Lecture04

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Lecture04

Linear Algebra

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Inverse Matrices

- The matrix A is invertible if there exists a matrix B such that $AB = BA = I$.

Theorem

The inverse exists if and only if elimination produces n pivots.

$$EPA = U = \begin{bmatrix} & \boxed{0} & \\ & \vdots & \\ & 0 & \end{bmatrix} \quad r_{r+1} \quad u_{r+1}$$

$$Ax = 0$$

$$EPA = U = \begin{bmatrix} u_{11} & \ddots & * \\ 0 & & u_{nn} \end{bmatrix}$$

$\underline{\underline{A}} \in M_n(\mathbb{R})$

$Ax = 0 \iff$ A

$A \longrightarrow U = \begin{bmatrix} \cdot \end{bmatrix}$

$I = A^{-1}Ax = 0$

$x = 0$

$$\rightarrow \underbrace{F_m \cdots F_1}_{F} EPA = F_m \cdots F_1 U = I$$

$$\begin{bmatrix} 1/d_1 & & 0 \\ & \ddots & \\ 0 & & 1/d_n \end{bmatrix} \qquad = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

Inverse Matrices

$\begin{bmatrix} 1/d_1 & & 0 \\ & \ddots & \\ 0 & & 1/d_n \end{bmatrix} \qquad = \begin{bmatrix} d_1 & & \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$

$\Rightarrow \underbrace{DFEP}_{A^{-1}}A = I$

Inverse Matrices

- The matrix A is invertible if there exists a matrix B such that $AB = BA = I$.

Theorem

The inverse exists if and only if elimination produces n pivots.

Corollary

A is invertible if and if the one and only solution t othe system equation $Ax = 0$ is $x = 0$.

Review: The Calculation of A^{-1}

The inverse of A is written A^{-1} in which $AA^{-1} = I$. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

We want to find A^{-1} such that $AA^{-1} = I$, so consider the columns of A^{-1} as x_1, x_2, x_3, that means $A^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$.

$$AA^{-1} = I$$
$$A\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}$$
$$\begin{bmatrix} Ax_1 & Ax_2 & Ax_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}$$

So $\quad Ax_1 = e_1 \qquad Ax_2 = e_2 \qquad Ax_3 = e_3$

A^{-1}

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

- We have

$$\left[\begin{array}{c|c} U & L^{-1} \end{array}\right]$$

A^{-1}

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ 0 & 1 & 0 & \frac{4}{8} & -\frac{3}{8} & -\frac{2}{8} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

A^{-1}

The inverse of 3 by 3 Hilbert matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{bmatrix}$$

The inverse of 3 by 3 Hilbert matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

The inverse of 3 by 3 Hilbert matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{bmatrix}$$

The inverse of 3 by 3 Hilbert matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{bmatrix}$$

The inverse of 3 by 3 Hilbert matrix

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{array}\right]$$

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$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{bmatrix}$$

The inverse of 3 by 3 Hilbert matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{bmatrix}$$

$$H^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

Hilbert matrix

The matrix H is an example of a family of matrices which are called **Hilbert** matrices. The n by n Hilbert matrix is

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

where $H_{ij} = \frac{1}{i+j+1}$. For every n, $n \times n$ Hilbert matrix is invertible and

its inverse has integer entries.

Application of inverse matrix!

Problem. Let $AB = 3A + 4B$. Show $AB = BA$.

Solution.

$AA^{-1} = A^{-1}A$

- $AB - 3A - 4B = 0$.
- $(A-?)(B-?) = ?$.
- $(A - 4I)(B - 3I) = 12I$
- $(A - 4I)(\frac{1}{12}B - \frac{1}{4}I) = I. \quad \Rightarrow \quad \underbrace{(A - 4I)}_{A}\underbrace{(\frac{1}{12}B - \frac{1}{4}I)}_{A^{-1}} = I$
- $\underbrace{(\frac{1}{12}B - \frac{1}{4}I)}_{A^{-1}}\underbrace{(A - 4I)}_{A} = I \Rightarrow BA = AB.$

$M, N \in M_n(\mathbb{R})$

$MN = I \Rightarrow NM = I$

$\wedge$

$B = a_m A^m + \cdots + a_1 A + a_0 I$

$AB = BA$

$AB = a_m A A^m + \cdots + a_1 A A + a_0 A$

$= (a_m A^m + \cdots + a_1 A + a_0 I) A$

$= BA$

Transpose Matrix

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 9 & 10 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 0 \\ -3 & 9 \\ 5 & 10 \end{bmatrix}$$

In general:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & \dots & a_{m1} \\ a_{12} & \dots & a_{m2} \\ \vdots & \vdots & \vdots \\ a_{1n} & \dots & a_{mn} \end{bmatrix}$$

$A^{-1}A = I \qquad A^T (A^{-1})^T = I$

i) $(AB)^T = B^T A^T$. $\Longrightarrow$ ii) $(A^T)^{-1} = (A^{-1})^T$.

$A \in M_{mn}(\mathbb{R})$

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$(A^T)_{ij} = A_{ji}$

$(AB)^T_{ij} = (BA)_{ji}$

$= \sum_{t=1}^{n} B_{jt} A_{ti}$

ij j

B

$(B^T A^T)_{ij} =$

Symmetric Matrices

- A **symmetric matrix** is a matrix that equals its own transpose: $A^T = A$; i.e. $A_{ij} = A_{ji}$.
- A symmetric matrix A is necessarily square.
- A symmetric matrix A need not be invertible.
- A is a symmetric matrix if and only if A^{-1} a symmetric matrix. (Why?)

$A^T = A$

$A^T = A$

$n \times m \quad m \times n \quad n = m$

Symmetric Products

- Symmetric Products $R^T R$, RR^T, and LDL are Symmetric.
- If A is invertible, then the LDU factorization and the LU factorization are uniquely determined by A.

$(R^T R)^T = R^T R$

Symmetric Products

- Symmetric Products $R^T R$, RR^T, and LDL are Symmetric.
- If A is invertible, then the LDU factorization and the LU factorization are uniquely determined by A.
- Suppose $A = A^T$ is factorized into $A = LDU$ without row exchanges.

Symmetric Products

- Symmetric Products R^TR, RR^T, and LDL are Symmetric.

- If A is invertible, then the LDU factorization and the LU factorization are uniquely determined by A.
- Suppose $A = A^T$ is factorized into $A = LDU$ without row exchanges.
  (1) Then U is the transpose of L, and the symmetric factorization becomes $A = LDL^T$.

$A = LDU$ $L = L'$

$A = L'D'U'$ $D = D'$

$U = U'$

$A = LDU = L'D'U'$

$\underbrace{L'^{-1}LD}_{} = \underbrace{D'U'U^{-1}}_{}$

$\underbrace{\begin{bmatrix} d_1 & & \\ * & \ddots & \\ & & d_n \end{bmatrix}}_{L'} \underbrace{\begin{bmatrix} I \end{bmatrix}}_{U^{-1}}$

$\Rightarrow D = D'$

$\Rightarrow \underbrace{L'^{-1}L}_{I} = \underbrace{U'U^{-1}}_{I}$

$\Rightarrow L = L', \quad U = U'$

Symmetric Products

- Symmetric Products R^TR, RR^T, and LDL are Symmetric.

- If A is invertible, then the LDU factorization and the LU factorization are uniquely determined by A.
- Suppose $A = A^T$ is factorized into $A = LDU$ without row exchanges.
  (1) Then U is the transpose of L, and the symmetric factorization becomes $A = LDL^T$.
  (2) If $D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$ where $d_i \geq 0$ for each $1 \leq i \leq n$, then A is factorized $A = LL^T$ in which L is a lower triangular matrix.

$A = LDU$

$A^T = U^TDL^T$

$A = LDU = U^TDL^T$

$\Rightarrow L^T = U$

$A = L \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} L^T$

$A = \underbrace{L \begin{bmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{bmatrix}}_{} \underbrace{\begin{bmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{bmatrix} L^T}_{}$

$A = \mathcal{L} \; \mathcal{L}^T$

Roundoff Error

Normally, we keep a fixed number of significant digits (say three, for an extremely weak computer). Then adding two numbers of different sizes raises an error:

Roundoff Error $\quad 0.456 + 0.00123 \rightarrow 0.457 \quad$ loses the digits $\underline{2}$ and $\underline{3}$.

Roundoff Error

$A = L$ L

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Normally, we keep a fixed number of significant digits (say three, for an extremely weak computer). Then adding two numbers of different sizes raises an error:

Roundoff Error $\quad 0.456 + 0.001\underline{23} \to 0.457 \quad$ loses the digits $\underline{2}$ and $\underline{3}$.

How do all these individual errors contribute to the final error in $Ax = b$?

Two simple examples

$$A = \underbrace{\begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 1.0001 \end{bmatrix}}_{\text{Nearly singular}} \qquad B = \underbrace{\begin{bmatrix} 0.0001 & 1.0 \\ 1.0 & 1.0 \end{bmatrix}}_{\text{far from sigular}}$$

Ill-conditioned

Two simple examples

$$A = \underbrace{\begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 1.0001 \end{bmatrix}}_{\text{Nearly singular}} \qquad B = \underbrace{\begin{bmatrix} 0.0001 & 1.0 \\ 1.0 & 1.0 \end{bmatrix}}_{\text{far from sigular}}$$

Ill-conditioned **Well-conditioned**

$Ax = b$

Consider two very close right-hand side b's:

$$\begin{array}{rcl} u + v &=& 2 \\ u + 1.0001v &=& 2.0000 \end{array} \qquad \begin{array}{rcl} u + v &=& 2 \\ u + 1.0001v &=& 2.0001 \end{array}$$

u=2 and v=0 u=1 and v=1

A change in the fifth digit of b was amplified to a change in the first digit of the solution. No numerical method can avoid this sensitivity to small perturbations.

Well-condition!

- Even a well-conditioned matrix like $B = \begin{bmatrix} 0.0001 & 1.0 \\ 1.0 & 1.0 \end{bmatrix}$ can be ruined by a poor algorithm.
- Suppose 0.0001 is accepted as the first pivot. Then 10000 times the first row is subtracted from the second. The lower right entry becomes -9999.

$$\begin{bmatrix} 0.0001 & 1.0 & 1 \\ 1.0 & 1.0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0.0001 & 1.0 & 1 \\ 0 & -9999 & -9998 \end{bmatrix}$$

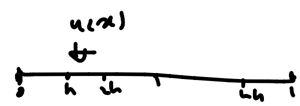
- $v = \frac{9998}{9999} \simeq \begin{cases} 0.9999 \Rightarrow u = 1 \\ \boxed{1} \quad \Rightarrow u = 0 \end{cases}$ for $\begin{cases} 0.0001\, u + v = 1 \\ u + v = 2 \end{cases}$
- B is well-conditioned but elimination is violently unstable.

Remedy Action!

- The small pivot 0.0001 brought instability.
- The remedy is clear —***exchange rows***.
- A small pivot forces a practical change in elimination. Normally we compare each pivot with all possible pivots in the same column. Exchanging rows to obtain the largest possible pivot is called ***partial pivoting***.

Special Matrices

- Consider differential equation $-\frac{d^2u}{dx^2} = f(x)$ for $0 \leq x \leq 1$ with the unknown function $u(x)$ which shows the temperature distribution in a rod with ends fixed at $0^{\cdot}C$ at each end of the interval: $u(0) = 0$ and $u(1) = 0$ (a *boundary condition*).
- We can only accept a finite amount of information about $f(x)$, say its values at n equally spaced points $x = h, x = 2h, \cdots, x = nh$.
- We compute approximate values $u_1, \ldots, u_n$ for the true solution $u(x)$ at these same points. At the ends $x = 0$ and $x = 1 = (n+1)h$, the boundary values are $u_0 = 0$ and $u_{n+1} = 0$.



$u(0) = 0$

$u(1) = 0$

$-\frac{d^2u}{dx^2} = f(x)$ $0 \leq x \leq 1$

$u(x) = ?$

$u(ih) = u_i$

$u_1, \cdots u_n$

Special Matrices

- How do we replace the derivative $\frac{d^2u}{dx^2}$ in $-\frac{d^2u}{dx^2} = f(x)$.
- The first derivative $\frac{du}{dx}$ can be approximated by stopping $\frac{\Delta u}{\Delta x}$ at a finite stepsize
- The difference Δu can be forward, backward:

$u((i+1)h) - u(ih)$ $\quad$ $u(ih) - u((i-1)h)$

ih

$$\frac{\Delta u}{\Delta x} = \frac{u(x+h) - u(x)}{h} \quad \text{or} \quad \frac{u(x) - u(x-h)}{h}$$

- $$\frac{d^2u}{dx^2} \approx \frac{\Delta^2 u}{\Delta x^2} = \frac{\Delta}{\Delta x}\left(\frac{\Delta u}{\Delta x}\right) = \frac{\frac{u(x+h)-u(x)}{h} - \frac{u(x)-u(x-h)}{h}}{h}$$
$$= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}. \Rightarrow$$

h $\quad$ $u_i = u(h)$

$2h$ $\quad$ u_2

$\vdots$

i $\quad$ $u_i = u(ih)$

$\vdots$

nh $\quad$ u_n

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = -f(ih)$$

$$-u_{i+1} + 2u_i - u_{i-1} = h^2 f(ih)$$

$i=1$

...

For $n = 5$, since $u_0 = u_{n+1} = 0$, we have

$$\begin{cases} 2u_1 - 1u_2 + & = h^2 f(h) \\ -1u_1 + 2u_2 - 1u_3 & = h^2 f(2h) \\ - 2u_2 + 2u_3 - 1u_4 & = h^2 f(3h) \\ - 1u_3 + 2u_4 - 1u_5 & = h^2 f(4h) \\ - 1u_4 + 2u_5 & = h^2 f(5h) \end{cases}$$

So

$$\underbrace{\begin{bmatrix} 2 & 1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = h^2 \begin{bmatrix} f(h) \\ f(2h) \\ f(3h) \\ f(4h) \\ f(5h) \end{bmatrix}$$

The fundamental properties of A

1) **The matrix A is tridiagonal** $\Rightarrow$ a tremendous simplification to Gaussian elimination.

$$\begin{bmatrix} 2 & 1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & & & \\ 0 & \frac{3}{2} & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \rightarrow \dots$$

$$\underbrace{\begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & -\frac{3}{4} & 1 & \\ & & & -\frac{4}{5} & 1 \end{bmatrix}}_{L} \begin{bmatrix} \frac{2}{1} & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \frac{5}{4} & \\ & & & & \frac{6}{5} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -\frac{1}{2} & & & \\ & 1 & -\frac{2}{3} & & \\ & & 1 & -\frac{3}{4} & \\ & & & 1 & -\frac{4}{5} \\ & & & & 1 \end{bmatrix}}_{U' = U}$$

$A = LU$

$[A \mid b]$

$EA = U$

$A = E^{-1} U$

$A = LDU$

$x_n = \frac{b_n}{u_{nn}}$

$A = LDL^T$

$Ax = b$

$EAx = Eb$

$Ux = c$

. . .

2) **The matrix is symmetric** $\Rightarrow A = LDL^T$.

$$\begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & -\frac{3}{4} & 1 & \\ & & & -\frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{1} & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \frac{5}{4} & \\ & & & & \frac{6}{5} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & & & \\ & 1 & -\frac{2}{3} & & \\ & & 1 & -\frac{3}{4} & \\ & & & 1 & -\frac{4}{5} \\ & & & & 1 \end{bmatrix}$$

2) **The matrix is symmetric** $\Rightarrow A = LDL^T$.

$$\begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & -\frac{3}{4} & 1 & \\ & & & -\frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{1} & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \frac{5}{4} & \\ & & & & \frac{6}{5} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & & & \\ & 1 & -\frac{2}{3} & & \\ & & 1 & -\frac{3}{4} & \\ & & & 1 & -\frac{4}{5} \\ & & & & 1 \end{bmatrix}$$

- **The matrix is positive definite**. This extra property says that the pivots are positive.

Thank You!