MODEL FOR TRANSMISSION OF SARS-COV-2 ON A UNIVERSITY CAMPUS

The system of ordinary differential equations for this model is

$$\begin{aligned}
\dot{S} &= -\beta\left(I + b_L L + b_A A\right) S, \\
\dot{L} &= (1-a)\,\beta\left(I + b_L L + b_A A\right) S - (\sigma + \xi)\, L, \\
\dot{A} &= a\beta\left(I + b_L L + b_A A\right) S - (\gamma_A + \xi)\, A, \\
\dot{I} &= \sigma L - \gamma_I I, \\
\dot{R} &= \gamma_I I + \gamma_A A + \xi L + \xi A.
\end{aligned}$$

We can simplify the notation by defining

$$\begin{aligned}
f &= (1-a)\,\beta\left(I + b_L L + b_A A\right), \\
g &= a\beta\left(I + b_L L + b_A A\right)
\end{aligned}$$

Let $N = S + L + A + I + R$ represent the total population. Then $\dot{N} = 0$. Thus the disease-free equilibrium is given by $S_0 = N(0) - R(0)$, and all other state variables are equal to zero.

The infectious compartments are represented by L, A, and I. Let x be the vector of state variables, ordered with infected compartments first: $x = (L, A, I, S, R)$. The DFE is then $x_0 = (0, 0, 0, S_0, R_0)$. Define $\mathscr{F}$ and $\mathscr{V}$ to be the rate of new infections and net rate out (for any other transition), for the infected compartments:

$$\mathscr{F}(x) = \begin{pmatrix} (1-a)\,\beta\left(I + b_L L + b_A A\right) S \\ a\beta\left(I + b_L L + b_A A\right) S \\ 0 \end{pmatrix}, \ \mathscr{V}(x) = \begin{pmatrix} (\sigma+\xi)\, L \\ (\gamma_A + \xi)\, A \\ \gamma_I I - \sigma L \end{pmatrix}$$

Let $F = D_x \mathscr{F}(x_0)$ and $V = D_x \mathscr{V}(x_0)$ where

$$\begin{aligned}
D_x \mathscr{F}(x) &= \begin{pmatrix} (1-a)\,\beta S b_L & (1-a)\,\beta S b_A & (1-a)\,\beta S \\ a\beta S b_L & a\beta S b_A & a\beta S \\ 0 & 0 & 0 \end{pmatrix}, \\
D_x \mathscr{V}(x) &= \begin{pmatrix} (\sigma+\xi) & 0 & 0 \\ 0 & (\gamma_A+\xi) & 0 \\ -\sigma & 0 & \gamma_I \end{pmatrix}
\end{aligned}$$

Thus

$$\begin{aligned}
F &= \begin{bmatrix} (1-a)\,\beta b_L S_0 & (1-a)\,\beta b_A S_0 & (1-a)\,\beta S_0 \\ a\beta S_0 b_L & a\beta S_0 b_A & a\beta S_0 \\ 0 & 0 & 0 \end{bmatrix}, \\
V &= \begin{bmatrix} (\sigma+\xi) & 0 & 0 \\ 0 & (\gamma_A+\xi) & 0 \\ -\sigma & 0 & \gamma_I \end{bmatrix}.
\end{aligned}$$

The basic reproduction number is defined as the spectral radius of the next generation matrix $K = FV^{-1}$ which is given by:

$$K = \begin{bmatrix} (1-a)\,\beta b_L S_0 & (1-a)\,\beta b_A S_0 & (1-a)\,\beta S_0 \\ a\beta S_0 b_L & a\beta S_0 b_A & a\beta S_0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\left(\sigma+\xi\right) & 0 & 0 \\ 0 & 1/\left(\gamma_A+\xi\right) & 0 \\ \sigma/\gamma_I\left(\sigma+\xi\right) & 0 & 1/\gamma_I \end{bmatrix}$$

$$= \beta S_0 \begin{bmatrix} (1-a)\,b_L & (1-a)\,b_A & (1-a) \\ ab_L & ab_A & a \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\left(\sigma+\xi\right) & 0 & 0 \\ 0 & 1/\left(\gamma_A+\xi\right) & 0 \\ \sigma/\gamma_I\left(\sigma+\xi\right) & 0 & 1/\gamma_I \end{bmatrix}$$

$$= \beta S_0 \begin{bmatrix} (1-a)\left[b_L\frac{1}{\sigma+\xi}+\frac{1}{\gamma_I}\left(\frac{\sigma}{\sigma+\xi}\right)\right] & (1-a)\,b_A\left(\frac{1}{\gamma_A+\xi}\right) & (1-a)\left(\frac{1}{\gamma_I}\right) \\ a\left[b_L\frac{1}{\sigma+\xi}+\frac{1}{\gamma_I}\left(\frac{\sigma}{\sigma+\xi}\right)\right] & ab_A\left(\frac{1}{\gamma_A+\xi}\right) & a\left(\frac{1}{\gamma_I}\right) \\ 0 & 0 & 0 \end{bmatrix}$$

To simplify this calculation, let $c_1 = \beta S_0\left[b_L\frac{1}{\sigma+\xi}+\frac{1}{\gamma_I}\left(\frac{\sigma}{\sigma+\xi}\right)\right]$, $c_2 = \beta S_0 b_A\left(\frac{1}{\gamma_A+\xi}\right)$ and $c_3 = \beta S_0\frac{1}{\gamma_I}$. Then K can be written simply as:

$$K = \begin{bmatrix} (1-a)\,c_1 & (1-a)\,c_2 & (1-a)\,c_3 \\ ac_1 & ac_2 & ac_3 \\ 0 & 0 & 0 \end{bmatrix}$$

and the eigenvalues of K are zero (multiplicity 2) and, the largest eigenvalue, $(1-a)\,c_1 + ac_2$. This last eigenvalue must be the spectral radius. Substituting back in the original parameters, we obtain:

$$\mathcal{R}_0 = \beta S_0\,(1-a)\left[b_L\frac{1}{(\sigma+\xi)}+\frac{1}{\gamma_I}\left(\frac{\sigma}{\sigma+\xi}\right)\right]+\beta S_0 ab_A\left(\frac{1}{\gamma_A+\xi}\right)$$

This can be decomposed to represent the contribution to the basic reproduction number of the symptomatic $(\mathcal{R}_L+\mathcal{R}_I)$ and asymptomatic classes $(\mathcal{R}_A)$:

$$\begin{aligned} \mathcal{R}_L &= \beta S_0\,(1-a)\,b_L\frac{1}{(\sigma+\xi)}, \\ \mathcal{R}_I &= \beta S_0\,(1-a)\,\frac{1}{\gamma_I}\left(\frac{\sigma}{\sigma+\xi}\right), \\ \mathcal{R}_A &= \beta S_0 ab_A\left(\frac{1}{\gamma_A+\xi}\right). \end{aligned}$$

Finding the critical value of ξ

The testing rate, ξ, is an important control parameter. The basic reproduction number can be brought below unity by increasing this value. We compute the value ξ_c at which $\mathcal{R}_0 = 1$.

$$\begin{aligned} 1 &= \beta S_0\,(1-a)\left[b_L\frac{1}{(\sigma+\xi_c)}+\frac{1}{\gamma_I}\left(\frac{\sigma}{\sigma+\xi_c}\right)\right]+\beta S_0 ab_A\left(\frac{1}{\gamma_A+\xi_c}\right) \\ (\sigma+\xi_c)\,(\gamma_A+\xi_c) &= \beta S_0\left[(\gamma_A+\xi_c)\,(1-a)\left(b_L+\frac{1}{\gamma_I}\sigma\right)+(\sigma+\xi_c)\,ab_A\right] \end{aligned}$$

Define the following reproduction numbers without control:

$$\begin{aligned} \mathcal{R}_{L0} &= \beta S_0\,(1-a)\,b_L\frac{1}{\sigma}, \\ \mathcal{R}_{I0} &= \beta S_0\,(1-a)\,\frac{1}{\gamma_I}, \\ \mathcal{R}_{A0} &= \beta S_0 ab_A\frac{1}{\gamma_A}, \\ \mathcal{R}_0 &= \mathcal{R}_{L0}+\mathcal{R}_{I0}+\mathcal{R}_{A0}. \end{aligned}$$

Then the above simplfies to the following equation:

$$\xi_c^2 + \left[\sigma\,(1-\mathcal{R}_0+\mathcal{R}_{A0})+\gamma_A\,(1-\mathcal{R}_{A0})\right]\xi_c + \sigma\gamma_A\,(1-\mathcal{R}_0) = 0$$

This can have two solutions:

$$\xi^{\pm} = -\frac{1}{2}\left[\sigma\left(1-\mathcal{R}_0+\mathcal{R}_{A0}\right)+\gamma_A\left(1-\mathcal{R}_{A0}\right)\right]\pm\frac{1}{2}\sqrt{\left[\sigma\left(1-\mathcal{R}_0+\mathcal{R}_{A0}\right)+\gamma_A\left(1-\mathcal{R}_{A0}\right)\right]^2+4\sigma\gamma_A\left(\mathcal{R}_0-1\right)},$$

only one of which is positive since $\xi^+\xi^- = \sigma\gamma_A\left(1-\mathcal{R}_0\right) < 0$. Hence the critical value of ξ is:

$$\xi_c = -\frac{1}{2}\left[\sigma\left(1-\mathcal{R}_0+\mathcal{R}_{A0}\right)+\gamma_A\left(1-\mathcal{R}_{A0}\right)\right]+\frac{1}{2}\sqrt{\left[\sigma\left(1-\mathcal{R}_0+\mathcal{R}_{A0}\right)+\gamma_A\left(1-\mathcal{R}_{A0}\right)\right]^2+4\sigma\gamma_A\left(\mathcal{R}_0-1\right)}.$$