# MOD 201: Modeling the Spread of an Infection

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## 1 The SI Model: $I_t = I_{t-1} + \beta I_{t-1}$

### 1.1 Modeling the infection

The SI Model, as described by the equation above, is a simple model for simulating the spread of an infection through a population. It assumes that the spread of the number of infected individuals at any time, $I_t$, is equal to the number of infected individuals at the previous timestep, $I_{t-1}$, plus the contagiousness parameter, $\beta$, times the number of infected individuals at $I_{t-1}$. The graph below shows the results of the model over 1 year, in a population of 100 people, with an inital number of infected persons, $I_0 = 3$, and a contagiousness parameter, $\beta = 0.02$.

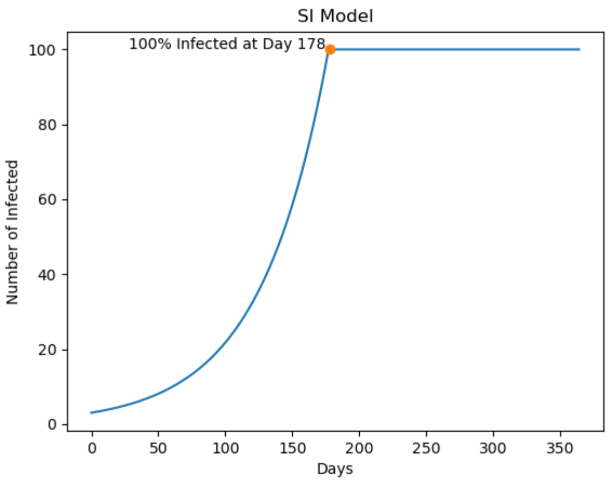
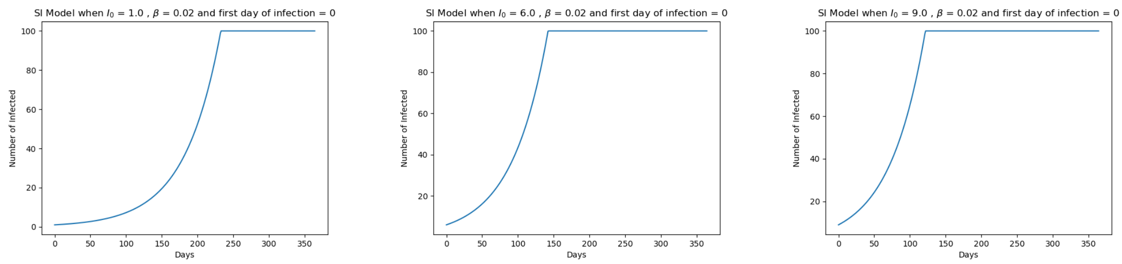



Figure 1: SI Model, over 1 year, with $I_0 = 3$, $\beta = 0.02$

Based on the plot, we can see that the number of infected individuals grows exponentially over time, until the entire population is infected. This is expected, since the model predicts that the number of newly infected individuals at each time step is dependent on the number of infected individuals at the previous time step. Thus, the number of newly infected will grow as the number of infected individuals grows. Based on this model, it will take 178 days for the entire population to be infected.

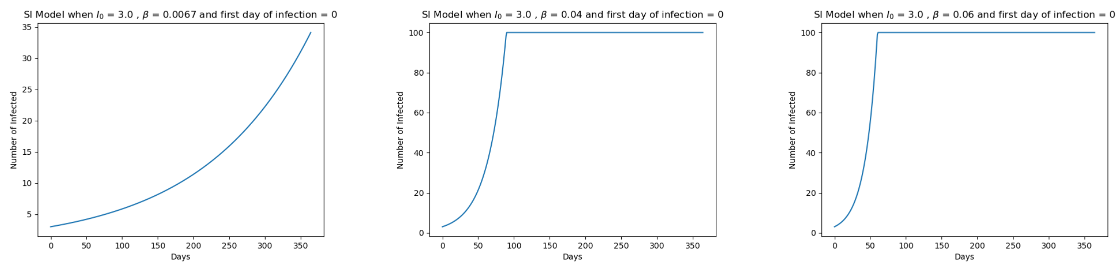
## 1.2 Varying the parameters




(a) $I_0 = 1$, $\beta = 0.02$, D = 0 (b) $I_0 = 6$, $\beta = 0.02$, D = 0 (c) $I_0 = 9$, $\beta = 0.02$, D = 0

Figure 2: Varying $I_0$

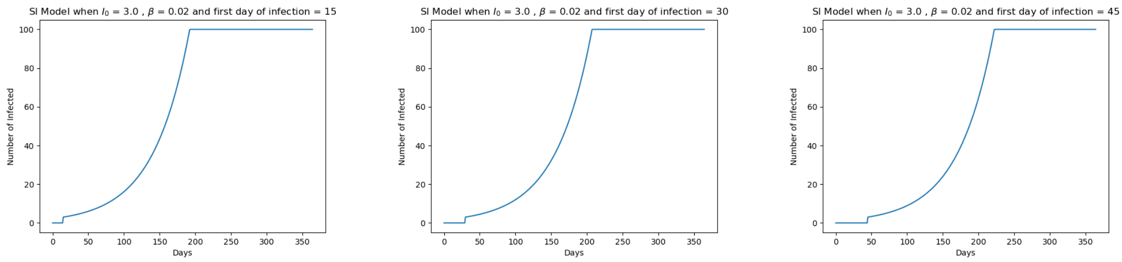
The above plots illustarate the behavior of the infection as predicted by the model for different numbers of initially infected persons, $I_0$. As the number of initial infected patients increases, the graph gets steeper, i.e., the number of infected individuals grows faster. Thus, the amount of time it takes for the entire population to be infected decreases. Consequently, as the number of initial infected decreases, the amount of time it takes for the entire population to be infected increases.




(a) $I_0 = 3$, $\beta = 0.0067$, D = 0 (b) $I_0 = 3$, $\beta = 0.04$, D = 0 (c) $I_0 = 3$, $\beta = 0.06$, D = 0

Figure 3: Varying $\beta$

The above plots illustrate the behavior of the infection predicted by the model for different contagiousness parameters, $\beta$. Varying $\beta$ provides similar results to varying $I_0$. As the contagiousness parameter increases, the curve gets steeper; as the the parameter decreases the curve gets shallower. One notable difference, however, is the magnitude of the effect a change in $\beta$ has on the steepness of the curve, compared to the magnitude of the effect a change in $I_0$ has. For example, dividing each value by 3 (relative to their initial values in Section 1.1), while holding the others constant (Fig. 2 (a), Fig. 3 (a)), we see that in Fig. 2 (a), the curve is shallower, yet the entire population is still infected in the period. Contrastingly, in Fig. 3 (a), the curve becomes so shallow that infection fails to reach the entire population within the period.

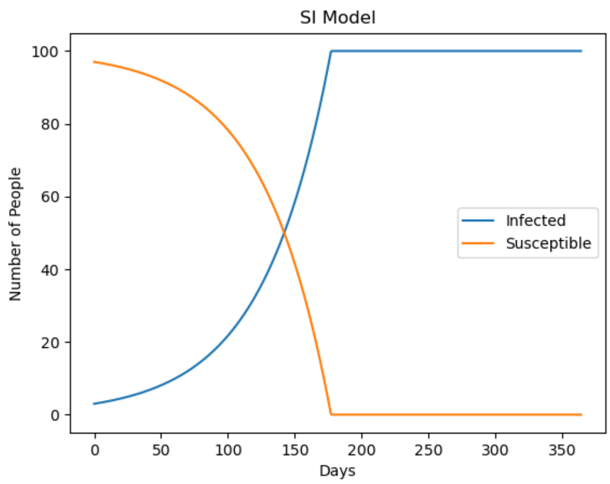



(a) $I_0 = 3$, $\beta = 0.02$, D = 15 (b) $I_0 = 3$, $\beta = 0.02$, D = 30 (c) $I_0 = 3$, $\beta = 0.02$, D = 45

Figure 4: Varying First Day of Infection (D)

The above graphs illustrate the behavior of the model for different first days of infection, D. Unlike the previous two cases, changes in the first day of infection does not effect the steepness of the curve. Rather, it shifts the point at which the number of infected is no longer 0 to the right. The result of this is that the day the entire population is infected is increased by the same amount. This expected, as the infection rate does not depend on when the first day of infection is.

## 1.3 S and I curve together




The above figure shows the predicted evoluton of both the number of susceptible individuals in the population, and infected individuals, as the virus spreads.

However, this model is not realistic. It does not account for the fact that in reality, there are more factors that can influence the spread of the infection than merely the amount of people infected the previous day and the contagiousness of the disease. For example, the probability of a susceptible individual encountering an infected individual would also influence the spread. Moreover, it does not account for the possibility that some individuals may recover from the disease, and thus be added back to the susceptible category.

### 1.4 Accounting for Probability of Contact: $I_t = I_{t-1} + \frac{\beta}{N}\ I_{t-1}S_{t-1}$

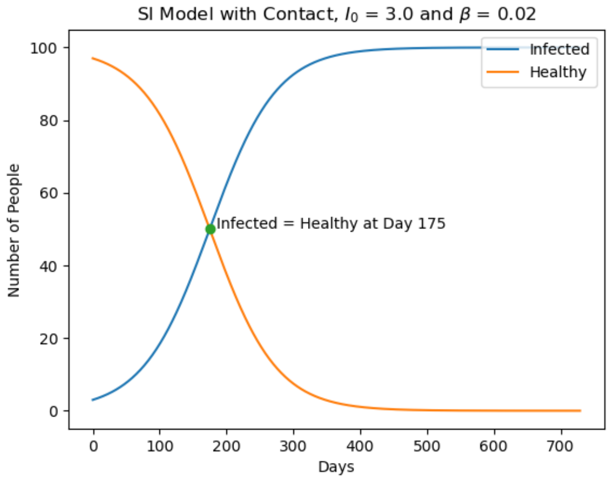
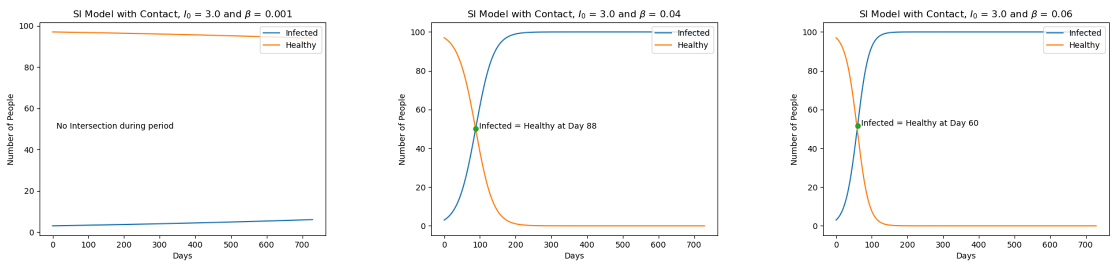



Figure 5: SI Model with Contact, over 2 years

In the previous section, the fact that the model did not take into account the probability of contact between an infected and susceptible was brought up as a point against the accuracy of the model. The above equation aims to ameliorate this concern by introducing this probability with the term $\frac{S}{N}$. This results in the above formula. When simulating this new model, the above graph is produced.

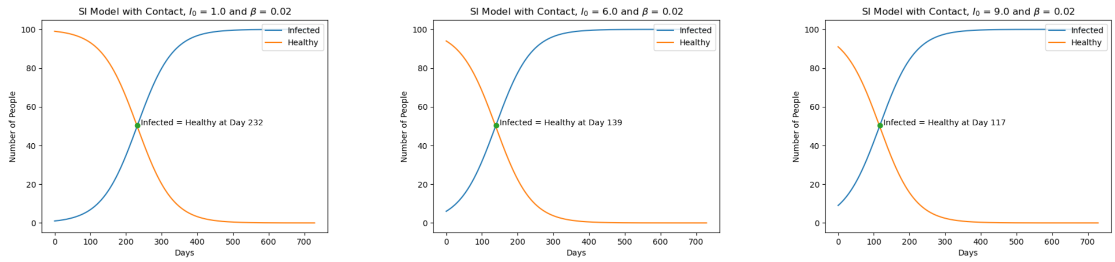
By accounting for the probability of contact, the number of infected individuals grows at a slower rate, given $\beta$ and $I_0$ parameters identical to the previous model. This is reflected in the number of days it takes for the number of infected individuals to be equivalent to the number of susceptible individuals. In the first model, this date was about 150 days, while in this model it is 175 days. Moreover, in the previous model, the whole population was infected after about 160 days. In this model it takes more than 400 days to infect the entire population.

### 1.5 Changing the parameters




(a) $I_0 = 3$, $\beta = 0.001$ (b) $I_0 = 3$, $\beta = 0.04$ (c) $I_0 = 3$, $\beta = 0.06$

Figure 6: SI Model with Contact, Varying $\beta$




(a) $I_0 = 1$, $\beta = 0.02$ (b) $I_0 = 6$, $\beta = 0.02$ (c) $I_0 = 9$, $\beta = 0.02$

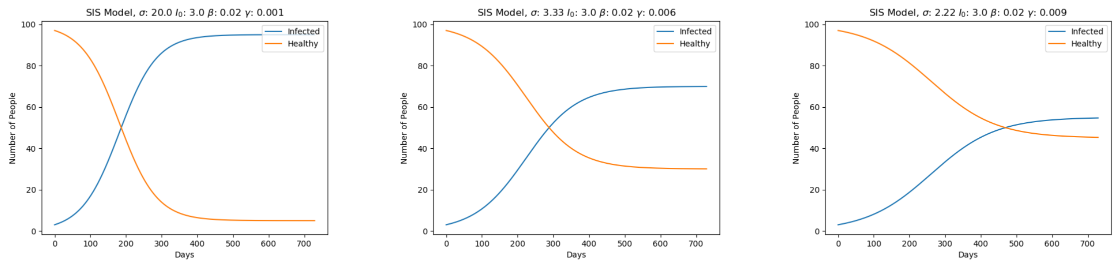
Figure 7: SI Model with Contact, Varying $I_0$

As in the previous model, changes in $\beta$ have a greater effect than similar changes in $I_0$ (Fig 6. (b), Fig 7. (b); Fig 6. (c), Fig 7. (c)). Increases in both parameters result in steeper curves, and thus faster infection rates. Likewise, decreases in both parameters result in shallower curves, and thus slower infection rates. However, compared to the simpler models with identical parameters, it takes longer for the entire population to be infected. For example, in Fig. 3 (b), it takes about 100 days to infect the entire population, while in Fig. 6 (b), it takes about 300 days.

Regarding different regimes, an infection rate of 0.001 is not enough to infect the entire population, 2 years. However, any non 0 value of $\beta$, in this model, will eventully result in the entire population being infected.

# 2 The SIS Model: $I_t = I_{t-1} + \frac{\beta}{N}\ I_{t-1}S_{t-1} - \gamma I_{t-1}$

## 2.1 Simulating the SIS Model




(a) $\beta = 0.02, I_0 = 3, \gamma = 0.001$ (b) $\beta = 0.02, I_0 = 3, \gamma = 0.006$ (c) $\beta = 0.02, I_0 = 3, \gamma = 0.009$

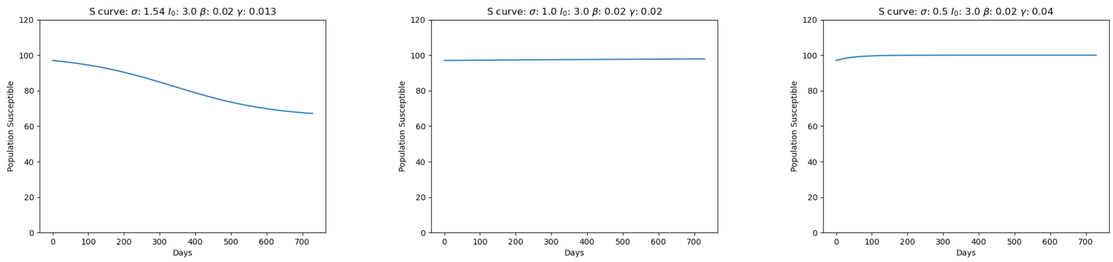
Figure 8: SIS Model with Contact, Varying $\gamma$

Another limitation of the SI model mentioned above was its failure to account for the possibility of individuals recovering from the disease. The SIS model addresses this concern by introducing a recovery parameter $\gamma$ into the equation (in addition to the probability of contact). The above plots illustrate the behavior of the infection as predicted by the SIS model for different values of $\gamma$.

Introducing this parameter results in model behavior that was previously impossible.While gamma is small, the model predicts that the entire population will eventully be infected (as before). However, as gamma increases, the maximum values of the two curves begin to converge.

Namely, for large enough values of gamma, the entire population will no longer be infected by the disease (Fig. 8 (b)). Moreover, Fig 8. (c) suggests that for even larger values of gamma, the two curves will never overlap, i.e., that the number of infected individuals will never equal (nor exceed) the number of susceptible individuals.

## 2.2 Varying the ratio $\sigma = \frac{\beta}{\gamma}$




(a) $\beta = 0.02, I_0 = 3, \gamma = 0.013, \sigma = 1.54$

(b) $\beta = 0.02, I_0 = 3, \gamma = 0.02, \sigma = 1$

(c) $\beta = 0.02, I_0 = 3, \gamma = 0.04, \sigma = 0.5$

Figure 9: S curve from the SIS Model, varying $\sigma$

The ratio $\sigma = \frac{\beta}{\gamma}$ is the ratio between the transmission rate and the recovery rate. The effect of different values for this ratio on the epidemic is illustrated in the above graphs. (a) illustrates the case when $\sigma > 1$, and thus the number of individuals in the susceptible group goes down to a point, as some individuals move in to the infected group. This is because the rate of infection is sufficiently greater than the rate of recovery. (b) illustrates the case when $\sigma = 1$. In this case, the number of infected individuals slowly approaches 0 over time. This dynamic is the result of the fact that $\beta$ is dependent on the total population number, whereas $\gamma$ is not. Finally, (c) illustrates the case when $\sigma < 1$. In this case, because the rate of recovery exceeds the rate of infection, the whole population eventually quickly moves into the susceptible group.

This dynamic is reflected in the equation for the stationary value of the curve. Given the model $I_t = I_{t-1} + \frac{\beta}{N} I_{t-1}S_{t-1} - \gamma I_{t-1}$, the stationary point, with respect to $S_t$, is the value for which $S_{t-1} = S_t$. Moreover, since it is always the case that $S_t + I_t = 100$, we can further assume that when $S_{t-1} = S_t = S$, $I_{t-1} = I_t = I$. Thus, $I = I + \frac{\beta}{N}IS - \gamma I$. Subtracting $I$ from both sides, and then dividing by $I$, we get: $0 = \frac{\beta}{N}S - \gamma$. Solving for $S$, and setting N to be our population of 100 we get our stationary value: $S = \frac{\gamma 100}{\beta}$

Solving this equation for the parameters in the above graphs, and taking into account that $S$ cannot be greater than the total population, i.e., 100, we get the following stationary values: (a) 65, (b) 100, (c) 100. Finally, there does exist a ratio for which the stationary value is $S_0$, i.e., where the curve is flat, given $I_0 = 3$. Solving our equation for $S_0 = N - I_0 = 100 - 3 = 97$; $97 = \frac{\gamma 100}{\beta}$; $\frac{\gamma}{\beta} = 97/100$. $\sigma = \frac{\gamma}{\beta}^{-1}$; $\sigma = \frac{97}{100}^{-1} \approx 1.03$. Given the parameters of our model, for any value of $\sigma$ greater than 1.03, the stationary value of the $S$ curve will be less than 97. For any value of $\sigma$ less than 1.03, but greater than 1, the stationary value of the $S$ curve is greater than 97, but does not reach 100. For any value of $\sigma$ equal to or less than 1, the stationary value of the $S$ curve will be 100. These values will vary as $I_0$ and $N$ vary.