EQUIVALENT TRANMISSION LINE MODELS FOR LAYERED STRUCTURES WITH SOURCES

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A method to extract zeros of a complex function based on the Argument principle method. The

$$f(z_k) = 0 \tag{1}$$

We seek to solve (1) in the complex plane to obtain z_k in a given search region $\mathbb{C}$. For the multilayer problem, f represents the dispersion relation or the characteristic equation of the Transmission Line Green's Functions (TLGF). Depending on the location of the zeros of the function f in the complex plane, z_k correspond to the surface wave poles of the TLGF.

Counting the zeros

In order to develop an efficient method of locating the zeros, we assume that the function f is holomorphic, and non-zero at the boundary, Γ of the region $\mathbb{C}$. Furthermore, the region is assumed rectangular. According to the Argument Principle Method (APM), the

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number of zeros, N inside a region with a boundary Γ can be found by [1, 2]:

$$N = \frac{1}{2\pi} \oint_{\Gamma} \mathrm{d}\{\arg f(z)\} \tag{2}$$

which states that each enclosed zero increments the argument by a factor of 2π. The integration around the contour is performed in a counter-clockwise manner and computed using a 15^{th} order adaptive Gauss-Konrod quadrature (*MATLAB*'s quadgk routine) with breakpoints defining the contour. In addition to (1), the APM can also be stated in terms of the Cauchy's Integral Theorem [3, pg. 71]:

$$N = \frac{1}{2\pi j} \oint_{\Gamma} \frac{f'(z)}{f(z)} \mathrm{d}z \tag{3}$$

Since finding an analytical derivative of f in the case of multilayer structure is cumbersome, we approximate it with a finite difference:

$$f'(z) = \frac{f(z+h) - f(z-h)}{2h} \tag{4}$$

where $h \sim \sqrt{\varepsilon_m} z$ with $\varepsilon_m = 2.2204 \times 10^{-16}$ as the double-precison machine accuracy [4, pg. 230].

Locating the zeros

Once the number of zeros is evaluated through (2), the next step involves the approximation the dispersion function f with an associated *formal orthogonal polynomial* (FOP), $\mathscr{P}$ of degree N with the condition that it has same roots, z_k where $k = 1, 2, \ldots, N$ as f [5, 6].

The Lagrange representation is:

$$\mathscr{P}_N(z) = \prod_{k=1}^{N} (z - z_k) \tag{5}$$

For orthogonality, we require the inner product,

$$\langle z^k, \mathscr{P}_N(z) \rangle = \frac{1}{2\pi j} \oint_{\Gamma} z^k \mathscr{P}_N(z) \frac{f'(z)}{f(z)} \, \mathrm{d}z, \quad \text{with} \quad k = 0, 1, ..., N-1. \tag{6}$$

to be zero [7]. The polynomial approximation of the original function reduces the complexity of the problem as techniques for finding polynomial roots are robust and well-established. To find the roots, we consider the sequence of integrals:

$$s_k = \frac{1}{2\pi j} \oint_{\Gamma} z^k \frac{f'(z)}{f(z)} \, \mathrm{d}z, \quad \text{with} \quad k = 0, 1, 2, ... \tag{7}$$

The summation notation polynomial $\mathscr{P}$ is more convenient than (5) for its solution:

$$\mathscr{P}_N(z) = \sum_{k=0}^{N} \alpha_k z^k \tag{8}$$

For a *monic polynomial*, $\alpha_N = 1$, and α_k are the sums of products of zeros, Z_k. The unknown α's in (8) can be found by applying the *Newton's Identities* [6].

$$\begin{aligned} s_1 + \alpha_1 &= 0 \\ s_2 + s_1\alpha_1 + 2\alpha_2 &= 0 \\ &\vdots \\ s_N + s_{N-1}\alpha_1 + ... + s_1\alpha_{N-1} + N\alpha_N &= 0 \end{aligned} \tag{9}$$

Next, we construct two Hankel matrices, $\mathbf{H}$ and $\mathbf{H}^{<}$ to set up the eigenvalue problem:

$$\mathbf{H} = \begin{bmatrix} s_1 & s_2 & \cdots & s_k \\ s_2 & s_3 & \cdots & s_{k+1} \\ s_3 & s_4 & \cdots & s_{k+2} \\ \vdots & \vdots & \ddots & \vdots \\ s_k & s_{k+1} & \cdots & s_{2k} \end{bmatrix} \tag{10}$$

and

$$\mathbf{H}^{<} = \begin{bmatrix} s_0 & s_1 & \cdots & s_{k-1} \\ s_1 & s_2 & \cdots & s_k \\ s_2 & s_3 & \cdots & s_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{k-1} & s_k & \cdots & s_{2k-2} \end{bmatrix} \tag{11}$$

The roots of the polynomial $\mathscr{P}$ are the generalized eigenvalues, λ of the matrix pencil:

$$\left(\mathbf{H} - \lambda \mathbf{H}^{<}\right) \mathbf{x} = 0 \tag{12}$$

0.1 Refining the roots

Approximating a function in a given contour with many zeros requires a higher-order polynomial that introduces computational problems. In addition, the integrals of the moments in (7) need to be evaluated with a higher-accuracy and the mapping between s_k and α_k (9) rseults in an ill-conditioned system. To overcome such pitfalls, a limit is enforced on the number of zeros in a given region. If the number of zeros exceeds a predetermined value M, the size of the search region is subdivided [5]. For problems pertinents to multilayer structures, a safe choice of M is 3.

The accuracy of the roots obtained from the eigenvalues,λ_k of (12) is not always high. However, λ_k's is an excellent inintial guess for any iterative root-search routine from the class of Householder's methods. We choose the *Halley's* method having cubic convergence and the iteration formula:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)} \tag{13}$$

with $f'(x)$ and $f''(x)$, the first and second order derivatives approximated by finite differences. In general the roots, z_k's lie in the complex plane. The iteration (13) needs to be performed on both the real and imaginary parts simultaneously.

0.2 Branch Cuts

For multilayer structures,

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