Lecture06

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Linear Algebra

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Chapter 2

Linear Spaces

The heart of linear algebra

$v_1+v_2 \in W_1+W_2$

$v_1+v_2 \in W_1+W_2$

$W_1 \cup W_2$ $\quad W_1 \subseteq W_2$

$A \in M_n(\mathbb{R})$ $\quad W_2 \subseteq W_1$

$Ax = b \iff b \in C(A)$

$$[A_1 \cdots A_n]\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = \underbrace{\begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix}}_{b}$$

$x_1 A_1 + \cdots + x_n A_n = b$

$C(A) = \{c_1A_1 + \cdots + c_nA_n \mid c_i \in \mathbb{R}\}$

$S = \{A_1, \dots, A_n\}$

$= \mathrm{span}(\{A_1, \dots, A_n\})$

Spanning subspaces

Definition

Let S be a set of a linear space V. The subspace spanned by S, denoted by $\mathrm{span}(S)$, is the set of all linear combinations of vectors in S.

- $\{0\} = \mathrm{span}(\varnothing)$.
- $\mathbb{R}^3 = \mathrm{span}(\{e_1, e_2, e_3\})$,
- $\mathbb{R}^2 = \mathrm{span}(\{e_1, e_2, e_1 + e_2\})$.

$$\begin{bmatrix}a\\b\\c\end{bmatrix} = a\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{e_1} + b\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{e_2} + c\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{e_3}$$

$\mathbb{R}^3 = \mathrm{span}(\{e_1, e_2, e_3\})$

$\mathbb{R}^2 = \mathrm{span}(\{e_1, e_2\})$

$\begin{bmatrix}a\\b\end{bmatrix} = ae_1 + be_2$

$= c_1e_1 + c_2e_2 + c_3(e_1+e_2)$

$c_1 + c_3 = a$

$c_2 + c_3 = b$

$= ae_1 + be_2 + 0(e_1+e_2)$

Spanning subspaces

$\mathrm{span}(S) =$

Theorem

Let W be a subspace of a linear space V such that $S \subseteq W$, then $\mathrm{span}(S) \subseteq W$.

- Note that the smallest subspace of V containing S is $\mathrm{span}(S)$.
- $\mathrm{span}(S)$ is the intersection of all subspaces W of V containing S, i.e. $\mathrm{span}(S) = \bigcap_{S \subseteq W \in \mathrm{sub}(V)} W$, where $\mathrm{sub}(V)$ is the set of all subspaces of V.
- The subspace $\mathrm{span}(S)$ contains no proper subspace including S.

$S \subseteq W$

$v_1, \dots, v_k \in S \subseteq W$

$c_1v_1 + \cdots + c_kv_k \in \mathrm{span}(S)$

$\in W$

$\bigcap_{S} W$

Examples

$W_1 \not\subseteq W_2, \quad W_2 \not\subseteq W_1$

$W_1 \cup W_2$ [illegible]

1. Let $W_1, \ldots, W_m$ be subspaces of a linear space V. The sum of them is

$W_1 + W_2 = \text{Span}(W_1 \cup W_2)$

$$W_1 + \cdots + W_m = \left\{ w_1 + \cdots + w_m \,\middle|\, w_i \in W_i \text{ for } 1 \le i \le m \right\}.$$

Thus

$$W_1 + \cdots + W_m = \text{span}\left(\bigcup_{j=1}^{m} W_j \right).$$

$Ax = b \Leftrightarrow b \in C(A)$

2. For $A \in M_{m,n}(\mathbb{R})$,

$$V = \{ Ax \mid x \in \mathbb{R}^n \}$$

$Ax = [A_1 \cdots A_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$= c_1 A_1 + \cdots + c_n A_n$

is $C(A)$, as it is generated by all columns of A.

فضای ستون

$\text{Span}(\{A_1, \ldots, A_n\})$

When does $Ax = b$ have a solution?

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

$$C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} \,\middle|\, c_i \in \mathbb{R} \right\}$$

$$= \text{span}\left(\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} \right\} \right)$$

When does $Ax = b$ have a solution?

$\mathbb{R}^2 = \text{Span}(\{e_1, e_2\})$

$\text{Span}(\{e_1, e_2, e_1 + e_2\})$

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \underbrace{\cancel{\begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}}}_{3\begin{bmatrix} 1 \\ 2 \\ \cdots \end{bmatrix}} + c_3 \underbrace{\cancel{\begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix}}}_{\begin{bmatrix} 1 \\ 2 \\ \cdots \end{bmatrix} + \begin{bmatrix} 2 \\ 7 \\ \cdots \end{bmatrix}} + c_4 \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

A

$C(A) = \text{Sp}(\{\overbrace{A_1, A_2, A_3, A_4}^{S}\})$

$$\overbrace{3\begin{bmatrix}1\\2\\-1\end{bmatrix} \quad \begin{bmatrix}1\\2\\-1\end{bmatrix}+\begin{bmatrix}2\\7\\4\end{bmatrix}}$$

$$\underline{(c_1+3c_2+c_3)}\begin{bmatrix}1\\2\\-1\end{bmatrix}+(c_3+c_4)\begin{bmatrix}2\\7\\4\end{bmatrix} \Rightarrow C(A)=\text{span}\left(\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}2\\7\\4\end{bmatrix}\right\}\right)$$

$$\begin{bmatrix}2c\\7c\\4c\end{bmatrix} = c\begin{bmatrix}2\\7\\4\end{bmatrix} = \begin{bmatrix}1\\2\\-1\end{bmatrix}$$

$$C(A) = \text{span}(\{A_1, A_2, A_3, A_4\})$$

$$= \text{span}(\{A_1, A_4\})$$

$$\text{span}(S) = \text{span}(T)$$

$$\subseteq T \subsetneq S$$

$$\Rightarrow \cancel{c = 1/2,\ c = 1/2,\ c = -1/4}$$

Linear dependent vectors

$$\text{span}\left(\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}2\\7\\4\end{bmatrix}\right\}\right) \supsetneq \text{span}\left(\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}\right)$$

Definition

Let V be a linear space and $\underline{S} \subseteq V$. We say that the elements of S are linearly dependent if there is some $\underline{s} \in S$ such that

$$\text{span}(S) = \text{span}(S \setminus \{s\}).$$

If the elements of S are not linearly dependent, then we say that they are linearly independent.

Linear Independence

Fact

The elements $v_1, \ldots, v_m$ *of the linear space* V *are linearly dependent, if there exist scalars* $c_1, \ldots, c_m \in \mathbb{R}$ *not all zero, such that*

$$c_1 v_1 + \cdots + c_m v_m = 0.$$

This implies that at least one of the scalars is nonzero.
Also, the elements $v_1, \ldots, v_m \in V$ *is linearly independent if it is not linearly dependent, that is, if the equation*

$$c_1 v_1 + \cdots + c_m v_m = 0$$

can only be satisfied by $c_1 = \cdots = c_m = 0$. *This implies that no element in the sequence can be represented as a linear combination of the remaining elements in the sequence.*

$S = \{v_1, \to v_m\}$

$c_1 v_1 + \cdots + c_m v_m = 0$

$\mathrm{Span}(S) = \mathrm{Span}(S \setminus \{s\})$

$s = v_1$

$v_1 = \alpha_2 v_2 + \cdots + \alpha_m v_m$

$1 v_1 - \alpha_2 v_2 - \cdots - \alpha_m v_m = 0$

$c_1 v_1 + c_2 v_2 + \cdots + c_m v_m = 0$

Example

Show that v_1, v_2, v_3 are independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

v_1, v_2, v_3

[illegible]

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$c_1 = c_2 = c_3 = 0$

$c_1 \neq 0$

$\cancel{v_1 = -\frac{c_2}{c_1} v_2 - \frac{c_3}{c_1} v_3}$

Example

- To show that the set of vectors $\{v_1, v_2, v_3\}$ is linearly independent, we should solve the following equation:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

Equivalently:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow N(A) = \{ c \mid Ac = 0 \} = \{0\}$$

$$\Updownarrow$$

مستقل اند $A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

Thus

$$c_1 = c_2 = c_3 = 0.$$

- The set of vectors $\{v_1, v_2, v_3\}$ is **linearly independent**.

$v_1, v_2, v_3, \underline{v_4}$ وابسته خطی اند

$v_4 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$1 v_4 - \alpha_1 v_1 - \alpha_2 v_2 - \alpha_3 v_3 = 0$

Example

For the set of vectors $\{v_1, v_2, v_3, v_4\}$:

فرم ستونی

$$\overset{1}{c_1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \overset{1}{c_2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \overset{-4}{c_3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \overset{1}{c_4} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$

- Equivalently:

ماتریسی

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{Ac = 0}$$

$$N(A) = \left\{ c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} c_4 \\ c_4 \\ -4c_4 \\ c_4 \end{bmatrix} \;\middle|\; c_4 \in \mathbb{R} \right\} \neq \{0\}$$

ستونهای A وابسته خطی اند

- By reduced echelon form, R:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}}_{R} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3×4

$c_3 + 4c_4 = 0 \Rightarrow c_3 = -4c_4$

$c_2 = c_4$

$c_1 = c_4$

$c_1 = c_2 = 1, \quad c_3 = -4, \quad c_4 = 1$

Checking procedure for linearly independence

Definition

The nullspace of a matrix consists of all vectors x such that $Ax = 0$. It is denoted by $N(A)$.

$N(A) = \{x \mid Ax = 0\}$

$Ax = 0 \iff N(A) = \{0\}$

- To check a set of vectors $\{v_1, \ldots, v_k\}$ is linearly independent:
  1. Put them in the columns of A.
  2. Solve the system $Ac = 0$.
  3. $N(A) = \{\mathbf{0}\}$ and the set of vector is linearly independent.
  4. If there is at least one free variable, then $N(A) \neq \{\mathbf{0}\}$ and the columns are dependent.

$c_1 v_1 + \cdots + c_k v_k = 0$

$$Ac = \begin{bmatrix} v_1 \cdots v_k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$N(A) = \{c \mid Ac = 0\}$

Echelon matrix U and Row Reduced matrix R

m معادله، n متغیر

$a_{11} x_1$

$\vdots$

$a_{m1} x_1$

$$\begin{array}{c} \\ \\ R = \end{array} \begin{array}{c} x_1 \\ \downarrow \\ \begin{bmatrix} 1 & 0 & * & 0 & * & * & * & * & * & 0 \\ 0 & 1 & * & 0 & * & * & * & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{m \times n} \end{array}$$

Example

$\underline{Ax = b}$

$C(A) = \{ b \mid \exists x \; Ax = b \}$

$C(A) = \mathrm{span}(\{ A_1, \dots, A_n \})$

$$Ax = \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 & 1 & -2 \end{bmatrix} \\ m \times n \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$\underline{Ax = 0}$

$C(A) = \{ Ax \mid x \in \mathbb{R}^n \}$

Echelon matrix U and Row matrix R

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 0 & b_1 \\ -1 & -1 & 2 & -3 & 1 & 0 & b_2 \\ 1 & 1 & -2 & 0 & 0 & 2 & b_3 \\ 0 & 0 & 0 & 3 & 1 & -2 & b_4 \end{bmatrix}$$

$$\begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 2 & -1 & 0 & 1 & 0 & b_1 \\ 0 & \mathbf{1} & 1 & -3 & 2 & 0 & b_1 + b_2 \\ 0 & 0 & 0 & \mathbf{-3} & 1 & 2 & b_2 + b_3 \\ 0 & 0 & 0 & 0 & \mathbf{2} & 0 & b_2 + b_3 + b_4 \end{bmatrix}$$

$$\begin{bmatrix} R & d \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 0 & -3 & 0 & 0 & 4 & \frac{2b_1 + 5b_3 + b_4}{2} \\ 0 & \mathbf{1} & 1 & 0 & 0 & -2 & \frac{2b_1 + b_2 - b_3 + b_4}{2} \\ 0 & 0 & 0 & \mathbf{1} & 0 & -2/3 & \frac{3b_2 - b_3 + b_4}{6} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \frac{b_2 + b_3 + b_4}{2} \end{bmatrix}$$

$Ax = 0$

$A \in M_n(\mathbb{R})$

$Ax = b$

$\underline{E_r \cdots E_1 A} x = \underline{E_r \cdots E_1 b}$

$Ux = c$

How does the reduced form R make the equation $Ax = b$ even clearer?

$$Ax = b \iff Rx = d.$$

$$Rx = \begin{array}{c} \begin{array}{cccccc} & & \times & & & \times \end{array} \\ \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 4 \\ 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & -2/3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}^{b} = \overbrace{\begin{bmatrix} \frac{2b_1+5b_3+b_4}{2} \\ \frac{2b_1+b_2-b_3+b_4}{2} \\ \frac{3b_2-b_3+b_4}{6} \\ \frac{b_2+b_3+b_4}{2} \end{bmatrix}}^{d}$$

$$\begin{aligned} x_5 &= \tfrac{b_2+b_3+b_4}{2} \\ x_4 &= \tfrac{3b_2-b_3+b_4}{6} + \tfrac{2}{3}x_6 \\ x_2 &= \tfrac{2b_1+b_2-b_3+b_4}{2} - x_3 + 2x_6 \\ x_1 &= \tfrac{2b_1+5b_3+b_4}{2} + 3x_3 + 4x_6 \end{aligned}$$

$Ax = b$

$x_1v_1 + \cdots + x_6v_6 = b$

$EA = R \Rightarrow A = E^{-1}R$

$\underbrace{E_r \cdots E_1}Ax = E_r \cdots E_1 b$

$Rx = d$

$x_1R_1 + \cdots + x_6R_6 = d$

R_3, R_6 را بسازند

$\Updownarrow$?

v_3, v_6 را بسازند

$R_3 =$

How does the reduced form R make this solution even clearer?

v.

$$\underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}} = \begin{bmatrix} \frac{2b_1+5b_3+b_4}{2} & +3x_3 & +4x_6 \\ \frac{2b_1+b_2-b_3+b_4}{2} & -x_3 & +2x_6 \\ 0 & +x_3 & \\ \frac{3b_2-b_3+b_4}{6} & & +\frac{2}{3}x_6 \\ \frac{b_2+b_3+b_4}{2} & & \\ 0 & & +x_6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2b_1+5b_3+b_4}{2} \\ \frac{2b_1+b_2-b_3+b_4}{2} \\ 0 \\ \frac{3b_2-b_3+b_4}{6} \\ \frac{b_2+b_3+b_4}{2} \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 4 \\ 2 \\ 0 \\ \frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$$

$Ax = b \iff C(A) =$

How does the reduced form R make the equation $Ax = b$ even clearer?

i. For every $b^T = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} \in \mathbb{R}^4$, the equation $Ax = b$ has a solution.
ii. So $C(A) = \mathbb{R}^4$.
iii. Also,

$$
\begin{aligned}
C(A) &= \left\{ x_1 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + x_2 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{v_2} + x_3 \underbrace{\begin{bmatrix} -1 \\ 2 \\ -2 \\ 0 \end{bmatrix}}_{v_3} + x_4 \underbrace{\begin{bmatrix} 0 \\ -3 \\ 0 \\ 3 \end{bmatrix}}_{v_4} + x_5 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{v_5} + x_6 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \end{bmatrix}}_{v_6} \;\middle|\; x_i \in \mathbb{R} \right\} \\
&= \operatorname{span}(\{v_1, v_2, \underline{v_3}, v_4, v_5, \underline{v_6}\}) \\
&= \operatorname{span}(\{v_1, v_2, v_4, v_5\}).
\end{aligned}
$$

Basis for a linear space

Definition

Let V be a linear space and $S \subseteq V$. The set S is a basis for V if

1. $V = \operatorname{span}(S)$,
2. $V \neq \operatorname{span}(T)$ for all $T \subsetneq S$.

- Trivially, a basis for a linear space is a linear independent set.
- A basis is a "minimal" spanning set for the linear space, in the sense that it has no "redundant" vector. At the same time, it is a "maximal" linearly independent set, in the sense that putting up a new vector makes it linearly dependent.
- There is no need for a basis to be finite! The linear space $\mathbb{R}[x]$ of all polynomials with real coefficients has no finite basis.
- A linear space may have more than one basis.

Finite basis for a linear space

Theorem

If $V = span(\{v_1, \dots, v_n\})$, then there is a subset of $\{v_1, \dots, v_n\}$ which is a basis for V.

Finite basis for a linear space

Theorem

If $V = span(\{v_1, \dots, v_n\})$, then there is a subset of $\{v_1, \dots, v_n\}$ which is a basis for V.

Theorem

Suppose that $V = span(\{v_1, \dots, v_n\})$. Then each independent set of V has at most n elements.

Dimension

Theorem

If $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ are both bases for a linear space V, then $m = n$.

Definition

Suppose that V has a finite basis. Then ***dimension*** of V denoted by $\dim V$ is the number of elements of any basis of V.

- Example. Assume the linear space $P_2(x) = \{a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R} \text{ for all } 0 \le i \le 2\}$.
  1. The sets $\{1, x, x^2\}$ is a basis for $P_2(x)$.
  2. $\dim(P_2(x)) = 3$.
  3. You can easily check that $\{1, x, x^2 - \frac{1}{3}\}$ is a basis for $P_2(x)$.

...

Thank You!