# Lecture03

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Lecture03

# Linear Algebra


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## Review

$Ax = b \Rightarrow Ux = c$

- Gaussian Elimination

$$\begin{array}{c} \\ -2R_1 \\ + \\ \\ \end{array}
\begin{array}{c}\text{pivot} \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -1 & 4 & 8 \\ -1 & 8 & 2 & 12 \end{array}\right]\end{array}
\quad \boxed{R_2 - 2R_1} \quad E_{12} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 + R_1 \rightarrow E_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -2 & -4 \\ 0 & 0 & 5 & 18 \end{array}\right] \quad R_3 + 2R_2 \quad E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow \underbrace{\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -2 & -4 \\ 0 & 0 & 1 & 10 \end{array}\right]}$$

$\underline{x_3} = 10, \quad \underline{x_2} = -\frac{16}{5}, \quad \underline{x_1} = -\frac{88}{5}$

$E_{23} E_{13} E_{12} A = U \Rightarrow A = \underbrace{E_{12}^{-1} E_{13}^{-1} E_{23}^{-1}}_{} U = LU$

بالامثلثی

وارون‌پذیر با درایه‌های ۱ روی قطر



$E_{12} A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ -1 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ -1 & 8 & 2 \end{bmatrix}$

c برابر سطر jام را به سطر iام اضافه کنیم

سطر iام — ضرب c در سطر jام

زدن

$E_{ij} = i\text{ام} \rightarrow \begin{bmatrix} 1 & & & \\ & 1 & & c \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ ($c$ in column $j$)

* اگر B, C وارون‌پذیر باشند آنگاه BC.

وارون‌پذیر

BC

## The Breakdown of Elimination

- Under what circumstances could the process break down?
- Example:

$Ax=b$

$$\begin{cases} x_1+x_2+x_3=b_1 \\ 2x_1+2x_2+5x_3=b_2 \\ 4x_1+6x_2+8x_3=b_3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & 2 & 5 & b_2 \\ 4 & 6 & 8 & b_3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 3 & -2b_1+b_2 \\ 0 & 2 & 4 & -4b_1+b_3 \end{array}\right].$$

$3x_3=-2b_1+b_2$

So

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 4 & 6 & 8 & b_3 \\ 2 & 2 & 5 & b_2 \end{array}\right] \nearrow \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 2 & 4 & -4b_1+b_3 \\ 0 & 0 & 3 & -2b_1+b_2 \end{array}\right]$$

- The system is now triangular, and back-substitution will solve it. The system is called Nonsingular system.

$BC$ $i<j$ $BC_{ij}=0$

$$A=L\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & 0 & 1 \end{bmatrix} = L\begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{bmatrix} = LDU$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix} = A \rightarrow PA = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 6 & 8 \\ 2 & 2 & 5 \end{bmatrix}$$

$$P=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{array}{c} \\ i \\ j \end{array}\begin{bmatrix} 1 & \vdots & \vdots \\ \cdots & 0 & 1 \\ & 1 & 0 \end{bmatrix}$$



## The Breakdown of Elimination

- Under what circumstances could the process break down?
- Example:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & 2 & 5 & b_2 \\ 4 & 4 & 8 & b_3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 3 & -2b_1+b_2 \\ 0 & 0 & 4 & -4b_1+b_3 \end{array}\right].$$

$$\begin{cases} x_1+x_2+x_3=b_1 \\ 2x_1+2x_2+5x_3=b_2 \\ 4x_1+4x_2+8x_3=b_3 \end{cases}$$

- The system is called singular system.

$x_3=1/4(-4b_1+b_3)$

$x_3=1/3(-2b_1+b_2)$

① $-4b_1+b_3=4/3(-2b_1+b_2)$

② $\neq$

## Permutation Matrices

- The simplest permutations exchange two rows.

$$\begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{matrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Other permutations exchange more rows.

$$\begin{matrix} & 1 & 2 & 3 \\ P & \downarrow & \downarrow & \downarrow \\ & 2 & 3 & 1 \\ P^2 & \downarrow & \downarrow & \downarrow \\ & 3 & 1 & 2 \\ P^3 & \downarrow & \downarrow & \downarrow \\ & 1 & 2 & 3 \end{matrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}}_{PA}$$

- A 3 by 3 permutation matrix $P$ with $P^3 = I$:

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^3}_{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- Some power of a permutation matrix is the identity.



## Permutation Matrices

**Definition**

$P$ be a square matrix of order $n$. If $P$ is a binary matrix (every entry in it is either 0 or 1, also called 0-1 matrix or (0, 1) matrix) and there is a unique "1" in its every row and every column, then $P$ is called a permutation matrix .

- Some power of a permutation matrix is the identity.

## Problem

$P_1 = P_{12}$ $\qquad\qquad$ $P_1 = P_{54}$

$P_2 = P_{23}$ $\qquad\qquad$ $P_2 = P_{67}$

- If $P_1$ and $P_2$ are permutation matrices, so is $P_1P_2$. This still has the rows of I in some order. Give examples with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$.



## The Breakdown of Elimination

- If elimination can be completed with the help of row exchanges, then:
  - Those exchanges are done first (by $P$).
  - The matrix PA will not need row exchanges.
  - $PA$ allows the standard factorization into $L$ times $U$.
- In the nonsingular case:
  - There is a permutation matrix $P$ that reorders the rows of $A$ to avoid zeros in the pivot positions.
  - Then $Ax = b$ has a unique solution.
  - With the rows reordered in advance, $PA$ can be factored into $LU$.
- In the singular case, no P can produce a full set of pivots: elimination fails.

درایه اول (1,1)

$PA =$

$P_1 \longrightarrow$ سطر ۳ و ۵

$P_2 \longrightarrow$ سطر ۴ و ۷

$$\begin{bmatrix} u_{11} & & \\ & \ddots & \\ & & u_{nn} \end{bmatrix}$$

$x_1 \cdots x_{n-1}, x_n$



## Uniqueness

نرمال شده

پایین مثلثی $\qquad$ بالامثلثی

- In the nonsingular case: The $LDU$ factorization and the $LU$ factorization are uniquely determined by A.

$A = LDU$

$A = L'D'U'$ $\Rightarrow L = L', \; D = D', \; U = U'$

$$A = LU = L\begin{bmatrix} d_{11} & \ddots & * \\ 0 & & u_{nn} \end{bmatrix}$$

$A = LDU$

$$D = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & u_{nn} \end{bmatrix}$$

$u_{11}$ تقسیم بر $\rightarrow [1 \quad u_{12}/u_{11} \cdots u_{1n}/u_{11}]$

$A = L'D'U'$ $\Rightarrow L = L'$, $D = D'$, $U = U'$

$LDU = L'D'U'$ $\Rightarrow$ ...



## Block Matrix Multiplication

- It is often convenient to partition a matrix $M$ into smaller matrices called blocks.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 1 \\ \hline 0 & 1 & 2 & 0 \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

- Matrix operations on block matrices can be carried out by treating the blocks as matrix entries.

$$M^2 = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A^2 + BC & AB + BD \\ \hline CA + DC & CB + D^2 \end{array}\right]$$

$$\left[\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & M_4 \end{array}\right] \left[\begin{array}{c|c} N_1 & N_2 \\ \hline N_3 & N_4 \end{array}\right]$$

$MN =$

$M_1 N_1 + M_2 N_3$

$r \times s$ $p \times t$

$p = s$

$BC = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{3\times 1} [0 \ 1 \ 2]_{1\times 3}$



## Inverse Matrices

- The matrix $A$ is invertible if there exists a matrix $B$ such that $AB = BA = I$.

$A^{-1} = B$

- Not all matrices have inverses.
- If $AB = I$ and $CA = I$, then $B = C$ (prove!). Therefore inverse matrix is unique. We denote it by $A^{-1}$.

$AB = I$

$\to CAB = C$

$\to B = C$

- The matrix $A$ is invertible if and only if $AX = b$ has one and only solution for a given $b$.

$Ax = b$

$x = A^{-1}b$

- The matrix $A$ is invertible if and only if $A = LU$ where $LU$ is a triangular factorization of $A$ with no zeros on the diagonal of $U$.

triangular factorization of $A$ with no zeros on the diagonal of $U$.

$U = \begin{bmatrix} & 0 & \end{bmatrix}$

$d_i$



## The Calculation of $A^{-1}$

The inverse of $A$ is written $A^{-1}$ in which $AA^{-1} = I$. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

We want to find $A^{-1}$ such that $AA^{-1} = I$, so consider the columns of $A^{-1}$ as $x_1, x_2, x_3$, that means $A^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$.

ستون

سطر ۱

$$AA^{-1} = I =$$

$$\underset{1\times1}{A} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}_{1\times3} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}$$

$$\begin{bmatrix} Ax_1 & Ax_2 & Ax_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}$$

So $\quad Ax_1 = e_1 \quad Ax_2 = e_2 \quad Ax_3 = e_3$



## The Gauss-Jordan Method

- Thus we have three systems of equations (or $n$ systems).
- They all have the same coefficient matrix $A$.
- The right-hand sides $e_1, e_2, e_3$ are different, but elimination is possible on all systems simultaneously.

## $A^{-1}$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right]$$

$A \quad e_1 \quad e_2 \quad e_3$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

- We have

$$\left[\, U \,\middle|\, L^{-1} \,\right]$$

$Ax_1 = e_1$

$Ax_2 = e_2$

$Ax_3 = e_3$

$A^{-1} = [x_1 \; x_2 \; x_3]$



## $A^{-1}$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ 0 & 1 & 0 & \frac{4}{8} & -\frac{3}{8} & -\frac{2}{8} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

## Problem

If $A$ is an $n \times n$ matrix, the followings are equivalent.

- A is invertible.
- The system of equations $AX = 0$ has only the trivial solution $X = 0$.
- The system of equations $AX = b$ has exactly one solution.
- $A$ is a product of elementary and permutation matrices.



## The inverse of 3 by 3 Hilbert matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{bmatrix}$$

## The inverse of 3 by 3 Hilbert matrix

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$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$



## The inverse of 3 by 3 Hilbert matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{bmatrix}$$

## The inverse of 3 by 3 Hilbert matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{bmatrix}$$



## The inverse of 3 by 3 Hilbert matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{bmatrix}$$

## The inverse of 3 by 3 Hilbert matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{bmatrix} \qquad H^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$



## Hilbert matrix

The matrix $H$ is an example of a family of matrices which are called **Hilbert** matrices. The $n$ by $n$ Hilbert matrix is

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

where $H_{ij} = \frac{1}{i+j+1}$. For every $n$, $n \times n$ Hilbert matrix is invertible and

its inverse has integer entries.

## Application of inverse matrix!

**Problem**. Let $AB = 3A + 4B$. Show $AB = BA$.

**Solution**.

- $AB - 3A - 4B = 0$.



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**Solution**.

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- $(A-?)(B-?) =?$.

## Application of inverse matrix!

**Problem**. Let $AB = 3A + 4B$. Show $AB = BA$.

**Solution**.

- $AB - 3A - 4B = 0$.
- $(A-?)(B-?) =?$.
- $(A - 4I)(B - 3I) = 12I$



## Application of inverse matrix!

**Problem**. Let $AB = 3A + 4B$. Show $AB = BA$.

**Solution**.

- $AB - 3A - 4B = 0$.
- $(A-?)(B-?) =?$.
- $(A - 4I)(B - 3I) = 12I$
- $(A - 4I)(\frac{1}{12}B - \frac{1}{4}I) = I$. $\quad \Rightarrow \quad \underbrace{(A - 4I)}_{\mathcal{A}}\underbrace{(\frac{1}{12}B - \frac{1}{4}I)}_{\mathcal{A}^{-1}} = I$

## Application of inverse matrix!

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**Solution**.

- $AB - 3A - 4B = 0$.
- $(A - ?)(B - ?) = ?$.
- $(A - 4I)(B - 3I) = 12I$
- $(A - 4I)(\frac{1}{12}B - \frac{1}{4}I) = I$. $\quad \Rightarrow \quad \underbrace{(A - 4I)}_{\mathcal{A}}\underbrace{(\frac{1}{12}B - \frac{1}{4}I)}_{\mathcal{A}^{-1}} = I$
- $\underbrace{(\frac{1}{12}B - \frac{1}{4}I)}_{\mathcal{A}^{-1}}\underbrace{(A - 4I)}_{\mathcal{A}} = I \Rightarrow BA = AB$.



## Problem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

i. $A^2 - (a + d)A + (ad - bc)I = 0$.

ii. If $ad - bc \neq 0$, then $A$ is invertible.

iii. Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, obtain $A^{100}$.

## Transpose Matrix

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 9 & 10 \end{bmatrix}$$



## Transpose Matrix

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 9 & 10 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 0 \\ -3 & 9 \\ 5 & 10 \end{bmatrix}$$

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## Transpose Matrix

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 9 & 10 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 0 \\ -3 & 9 \\ 5 & 10 \end{bmatrix}$$

In general:



## Transpose Matrix

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In general:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & \dots & a_{m1} \\ a_{12} & \dots & a_{m2} \\ \vdots & \vdots & \vdots \\ a_{1n} & \dots & a_{mn} \end{bmatrix}$$

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i) $(AB)^T = B^T A^T$.



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i) $(AB)^T = B^T A^T$.

ii) $(A^T)^{-1} = (A^{-1})^T$.

## Symmetric Matrices

- A **symmetric matrix** is a matrix that equals its own transpose: $A^T = A$; i.e. $A_{ij} = A_{ji}$.



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## Symmetric Matrices

- A **symmetric matrix** is a matrix that equals its own transpose: $A^T = A$; i.e. $A_{ij} = A_{ji}$.
- A symmetric matrix $A$ is necessarily square.
- A symmetric matrix $A$ need not be invertible.



## Symmetric Matrices

- A **symmetric matrix** is a matrix that equals its own transpose: $A^T = A$; i.e. $A_{ij} = A_{ji}$.
- A symmetric matrix $A$ is necessarily square.
- A symmetric matrix $A$ need not be invertible.
- $A$ is a symmetric matrix if and only if $A^{-1}$ a symmetric matrix. (Why?)

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  (1) Then $U$ is the transpose of $L$, and the symmetric factorization becomes $A = LDL^T$.

  (2) If $D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$ where $d_i \geq 0$ for each $1 \leq i \leq n$, then $A$ is factorized $A = LL^T$ in which $L$ is a lower triangular matrix.

## Special Matrices

- Consider differential equation $-\frac{d^2u}{dx^2} = f(x)$ for $0 \leq x \leq 1$ with the unknown function $u(x)$ which shows the temperature distribution in a rod with ends fixed at $0^{\cdot}C$ at each end of the interval: $u(0) = 0$ and $u(1) = 0$ ( a *boundary condition*).



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- We can only accept a finite amount of information about $f(x)$, say its values at $n$ equally spaced points $x = h, x = 2h, \cdots, x = nh$.
- We compute approximate values $u_1, \ldots, u_n$ for the true solution $u(x)$ at these same points. At the ends $x = 0$ and $x = 1 = (n+1)h$, the boundary values are $u_0 = 0$ and $u_{n+1} = 0$.



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- $$\begin{aligned}\frac{d^2u}{dx^2} \approx \frac{\Delta^2 u}{\Delta x^2} = \frac{\Delta}{\Delta x}\left(\frac{\Delta u}{\Delta x}\right) &= \frac{\frac{u(x+h)-u(x)}{h} - \frac{u(x)-u(x-h)}{h}}{h} \\ &= \frac{u(x+h)-2u(x)+u(x-h)}{h^2}.\end{aligned}$$



# ...

For $n = 5$, since $u_0 = u_{n+1} = 0$, we have

$$\begin{cases} 2u_1 - 1u_2 + & = h^2 f(h) \\ -1u_1 + 2u_2 - 1u_3 & = h^2 f(2h) \\ \quad - 2u_2 + 2u_3 - 1u_4 & = h^2 f(3h) \\ \quad\quad - 1u_3 + 2u_4 - 1u_5 & = h^2 f(4h) \\ \quad\quad\quad - 1u_4 + 2u_5 & = h^2 f(5h) \end{cases}$$

So

$$\underbrace{\begin{bmatrix} 2 & 1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = h^2 \begin{bmatrix} f(h) \\ f(2h) \\ f(3h) \\ f(4h) \\ f(5h) \end{bmatrix}$$

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1) **The matrix $A$ is tridiagonal** $\Rightarrow$ a tremendous simplification to Gaussian elimination.

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$$\begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & -\frac{3}{4} & 1 & \\ & & & -\frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{1} & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \frac{5}{4} & \\ & & & & \frac{6}{5} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & & & \\ & 1 & -\frac{2}{3} & & \\ & & 1 & -\frac{3}{4} & \\ & & & 1 & -\frac{4}{5} \\ & & & & 1 \end{bmatrix}$$



# . . .

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- **The matrix is positive definite**. This extra property says that the pivots are positive.



## . . .

*Thank You!*