✓ Chain Rule, Jacobian

✗ Implicit funⁿ

Chain Rule for Function of several variables —

I. Let $f(u)$ be a differentiable funⁿ of u and u is a funⁿ of x. Then f becomes a funⁿ of x.

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

e.g. $y = \log_e(x^2+1)$: $y = f(x)$ and $x = g(z)$

$x = \sin z$

now, $$\frac{dy}{dz} = \frac{d}{dz}\left[\log_e(x^2+1)\right] = \frac{d}{dx}\left[\log_e(x^2+1)\right]\frac{dx}{dz}$$

$$= \frac{1}{x^2+1} \cdot 2x \, \frac{d}{dz}(\sin z)$$

$$= \frac{2x\cos z}{1+x^2} = \frac{2\sin z \cos z}{1+\sin^2 z}$$

$$= \frac{\sin 2z}{1+\sin^2 z}$$

$$\therefore \boxed{\frac{dy}{dz} = \frac{\sin 2z}{1+\sin^2 z}}$$

Note – Suppose $u = f(v)$, $v = g(w)$, $w = h(t)$

Then $$\frac{du}{dt} = \frac{du}{dv}\frac{dv}{dw}\frac{dw}{dt}$$

II. Let $f(u)$ is a funⁿ of u and u is a funⁿ of x and y. Then f becomes a funⁿ of x and y. The two partial derivatives of f is given by

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y}$$

e.g. $v = \sin^2 u, \quad u = x^3 y^4$

$\therefore v = f(u)$ and $u = g(x, y)$

$$\therefore \frac{\partial v}{\partial x} = \frac{\partial v}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial}{\partial u}(\sin^2 u)\frac{\partial}{\partial x}(x^3 y^4)$$

$$= 2\sin u \cos u \cdot 3x^2 y^4$$

$$= 3x^2 y^4 \sin 2u$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial}{\partial u}(\sin^2 u)\frac{\partial}{\partial y}(x^3 y^4)$$

$$= 2\sin u \cos u \cdot 4x^3 y^3$$

$$= 4x^3 y^3 \sin 2u$$

III. Let $f(u, v)$ be a fu^n of u and v; u, v are fu^n of x and y. The f will be fu^n of x and y.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

e.g. $z = u^2 + v^3, \quad u = \sin xy, \quad v = y^2$

$z = f(u, v)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial}{\partial u}(u^2 + v^3)\frac{\partial}{\partial x}(\sin xy) + \frac{\partial}{\partial v}(u^2 + v^3)\frac{\partial}{\partial x}(y^2)$$

$$= 2u \cdot \cos xy \cdot y + 3v^2 \cdot 0$$

$$= 2y \sin xy \cos xy = y \sin 2xy$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial}{\partial u}(u^2+v^3)\frac{\partial}{\partial y}(\sin xy) + \frac{\partial}{\partial v}(u^2+v^3)\frac{\partial}{\partial y}(y^2)$$

$$= 2u \cdot \cos xy \cdot x + 3v^2 \cdot 2y$$

$$= 2x \sin xy \cos xy + 6y^4 \cdot y$$

$$= x \sin 2xy + 6y^5$$

Note – If $z = f(u, v)$ and u, v are fu^n of x only Then z becomes a fu^n of x only. So ordinary derivative of z will be

$$\frac{dz}{dx} = \frac{df}{dx} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

e.g. $z = \sin uv$, $u = 3x^2$, $v = \log_e x$

$\therefore$ $z = fu^n$ of x only.

$$\frac{dz}{dx} = \frac{d}{dx}(\sin uv) = \frac{\partial}{\partial u}(\sin uv)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}(\sin uv)\frac{\partial v}{\partial x}$$

$$= \cos uv \cdot v\frac{\partial}{\partial x}(3x^2) + \cos uv \cdot u\frac{\partial}{\partial x}(\log_e x)$$

$$= 6x\, v \cos uv + \frac{u}{x}\cos uv$$

Q. Let $u = f(y-z,\ z-x,\ x-y)$. Then S.T. $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$

Let $a = y - z$

$b = z - x \qquad\qquad u = f(a, b, c)$

$c = x - y$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial u}{\partial b}\frac{\partial b}{\partial x} + \frac{\partial u}{\partial c}\frac{\partial c}{\partial x}$$

$$= \frac{\partial u}{\partial a} \times 0 + \frac{\partial u}{\partial b}(-1) + \frac{\partial u}{\partial c}(1)$$

$$= -\frac{\partial u}{\partial b} + \frac{\partial u}{\partial c} \quad \text{—— (i)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial u}{\partial b}\frac{\partial b}{\partial y} + \frac{\partial u}{\partial c}\frac{\partial c}{\partial y}$$

$$= \frac{\partial u}{\partial a}(1) + \frac{\partial u}{\partial b}(0) + \frac{\partial u}{\partial c}(-1)$$

$$= \frac{\partial u}{\partial a} - \frac{\partial u}{\partial c} \quad \text{—— (ii)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial a}\frac{\partial a}{\partial z} + \frac{\partial u}{\partial b}\frac{\partial b}{\partial z} + \frac{\partial u}{\partial c}\frac{\partial c}{\partial z}$$

$$= \frac{\partial u}{\partial a}(-1) + \frac{\partial u}{\partial b}(1) + \frac{\partial u}{\partial c}(0)$$

$$= -\frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} \quad \text{—— (iii)}$$

adding (i), (ii), (iii) we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = -\cancel{\frac{\partial u}{\partial b}} + \cancel{\frac{\partial u}{\partial c}} + \cancel{\frac{\partial u}{\partial a}} - \cancel{\frac{\partial u}{\partial c}} - \cancel{\frac{\partial u}{\partial a}} + \cancel{\frac{\partial u}{\partial b}}$$

$$= 0$$

9. If $x = u + v + w$, $y = vw + wu + uv$, $z = uvw$

$F = (x, y, z)$. S.T. $u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w}$

$$= x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$$

$\Rightarrow$
$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial u}$$
$$= \frac{\partial F}{\partial x} + (w+v)\frac{\partial F}{\partial y} + vw\frac{\partial F}{\partial z}$$

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial v}$$
$$= \frac{\partial F}{\partial x} + (w+u)\frac{\partial F}{\partial y} + uw\frac{\partial F}{\partial z}$$

$$\frac{\partial F}{\partial w} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial w}$$
$$= \frac{\partial F}{\partial x} + (v+u)\frac{\partial F}{\partial y} + uv\frac{\partial F}{\partial z}$$

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w}$$
$$= (u+v+w)\frac{\partial F}{\partial x} + (uw + uv + vw + uv + vw + uw)\frac{\partial F}{\partial y}$$
$$+ (uvw + uvw + uvw)\frac{\partial F}{\partial z}$$

$$= x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$$

Jacobians —

If $F(x, y)$ and $G(x, y)$ are two differentiable fu^n in the region then the Jacobian of F and G w.r.t. x and y is

$$J\left(\frac{F, G}{x, y}\right) = \frac{\partial(F, G)}{\partial(x, y)}$$

$$= \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{vmatrix}$$

If F, G, H are fu^n of x, y, z, then

$$J\left(\frac{F, G, H}{x, y, z}\right) = \frac{\partial(F, G, H)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{vmatrix}$$

e.g. $u = x - y$
$v = x^2 - y^2$

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 \\ 2x & -2y \end{vmatrix} = -2y + 2x$$

$$= 2(x - y)$$

e.g. $x = r\cos\theta$
$y = r\sin\theta$

$$J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r$$

$r^2 = x^2 + y^2$

$\theta = \tan^{-1}\frac{y}{x} \Rightarrow \tan\theta = \frac{y}{x}$

$\therefore 2r\frac{\partial r}{\partial x} = 2x$

$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$

Similary, $\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$

$$\frac{\partial \theta}{\partial x} = \frac{-y/x^2}{1 + \frac{y^2}{x^2}}$$

$$= -\frac{y}{x^2 + y^2} = \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1/x}{1 + y^2/x^2} = \frac{x}{x^2 + y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r^2}$$

$$\therefore \quad J\left(\frac{r, \theta}{x, y}\right) = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix}$$

$$= 1/r$$

$\therefore J\left(\frac{x, y}{r, \theta}\right) = r, \quad J\left(\frac{r, \theta}{x, y}\right) = 1/r$

8. $u = x^2 + y^2 \qquad x = r\cos\theta$

$v = x^2 - y^2 \qquad y = r\sin\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (-4xy - 4xy)\, r = -8xyr$$