Lecture16

Saturday, November 13, 2021 4:00 PM

Lecture16

Linear Algebra

Samira Hossein Ghorban
s.hosseinghorban@ipm.ir

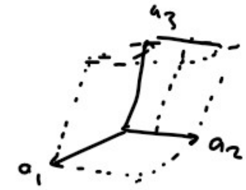
Fall, 2021

. . .

Determinant

What is volume?

n-خطی $\leftarrow \Phi : \underbrace{V \times \cdots \times V}_{n \text{ بار}} \longrightarrow \mathbb{R}$



- Every $n-$dimensional parallelepiped with $\{a_1, \ldots, a_n\}$ as legs is associated with a real number, called its volume which has the following properties:

  1. If we stretch a parallelepiped by multiplying one of its legs by a scalar λ, its volume gets multiplied by λ.
  2. If we add a vector w to i-th legs of a $n-$dimensional parallelepiped with $\{a_1, \ldots, a_i, a_{i+1}, \ldots, a_n\}$, then its volume is the sum of the volume from $\{a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n\}$ and the volume of $\{a_1, \ldots, a_{i-1}, w, a_{i+1}, \ldots, a_n\}$.
  3. The volume changes sign when two legs are exchanged.
  4. The volume of the parallelepiped with $\{e_1, \ldots, e_n\}$ is one.

$\Phi(a_1, \ldots, a_{i-1}, a_i + w, a_{i+1}, \ldots, a_n) = \Phi(a_1, \ldots, a_i, \ldots, a_n) + \Phi(a_1, \ldots, a_{i-1}, w, a_{i+1}, \ldots, a_n)$

$\Phi(a_1, \ldots, a_{i-1}, a_i + \lambda w, a_{i+1}, \ldots, a_n) = \Phi(a_1, \ldots, a_n) + \lambda \Phi(a_1, \ldots, a_{i-1}, w, \ldots, a_n)$

Φ نسبت به تک تک آرگ‌ها خطی است.

Volume map as an n-alternating multilinear map

$\Phi : \underbrace{V \times \cdots \times V}_{1 \text{ تا } n} \longrightarrow \mathbb{R}$

$\Phi(a_1, \ldots, a_i, a_{i+1}, \ldots, a_j, \ldots, a_n) = -\Phi(a_1, \ldots, a_j, a_{i+1}, \ldots, a_i, a_{j+1}, \ldots, a_n)$

- More formally, the volume is a n-alternating multilinear map on all n-parallelepipeds such that the volume of the standard unit parallelepiped is one.

نکته:

$\Phi(a_1, \ldots, v, a_{i+1}, \ldots, a_{j-1}, v, a_{j+1}, \ldots, a_n) = 0$

- Thus, the volume is n-alternating multilinear map

$$\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

such that V is a linear space with dimension n and $\phi(e_1, \ldots, e_n) = 1$.

فرض $\{e_1, \ldots, e_n\}$ پایه استاندارد $V = \mathbb{R}^n$

و $a_1, \ldots, a_n \in \mathbb{R}^n \Rightarrow a_i = \sum_{j_i=1}^{n} a_{i j_i} e_{j_i}$

$\Phi(a_1, \ldots, a_n) = \Phi\left(\sum_{j_1=1}^{n} a_{1 j_1} e_{j_1}, \ldots, \sum_{j_n=1}^{n} a_{n j_n} e_{j_n}\right) = \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1 j_1} \cdots a_{n j_n} \Phi(e_{j_1}, \ldots, e_{j_n})$

$n^n \quad n!$

$\Phi(e_{j_1}, \ldots, e_{j_n}) \neq 0 \iff \{j_1, \ldots, j_n\} = \{1, \ldots, n\}$

$\sigma = \begin{pmatrix} 1 & \cdots & n \\ \downarrow & & \downarrow \\ j_1 & & j_n \end{pmatrix}$

n-alternating multilinear maps

$3, 2, 1 \qquad 1, 2, 3$

$\Phi(e_3, e_2, e_1)$

$\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$

- Let $\phi : \underbrace{V \times \cdots \times V}_{5} \to \mathbb{R}$ be an 5-alternating multilinear map. $\{a_1, \ldots, a_5\}$

ترتیب $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$

- So, $\phi(a_2, a_1, \ldots, a_5) = -\phi(a_1, a_2, \ldots, a_5)$.

یک جابجایی است اگر تنها دو تا را σ تغییر دهد

- Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$$

S_n = مجموعه همه جایگشت‌های $\{1, \ldots, n\}$

n-alternating multilinear maps

$3, 2, 1 \qquad 1, 2, 3$

$\phi(e_3, e_2, e_1)$

$\{a_1, \ldots, a_5\}$

- Let $\phi : \underbrace{V \times \cdots \times V}_{5} \to \mathbb{R}$ be an 5-alternating multilinear map.
- So, $\phi(a_2, a_1, \ldots, a_5) = -\phi(a_1, a_2, \ldots, a_5)$.
- Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$$

- That means that σ satisfies

$$\sigma(1) = 3 \quad \sigma(2) = 4 \quad \sigma(3) = 1 \quad \sigma(4) = 5 \quad \sigma(5) = 2.$$

- $\phi(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}, e_{\sigma(4)}, e_{\sigma(5)}) =? \phi(e_1, e_2, e_3, e_4, e_5)$.

$=$

$\sigma = \begin{pmatrix} 1 & \cdots & n \\ j_1 & \cdots & j_n \end{pmatrix}$

$\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$

ترتیب $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$

یک جایگشت است = اگر و تنها اگر σ یک به یک و پوشا e_3

$S_n =$ مجموعه همه جایگشت‌های $\{1, \ldots, n\}$

$|S_n| = n!$

Classification of n-alternating multilinear maps

- Let $\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$ be an n-alternating multilinear map.
- $\phi(a_1, \ldots, a_{i-1}, v, a_{i+1}, \ldots, a_{j-1}, v, a_{j+1}, \ldots, a_n) = 0$ (why?)
- Since the map is n- multilinear map, we have

$\phi(e_1, e_2, e_3, e_4)$

$\phi(e_2, e_1, e_4, e_3)$

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1\,2)(3\,4)$

$$\phi(a_1, \ldots, a_n) = \phi\left(\sum_{j_1=1}^{n} a_{1j_1} e_{j_1}, \ldots, \sum_{j_n=1}^{n} a_{nj_n} e_{j_n}\right)$$

$$= \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} \cdots a_{nj_n} \phi(e_{j_1}, \ldots, e_{j_n}) = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \phi(e_{\sigma(1)}, \ldots, e_{\sigma(n)})$$

n^n

$\sigma = \begin{pmatrix} 1 & \cdots & n \\ j_1 & \cdots & j_n \end{pmatrix}$

$\sigma(i) = j_i \quad \sigma(1) = j_1$

$\phi(e_1, \ldots, e_n)$ $\pm$ یا

- Now, we note that if any two of the e_{j_k}'s are equal, then

$$\phi(e_{j_1}, \ldots, e_{j_n}) = 0.$$

$\overset{?}{=}$

$\phi(e_1, \ldots, e_n) = 1$

$\phi(e_1, e_2, e_3) = -\phi(e_2, e_1, e_3)$

- So, we may remove the corresponding term from the sum.

Classification of n-alternating multilinear maps

$$\phi(a_1, \ldots, a_n) = \phi\left(\sum_{j_1=1}^{n} a_{1j_1} e_{j_1}, \ldots, \sum_{j_n=1}^{n} a_{nj_n} e_{j_n}\right)$$

$$= \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} \cdots a_{nj_n} \phi(e_{j_1}, \ldots, e_{j_n})$$

- We want all tuples of j_k's such that each pair of j_k's are mutually distinct.

Classification of n-alternating multilinear maps

$$\phi(a_1, \ldots, a_n) = \phi\left(\sum_{j_1=1}^{n} a_{1j_1} e_{j_1}, \ldots, \sum_{j_n=1}^{n} a_{nj_n} e_{j_n}\right)$$
$$= \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} \cdots a_{nj_n} \phi(e_{j_1}, \ldots, e_{j_n})$$

- We want all tuples of j_k's such that each pair of j_k's are mutually distinct.
- That is, $(j_1, \ldots, j_n)$ must be a permutation of $(1, \ldots, n)$.
- We denote the set of all permutations of $(1, \ldots, n)$ by S_n.
- $(j_1, \ldots, j_n)$ is a permutation of $(1, \ldots, n)$ means that there is $\sigma \in S_n$ such that $j_k = \sigma(k)$.

Classification of n-alternating multilinear maps

- For an n-alternating multilinear map

$$\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

we have

$$\phi(a_1, \ldots, a_n) = \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} \cdots a_{nj_n} \phi(e_{j_1}, \ldots, e_{j_n})$$
$$= \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \phi\left(e_{\sigma(1)}, \ldots, e_{\sigma(n)}\right)\right)$$

n-alternating multilinear maps

- Let $\phi : \underbrace{V \times \cdots \times V}_{5} \to \mathbb{R}$ be an 5-alternating multilinear map.
- So, $\phi(e_2, e_1, \ldots, e_5) = -\phi(e_1, e_2, \ldots, e_5)$.
- Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$$

- That means that σ satisfies

$$\sigma(1) = 3 \quad \sigma(2) = 4 \quad \sigma(3) = 1 \quad \sigma(4) = 5 \quad \sigma(5) = 2.$$

- $\phi(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}, e_{\sigma(4)}, e_{\sigma(5)}) =? \phi(e_1, e_2, e_3, e_4, e_5)$.

n-alternating multilinear maps

$\phi(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}, e_{\sigma(4)}, e_{\sigma(5)})$

$= + (e_1, e_2, e_3, e_4, e_5)$

- Look at the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}.$$

$\sigma = \begin{pmatrix} 2 & 4 & 5 \\ 4 & 5 & 2 \end{pmatrix}$

- 1 is sent to 3. But 3 is sent back to 1.This is given by the transposition $(1, 3)$.
- Now look at what is left $\{2, 4, 5\}$.
  1. Look at 2. Then 2 is sent to 4.
  2. Now, 4 is sent to 5.
  3. Finally 5 is sent to 2.
  4. So another part of the cycle type is given by the 3-cycle $(2, 4, 5)$.
  5. $(2, 4, 5) = (2, 4)(4, 5)$.
- $\sigma = (2, 4)(4, 5)(1, 3)$.
- $\phi(e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(5)}) = \phi(e_1, e_2, \ldots, e_5)$.

$2 \to 4 \to 5$

$-\underbrace{(2\ 4}_{6}\ 5) = \underbrace{\overset{\tau_2}{(2\ 4)}\ \overset{\tau_1}{(4\ 5)}}$

$4 \to 5 \to 2$

$\tau_1 \begin{pmatrix} 2 & 4 & 5 \\ 2 & 5 & 4 \end{pmatrix}$

$\tau_2 \begin{pmatrix} 4 & 5 & 2 \end{pmatrix}$

Permutations

Definition

A permutation of a finite set X is **even** if it can be written as the product of an even number of transpositions, and it is **odd** if it can be written as a product of an odd number of transpositions.

$\sigma = (1\,3)(2\,4\,5)$ دوتا فرد

Example

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} = (2,4)(4,5)(1,3) \text{ is an odd permutation}$$

$\sigma = (1\,3)(2\,4\,5)$

$\sigma \quad \begin{matrix}1\\3\end{matrix}$

$\sigma = \sigma_2 \sigma_1 \cdots \sigma_r$

$\sigma = \sigma_1 \cdots \sigma_r$

قضیه: هر جایگشت $\sigma \in S_n$ را می‌توان به صورت حاصلضرب سیکل‌های مجزا بنویسیم. صرف‌نظر از ترتیب دورهای این تجزیه، تجزیه به حاصلضرب دورهای مجزا یکتاست.

The sign of a permutation

Definition

The sign of a permutation σ, denoted by $\text{sgn}(\sigma)$ and

$\sigma = (\quad)(\quad)(\quad)(\quad) \quad +1$

$$\text{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even}, \\ -1 & \text{if } \sigma \text{ is odd}. \end{cases}$$

$i \in \{1, \dots, n\}$

$\sigma^r(i) = i$

$\{i, \sigma(i), \dots, \sigma^{r-1}(i)\} \subseteq \{1, \dots, n\}$

$(i, \sigma(i), \dots, \sigma^{r-1}(i))$

$\{1, \dots, n\} \setminus \{i, \sigma(i), \dots, \sigma^{r-1}(i)\}$

Example

Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} = (2,4)(4,5)(1,3)$. Then $\text{sgn}(\sigma) = -1$ and

$$\phi\left(e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(5)}\right) = \text{sgn}(\sigma)\phi(e_1, e_2, \dots, e_5)$$

نتیجه. هر جایگشتی را می‌توان به حاصلضرب از ترانهش‌ها تجزیه کرد.

$(a_1, \dots, a_m) =$

$(a_1\, a_2)(a_2\, a_3) \cdots (a_{m-1}\, a_m)$

Classification of n-alternating multilinear maps

- For an n-alternating multilinear map

$$\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

$(a_1, \dots, a_m) =$

$(a_1\, a_m)(a_1\, a_{m-1}) \cdots (a_1\, a_2)$

we have

$$\phi(a_1, \dots, a_n) = \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} \cdots a_{nj_n} \phi\left(e_{j_1}, \dots, e_{j_n}\right)$$

$$= \sum \left(\prod^{n} a_{i\sigma(i)} \phi\left(e_{\sigma(1)}, \dots, e_{\sigma(n)}\right) \right)$$

قضیه. اگر $\sigma \in S_n$ را بتوان هم به حاصلضرب r ترانهش و هم حاصلضرب s ترانهش تجزیه کنید

$$
\begin{aligned}
&= \sum_{\sigma\in S_n}\left(\prod_{i=1}^{n} a_{i\sigma(i)}\phi\left(e_{\sigma(1)},\dots,e_{\sigma(n)}\right)\right)\\
&= \left(\sum_{\sigma\in S_n}\prod_{i=1}^{n} a_{i\sigma(i)}\right)\operatorname{sgn}(\sigma)\phi\left(e_1,\dots,e_n\right)\\
&= \left(\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{i\sigma(i)}\right)\phi\left(e_1,\dots,e_n\right)
\end{aligned}
$$

n ترانهش، هم حاصلضرب S ترانهش بنویسید
آنگاه ۰ یا ۲ یک هر مرحله جابجا می‌شود.

The result of classification of n-alternating multilinear map

$T: \mathbb{R}^n \longrightarrow \mathbb{R}$ $\quad [T(e_i)]\,[\;]$

$T(e_1), \dots, T(e_n)$

- An n-alternating multilinear map

$$\phi : \underbrace{V\times\cdots\times V}_{n}\to\mathbb{R}$$

$\phi(e_1, \dots, e_n)$

$\langle\;\rangle : \mathbb{R}^n\times\mathbb{R}^n \longrightarrow \mathbb{R}$

is given with

$\langle e_i, e_j\rangle$

$$\underline{\underline{A}} = [a_1 \dots a_n] \quad \phi(a_1,\dots,a_n) = \left(\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}a_{i\sigma(i)}\right)\underbrace{\phi(e_1,\dots,e_n)}_{1}$$

$a_i = \sum_{j=1}^{n} a_{ij}\, e_j$

$\det A = \phi(a_1, \dots, a_n) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod a_{i\sigma(i)}$

$X^T G X$

Determinant

- Let a_i be the i-th row of $A = [a_{ij}]$.
- The determinant of A is defined by

$$\det A = \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}a_{i\sigma(i)}.$$

$\det : \underbrace{\mathbb{R}^n\times\dots\times\mathbb{R}^n}_{\underline{\underline{M_n(\mathbb{R})}}} \longrightarrow \mathbb{R}$

- Thus, The determinant is an n-alternating multilinear map $\det : M_n(\mathbb{R}) \to \mathbb{R}$ such that $\det I = 1$. That means:
  1. The determinant changes sign when two rows are exchanged.
  2. The determinant of the identity matrix is 1.
  3. The determinant depends linearly on the each row.

Properties of the Determinant

- For 2 by 2 matrix

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

4. If two rows of A are equal, then $\det A = 0$ (why?)
5. Subtracting a multiple of one row from another row leaves the same determinant.(why?)
6. If A has a row of zeros, then $\det A = 0$ since the map det is n-multilinear.

Properties of the Determinant

7. If A is triangular then $\det A = a_{11}a_{22}\cdots a_{nn}$ (why?)
8. If A is singular, then $\det A = 0$. If A is invertible, then $\det A \neq 0$ (why?)

Properties of the Determinant

9. The transpose of A has the same determinant as A itself: $\det A = \det A^T$ (why?)

10. The determinant of AB is the product of $\det A$ times $\det B$ (why?)

11. Let A be an invertible matrix. Then $\det A \neq 0$.

Properties of the Determinant

12. Let $A \in M_r(\mathbb{R}), B \in M_{rs}(\mathbb{R})$ and $C \in M_s(\mathbb{R})$, then

$$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det A \det C.$$

Proof:

Properties of the Determinant

13. Let $A, B, C, D \in M_n(\mathbb{R})$. If $CD = DC$ then

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BC).$$

- Note that it is also true if $AC = CA$ or $AB = BA$ or $BD = DB$.
- Proof:

Properties of the Determinant

14. (**Schur formula**) Let $A \in M_n(\mathbb{R})$. Let

$$A \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square matrices. Then

$$\det A = \big(\det A_{11}\big) \det \big(A_{22} - A_{21}A_{11}^{-1}A_{12}\big).$$

Proof. The following identity is easy verified:

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}.$$

Thank You!