Test case for WEC:
A propagating wave packet

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June 26, 2019

We check the WEC implementation in ROMS with a test case of a propagating wave packet as described by McWilliams & al. (2004), Sect. 6.5.

The set up for this test case is a simple x, y box where $y \ll x$, an elongated West-East box, with periodic boundary conditions in every direction. A wave packet propagates Eastward, $\mathbf{k} = k\mathbf{e_x}$, with a group velocity C_g and an homogeneous amplitude A in the y direction.

Without atmospheric forcing neither stratification or rotation, we expect the dynamics to be entirely controlled by the wave effects.

The long-wave asymptotic theory predicts that the velocity field that arises at the passage of the wave packet is also purely Eastward, opposed to the vertically integrated Stokes drifts and with a set-up, $\hat{\zeta}$, component.

We describe hereafter the equations and relevant expressions of the wave packet propagation test case.

1 The test case description

1.1 Expressions of the variables in the general case and in the shallow waters approximation

The depth h, the frequency σ and the wavenumber k are constant. The wave amplitude A is homogeneous in the y direction.

1.1.1 Dispersion relation, group and phase velocities

The dispersion relation of gravity waves reads

$$\sigma = 2\pi f = \sqrt{gk \tanh kD} \ , \tag{1}$$

with $D \hat{=} h$ for simplicity.

By definition, the group velocity is

$$C_g = \partial_k \sigma = \frac{\sigma}{2k}\left(1 + \frac{2kh}{\sinh 2kh}\right) \ , \tag{2}$$

and the phase velocity is defined as

$$C_\phi = \frac{\sigma}{k} \ . \tag{3}$$

In shallow waters $kh \to 0$, the dispersion relation becomes

$$\sigma \sim \sqrt{gk(kh(1-(kh)^2/3))} \sim k\sqrt{gh} \ . \tag{4}$$

Then,

$$C_\phi = C_g \sim \sqrt{gh} \ . \tag{5}$$

In shallow waters group and phase velocities are equal leading to a non dispersive wave propagation.

1.1.2 Vertically integrated Stokes drifts

The Stokes drifts in the x direction is

$$u^{\mathrm{St}} = \frac{a^2 \sigma k}{2\sinh^2 kh} \cosh 2k(z+h) \ , \tag{6}$$

integrating vertically

$$T^{\mathrm{St}} = \frac{a^2 \sigma k}{2\sinh^2 kh} \int_{-h}^{0} \cosh 2k(z+h) dz \ , \tag{7}$$

substituting $z' = 2k(z+h)$

$$= \frac{a^2 \sigma k}{2\sinh^2 kh} \times \frac{1}{2k} \int_{0}^{2kh} \cosh z' dz' \ , \tag{8}$$

$$= \frac{a^2 \sigma}{4\sinh^2 kh} \sinh 2kh \ , \tag{9}$$

noting $\sinh 2kh = 2\cosh kh \sinh kh$, we finally get

$$T^{\mathrm{St}} = \frac{a^2 \sigma}{2\tanh kh} \ . \tag{10}$$

In the shallow water approximation, using (4), this expression reduces to

$$T^{\mathrm{St}} \sim \frac{a^2 \sigma}{2kh} \sim \frac{a^2}{2}\sqrt{\frac{g}{h}} \ . \tag{11}$$

1.1.3 The set-up/down $\hat{\zeta}$

The set-up/down $\hat{\zeta}$ is the quasi-static sea-level component defined as

$$\hat{\zeta} = -\frac{a^2 k}{2\sinh 2kh} \ . \tag{12}$$

In shallow waters, it becomes

$$\hat{\zeta} \sim -\frac{a^2 k}{2(2kh(1+(2kh)^2/6)} \sim -\frac{a^2}{4h} \ . \tag{13}$$

1.1.4 The long-wave velocity

Arising from the propagation of the wave packet, the long-wave steady state flow in the x direction is

$$u^{\mathrm{lw}} = -\frac{(C^{\mathrm{lw}})^2}{(C^{\mathrm{lw}})^2 - (C_g)^2} \left(\frac{T^{\mathrm{St}}}{h} - C_g \frac{\hat{\zeta}}{h} \right) \ , \tag{14}$$

where the long (non-rotating, shallow-water) gravity-wave speed is $C^{\mathrm{lw}} = \sqrt{gh}$. Note that it does not depend on depth. In shallow waters, using (11) and (13)

$$T^{\mathrm{St}} - C_g \hat{\zeta} \sim \frac{a^2}{2}\sqrt{\frac{g}{h}} + C_g \frac{a^2}{4h} \ , \tag{15}$$

$$\sim \frac{a^2}{2}\left(\sqrt{\frac{g}{h}} + \frac{\sqrt{gh}}{2h} \right) \sim \frac{3a^2}{4}\sqrt{\frac{g}{h}} \ , \tag{16}$$

but also $C^{\mathrm{lw}} = C_g$ and therefor u^{lw} diverges. We need to resort to the next order on C_g. For that we keep the next order in the dispersion equation

$$\sigma^2 = gk \tanh kh \sim gk\left(kh - \frac{(kh)^3}{3} \right) \sim ghk^2 \left(1 - \frac{(kh)^2}{3} \right) \sim (C^{\mathrm{lw}})^2 k^2 \left(1 - \frac{(kh)^2}{3} \right) \ , \tag{17}$$

$$\sigma \sim C^{\mathrm{lw}} k \left(1 - \frac{(kh)^2}{6} \right) \ , \tag{18}$$

and then

$$C_g = \partial_k \sigma \sim C^{\mathrm{lw}} \left(1 - \frac{(kh)^2}{2} \right) \ , \tag{19}$$

leading to

$$\frac{(C^{\mathrm{lw}})^2}{(C^{\mathrm{lw}})^2 - (C_g)^2} = \frac{1}{(kh)^2} \ , \tag{20}$$

and finally

$$u^{\mathrm{lw}} \sim -\frac{1}{h(kh)^2} \frac{3a^2}{4}\sqrt{\frac{g}{h}} \sim -\frac{3a^2\sqrt{g}}{4k^2 h^{7/2}} \ . \tag{21}$$

1.1.5 The sea surface elevation ζ

$$\zeta = -\frac{\partial_t \varphi}{g} = \frac{C_g}{g} u^{\text{lw}} = -\frac{C_g}{gh} \frac{(C^{\text{lw}})^2}{(C^{\text{lw}})^2 - (C_g)^2} \left(T^{\text{St}} - C_g \hat{\zeta} \right) , \tag{22}$$

since the long-wave velocity derives from the potential velocity $u^{\text{lw}} = \partial_x \varphi$. In shallow waters,

$$\zeta \sim \sqrt{\frac{h}{g}} \times -\frac{3a^2\sqrt{g}}{4k^2 h^{7/2}} \sim -\frac{3a^2}{4k^2 h^3} . \tag{23}$$

We can quantify the ratio

$$\frac{\hat{\zeta}}{\zeta} \sim \frac{a^2}{4h} \frac{4k^2h^3}{3a^2} \sim \frac{(kh)^2}{3} \ll 1 . \tag{24}$$

1.2 Summary of the expressions of relevant variables depending on the depth

Table 1: Expressions of relevant variables to the wave packet propagation test case depending on the depth

Variables	Shallow waters	Intermediate/finite depth	Deep waters
σ	$k\sqrt{gh}$	$\sqrt{gk \tanh kh}$	$\sqrt{gk}$
C_g	$\sqrt{gh}$	$\partial_k \sigma$	$\frac{1}{2}\sqrt{\frac{g}{k}}$
C_ϕ	$\sqrt{gh}$	$\frac{\sigma}{k} = 2C_g$	$\sqrt{\frac{g}{k}}$
T^{St}	$\frac{a^2}{2}\sqrt{\frac{g}{h}}$	$\frac{a^2\sigma}{2\tanh kh}$	$\frac{a^2\sqrt{gk}}{2}$
$\hat{\zeta}$	$-\frac{a^2}{4h}$	$-\frac{a^2 k}{2\sinh 2kh}$	0
u^{lw}	$-\frac{3a^2\sqrt{g}}{4k^2h^{7/2}}$	$-\frac{(C^{\text{lw}})^2}{(C^{\text{lw}})^2-(C_g)^2}\left(\frac{T^{\text{St}}}{h} - C_g\frac{\hat{\zeta}}{h}\right)$	$-\frac{(C^{\text{lw}})^2}{(C^{\text{lw}})^2-(C_g)^2}\frac{T^{\text{St}}}{h}$
ζ	$-\frac{3a^2}{4k^2h^3}$	$\frac{C_g}{g}u^{\text{lw}}$	$-\frac{a^2}{4h}\frac{(C^{\text{lw}})^2}{(C^{\text{lw}})^2-(C_g)^2}$

1.3 Offline Python validation

We test the validity of the above derived shallow waters expressions with an offline python script. We also would like $\hat{\zeta}$ to be none negligible, we therefor set the depth h to be small compared to the wavelength (shallow water approximation and long-wave dynamics). Under these circumstances, the propagation of the wave packet is non dispersive i.e. the group and phase velocities are equal.

1.3.1 Parameters

We chose the following parameters: the amplitude of the envelope of the wave packet A, the frequency f and the depth h in order to be in the shallow water approximation ($h < 0.05\lambda$, $kh \ll 1$ leading to $C_g \sim C_\phi$).

Table 2: The wave bulk parameters

Variables	
Amplitude A (m)	1
Frequency f (Hz)	0.01
Depth h (m)	50
Wavelength λ (m)	2207.3
kh	0.1423
k (m^{-1})	0.0028
C_g (ms^{-1})	21.93
C_ϕ (ms^{-1})	22.07

1.3.2 Propagating a wave packet

We propagate a non dispersive wave packet whose amplitude evolve as

$$a_w(x,t) = Ae^{-x_\star^2}\ , \tag{25}$$

where $x_\star = x - C_g t$.

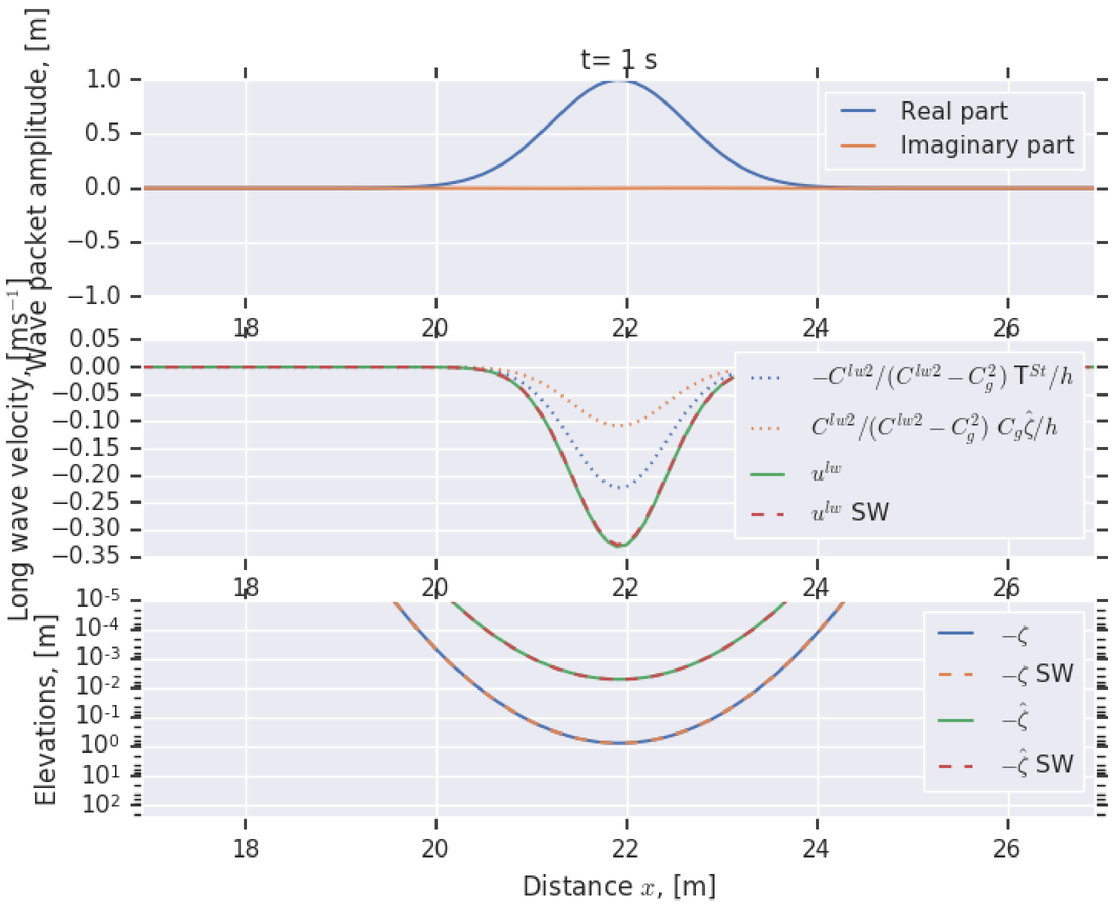

Figure 1: First line: Wave amplitude $a(x,t)$. Second line: Long-wave velocities. Third line: Sea-surface elevations. We multiply them by -1 in order to use the logarithmic scaling ($\hat{\zeta}$ being very small compared to ζ) and we reverse the y axis to restore the natural reading sense of the figure: the wave packet leads to a set-down of the sea surface elevation. SW denotes quantities computed in the shallow water approximation while the other quantities are computed using their not approximated formulation.

The shallow water expressions we derived agrees with the original formulation.

2 ROMS implementation

2.1 Options and grid

2.1.1 Defined options in cpp_defs.h

We create a case called WAVE_PACKET in cpp_defs.h (see cppdefs.h_DD_WAVE_PACKET). We would solve the equations in 2D as the solution does not depend on depth but it seems that it is currently not possible to run analytical WEC cases in 2D so we solve the equations in 3D.

NB: Never turn on the option MRL_WCI without the SOLVE3D option.

It contains the following options:

- define ANA_GRID
- define ANA_INITIAL

- define ANA_SMFLUX
  This option sets the kinematic surface momentum flux (wind stress) components in XI- and ETA-directions sustr, svstr to zero at first (default values) (see analytical.F l218).
- define ANA_SRFLUX
- define ANA_STFLUX
  zero by default.
- define ANA_SSFLUX

- define SOLVE3D
  It appears that waves in 2D may have never been performed.
- define UV_ADV
- define EW_PERIODIC
- define NS_PERIODIC
  Since, we set a symmetry in the y direction, it seems reasonable to use periodic boundary conditions on the North-South boundaries.
- define NONLIN_EOS
- define SPLIT_EOS
- define MRL_WCI
- define ANA_WWAVE
- define BRK0

2.1.2 The grid

First, we set the following parameters in **param.h**:

Table 3: Grid parameters

Parameters		where in the code
LLm	1400	param.h l15
MMm	2	l15
N	2	l15
NSUB_X	1	else case l187
NSUB_E	1	else case l187
	Nodes	
	TN	ND (l188)
NP_XI	8	12
NP_ETA	1	1

MMm must be at least twice the number of nodes. Using only 1, we set MMm to two. The other parameters of the grid are then defined in **ana_grid.F** (key search with DD lines 85-89):

Table 4: The grid parameters

Parameters	
size_XI (m)	1400
size_ETA (m)	2
Depth h (m)	10
f_0	0
β	0

We have a resolution at $dx = 1$m by setting

$$\text{size\_XI} = \text{LLm} \ . \tag{26}$$

2.1.3 Tiles

The domain in the x direction is divided in $NP\_XI = 8$ subdomains or tiles (respectively in NP_ETA in the y direction, in this case only one). Each tile has the South-West corner coordinate stored in the variable iSW_corn defined in mpi_setup.F (the option VERBOSE gives details) as

$$iSW\_corn = inode \times Lm - \frac{off\_XI}{2} \ , \tag{27}$$

where $Lm = LLm/NP\_XI = 175$, $off\_XI = NP\_XI \times Lm - LLm = 0$ and $inode$ is the node number variable ($\in [0, 7]$, $\in \mathbb{N}$). See the python script tile_wave_packet.py. With our settings, the tiles South West corner i coordinate is a geometrical series with the common ratio Lm and first term $i = 1$.

2.2 Initial conditions

The model gets its initial conditions in **ana_init.F**. We call forces.h (line 61 along with boundaries.h) to declare the wave variables that need to be initialized: $whrm$ the RMS wave height, $wfrq$ the frequency, $wdrx$ the wave direction to xi (x axis), $wdre$ to eta (y axis), sup the set-up and $wdsp$.

The wave packet propagation starts at mid domain $LLm/2$ on the x axis (i index) and does not depend on y (j index respectively). The associated wave averaged RMS wave height (the wave amplitude a is related to the RMS wave height, used by ROMS, by $whrm(i, j) = 2a(i, j)$, so $a(i, j)^2 = 0.25 whrm(i, j)^2$) is initialized with

$$whrm(i, j) = 2Ae^{-env(i+iSW\_corn-(LLm/2))^2} \ , \tag{28}$$

where env^{-1} is the order of magnitude of the length scale of the envelope of the wave packet. Then, we initialize σ ($wfrq(i, j)$), $\bar{u}$ ($ubar(i, j)$), T^{St} ($h \times ust2d(i, j)$[1]), ζ ($zeta(i, j)$) and $\hat{\zeta}$ ($sup(i, j)$) with the solution summarized in table (1) in the finite depth case. The v components are set to zero. We specify the variables env and k and compute locally C^{lw} and C_g accordingly.

The wave propagation direction is purely Eastward, i.e. along the x axis, so $wdrx = \cos 0 = 1$ and $wdre = \sin 0 = 0$. The temperature is set homogeneous, using the default $t(i, j, k, 1, itemp) = 18.0\ ^oC$. The temperature initialization is required when the option SOLVE3D is on but it does not evolve. Other field are needed and all set to zero: $wdsp$, $brk2dx$, $brk2de$ (breaking terms), $frc2dx$ and $frc2de$ (friction terms). Note that because the solution does not depend on z we initialize the barotropic variables but the code actually needs the 3D variables (u, v, ust and vst) to be initialized consistently (set them equal to their barotropic counterpart at every z level).

2.3 Evolution of the wave amplitude at each time step

At each time step, the wave variable $whrm$ needs to be updated in **analytical.F** in order to produce a propagation. To respect the periodicity of the domain, we take the modulo of the time with respect to the time needed by the wave packet to propagate along the entire domain LLm/C_g such as $t' = modulo(time, (LLm/Cg))$. In each tile, since $dx = 1$m, we compute $x_\star$ as $i - C_g \times t$ with $t = t' - \frac{iSW\_corn}{C_g}$. Taking into account the initial position of the wave packet $LLm/2$, it leads to

$$\begin{aligned} whrm(i, j) = 2A(&e^{-env(i-C_g t'+iSW\_corn-(LLm/2))^2} \\ &+e^{-env(i-C_g t'+iSW\_corn-(LLm/2)+LLm)^2}) \ . \end{aligned} \tag{29}$$

The second exponential is to insure the periodicity of the wave packet propagation. Dissipative coefficients ε_d and ε_b are needed and set to zero, i.e. no bottom drag nor wave breaking are included.

2.4 Running parameters

We run with timestep of one second or less. We need to output zeta, u bar, the set-up and the vertically integrated Stokes drift. We set the root for file name to wpp for Wave Packet Propagation. We set a total time of running to approximately ten wave propagation across the domain and save the solution at each second (history files).

[1] T^{St} is the depth integrated Stokes drift while $ust2d$ is the depth averaged.

Accordingly, we edit the roms.in file:
title:
Wave packet propagation 1m 2 levels

time_stepping: NTIMES dt[sec] NDTFAST NINFO
200 1 50 1

S-coord: THETA_S, THETA_B, hc (m)
6.0D0 6.0D0 50.0D0

initial: NRREC filename
0
none

restart: NRST, NRPFRST / filename
9999999 1
wpp_rst.nc

history: LDEFHIS, NWRT, NRPFHIS / filename
T 1 200
wpp_his.nc

primary_history_fields: zeta U,VBAR U,V wrtT(1:NT)
T T F F F T T T T

auxiliary_history_fields: rho Omega W Akv Akt Aks HBL HBBL
F F F F F F F F

wci_history_fields: sup ust2d vst2d ust vst wst Akvb Akvw kvf calP KapSrf
T T F F F F F F F F F

lin_EOS_cff:
1027.5 18.0 32.0 0.14 0.80
vertical_mixing:
0.e-6 0.e-6 0.e-6 0.e-6 0.e-6 0.e-6

bottom_drag: RDRG [m/s], RDRG2, Zob [m], Cdb_min, Cdb_max
0.E-4 0.0E-3 0.E-2 0.E-4 0.E-2

The equation of state, bottom drag and vertical mixing parameters are needed without no additional option are specified and set to zero. NRREC (initial) must be set to zero otherwise read_inp.F is looking for an input file to read. NDTFAST is the number of barotropic timesteps between each baroclinic timestep (it is not a timestep).

3 ROMS results

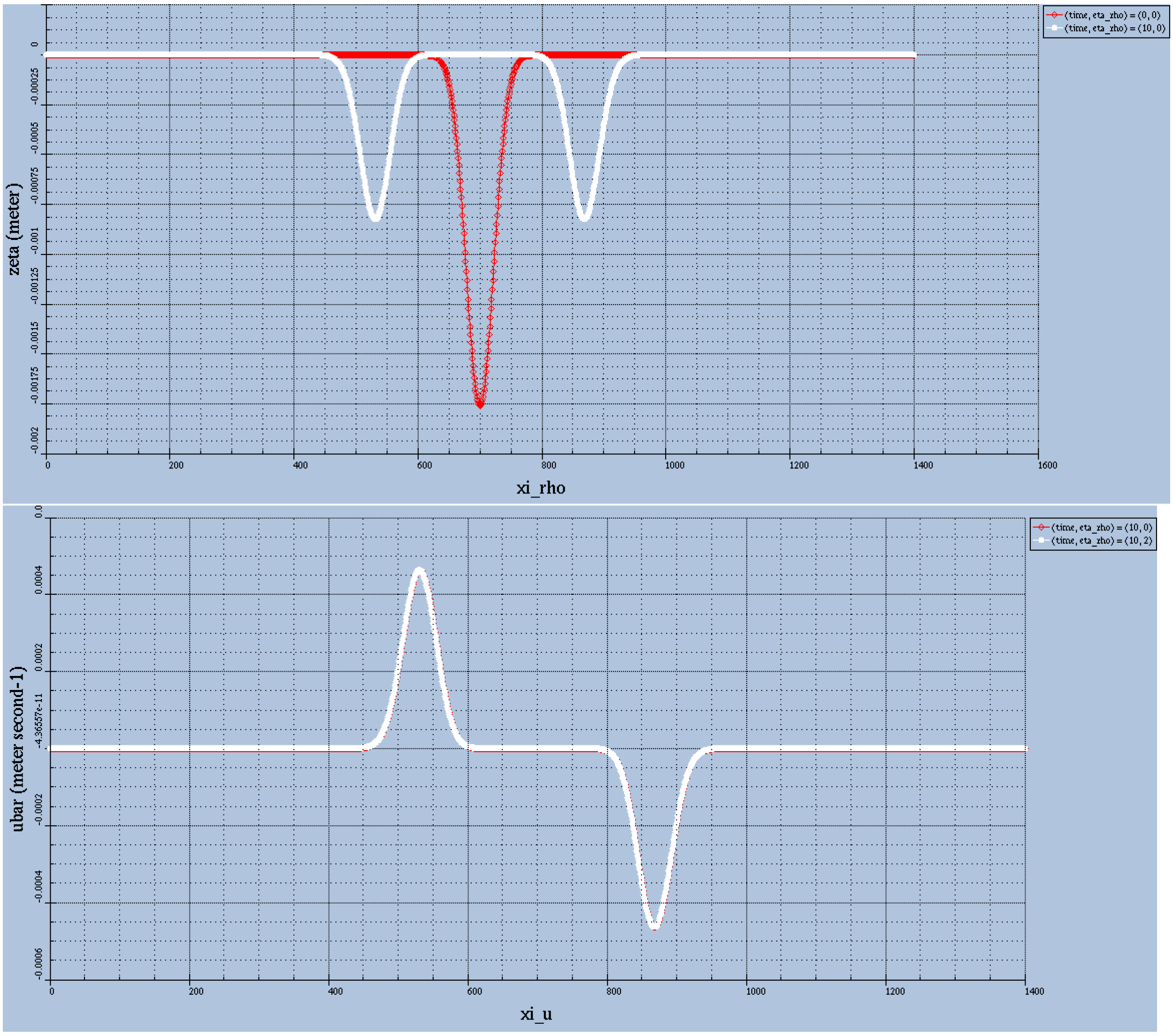

Figure 2: Free waves propagation, the option mrl_wci is off. Top panel: Sea surface elevation. Bottom panel: Barotropic u velocity component. After 10s (white lines), the peaks are displaced of $10 \times C^{\mathrm{lw}} \sim 172$ m. In this case, we set $h = 30$ m.

An initial zeta gradient, as set in the initial conditions above, see the red line of the top panel of the Fig. (2), generate free waves. They propagate Eastward and Westward with the velocity C^{lw}, see the white lines, same figure. These waves are characterized by a negative sea surface elevation whose amplitude decreases slowly in time. The Westward component is characterized by $\bar{u} > 0$ and the Eastward one by $\bar{u} < 0$.

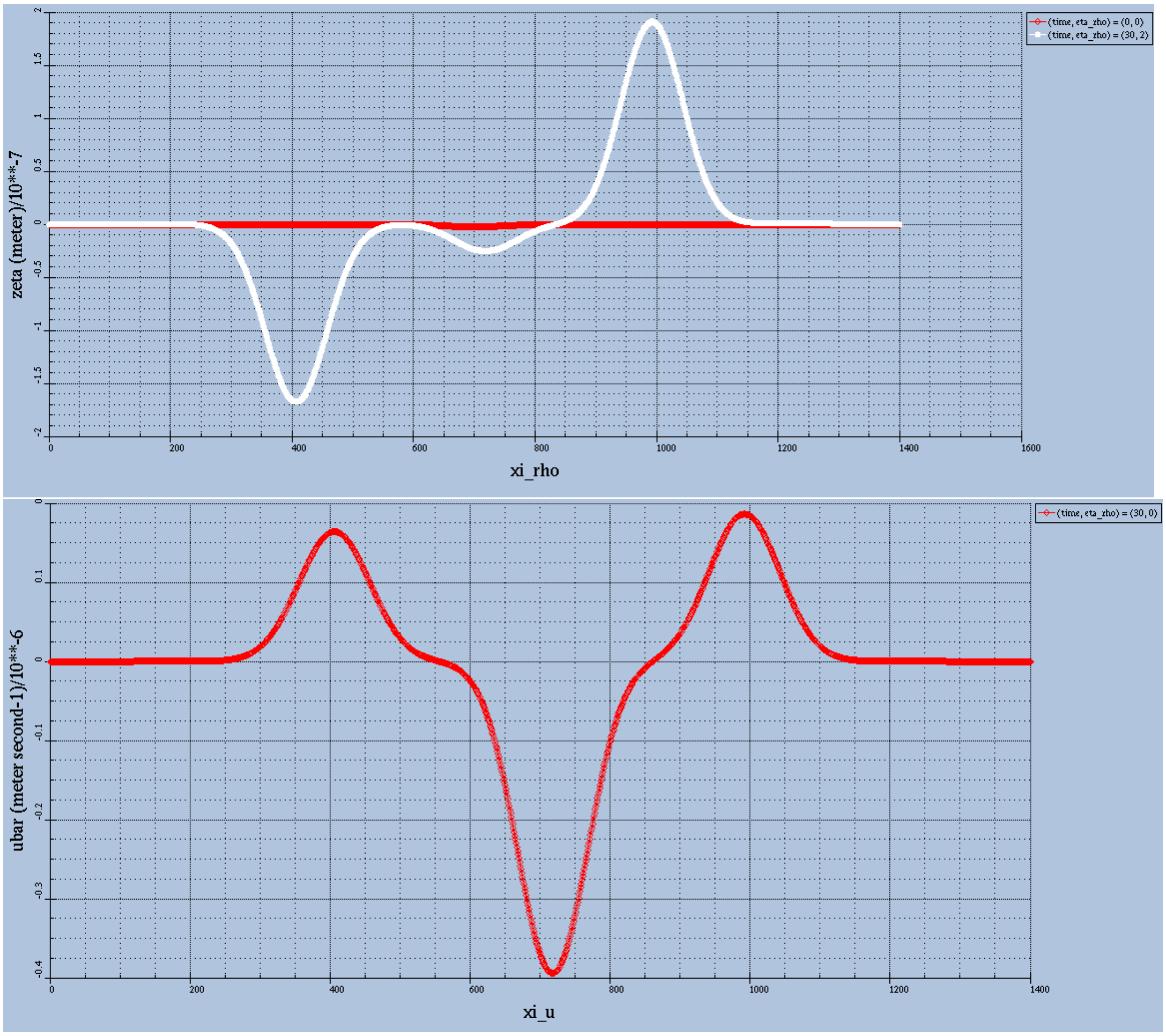

Figure 3: Free waves propagation, the option mrl_wci is on. Top panel: Sea surface elevation. Bottom panel: Barotropic u velocity component. After 30s (white lines), the peaks of the free waves are displaced of $30 \times C^{\mathrm{lw}} \sim 297$ m. In this case, we set $h = 10$ m. The peak of the wave packet envelope is displaced by $30 * C_g \sim 19$ m

We force the wave packet propagation at the velocity C_g. It is therefor necessary that C^{lw} and C_g to be different enough to separate these two signals: the free waves and the forced ones which are the ones we are interested in. We choose the variables $k = 2\pi$ m^{-1} corresponding to a wavelength $\lambda = 1$m, i.e. the short wind waves see deep waters. But we set $env = 0.0001$ for the envelope length scale to be very large compared to the depth h. It allows to have C^{lw} larger than C_g enough (for $h = 10$ m, $C^{\mathrm{lw}} = 9.9$ ms^{-1} and $C_g = 0.6$ ms^{-1}). The asymptotic parameter of McWilliams & al. 2004 theory is the wave steepness $\varepsilon = Ak$ so we use a small wave amplitude $A = 1$mm.

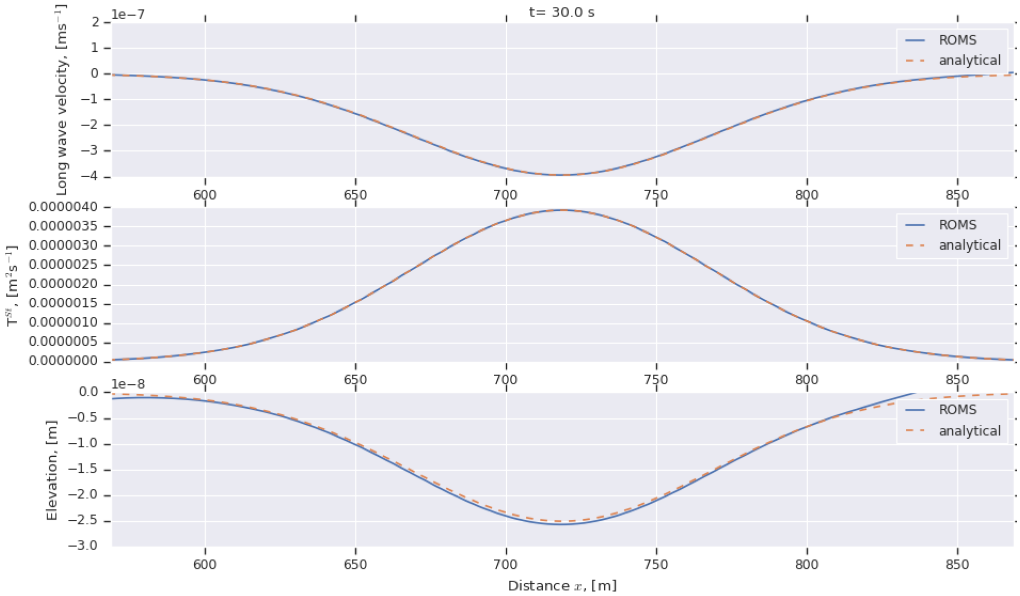

Figure 4: First line: Long-wave velocity. Second line: vertically intergrated Stokes drifts. Third line: Sea-surface elevation. Blue lines are the solutions from ROMS and dashed orange lines are the analytical solutions.

The Fig. (4) shows that we reached an asymptotical agreement between the numerical ROMS solution and the analytical solutions.

Appendix

3.1 Useful properties of the hyperbolic functions

$$\sinh x = x + \frac{x^3}{3!} + \mathcal{O}(x^5)\ , \quad \cosh x = 1 + \frac{x^2}{2!} + \mathcal{O}(x^4) \tag{30}$$

$$\tanh x = x - \frac{x^3}{3} + \mathcal{O}(x^5)\ , \quad \sinh^2 x = x^2 + \frac{x^4}{3} + \mathcal{O}(x^6) \tag{31}$$