Lecture09

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Lecture09

Linear Algebra

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Linear functions

- Suppose that $A \in M_{mn}(\mathbb{R})$. Let $x \in \mathbb{R}^n$:
- Consider the mapping $x \longmapsto Ax$

$$\begin{cases} T : \overset{\text{دامنه}}{\mathbb{R}^n} \to \mathbb{R}^m \\ T(x) = \underset{m\times n}{A}x \end{cases}$$

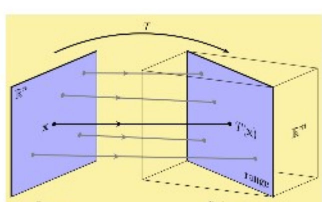
$T(x+y) = T(x) + T(y)$

$T(cx) = cT(x)$

تابع خطی

$N(A) = \underline{\underline{N(T)}} = \{x \mid T(x) = 0\} \subseteq \mathbb{R}^2$

$Im(T) = C(A)$



$[A_1 \cdots A_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

- $\{Ax \mid x \in \mathbb{R}^n\} = \{x_1A_1 + \cdots + x_nA_n \mid x \in \mathbb{R}^n\} = C(A) \subseteq \mathbb{R}^m$.
- $\{x \in \mathbb{R}^n \mid Ax = 0\} = N(A) \subseteq \mathbb{R}^n$.

Linear functions

- A function

$$\begin{cases} T : \mathbb{R}^n \to \mathbb{R}^m \\ T(x) = Ax \end{cases}$$

has two properties:

1. $T(x+y) = T(x) + T(y)$, for each $x, y \in \mathbb{R}^n$.
2. $T(cx) = cT(x)$, for each $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Linear functions

Linear function

Let V and W be two linear spaces. Every function $T : V \to W$ that meets two below requirements is a linear function (transformation):

1. $T(x+y) = T(x) + T(y)$, for each $x, y \in V$.
2. $T(cx) = cT(x)$, for each $x \in V$ and $c \in \mathbb{R}$.

Linear function

- Let $T : V \to W$ be a linear function.
- $Im(T) = \{T(x) \mid x \in V\}$ is a linear subspace of W and is called image of T.

[illegible]

$T(x), T(y) \in Im(T)$, $c \in R$

$\Rightarrow cT(x) + T(y) = T(cx+y) \in Im(T)$

- $N(T) = \{x \in V \mid T(x) = 0\}$ is a linear subspace of $V$and is called nullspace of T.
- The dimension of $Im(T)$ is called **rank** of T and denoted by rank(T).

$A \in M_{mn}(\mathbb{R})$

$T : V \to W$

$T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$

$T(x) = \underline{Ax}$ | T خطی

$N(A)$ | $N(T)$

rank (A) | rank T

$\dim \underline{C(A)}$

$= \dim I(T)$

Injective and surjective linear functions

مثال

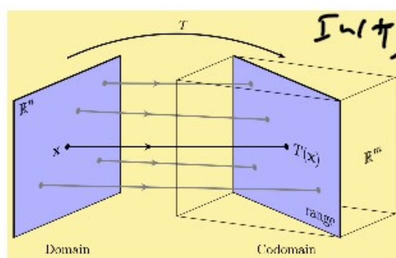
$$\begin{cases} T : \mathbb{R}^n \to \mathbb{R}^m \\ T(x) = Ax \end{cases}$$

پوشا

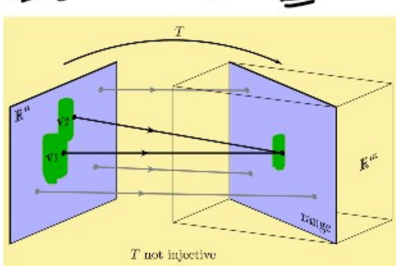
$Im(T) = \mathbb{R}^m$

یک‌به‌یک

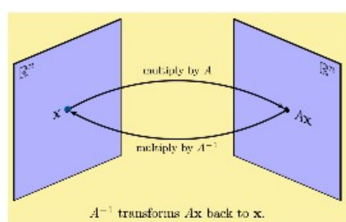
$x \longmapsto \underline{Ax}$



$Im(T)$



T not injective



A^{-1} transforms Ax back to x.

$T : \mathbb{V} \longrightarrow \mathbb{W}$

$\dim V < \infty$

$\dim V = n$

$v \in V$ $[v]_B$

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$T : \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$

$\mathbb{R}^n \xleftarrow{T^{-1}} \mathbb{R}^m$

$T : \mathbb{R}^n \longrightarrow \underline{(Im\,T)} \subseteq \mathbb{R}^m$

$\mathbb{R}^n \xleftarrow{T^{-1}} Im\,T$

[illegible]

[illegible]

Linear functions

یک‌به‌یک

- $T : V \to W$ is injective if and only if $\underline{N(T) = \{0\}}$.

$a \in N(T)$ $T(a) = 0$

$\forall b \in V$ $T(a+b) = T(b) \Rightarrow a+b = b \Rightarrow a = 0$

$e \xleftarrow{T^{-1}} \operatorname{Im} T$

Linear functions

یک به یک

- $T : V \to W$ is injective if and only if $N(T) = \{0\}$.

$a \in N(T) \quad T(a) = 0$

$\forall b \in V \quad T(a+b) = T(b) \Rightarrow a+b = b \Rightarrow a = 0$

$\Leftarrow \; T(a) = T(b) \Rightarrow T(a-b) = 0 \Rightarrow a-b \in N(T) \Rightarrow a = b$

$\Rightarrow$ یک به یک

Example

What linear function takes e_1 and e_2 to Ae_1 and Ae_2, respectively?

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to Te_1 = \underbrace{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}_{w_1} \qquad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to Te_2 = \underbrace{\begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}}_{w_2}$$

$B = \{e_1, e_2\} \quad \{w_1, w_2\}$

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ with the above conditions. Thus $x = x_1e_1 + x_2e_2$ for $x \in \mathbb{R}^2$.

$$T(x) = x_1T(e_1) + x_2T(e_2) = x_1 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}}_{A} x = Ax.$$

$T : \mathbb{R}^2 \to \mathbb{R}^3$

$T(e_1)$

$T(e_2)$

$T : \mathbb{R}^2 \to \mathbb{R}^3$

$Te_1 \quad Te_2$

Linear functions

- Suppose that V is a finite linear space with dimension n. Let $B = \{v_1, \ldots, v_n\}$ be a basis for V and $\{w_1, \ldots, w_n\} \subseteq W$. Then there is a unique linear function $T : V \to W$ such that

$$T(v_i) = w_i$$

$T : V \to W$

$T \to$ ماتریس نظیر

$\dim V = n$

$B = \{v_1, \ldots, v_n\}$

$v_1 \to w_1$

$\vdots$

$v_n \to w_n$

$T(v) = \underline{T}(\sum c_i v_i) = \sum c_i T(v_i) = \sum c_i w_i$

$= \underbrace{[w_1 \cdots w_n]}_{A} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$= Ac = A[v]_\beta$

$\vdots$

$v_n \to w_n$

$v \in V \quad v = \sum c_i v_i$

Linear functions

$Tv \in Im(T)$, $v \in V$: $v = \sum_{j=1}^{n} c_j v_j \Rightarrow T(v) = \cancel{\sum_{j=1}^{r} c_j T(v_j)} + \sum_{j=r+1}^{n} c_j T(v_j) \in Span(\{*\})$

$\underbrace{\dim C(A)} + \dim N(A) = n$

$\text{rank}(A) + \dim N(A) = n$

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$A \longrightarrow R$

- If V is a finite dimensional linear space and W is a linear space and

$$T : V \to W$$

$\dim V = n$

then

$$\underline{\dim(ImT)} + \dim(N(T)) = n$$

$\dim N(T) = r$

$\{v_1, \ldots, v_r\} \longrightarrow N(T)$ پایه برای

$\{v_1, \ldots, v_r, \underline{v_{r+1}, \ldots, v_n}\}$

ادعا: $\{T(v_{r+1}), \ldots, T(v_n)\} \longrightarrow Im(T)$ پایه

$*$ ثابت: $\sum c_i T(v_{r+i}) = 0 \overset{\text{خطی}}{\Rightarrow} T\left(\sum_{i=1}^{n-r} c_i v_{r+i}\right) = 0 \Rightarrow \sum_{i=1}^{n-r} c_i v_{r+i} \in N(T)$

$\Rightarrow \sum c_i v_{r+i} = 0 \Rightarrow c_1 = \cdots = c_{n-r} = 0$

$* \ \underline{Im(T) \overset{2}{=} Span(\{Tv_{r+1}, \ldots, Tv_n\})}$ $*$

Does every linear function lead to a matrix?

Linear functions Represented by Matrices

$B' = \{w_1, \ldots, w_m\}$

- Assume $\dim(V) = n$ and $T : V \to W$. $T(v_i) \in W$

  $B = \{v_1, \ldots, v_n\}$

- Let $\{v_1, \ldots, v_n\}$ be a basis for V. $[T(v_i)]_{B'}$

- If $v \in V$, then there are $c_i \in \mathbb{R}$ such that $v = c_1 v_1 + \cdots + c_n v_n$.

- By Linearity,

$$T(v) = c_1 T(v_1) + \cdots + c_n T(v_n)$$

$$= \underbrace{\begin{bmatrix} T(v_1) & \cdots & T(v_n) \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

- $A = \begin{bmatrix} T(v_1) & \cdots & T(v_n) \end{bmatrix}$ is a matrix representation of T.

r

$T : \mathbb{R}^n \to \mathbb{R}^m$

$T(v_i) \in \mathbb{R}^m$

$\dim(\operatorname{Im} T) + \dim N(T) = n$

Example (the operation of **differentiation**)

- Suppose that

$$P_n = \{p(t) = a_0 + a_1 t + \cdots + a_n t^n \mid a_i \in \mathbb{R}, 0 \leqslant i \leqslant n\}.$$

- $\dim(P_n) = n + 1.$
- Consider

$$\frac{d}{dt} : P_3(x) \to P_3(x)$$

$$\frac{d}{dt}(a_3x^3 + a_2x^2 + a_1x + a_0) = 3a_3x^2 + 2a_2x + a_1$$

- This is a linear function.
- It is surjective since $Im(\frac{d}{dt}) = P_2$.
- It is not injective since $N(\frac{d}{dt}) = \mathbb{R} \neq \{0\}$.   یک به یک نیست.
-

$$\dim(Im(\tfrac{d}{dt})) + \dim(N(\tfrac{d}{dt})) = 3 + 1 = \underline{4.}$$

Representation matrix of differentiation

$$p(x) = 3a_3x^2 + 2a_2x + a_1$$

$$[p(x)]_B = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{bmatrix} \begin{matrix} 1 \\ x \\ x^2 \\ x^3 \end{matrix}$$

$$\begin{cases} \frac{d}{dx} : P_3 \to P_3 \\ \frac{d}{dx}(a_3x^3 + a_2x^2 + a_1x + a_0) = \underline{3a_3x^2 + 2a_2x + a_1} \end{cases}$$

$B = \{1, x, x^2, x^3\}$

- Take that $\{1, t, t^2, t^3\}$ as a basis for P_3.
- The representation matrix for $\frac{d}{dt}$ is:

$B' = \{1, x, x^2, \dots, x^{n-1}\}$

$$A_{\text{diff}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \Big[[T(1)]_B \; [T(x)]_B \; [T(x^2)]_B \; [T(x^3)]_B \Big]$$

$T(1) = 0 \to [T(1)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$T(x) = 1 \to [T(x)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$T(x^2) = 2x \quad [T(x^2)]_B = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

$T(x^3) = 3x^2 \quad [T(x^3)]_B = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$

Example (the operation of **integration**)

- Consider $\int_0^t : P_n \to P_{n+1}$ with

$$\int_0^t (a_0 + a_1 t + \cdots + a_n t^n)dt = a_0 t + \frac{a_1}{2} t^2 + \cdots + \frac{a_n}{n+1} t^{n+1}$$

- This is a linear function.
- It is not surjective since $c \in \mathbb{R} \subseteq P_{n+1}$ but $c \notin Im(\int_0^t)$.
- It is injective since $N(\int_0^t) = \{0\}$

Representation matrix of integration

$T \begin{cases} \int_0^t : P_3 \to P_4 \quad B' = \{1, t, t^2, t^3, t^4\} \\ \int_0^t (a_0 + a_1 t + a_2 t^2 + a_3 t^3)dt = \underline{a_0 t + \frac{a_1}{2} t^2 + \frac{a_2}{3} t^3 + \frac{a_3}{4} t^4} \end{cases}$

$T : \overset{B}{V} \to \overset{B'}{W}$

$\left[[T(v_1)]_{B'}, \ldots, [T(v_n)]_{B'} \right]$

$v_1, \ldots, v_n$

- Take $B = \{1, t, t^2, t^3\}$ as a basis for P_3.
- The representation matrix for $\frac{d}{dt}$ is:

$$A_{\text{int}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$\int^t t = \frac{1}{2} t^2$

$[T(t)]_{B'} = \left[\int^t t\right]_{B'} = \left[\frac{1}{2} t^2\right]_{B'} = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{array}$

$\left[\int^t t^2\right] = \left[\frac{t^3}{3}\right]$

$\left[\int t^3\right]_{B'} = \left[\frac{t^4}{4}\right]_{B'}$

$\left[\int^t 1\right]_B = [t]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Composition

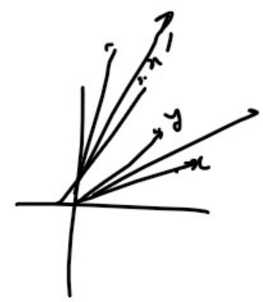
Function Composition

Suppose that $T : V \to W$ is a linear function with representation matrix A and $S : W \to U$ representation matrix B , then $S \circ T : V \to U$ is a linear function with representation matrix BA

BA

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, "S \circ T"'] & W \arrow[d, "S"] \\ & U \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "A"] \arrow[dr, "BA"'] & W \arrow[d, "B"] \\ & U \end{tikzcd}$$

$S \circ T(x) = S(T(x)) = S(Ax) = BAx$



θ rotations

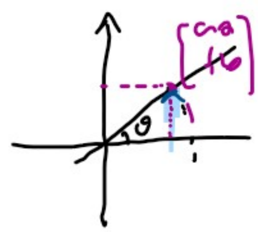
θ **rotation**

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

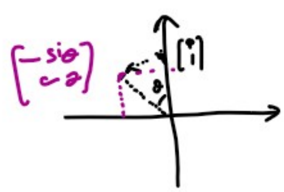
$$\begin{bmatrix} \cos\theta & ? \\ \sin\theta & ? \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

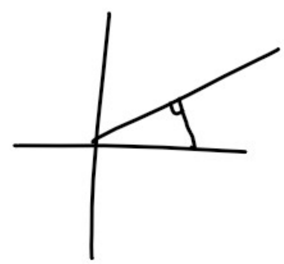
$$\begin{bmatrix} \cos\theta & ? \\ \sin\theta & ? \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{Q_\theta} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$



$| - \theta$





V

$W \subseteq V$

$V \subseteq V$

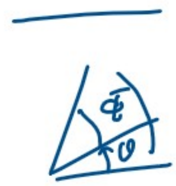
Notes

- Does the inverse of Q_θ equal $Q_{-\theta}$ (rotation backward through θ)?
- Yes. $Q_\theta^{-1} = Q_{-\theta}$

$$Q_\theta Q_{-\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Does the square of Q_θ equal $Q_{2\theta}$ (rotation through a double angle)? Yes.

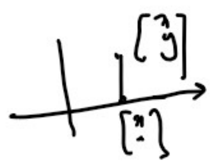
$$Q_{2\theta} = Q_\theta^2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$



- Does the product of Q_θ and Q_ϕ equal $Q_{\theta-\phi}$ (rotation through θ then ϕ)? Yes. $Q_\theta Q_\phi = Q_{\theta-\phi}$

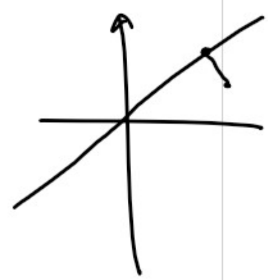
Projections onto the x-axis

- Projections onto the x-axis $\begin{bmatrix} c \\ s \end{bmatrix}$



$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$$

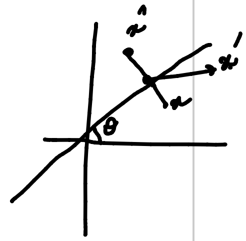


Projections onto the θ-lines

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ \cos\theta\sin\theta \end{bmatrix}$$

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\theta \\ \sin^2\theta \end{bmatrix}$$

$$P = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$



انعکاس x
روی خط
θ

Projections onto the θ-lines

- The linear function has no inverse. (Why?)
- Points on the θ-line are projected to themselves.
- Projecting twice is the same as projecting once, and $P^2 = P$

Reflections through the 45° line

- Reflections through the 45° line.

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} s \\ c \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} c \\ s \end{bmatrix}$$

Reflection in the θ-line

- The reflection of $\begin{bmatrix} x \\ y \end{bmatrix}$ in the θ-lines.

$$H = \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta^2 - 1 \end{bmatrix}$$

$$H = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

Reflection in the θ-line

- $H^2 = I$: Two reflections bring back the original.
- $H^2 = I \Rightarrow H^{-1} = H$.
- Two reflections bring back the original which is clear from the geometry but less clear from the matrix.
- To show that $H^2 = I$, we use $P^2 = P$:

  We have $H = 2P - I$, thus

$$H^2 = (2P - I)^2 = 4P^2 - 4P + I = I$$

Review: Change of basis

- Suppose that V be a linear space with finite dimension.
- Let $B = \{v_1, \ldots, v_n\}$ be a ordered basis for V.
- The coordinate representation of $v \in V$ is denoted by $[v]_B$.
- Let $B' = \{v'_1, \ldots, v'_n\}$ be another ordered basis for V.
- What is relation between $[v]_B$ and $[v]'_B$ for any vector $v \in V$?

Representation Matrix and change basis

- Suppose that V be a linear space with finite dimension.
- Let $B = \{v_1, \dots, v_n\}$ be a ordered basis for V.
- Consider $T : V \to V$ be a linear function. The representation matrix of T with respect to B is denoted by $[T]_B$.
- Let $B' = \{v'_1, \dots, v'_n\}$ be another ordered basis for V.
- What is relation between $[T]_B$ and $[T]_{B'}$?

Change of basis for a linear function

- The matrix

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is invertible, so its columns is a basis for $\mathbb{R}^2$ and denote it by B.

- The representation matrix of reflections by considering the new basis is

$$[H]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

While in the standard basis:

$$[H]_E = \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta^2 - 1 \end{bmatrix}$$

What is a relationship between $[H]_B$ and $[H]_B$.

Example

- The matrix
$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
is inve
rtible, so its columns is a basis for $\mathbb{R}^2$ and denote it by B.

- The representation matrix of reflections by considering the new basis is
$$[H]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$[H]_E = \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta^2 - 1 \end{bmatrix} = Q_\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} Q_{-\theta}$$

Existence of Inverses

Definition

For $A \in M_{mn}(\mathbb{R})$, if there is a $C \in M_{nm}(\mathbb{R})$ such that $AC = I$, then C is a right-inverse for A.

Definition

For $A \in M_{mn}(\mathbb{R})$, if there is a $B \in M_{nm}(\mathbb{R})$ such that $BA = I$, then B is a left-inverse for A.

Fact

Only a square matrix can have a two-sided inverse.

Right-inverse

- Suppose that $A \in M_{mn}$ has a right inverse. That means there is a matrix $C \in M_{nm}(\mathbb{R})$ such that $AC = I_m$.
- Let C_i be the i-th column of C.
- We have
$$AC = A\left[C_1 \cdots C_m\right] = \left[AC_1 \cdots AC_m\right] = I_m = \left[e_1 \cdots e_m\right]$$
- Thus, $AC_i = e_i$ for each $1 \leqslant i \leqslant m$.
- For every $b \in \mathbb{R}^m$, we have $b = b_1 AC_1 + \ldots + b_m AC_m$.
- Consequently, $\dim \left(\underbrace{C(A)}_{\text{Column space of } A} \right) = m$.
- As a result, $\operatorname{rank}(A) = r = m$, Full row rank.

Left-inverse

- Suppose that $A \in M_{mn}$ has a left inverse. That means there is a matrix $B \in M_{nm}(\mathbb{R})$ such that $BA = I_n$.
- Let B_i be the i-th row of B.
- $BA = \begin{bmatrix} B_1 A \\ \vdots \\ B_n A \end{bmatrix} = I = \begin{bmatrix} e_1^T \\ \vdots \\ e_n^T \end{bmatrix} \Rightarrow A^T B_i^T = e_i$, for each $1 \leqslant i \leqslant n$.
- For every $x \in \mathbb{R}^n$,, we have $x = x_1 A^T B_1^T + \ldots + x_n A^T B_n^T$.
- Consequently, $\dim \left(\underbrace{C(A^T)}_{\text{Row space of } A} \right) = n$.
- As a result, $\operatorname{rank}(A) = r = n$, Full column rank.

When a matrix has a left-inverse (right-inverse)?

Right-inverse

A matrix A has a right-inverse if and only if $r = m$, full row rank.

Left-inverse

A matrix A has a left-inverse if and only if $r = n$, full column rank.

The condition for invertibility is full rank: $r = m = n$.

Corollary

Only a square matrix can have a two-sided inverse.

Example

- Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$.
- $\text{rank}(A) = r = m = 2$ shows that A has a right-inverse.
- $AC = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{cases} c_{11} = \frac{1}{4} & c_{12} = 0 \\ c_{21} = 0 & c_{22} = \frac{1}{5} \end{cases}$
- There are many right-inverses because the last row of C is **completely arbitrary**.
- This is a case of **existence** but not **uniqueness**.

Example

- Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$.
- $C = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ c_{31} & c_{32} \end{bmatrix}$ is a right-inverse for A for every $c_{31}, c_{32} \in \mathbb{R}$.
- $AA^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 25 \end{bmatrix}$ and $(AA^T)^{-1} = \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{25} \end{bmatrix}$
- $A^T(AA^T)^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix} = C$, pseudo-inverse.

Example

- The transpose of A yields an example with infinitely many left-inverses:

$$BA^T = \begin{bmatrix} \frac{1}{4} & 0 & b_{13} \\ 0 & \frac{1}{5} & b_{23} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Now, it is the last column of B that is completely arbitrary.

- The pseudo-inverse: $b_{13} = b_{23} = 0$. That means

$$\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \end{bmatrix} = \left(A^T(AA^T)^{-1}\right)^T$$

Review: Two-sided inverse

- The matrix A is invertible if there exists a matrix B such that $AB = BA = I$.
- Not all matrices have inverses.
- If $AB = I$ and $CA = I$, then $B = C$ (prove!). Therefore inverse matrix is unique. We denote it by A^{-1}.
- The matrix A is invertible if and only if $AX = b$ has one and only solution for a given b.
- The matrix A is invertible if and only if $A = LU$ where LU is a triangular factorization of A with no zeros on the diagonal of U.

When does a square matrix have inverse?

Each of these conditions is a necessary and sufficient test:

1. The columns span $\mathbb{R}^n$, so $Ax = b$ has at least one solution for every b.
2. The columns are independent, so $Ax = 0$ has only the solution $x = 0$.
3. The rows of A span $\mathbb{R}^n$.
4. The rows are linearly independent.
5. Elimination can be completed: $A = LDU$, with all n pivots.
6. (In Future) The determinant of A is not zero.
7. (In Future) Zero is not an eigenvalue of A.
8. (In Future) $A^T A$ is positive definite.

Thank You!