9. By integration, for x, y > 0,

$$f_Y(y) = \int_0^y f(x, y) dx = \frac{1}{6} c y^3 e^{-y}, \quad f_X(x) = \int_x^\infty f(x, y) dy = c x e^{-x},$$

whence c=1. It is simple to check the values of $f_{X|Y}(x\mid y)=f(x,y)/f_Y(y)$ and $f_{Y|X}(y\mid x)$, and then deduce by integration that $\mathbb{E}(X\mid Y=y)=\frac{1}{2}Y$ and $\mathbb{E}(Y\mid X=x)=x+2$.

A non-negative random variable with a zero expected value is almost surely equal to 0. This comes from the properties of the integral.

If you apply this to the random variable $(X-E(X))^2$, which is nonnegative, assuming that $V(X)=E\left[(X-E(X))^2\right]=0$ implies that $(X-E(X))^2=0$ a.s., which means that X=E(X) a.s.

- 35. (§) Judit plays in a total of N ~ Geom(s) chess tournaments in her career. Suppose that in each tournament she has probability p of winning the tournament, independently. Let T be the number of tournaments she wins in her career.
 - (a) Find the mean and variance of T.
 - (b) Find the MGF of T. What is the name of this distribution (with its parameters)?
 Solution:
 - (a) We have $T|N \sim \text{Bin}(N, p)$. By Adam's Law,

$$E(T) = E(E(T|N)) = E(Np) = p(1-s)/s.$$

By Eve's Law,

$$Var(T) = E(Var(T|N)) + Var(E(T|N))$$

= $E(Np(1-p)) + Var(Np)$
= $p(1-p)(1-s)/s + p^2(1-s)/s^2$
= $\frac{p(1-s)(s+(1-s)p)}{s^2}$.

(b) Let $I_j \sim \text{Bern}(p)$ be the indicator of Judit winning the jth tournament. Then

$$E(e^{tT}) = E(E(e^{tT}|N))$$

$$= E((pe^{t} + q)^{N})$$

$$= s \sum_{n=0}^{\infty} (pe^{t} + 1 - p)^{n} (1 - s)^{n}$$

$$= \frac{s}{1 - (1 - s)(pe^{t} + 1 - p)}.$$

This is reminiscent of the Geometric MGF, which was derived in Example 6.4.3. If

 $T \sim \text{Geom}(\theta)$, we have $\theta = \frac{s}{s+p(1-s)}$, as found by setting $E(T) = \frac{1-\theta}{\theta}$ or by finding Var(T)/E(T). Writing the MGF of T as

$$E(e^{tT}) = \frac{s}{s + (1-s)p - (1-s)pe^t} = \frac{\frac{s}{s + (1-s)p}}{1 - \frac{(1-s)p}{s + (1-s)p}e^t},$$

we see that $T \sim \text{Geom}(\theta)$, with $\theta = \frac{s}{s + (1 - s)p}$. Note that this is consistent with (a).

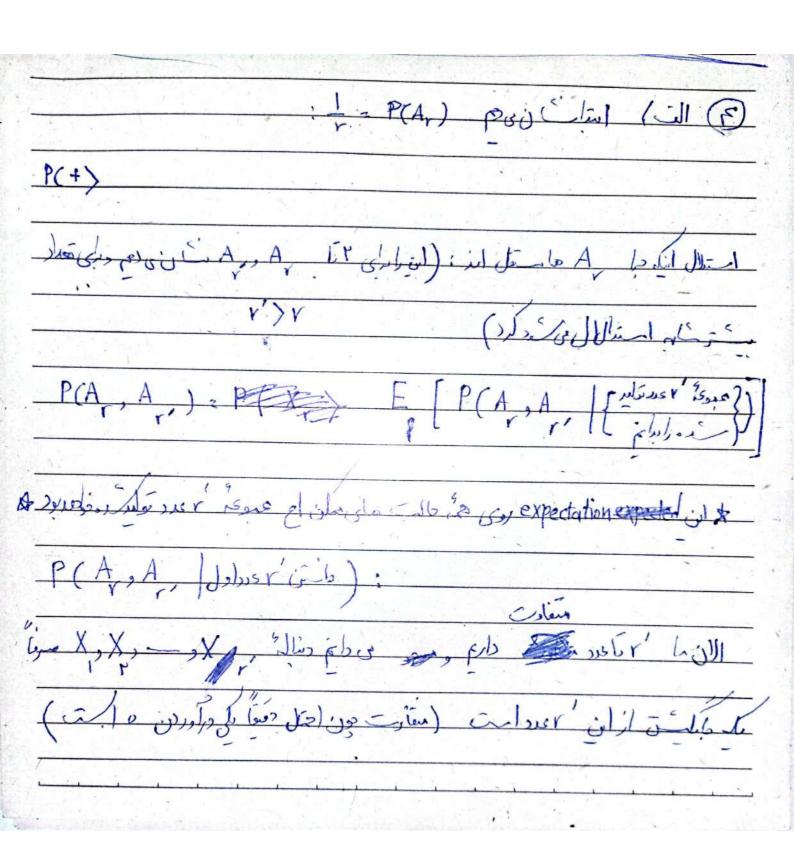
The distribution of T can also be obtained by a story proof. Imagine that just before each tournament she may play in, Judit retires with probability s (if she retires, she does not play in that or future tournaments). Her tournament history can be written as a sequence of W (win), L (lose), R (retire), ending in the first R, where the probabilities of W, L, R are (1-s)p, (1-s)(1-p), s respectively. For calculating T, the losses can be ignored: we want to count the number of W's before the R. The probability that a result is R given that it is W or R is $\frac{s}{s+(1-s)p}$, so we again have

$$T \sim \text{Geom}\left(\frac{s}{s + (1 - s)p}\right).$$

$$P(t > \max\{X_1, \dots, X_{n-1}\}) = \prod_{i=1}^{n-1} P(X_i < t) = F(t)^{n-1}.$$

The above quantity appears if you take the probability you are interested in and condition on X_n .

$$P(X_n > \max\{X_1, \dots, X_{n-1}\}) = E[P(X_n > \max\{X_1, \dots, X_{n-1}\})]$$
 $= E[F(X_n)^{n-1}]$
 $= \int_{-\infty}^{\infty} F(t)^{n-1} f(t) dt$
 $= \left[\frac{1}{n} F(u)^n\right]_{u=-\infty}^{\infty}$
 $= \frac{1}{n}.$



all de de la عدلت. با بزرات عدر وا با هسرو روی ا مال مان بای ۱-۲ عد دلم هرطوری تاكونولفًا بن على كون بزركة بن الع اونو طالما در مانى دوبال ما و مانى دوبال بعين (r'-1)(r-1)!(r'-r-1)! (r'-1)!(r'-r-1)! (r'-1)!(r'-r-1)! $\begin{array}{c|c}
R & \sum A \\
r & |cA| \\
\hline
\end{array}$ VAR(R) 2 > VAR(A) 2 > (\frac{1}{r} - \frac{1}{rr}) T. ZA

T. ZA

Arrow

F(T) F(A)

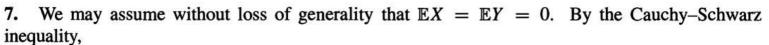
M

TE(A)

M

TE(A)

D E(E(X|Y,Z)|Y) ty (y)



$$\mathbb{E}(XY)^2 = \mathbb{E}\big(X\mathbb{E}(Y\mid X)\big)^2 \leq \mathbb{E}(X^2)\mathbb{E}\big(\mathbb{E}(Y\mid X)^2\big).$$

Hence,

$$\mathbb{E}\left(\operatorname{var}(Y\mid X)\right) = \mathbb{E}(Y^2) - \mathbb{E}\left(\mathbb{E}(Y\mid X)^2\right) \le \mathbb{E}Y^2 - \frac{\mathbb{E}(XY)^2}{\mathbb{E}(X^2)} = (1 - \rho^2)\operatorname{var}(Y).$$