

CURRENT ON A PLANAR DIELECTRIC PLATE

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We apply the surface equivalence theorem to find the electric and magnetic currents on the surface of a planar dielectric sheet. A plane wave propagating along the direction \mathbf{k} with electric field \mathbf{E} polarized along the z direction as shown in Fig. ?? is incident on the dielectric surface at an angle ϕ_i .

$$\mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H} = \hat{\mathbf{z}}J(\zeta), \quad (1a)$$

$$\mathbf{M}_s = -\hat{\mathbf{n}} \times \mathbf{E} = \hat{\mathbf{x}}M(\zeta) \quad (1b)$$

where the normal unit vector $\hat{\mathbf{n}}$ is in the y direction and ζ depends on x and y . To find the surface currents, we set up an homogeneous equivalent problem first for the region outside the dielectric sheet as depicted in Fig. ?. The total field can be written as:

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_1^{scat} \quad (2)$$

where \mathbf{E}_i is the incident electric field due to the plane wave,

$$\mathbf{E}_i = \hat{\mathbf{z}} E^0 e^{-jk_0(x \cos \phi_i - y \sin \phi_i)} \quad (3)$$

with k_0 as the propagation constant of air and E^0 the amplitude of the incoming plane wave. The scattered field in (2) can be expressed as:

$$\mathbf{E}_1^{scat} = \left(k_0^2 + \nabla \nabla \cdot \right) \mathbf{A} - \frac{1}{\varepsilon_1} \nabla \times \mathbf{F} \quad (4)$$

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where \mathbf{A} and \mathbf{F} are the magnetic and electric vector potentials respectively, given by:

$$\mathbf{A} = \frac{\mu}{4\pi} \iint_S \mathbf{J}_s(\mathbf{r}') \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dS', \quad (5a)$$

$$\mathbf{F} = \frac{\varepsilon}{4\pi} \iint_S \mathbf{M}_s(\mathbf{r}') \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dS'. \quad (5b)$$

with the position vectors \mathbf{r} and \mathbf{r}' illustrated in Fig. ?? . For a sheet structure extending to infinity in the z direction, (5) can be re-written as:

$$\mathbf{A} = \frac{\mu}{4j} \int_l \mathbf{J}_s(\rho') H_0^{(2)}(k_0|\rho - \rho'|) dl', \quad (6a)$$

$$\mathbf{F} = \frac{\varepsilon}{4j} \int_l \mathbf{M}_s(\rho') H_0^{(2)}(k_0|\rho - \rho'|) dl', \quad (6b)$$

where $H_0^{(2)}(k_0|\rho - \rho'|)$ is the Hankel function of order 0 and the second kind. For a z -directed source, the scattered electric field in (4) can be simply written as:

$$\begin{aligned} \mathbf{E}_1^{scat} &= -\hat{\mathbf{z}} j\omega \mathbf{A}_z \\ &= -\hat{\mathbf{z}} \frac{\omega\mu}{4j} \int_l J_z(\rho') H_0^{(2)}(k_0|\rho - \rho'|) dl' \end{aligned} \quad (7)$$

The scattered magnetic field can similarly be expressed in terms of the vector potentials. For the case in consideration, we obtain:

$$\begin{aligned} \mathbf{H}_1^{scat} &= -\hat{\mathbf{x}} \frac{j\omega}{k_0^2} \left(k_0^2 + \frac{\partial^2}{\partial x^2} \right) F_x \\ &= -\hat{\mathbf{x}} \frac{j\omega}{k_0^2} \left(k_0^2 + \frac{\partial^2}{\partial x^2} \right) \int_l M_x(\rho') H_0^{(2)}(k_0|\rho - \rho'|) dl' \end{aligned} \quad (8)$$

For the region inside the dielectric, an interior equivalent is set up with the currents reversing the signs. The total fields for the interior region only contain the scattered fields.

$$\mathbf{E}_2^{scat} = -\hat{\mathbf{z}} \frac{\omega\mu}{4j} \int_l -J_z(\rho') H_0^{(2)}(k_2|\rho - \rho'|) dl' \quad (9a)$$

$$\mathbf{H}_2^{scat} = -\hat{\mathbf{x}} \frac{j\omega}{k_2^2} \left(k_2^2 + \frac{\partial^2}{\partial x^2} \right) \int_l -M_x(\rho') H_0^{(2)}(k_2|\rho - \rho'|) dl' \quad (9b)$$

In order to find the electric and magnetic currents, we apply the boundary conditions at the

interface ensuring the continuity of tangential component of the fields. At the interface:

$$\hat{\mathbf{n}} \times (\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{0} \quad (10a)$$

$$\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{0} \quad (10b)$$

Since the electric field is only z-directed, we obtain a scalar equation by the application of (10a):

$$E_i = \frac{\omega\mu}{4} \int_c J_z(\rho') \left[H_0^{(2)}(k_0|\rho - \rho'|) + H_0^{(2)}(k_2|\rho - \rho'|) \right] dl' \quad (11)$$

Similarly, the magnetic field can be written as:

$$\begin{aligned} H_i = & \frac{j\omega}{k_0^2} \left(k_0^2 + \frac{\partial^2}{\partial x^2} \right) \int_l -M_x(\rho') H_0^{(2)}(k_0|\rho - \rho'|) dl' \\ & + \frac{j\omega}{k_2^2} \left(k_2^2 + \frac{\partial^2}{\partial x^2} \right) \int_l -M_x(\rho') H_0^{(2)}(k_2|\rho - \rho'|) dl' \end{aligned} \quad (12)$$

(12) represents an integro-differential equation in which the differential and integral operators on the right hand side may be interchanged, thereby obtaining:

$$\begin{aligned} H_i = & \frac{j\omega}{k_0^2} \int_l -M_x(\rho') \left(k_0^2 + \frac{\partial^2}{\partial x^2} \right) H_0^{(2)}(k_0|\rho - \rho'|) dl' \\ & + \frac{j\omega}{k_2^2} \int_l -M_x(\rho') \left(k_2^2 + \frac{\partial^2}{\partial x^2} \right) H_0^{(2)}(k_2|\rho - \rho'|) dl' \end{aligned} \quad (13)$$

Operators with the order as in (13) represent *Pocklington's* integro-differential equation []. The second order derivative can be removed by expressing in terms of other Hankel functions through the recurrence relations [, p. 361].

$$\frac{dH_0^{(2)}(x)}{dx} = -H_1^{(2)}(x) + \frac{1}{x} H_0^{(2)}(x) \quad (14a)$$

$$H_1^{(2)}(x) = \frac{x}{2} \left[H_0^{(2)}(x) + H_2^{(2)}(x) \right] \quad (14b)$$

Furthermore, A Hankel with an argument $k_i r = k_i |\rho - \rho'|$, where $i = 0, 2$ can be differentiated by the chain-rule:

$$\begin{aligned} \frac{\partial H_0^{(2)}(k_i r)}{\partial x} &= \frac{dH_0^{(2)}(k_i r)}{dk_i r} \frac{\partial k_i r}{\partial x} \\ &= \frac{dH_0^{(2)}(k_i r)}{dk_i r} \times \frac{k_i(x - x')}{r} \end{aligned} \quad (15)$$

By differentiating (15), we obtain:

$$\frac{\partial^2 H_0^{(2)}(k_i r)}{\partial x^2} = \frac{k_i}{r} \left[H_2^{(2)}(k_i r) \frac{k_i(x - x')^2}{r} - H_1^{(2)}(k_i r) \right] \quad (16)$$

The differential operator in (13) can now be removed by applying the recurrence relations (14) and the expression is rewritten as:

$$\begin{aligned}
\left(k_i^2 + \frac{\partial^2}{\partial x^2}\right) H_0^{(2)}(k_i r) &= \frac{k_i^2}{2} H_0^{(2)}(k_i r) + k_i^2 \left[\frac{(x - x')^2}{r^2} - \frac{1}{2} \right] H_2^{(2)}(k_i r) \\
&= \frac{k_i^2}{2} H_0^{(2)}(k_i r) + k_i^2 \left(\cos \zeta - \frac{1}{2} \right) H_2^{(2)}(k_i r) \\
&= \frac{k_i^2}{2} H_0^{(2)}(k_i r) + k_i^2 \cos(2\zeta) H_2^{(2)}(k_i r)
\end{aligned} \tag{17}$$

where $\cos \zeta = (x - x')/r$. The magnetic field in (13) can be re-expressed as:

$$\begin{aligned}
H_i &= \frac{-j\omega}{2} \int_l M_x(\rho') \left[H_0^{(2)}(k_0 r) + \cos(2\zeta) H_2^{(2)}(k_0 r) \right. \\
&\quad \left. + H_0^{(2)}(k_2 r) + \cos(2\zeta) H_2^{(2)}(k_2 r) \right] dl'
\end{aligned} \tag{18}$$

1 Flat Dielectric Plate

Consider a plane wave given by (3) incident on an infinitesimally thin planar dielectric strip lying along the x-axis, that is phase referenced to origin. The field expressions (11), (18) can be reduced for the pertinent structure:

$$E_i = \frac{\omega\mu}{4} \int_c J_z(x') \left[H_0^{(2)}(k_0 |x - x'|) + H_0^{(2)}(k_2 |x - x'|) \right] dl' \tag{19a}$$

$$\begin{aligned}
H_i &= \frac{-j\omega}{2} \int_l M_x(x') \left[H_0^{(2)}(k_0 |x - x'|) + H_2^{(2)}(k_0 |x - x'|) \right. \\
&\quad \left. + H_0^{(2)}(k_2 |x - x'|) + H_2^{(2)}(k_2 |x - x'|) \right] dl'
\end{aligned} \tag{19b}$$