EQUIVALENT TRANMISSION LINE MODELS FOR LAYERED STRUCTURES WITH SOURCES

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We seek to solve (1) in the complex plane to obtain the zeros,z_k in a given search region $\mathbb{C}$. For the multilayer problem, f represents the dispersion relation which is the characteristic equation of the Transmission Line Green's Functions (TLGF). Depending on the location of the zeros of the function f in the complex plane, z_k correspond to the surface wave poles of the TLGF.

$$f(z_k) = 0 \tag{1}$$

Counting the zeros

In order to develop an efficient method of locating the zeros, we assume that the function f is holomorphic, and non-zero at the boundary, Γ of the region $\mathbb{C}$. Furthermore, the region is assumed rectangular. According to the Argument Principle Method (APM), the number of zeros, N inside a region with a boundary Γ can be found by [1, 2]:

$$N = \frac{1}{2\pi} \oint_{\Gamma} \mathrm{d}\{\arg f(z)\} \tag{2}$$

which states that each enclosed zero increments the argument by a factor of 2π. The integration around the contour is performed in a counter-clockwise manner and computed

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using a 15^{th} order adaptive Gauss-Konrod quadrature (*MATLAB*'s quadgk routine) with breakpoints defining the contour. In addition to (1), the APM can also be stated in terms of the Cauchy's Integral Theorem [3, pg. 71]:

$$N = \frac{1}{2\pi j} \oint_{\Gamma} \frac{f'(z)}{f(z)} \,\mathrm{d}z \tag{3}$$

Since finding an analytical derivative of f in the case of multilayer structure is cumbersome, we approximate it with a finite difference:

$$f'(z) = \frac{f(z+h) - f(z-h)}{2h} \tag{4}$$

where $h \sim \sqrt{\varepsilon_m} z$ with $\varepsilon_m = 2.2204 \times 10^{-16}$ as the double-precison machine accuracy [4, pg. 230].

Locating the zeros

Once the number of zeros is evaluated through (2), the next step involves the approximation the dispersion function f with an associated *formal orthogonal polynomial* (FOP), $\mathscr{P}$ of degree N with the condition that it has same roots, z_k where $k = 1, 2, \ldots, N$ as f [5, 6]. The Lagrange representation is:

$$\mathscr{P}_N(z) = \prod_{k=1}^{N} (z - z_k) \tag{5}$$

For orthogonality, we require the inner product,

$$\langle z^k, \mathscr{P}_N(z) \rangle = \frac{1}{2\pi j} \oint_{\Gamma} z^k \mathscr{P}_N(z) \frac{f'(z)}{f(z)} \,\mathrm{d}z, \quad \text{with} \quad k = 0, 1, \ldots, N-1. \tag{6}$$

to be zero [7]. The polynomial approximation of the original function reduces the complexity of the problem as techniques for finding polynomial roots are robust and well-established. To find the roots, we consider the sequence of integrals:

$$s_k = \frac{1}{2\pi j} \oint_{\Gamma} z^k \frac{f'(z)}{f(z)} \,\mathrm{d}z, \quad \text{with} \quad k = 0, 1, 2, \ldots \tag{7}$$

The summation notation polynomial $\mathscr{P}$ is more convenient than (5) for its solution:

$$\mathscr{P}_N(z) = \sum_{k=0}^{N} \alpha_k z^k \tag{8}$$

For a *monic polynomial*, $\alpha_N = 1$, and α_k are the sums of products of zeros, Z_k. The unknown α's in (8) can be found by applying the *Newton's Identities* [6].

$$\begin{aligned} s_1 + \alpha_1 &= 0 \\ s_2 + s_1\alpha_1 + 2\alpha_2 &= 0 \\ &\vdots \\ s_N + s_{N-1}\alpha_1 + ... + s_1\alpha_{N-1} + N\alpha_N &= 0 \end{aligned} \tag{9}$$

Next, we construct two Hankel matrices, $\mathbf{H}$ and $\mathbf{H}^{<}$ to set up the eigenvalue problem:

$$\mathbf{H} = \begin{bmatrix} s_1 & s_2 & \cdots & s_k \\ s_2 & s_3 & \cdots & s_{k+1} \\ s_3 & s_4 & \cdots & s_{k+2} \\ \vdots & \vdots & \ddots & \vdots \\ s_k & s_{k+1} & \cdots & s_{2k} \end{bmatrix} \tag{10}$$

and

$$\mathbf{H}^{<} = \begin{bmatrix} s_0 & s_1 & \cdots & s_{k-1} \\ s_1 & s_2 & \cdots & s_k \\ s_2 & s_3 & \cdots & s_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{k-1} & s_k & \cdots & s_{2k-2} \end{bmatrix} \tag{11}$$

The roots of the polynomial $\mathscr{P}$ are the generalized eigenvalues, λ of the matrix pencil:

$$\left(\mathbf{H} - \lambda \mathbf{H}^{<}\right)\mathbf{x} = 0 \tag{12}$$

where the column vector $\mathbf{x}$ is the eigenvector.

Refining the roots

Approximating a function in a given contour with many zeros requires a higher-order polynomial that introduces computational problems. In addition, the integrals of the moments in (7) need to be evaluated with a higher-accuracy and the mapping between s_k and α_k (9) rseults in an ill-conditioned system. To overcome such pitfalls, a limit is enforced on the number of zeros in a given region. If the number of zeros exceeds a predetermined value M, the size of the search region is subdivided [5]. For problems pertinents to multilayer structures, a safe choice of M is 3.

The accuracy of the roots obtained from the eigenvalues,λ_k of (12) is not always high. However, λ_k's is an excellent inintial guess for any iterative root-search routine from the class of Householder's methods. We choose the *Halley's* method having cubic convergence and the iteration formula:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)} \tag{13}$$

with $f'(x)$ and $f''(x)$, the first and second order derivatives approximated by finite differences. In general the roots, z_k's lie in the complex plane. The iteration (13) needs to be performed on both the real and imaginary parts simultaneously.

Branch Cuts

The TLGF for a multilayer possesses two types singularities in the complex plane. We are primarily interested in finding the pole singularities that correspond to guided modes. In addition, the branch point singularities correspond to the radiation modes. Branch points arise due to the propagation constant, k_{zi} of the layers having semi-infinite dimensions. The dispersion relation shows an even dependence on k_{zi} of the layers that have finite width, hence there is no corresponding branch point. [8, Section 5.3a]. In order to satisfy

Sommerfeld radiation condition that all fields must decay to zero at infinity, the proper sheet of the Riemann surface of the square-root function needs to be selected. This is accomplished by enforcing that the imaginary part of k_{zi} that contributes a branch point, is negative everywhere.

$$\mathrm{Im}(k_{zi}) < 0 \tag{14}$$

References

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