Gradient of a function –

The gradient of a scalar valued multivariable function $f(x, y, \dots)$, denoted by ∇f, is the collection of all its partial derivatives into a vector.

$$\nabla f(\overbrace{x_0, y_0, \dots}^{f \text{ is a scalar valued multivariate}}) = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0, \dots) \\ \frac{\partial f}{\partial y}(x_0, y_0, \dots) \\ \vdots \end{bmatrix}}_{\nabla f \text{ outputs a vector with all possible partial derivatives of } f}$$

notation for gradient, called 'nabla'

e.g. If $f(x, y) = x^2 - xy$, then find ∇f:

→ $$\nabla f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x - y \\ -x \end{bmatrix}$$

at some point (x_0, y_0), $$\nabla f(x_0, y_0) = \begin{bmatrix} 2x_0 - y_0 \\ -x_0 \end{bmatrix}$$

$$= (2x_0 - y_0)\hat{i} - x_0\hat{j}$$

Note – ∇f is a vector valued fu^n (can be denoted by $\vec{\nabla} f$ also). If you imagine standing at a point $(x_0, y_0, \dots)$ in the input space of f, the vector

$\nabla f(x_0, y_0, \cdots)$ tells you which direction you should travel to increase the value of f most rapidly. These gradient vectors $\nabla f(x_0, y_0, \cdots)$ are also $\perp^r$ to the contour lines of f.

Divergence of a Vector Point Function –

The divergence of a differentiable vector point fu^n $\vec{v}(x, y, z)$ is denoted by $div(\vec{v})$ and is defined as $div(\vec{v}) = div\,\vec{v} = \nabla \cdot \vec{v}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}\right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

where $v_1 = v_x = x$ comp of v

$v_2 = v_y = y$ Comp. of v

$v_3 = v_z = z$ comp of v

$$\therefore \boxed{div\,\vec{v} = \nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \cdots}$$

e.g. If $\vec{v} = \begin{bmatrix} 2x - y \\ y^2 \end{bmatrix} = (2x - y)\hat{i} + y^2\hat{j}$. then find $div(\vec{v})$

$\rightarrow$ $$div(\vec{v}) = \nabla \cdot \vec{v} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} 2x - y \\ y^2 \end{bmatrix}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right) \cdot \left((2x - y)\hat{i} + y^2\hat{j}\right)$$

$$= \frac{\partial}{\partial x}(2x - y) + \frac{\partial}{\partial y}(y^2)$$

$$= 2 + 2y$$

Note — The divergence is an operator, which takes in the vector-valued function defining this vector field, and outputs a scalar-valued function measuring the change in the density of the fluid at each point, if we represent vector field as a fluid flow.

Curl of a Vector Point Function —

If a 3-dimensional vector-valued function $\vec{v}(x, y, z)$ has component functions $v_1(x, y, z)$, $v_2(x, y, z)$ and $v_3(x, y, z)$, then the curl is computed as

$$\text{Curl } \vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

∴ Curl is an operator which takes in a fu^n representing a 3-dimensional vector field and gives another fu^n representing a different 3-dimensional vector field.

- Solenoidal Vector – A vector point fu^n $\vec{v}$ is said to be solenoidal if div $\vec{v} = 0$ i.e. $\vec{\nabla} \cdot \vec{v} = 0$

- Irrotational Vector – A vector point fu^n $\vec{v}$ is said to be irrotational if curl $\vec{v} = 0$ i.e. $\vec{\nabla} \times \vec{v} = \vec{0}$

Q. S.T. the vector $\vec{v}(x,y,z) = -x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\hat{k}$ is Solenoidal.

$\rightarrow$ $$\vec{\nabla} \cdot \vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(-x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\hat{k}\right)$$

$$= -\frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(2xz) + \frac{\partial}{\partial z}(2yz)$$

$$= -2xy - 0 + 2xy = 0$$

Q. Check whether the vector $\vec{v}(x,y,z) = (x^2 + yz)\hat{i} + (y^2 + zx)\hat{j} + (z^2 + xy)\hat{k}$ is irrotational or not.

$\rightarrow$ $$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + yz & y^2 + zx & z^2 + xy \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(z^2 + xy) - \frac{\partial}{\partial z}(y^2 + zx)\right]\hat{i} + \left[\frac{\partial}{\partial z}(x^2 + yz) - \frac{\partial}{\partial x}(z^2 + xy)\right]\hat{j} + \left[\frac{\partial}{\partial x}(y^2 + zx) - \frac{\partial}{\partial y}(x^2 + yz)\right]\hat{k}$$

$$= (x - x)\hat{i} + (y - y)\hat{j} + (z - z)\hat{k}$$

$$= \vec{0}$$ $\therefore$ Hence irrotational

Q.1 If $\vec{f} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$ Find curl curl $\vec{f}$

$\rightarrow$

$$\text{curl } \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix} = \hat{i}(2z+2x) - \hat{j}(0-0) + \hat{k}(-2z-x^2)$$

$$\text{curl curl } \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z+2x & 0 & -2z-x^2 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(-2x-2) + \hat{k}(0-0)$$

$$= 2(x+1)\hat{j}$$

Q.2 Find div $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

$\rightarrow$ $\text{grad}(x^3+y^3+z^3-3xyz) = \vec{F}$

$$= \hat{i}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + \hat{j}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz) + \hat{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$= \hat{i}(3x^2-3yz) + \hat{j}(3y^2-3xz) + \hat{k}(3z^2-3xy)$$

$$\text{div } \vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\vec{F}$$

$$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$$

$$= 6x+6y+6z = 6(x+y+z)$$

$$\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$= \hat{i}(-3x + 3x) + \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z)$$

$$= \vec{0}$$

Q.3 Determine the constant m so that the vector $\vec{v} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+mz)\hat{k}$ is solenoidal.

$\rightarrow$ $\text{div } \vec{v} = \vec{\nabla} \cdot \vec{v}$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+mz)$$

$$= 1 + 1 + m = m + 2$$

as $\vec{v}$ is solenoidal, then $\text{div } \vec{v} = 0$

$$\Rightarrow m = -2$$

Q.4 Show that curl grad $f = 0$ where $f = x^2 y + 2xy + z^2$

$\rightarrow$ $\text{grad } \vec{f} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= (2xy + 2y)\hat{i} + (x^2 + 2x)\hat{j} + 2z\hat{k}$$

curl grad $\vec{f}$ $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+2y & x^2+2x & 2z \end{vmatrix}$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(\cancel{2x} + \cancel{2} - \cancel{2x} - \cancel{2})$$

$$= \vec{0}$$

Q.5 Show that $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. Find the scalar fu^n ϕ such that $\vec{A} = \vec{\nabla}\phi$

$\Rightarrow$ curl $\vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$

$$= \hat{i}(-\cancel{1} + \cancel{1}) - \hat{j}(\cancel{3z^2} - \cancel{3z^2}) + \hat{k}(\cancel{6x} - \cancel{6x})$$

$$= \vec{0}$$

$\therefore$ vector $\vec{A}$ is irrotational.

Since, $\vec{A} = \vec{\nabla}\phi$

$$\Rightarrow (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$$

$$= \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

now equating both side,

$$\frac{\partial \phi}{\partial x} = 6xy + z^3 \quad\Big|\quad \frac{\partial \phi}{\partial y} = 3x^2 - z \quad\Big|\quad \frac{\partial \phi}{\partial z} = 3xz^2 - y$$

$$\Rightarrow \phi = 3x^2 y + xz^3 + f(y, z) \quad \text{— (i)}$$

$$\Rightarrow \phi = 3x^2 y - yz + g(x, z) \quad \text{— (ii)}$$

$$\Rightarrow \phi = xz^3 - yz + h(x, y) \quad \text{— (iii)}$$

where f, g, h are arbitrary functions of variables as written.

from eq^n (i), (ii), (iii), by comparing we get

$$f(y, z) = -yz$$
$$g(x, z) = xz^3$$
$$h(x, y) = 3x^2 y$$

Hence $\phi(x, y, z)$

$$= 3x^2 y + xz^3 - yz + \underbrace{\text{constant}}_{\text{arbitrary cons independent of } x, y, z.}$$