:**

if $(a, b) = 1$, they are called relatively prime

**Problem:** Among 5 consecutive numbers, there is one which is relatively prime to the other four numbers

**Solution:**

for any $|a - b| < 5$, we know $(a, b) = 1,2,3, or\ 4$

It suffices to show there is a number $x$ s.t. $(x, 6) = 1$

Between 5 consecutive numbers, there are two consecutive odd numbers. One of these two is not divisible by 3; otherwise their difference which is 2 must be divisible by 3. This number is the answer.

**Problem:** Prove problem for 16 consecutive numbers

# Least Common Multiples

**Definition:**

- $LCM(a,b) = L$ iff

1. $L > 0$
2. $a|L, b|L$
3. $\forall L': a|L', b|L' \rightarrow L \leq L'$

**Properties (Let 's denote $LCM(a,b)$ by $[a,b]$):**

1. $[a,b] = [b,a] = [-a,b] = [a,-b] = [-a,-b]$
2. $[a^n, b^n] = [a,b]^n$
3. $[ka,kb] = |k|[a,b]$
4. $[a,b] = |ab|/(a,b)$

# Some Problems

**Problem:** $[a,b,c]=\frac{abc}{(ab,ac,bc)}$

**Solution:**

$$[a,b,c]=[[a,b],c]=\left[\frac{ab}{(a,b)},c\right]=\frac{\frac{abc}{(a,b)}}{(\frac{ab}{(a,b)},c)}$$

$$=\frac{\frac{abc}{(a,b)}}{\frac{(ab,ac,bc)}{(a,b)}}=\frac{abc}{(ab,ac,bc)}$$

# Division Algorithm

*Theorem:*

$$\forall a, b \neq 0\ \exists q, r: a = bq + r, 0 \leq r < |b|$$

Proof:

- For simplicity assume $a, b > 0$
- Consider $\mathrm{R} = \{a - bq | a - bq \geq 0\}$
- R has a least element; called it $r$; $r = a - bq$ for some $q$
- $r$ must be smaller than $b$, otherwise
- $0 \leq r - b = a - bq - b = a - b(q + 1) \rightarrow r - b\ \in R$

# Some Problems

Any number can be written in any of the following format

- $2k, 2k+1$
- $3k, 3k+1, 3k+2$
- $4k, 4k+1, 4k+2, 4k+3,$
- …

Problem: show $120|n^5-n$ for odd n

Solution:

$3\times5\times8|n(n-1)(n+1)(n^2+1)$

We know $3|n(n-1)(n+1)$

$5\nmid n(n-1)(n+1)\rightarrow n=5k\mp2\rightarrow 5|n^2+1$

$n=2k+1\rightarrow 8|(n-1)(n+1)(n^2+1)$

# Euclidean Algorithm

Assume $a, b > 0,\ r_0 = a, r_1 = b$

- $r_0 = r_1 q_0 + r_2, 0 < r_2 < r_1$
- $r_1 = r_2 q_1 + r_3, 0 < r_3 < r_2$
- $r_2 = r_3 q_2 + r_4, 0 < r_4 < r_3$
- …
- $r_n = r_{n+1} q_n + r_{n+2}, 0 < r_{n+2} < r_{n+1}$
- $r_{n+1} = r_{n+2} q_{n+1}$

Then $(a, b) = (r_0, r_1) = (r_1, r_2) = \cdots = (r_n, r_{n+1}) = (r_{n+1}, r_{n+2}) = r_{n+2}$

# Representation of Integers

Let $b > 1$. Any positive integer $n$ can be written in form of $n = a_k b^k + \cdots + a_1 b + a_0 = (a_k \ldots a_1 a_0)_b$ where $0 \le a_i < b$, $a_i$ is called a digit in base $b$

Examples:

$859 = 8 \times 10^2 + 5 \times 10 + 9$

$(10110)_2 = 1 \times 2^4 + 1 \times 2^2 + 1 \times 2 = (22)_{10}$

$(3A0F)_{16} = 3 \times 16^3 + 10 \times 16^2 + 15 \times 16 = (14863)_{10}$

How to compute digits of $n$ base $b$:

- simply apply division algorithm
- $n = bq_0 + a_0, q_0 = bq_1 + a_1, \ldots$

# Prime Numbers

**Definition:**

Any number greater than 1 whose factors are only 1 and itself is called a prime number. Otherwise; it is called composite.

**Properties:**

1. $p|ab \rightarrow p|a \vee p|b$
2. $(a,p) = 1 \vee p$
3. Any number has a prime factor
4. Any composite $n$ has $a$ prime factor $p$ s.t. $p \leq \sqrt{n}$
5. #primes is infinity
6. #primes in form of ak+b where (a,b)=1 is infinity
7. $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \ldots {p_k}^{\alpha_k}, \alpha_i \geq 1$
8. At least a prime number exists between $n$ and $2n$

# Some Proofs

Proof (3)

using strong induction

If $n$ is prime, we are done. Otherwise $n = ab$ where $a, b > 1$. Now consider the prime factor of $a$ which is a factor of $n$

Proof (4)

$n = ab$ where $a, b > 1$. then $\min(a, b) \leq \sqrt{n}$. Now consider a prime factor of $\min(a, b)$

Proof (5)

Assume all prime numbers are $\{p_1, \ldots, p_k\}$

Consider $N = p_1 \ldots p_k + 1$ which has a prime factor $p$ .

$p|N \rightarrow (p, p_i) = 1 \rightarrow p$ is a new prime number.

# Some Problems

Problem: find $n$ s.t. $n \nmid (n-1)!$

Solution:

- If $n$ is in form of $n = ab, a, b > 1, a \neq b$, then $n \mid (n-1)!$. Otherwise $n = p$ or $p^2$ where p is prime
- $n = p$ is of course is answer. If $n = p^2$, number $p, 2p$ exist in $(n-1)!$ for $p > 2$. Just check $n = 2^2$.

Problem: find $n$ s.t. $n^2 \nmid (n-1)!$

# Some Problems

Theorem:

The power of $p$ in $n! = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$

Problem: show $\frac{(n+m)!}{n!m!}$ is integer

Solution:

We have to show for any $p$, the power of $p$ in $(n+m)!$ is at least the power of $p$ in $n!\,m!$

It is sufficient to show $\left\lfloor \frac{n+m}{p^i} \right\rfloor \geq \left\lfloor \frac{m}{p^i} \right\rfloor + \left\lfloor \frac{n}{p^i} \right\rfloor$ for any $i$

Problem: show $k!^{k^2+k+1} | k^3!$

Solution: show $(k^2+k+1) \left\lfloor \frac{k}{p^i} \right\rfloor \leq \left\lfloor \frac{k^3}{p^i} \right\rfloor$ for any prime $p$

# Some Problems

Problem:

At least a prime number exists between $n$ and $n!$

Solution:

One way is to show $n! \geq 2n$. The other way is to look at $n! - 1$ which is relatively prime to any number equal or less than $n$. So, this has a prime factor which is greater than $n$ and of course less than $n!$.

Problem:

if $p$ and $p^2 + 2$ are prime, then $p^3 + 2$ is prime

Solution:

$p = 3$ is the answer. Other prime numbers are of form $3k + 1$ or $3k + 2$. For both $3|p^2 + 2$

# Congruence

**Definition:**

$$a \equiv b \ (mod\ m) \leftrightarrow m | a - b$$

**Properties:**

1. $a \equiv a \ (mod\ m)$
2. $a \equiv b \ (mod\ m), b \equiv c \ (mod\ m) \rightarrow a \equiv c \ (mod\ m)$
3. $a \equiv b \ (mod\ m) \leftrightarrow \mathrm{a} + \mathrm{c} \equiv b + c \ (mod\ m)$
4. $a \equiv b \ (mod\ m) \rightarrow \mathrm{ac} \equiv bc \ (mod\ m)$
5. $\mathrm{ac} \equiv bc \ (mod\ m) \rightarrow a \equiv b \ (mod\ (m/(m, c))$
6. $a \equiv b \ (mod\ m) \rightarrow a^n \equiv b^n \ (mod\ m)$
7. $a \equiv b \ (mod\ m) \rightarrow \mathrm{P(a)} \equiv \mathrm{P(b)}$ where $P$ is a polynomial
8. $a \equiv r (mod\ m)$ where $a = mq + r, 0 \leq r < m$
9. $0 \leq i \neq j < m \rightarrow [i] \cap [j] = \emptyset$ where $[i] = \{x | x \equiv i \ (mod\ m)\}$

# Some Proofs

**Proof (5)**

$(c, m) = d \rightarrow c = c'd, m = m'd, (c', m') = 1$

$ac \equiv bc \ (mod\ m) \rightarrow m|c(a - b) \rightarrow m'd|c'd(a - b)$

$\rightarrow m'|c'(a - b), (m', c') = 1 \rightarrow m'|a - b \rightarrow a \equiv b \ (mod\ m')$

**Proof (7)**

Let $P(x) = p_k x^k + \cdots + p_1 x + p_0$

$a \equiv b \ (mod\ m) \rightarrow \forall i{:}\ p_i a^i \equiv p_i b^i (mod\ m) \rightarrow P(a) \equiv P(b) \ (mod\ m)$

**Proof (9)**

$x \in [i] \cap [j] \rightarrow x \equiv i \equiv j \ (mod\ m) \rightarrow m|i - j, 0 \leq i, j < m$
$\rightarrow i = j$

# Some Problems

**Problem:** $x \equiv 1\ (mod\ 2) \rightarrow x^2 \equiv 1\ (mod\ 8)$

**Solution:** $x \equiv 1\ (mod\ 2) \rightarrow x = 4k + 1 \vee 4k + 3 \rightarrow x^2 \equiv 1\ (mod\ 8)$

**Problem:** *Compute the remainder of* $3 \times 2^{1399}$ *to* $7$

**Solution:**

$2^3 \equiv 1\ (mod\ 7) \rightarrow 2^{3\times466} \equiv 1\ (mod\ 7) \rightarrow 2^{1399} \equiv 2\ (mod\ 7) \rightarrow 3 \times 2^{1399} \equiv 6\ (mod\ 7)$ *or simply write* $3 \times 2^{1399} \equiv 3 \times 2 \times 2^{3\times466} \equiv 6 \times 1^{466} \equiv 6\ (mod\ 7)$

**Problem:**

*Compute the rightmost digit of* $1398^{1399}$ *base* $10$

**Solution:**

$1398^{1399} \equiv 8^{1399} \equiv (-2)^{1399} \equiv -2^{1399}\ (mod\ 10)$

$2^{1398} \equiv 2^{4\times349+2} \equiv 1^{349} \times 2^2 \equiv 4\ (mod\ 5) \rightarrow 2^{1399} \equiv 8 (mod\ 10) \rightarrow 1398^{1399} \equiv 2\ (mod\ 10)$

# Some Problems

**Problem:** $n = (a_k \dots a_0)_{10} \rightarrow n \equiv a_k + \cdots + a_0 \ (mod\ 9)$

**Solution:** $n = a_k 10^k + \cdots + a_1 10 + a_0, 10 \equiv 1 \ (mod\ 9)$

**Problem:** Find all prime $p$ and $q$ s.t. $p^2 + 2q^2 = x^2$

**Solution:**

$p = 2 \rightarrow x = 2k \rightarrow q = 2$ but $(2,2)$ is not the answer

Otherwise, $p = 2k + 1 \rightarrow x = 2k' + 1 \rightarrow p^2 \equiv x^2 \equiv 1 \ (mod\ 8), p^2 + 2q^2 \equiv x^2 \ (mod\ 8) \rightarrow 2q^2 \equiv 0 (mod\ 8) \rightarrow$
$q = 2 \rightarrow (x - p)(x + p) = 8 \rightarrow$
$(x - p = 1 \wedge x + p = 8) \vee (x - p = 2 \wedge x + p = 4) \rightarrow$
$x = 3, p = 1$

but 1 is not prime

# Euler's Totient Function

**Definition:**
$\Phi(n) = \{x|(x,n) = 1, 1 \le x \le n\}, \varphi(n) = |\Phi(n)|$
$\varphi(1) = 1, \varphi(2) = 1, \varphi(3) = 2, \varphi(4) = 2, \varphi(p) = p - 1$

**Theorem:** $n = {p_1}^{\alpha_1} \dots {p_k}^{\alpha_k} \rightarrow \varphi(n) = n(1 - \frac{1}{p_1}) \dots (1 - \frac{1}{p_k})$

**Proof:** show that $(m,n) = 1 \rightarrow \varphi(mn) = \varphi(m)\varphi(n)$

**Lemma:** $(a,n) = 1, i \ne j \in \Phi(n) \rightarrow ai \not\equiv aj \ (mod\ n)$

**Lemma:** $(a,n) = 1, i \in \Phi(n) \rightarrow \exists j\colon ai \equiv j \ (mod\ n)$

Then, there is a one-to-one correspondence between $\Phi(n)$ and $\{ax|x \in \Phi(n)\}$ mod $n$. Therefore,
$\prod_{i\in\Phi(n)} i \equiv \prod_{i\in\Phi(n)} ai \ (mod\ n)$ and $(\prod_{i\in\Phi(n)} i, n) = 1 \rightarrow$

**Theorem:** $(\mathrm{a}, \mathrm{n}) = 1 \rightarrow a^{\varphi(n)} \equiv 1 \ (mod\ n)$

# Some Problems

**Problem:** if $d$ is the smallest natural number s.t. $a^d \equiv 1 \ (mod\ n)$ and $a^m \equiv 1 \ (mod\ n)$, then $d|m$

**Solution:**

$m = dq + r, 0 \leq r < d, a^m \equiv a^d \equiv 1 \ (mod\ n) \rightarrow$
$a^r \equiv 1 \ (mod\ n) \rightarrow r = 0 \rightarrow d|m$

**Problem:** $n|\varphi(2^n - 1)$

**Solution:**

$n$ is the smallest number s.t. $2^n \equiv 1 \ (mod\ 2^n - 1)$

Since $2^{\varphi(2^n-1)} \equiv 1 (mod\ 2^n - 1)$, then $n|\varphi(2^n - 1)$

# Wilson's Theorem

Definition:

$a^*$ is called inverse of $a$ mod $n$ iff $aa^* \equiv 1 \ (mod\ n)$

Lemma: ($a^*$ exists iff $(a, n) = 1$) and $a^* \equiv a^{\varphi(n)-1} \ (mod\ n)$

Theorem: if $p$ is prime, then $(p-1)! \equiv -1 \ (mod\ p)$

Proof:

- For any $a \in \{1, \dots, p-1\}$, inverse exists.
- If $a^* = \mathrm{a} \rightarrow a^2 \equiv 1 \ (mod\ p) \rightarrow p | (a-1)(a+1) \rightarrow a = 1 \vee a = p-1$
- For other $a$, we have $a^* \neq a$
- Set $\{2,3, \dots, p-2\}$ can be decomposed into disjoint pairs $(a, b)$ ($a \neq b$) s.t. $ab \equiv 1 \ (mod\ p)$

# Chinese Remainder Theorem

Theorem:

$\forall a, b\ \forall m, n\ s.t. (m, n) = 1$ *we have*

$\exists x \left(x \equiv a\ (mod\ n) \wedge\ x \equiv b\ (mod\ m)\right)$

There is an unique $x$ in $[0..mn-1]$ satisfying above

Solution: $x \equiv bn^*n + am^*m\ (mod\ mn)$ where

$nn^* \equiv 1\ (mod\ m)$ and $mm^* \equiv 1\ (mod\ n)$

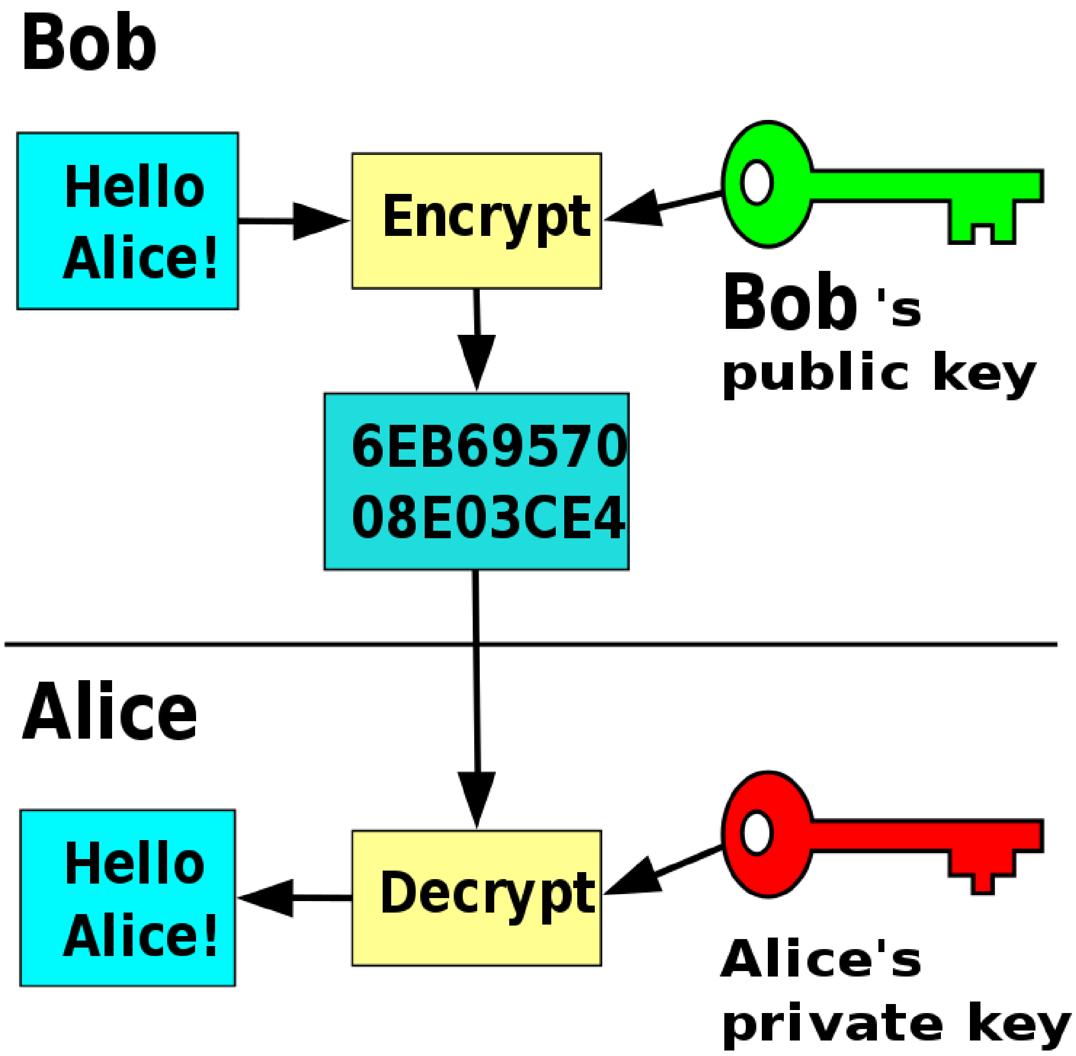
Theorem can be extended to $k$ linear equations:

$x \equiv a_1\ (mod\ n_1)\ \ldots x \equiv a_k\ (mod\ n_k)$ where $(n_i, n_j) = 1$

Example:

$x \equiv 3\ (mod\ 7) \wedge\ x \equiv 5\ (mod\ 9) \rightarrow \mathrm{x} \equiv 59\ (mod\ 63)$

# Public Key Encryption



# Public Key Encryption

Let $n = pq$ where $p$ and $q$ are large prime numbers

Bob's key is the pair $n$ and $e$ where $e$ is the number s.t. $(e, (p-1)(q-1)) = 1$. Anybody else may have this key (indeed it is a public key).

Alice's key is the pair $p$ and $q$ and a number $d$ s.t. $de \equiv 1\ (mod\ (p-1)(q-1))$ (the key is private)

See message M as an integer number

Bob sends $C = M^e\ mod\ n$ instead of sending $M$

Alice computes $C^d \equiv M^{de} \equiv M^{k(p-1)(q-1)+1} \equiv M\ (mod\ n)$

To uniquely decrypt $M$, we need $M < n$. Then decompose original message into smaller pieces; each smaller than $n$

Without knowing p and q is hard to decrypt $M^e\ mod\ n$

It is hard (time-consuming) to decompose $n$ to $pq$.

# Miscellaneous Problems

**Problem:**

$x^2 + y^2 = z^2 \leftrightarrow \exists m, n, \mathrm{d}: x = (m^2 - n^2)d, y = 2\mathrm{mnd},$
$z = (m^2 + n^2)d$

**Solution:**

We can assume $(x, y) = (x, z) = (y, z) = 1$, $x$ and $z$ are odd and $y$ is even.

$y^2 = (z - x)(z + x), (z - x, z + x) = 2 \rightarrow z - x = 2m^2, z + x = 2n^2 \rightarrow z = m^2 + n^2, x = m^2 - n^2, y = 2mn$

We use the fact that $ab = x^2, (a, b) = 1 \rightarrow a = m^2, b = n^2$

The reverse is obvious. Just replace.

# Miscellaneous Problems

Problem: $(2^m - 1, 2^n - 1) = 2^{(m,n)} - 1$

Solution:

Let $(2^m - 1, 2^n - 1) = \mathrm{d}$

$2^{(m,n)} - 1 | 2^m - 1, 2^{(m,n)} - 1 | 2^n - 1 \rightarrow 2^{(m,n)} - 1 | \mathrm{d}$

Let r be the smallest number s.t. $2^r \equiv 1 \ (mod\ d)$

We know $2^n \equiv 1 \ (mod\ d)$, $2^m \equiv 1 \ (mod\ d)$.

So $r|n, r|m \rightarrow r|(m,n) \rightarrow 2^{(m,n)} \equiv 1 \ (mod\ d) \rightarrow d|2^{(m,n)} - 1$

Therefore, $d = 2^{(m,n)} - 1$

# Miscellaneous Problems

Problem:

$f_n = f_{n-1} + f_{n-2}, f_2 = f_1 = 1 \to (f_m, f_n) = f_{(m,n)}$

Solution:

$f_n$ can be extended for negative $n$.

$f_0 = 0, f_{-1} = 1, f_{-2} = -1, \ldots$

We can show $f_{-2n} = -f_{2n}$ and $f_{-2n+1} = f_{2n-1}$

We can also show $\forall n, m \in Z$: $f_{n+m} = f_{n+1} f_m + f_n f_{m-1}$

Using this and induction, we can show $k|n \to f_k|f_n$ (assume $n = ki$ and run induction on $i$)

Let $(f_m, f_n) = \mathrm{d}$

$(m,n)|n, (m,n)|m \to f_{(m,n)}|f_m, f_{(m,n)}|f_n \to f_{(m,n)}|d$

$\exists x, y: mx + ny = (m,n) \to f_{(m,n)} = f_{mx+ny}$

$= f_{mx+1} f_{ny} + f_{mx} f_{ny-1}, \mathrm{d}|f_n|f_{ny}, \mathrm{d}|f_m|f_{mx} \to d|f_{(m,n)}$