CURRENT ON A PLANAR DIELETRIC PLATE

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We apply the surface equiavalence theorem to find the electric and magnetic currents on the surface of a planar dielectric sheet. A plane wave propagating along the direction $\mathbf{k}$ with electric field $\mathbf{E}$ polarized along the z direction as shown in Fig. **??** is incident on the dielctric surface at an angle ϕ_i.

$$\mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H} = \hat{\mathbf{z}}\mathrm{J}(\zeta), \tag{1a}$$

$$\mathbf{M}_s = -\hat{\mathbf{n}} \times \mathbf{E} = \hat{\mathbf{x}}\mathrm{M}(\zeta) \tag{1b}$$

where the normal unit vector $\hat{\mathbf{n}}$ is in the y direction and ζ depends on x and y. To find the surface currents, we set up an homogeneous equivalent problem first for the region outside the dielectric sheet as depicted in Fig. **??**. The total field can be written as:

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_1^{scat} \tag{2}$$

where $\mathbf{E}_i$ is the incident electric field due to the plane wave,

$$\mathbf{E}_i = \hat{\mathbf{z}}\ E^0 e^{-jk_0(x\cos\phi_i - y\sin\phi_i)} \tag{3}$$

with k_0 as the propagation constant of air and E^0 the amplitude of the incoming plane wave. The scattered field in (2) can be expressed as:

$$\mathbf{E}_1^{scat} = \left(k_0^2 + \nabla\nabla\cdot\right)\mathbf{A} - \frac{1}{\varepsilon_1}\nabla \times \mathbf{F} \tag{4}$$

*Last Modified: 00:02, Tuesday 22nd November, 2016.

where $\mathbf{A}$ and $\mathbf{F}$ are the magnetic and electric vector potentials respectively, given by:

$$\mathbf{A} = \frac{\mu}{4\pi} \iint\limits_S \mathbf{J}_s(\mathbf{r}') \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \, \mathrm{d}S', \tag{5a}$$

$$\mathbf{F} = \frac{\varepsilon}{4\pi} \iint\limits_S \mathbf{M}_s(\mathbf{r}') \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \, \mathrm{d}S'. \tag{5b}$$

with the position vectors $\mathbf{r}$ and $\mathbf{r}'$ illustrated in Fig. **??**. For a sheet structure extending to infiniity in the z direction, (5) can be re-written as:

$$\mathbf{A} = \frac{\mu}{4j} \int\limits_l \mathbf{J}_s(\rho') H_0^{(2)}(k_0|\rho-\rho'|) \, \mathrm{d}l', \tag{6a}$$

$$\mathbf{F} = \frac{\varepsilon}{4j} \int\limits_l \mathbf{M}_s(\rho') H_0^{(2)}(k_0|\rho-\rho'|) \, \mathrm{d}l', \tag{6b}$$

where $H_0^{(2)}(k_0|\rho-\rho'|)$ is the Hankel function of order 0 and the second kind. For a z-directed source, the scattered electric field in (4) can be simply written as:

$$\begin{aligned} \mathbf{E}_1^{scat} &= -\hat{\mathbf{z}} j\omega \mathrm{A}_z \\ &= -\hat{\mathbf{z}} \frac{\omega\mu}{4j} \int\limits_l J_z(\rho') H_0^{(2)}(k_0|\rho-\rho'|) \, \mathrm{d}l' \end{aligned} \tag{7}$$

The scattered magnetic field can similarly be expressed in terms of the vector potenatials. For the case in consideration, we obtain:

$$\begin{aligned} \mathbf{H}_1^{scat} &= -\hat{\mathbf{x}} \frac{j\omega}{k_0^2} \left(k_0^2 + \frac{\partial^2}{\partial x^2} \right) \mathrm{F}_x \\ &= -\hat{\mathbf{x}} \frac{j\omega}{k_0^2} \left(k_0^2 + \frac{\partial^2}{\partial x^2} \right) \int\limits_l M_x(\rho') H_0^{(2)}(k_0|\rho-\rho'|) \, \mathrm{d}l' \end{aligned} \tag{8}$$

For the region inside the dielectric, an interior equivalent is set up with the currents reversing the signs. The total fields for the interior region only contain the scattered fields.

$$\mathbf{E}_2^{scat} = -\hat{\mathbf{z}} \frac{\omega\mu}{4j} \int\limits_l -J_z(\rho') H_0^{(2)}(k_2|\rho-\rho'|) \, \mathrm{d}l' \tag{9a}$$

$$\mathbf{H}_2^{scat} = -\hat{\mathbf{x}} \frac{j\omega}{k_2^2} \left(k_2^2 + \frac{\partial^2}{\partial x^2} \right) \int\limits_l -M_x(\rho') H_0^{(2)}(k_2|\rho-\rho'|) \, \mathrm{d}l' \tag{9b}$$

In order to find the electric and magnetic currents, we apply the boundary conditions at the interface ensuring the continuity of tangential component of the fields. At the interface:

$$\hat{\mathbf{n}} \times (\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{0} \tag{10a}$$

$$\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{0} \tag{10b}$$

Since the electric field is only z-directed, we obtain a scalar equation by the application of (10a):

$$E_i = \frac{\omega\mu}{4} \int\limits_c J_z(\rho') \left[H_0^{(2)}(k_0|\rho - \rho'|) + H_0^{(2)}(k_2|\rho - \rho'|) \right] \mathrm{d}l' \tag{11}$$

Similarly, the magnetic field can be written as:

$$\begin{aligned} H_i = \frac{j\omega}{k_0^2} \left(k_0^2 + \frac{\partial^2}{\partial x^2} \right) \int\limits_l -M_x(\rho') H_0^{(2)}(k_0|\rho - \rho'|) \, \mathrm{d}l' \\ + \frac{j\omega}{k_2^2} \left(k_2^2 + \frac{\partial^2}{\partial x^2} \right) \int\limits_l -M_x(\rho') H_0^{(2)}(k_2|\rho - \rho'|) \, \mathrm{d}l' \end{aligned} \tag{12}$$

(12) represents an integro-differential equation in which the differential and integral operators on the right hand side may be interchanged, thereby obtaining:

$$\begin{aligned} H_i = \frac{j\omega}{k_0^2} \int\limits_l -M_x(\rho') \left(k_0^2 + \frac{\partial^2}{\partial x^2} \right) H_0^{(2)}(k_0|\rho - \rho'|) \, \mathrm{d}l' \\ + \frac{j\omega}{k_2^2} \int\limits_l -M_x(\rho') \left(k_2^2 + \frac{\partial^2}{\partial x^2} \right) H_0^{(2)}(k_2|\rho - \rho'|) \, \mathrm{d}l' \end{aligned} \tag{13}$$

Operators with the order as in (13) represent *Pocklington's* integro-differential equation []. The second order derivative can be removed by expressing in terms of other Hankel functions through the recurrence relations [, p. 361].

$$\frac{\mathrm{d}H_0^{(2)}(x)}{\mathrm{d}x} = -H_1^{(2)}(x) + \frac{1}{x} H_0^{(2)}(x) \tag{14a}$$

$$H_1^{(2)}(x) = \frac{x}{2} \left[H_0^{(2)}(x) + H_2^{(2)}(x) \right] \tag{14b}$$

Furthermore, A Hankel with an argument $k_i r = k_i|\rho - \rho'|$, where $i = 0, 2$ can be differentiated by the chain-rule:

$$\begin{aligned} \frac{\partial H_0^{(2)}(k_i r)}{\partial x} &= \frac{\mathrm{d}H_0^{(2)}(k_i r)}{\mathrm{d}k_i r} \frac{\partial k_i r)}{\partial x} \\ &= \frac{\mathrm{d}H_0^{(2)}(k_i r)}{\mathrm{d}k_i r} \times \frac{k_i(x - x')}{r} \end{aligned} \tag{15}$$

By differentiating (15), we obtain:

$$\frac{\partial^2 H_0^{(2)}(k_i r)}{\partial x^2} = \frac{k_i}{r} \left[H_2^{(2)}(k_i r) \frac{k_i(x - x')^2}{r} - H_1^{(2)}(k_i r) \right] \tag{16}$$

The differential operator in (13) can now removed by applying the recurrence relations (14) and the expression is rewritten as:

$$\begin{aligned}
\left(k_i^2 + \frac{\partial^2}{\partial x^2}\right) H_0^{(2)}(k_i r) &= \frac{k_i^2}{2} H_0^{(2)}(k_i r) + k_i^2 \left[\frac{(x-x')^2}{r^2} - \frac{1}{2}\right] H_2^{(2)}(k_i r) \\
&= \frac{k_i^2}{2} H_0^{(2)}(k_i r) + k_i^2 \left(\cos\zeta - \frac{1}{2}\right) H_2^{(2)}(k_i r) \qquad (17) \\
&= \frac{k_i^2}{2} H_0^{(2)}(k_i r) + k_i^2 \cos(2\zeta) H_2^{(2)}(k_i r)
\end{aligned}$$

where $\cos\zeta = (x-x')/r$. The magnetic field in (13) can be re-expressed as:

$$\begin{aligned}
H_i = \frac{j\omega}{2} \int\limits_l -M_x(\rho') \Big[& H_0^{(2)}(k_0 r) + \cos(2\zeta) H_2^{(2)}(k_0 r) \\
& + H_0^{(2)}(k_2 r) + \cos(2\zeta) H_2^{(2)}(k_2 r) \Big] \, \mathrm{d}l' \qquad (18)
\end{aligned}$$

1 x-directed plate

For