MOD 202 - Excercise 5

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1 Covariance and Correlation

a) The variance of a sample is given by:

$$\sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2 \tag{1}$$

Where x_i is the ith data point, and $\bar{x}$ is the sample mean. The vector $\vec{x}$ stores the deviation of each firing rate of neuron 1 from its mean:

$$\vec{x} = \begin{pmatrix} r_{1,1} - \bar{r}_1 \\ r_{1,2} - \bar{r}_2 \\ \vdots \\ r_{1,N} - \bar{r}_1 \end{pmatrix} \tag{2}$$

Where $r_{1,i}$ is the firing rate of neuron 1 at trial i, and $\bar{r}_1$ is the mean firing rate of neuron 1. Thus, the variance of the firing rate of neuron 1 can be rewritten as:

$$\begin{aligned} \sigma_1^2 &= \frac{1}{N-1}\sum_{i=1}^{N}(r_{1,i} - \bar{r}_1)^2 \\ &= \frac{1}{N-1}\sum_{i=1}^{N}\vec{x}_i^2 \end{aligned} \tag{3}$$

Where $\vec{x}_i$ is the ith element of the vector $\vec{x}$. The Euclidean norm of the vector $\vec{x}$ is given by:

$$||\vec{x}|| = \sqrt{\sum_{i=1}^{N}\vec{x}_i^2} \tag{4}$$

Therefore,

$$||\vec{x}||^2 = \sum_{i=1}^{N} \vec{x}_i^2 \tag{5}$$

So, the variance of the firing rate of neuron 1 can be written as:

$$\sigma_{r_1}^2 = \frac{1}{N-1} ||\vec{x}||^2 \tag{6}$$

b) The cosine of the angle between two vectors $\vec{x}$ and $\vec{y}$, where $\vec{y}$ is similary defined as a vector of deviations from the mean of the firing rate of neuron 2 is given by:

$$\cos(\theta) = \frac{\vec{x} \cdot \vec{y}}{||\vec{x}|| \cdot ||\vec{y}||} \tag{7}$$

The dot product of these two vectors is given by:

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{N} \vec{x}_i \vec{y}_i \tag{8}$$

But, since both $\vec{x}$ and $\vec{y}$ are comprised of deviations from the mean of their respective firing rates, the dot product can be written as:

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{N} (r_{1,i} - \bar{r}_1)(r_{2,i} - \bar{r}_2) \tag{9}$$

so,

$$\cos\theta = \frac{\sum_{i=1}^{N} (r_{1,i} - \bar{r}_1)(r_{2,i} - \bar{r}_2)}{||\vec{x}|| \cdot ||\vec{y}||}$$

Moreover, from the previous excerise, we know that:

$$\sigma_{r_1}^2 = \frac{1}{N-1} ||\vec{x}||^2$$

Therefore, the Euclidean norm of the vector $\vec{x}$ is given by:

$$||\vec{x}|| = \sqrt{\sigma_{r_1}^2 (N-1)}$$

Since $\vec{y}$ is similary defined, and assuming an equal amount of trials for neuron 2, we have that its Euclidean norm is given by:

$$||\vec{y}|| = \sqrt{\sigma_{r_2}^2 (N-1)}$$

Therefore, their product is:

$$\begin{aligned} ||\vec{x}|| \cdot ||\vec{y}|| &= \sqrt{(N-1)\sigma_{r_1}^2}\sqrt{(N-1)\sigma_{r_2}^2} \\ &= (N-1)\sqrt{\sigma_{r_1}^2\sigma_{r_2}^2} \\ &= (N-1)\sigma_{r_1}\sigma_{r_2} \end{aligned} \tag{10}$$

Therefore, the cosine of the angle between the two vectors is given by:

$$\cos\theta = \frac{\sum_{i=1}^{N}(r_{1,i}-\bar{r}_1)(r_{2,i}-\bar{r}_2)}{(N-1)\sigma_{r_1}\sigma_{r_2}} \tag{11}$$

Notice, however, that the sample covariance is defined as:

$$\mathrm{Cov}(\vec{x},\vec{y}) = \frac{1}{N-1}\sum_{i=1}^{N}(x_i-\bar{x})(y_i-\bar{y}) \tag{12}$$

Therefore, the cosine of the angle between the two vectors can be written as:

$$\cos\theta = \frac{\mathrm{Cov}(\vec{x},\vec{y})}{\sigma_{r_1}\sigma_{r_2}} \tag{13}$$

Interestingly, this is the same formula as the Correlation Coefficient. Thus, the cosine of the angle between two vectors measures the correlation between the two samples.

c) The maximum and minimum values for the correlation coefficient between r_1 and r_2 are 1 and -1 respectively. We know this since in the previous excersie we showed that the $\cos\theta_{x,y} = \mathrm{Corr}(r_1, r_2)$. Therefore, the correlation coefficient is bounded by the possible values of the cosine function. Since cosine ranges between -1 and 1, so does the correlation coefficient.

d) For data to be 'centered' means that it is manipulated such that the mean of the data points is 0. This is done by subtracting the mean from each data point. This allows us to compare data from neuron 1 and neuron 2 independent of the mean firing rate of each neuron, while preserving the shape of the distribution of firing rates (variance, standard deviation, relative distance between points).

2 Bayes' Theorem

a) Since there are only two possible stimuli, s_1 and s_2, the probability that there will be a response r is given by:

$$\begin{aligned} P(r) &= \sum_{i=1}^{2} p(r|s_i)p(s_i) \\ P(r) &= p(r|s_1)p(s_1) + p(r|s_2)p(s_2) \end{aligned} \tag{14}$$

b) The posterior probability $p(s|r)$ is given by Bayes' Theorem:

$$p(s|r) = \frac{p(r|s)p(s)}{p(r)} \tag{15}$$

To find the the probability that the stimulus was to the left, i.e., $p(s_1|r) = p(\leftarrow |r)$, when the prior probability of the stimulus being to the left is 0.5, and the prior probability fot he stimulus being to the right is 0.5 we can rewrite the above equation as:

$$\begin{aligned} p(\leftarrow |r) &= \frac{p(r| \leftarrow)p(\leftarrow)}{p(r| \leftarrow)p(\leftarrow) + p(r| \rightarrow)p(\rightarrow)} \\ &= \frac{p(r| \leftarrow)0.5}{p(r| \leftarrow)0.5 + p(r| \rightarrow)0.5} \\ &= \frac{p(r| \leftarrow)}{p(r| \leftarrow) + p(r| \rightarrow)} \end{aligned} \tag{16}$$

Morever, since we know that $p(r|s)$ follows a Gaussian probability density:

$$p(\leftarrow |r) = \frac{\varphi(\mu_{\leftarrow}, \sigma_{\leftarrow})}{\varphi(\mu_{\leftarrow}, \sigma_{\leftarrow}) + \varphi(\mu_{\rightarrow}, \sigma_{\rightarrow})} \tag{17}$$

Figure 1. below shows the posterior probabilities of the leftward and rightward stimulus as a function of the response r. The plot shows the behavior of the posterior when the firing rate of the neuron ranges from $0Hz$ to $100Hz$, the mean for the leftward stimulus is $25Hz$, and the mean for the rightward stimulus is $75Hz$. The standard deviation for both stimuli is $15Hz$.

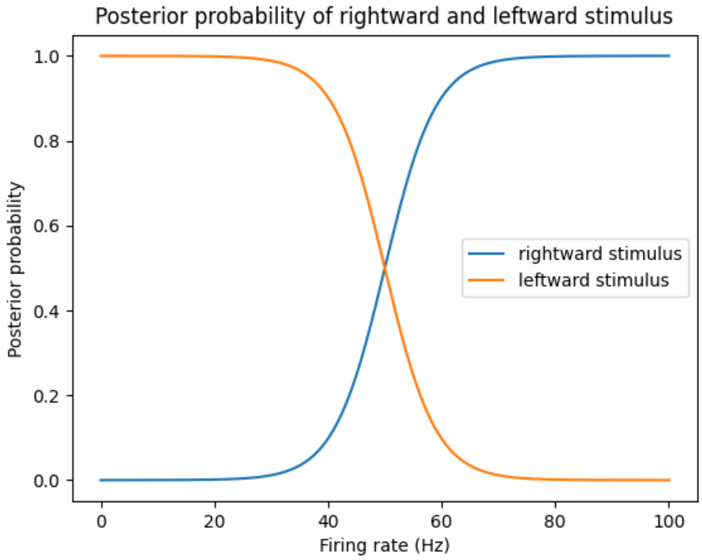

Figure 1: Posterior probaility functions of the leftward and rightward stimulus as a function of the response r when the prior probability of the stimulus being to the left is 0.5

We see that the posteriro probability of the leftward stimulus is higher than the rightward stimulus when the firing rate of the neuron is less than $50Hz$. Conversely, the posterior probability of the rightward stimulus is higher than the leftward stimulus when the firing rate of the neuron is greater than $50Hz$. This indicates that the stimulus is more likely to be to the left when the firing rate of the neuron is less than $50Hz$, and more likely to be to the right when the firing rate of the neuron is greater than $50Hz$. Moreover, the posterior probabilities of the two stimuli sum to 1 for all values of the firing rate of the neuron.

c) Figure 2. shows the posterior probabilities when the prior probabilities for the two stimuli are different.

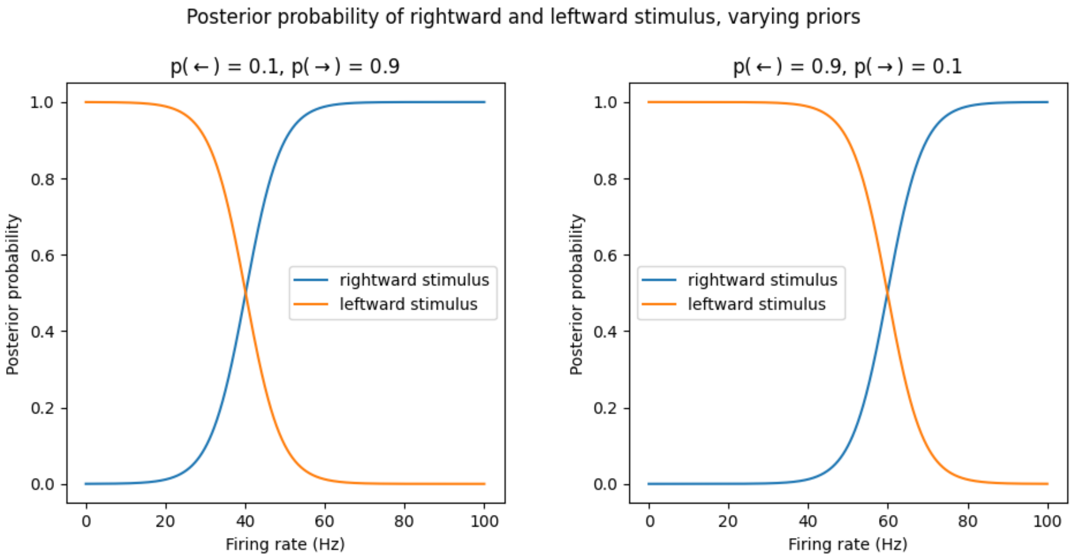

Figure 2: Posterior probaility functions given varying prior probabilities of the leftward and rightward stimulus

We see that the as the prior probability changes, the posterior probability of the two stimuli change as well. Specifically, as the prior for the leftward stimulus decreases, there is downward pressure on the range of firing rates for which the posterior probability of the leftward stimulus is higher than the rightward stimulus (and non-zero). Correspondingly, as the prior for the rightward stimulus increases, there is upward pressure on the range of firing rates for which the posterior probability of the rightward stimulus is higher than the leftward stimulus (and non-zero). And vice versa. The posterior probabilities of the two stimuli still sum to 1 for all values of the firing rate of the neuron.