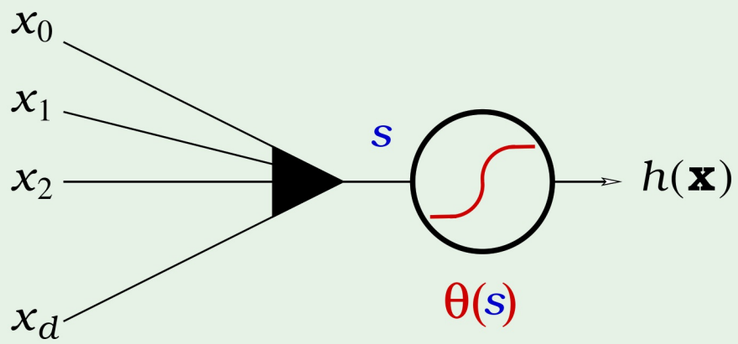
$\underline{W} = \{\underline{w}^{(1)}, \underline{w}^{(2)}, \ldots, \underline{w}^{(K)}\}$ (Backpropagation)

Review of Lecture 9

- Logistic regression

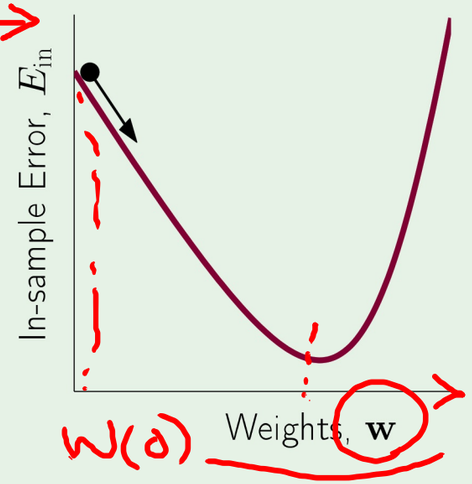


- Likelihood measure

$$\prod_{n=1}^{N} P(y_n \mid \mathbf{x}_n) = \prod_{n=1}^{N} \theta(y_n \mathbf{w}^{\mathsf{T}} \mathbf{x}_n)$$

- Gradient descent

empirical error $\rightarrow$ $\frac{1}{N}\sum_{i=1}^{N}(f_{\underline{w}}(x_i) - y_i)^2$



- Initialize $\mathbf{w}(0)$
- For $t = 0, 1, 2, \cdots$ [to termination]

$$\mathbf{w}(t+1) = \mathbf{w}(t) - \eta \, \nabla E_{\text{in}}(\mathbf{w}(t))$$

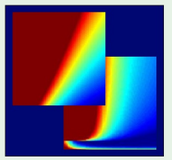
- Return final $\mathbf{w}$

Learning From Data

Yaser S. Abu-Mostafa
California Institute of Technology

Lecture 10: Neural Networks

Sponsored by Caltech's Provost Office, E&AS Division, and IST • Thursday, May 3, 2012



Outline

- Stochastic gradient descent
- Neural network model
- Backpropagation algorithm

Stochastic gradient descent

GD minimizes:

$$E_{\text{in}}(\mathbf{w}) = \frac{1}{N}\sum_{n=1}^{N} \underbrace{\mathbf{e}\left(h(\mathbf{x}_n), y_n\right)}_{\ln\left(1+e^{-y_n \mathbf{w}^{\mathsf{T}}\mathbf{x}_n}\right)} \longleftarrow \text{in logistic regression}$$

by iterative steps along $-\nabla E_{\text{in}}$:

$$\Delta\mathbf{w} = -\eta\, \nabla E_{\text{in}}(\mathbf{w})$$

∇E_{in} is based on all examples $(\mathbf{x}_n, y_n)$

"**batch**" GD

The stochastic aspect

Pick one $(\mathbf{x}_n, y_n)$ at a time. Apply GD to $\mathbf{e}\left(h(\mathbf{x}_n), y_n\right)$

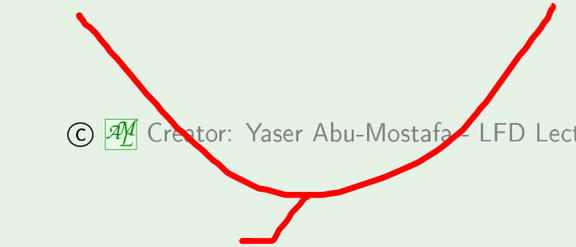
"Average" direction:

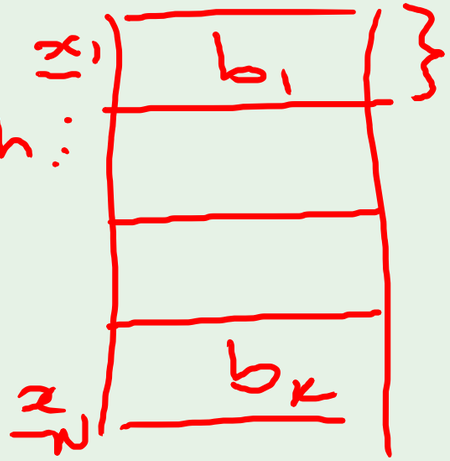
$$\mathbb{E}_n\left[-\nabla \mathbf{e}\left(h(\mathbf{x}_n), y_n\right)\right] = \frac{1}{N}\sum_{n=1}^{N} -\nabla \mathbf{e}\left(h(\mathbf{x}_n), y_n\right)$$

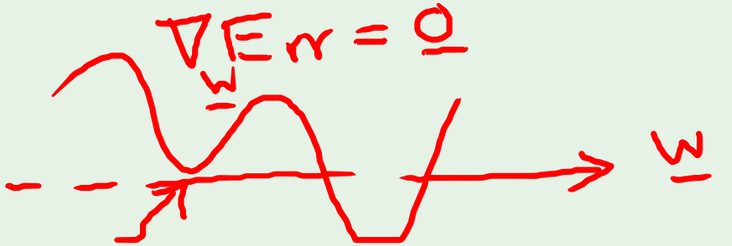
$$= -\nabla E_{\text{in}}$$

randomized version of GD

stochastic gradient descent (SGD)





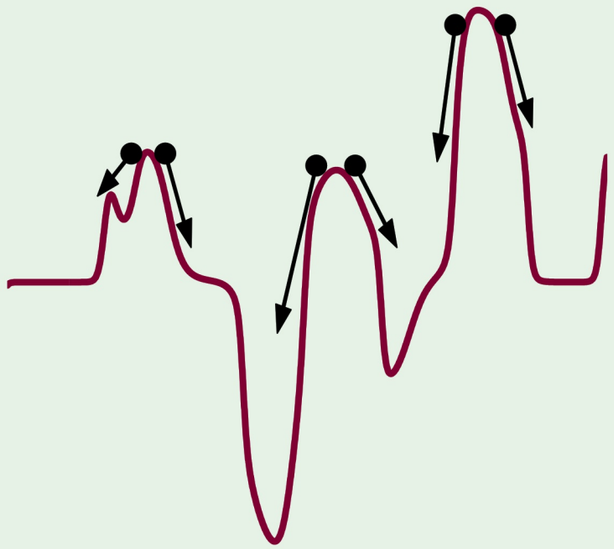


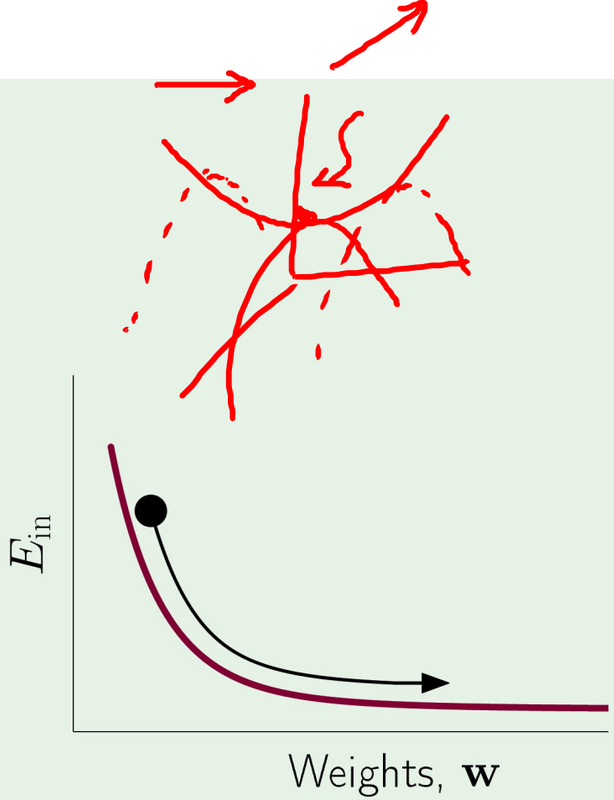
Benefits of SGD

1. cheaper computation
2. randomization
3. simple

Rule of thumb:

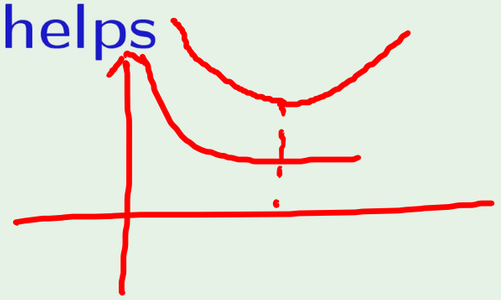
$\eta = 0.1$ works





randomization helps

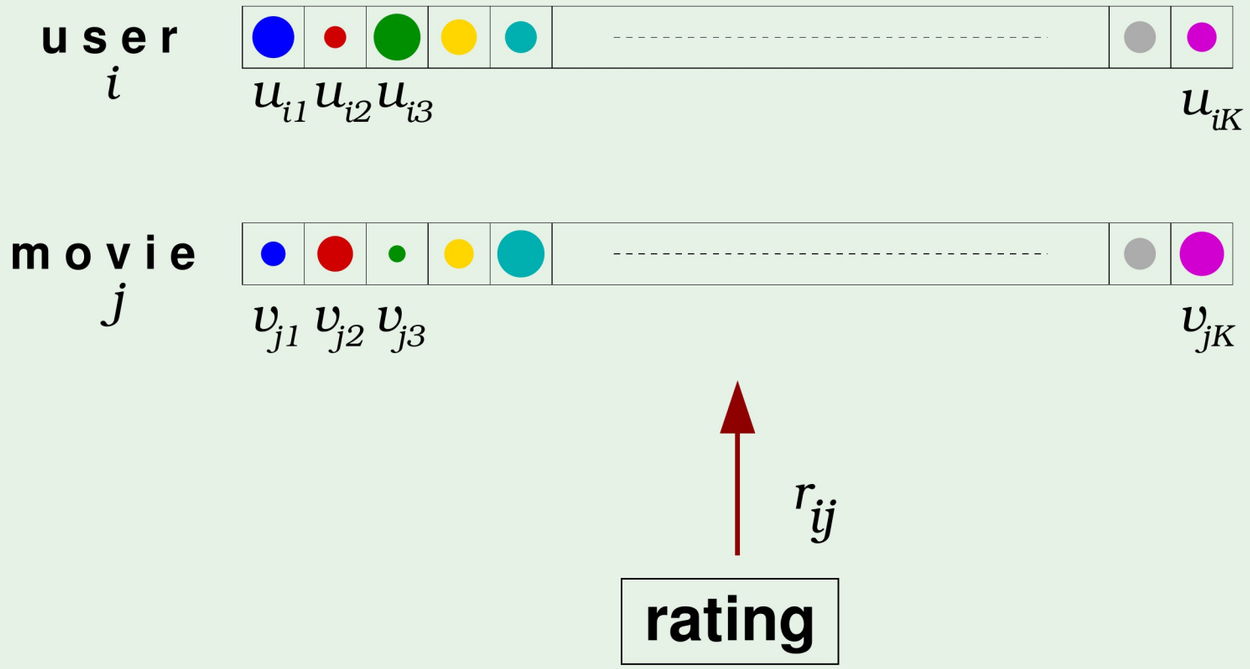
Early Stopping



SGD in action

Remember movie ratings?

$$\mathbf{e}_{ij} = \left(r_{ij} - \sum_{k=1}^{K} u_{ik} v_{jk} \right)^2$$

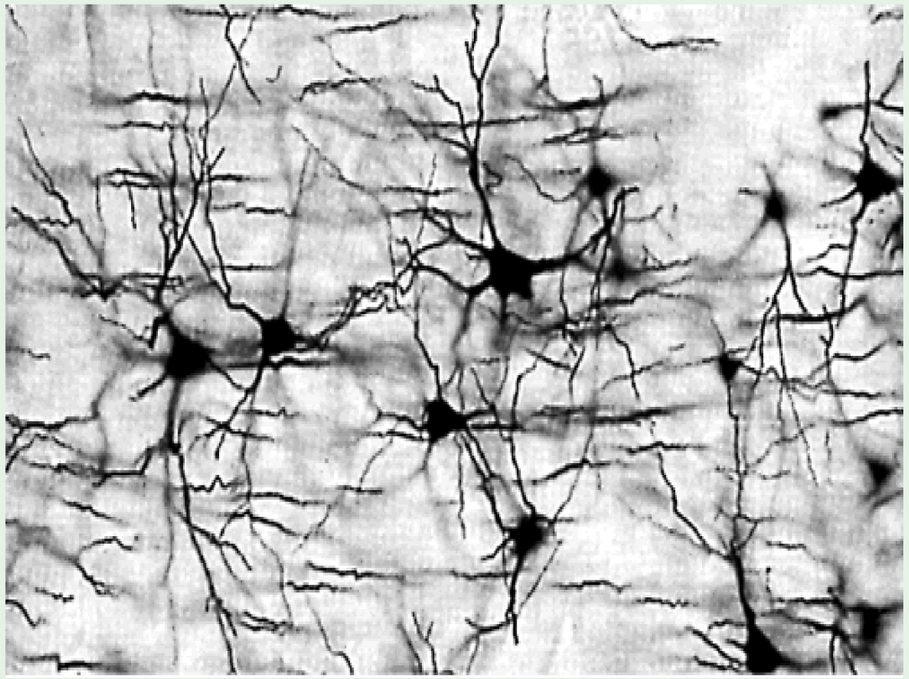


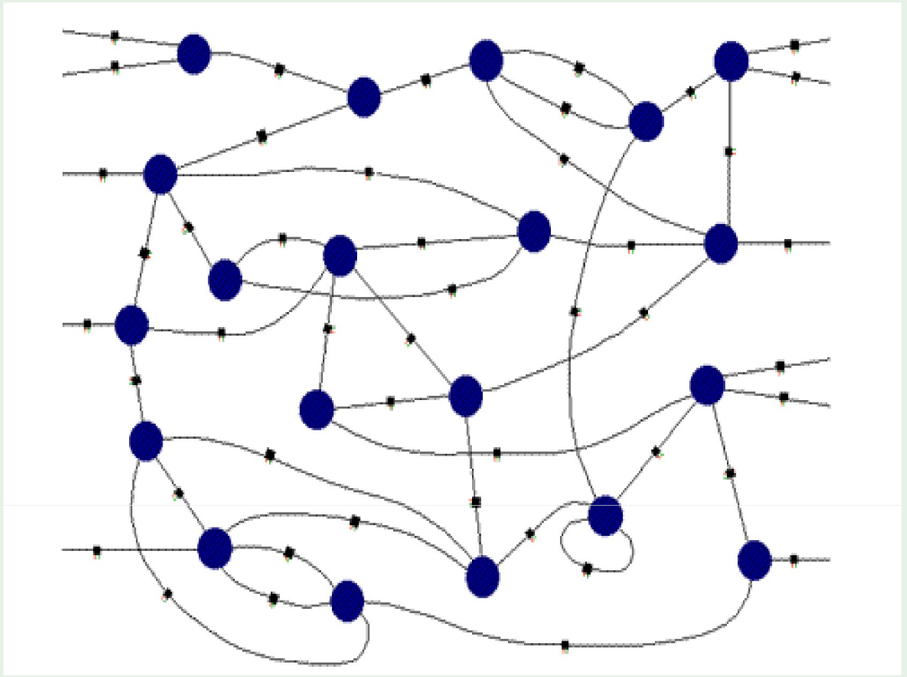
Outline

- Stochastic gradient descent
- Neural network model
- Backpropagation algorithm

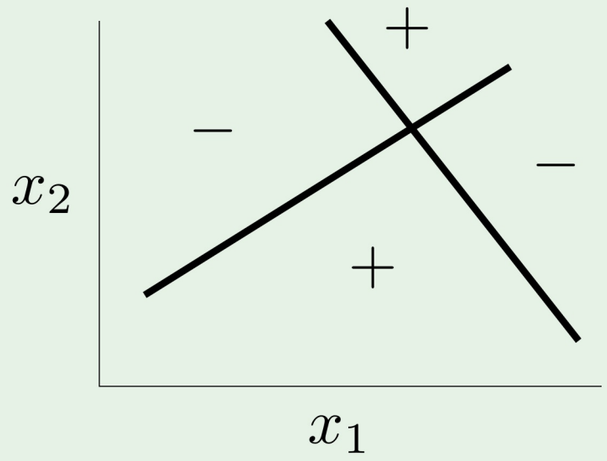
Biological inspiration

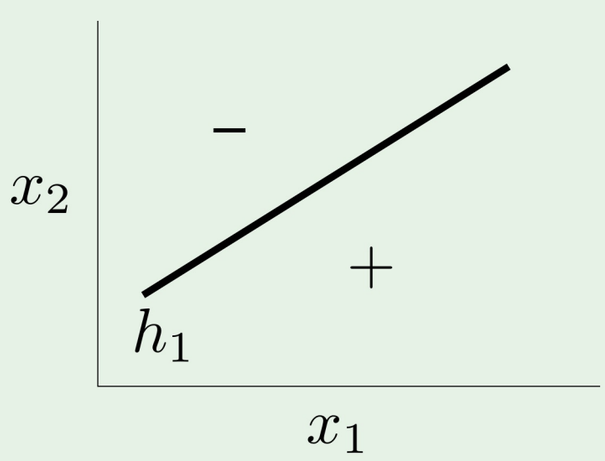
biological function $\longrightarrow$ biological structure

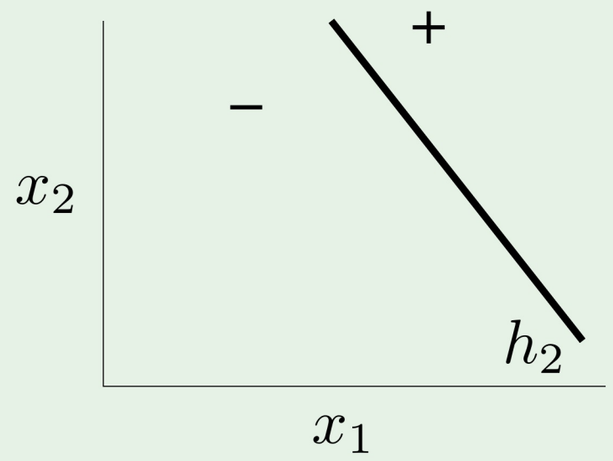


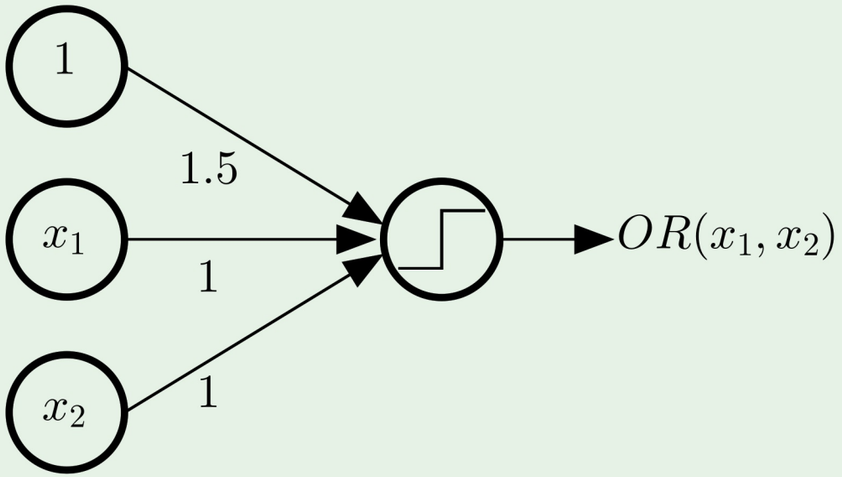


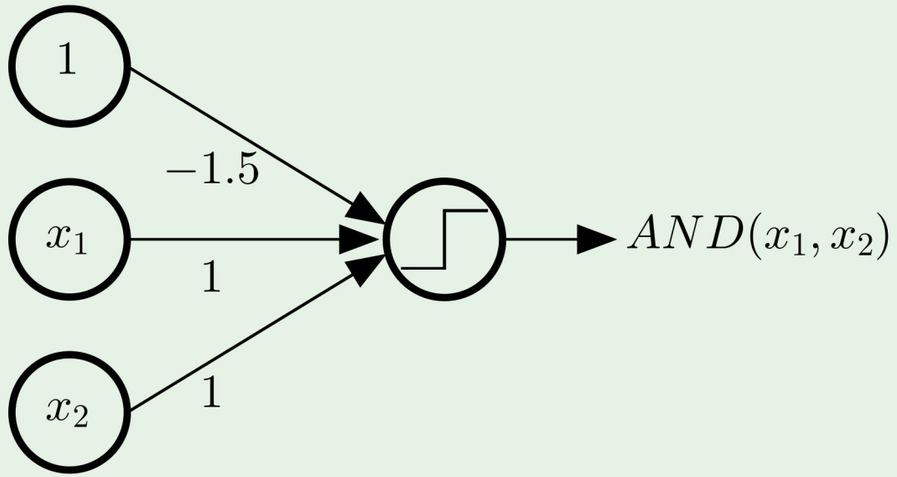
Combining perceptrons



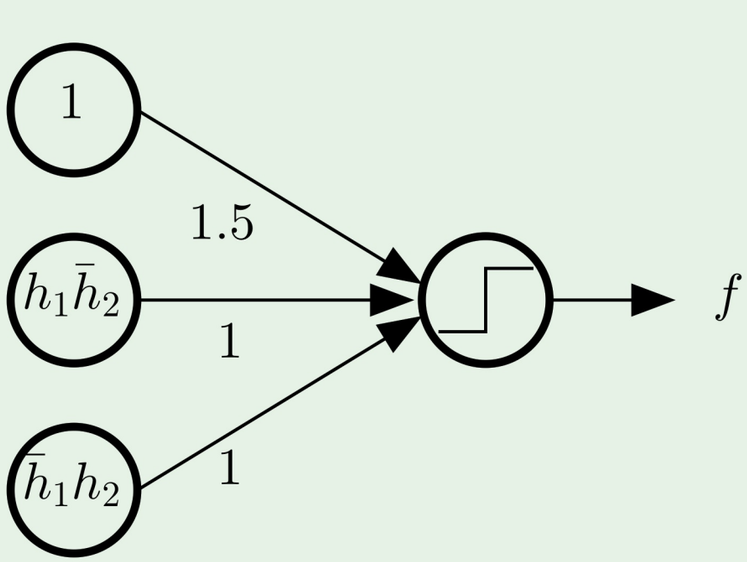


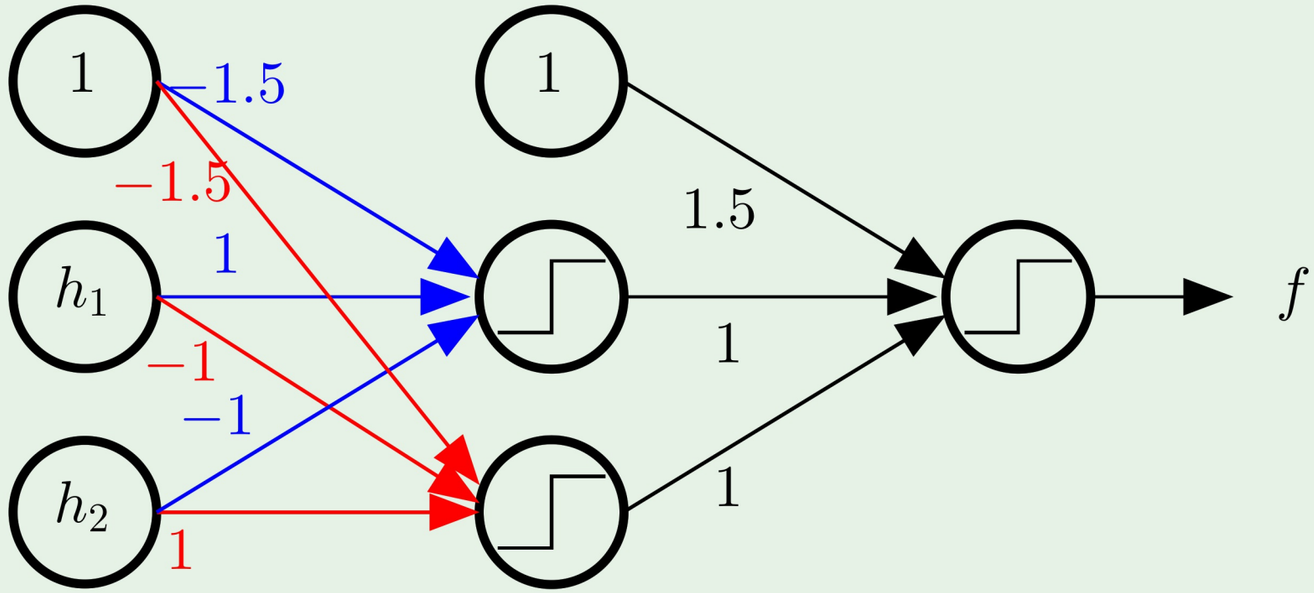




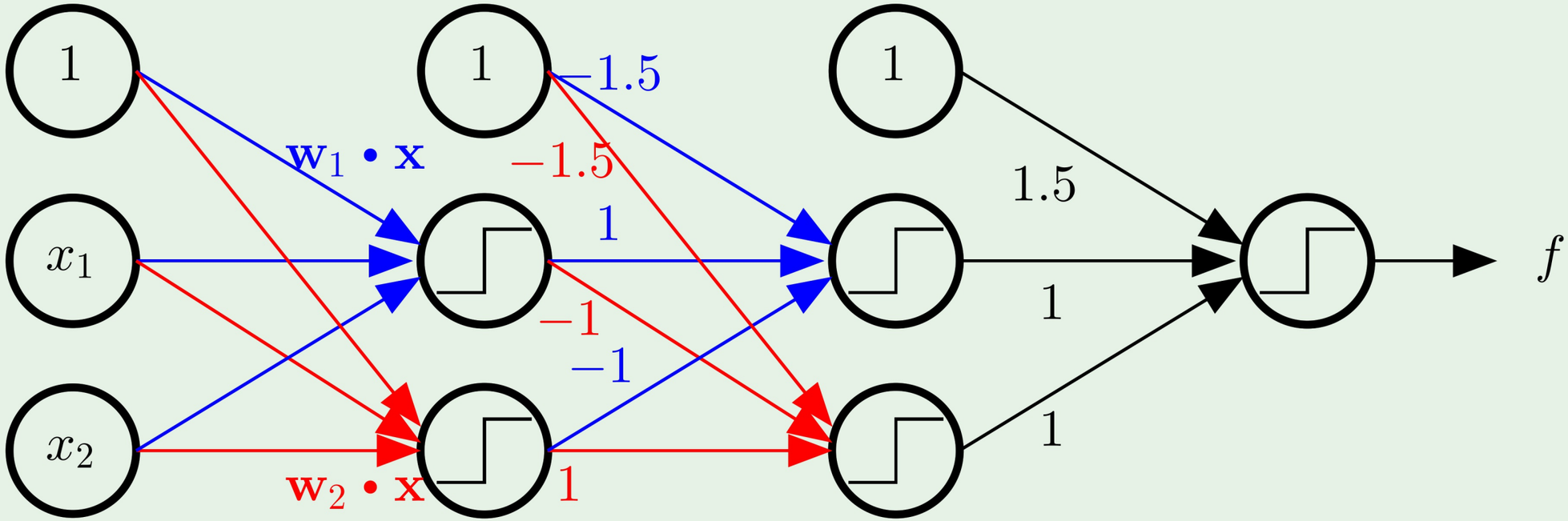


Creating layers



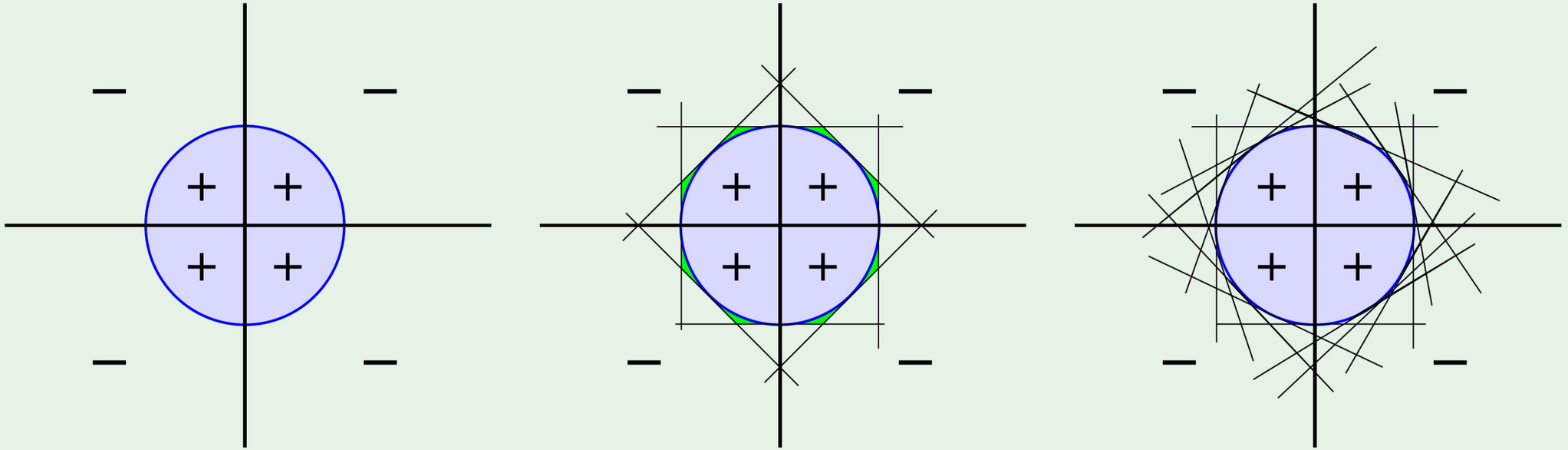


The multilayer perceptron



3 layers “feedforward”

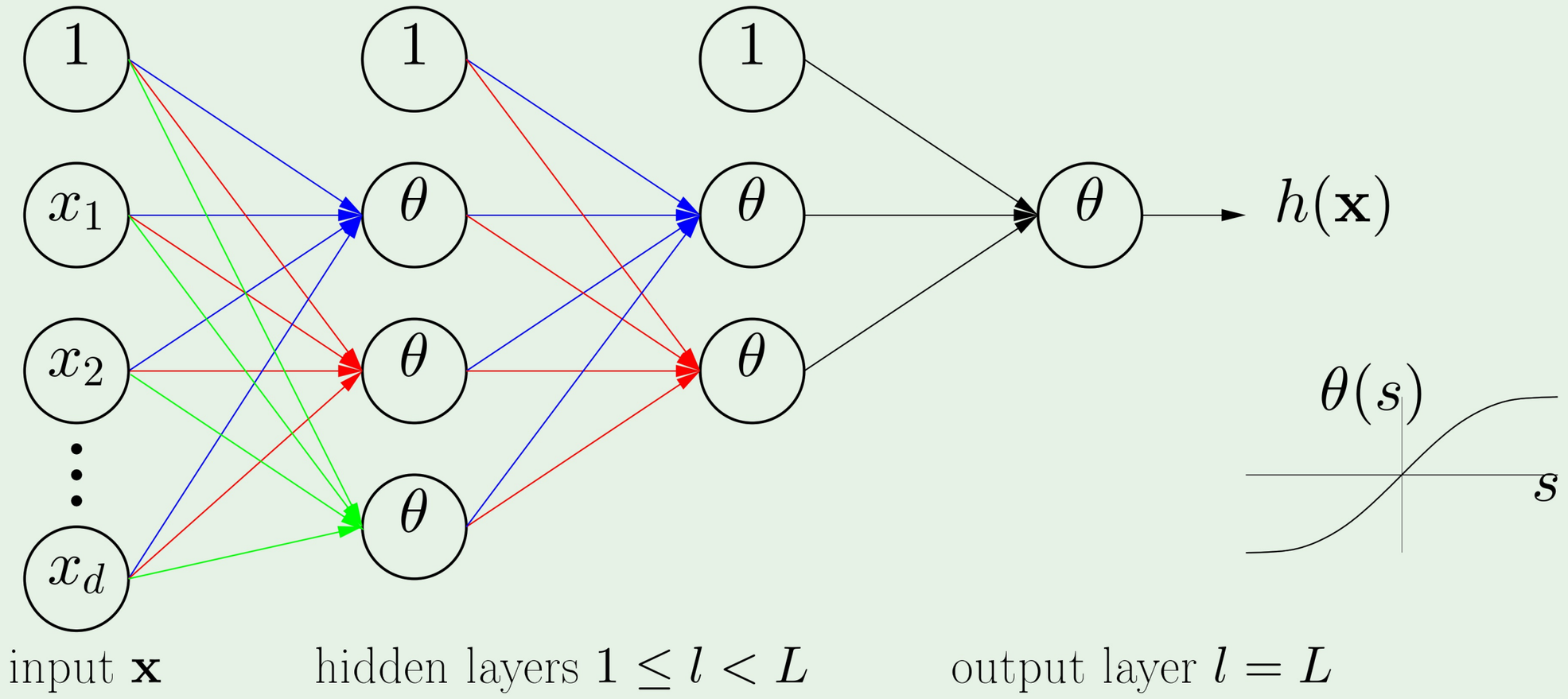
A powerful model



Target | 8 perceptrons | 16 perceptrons

2 red flags for **generalization** and **optimization**

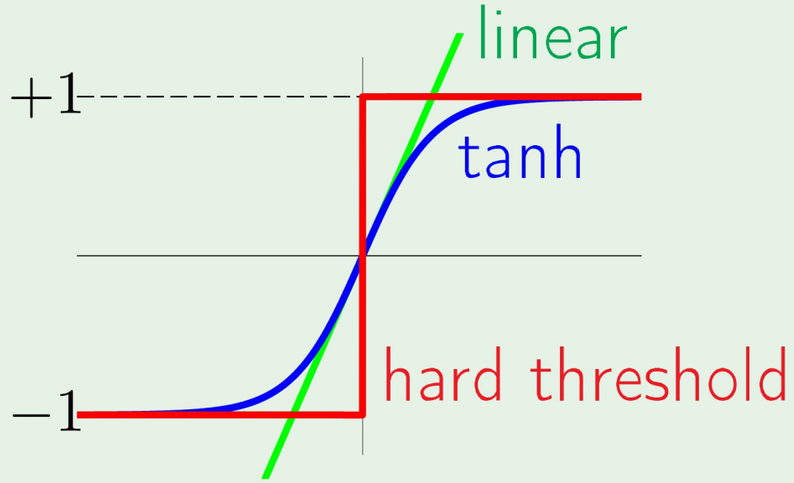
The neural network



How the network operates

$$w_{ij}^{(l)} \quad \begin{cases} 1 \leq l \leq L & \text{layers} \\ 0 \leq i \leq d^{(l-1)} & \text{inputs} \\ 1 \leq j \leq d^{(l)} & \text{outputs} \end{cases}$$

$$x_j^{(l)} = \theta(s_j^{(l)}) = \theta\left(\sum_{i=0}^{d^{(l-1)}} w_{ij}^{(l)}\, x_i^{(l-1)}\right)$$



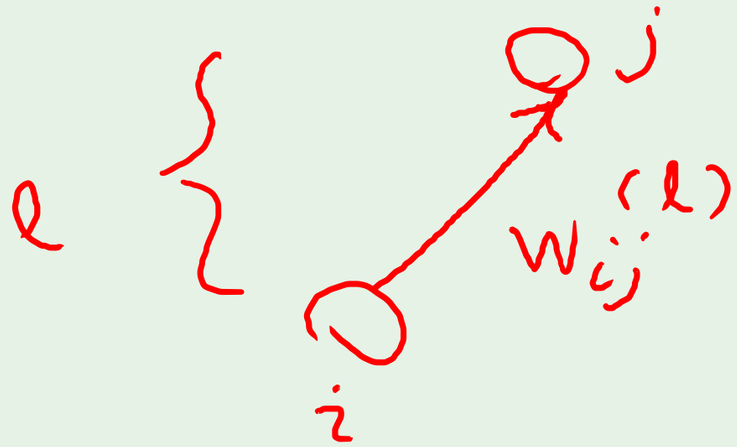
$$\theta(s) = \tanh(s) = \frac{e^s - e^{-s}}{e^s + e^{-s}}$$

Apply $\mathbf{x}$ to $x_1^{(0)} \cdots x_{d^{(0)}}^{(0)} \rightarrow\rightarrow x_1^{(L)} = h(\mathbf{x})$

Outline

- Stochastic gradient descent
- Neural network model
- Backpropagation algorithm

Applying SGD



All the weights $\mathbf{w} = \{w_{ij}^{(l)}\}$ determine $h(\mathbf{x})$

Error on example $(\mathbf{x}_n, y_n)$ is

$$\mathbf{e}\left(h(\mathbf{x}_n), y_n\right) = \mathbf{e}(\mathbf{w})$$

To implement SGD, we need the gradient

$$\nabla \mathbf{e}(\mathbf{w}): \quad \frac{\partial\ \mathbf{e}(\mathbf{w})}{\partial\ w_{ij}^{(l)}} \quad \text{for all } i, j, l$$

$f(x_1, \cdots, x_n) \in \mathbb{R}$ $\quad \underline{x} = (x_1, \cdots x_n)$

$$\frac{\partial}{\partial x_j} f(g_1(\underline{x}), \cdots, g_n(\underline{x}))$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial g_i} \frac{\partial g_i}{\partial x_j}$$

$$s_j^{(l)} = \sum_{i=1}^{n^{(l-1)}} w_{ij}^{(l)} x_i^{(l-1)} \qquad x_j^{(l)} = \theta(s_j^{(l)})$$

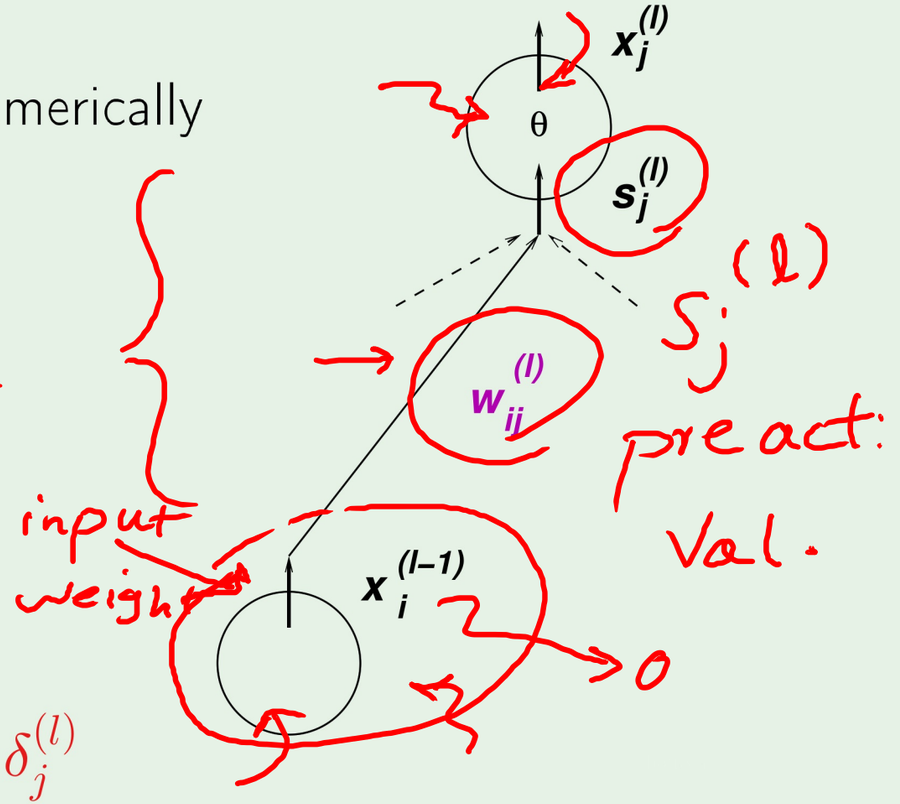
Computing $\dfrac{\partial\ \mathbf{e}(\mathbf{w})}{\partial\ w_{ij}^{(l)}}$

We can evaluate $\dfrac{\partial\ \mathbf{e}(\mathbf{w})}{\partial\ w_{ij}^{(l)}}$ one by one: analytically or numerically

A trick for efficient computation:

$$\frac{\partial\ \mathbf{e}(\mathbf{w})}{\partial\ w_{ij}^{(l)}} = \frac{\partial\ \mathbf{e}(\mathbf{w})}{\partial\ s_j^{(l)}} \times \frac{\partial\ s_j^{(l)}}{\partial\ w_{ij}^{(l)}}$$

We have $\dfrac{\partial\ s_j^{(l)}}{\partial\ w_{ij}^{(l)}} = x_i^{(l-1)}$ We only need: $\dfrac{\partial\ \mathbf{e}(\mathbf{w})}{\partial\ s_j^{(l)}} = \delta_j^{(l)}$



Credit Assignment

the act. val. of the input neuron

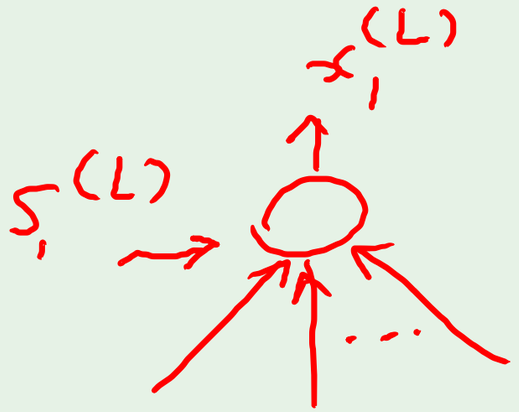
δ for the final layer

$L \rightarrow$ last layer

$$\delta_j^{(l)} = \frac{\partial\, \mathbf{e}(\mathbf{w})}{\partial\, s_j^{(l)}}$$

For the final layer $l = L$ and $j = 1$:

$$\delta_1^{(L)} = \frac{\partial\, \mathbf{e}(\mathbf{w})}{\partial\, s_1^{(L)}}$$



$$\mathbf{e}(\mathbf{w}) = (x_1^{(L)} - y_n)^2$$

$$x_1^{(L)} = \theta(s_1^{(L)})$$

$$\frac{\partial e}{\partial s_1^{(L)}} = \frac{\partial}{\partial s_1^{(L)}} (\theta(s_1^{(L)}) - y)^2$$

$$= 2(x_1^{(L)} - y)\,\theta'(s_1^{(L)})$$

$$= 2(x_1^{(L)} - y)(1 - \theta^2(s_1^{(L)}))$$

$$= 2(x_1^{(L)} - y)(1 - x_1^{(L)2})$$

$$\theta'(s) = 1 - \theta^2(s) \quad \text{for the tanh}$$

$$s_j^{(l)} = \sum_{i=1}^{n^{(l-1)}} w_{ij}^{(l)} \, x_i^{(l-1)} \qquad x_j^{(l)} = \theta\left(s_j^{(l)}\right)$$

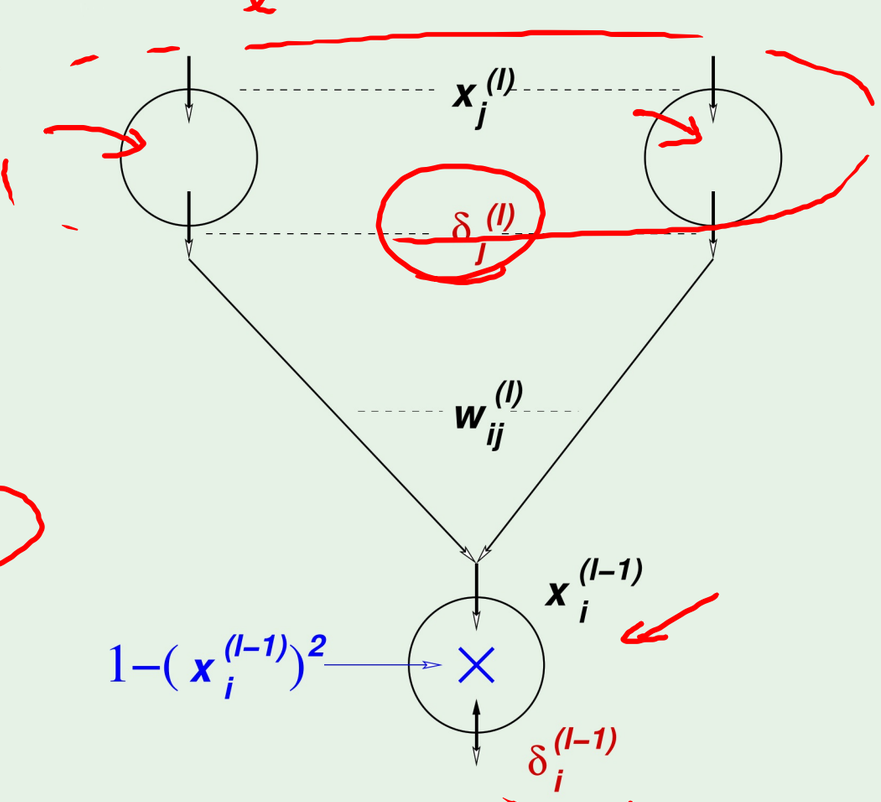
Back propagation of δ

$$\delta_i^{(l-1)} = \frac{\partial \, \mathbf{e}(\mathbf{w})}{\partial \, s_i^{(l-1)}}$$

$$= \sum_{j=1}^{d^{(l)}} \frac{\partial \, \mathbf{e}(\mathbf{w})}{\partial \, s_j^{(l)}} \times \frac{\partial \, s_j^{(l)}}{\partial \, x_i^{(l-1)}} \times \frac{\partial \, x_i^{(l-1)}}{\partial \, s_i^{(l-1)}}$$

$$= \sum_{j=1}^{d^{(l)}} \delta_j^{(l)} \times w_{ij}^{(l)} \times \theta'(s_i^{(l-1)})$$

$$\delta_i^{(l-1)} = (1 - (x_i^{(l-1)})^2) \sum_{j=1}^{d^{(l)}} w_{ij}^{(l)} \, \delta_j^{(l)}$$



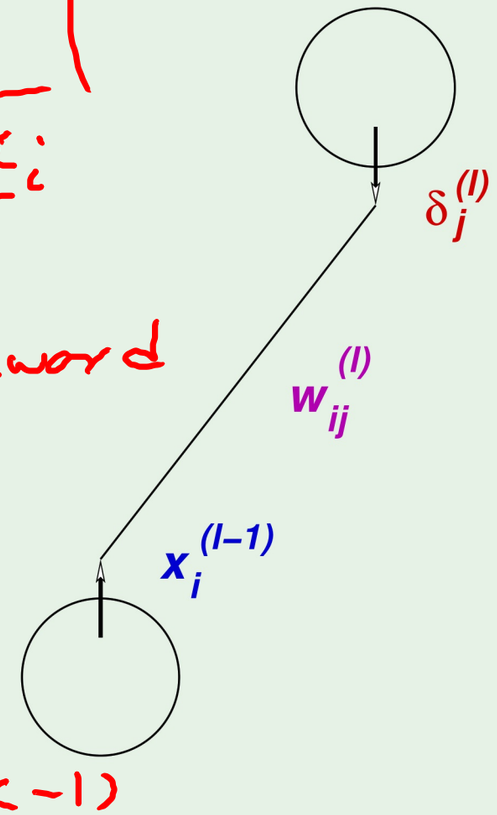
Backpropagation algorithm

ResNet

1: Initialize all weights $w_{ij}^{(l)}$ at random
2: **for** $t = 0, 1, 2, \ldots$ **do**
3: Pick $n \in \{1, 2, \cdots, N\}$
4: *Forward:* Compute all $x_j^{(l)}$
5: *Backward:* Compute all $\delta_j^{(l)}$
6: Update the weights: $w_{ij}^{(l)} \leftarrow w_{ij}^{(l)} - \eta\, x_i^{(l-1)} \delta_j^{(l)}$
7: Iterate to the next step until it is time to stop
8: Return the final weights $w_{ij}^{(l)}$

forward backward

x_i



$v^{(k)} = \alpha \nabla e + (1-\alpha) v^{(k-1)}$

$w^{(k)} \leftarrow w^{(k-1)} - \eta v^{(k-1)}$ w

Final remark: hidden layers

learned nonlinear transform

interpretation?