NUMERICAL INTEGRATION OF SOMMERFELD INTEGRALS

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The fields of sources in a layered media environment involves computation of Sommerfeld Integrals which are reminiscent of a Hankel transform (a Fourier transform in cylindrical coordinate system):

$$\int_0^\infty \tilde{G}(k_\rho;\mathbf{r}|\mathbf{r}')J_n(k_\rho\rho)k_\rho\,\mathrm{d}k_\rho \tag{1}$$

where $\tilde{G}$ is a spectral domain Green's function of the structure, $\mathbf{r}$ and $\mathbf{r}'$ are the observation and source locations respectively, and J_n is bessel's function of order n. In most cases, the integral in (1) cannot be solved analytically. The integrand exhibits branch point singularities and very slow rate of convergence due to oscillaotory behaviour of the bessel function. Therefore, intelligent methods need to be implemented to expediete the integration of such integrals. By invoking the Cauchy's Integral Theoerem, [1, p. 377] we choose the path of integration along the positive real axis in the complex k_ρ plane as shown in Fig. 1 with slight indentation into the first quadrant around possible singularities. [2, 3]

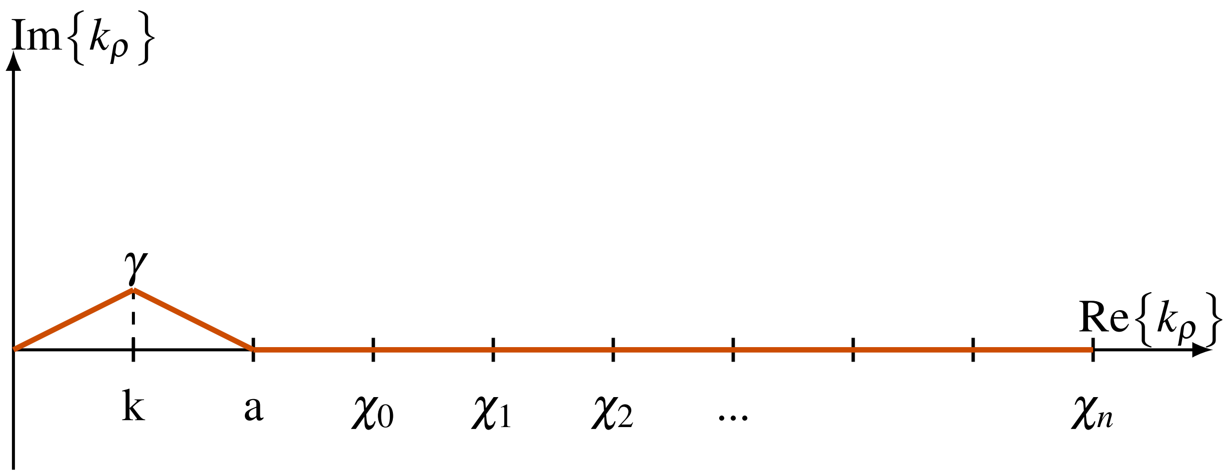

Figure 1: Integration Path along the real axis

The integral in (1) is computed by splitting into three sections:

$$\int_0^\infty \tilde{G}(k_\rho;\mathbf{r}|\mathbf{r}')J_n(k_\rho\rho)k_\rho\,\mathrm{d}k_\rho = \left(\int_0^\gamma + \int_\gamma^a + \int_a^\infty\right)\tilde{G}(k_\rho;\mathbf{r}|\mathbf{r}')J_n(k_\rho\rho)k_\rho\,\mathrm{d}k_\rho \tag{2}$$

where γ represents the detour from the real axis. The distance between γ and the real axis should be kept small in order to avoid wild growth of the bessel function with increasing imaginary part as illustrated in Fig. 2. The path detour ends at the breakpoint a, where the method of numerical integration is switched. The two

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finite interval integrals in (2) are computed using Double Exponential (DE) quadrature rule. [4]. Unlike other quadratures, the DE rule works efficiently on integrands with singularity at one or both of the endpoints. [5, 6] The remaining infinite interval along the real axis is computed through Partition Extrapolation (PE) method

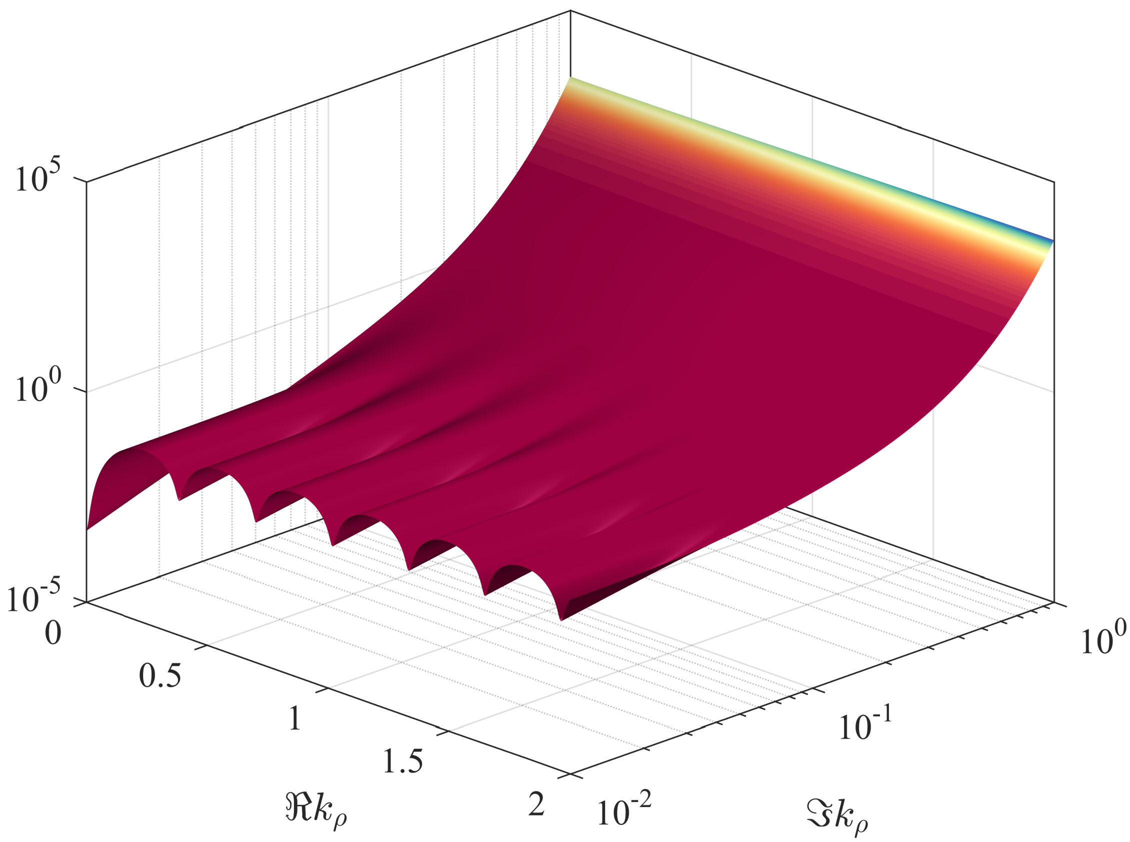

Figure 2: Exponential Growth of Bessel function $J_1(10k_\rho)$ along the imaginary axis

1 Overview of Numerical Integration

Most numerical integration routines are based on polynomial interpolation where the integral is approximated by:

$$\int_a^b f(x)\,\mathrm{d}(x) \approx w_0 f(x_0) + w_1 f(x_1) + \ldots + w_n f(x_n) = \sum_{i=0}^{n} w_i f(x_i) \tag{3}$$

References

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[6] M. Mori, “Discovery of the double exponential transformation and its developments,” *Publications of the Research Institute for Mathematical Sciences*, vol. 41, no. 4, pp. 897–935, 2005.