9. By integration, for $x, y > 0$,

$$f_Y(y) = \int_0^y f(x, y)\,dx = \tfrac{1}{6}cy^3e^{-y}, \quad f_X(x) = \int_x^\infty f(x, y)\,dy = cxe^{-x},$$

whence $c = 1$. It is simple to check the values of $f_{X|Y}(x \mid y) = f(x, y)/f_Y(y)$ and $f_{Y|X}(y \mid x)$, and then deduce by integration that $\mathbb{E}(X \mid Y = y) = \frac{1}{2}Y$ and $\mathbb{E}(Y \mid X = x) = x + 2$.

A non-negative random variable with a zero expected value is almost surely equal to 0. This comes from the properties of the integral.

If you apply this to the random variable $(X - E(X))^2$, which is non-negative, assuming that $V(X) = E\left[(X - E(X))^2\right] = 0$ implies that $(X - E(X))^2 = 0$ a.s., which means that $X = E(X)$ a.s.

35. Ⓢ Judit plays in a total of $N \sim \text{Geom}(s)$ chess tournaments in her career. Suppose that in each tournament she has probability p of winning the tournament, independently. Let T be the number of tournaments she wins in her career.

(a) Find the mean and variance of T.

(b) Find the MGF of T. What is the name of this distribution (with its parameters)?

Solution:

(a) We have $T|N \sim \text{Bin}(N, p)$. By Adam's Law,

$$E(T) = E(E(T|N)) = E(Np) = p(1-s)/s.$$

By Eve's Law,

$$\begin{aligned}
\text{Var}(T) &= E(\text{Var}(T|N)) + \text{Var}(E(T|N)) \\
&= E(Np(1-p)) + \text{Var}(Np) \\
&= p(1-p)(1-s)/s + p^2(1-s)/s^2 \\
&= \frac{p(1-s)(s+(1-s)p)}{s^2}.
\end{aligned}$$

(b) Let $I_j \sim \text{Bern}(p)$ be the indicator of Judit winning the jth tournament. Then

$$\begin{aligned}
E(e^{tT}) &= E(E(e^{tT}|N)) \\
&= E((pe^t + q)^N) \\
&= s\sum_{n=0}^{\infty}(pe^t + 1 - p)^n(1-s)^n \\
&= \frac{s}{1-(1-s)(pe^t + 1 - p)}.
\end{aligned}$$

This is reminiscent of the Geometric MGF, which was derived in Example 6.4.3. If

$T \sim \text{Geom}(\theta)$, we have $\theta = \frac{s}{s+p(1-s)}$, as found by setting $E(T) = \frac{1-\theta}{\theta}$ or by finding $\text{Var}(T)/E(T)$. Writing the MGF of T as

$$E(e^{tT}) = \frac{s}{s + (1-s)p - (1-s)pe^t} = \frac{\frac{s}{s+(1-s)p}}{1 - \frac{(1-s)p}{s+(1-s)p}e^t},$$

we see that $T \sim \text{Geom}(\theta)$, with $\theta = \frac{s}{s+(1-s)p}$. Note that this is consistent with (a).

The distribution of T can also be obtained by a story proof. Imagine that just before each tournament she may play in, Judit retires with probability s (if she retires, she does not play in that or future tournaments). Her tournament history can be written as a sequence of W (win), L (lose), R (retire), ending in the first R, where the probabilities of W, L, R are $(1-s)p, (1-s)(1-p), s$ respectively. For calculating T, the losses can be ignored: we want to count the number of W's before the R. The probability that a result is R given that it is W or R is $\frac{s}{s+(1-s)p}$, so we again have

$$T \sim \text{Geom}\left(\frac{s}{s+(1-s)p}\right).$$

$$P(t > \max\{X_1, \ldots, X_{n-1}\}) = \prod_{i=1}^{n-1} P(X_i < t) = F(t)^{n-1}.$$

The above quantity appears if you take the probability you are interested in and condition on X_n.

$$\begin{aligned} P(X_n > \max\{X_1, \ldots, X_{n-1}\}) &= E[P(X_n > \max\{X_1, \ldots, X_{n-1}\} \\ &= E[F(X_n)^{n-1}] \\ &= \int_{-\infty}^{\infty} F(t)^{n-1} f(t)\, dt \\ &= \left[\frac{1}{n} F(u)^n\right]_{u=-\infty}^{\infty} \\ &= \frac{1}{n}. \end{aligned}$$

۴) الف) استنباط نمی‌دیم $P(A_r) = \frac{1}{r}$:

$P(+)$

استدلال اینکه چرا A_r ها مستقل اند: (این را برای ۲ تا A_r و $A_{r'}$ نشان می‌دهیم و برای تعداد

$r' > r$

بیشتر مشابه استدلال می‌شود کرد)

$$P(A_r, A_{r'}) = \cancel{P(X_r \ldots)} \; E\left[P\left(A_r, A_{r'} \;\middle|\; \left\{ \text{مجموعه } r' \text{ عدد تولید شده را بدانیم} \right\} \right) \right]$$

* این expectation ~~expected~~ روی همه حالت های ممکن از مجموعه r' عدد تولید شده خواهد بود *

$P(A_r, A_{r'} \mid \text{داشتن } r' \text{ عدد اول})$:

الان ما r' تا عدد ~~[illegible]~~ متفاوت داریم و می‌دانیم دنباله $X_1, X_2, \ldots, X_{r'}$ صرفاً

یک جایگشت از این r' عدد است (متفاوت چون احتمال دقیقاً یکی درآوردن ۰ است)

حال ما ~~[illegible]~~ شرط گذاشتن برداشتن عمومیه r' عدد A_r و $A_{r'}$ صرفا معادل یافتن تعداد همه جایگشت هایی است که عدد r ام از $r-1$ عدد قبلی‌ش بزرگتر و عدد r' ام بزرگترین عدد است. یا بزرگترین عدد رو جایگاهش رو روی r' فیکس کن برای $r'-1$ عدد دیگم هرطوری r تاشونو انتخاب کن یکی شون بزرگترینه اونو جایگاهش رو بذار r و باقی رو هرطوری خواهی بچین.

$$\frac{\binom{r'-1}{r}(r-1)!\;(r'-r-1)!}{r'!} = \frac{1}{r'} \times \frac{1}{r}$$

ب)

$$R_n = \sum_{r=1}^{n} A_r \longrightarrow$$

A_r ها مستقل

$$VAR(R_n) = \sum VAR(A_r) = \sum \left(\frac{1}{r} - \frac{1}{r^2}\right)$$

indicator var

ج)

$$T = \sum_{r=2}^{\infty} A_r \qquad A_r > 0 \longrightarrow$$

$$E(T) > \sum_{r=2}^{m} E(A_r) \qquad \forall m$$

$$\sum_{r=2}^{m} E(A_r) = \sum_{r=2}^{m} \frac{1}{r} = \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m}$$

که واضح میدانیم این سری diverge می‌کند (با این سری مقایسه‌ش کن $\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\dots\right)$

$E(E(X|Y,Z)|Y) =$ (الف) 8

$$\int \left(\int x \frac{f_{X,Y,Z}(x,y,z)}{f_{Y,Z}(y,z)} dx \right) \frac{f_{Z,Y}(z,y)}{f_Y(y)} dz =$$

$$\iint x \times \frac{f_{X,Y,Z}(x,y,z)}{f_{Y,Z}(y,z)} \times \frac{f_{Z,Y}(z,y)}{f_Y(y)} dz\, dx =$$

$$\iint x \frac{f_{X,Y,Z}(x,y,z)}{f_Y(y)} dz\, dx = \int x \frac{f_{X,Y}(x,y)}{f_Y(y)} dx =$$

$E(X|Y)$

7. We may assume without loss of generality that $\mathbb{E}X = \mathbb{E}Y = 0$. By the Cauchy–Schwarz inequality,

$$\mathbb{E}(XY)^2 = \mathbb{E}\big(X\mathbb{E}(Y \mid X)\big)^2 \le \mathbb{E}(X^2)\mathbb{E}\big(\mathbb{E}(Y \mid X)^2\big).$$

Hence,

$$\mathbb{E}\big(\mathrm{var}(Y \mid X)\big) = \mathbb{E}(Y^2) - \mathbb{E}\big(\mathbb{E}(Y \mid X)^2\big) \le \mathbb{E}Y^2 - \frac{\mathbb{E}(XY)^2}{\mathbb{E}(X^2)} = (1 - \rho^2)\,\mathrm{var}(Y).$$