Lecture05

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Lecture05

Linear Algebra

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Chapter 2

Linear Spaces

The heart of linear algebra

Linear Spaces

$$\mathbb{R}^n = \left\{ \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \mid v_i \in \mathbb{R} \text{ for } 0 \leq i \leq n \right\}.$$

$(\mathbb{R}^n, +, \cdot)$

$$c \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix}$$

a set of vectors together with rules for vector addition and multiplication by real numbers such that

1. جابجایی **commutativity** $u + v = v + u$ for all $u, v \in \mathbb{R}^n$;
2. شرکت پذیری **associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in \mathbb{R}^n$ and all $a, b \in \mathbb{R}$;
3. عضو خنثی جمع **additive identity** there exists an element $0 \in \mathbb{R}^n$ such that $v + 0 = v$ for all $v \in \mathbb{R}^n$;
4. وارون جمعی **additive inverse** for every $v \in \mathbb{R}^n$, there exists $w \in \mathbb{R}^n$ such that $v + w = 0$;

   $v + (-1)v = 0$
5. عضو خنثی ضرب **multiplicative identity** $1v = v$ for all $v \in \mathbb{R}^n$;
6. توزیع پذیری **distributive properties** $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $a, b \in \mathbb{R}$ and all $u, v \in \mathbb{R}^n$.

Linear Spaces

$$M_{m,n}(\mathbb{R}) = \left\{ \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \cdots & \vdots \\ v_{m1} & \cdots & v_{mn} \end{bmatrix}_{m \times n} \mid v_{ij} \in \mathbb{R} \right\},$$

$(M_{m,n}(\mathbb{R}), +, \cdot)$ has the following properties:

1. **commutativity** $u + v = v + u$ for all $u, v \in M_{m,n}$;
2. **associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in M_{m,n}$ and all $a, b \in \mathbb{R}$;
3. **additive identity** there exists an element $0 \in M_{m,n}$ such that $v + 0 = v$ for all $v \in M_{m,n}$;
4. **additive inverse** for every $v \in \mathbb{R}^n$, there exists $w \in M_{m,n}$ such that $v + w = 0$;
5. **multiplicative identity** $1v = v$ for all $v \in M_{m,n}$;
6. **distributive properties** $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $a, b \in \mathbb{R}$ and all $u, v \in M_{m,n}$.

Linear Spaces

The infinite-dimensional space $\mathbb{R}^\infty$ whose vectors have infinitely many components, as in $v = (1, 2, 1, 2, ...)$ has the following properties:

1. **commutativity** $u + v = v + u$ for all $u, v \in \mathbb{R}^\infty$;
2. **associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in \mathbb{R}^\infty$ and all $a, b \in \mathbb{R}$;
3. **additive identity** there exists an element $0 \in \mathbb{R}^\infty$ such that $v + 0 = v$ for all $v \in \mathbb{R}^\infty$;
4. **additive inverse** for every $v \in \mathbb{R}^\infty$, there exists $w \in \mathbb{R}^\infty$ such that $v + w = 0$;
5. **multiplicative identity** $1v = v$ for all $v \in \mathbb{R}^\infty$;
6. **distributive properties** $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $a, b \in \mathbb{R}$ and all $u, v \in \mathbb{R}^\infty$.

Linear Spaces

$V = \{ f : [0,1] \to \mathbb{R} \}$

The space of functions V that consists of all functions $f : \underline{[0, 1]} \to \mathbb{R}$ has the following properties :

1. **commutativity** $u + v = v + u$ for all $u, v \in V$;
2. **associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbb{R}$;
3. **additive identity** there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;
4. **additive inverse** for every $v \in \mathbb{R}^n$, there exists $w \in V$ such that $v + w = 0$;
5. **multiplicative identity** $1v = v$ for all $v \in V$;
6. **distributive properties** $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $a, b \in \mathbb{R}$ and all $u, v \in V$.

$f + g \to g + f$

$f : [0,1] \to \mathbb{R}$

$f(x) = 0$

$a(f + g) = af + ag$

$V = \{ f : \mathbb{R} \to \mathbb{R} \}$

Linear Spaces

$(V, +, \cdot)$

$$\boxed{P_n(x) = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{R} \text{ for all } 0 \le i \le n\}}$$

$\boxed{\mathbb{R}[x]}$

has the following properties:

1. **commutativity** $u + v = v + u$ for all $u, v \in P_n(x)$;
2. **associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in P_n(x)$ and all $a, b \in \mathbb{R}$;
3. **additive identity** there exists an element $0 \in P_n(x)$ such that $v + 0 = v$ for all $v \in P_n(x)$;
4. **additive inverse** for every $v \in P_n(x)$, there exists $w \in P_n(x)$ such that $v + w = 0$;
5. **multiplicative identity** $1v = v$ for all $v \in P_n(x)$;
6. **distributive properties** $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $a, b \in \mathbb{R}$ and all $u, v \in P_n(x)$.

V فضای خطی روی $\mathbb{R}$

فضای خطی حقیقی

مثال. $(\mathbb{R}, \times, +)$

$a, b \in \mathbb{R} = V$

$a + b = ab$

$a \in V$, $r \in \mathbb{R}$

$r.a = r + a$

$r.(a+b) = r.(ab) = r + ab$

$r.a + r.b = (r.a)(r.b) = (r+a)(r+b)$

Remark

Why for every $v \in V$, we have

- $0 \cdot v = \overset{0}{\cancel{v}}$ for every $v \in V$.
- $c \cdot 0 = 0$ for every $c \in \mathbb{R}$.
- $(-1) \cdot v = -v$ for every $v \in V$.

$0v = (0+0)v = 0.v + 0.v \Rightarrow 0.v = 0$

$c.(0+0) = c.0 + c.0 \Rightarrow c.0 = 0$

$0 \in V$

$1.v + (-1).v = (1 + (-1)).v = 0.v = 0 \Rightarrow (-1)v = -v$

Linear Spaces

Definition

A set V with an **addition** and a **scalar multiplication**, $\big(V, +, \cdot\big)$, is a linear space if it has the following properties:

1. **commutativity** $u + v = v + u$ for all $u, v \in V$;
2. **associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbb{R}$;
3. **additive identity** there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;
4. **additive inverse** for every $v \in V$, there exists $w \in V$ such that $v + w = 0$;
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6. **distributive properties** $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $a, b \in \mathbb{R}$ and all $u, v \in V$.

Remark

Why for every $v \in V$, we have

- $0 \cdot v = v$ for every $v \in V$.
- $c \cdot 0 = 0$ for every $c \in \mathbb{R}$.
- $(-1) \cdot v = -v$ for every $v \in V$.

Linear combinations

ترکیب خطی

Definition

Let V be a real linear space. An element $w \in V$ is a linear combination of $v_1, \cdots, v_m \in V$ if and only if there exit scalars $c_1, \cdots, c_m \in \mathbb{R}$ as coefficients such that

$$w = c_1 v_1 + \ldots + c_m v_m.$$

Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 8 \\ 4 & 9 \end{bmatrix}$. $\longrightarrow C(A) =$

Consider vectors $\cancel{v_1}$ A_1 and $\cancel{v_2}$ A_2, the first and the second column of A.

Let $\underline{V} = \{c_1 \overset{A_1}{v_1} + c_2 \overset{A_2}{v_2} \mid c_1, c_2 \in \mathbb{R}\}$ $\underline{C(A)} = \left\{ c_1 \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 8 \\ 9 \end{bmatrix} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$

Is V a linear space?

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$A = [A_1 \cdots A_n]$

$Ax = b$ —

سطری —

ستونی —

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$x_1 A_1 + \cdots + x_n A_n = b$$

$b \in \mathbb{R}^m$

Definition of Linear sub-Spaces

Definition

Let a set V is a linear space along with an **addition** on V and a **scalar multiplication** on V

$$(V, +, \cdot).$$

A non-empty subset $W \subseteq V$ is called a linear sub-space if it is a linear space under the addition and the scalar multiplication of V.

Note. The zero vector belongs to every subspace (Why?).

Note. The smallest subspace contains only one vector 0.

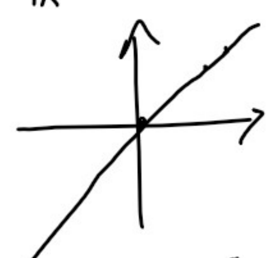
$W \neq \emptyset$

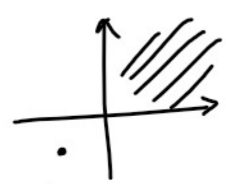
$v \in W$

$-v \in W$

$v + (-v) = 0 \in W$

$\mathbb{R}^2$:





Examples

زیرفضا نیست $S_1 = \{(x, y) \mid x \geq 0,\ y \geq 0\}$

Example. Construct a subset of $\mathbb{R}^2$ that is

$-\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

i. **closed under vector addition** and subtraction, but **not scalar multiplication** on $\mathbb{R}$.

Examples

Example. Construct a subset of $\mathbb{R}^2$ that is

زیرفضا نیست $S_1 = \{(x,y) \mid x \geq 0, \; y \geq 0\}$

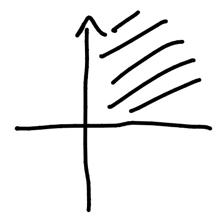
$-\begin{bmatrix}1\\1\end{bmatrix}$

i. **closed under vector addition** and subtraction, but **not scalar multiplication** on $\mathbb{R}$.

Examples

Example. Construct a subset of $\mathbb{R}^2$ that is

$v_1, v_2 \in W, \; c \in \mathbb{R}$
$cv_1 + v_2 \in W$



i. **closed under vector addition** and subtraction, but **not scalar multiplication** on $\mathbb{R}$.

$\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}a\\b\end{bmatrix}$

ii. **closed under scalar multiplication** but **not under vector addition**.

$S_2 = \{(x,y) \mid x \geq 0, \; y \geq 0$
$x \leq 0, \; y \leq 0\}$

Linear Subspace

Example.

- What is the **smallest subspace** of $\mathbb{R}^2$ which contains $\boldsymbol{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a_1 e_1 + a_2 e_2$

$W \subseteq \mathbb{R}^2 \qquad e_1 \in W, \quad e_2 \in W$

$a = a_1 e_1 + a_2 e_2 \in W = \mathbb{R}^2$

Linear Subspace

Example.

- What is the **smallest subspace** of $\mathbb{R}^2$ which contains $\boldsymbol{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

$W \subseteq \mathbb{R}^3$: 1) $e_1, e_2 \in W$

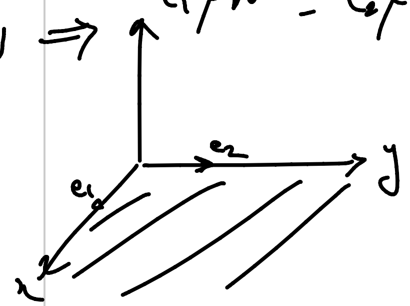
2) $W' \subsetneq W$ وجود ندارد.

$\Rightarrow e_1 \notin W'$ یا $e_2 \notin W'$

- What is the **smallest subspace** of $\mathbb{R}^3$ which contains $\boldsymbol{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$?

b

$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 1e_1 + 2e_2$



$e_1, e_2, b \in W = \mathbb{R}^3$

Linear Subspace

زیرفضا $W \neq \phi \iff \forall\, v_1, v_2 \in W,\ c \in \mathbb{R} \quad cv_1 + v_2 \in W$

$W \subseteq V$

زیرفضا

1) $W \neq \phi$

2) ۷ خاصیت

$\Downarrow$

~~W~~ فضا را دارد

Theorem

A non-empty subset W of a real linear space V is a sub-space if and only if $cv_1 + v_2 \in W$ for every $v_1, v_2 \in W$ and $c \in \mathbb{R}$.

$\Leftarrow$ W زیرفضا باشد: $c_1v_1 + v_2 \in W$

$(\Rightarrow)$ اگر $W \neq \phi$ و برای هر $v_1, v_2 \in W$ و $c \in \mathbb{R}$: $cv_1 + v_2 \in W \Rightarrow W$ زیرفضا

W ۷ خاصیت دارد. جابجایی ✓ شرکت پذیری ✓ عنصر خنثی را دارد ($0 \in W$)

داشتن قرینه: $v \in W \quad -v = 0 + (-1)v \in W$

توزیع پذیری ✓

W زیرفضاست $\Downarrow$ $c_1v_1 + c_2v_2 \in W$, $c_2 = 1$

زیرفضاست. $\Uparrow$ $cv_1 + v_2 \in W$

Linear Subspace

Theorem

A non-empty subset W of a real linear space V is a sub-space if and only if $cv_1 + v_2 \in W$ for every $v_1, v_2 \in W$ and $c \in \mathbb{R}$.

$$\underbrace{\text{Linear combinations stay in the subspace.}}_{\text{“closed” under addition and scalar multiplication.}}$$

Theorem

The intersection of each family of sub-spaces of a linear space V is a subspace of V?

$\{W_i\}_{i\in I}$ $\qquad W = \bigcap_{i\in I} W_i$

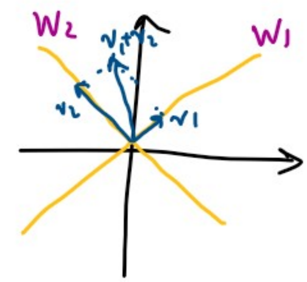
$v_1, v_2 \in W,\ c \in \mathbb{R} \Rightarrow cv_1 + v_2 \in W$

$\forall i$ W_i زیرفضا $\xRightarrow{\text{قضیه}}$ $cv_1 + v_2 \in W_i \Rightarrow cv_1 + v_2 \in \bigcap_{i\in I} W_i = W$

$\Rightarrow$ W زیرفضاست.

Theorem

The intersection of each family of sub-spaces of a linear space V is a subspace of V?



$W_1 \cup W_2$

Fact

The union of two sub-spaces is not a subspace unless one is contained in the other.

قضیه: $W_1, W_2 \subseteq V$ زیرفضای V باشند. $W_1 \cup W_2$ زیرفضاست اگر و تنها اگر

$W_1 \subseteq W_2$ یا $W_2 \subseteq W_1$

$\Rightarrow$ اگر $W_1 \subseteq W_2$ یا $W_2 \subseteq W_1$ آنگاه $W_1 \cup W_2 = W_2$ یا $W_1 \cup W_2 = W_1$.

$v_1 \in W_1 \quad , \quad v_2 \in W_2$

$(\Leftarrow)$ اگر $W_1 \cup W_2$ زیرفضا باشد $\Leftarrow$ $W_1 \subseteq W_2$ یا $W_2 \subseteq W_1$

$v_1 + v_2 \notin W_1 \cup W_2$

برهان خلف: $W_2 \not\subseteq W_1$ و $W_1 \not\subseteq W_2$

$\Rightarrow v_1 \in W_1 \setminus W_2,\ v_2 \in W_2 \setminus W_1$ و $v_1 + v_2 \in W_1 \cup W_2 \Rightarrow v_1 + v_2 \in W_1$ یا $v_1 + v_2 \in W_2$

$\Downarrow \qquad\qquad \Downarrow$

$v_2 \in W_1 \qquad v_1 \in W_2$

$\Downarrow \qquad\qquad \Downarrow$

تناقض $\qquad$ تناقض

$\underbrace{\qquad\qquad}$

$W_1 \subseteq W_2$ یا $W_2 \subseteq W_1$

Theorem

The intersection of each family of sub-spaces of a linear space V is a subspace of V?

W_1 , W_2

Fact

The union of two sub-spaces is not a subspace unless one is contained in the other.

$W_1 \subseteq \cancel{\boxed{W_1 \cup W_2}}$

$W_2 \subseteq \boxed{W_1 + W_2}$

Definition

in the other.

Definition

Let $W_1, \dots, W_m$ be sub-spaces of a linear space V. The sum of them is defined as

$$W_1 + \cdots + W_m = \left\{ w_1 + \cdots + w_m \,\middle|\, w_i \in W_i \text{ for } 1 \le i \le m \right\}.$$

- Note that $W_1 + \ldots + W_m$ is a subspace of V which contains W_i for each $1 \le i \le m$.

$W_2 \subseteq \boxed{W_1 + W_2}$

$W_i \subseteq \underbrace{W_1 + \cdots + W_m}_{\text{زیرفضا؟}}$

$\underbrace{0 + \cdots + v_i + 0 + \cdots + 0}_{\in W_1 + \cdots + W_m} = v_i \in W_i$

...

ثابت: اگر $W_1, \dots, W_m$ زیرفضاهای V باشند آنگاه $W_1 + \cdots + W_m$ زیرفضا

$W_1 + \cdots + W_m \neq \emptyset \Leftarrow 0 + \cdots + 0 \in W_1 + \cdots + W_m$

$v = v_1 + \cdots + v_m \in W_1 + \cdots + W_m$, $w = w_1 + \cdots + w_m \in W_1 + \cdots + W_m$, $c \in \mathbb{R}$

Thank You!

$$cv + w = (cv_1 + \cdots + cv_m) + (w_1 + \cdots + w_m) = \underbrace{(cv_1 + w_1)}_{\substack{\in W_1 \\ \text{دارزیرفضاست}}} + \cdots + \underbrace{(cv_m + w_m)}_{\in W_m}$$

$\in W_1 + \cdots + W_m$

حکم: زیرفضاست.

$A = L_1 D_1 U_1$

$A = L_2 D_2 U_2$

$\Rightarrow L_1 D_1 U_1 = L_2 D_2 U_2$

$\underbrace{L_2^{-1} L_1 D_1}_{\text{پایین‌مثلثی با درایه قطری ناصفر}} = \underbrace{D_2 U_2 U_1^{-1}}_{\text{بالامثلثی با قطر ناصفر}}$

D_1 قطری $\qquad D_2$ قطری

$\Rightarrow D_1 = D_2$

$\Rightarrow \underset{\substack{\| \\ I_n \\ \Downarrow \\ L_2 = L_1}}{L_2^{-1} L_1} = \underset{\substack{\| \\ I_n \\ \Downarrow \\ U_2 = U_1}}{U_2 U_1^{-1}}$

$U_i = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

چون تنها جواب $A_i x = 0$ است $\Rightarrow$ U_i وارون‌پذیر $\quad i = 1, 2$