MOD202 Excercise 3
Signal Detection Theory

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(a) **Gaussian Distribtuions**

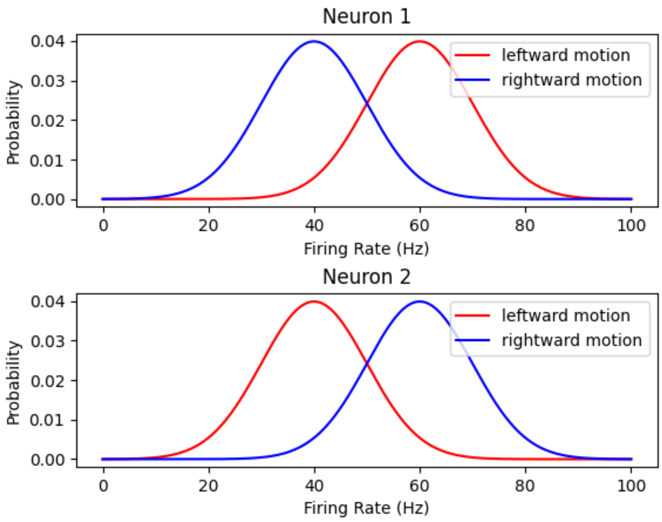

Figure 1: Neuronal firing rate distributions for leftward and rightward motion.

Assuming that the firing rate distributions $p(r|+)$ and $p(r|-)$ are Gaussian, Figure 1 illustrates the distributions of the neuronal firing rates given the direction of motion, where the mean of $p(r|+)$ is 60hz, the mean of $p(r|-)$ is 40hz, and the standard deviation of both distributions is 10hz.

Neuron 1's firing rate for leftward motion follows the distribution $p(r|+)$ since it's tuning is such that it fires more strongly when the stimulus is moving to the left. Conversely, when the stimulus is rightward motion, it's firing rate follows the distribution $p(r|-)$. This is illustrated in the top half of Figure 1.

Neuron 2's firing rate follows the distribution $p(r|+)$ when the motion is rightwards, since it's tuning is such that it fires more strongly when the stimulus is moving to the right. Conversely, when the stimulus is moving leftward, it's firing rate follows the distribution $p(r|-)$. This situation is illustrated in the bottom half of Figure 1.

(b) **Hit Rate and False Alarm Rate**

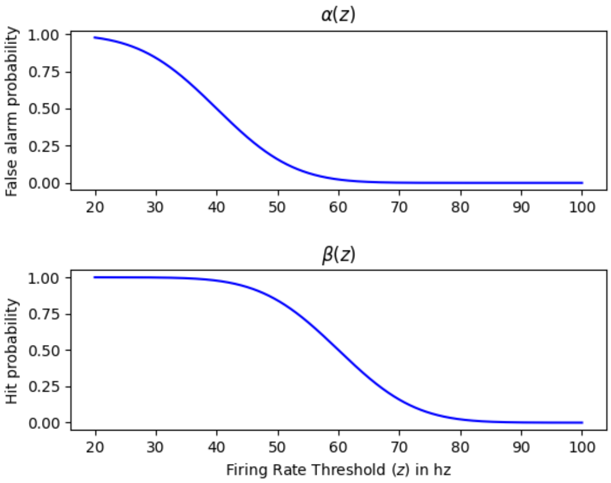

Figure 2: α and β as a function of the decision criterion z.

The function:

$$\alpha(z) = p(r \geq z|-) = \int_z^\infty dr' p(r'|-) \tag{1}$$

is illustrated in the top half of Figure 2. It gives the area under the curve of the firing rate distribution $p(r|-)$ to the right of the decision criterion z, i.e., the complementary cumulative distribution function, $1 - \Phi(z)$, of the gaussain $p(r|-)$, evaluated from z to ∞, where Φ is the (normal) cumulative distribution function. Note, however, that since we have assumed $\mu = 40$ and $\sigma = 10$, the figure above shows the standardized values of $a(z)$, i.e.,$1 - \Phi(\frac{z-40}{10})$.

In this context, it indicates the **false alarm rate** for a given decision criterion z. That is: for each neuron, it is the probability that it will report the stiumulus is moving according to its tuning when it is in fact moving in the opposite direction, given a certain decision threshold z.

The function:

$$\beta(z) = p(r \geq z|+) = \int_z^\infty dr' p(r'|+) \tag{2}$$

is illustrated in the bottom half of Figure 2. It gives the area under the curve of the firing rate distribution $p(r|+)$ to the right of the decision criterion z, i.e., the complementary cumulative distribution function, $1 - \Phi(\frac{z-\mu}{\sigma})$, of the gaussain $p(r|+)$ evaluated from z to ∞, where Φ is the (normal) cumulative distribution funciton. In this case, $\mu = 60$ and $\sigma = 10$, so the figure above shows the standardized values of $b(z)$, i.e., $1 - \Phi(\frac{z-60}{10})$.

In this context, it indicates the **hit rate** for a given decision criterion z. That is: for each neuron, the that it will report the stimulus is moving according to its tuning when it is in fact moving according to its tuning, given a certain decision threshold z.

(c) **Their respective derivatives**

The fundamental theorem of calculus tells us that if:

$$F(x) = \int_a^b f(t)dt \tag{3}$$

then:

$$F'(x) = f(x) \tag{4}$$

Since $\alpha(z)$ and $\beta(z)$ are both integrals of the firing rate distributions $p(r|-)$ and $p(r|+)$, respectively, their derivatives will be the firing rate distributions themselves.

In the case of α, it is the integral of the distribution $p(r|-)$, which has a mean $\mu = 40$, and a standard deviation $\sigma = 10$. So the standardized derivative of $\alpha(z)$ is:

$$z'_\alpha = \frac{z - \mu_\alpha}{\sigma_\alpha} = \frac{z - 40}{10}$$

$$\alpha(z'_\alpha) = 1 - \Phi(z'_\alpha) = 1 - \int_{-\infty}^{z'_\alpha} \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}(z'_\alpha)^2} dt$$

$$\frac{d\alpha(z'_\alpha)}{dz} = -\frac{d\Phi(z'_\alpha)}{dz}$$

$$-\frac{d\Phi(z'_\alpha)}{dz} = -\frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}(z'_\alpha)^2} = -\frac{1}{10\sqrt{2\pi}} e^{\frac{-(z-40)^2}{200}}$$

Thus, the derivative of the complementary cumulative distribution function of the gaussian $p(r|-)$ is the negative probability density function of that gaussian.

A similar argument can be made for $\beta(z)$, which is the integral of the distribution $p(r|+)$, which has a mean $\mu = 60$, and a standard deviation $\sigma = 10$. So the standardized derivative of $\beta(z)$ is:

$$z'_\beta = \frac{z - \mu_\beta}{\sigma_\beta} = \frac{z - 60}{10}$$

$$\beta(z'_\beta) = 1 - \Phi(z'_\beta) = 1 - \int_{-\infty}^{z'_\beta} \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}(z'_\beta)^2} dt$$

$$\frac{d\beta(z'_\beta)}{dz} = -\frac{d\Phi(z'_\beta)}{dz}$$

$$-\frac{d\Phi(z'_\beta)}{dz} = -\frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}(z'_\beta)^2} = -\frac{1}{10\sqrt{2\pi}} e^{\frac{-(z-60)^2}{200}}$$

Thus, the derivative of the complementary cumulative distribution function of the gaussian $p(r|+)$ is the negative probability density function of that gaussian.

(d) **Probabilities given leftward motion**

Given that the neurons are independant from each other, the probability that both neurons fire given that the stimulus is moving to the left is the product of the probabilities that each neuron fires given that the stimulus is moving to the left. Where r_1 is the firing rate for neuron 1, and r_2 is the firing rate for neuron 2, we have:

$$p(r_1, r_2|+) = p(r_1|+) * p(r_2|-) \tag{5}$$

We already know the firing rate distributions $p(r_1|+)$ and $p(r_2|-)$ from above. Namely,

$$p(r_1|+) = \frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{r_1-\mu_1}{\sigma_1})^2}$$

$$p(r_2|-) = \frac{1}{\sigma_2\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{r_2-\mu_2}{\sigma_2})^2}$$

Where $\mu_1 = 60$, $\sigma_1 = 10$, $\mu_2 = 40$, and $\sigma_2 = 10$.

Thus, the probability that both neurons fire given that the stimulus is moving to the left is:

$$p(r_1|+) * p(r_2|-) = \frac{1}{2\pi 100}e^{-\frac{(r_1-60)^2+(r_2-40)^2}{200}} \tag{6}$$

(e) **Accuracy of the observer**

The decision that "the motion was leftwards" is made whenever $r_1 > r_2$, i.e., whenever the firing rate of neuron 1 is greater than the firing rate of neuron 2. Therefore, to find the accuracy of the observer, we need to find the probabiltiy that $r_1 > r_2$ when the stimulus is leftwards, i.e.,

$$p(r_1 > r_2|+) = p(r_1 - r_2 > 0|+)$$

Since both r_1 and r_2 are normally distibuted, we can use the fact that the difference of two normally distributed variables is also normally distributed.

i.e,

$$r_1 \sim N(\mu_1, \sigma_1)$$

and

$$r_2 \sim N(\mu_2, \sigma_2)$$

Then, the difference $r_1 - r_2$ is also normally distributed with mean $\mu_1 - \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

$$\begin{aligned} r_1 - r_2 &\sim N(60 - 40, 10^2 + 10^2) \\ r_1 - r_2 &\sim N(20, 200) \end{aligned} \tag{7}$$

To find the accuracy of the observer, we need to find the probability that $r_1 - r_2 > 0$ when the stimulus is moving to the left, i.e.,

$$p(r_1 - r_2 > 0|+)$$

Here, we can use the normal complementary cumulative distribution function again, i.e.,

$$p(r_1 > r_2|+) = p(r_1 - r_2 > 0|+) = 1 - \Phi\left(\frac{0-20}{\sqrt{200}}\right)$$

Solving the integral, we get that the accuracy of the observer given the above distributions is **0.92135** or **92.14%**.

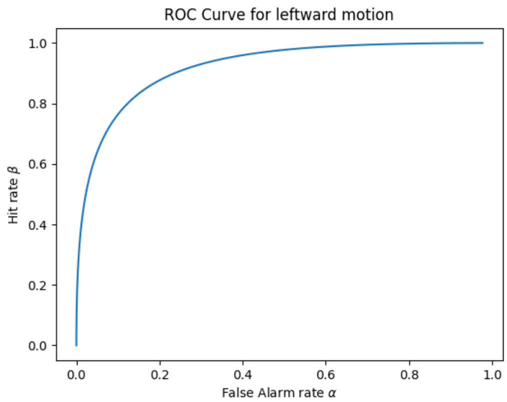

Figure 3: ROC curve for the observer.

The ROC curve is the plot of the hit rate, $\beta(z)$ against the false alarm rate, $\alpha(z)$, for all possible decision criteria z. Thus, the equation of the curve is $\beta(\alpha(z))$, so the area under the ROC curve, the AUC, is given by the integral of the ROC curve, i.e., $\int_0^1 \beta(\alpha)d\alpha$.
In d), we found that the probability density function for $p(r_1, r_2|+) = p(r_1|+) * p(r_2|-)$, where $p(r_1|+)$ and $p(r_2|-)$ are respectively probability denstiy functions $\phi(r_1)$ and $\phi(r_2)$. To find the exact probability for any range of r, we need to compute the integral of this function.

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{r_2}^{\infty} \phi(r_1)\phi(r_2)dr_1dr_2 \\
&= \int_{-\infty}^{\infty} \phi(r_2)\int_{r_2}^{\infty} \phi(r_1)dr_1dr_2 \\
&= \int_{-\infty}^{\infty} \phi(r_2)(1-\Phi(r_2))dr_2 \\
&= \int_{-\infty}^{\infty} \phi(r_2)\beta(r_2)dr_2 \\
&= -\int_{-\infty}^{\infty} \alpha(r_2)\beta(r_2)dr_2 \\
&= \int_{\infty}^{-\infty} \alpha(r_2)\beta(r_2)dr_2 \\
&?\ldots? \\
&= \int_0^1 \beta(\alpha)d\alpha
\end{aligned} \tag{8}$$