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Modelling Firm-Product Level Trade: A Multi-Dimensional Random Effects Panel Data Approach

by

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Abstract

The paper deals with the problems of formalizing econometric models on firm-product level trade data sets, or similar economic flows. A multi-dimensional random effects panel data approach is adopted. Several models are introduced taking into account different types of specific effects, interactions and cross correlations. The respective covariance matrixes are derived, as well as procedures to estimate the unknown variance and covariance components, in order to make the Feasible Generalized Least Squares estimation operational. Whenever possible, the spectral decomposition of the covariance matrixes is also provided to make the estimation procedure simpler to implement. Both balanced and unbalanced data sets are considered.

Key words: panel data, multidimensional panel data, random effects, error components model, trade model, gravity model, firm and product level data, micro trade data.

JEL classification: C1, C2, C4, F17, F47

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1. Introduction

The last decade or so, many new, large and very large socio-economic data sets have started to emerge. One area, where these large data sets have been compiled, relates to trade, where micro, firm, product and shipment level observations have become more readily available. Such data sets often present themselves in the form higher dimensional panels. While three-dimensional panel data, mostly related to macro trade, and other macro economic flows are now better understood (see, for example, *Matyas and Balazsi* [2012] and *Matyas, Hornok and Pus* [2012]), in higher dimensions, with potentially extremely large number of observations, the models and, even the simplest, estimation methods can quickly become very complex.

This paper focuses on four-dimensional (4D) panel data models and their estimation, with some outlook on higher dimensions as well. These 4D data sets, in a trade context, typically consist of firm or disaggregated sector (called in short firm) and product (or, again sector, called in short product) level observations, best formalized in a four-dimensional panel data model framework (see, for example, *Bekes and Murakozi* [2012], *Corcos et al.* [2012], *Gorg et al.* [2010], *Berthou and Fontagne* [2009], and *Defever and Toubal* [2007]). As the numbers of observation usually in these cases is (very) large, a fixed effects approach would mean the explicit or implicit inclusion into the model of tens of thousands of additional parameters and dummy variables. This would look in fact very much like a case of over-fitting, mostly "destroying" the explanatory power of any other variables. And also, in these higher dimension, different fixed effects specifications can substantially change the estimation results (see, for example, *Arkolakis and Muendler* [2009]), so instead, in this paper, we introduce and analyse several appropriate random effects model specifications. The models considered are extended to include across firms or products cross correlations, and are adapted to deal with problems related to unbalanced data as well.

The framework of the analysis remains, throughout the paper, within the standard panel data approach. While it can be argued, for example, that the number of products traded by a firm is in itself endogenous, as this is in fact one of our four data dimensions, it will be tread as given. In general, all data dimensions are going to be treated as fixed and given. Similarly, the balanced and unbalanced data panels can be defined, unlike in the "usual" panel data case, in several different ways. So we are going to apply some restrictions on the assumed data structure which best characterize these micro trade data sets.

2. The Model Specifications Considered

In this paper we introduce different types of random effects model specifications suited for this four-dimensional panel data approach, derive proper estimation methods for each of them and analyze their properties under different data structures.

The baseline model to be considered is

$$y_{ijst} = \beta' x_{IS,t} + u_{ijst}$$

where x are the explanatory variables of the model, β are the the unknown parameters, u are the idiosyncratic disturbance terms (which are assumed to be uncorrelated with the explanatory variables x), IS is the time-invariant Index Set (it can be: ijs, ij, is, js or just have a single index for the explanatory variables, and can be different for different for each one of them). It is important to be specific about the sample size and sample structure. First of all, the baseline sample structure, let us call it the balanced one, is when $i = 1, \ldots, N^{(1)}$, $j = 1, \ldots, N^{(2)}$, $s = 1, \ldots, N^{(3)}$ and $t = 1, \ldots, T, \forall i, s, j$. Let us note here that a "real" balanced sample would mean, as in *Davis* [2002], that $N^{(1)} = N^{(2)} = N^{(3)}$, but in our case this makes little sense, as essentially we are dealing here with micro trade data. Typically i stands for firms, j for products (or disaggregated sectors like ISIC level 1, 2 or 3 classifications), and s for the trade destination countries. The unbalanced data structure considered in this paper is quite general as most of these data sets present themselves in "granular" form (looking like an Emmental cheese with high density of missing data bubbles). Individual time series may not only be of different length, but may also have no-observation holes in it. This can only be handled by the introduction of different index sets, referring to different groups of observations, which makes the analytical treatment of these cases quite complex.

As we are using here a random effects approach, the different model specifications are characterized by different structures of the disturbance terms u_{ijst}. *Moulton* [1990] already pointed out more than two decades ago, how important these structures are in fact from a practical point of view. For each of them we derive the covariance matrix of the model, and then proper estimators for its variance and covariance components in order to be able to use the Feasible GLS (FGLS) estimator to estimate the unknown parameters of the model.

The simplest model considered is a straight generalization of the "usual" error components panel data model (see, for example, *Matyas* [1997], and *Baltagi et al.* [2008])

$$u_{ijst} = \mu_i + \gamma_j + \alpha_s + \lambda_t + \varepsilon_{ijst} \tag{1}$$

where the error components are pair-wise uncorrelated, have zero expected values, and their variances are

$$E(\mu_i\mu_{i'}) = \begin{cases} \sigma_\mu^2 & \text{if } i = i' \\ 0 & \text{otherwise} \end{cases} ; \qquad E(\gamma_j\gamma_{j'}) = \begin{cases} \sigma_\gamma^2 & \text{if } j = j' \\ 0 & \text{otherwise} \end{cases}$$

$$E(\alpha_s\alpha_{s'}) = \begin{cases} \sigma_\alpha^2 & \text{if } s = s' \\ 0 & \text{otherwise} \end{cases} ; \qquad E(\lambda_t\lambda_{t'}) = \begin{cases} \sigma_\lambda^2 & \text{if } t = t' \\ 0 & \text{otherwise} \end{cases}$$

The next model to be considered is

$$u_{ijst} = \mu_{ijs} + \varepsilon_{ijst} \tag{2}$$

$$E(\mu_{ijs}\mu_{i'j's'}) = \begin{cases} \sigma_\mu^2 & \text{if } i = i',\ j = j' \text{ and } s = s' \\ 0 & \text{otherwise} \end{cases}$$

This is the "usual" panel data model with random individual effects, where the individual effects correspond to the (ijs) triplets. An extended version of this model is

$$u_{ijst} = \mu_{ijs} + \lambda_t + \varepsilon_{ijst} \tag{3}$$

where

$$E(\lambda_t\lambda_{t'}) = \begin{cases} \sigma_\lambda^2 & \text{if } t = t' \\ o & \text{otherwise} \end{cases}$$

The next model to be considered is with pair-wise interaction effects

$$u_{ijst} = \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + \varepsilon_{ijst} \tag{4}$$

where $\mu_{ij}^{(1)}$, $\mu_{is}^{(2)}$, $\mu_{js}^{(3)}$ and ε_{ijst} are pair-wise uncorrelated, have zero expected value, and

$$E(\mu_{ij}\mu_{i'j'}) = \begin{cases} \sigma_\mu^{(1)^2} & \text{if } i = i', \text{ and } j = j' \\ 0 & \text{otherwise} \end{cases}$$

$$E(\mu_{is}\mu_{i's'}) = \begin{cases} \sigma_\mu^{(2)^2} & \text{if } i = i', \text{ and } s = s' \\ 0 & \text{otherwise} \end{cases}$$

$$E(\mu_{js}\mu_{j's'}) = \begin{cases} \sigma_\mu^{(3)^2} & \text{if } j = j', \text{ and } s = s' \\ 0 & \text{otherwise} \end{cases}$$

The next model is the extension of model (4) with a time effect

$$u_{ijst} = \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + \lambda_t + \varepsilon_{ijst} \tag{5}$$

Another form of heterogeneity is to use individual-time-varying effects. This in fact is the generalization of the approach used in multilevel modeling, see for example, *Snijders and Boske* [1999], *Ebbes, Bockenholt and Wedel* [2004], *Hubler* [2006] or *Gelman* [2006]. In this case model (2) is extended with pair-wise split individual specific time effects

$$u_{ijst} = \mu_{ijs} + v^{(1)}_{it} + v^{(2)}_{jt} + v^{(3)}_{st} + \varepsilon_{ijst} \tag{6}$$

where $v^{(1)}_{it}$, $v^{(2)}_{jt}$, $v^{(3)}_{st}$ and ε_{ijst} are pair-wise uncorrelated, have zero expected value, and

$$E(v_{it}v_{i't'}) = \begin{cases} \sigma_v^{(1)^2} & \text{if } i = i', \text{ and } t = t' \\ 0 & \text{otherwise} \end{cases}$$

$$E(v_{jt}v_{j't'}) = \begin{cases} \sigma_v^{(2)^2} & \text{if } j = j', \text{ and } t = t' \\ 0 & \text{otherwise} \end{cases}$$

$$E(v_{st}v_{s't'}) = \begin{cases} \sigma_v^{(3)^2} & \text{if } s = s', \text{ and } t = t' \\ 0 & \text{otherwise} \end{cases}$$

And finally, the last model to be considered is an all-encompassing model with

$$u_{ijst} = \mu^{(1)}_{ij} + \mu^{(2)}_{is} + \mu^{(3)}_{js} + v^{(1)}_{it} + v^{(2)}_{jt} + v^{(3)}_{st} + \varepsilon_{ijst} \tag{7}$$

In some cases it is important to deal with cross-correlations as well. The cross-correlations to be considered are for models (2), (3) and (6)

$$E(\mu_{ijs}\mu_{i'j's'}) = \begin{cases} \sigma_\mu^2 & i = i',\ j = j' \text{ and } s = s' \\ \rho_{(1)} & i \neq i',\ j = j' \text{ and } s = s' \\ \rho_{(2)} & i = i',\ j \neq j' \text{ and } s = s' \\ \rho_{(3)} & i = i',\ j = j' \text{ and } s \neq s' \\ 0 & \text{otherwise} \end{cases}$$

and for models (4), (5) and (7)

$$E(\mu_{ij}\mu_{i'j'}) = \begin{cases} \sigma_\mu^{(1)^2} & i = i' \text{ and } j = j' \\ \rho^{(1)}_{(1)} & i \neq i' \text{ and } j = j' \\ \rho^{(1)}_{(2)} & i = i' \text{ and } j \neq j' \\ 0 & \text{otherwise} \end{cases}$$

$$E(\mu_{is}\mu_{i's'}) = \begin{cases} \sigma_\mu^{(2)^2} & i = i' \text{ and } s = s' \\ \rho^{(2)}_{(1)} & i \neq i' \text{ and } s = s' \\ \rho^{(2)}_{(2)} & i = i' \text{ and } s \neq s' \\ 0 & \text{otherwise} \end{cases}$$

and

$$E(\mu_{js}\mu_{j's'}) = \begin{cases} \sigma_{\mu}^{(3)^2} & j = j' \text{ and } s = s' \\ \rho_{(1)}^{(3)} & j \neq j' \text{ and } s = s' \\ \rho_{(2)}^{(3)} & j = j' \text{ and } s \neq s' \\ 0 & \text{otherwise} \end{cases}$$

3. Covariance Matrixes and the Estimation of the Variance Components

As it is well known, the most efficient way to estimate the models introduced in Section 2 is through the Feasible GLS estimator. First, starting with the balanced case, we need to derive the covariance matrix of each model (1) - (7). However, given the four dimensions, the sample size can become very large quite quickly, meaning that the inverse of the covariance matrix (needed to perform a the FGLS estimation) frequently cannot easily be calculated in practice. To overcome this problem we also derive the spectral decomposition for each covariance matrix, which makes the inverse operation much more handy to perform. Then, we estimate the unknown variance and covariance components of the respective covariance matrixes in order to make the FGLS operational.

3.1 Covariance Matrixes of the Different Models

As the four dimensional setup makes the matrix algebra a bit more complex, we need to introduce some new notations upfront. Let us make the following definitions:

$$\begin{aligned} B_i &= I_{N^{(1)}} \otimes \frac{J_{N^{(2)}N^{(3)}T}}{N^{(2)}N^{(3)}T} \\ B_j &= \frac{J_{N^{(1)}}}{N^{(1)}} \otimes I_{N^{(2)}} \otimes \frac{J_{N^{(3)}T}}{N^{(3)}T} \\ B_s &= \frac{J_{N^{(1)}N^{(2)}}}{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}} \otimes \frac{J_T}{T} \\ B_t &= \frac{J_{N^{(1)}N^{(2)}N^{(3)}}}{N^{(1)}N^{(2)}N^{(3)}} \otimes I_T \\ B_{ij} &= I_{N^{(1)}N^{(2)}} \otimes \frac{J_{N^{(3)}T}}{N^{(3)}T} \\ B_{is} &= I_{N^{(1)}} \otimes \frac{J_{N^{(2)}}}{N^{(2)}} \otimes I_{N^{(3)}} \otimes \frac{J_T}{T} \\ B_{js} &= \frac{J_{N^{(1)}}}{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes \frac{J_T}{T} \\ B_{it} &= I_{N^{(1)}} \otimes \frac{J_{N^{(2)}N^{(3)}}}{N^{(2)}N^{(3)}} \otimes I_T \end{aligned}$$

$$B_{jt} = \frac{J_{N^{(1)}}}{N^{(1)}} \otimes I_{N^{(2)}} \frac{J_{N^{(3)}}}{N^{(3)}} \otimes I_T$$
$$B_{st} = \frac{J_{N^{(1)}N^{(2)}}}{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}T}$$
$$B_{ijs} = I_{N^{(1)}N^{(2)}N^{(3)}} \otimes \frac{J_T}{T}$$
$$B_{ijt} = I_{N^{(1)}N^{(2)}} \otimes \frac{J_{N^{(3)}}}{N^{(3)}} \otimes I_T$$
$$B_{jst} = \frac{J_{N^{(1)}}}{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}T}$$
$$B_{ist} = I_{N^{(1)}} \otimes \frac{J_{N^{(2)}}}{N^{(2)}}$$
$$J = \frac{J_{N^{(1)}N^{(2)}N^{(3)}T}}{N^{(1)}N^{(2)}N^{(3)}T}$$
$$I = I_{N^{(1)}N^{(2)}N^{(3)}T}$$
$$B_i^* = I_{N^{(1)}} \otimes \frac{J_{N^{(2)}N^{(3)}}}{N^{(2)}N^{(3)}}$$
$$B_j^* = \frac{J_{N^{(1)}}}{N^{(1)}} \otimes I_{N^{(2)}} \otimes \frac{J_{N^{(3)}}}{N^{(3)}}$$
$$B_s^* = \frac{J_{N^{(1)}N^{(2)}}}{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}}$$
$$B_{ij}^* = I_{N^{(1)}N^{(2)}} \otimes \frac{J_{N^{(3)}}}{N^{(3)}}$$
$$B_{is}^* = I_{N^{(1)}} \otimes \frac{J_{N^{(2)}}}{N^{(2)}} \otimes I_{N^{(3)}}$$
$$B_{js}^* = \frac{J_{N^{(1)}}}{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}}$$
$$J^* = \frac{J_{N^{(1)}N^{(2)}N^{(3)}}}{N^{(1)}N^{(2)}N^{(3)}}$$
$$I^* = I_{N^{(1)}N^{(2)}N^{(3)}}$$

Model (1)

To derive the covariance matrix of model (1) we start from composite disturbance term

$$u_{ijst} = \mu_i + \gamma_j + \alpha_s + \lambda_t + \epsilon_{ijst}$$

For all T observations we get

$$\begin{aligned} u_{ijs} &= \mu_i \otimes l_T + \gamma_j \otimes l_T + \alpha_s \otimes l_T + \lambda + \epsilon_{ijs} \\ E[u_{ijs}u'_{ijs}] &= E[(\mu_i \otimes l_T)(\mu_i \otimes l_T)'] + E[(\gamma_j \otimes l_T)(\gamma_j \otimes l_T)'] + \\ &\quad + E[(\alpha_s \otimes l_T)(\alpha_s \otimes l_T)'] + E[\lambda\lambda'] + E[\epsilon_{ijs}\epsilon'_{ijs}] = \\ &= \sigma_\mu^2 J_T + \sigma_\gamma^2 J_T + \sigma_\alpha^2 J_T + \sigma_\lambda^2 I_T + \sigma_\epsilon^2 I_T \end{aligned}$$

Continuing this building up of the observations for the s index and then the j and i indexes as well, we get

$$u_{ij} = \mu_i \otimes l_T \otimes l_{N^{(3)}} + \gamma_j \otimes l_T \otimes l_{N^{(3)}} + \alpha \otimes l_T + l_{N^{(3)}} \otimes \lambda + \epsilon_{ij}$$
$$E[u_{ij}u'_{ij}] = \sigma^2_\mu J_{N^{(3)}T} + \sigma^2_\gamma J_{N^{(3)}T} + \sigma^2_\alpha I_{N^{(3)}} \otimes J_T + \sigma^2_\lambda J_{N^{(3)}} \otimes I_T + \sigma^2_\epsilon I_{N^{(3)}T}$$

$$u_i = \mu_i \otimes l_T \otimes l_{N^{(3)}} \otimes l_{N^{(2)}} + \gamma \otimes l_T \otimes l_{N^{(3)}} + l_{N^{(2)}} \otimes \alpha \otimes l_T + l_{N^{(2)}} \otimes l_{N^{(3)}} \otimes \lambda + \epsilon_i$$
$$E[u_iu'_i] = \sigma^2_\mu J_{N^{(2)}N^{(3)}T} + \sigma^2_\gamma I_{N^{(2)}} \otimes J_{N^{(3)}T} + \sigma^2_\alpha J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T +$$
$$+ \sigma^2_\lambda J_{N^{(2)}N^{(3)}} \otimes I_T + \sigma^2_\epsilon I_{N^{(2)}N^{(3)}T}$$

So this gives finally

$$u = \mu \otimes l_T \otimes l_{N^{(3)}} \otimes l_{N^{(2)}} + l_{N^{(1)}} \otimes \gamma \otimes l_T \otimes l_{N^{(3)}} + l_{N^{(1)}} \otimes l_{N^{(2)}} \otimes \alpha \otimes l_T$$
$$+ l_{N^{(1)}} \otimes l_{N^{(2)}} \otimes l_{N^{(3)}} \otimes \lambda + \epsilon$$
$$E[uu'] = \sigma^2_\mu I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}T} + \sigma^2_\gamma J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}T} + \sigma^2_\alpha J_{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T +$$
$$+ \sigma^2_\lambda J_{N^{(1)}N^{(2)}N^{(3)}} \otimes I_T + \sigma^2_\epsilon I_{N^{(1)}N^{(2)}N^{(3)}T} = \Omega$$

where l is the vector of ones (all elements being 1) with its size in the index, J is the matrix of ones with its size in the index and I is the identity matrix, with its size in the index, and μ, α, γ, λ and ϵ are the vectors containing the elements of μ_i, γ_j, α_s, λ_t, and ϵ_{ijst} respectively.

Like in the usual panel data case let us work out the spectral decomposition of this matrix to simplify the inverse needed for the FGLS. Using the notation

$$C_{11} = B_i - J$$
$$C_{12} = B_j - J$$
$$C_{13} = B_s - J$$
$$C_{14} = B_t - J$$
$$W_1 = I - C_{11} - C_{12} - C_{13} - C_{14} - J$$

we get

$$\Omega = N^{(2)}N^{(3)}T\sigma^2_\mu (C_{11} + J) + N^{(1)}N^{(3)}T\sigma^2_\gamma (C_{12} + J) + N^{(2)}N^{(3)}T\sigma^2_\alpha (C_{13} + J) +$$
$$+ N^{(1)}N^{(2)}N^{(3)}\sigma^2_\lambda (C_{14} + J) + +\sigma^2_\epsilon (W_1 + C_{11} + C_{12} + C_{13} + C_{14} + J) =$$
$$= \left(N^{(2)}N^{(3)}T\sigma^2_\mu + N^{(1)}N^{(3)}T\sigma^2_\gamma + N^{(2)}N^{(3)}T\sigma^2_\alpha + N^{(1)}N^{(2)}N^{(3)}\sigma^2_\lambda + \sigma^2_\epsilon\right) J +$$
$$+ \left(N^{(2)}N^{(3)}T\sigma^2_\mu + \sigma^2_\epsilon\right) C_{11} + \left(N^{(1)}N^{(3)}T\sigma^2_\gamma + \sigma^2_\epsilon\right) C_{12} +$$
$$+ \left(N^{(2)}N^{(3)}T\sigma^2_\alpha + \sigma^2_\epsilon\right) C_{13} + \left(N^{(1)}N^{(2)}N^{(3)}\sigma^2_\lambda + \sigma^2_\epsilon\right) C_{14} + \sigma^2_\epsilon W_1$$

Now using

$$
\begin{aligned}
\theta_1 &= \frac{\sigma_\epsilon^2}{N^{(2)}N^{(3)}T\sigma_\mu^2 + \sigma_\epsilon^2} \\
\theta_2 &= \frac{\sigma_\epsilon^2}{N^{(1)}N^{(3)}T\sigma_\gamma^2 + \sigma_\epsilon^2} \\
\theta_3 &= \frac{\sigma_\epsilon^2}{N^{(1)}N^{(2)}T\sigma_\alpha^2 + \sigma_\epsilon^2} \\
\theta_4 &= \frac{\sigma_\epsilon^2}{N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2} \\
\theta_5 &= \frac{\sigma_\epsilon^2}{N^{(2)}N^{(3)}T\sigma_\mu^2 + N^{(1)}N^{(3)}T\sigma_\gamma^2 + N^{(1)}N^{(2)}T\sigma_\alpha^2 + N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2}
\end{aligned}
$$

we get for the inverse of the covariance matrix

$$
\begin{aligned}
\sigma_\epsilon^2\Omega^{-1} &= \theta_1 C_{11} + \theta_2 C_{12} + \theta_3 C_{13} + \theta_4 C_{14} + \theta_5 J + W_1 = \\
&= I - (1-\theta_1)\, B_i - (1-\theta_2)\, B_j - (1-\theta_3)\, B_s - (1-\theta_4)\, B_t + \\
&+ (3 - \theta_1 - \theta_2 - \theta_3 - \theta_4 + \theta_5)\, J
\end{aligned}
$$

Now this model is suited to deal with purely cross sectional data as well, that is when $T = 1$. In this case

$$
E\left[uu'\right] = \sigma_\mu^2 I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}} + \sigma_\gamma^2 J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}} + \sigma_\alpha^2 J_{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}} + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}} = \Omega
$$

with

$$
\begin{aligned}
C_{11}^* &= B_i^* - J^* \\
C_{12}^* &= B_j^* - J^* \\
C_{13}^* &= B_s^* - J^* \\
W_1^* &= I^* - C_{11}^* - C_{12}^* - C_{13}^* - J^*
\end{aligned}
$$

and we get

$$
\begin{aligned}
\Omega = N^{(2)}N^{(3)}\sigma_\mu^2 \left(C_{11}^* + J^*\right) + N^{(1)}N^{(3)}\sigma_\gamma^2 \left(C_{12}^* + J^*\right) + N^{(2)}N^{(3)}\sigma_\alpha^2 \left(C_{13}^* + J^*\right) + \\
+ \sigma_\epsilon^2 \left(W_1^* + C_{11}^* + C_{12}^* + C_{13}^* + J^*\right) = \\
= \left(N^{(2)}N^{(3)}\sigma_\mu^2 + N^{(1)}N^{(3)}\sigma_\gamma^2 + N^{(2)}N^{(3)}\sigma_\alpha^2 + \sigma_\epsilon^2\right) J^* + \left(N^{(2)}N^{(3)}\sigma_\mu^2 + \sigma_\epsilon^2\right) C_{11}^* + \\
+ \left(N^{(1)}N^{(3)}\sigma_\gamma^2 + \sigma_\epsilon^2\right) C_{12}^* + \left(N^{(2)}N^{(3)}\sigma_\alpha^2 + \sigma_\epsilon^2\right) C_{13}^* + \sigma_\epsilon^2 W_1^*
\end{aligned}
$$

Proceeding like in the panel data case above, with the notation

$$\theta_1^* = \frac{\sigma_\epsilon^2}{N^{(2)}N^{(3)}\sigma_\mu^2 + \sigma_\epsilon^2}$$
$$\theta_2^* = \frac{\sigma_\epsilon^2}{N^{(1)}N^{(3)}\sigma_\gamma^2 + \sigma_\epsilon^2}$$
$$\theta_3^* = \frac{\sigma_\epsilon^2}{N^{(1)}N^{(2)}\sigma_\alpha^2 + \sigma_\epsilon^2}$$
$$\theta_4^* = \frac{\sigma_\epsilon^2}{N^{(2)}N^{(3)}\sigma_\mu^2 + N^{(1)}N^{(3)}\sigma_\gamma^2 + N^{(1)}N^{(2)}\sigma_\alpha^2 + \sigma_\epsilon^2}$$

we get

$$\begin{aligned}\sigma_\epsilon^2\Omega^{-1} &= \theta_1^* C_{11}^* + \theta_2^* C_{12}^* + \theta_3^* C_{13}^* + \theta_4^* J^* + W_1^* = \\ &= I - (1-\theta_1^*)\,B_i^* - (1-\theta_2^*)\,B_j^* - (1-\theta_3^*)\,B_s^* + (2-\theta_1^*-\theta_2^*-\theta_3^*+\theta_4^*)\,J^*\end{aligned}$$

Model (2)

Proceeding likewise for model (2) we first build up the covariance matrix

$$\begin{aligned}u_{ijst} &= \mu_{ijs} + \epsilon_{ijst}\\ u_{ijs} &= \mu_{ijs} \otimes l_T + \epsilon_{ijs}\\ E[u_{ijs}u_{ijs}'] &= \sigma_\mu^2 J_T + \sigma_\epsilon^2 I_T\\ u_{ij} &= \mu_{ij} \otimes l_T + \epsilon_{ij}\\ E[u_{ij}u_{ij}'] &= \sigma_\mu^2 I_{N^{(3)}} J_T + \sigma_\epsilon^2 I_{N^{(3)}T}\\ u_i &= \mu_i \otimes l_T + \epsilon_i\\ E[u_i u_i'] &= \sigma_\mu^2 I_{N^{(2)}N^{(3)}} \otimes J_T + \sigma_\epsilon^2 I_{N^{(2)}N^{(3)}T}\\ u &= \mu \otimes l_T + \epsilon\\ E[uu'] &= \sigma_\mu^2 I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T} = \Omega\end{aligned}$$

Then using

$$C_2 = B_{ijs} - J$$
$$W_2 = I - B_{ijs}$$

we get for the covariance matrix

$$\Omega = T\sigma_\mu^2\,(C_2 + J) + \sigma_\epsilon^2\,(W_2 + C_2 + J) = \left(T\sigma_\mu^2 + \sigma_\epsilon^2\right) J + \left(T\sigma_\mu^2 + \sigma_\epsilon^2\right) C_2 + \sigma_\epsilon^2 W_2$$

and for the spectral decomposition

$$\sigma_\epsilon^2 \Omega^{-1} = \theta J + \theta C_2 + W_2 = I - (1-\theta) B_{ijs}$$

with

$$\theta = \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + \sigma_\epsilon^2}$$

The GLS estimator then is

$$\hat{\beta}_{GLS} = \left[X' \left(I - (1-\theta) B_{ijs}\right) X\right]^{-1} X' \left(I - (1-\theta) B_{ijs}\right) y$$

The GLS estimator is in fact an OLS estimator on the transformed model, where all the variable of the model are transformed like $\tilde{y}_{ijst} = y_{ijst} - (1-\theta)\sum_{t=1}^{T} \frac{1}{T} y_{ijst}$.

Model (3)

Proceeding in the same way as above for model (3) we get

$$\begin{aligned} u_{ijst} &= \mu_{ijs} + \lambda_t + \epsilon_{ijst} \\ E[uu'] &= \sigma_\mu^2 I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T + \sigma_\lambda^2 J_{N^{(1)}N^{(2)}N^{(3)}} \otimes I_T + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T} = \Omega \end{aligned}$$

and so

$$\begin{aligned} \Omega &= T\sigma_\mu^2 \left(C_{32} + J\right) + N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 \left(C_{31} + J\right) + \sigma_\epsilon^2 \left(W_3 + C_{31} + C_{32} + J\right) = \\ &= \left(T\sigma_\mu^2 + N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2\right) J + \left(T\sigma_\mu^2 + \sigma_\epsilon^2\right) C_{32} + \left(N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2\right) C_{31} + \sigma_\epsilon^2 W_3 \end{aligned}$$

with

$$\begin{aligned} C_{31} &= B_t - J \\ C_{32} &= B_{ijs} - J \\ W_3 &= I - C_{31} - C_{32} - J \end{aligned}$$

For the spectral decomposition we get

$$\sigma_\epsilon^2 \Omega^{-1} = \theta_1 J + \theta_2 C_{32} + \theta_3 C_{31} + W_3 = I - (1-\theta_2) B_{ijs} - (1-\theta_3) B_t + (1 + \theta_1 - \theta_2 - \theta_3) J$$

with

$$\begin{aligned} \theta_1 &= \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2} \\ \theta_2 &= \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + \sigma_\epsilon^2} \\ \theta_3 &= \frac{\sigma_\epsilon^2}{N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2} \end{aligned}$$

Model (4)

For model (4) we get

$$
\begin{aligned}
u_{ijst} &= \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + \epsilon_{ijst} \\
E[uu'] &= \sigma_\mu^{(1)^2} I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} + \sigma_\mu^{(2)^2} I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T + \\
&\quad + \sigma_\mu^{(3)^2} J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}
\end{aligned}
$$

and

$$
\begin{aligned}
\Omega &= N^{(3)}T\sigma_\mu^{(1)^2}\left(C_{41} + B_i + B_j - J\right) + N^{(2)}T\sigma_\mu^{(2)^2}\left(C_{42} + B_i + B_s - J\right) + \\
&\quad + N^{(1)}T\sigma_\mu^{(3)^2}\left(C_{43} + B_j + B_s - J\right) + \sigma_\epsilon^2\left(W_4 + B_i + B_j + B_s + C41 + C_{42} + C_{43} - 2J\right) \\
&= \sigma_\epsilon^2 W_4 + + \left(N^{(3)}T\sigma_\mu^{(1)^2} + \sigma_\epsilon^2\right) C_{41} + \left(N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right) C_{42} + \\
&\quad + \left(N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) C_{43} + \left(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right) B_i + \\
&\quad + \left(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) B_j + \left(N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) B_s - \\
&\quad - \left(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + 2\sigma_\epsilon^2\right) J
\end{aligned}
$$

with

$$
\begin{aligned}
C_{41} &= B_{ij} - B_i - B_j + J \\
C_{42} &= B_{is} - B_i - B_s + J \\
C_{43} &= B_{js} - B_j - B_s + J \\
W_4 &= I - B_i - B_j - B_s - C_{41} - C_{42} - C_{43} + 2J
\end{aligned}
$$

The spectral decomposition now is

$$
\begin{aligned}
\sigma_\epsilon^2\Omega^{-1} &= W_4 + \theta_1 C_{41} + \theta_2 C_{42} + \theta_3 C_{43} + \theta_4 B_i + \theta_5 B_j + \theta_6 B_s - \theta_7 J = \\
&= I - (1-\theta_1) B_{ij} - (1-\theta_2) B_{is} - (1-\theta_3) B_{js} + (1-\theta_1-\theta_2+\theta_4) B_i + \\
&\quad + (1-\theta_1-\theta_3+\theta_5) B_j + (1-\theta_2-\theta_3+\theta_6) B_s - (1-\theta_1-\theta_2-\theta_3+\theta_7) J
\end{aligned}
$$

with

$$\theta_1 = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + \sigma_\epsilon^2}$$

$$\theta_2 = \frac{\sigma_\epsilon^2}{N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2}$$

$$\theta_3 = \frac{\sigma_\epsilon^2}{N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_4 = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2}$$

$$\theta_5 = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_6 = \frac{\sigma_\epsilon^2}{N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_7 = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + 2\sigma_\epsilon^2}$$

This model seems to be the perfect choice when one is dealing with cross sectional data. In this case

$$\begin{aligned} E[uu'] = \sigma_\mu^{(1)^2} I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}} + \sigma_\mu^{(2)^2} I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} + \\ + \sigma_\mu^{(3)^2} J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}} = \Omega \end{aligned}$$

with the notation

$$\begin{aligned} C_{41}^* &= B_{ij}^* - B_i^* - B_j^* + J^* \\ C_{42}^* &= B_{is}^* - B_i^* - B_s^* + J^* \\ C_{43}^* &= B_{js}^* - B_j^* - B_s^* + J^* \\ W_4^* &= I^* - B_i^* - B_j^* - B_s^* - C_{41}^* - C_{42}^* - C_{43}^* + 2J^* \end{aligned}$$

we get

$$\begin{aligned} \Omega = {} & N^{(3)}\sigma_\mu^{(1)^2}\left(C_{41}^* + B_i^* + B_j^* - J^*\right) + N^{(2)}\sigma_\mu^{(2)^2}\left(C_{42}^* + B_i^* + B_s^* - J^*\right) + \\ & + N^{(1)}\sigma_\mu^{(3)^2}\left(C_{43}^* + B_j^* + B_s^* - J^*\right) + \\ & + \sigma_\epsilon^2\left(W_4^* + B_i^* + B_j^* + B_s^* + C_{41}^* + C_{42}^* + C_{43}^* - 2J^*\right) \\ = {} & \sigma_\epsilon^2 W_4^* + \left(N^{(3)}\sigma_\mu^{(1)^2} + \sigma_\epsilon^2\right) C_{41}^* + \left(N^{(2)}\sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right) C_{42}^* + \\ & + \left(N^{(1)}\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) C_{43}^* + \left(N^{(3)}\sigma_\mu^{(1)^2} + N^{(2)}\sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right) B_i^* + \\ & + \left(N^{(3)}\sigma_\mu^{(1)^2} + N^{(1)}\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) B_j^* + \left(N^{(2)}\sigma_\mu^{(2)^2} + N^{(1)}\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) B_s^* - \\ & - \left(N^{(3)}\sigma_\mu^{(1)^2} + N^{(2)}\sigma_\mu^{(2)^2} + N^{(1)}\sigma_\mu^{(3)^2} + 2\sigma_\epsilon^2\right) J^* \end{aligned}$$

Introducing a similar notation than earlier

$$\theta_1^* = \frac{\sigma_\epsilon^2}{N^{(3)}\sigma_\mu^{(1)^2} + \sigma_\epsilon^2}$$

$$\theta_2^* = \frac{\sigma_\epsilon^2}{N^{(2)}\sigma_\mu^{(2)^2} + \sigma_\epsilon^2}$$

$$\theta_3^* = \frac{\sigma_\epsilon^2}{N^{(1)}\sigma_\mu^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_4^* = \frac{\sigma_\epsilon^2}{N^{(3)}\sigma_\mu^{(1)^2} + N^{(2)}\sigma_\mu^{(2)^2} + \sigma_\epsilon^2}$$

$$\theta_5^* = \frac{\sigma_\epsilon^2}{N^{(3)}\sigma_\mu^{(1)^2} + N^{(1)}\sigma_\mu^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_6^* = \frac{\sigma_\epsilon^2}{N^{(2)}\sigma_\mu^{(2)^2} + N^{(1)}\sigma_\mu^{(3)^2} + \sigma_\epsilon^2}$$

$$\theta_7^* = \frac{\sigma_\epsilon^2}{N^{(3)}\sigma_\mu^{(1)^2} + N^{(2)}\sigma_\mu^{(2)^2} + N^{(1)}\sigma_\mu^{(3)^2} + 2\sigma_\epsilon^2}$$

we get

$$\begin{aligned}
\sigma_\epsilon^2\Omega^{-1} &= W_4^* + \theta_1^* C_{41}^* + \theta_2^* C_{42}^* + \theta_3^* C_{43}^* + \theta_4^* B_i^* + \theta_5^* B_j^* + \theta_6^* B_s^* - \theta_7^* J^* = \\
&= I^* - (1-\theta_1^*)\, B_{ij}^* - (1-\theta_2^*)\, B_{is}^* - (1-\theta_3^*)\, B_{js}^* + (1-\theta_1^* - \theta_2^* + \theta_4^*)\, B_i^* + \\
&\quad\; + (1-\theta_1^* - \theta_3^* + \theta_5^*)\, B_j^* + (1-\theta_2^* - \theta_3^* + \theta_6^*)\, B_s^* - \\
&\quad\; - (1-\theta_1^* - \theta_2^* - \theta_3^* + \theta_7^*)\, J^*
\end{aligned}$$

Model (5)

For model (5) we get

$$\begin{aligned}u_{ijst} &= \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + \lambda_t + \epsilon_{ijst}\\ E[uu'] &= \Omega = \sigma_\mu^{(1)^2} I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} + \sigma_\mu^{(2)^2} I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T\\ &\quad + \sigma_\mu^{(3)^2} J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T + \sigma_\lambda^2 J_{N^{(1)}N^{(2)}N^{(3)}} \otimes I_T + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}\end{aligned}$$

This leads to

$$\begin{aligned}\Omega &= N^{(3)}T\sigma_\mu^{(1)^2}\left(C_{51} + B_i + B_j\right) + N^{(2)}T\sigma_\mu^{(2)^2}\left(C_{52} + B_i + B_s\right) + N^{(1)}T\sigma_\mu^{(3)^2}\left(C_{53} + B_s + B_j\right) +\\ &+ N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 C_{54} + \sigma_\epsilon^2\left(W_5 + C_{51} + C_{52} + C_{53} + C_{54} + B_i + B_j + B_s\right) =\\ &= \sigma_\epsilon^2 W_5 + \left(N^{(3)}T\sigma_\mu^{(1)^2} + \sigma_\epsilon^2\right)C_{51} + \left(N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right)C_{52} + \left(N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right)C_{53} +\\ &+ \left(N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2\right)C_{54} + \left(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right)B_i +\\ &+ \left(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right)B_j + \left(N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right)B_s\end{aligned}$$

with

$$\begin{aligned}C_{51} &= B_{ij} - B_i - B_j\\ C_{52} &= B_{is} - B_s - B_i\\ C_{53} &= B_{js} - B_s - B_j\\ C_{54} &= B_t\\ W_5 &= I - C_{51} - C_{52} - C_{53} - C_{54} - B_i - B_j - B_s\end{aligned}$$

and the spectral decomposition is

$$\begin{aligned}\sigma_\epsilon^2\Omega^{-1} &= W_5 + \theta_1 C_{51} + \theta_2 C_{52} + \theta_3 C_{53} + \theta_4 C_{54} + \theta_5 B_i + \theta_6 B_j + \theta_7 B_s =\\ &= I - (1-\theta_1)B_{ij} - (1-\theta_2)B_{is} - (1-\theta_3)B_{js} - (1-\theta_4)B_t + (1-\theta_1-\theta_2+\theta_5)B_i +\\ &+ (1-\theta_1-\theta_3+\theta_6)B_j + (1-\theta_2-\theta_3+\theta_7)B_s\end{aligned}$$

with

$$\theta_1 = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + \sigma_\epsilon^2}$$
$$\theta_2 = \frac{\sigma_\epsilon^2}{N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2}$$
$$\theta_3 = \frac{\sigma_\epsilon^2}{N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2}$$
$$\theta_4 = \frac{\sigma_\epsilon^2}{N^{(1)}N^{(2)}N^{(3)}\sigma_\lambda^2 + \sigma_\epsilon^2}$$
$$\theta_5 = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2}$$
$$\theta_6 = \frac{\sigma_\epsilon^2}{N^{(3)}T\sigma_\mu^{(1)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + +\sigma_\epsilon^2}$$
$$\theta_7 = \frac{\sigma_\epsilon^2}{N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2}$$

Model (6)

For model (6) we get

$$\begin{aligned}
u_{ijst} &= \mu_{ijs} + v_{it}^{(1)} + v_{jt}^{(2)} + v_{st}^{(3)} + \epsilon_{ijst} \\
E[uu'] &= \Omega = \sigma_\mu^2 I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T + \sigma_v^{(1)^2} I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}} \otimes I_T + \\
&\quad + \sigma_v^{(2)^2} J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}} \otimes I_T + \sigma_v^{(3)^2} J_{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}T} + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}
\end{aligned}$$

leading to

$$\begin{aligned}
\Omega &= T\sigma_\mu^2 \left(C_{61} + B_i + B_j + B_s - J\right) + N^{(2)}N^{(3)}\sigma_v^{(1)^2} \left(C_{62} + B_i + B_t - J\right) + \\
&+ N^{(1)}N^{(3)}\sigma_v^{(2)^2} \left(C_{63} + B_j + B_t - J\right) + N^{(1)}N^{(2)}\sigma_v^{(3)^2} \left(C_{64} + B_s + B_t - J\right) + \\
&+ \sigma_\epsilon^2 \left(W_6 + C_{61} + C_{62} + C_{63} + C_{64} + B_i + B_j + B_s + B_t - 2J\right) = \\
&= \left(T\sigma_\mu^2 + \sigma_\epsilon^2\right) C_{61} + \left(N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \sigma_\epsilon^2\right) C_{62} + \left(N^{(1)}N^{(3)}\sigma_v^{(2)^2} + \sigma_\epsilon^2\right) C_{63} + \\
&+ \left(N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2\right) C_{64} + \left(T\sigma_\mu^2 + N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \sigma_\epsilon^2\right) B_i + \\
&+ \left(T\sigma_\mu^2 + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + \sigma_\epsilon^2\right) B_j + \left(T\sigma_\mu^2 + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2\right) B_s + \\
&+ \left(N^{(2)}N^{(3)}\sigma_v^{(1)^2} + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2\right) B_t - \\
&- \left(T\sigma_\mu^2 + N^{(2)}N^{(3)}\sigma_v^{(1)^2} + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + 2\sigma_\epsilon^2\right) J + \sigma_\epsilon^2 W_6
\end{aligned}$$

with

$$
\begin{aligned}
C_{61} &= B_{ijs} - B_i - B_j - B_s + J \\
C_{62} &= B_{it} - B_i - B_t + J \\
C_{63} &= B_{jt} - B_j - B_t + J \\
C_{64} &= B_{st} - B_s - B_t + J \\
W_6 &= I - C_{61} - C_{62} - C_{63} - C_{64} - B_i - B_j - B_s - B_t + 2J
\end{aligned}
$$

The spectral decomposition now is

$$
\begin{aligned}
\sigma_\epsilon^2 \Omega^{-1} &= W_6 + \theta_1 C_{61} + \theta_2 C_{62} + \theta_3 C_{63} + \theta_4 C_{64} + \theta_5 B_i + \theta_6 B_j + \theta_7 B_s + \theta_8 B_t - \theta_9 J = \\
&= I - (1-\theta_1) B_{ijs} - (1-\theta_2) B_{it} - (1-\theta_3) B_{jt} - (1-\theta_4) B_{st} + (1-\theta_1-\theta_2+\theta_5) B_i + \\
&+ (1-\theta_1-\theta_3+\theta_6) B_j + (1-\theta_1-\theta_4+\theta_7) B_s + (2-\theta_2-\theta_3-\theta_4+\theta_8) B_t - \\
&- (2-\theta_1-\theta_2-\theta_3-\theta_4+\theta_9) J
\end{aligned}
$$

with

$$
\begin{aligned}
\theta_1 &= \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + \sigma_\epsilon^2} \\
\theta_2 &= \frac{\sigma_\epsilon^2}{N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \sigma_\epsilon^2} \\
\theta_3 &= \frac{\sigma_\epsilon^2}{N^{(1)}N^{(3)}\sigma_v^{(2)^2} + \sigma_\epsilon^2} \\
\theta_4 &= \frac{\sigma_\epsilon^2}{N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2} \\
\theta_5 &= \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \sigma_\epsilon^2} \\
\theta_6 &= \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + \sigma_\epsilon^2} \\
\theta_7 &= \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2} \\
\theta_8 &= \frac{\sigma_\epsilon^2}{N^{(2)}N^{(3)}\sigma_v^{(1)^2} + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2} \\
\theta_9 &= \frac{\sigma_\epsilon^2}{T\sigma_\mu^2 + N^{(2)}N^{(3)}\sigma_v^{(1)^2} + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + 2\sigma_\epsilon^2}
\end{aligned}
$$

Model (7)

And finally, for model (7) we get

$$
\begin{aligned}
u_{ijst} &= \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + v_{it}^{(1)} + v_{jt}^{(2)} + v_{st}^{(3)} + \epsilon_{ijst} \\
E[uu'] &= \sigma_\mu^{(1)^2} I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} + \sigma_\mu^{(2)^2} I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T + \\
&\quad + \sigma_\mu^{(3)^2} J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T + \sigma_v^{(1)^2} I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}} \otimes I_T + \\
&\quad + \sigma_v^{(2)^2} J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}} \otimes I_T + \sigma_v^{(3)^2} J_{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}T} + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}
\end{aligned}
$$

leading to

$$
\begin{aligned}
\Omega &= N^{(3)}T\sigma_\mu^{(1)^2}\left(C_{71} + B_i + B_j - J\right) + N^{(2)}T\sigma_\mu^{(2)^2}\left(C_{72} + B_i + B_s - J\right) + \\
&+ N^{(1)}T\sigma_\mu^{(3)^2}\left(C_{73} + B_j + B_s - J\right) + N^{(2)}N^{(3)}\sigma_v^{(1)^2}\left(C_{74} + B_i + B_t - J\right) + \\
&+ N^{(1)}N^{(3)}\sigma_v^{(2)^2}\left(C_{75} + B_j + B_t - J\right) + N^{(1)}N^{(2)}\sigma_v^{(3)^2}\left(C_{76} + B_s + B_t - J\right) + \\
&+ \sigma_\epsilon^2\left(W_7 + C_{71} + C_{72} + C_{73} + C_{75} + C_{76} + B_i + B_j + B_s + B_t - 3J\right) = \\
&= \sigma_\epsilon^2 W_7 + C_{71}\left(N^{(3)}T\sigma_\mu^{(1)^2} + \sigma_\epsilon^2\right) + C_{72}\left(N^{(2)}T\sigma_\mu^{(2)^2} + \sigma_\epsilon^2\right) + C_{73}\left(N^{(1)}T\sigma_\mu^{(3)^2} + \sigma_\epsilon^2\right) + \\
&+ C_{74}\left(N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \sigma_\epsilon^2\right) + C_{75}\left(N^{(1)}N^{(3)}\sigma_v^{(2)^2} + \sigma_\epsilon^2\right) + \\
&+ C_{76}\left(N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2\right) + B_i\left(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \sigma_\epsilon^2\right) + \\
&+ B_j\left(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + \sigma_\epsilon^2\right) + \\
&+ B_s\left(N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2\right) \\
&+ B_t\left(N^{(2)}N^{(3)}\sigma_v^{(1)^2} + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + \sigma_\epsilon^2\right) - \\
&- J\Bigg(N^{(3)}T\sigma_\mu^{(1)^2} + N^{(2)}T\sigma_\mu^{(2)^2} + N^{(1)}T\sigma_\mu^{(3)^2} + N^{(2)}N^{(3)}\sigma_v^{(1)^2} + \\
&\qquad + N^{(1)}N^{(3)}\sigma_v^{(2)^2} + N^{(1)}N^{(2)}\sigma_v^{(3)^2} + 3\sigma_\epsilon^2\Bigg)
\end{aligned}
$$

with

$$
\begin{aligned}
C_{71} &= B_{ij} - B_i - B_j + J \\
C_{72} &= B_{is} - B_i - B_s + J \\
C_{73} &= B_{js} - B_j - B_s + J \\
C_{74} &= B_{it} - B_i - B_t + J \\
C_{75} &= B_{jt} - B_j - B_t + J \\
C_{76} &= B_{st} - B_s - B_t + J \\
W_7 &= I - C_{71} - C_{72} - C_{73} - C_{74} - C_{75} - C_{76} - B_i - B_j - B_s - B_t + 3J
\end{aligned}
$$

The spectral decomposition now is

$$
\begin{aligned}
\sigma_\epsilon^2 \Omega^{-1} = & W_7 + \theta_1 C_{71} + \theta_2 C_{72} + \theta_3 C73 + \theta_4 C_{74} + \theta_5 C_{75} + \theta_6 C_{76} + \theta_7 B_i + \theta_8 B_j + \\
& + \theta_9 B_s + \theta_{10} B_t - \theta_{11} J = \\
= & I - (1-\theta_1) B_{ij} - (1-\theta_2) B_{is} - (1-\theta_3) B_{js} - (1-\theta_4) B_{it} - (1-\theta_5) B_{jt} - \\
& - (1-\theta_6) B_{st} + (2-\theta_1-\theta_2-\theta_4+\theta_7) B_i + (2-\theta_1-\theta_3-\theta_5+\theta_8) B_j + \\
& + (2-\theta_2-\theta_3-\theta_6+\theta_9) B_s + (2-\theta_4-\theta_5-\theta_6+\theta_{10}) B_t - \\
& - (3-\theta_1-\theta_2-\theta_3-\theta_4-\theta_5-\theta_6+\theta_{11}) J
\end{aligned}
$$

with

$$
\begin{aligned}
\theta_1 &= \frac{\sigma_\epsilon^2}{N^{(3)} T \sigma_\mu^{(1)^2} + \sigma_\epsilon^2} \\
\theta_2 &= \frac{\sigma_\epsilon^2}{N^{(2)} T \sigma_\mu^{(2)^2} + \sigma_\epsilon^2} \\
\theta_3 &= \frac{\sigma_\epsilon^2}{N^{(1)} T \sigma_\mu^{(3)^2} + \sigma_\epsilon^2} \\
\theta_4 &= \frac{\sigma_\epsilon^2}{N^{(2)} N^{(3)} \sigma_v^{(1)^2} + \sigma_\epsilon^2} \\
\theta_5 &= \frac{\sigma_\epsilon^2}{N^{(1)} N^{(3)} \sigma_v^{(2)^2} + \sigma_\epsilon^2} \\
\theta_6 &= \frac{\sigma_\epsilon^2}{N^{(1)} N^{(2)} \sigma_v^{(3)^2} + \sigma_\epsilon^2} \\
\theta_7 &= \frac{\sigma_\epsilon^2}{N^{(3)} T \sigma_\mu^{(1)^2} + N^{(2)} T \sigma_\mu^{(2)^2} + N^{(2)} N^{(3)} \sigma_v^{(1)^2} + \sigma_\epsilon^2} \\
\theta_8 &= \frac{\sigma_\epsilon^2}{N^{(3)} T \sigma_\mu^{(1)^2} + N^{(1)} T \sigma_\mu^{(3)^2} + N^{(1)} N^{(3)} \sigma_v^{(2)^2} + \sigma_\epsilon^2} \\
\theta_9 &= \frac{\sigma_\epsilon^2}{N^{(2)} T \sigma_\mu^{(2)^2} + N^{(1)} T \sigma_\mu^{(3)^2} + N^{(1)} N^{(2)} \sigma_v^{(3)^2} + \sigma_\epsilon^2} \\
\theta_{10} &= \frac{\sigma_\epsilon^2}{N^{(2)} N^{(3)} \sigma_v^{(1)^2} + N^{(1)} N^{(3)} \sigma_v^{(2)^2} + N^{(1)} N^{(2)} \sigma_v^{(3)^2} + \sigma_\epsilon^2} \\
\theta_{11} &= \frac{\sigma_\epsilon^2}{N^{(3)} T \sigma_\mu^{(1)^2} + N^{(2)} T \sigma_\mu^{(2)^2} + N^{(1)} T \sigma_\mu^{(3)^2} + N^{(2)} N^{(3)} \sigma_v^{(1)^2} + A} \\
A &= N^{(1)} N^{(3)} \sigma_v^{(2)^2} + N^{(1)} N^{(2)} \sigma_v^{(3)^2} + 3\sigma_\epsilon^2
\end{aligned}
$$

3.2 Estimation of the Variance Components

In order to make the GLS estimator feasible we need to estimate the variance components of the different models. Given the four dimensions this quite tedious and unfortunately there is no way to go around. This is done below for all models in two steps. First, using the appropriate Within transformation for each model which cancels out the specific effects (see *Matyas and Balazsi* [2012]), identifying equations are derived for the unknown variance components. Then, using these identifying equations, estimators for the variance components are derived one by one.

Model (1)

The Within transformation that cancels out the specific effects for this model is

$$u_{ijst} - \bar{u}_i - \bar{u}_j - \bar{u}_s - \bar{u}_t + 3\bar{u} = \epsilon_{ijst} - \bar{\epsilon}_i - \bar{\epsilon}_j - \bar{\epsilon}_s - \bar{\epsilon}_t + 3\bar{\epsilon}$$

which leads to the following identifying equations

$$E\left[(u_{ijst} - \bar{u}_i - \bar{u}_j - \bar{u}_s - \bar{u}_t + 3\bar{u})^2\right] = \sigma_\epsilon^2 \frac{N^{(1)}N^{(2)}N^{(3)}T - N^{(1)} - N^{(2)} - N^{(3)} - T + 3}{N^{(1)}N^{(2)}N^{(3)}T}$$

$$E\left[u_{ijst}^2\right] = E\left[(\mu_i + \gamma_j + \alpha_s + \lambda_t + \epsilon_{ijst})^2\right] = \sigma_\mu^2 + \sigma_\gamma^2 + \sigma_\alpha^2 + \sigma_\lambda^2 + \sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{T}\sum_{t=1}^{T} u_{ijst}\right)^2\right] = E\left[\left(\frac{1}{T}\sum_{t=1}^{T}(\mu_i + \gamma_j + \alpha_s + \lambda_t + \epsilon_{ijst})\right)^2\right] =$$

$$= E\left[\mu_i^2\right] + E\left[\gamma_j^2\right] + E\left[\alpha_s^2\right] + \frac{1}{T^2}E\left[\sum_{t=1}^{T}\lambda_t^2\right] + \frac{1}{T^2}E\left[\sum_{t=1}^{T}\epsilon_{ijst}^2\right] =$$

$$= \sigma_\mu^2 + \sigma_\gamma^2 + \sigma_\alpha^2 + \frac{1}{T}\sigma_\lambda^2 + \frac{1}{T}\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(1)}}\sigma_\mu^2 + \sigma_\gamma^2 + \sigma_\alpha^2 + \sigma_\lambda^2 + \frac{1}{N^{(1)}}\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}} u_{ijst}\right)^2\right] = \sigma_\mu^2 + \frac{1}{N^{(2)}}\sigma_\gamma^2 + \sigma_\alpha^2 + \sigma_\lambda^2 + \frac{1}{N^{(2)}}\sigma_\epsilon^2$$

So the appropriate estimators for the variance components are

$$\hat{\sigma}_\epsilon^2 = \frac{N^{(1)}N^{(2)}N^{(3)}T}{N^{(1)}N^{(2)}N^{(3)}T - N^{(1)} - N^{(2)} - N^{(3)} - T + 3}\hat{u}'_{within}\hat{u}_{within}$$

$$\hat{\sigma}_\mu^2 = \frac{1}{\left(N^{(1)}-1\right)N^{(2)}N^{(3)}T}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^{T}\left(\sum_i^{N^{(1)}}\hat{u}_{ijst}^2 - \frac{1}{N^{(1)}}\left(\sum_i^{N^{(1)}}\hat{u}_{ijst}\right)^2\right) - \hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_\gamma^2 = \frac{1}{N^{(1)}\left(N^{(2)}-1\right)N^{(3)}T}\sum_i^{N^{(1)}}\sum_s^{N^{(3)}}\sum_t^{T}\left(\sum_j^{N^{(2)}}\hat{u}_{ijst}^2 - \frac{1}{N^{(2)}}\left(\sum_j^{N^{(2)}}\hat{u}_{ijst}\right)^2\right) - \hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_\lambda^2 = \frac{1}{N^{(1)}N^{(2)}N^{(3)}(T-1)}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\left(\sum_t^{T}\hat{u}_{ijst}^2 - \frac{1}{T}\left(\sum_t^{T}\hat{u}_{ijst}\right)^2\right) - \hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_\alpha^2 = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^{T}\hat{u}_{ijst}^2 - \hat{\sigma}_\mu^2 - \hat{\sigma}_\gamma^2 - \hat{\sigma}_\lambda^2 - \hat{\sigma}_\epsilon^2$$

where the "within" index means that it is the residual obtained from the appropriate Within estimation of the model.

For cross sectional data only, the Within transformation that cancels out the specific effects for this model is

$$u_{ijs} - \bar{u}_i - \bar{u}_j - \bar{u}_s + 2\bar{u} = \epsilon_{ijs} - \bar{\epsilon}_i - \bar{\epsilon}_j - \bar{\epsilon}_s + 2\bar{\epsilon}$$

which leads us to the following identification equations

$$\begin{aligned}
E\left[\left(u_{ijs} - \bar{u}_i - \bar{u}_j - \bar{u}_s + 2\bar{u}\right)^2\right] &= \sigma_\epsilon^2\frac{N^{(1)}N^{(2)}N^{(3)} - N^{(1)} - N^{(2)} - N^{(3)} + 2}{N^{(1)}N^{(2)}N^{(3)}} \\
E\left[u_{ijs}^2\right] &= \sigma_\mu^2 + \sigma_\gamma^2 + \sigma_\alpha^2 + \sigma_\epsilon^2 \\
E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}}u_{ijs}\right)^2\right] &= \frac{1}{N^{(1)}}\sigma_\mu^2 + \sigma_\gamma^2 + \sigma_\alpha^2 + \sigma_\lambda^2 + \frac{1}{N^{(1)}}\sigma_\epsilon^2 \\
E\left[\left(\frac{1}{N^{(2)}}\sum_{i=1}^{N^{(2)}}u_{ijs}\right)^2\right] &= \sigma_\mu^2 + \frac{1}{N^{(2)}}\sigma_\gamma^2 + \sigma_\alpha^2 + \sigma_\lambda^2 + \frac{1}{N^{(2)}}\sigma_\epsilon^2
\end{aligned}$$

Thus the estimators for the variance components are

$$\hat{\sigma}^2_\epsilon = \frac{N^{(1)}N^{(2)}N^{(3)}}{N^{(1)}N^{(2)}N^{(3)} - N^{(1)} - N^{(2)} - N^{(3)} + 2}\hat{u}'_{within}\hat{u}_{within}$$

$$\hat{\sigma}^2_\mu = \frac{1}{\left(N^{(1)}-1\right)N^{(2)}N^{(3)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\left(\sum_i^{N^{(1)}}\hat{u}^2_{ijs} - \frac{1}{N^{(1)}}\left(\sum_i^{N^{(1)}}\hat{u}_{ijs}\right)^2\right) - \hat{\sigma}^2_\epsilon$$

$$\hat{\sigma}^2_\gamma = \frac{1}{N^{(1)}\left(N^{(2)}-1\right)N^{(3)}}\sum_i^{N^{(1)}}\sum_s^{N^{(3)}}\left(\sum_j^{N^{(2)}}\hat{u}^2_{ijs} - \frac{1}{N^{(2)}}\left(\sum_j^{N^{(2)}}\hat{u}_{ijs}\right)^2\right) - \hat{\sigma}^2_\epsilon$$

$$\hat{\sigma}^2_\alpha = \frac{1}{N^{(1)}N^{(2)}N^{(3)}}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\hat{u}^2_{ijs} - \hat{\sigma}^2_\mu - \hat{\sigma}^2_\gamma - \hat{\sigma}^2_\epsilon$$

Model (2)

The Within transformation now is

$$u_{ijst} - \bar{u}_{ijs} = \epsilon_{ijst} - \bar{\epsilon}_{ijs}$$

The identifying equations are

$$E\left[(u_{ijst} - \bar{u}_{ijs})^2\right] = E\left[(\epsilon_{ijst} - \bar{\epsilon}_{ijs})^2\right] = E\left[\epsilon^2_{ijst} - 2\epsilon_{ijst}\bar{\epsilon}_{ijs} + \bar{\epsilon}^2_{ijs}\right] =$$
$$= \sigma^2_\epsilon - \frac{2}{T}\sigma^2_\epsilon + \frac{1}{T}\sigma^2_\epsilon = \sigma^2_\epsilon\frac{T-1}{T}$$

$$E\left[{u^\star}^2_{ijst}\right] = E\left[(\mu_{ij} + \epsilon_{ijst})^2\right] = E\left[\mu^2_{ij}\right] + E\left[\epsilon^2_{ijst}\right] = \sigma^2_\mu + \sigma^2_\epsilon$$

So the estimators for the variance components are

$$\hat{\sigma}^2_\epsilon = \frac{T}{T-1}\hat{u}'_{within}\hat{u}_{within}$$
$$\hat{\sigma}^2_\mu = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^{T}\hat{u}^2_{ijst} - \hat{\sigma}^2_\epsilon$$

Model (3)

The Within transformation is

$$u_{ijst} - \bar{u}_{ijs} - \bar{u}_t + \bar{u} = \epsilon_{ijst} - \bar{\epsilon}_{ijs} - \bar{\epsilon}_t + \bar{\epsilon}$$

which leads to the following identifying equations

$$
\begin{aligned}
&E\left[(u_{ijst}-\bar{u}_{ijs}-\bar{u}_t+\bar{u})^2\right]=E\left[(\epsilon_{ijst}-\bar{\epsilon}_{ijs}-\bar{\epsilon}_t+\bar{\epsilon})^2\right]=\\
&\quad=E\left[\epsilon_{ijst}^2\right]+E\left[\bar{\epsilon}_{ijs}^2\right]+E\left[\bar{\epsilon}_t^2\right]+E\left[\bar{\epsilon}^2\right]-2E\left[\epsilon_{ijst}\bar{\epsilon}_{ijs}\right]-2E\left[\epsilon_{ijst}\bar{\epsilon}_t\right]+\\
&\quad+2E\left[\epsilon_{ijst}\bar{\epsilon}\right]+2E\left[\bar{\epsilon}_{ijs}\bar{\epsilon}_t\right]-2E\left[\bar{\epsilon}_{ijs}\bar{\epsilon}\right]-2E\left[\bar{\epsilon}_t\bar{\epsilon}\right]=\\
&\quad=\sigma_\epsilon^2+\frac{1}{T}\sigma_\epsilon^2+\frac{1}{N^{(1)}N^{(2)}N^{(3)}}\sigma_\epsilon^2+\frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sigma_\epsilon^2-\frac{2}{T}\sigma_\epsilon^2-\frac{2}{N^{(1)}N^{(2)}N^{(3)}}\sigma_\epsilon^2+\\
&\quad+\frac{2}{N^{(1)}N^{(2)}N^{(3)}T}\sigma_\epsilon^2+\frac{2}{N^{(1)}N^{(2)}N^{(3)}T}\sigma_\epsilon^2-\frac{2}{N^{(1)}N^{(2)}N^{(3)}T}\sigma_\epsilon^2-\\
&\quad-\frac{2}{N^{(1)}N^{(2)}N^{(3)}T}\sigma_\epsilon^2=\sigma_\epsilon^2\frac{\left(N^{(1)}N^{(2)}N^{(3)}-1\right)(T-1)}{N^{(1)}N^{(2)}N^{(3)}T}
\end{aligned}
$$

The estimators for the variance components now are

$$
\begin{aligned}
\hat{\sigma}_\epsilon^2&=\frac{N^{(1)}N^{(2)}N^{(3)}T}{\left(N^{(1)}N^{(2)}N^{(3)}-1\right)(T-1)}\hat{u}'_{within}\hat{u}_{within}\\
\hat{\sigma}_\mu^2&=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T(T-1)}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\left(\left(\sum_t^T\hat{u}_{ijst}\right)^2-\sum_t^T\hat{u}_{ijst}^2\right)\\
\hat{\sigma}_\lambda^2&=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^T\hat{u}_{ijst}^2-\hat{\sigma}_\mu^2-\hat{\sigma}_\epsilon^2
\end{aligned}
$$

Model (4)

The Within transformation now is

$$u_{ijst}-\bar{u}_{ij}-\bar{u}_{is}-\bar{u}_{js}+\bar{u}_i+\bar{u}_j+\bar{u}_s-\bar{u}$$

The identifying equations are

$$
\begin{aligned}
&E\left[(u_{ijst}-\bar{u}_{ij}-\bar{u}_{is}-\bar{u}_{js}+\bar{u}_i+\bar{u}_j+\bar{u}_s-\bar{u})^2\right]=\\
&=\sigma_\epsilon^2\frac{N^{(1)}N^{(2)}N^{(3)}T-N^{(1)}N^{(2)}-N^{(1)}N^{(3)}-N^{(2)}N^{(3)}+N^{(1)}+N^{(2)}+N^{(3)}-1}{N^{(1)}N^{(2)}N^{(3)}T}\\
&E\left[u_{ijst}^2\right]=\sigma_\mu^{(1)^2}+\sigma_\mu^{(2)^2}+\sigma_\mu^{(3)^2}+\sigma_\epsilon^2\\
&E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(1)}}\sigma_\mu^{(2)^2}+\sigma_\mu^{(3)^2}+\frac{1}{N^{(1)}}\sigma_\epsilon^2\\
&E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(2)}}\sigma_\mu^{(1)^2}+\sigma_\mu^{(2)^2}+\frac{1}{N^{(2)}}\sigma_\mu^{(3)^2}+\frac{1}{N^{(2)}}\sigma_\epsilon^2
\end{aligned}
$$

The estimators of the variance components now are

$$\hat{\sigma}_\epsilon^2 = \frac{N^{(1)}N^{(2)}N^{(3)}T}{N^{(1)}N^{(2)}N^{(3)}T - N^{(1)}N^{(2)} - N^{(1)}N^{(3)} - N^{(2)}N^{(3)} + N^{(1)} + N^{(2)} + N^{(3)} - 1} \hat{u}'_{with}\hat{u}_{with}$$

$$\hat{\sigma}_\mu^{(3)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)} \sum_j^{N^{(2)}} \sum_s^{N^{(3)}} \sum_t^{T} \left(\left(\sum_i^{N^{(1)}} \hat{u}_{ijst} \right)^2 - \sum_i^{N^{(1)}} \hat{u}_{ijst}^2 \right)$$

$$\hat{\sigma}_\mu^{(2)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)} \sum_i^{N^{(1)}} \sum_s^{N^{(3)}} \sum_t^{T} \left(\left(\sum_j^{N^{(2)}} \hat{u}_{ijst} \right)^2 - \sum_j^{N^{(2)}} \hat{u}_{ijst}^2 \right)$$

$$\hat{\sigma}_\mu^{(1)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T} \sum_i^{N^{(1)}} \sum_j^{N^{(2)}} \sum_s^{N^{(3)}} \sum_t^{T} \hat{u}_{ijst}^2 - \hat{\sigma}_\mu^{(2)^2} - \hat{\sigma}_\mu^{(3)^2} - \hat{\sigma}_\epsilon^2$$

For cross sectional data only, the Within transformation now is

$$u_{ijs} - \bar{u}_{ij} - \bar{u}_{is} - \bar{u}_{js} + \bar{u}_i + \bar{u}_j + \bar{u}_s - \bar{u}$$

The identifying equations are

$$E\left[\left(u_{ijs} - \bar{u}_{ij} - \bar{u}_{is} - \bar{u}_{js} + \bar{u}_i + \bar{u}_j + \bar{u}_s - \bar{u}\right)^2\right] =$$

$$= \sigma_\epsilon^2 \frac{N^{(1)}N^{(2)}N^{(3)} - N^{(1)}N^{(2)} - N^{(1)}N^{(3)} - N^{(2)}N^{(3)} + N^{(1)} + N^{(2)} + N^{(3)} - 1}{N^{(1)}N^{(2)}N^{(3)}}$$

$$E\left[u_{ijs}^2\right] = \sigma_\mu^{(1)^2} + \sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2} + \sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(1)}} \sum_{i=1}^{N^{(1)}} u_{ijs}\right)^2\right] = \frac{1}{N^{(1)}}\sigma_\mu^{(1)^2} + \frac{1}{N^{(1)}}\sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2} + \frac{1}{N^{(1)}}\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(2)}} \sum_{i=1}^{N^{(2)}} u_{ijs}\right)^2\right] = \frac{1}{N^{(2)}}\sigma_\mu^{(1)^2} + \sigma_\mu^{(2)^2} + \frac{1}{N^{(2)}}\sigma_\mu^{(3)^2} + \frac{1}{N^{(2)}}\sigma_\epsilon^2$$

And so the estimators of variance components now are

$$\hat{\sigma}_\epsilon^2 = \frac{N^{(1)}N^{(2)}N^{(3)}}{N^{(1)}N^{(2)}N^{(3)} - N^{(1)}N^{(2)} - N^{(1)}N^{(3)} - N^{(2)}N^{(3)} + N^{(1)} + N^{(2)} + N^{(3)} - 1} \times$$
$$\times \hat{u}'_{within}\hat{u}_{within}$$

$$\hat{\sigma}_\mu^{(3)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}\left(N^{(1)}-1\right)} \sum_j^{N^{(2)}} \sum_s^{N^{(3)}} \left(\left(\sum_i^{N^{(1)}} \hat{u}_{ijs} \right)^2 - \sum_i^{N^{(1)}} \hat{u}_{ijs}^2 \right)$$

$$\hat{\sigma}_\mu^{(2)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}\left(N^{(2)}-1\right)} \sum_i^{N^{(1)}} \sum_s^{N^{(3)}} \left(\left(\sum_j^{N^{(2)}} \hat{u}_{ijs} \right)^2 - \sum_j^{N^{(2)}} \hat{u}_{ijs}^2 \right)$$

$$\hat{\sigma}_\mu^{(1)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}} \sum_i^{N^{(1)}} \sum_j^{N^{(2)}} \sum_s^{N^{(3)}} \hat{u}_{ijs}^2 - \hat{\sigma}_\mu^{(2)^2} - \hat{\sigma}_\mu^{(3)^2} - \hat{\sigma}_\epsilon^2$$

Model (5)

The Within transformation now is

$$u_{ijst} - \bar{u}_{ij} - \bar{u}_{is} - \bar{u}_{js} - \bar{u}_t + \bar{u}_i + \bar{u}_j + \bar{u}_s$$

The identifying equations are

$$E\left[\left(u_{ijst} - \bar{u}_{ij} - \bar{u}_{is} - \bar{u}_{js} - \bar{u}_t + \bar{u}_i + \bar{u}_j + \bar{u}_s \right)^2 \right] =$$
$$= \sigma_\epsilon^2 \frac{N^{(1)}N^{(2)}N^{(3)}T - N^{(1)}N^{(2)} - N^{(1)}N^{(3)} - N^{(2)}N^{(3)} - T + N^{(1)} + N^{(2)} + N^{(3)}}{N^{(1)}N^{(2)}N^{(3)}T}$$
$$E\left[u_{ijst}^2\right] = \sigma_\mu^{(1)^2} + \sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2} + \sigma_\lambda^2 + \sigma_\epsilon^2$$
$$E\left[\left(\frac{1}{T} \sum_{t=1}^{T} u_{ijst} \right)^2 \right] = \sigma_\mu^{(1)^2} + \sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2} + \frac{1}{T}\sigma_\lambda^2 + \frac{1}{T}\sigma_\epsilon^2$$
$$E\left[\left(\frac{1}{N^{(1)}} \sum_{i=1}^{N^{(1)}} u_{ijst} \right)^2 \right] = \frac{1}{N^{(1)}}\sigma_\mu^{(1)^2} + \frac{1}{N^{(1)}}\sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2} + \sigma_\lambda^2 + \frac{1}{N^{(1)}}\sigma_\epsilon^2$$
$$E\left[\left(\frac{1}{N^{(2)}} \sum_{j=1}^{N^{(2)}} u_{ijst} \right)^2 \right] = \frac{1}{N^{(2)}}\sigma_\mu^{(1)^2} + \sigma_\mu^{(2)^2} + \frac{1}{N^{(2)}}\sigma_\mu^{(3)^2} + \sigma_\lambda^2 + \frac{1}{N^{(2)}}\sigma_\epsilon^2$$

The estimators of the variance components now are

$$\hat{\sigma}_{\epsilon}^{2} = \frac{N^{(1)}N^{(2)}N^{(3)}T}{N^{(1)}N^{(2)}N^{(3)}T - N^{(1)}N^{(2)} - N^{(1)}N^{(3)} - N^{(2)}N^{(3)} - T + N^{(1)} + N^{(2)} + N^{(3)}} \hat{u}_{with}' \hat{u}_{with}$$

$$\hat{\sigma}_{\lambda}^{2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}(T-1)} \sum_{i}^{N^{(1)}} \sum_{j}^{N^{(2)}} \sum_{s}^{N^{(3)}} \left(\sum_{t}^{T} \hat{u}_{ijst}^{2} - \frac{1}{T} \left(\sum_{t}^{T} \hat{u}_{ijst} \right)^{2} \right) - \hat{\sigma}_{\epsilon}^{2}$$

$$\hat{\sigma}_{\mu}^{(3)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)} \sum_{j}^{N^{(2)}} \sum_{s}^{N^{(3)}} \sum_{t}^{T} \left(\left(\sum_{i}^{N^{(1)}} \hat{u}_{ijst} \right)^{2} - \sum_{i}^{N^{(1)}} \hat{u}_{ijst}^{2} \right) - \hat{\sigma}_{\lambda}^{2}$$

$$\hat{\sigma}_{\mu}^{(2)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)} \sum_{i}^{N^{(1)}} \sum_{s}^{N^{(3)}} \sum_{t}^{T} \left(\left(\sum_{j}^{N^{(2)}} \hat{u}_{ijst} \right)^{2} - \sum_{j}^{N^{(2)}} \hat{u}_{ijst}^{2} \right) - \hat{\sigma}_{\lambda}^{2}$$

$$\hat{\sigma}_{\mu}^{(1)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T} \sum_{i}^{N^{(1)}} \sum_{j}^{N^{(2)}} \sum_{s}^{N^{(3)}} \sum_{t}^{T} \hat{u}_{ijst}^{2} - \hat{\sigma}_{\mu}^{(2)^2} - \hat{\sigma}_{\mu}^{(3)^2} - \hat{\sigma}_{\lambda}^{2} - \hat{\sigma}_{\epsilon}^{2}$$

Model (6)

The Within transformation now is

$$u_{ijst} - \bar{u}_{ijs} - \bar{u}_{it} - \bar{u}_{jt} - \bar{u}_{st} + \bar{u}_{i} + \bar{u}_{j} + \bar{u}_{s} + 2\bar{u}_{t} - 2\bar{u}$$

and the identifying equations are

$$E\left[\left(u_{ijst} - \bar{u}_{ijs} - \bar{u}_{it} - \bar{u}_{jt} - \bar{u}_{st} + \bar{u}_{i} + \bar{u}_{j} + \bar{u}_{s} + 2\bar{u}_{t} - 2\bar{u} \right)^{2} \right] =$$
$$= \sigma_{\epsilon}^{2} \frac{N^{(1)}N^{(2)}N^{(3)}T - N^{(1)}N^{(2)}N^{(3)} - N^{(1)}T - N^{(2)}T - N^{(3)}T + N^{(1)} + N^{(2)} + N^{(3)} + 2T - 2}{N^{(1)}N^{(2)}N^{(3)}T}$$

$$E\left[u_{ijst}^{2} \right] = \sigma_{\mu}^{2} + \sigma_{v}^{(1)^2} + \sigma_{v}^{(2)^2} + \sigma_{v}^{(3)^2} + \sigma_{\epsilon}^{2}$$

$$E\left[\left(\frac{1}{T} \sum_{t=1}^{T} u_{ijst} \right)^{2} \right] = \sigma_{\mu}^{2} + \frac{1}{T}\sigma_{v}^{(1)^2} + \frac{1}{T}\sigma_{v}^{(2)^2} + \frac{1}{T}\sigma_{v}^{(3)^2} + \frac{1}{T}\sigma_{\epsilon}^{2}$$

$$E\left[\left(\frac{1}{N^{(1)}} \sum_{i=1}^{N^{(1)}} u_{ijst} \right)^{2} \right] = \frac{1}{N^{(1)}}\sigma_{\mu}^{2} + \frac{1}{N^{(1)}}\sigma_{v}^{(1)^2} + \sigma_{v}^{(2)^2} + \sigma_{v}^{(3)^2} + \frac{1}{N^{(1)}}\sigma_{\epsilon}^{2}$$

$$E\left[\left(\frac{1}{N^{(2)}} \sum_{j=1}^{N^{(2)}} u_{ijst} \right)^{2} \right] = \frac{1}{N^{(2)}}\sigma_{\mu}^{2} + \sigma_{v}^{(1)^2} + \frac{1}{N^{(2)}}\sigma_{v}^{(2)^2} + \sigma_{v}^{(3)^2} + \frac{1}{N^{(2)}}\sigma_{\epsilon}^{2}$$

The estimators of the variance components now are

$$\hat{\sigma}_\epsilon^2 = \frac{N^{(1)}N^{(2)}N^{(3)}T}{N^{(1)}N^{(2)}N^{(3)}T - N^{(1)}N^{(2)}N^{(3)} - N^{(1)}T - N^{(2)}T - N^{(3)}T + A}\hat{u}'_{within}\hat{u}_{within}$$
$$\text{with } A = N^{(1)} + N^{(2)} + N^{(3)} + 2T - 2$$
$$\hat{\sigma}_\mu^2 = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T(T-1)} \sum_i^{N^{(1)}} \sum_j^{N^{(2)}} \sum_s^{N^{(3)}} \left(\left(\sum_t^T \hat{u}_{ijst} \right)^2 - \sum_t^T \hat{u}_{ijst}^2 \right)$$
$$\hat{\sigma}_v^{(1)^2} = \frac{1}{\left(N^{(1)}-1\right)N^{(2)}N^{(3)}T} \sum_j^{N^{(2)}} \sum_s^{N^{(3)}} \sum_t^T \left(\sum_i^{N^{(1)}} \hat{u}_{ijst}^2 - \frac{1}{N^{(1)}} \left(\sum_i^{N^{(1)}} \hat{u}_{ijst} \right)^2 \right) - \hat{\sigma}_\mu^2 - \hat{\sigma}_\epsilon^2$$
$$\hat{\sigma}_v^{(2)^2} = \frac{1}{N^{(1)}\left(N^{(2)}-1\right)N^{(3)}T} \sum_i^{N^{(1)}} \sum_s^{N^{(3)}} \sum_t^T \left(\sum_j^{N^{(2)}} \hat{u}_{ijst}^2 - \frac{1}{N^{(2)}} \left(\sum_j^{N^{(2)}} \hat{u}_{ijst} \right)^2 \right) - \hat{\sigma}_\mu^2 - \hat{\sigma}_\epsilon^2$$
$$\hat{\sigma}_v^{(3)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T} \sum_i^{N^{(1)}} \sum_j^{N^{(2)}} \sum_s^{N^{(3)}} \sum_t^T \hat{u}_{ijst}^2 - \hat{\sigma}_\mu^2 - \hat{\sigma}_v^{(1)^2} - \hat{\sigma}_v^{(2)^2} - \hat{\sigma}_\epsilon^2$$

Model (7)

The Within transformation for this last model is

$$u_{ijst} - \bar{u}_{ij} - \bar{u}_{is} - \bar{u}_{js} - \bar{u}_{it} - \bar{u}_{jt} - \bar{u}_{st} + 2\bar{u}_i + 2\bar{u}_j + 2\bar{u}_s + 2\bar{u}_t - 3\bar{u}$$

The identifying equations are

$$E\left[\left(u_{ijst} - \bar{u}_{ij} - \bar{u}_{is} - \bar{u}_{js} - \bar{u}_{it} - \bar{u}_{jt} - \bar{u}_{st} + 2\bar{u}_i + 2\bar{u}_j + 2\bar{u}_s + 2\bar{u}_t - 3\bar{u}\right)^2\right] =$$
$$= \sigma_\epsilon^2 \frac{N^{(1)}N^{(2)}N^{(3)}T - N^{(1)}N^{(2)} - N^{(1)}N^{(3)} - N^{(2)}N^{(3)} - N^{(1)}T - N^{(2)}T - N^{(3)}T + A}{N^{(1)}N^{(2)}N^{(3)}T}$$
$$\text{with } A = 2N^{(1)} + 2N^{(2)} + 2N^{(3)} + 2T - 3$$
$$E\left[u_{ijst}^2\right] = \sigma_\mu^{(1)^2} + \sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2} + \sigma_v^{(1)^2} + \sigma_v^{(2)^2} + \sigma_v^{(3)^2} + \sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(1)}} \sum_{i=1}^{N^{(1)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(1)}}\sigma_\mu^{(1)^2} + \frac{1}{N^{(1)}}\sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2} + \frac{1}{N^{(1)}}\sigma_v^{(1)^2} + \sigma_v^{(2)^2} +$$
$$+ \sigma_v^{(3)^2} + \frac{1}{N^{(1)}}\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(2)}}\sigma_\mu^{(1)^2}+\sigma_\mu^{(2)^2}+\frac{1}{N^{(2)}}\sigma_\mu^{(3)^2}+\sigma_v^{(1)^2}+\frac{1}{N^{(2)}}\sigma_v^{(2)^2}+$$

$$+\sigma_v^{(3)^2}+\frac{1}{N^{(2)}}\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(3)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\sigma_\mu^{(1)^2}+\frac{1}{N^{(3)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(3)}}\sigma_\mu^{(3)^2}+\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+$$

$$+\frac{1}{N^{(3)}}\sigma_v^{(3)^2}+\frac{1}{N^{(3)}}\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(1)}N^{(2)}}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}N^{(2)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(1)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(2)}}\sigma_\mu^{(3)^2}+$$

$$+\frac{1}{N^{(1)}}\sigma_v^{(1)^2}+$$

$$+\frac{1}{N^{(2)}}\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+\frac{1}{N^{(1)}N^{(2)}}\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(1)}N^{(3)}}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(1)}N^{(3)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(3)}}\sigma_\mu^{(3)^2}+$$

$$+\frac{1}{N^{(1)}}\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+\frac{1}{N^{(3)}}\sigma_v^{(3)^2}+\frac{1}{N^{(1)}N^{(3)}}\sigma_\epsilon^2$$

Finally, the estimators of the variance components are

$$\hat{\sigma}_\epsilon^2=\frac{N^{(1)}N^{(2)}N^{(3)}T}{N^{(1)}N^{(2)}N^{(3)}T-N^{(1)}N^{(2)}-N^{(1)}N^{(3)}-N^{(2)}N^{(3)}+A}\hat{u}'_{within}\hat{u}_{within}$$

$$\text{with } A=-N^{(1)}T-N^{(2)}T-N^{(3)}T+2N^{(1)}+2N^{(2)}+2N^{(3)}+2T-3$$

$$\hat{\sigma}_v^{(3)^2}=\frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)\left(N^{(2)}-1\right)}\times$$

$$\times\sum_s^{N^{(3)}}\sum_t^T\left(\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\hat{u}_{ijst}^2-\sum_i^{N^{(1)}}\left(\sum_j^{N^{(2)}}\hat{u}_{ijst}\right)^2-\sum_j^{N^{(2)}}\left(\sum_i^{N^{(1)}}\hat{u}_{ijst}\right)^2+\left(\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\hat{u}_{ijst}\right)^2\right)$$

$$\hat{\sigma}_v^{(2)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)\left(N^{(3)}-1\right)}\times$$
$$\times\sum_j^{N^{(2)}}\sum_t^{T}\left(\sum_i^{N^{(1)}}\sum_s^{N^{(3)}}\hat{u}_{ijst}^2-\sum_i^{N^{(1)}}\left(\sum_s^{N^{(3)}}\hat{u}_{ijst}\right)^2-\sum_s^{N^{(3)}}\left(\sum_i^{N^{(1)}}\hat{u}_{ijst}\right)^2+\left(\sum_i^{N^{(1)}}\sum_s^{N^{(3)}}\hat{u}_{ijst}\right)^2\right)$$
$$\hat{\sigma}_v^{(1)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)\left(N^{(3)}-1\right)}\times$$
$$\times\sum_i^{N^{(2)}}\sum_t^{T}\left(\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\hat{u}_{ijst}^2-\sum_j^{N^{(2)}}\left(\sum_s^{N^{(3)}}\hat{u}_{ijst}\right)^2-\sum_s^{N^{(3)}}\left(\sum_j^{N^{(2)}}\hat{u}_{ijst}\right)^2+\left(\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\hat{u}_{ijst}\right)^2\right)$$
$$\hat{\sigma}_\mu^{(3)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^{T}\left(\left(\sum_i^{N^{(1)}}\hat{u}_{ijst}\right)^2-\sum_i^{N^{(1)}}\hat{u}_{ijst}^2\right)-\hat{\sigma}_v^{(2)^2}-\hat{\sigma}_v^{(3)^2}$$
$$\hat{\sigma}_\mu^{(2)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)}\sum_i^{N^{(1)}}\sum_s^{N^{(3)}}\sum_t^{T}\left(\left(\sum_j^{N^{(2)}}\hat{u}_{ijst}\right)^2-\sum_j^{N^{(2)}}\hat{u}_{ijst}^2\right)-\hat{\sigma}_v^{(1)^2}-\hat{\sigma}_v^{(3)^2}$$
$$\hat{\sigma}_\mu^{(1)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sum_i^{N^{(1)}}\sum_j^{N^{(2)}}\sum_s^{N^{(3)}}\sum_t^{T}\hat{u}_{ijst}^2-\hat{\sigma}_\mu^{(2)^2}-\hat{\sigma}_\mu^{(3)^2}-\hat{\sigma}_v^{(1)^2}-\hat{\sigma}_v^{(2)^2}-\hat{\sigma}_v^{(3)^2}-\hat{\sigma}_\epsilon^2$$

3.3 Covariance Matrixes of the Models with Cross Correlations

Models (2), (3) and (6)

Both for models (2), (3) and (6) we have

$$\begin{aligned}
E[\mu_{ij}\mu_{ij}'] &= \sigma_\mu^2 I_{N^{(3)}}\otimes J_T+\rho_{(3)}\left(J_{N^{(3)}T}-I_{N^{(3)}}\otimes J_T\right)\\
E[\mu_i\mu_i'] &= \sigma_\mu^2 I_{N^{(2)}N^{(3)}}\otimes J_T+\rho_{(3)}\left(I_{N^{(2)}}\otimes J_{N^{(3)}T}-I_{N^{(2)}N^{(3)}}\otimes J_T\right)+\\
&\quad+\rho_{(2)}\left(\left(J_{N^{(2)}}-I_{N^{(2)}}\right)\otimes\left(I_{N^{(3)}}\otimes J_T\right)\right)\\
E[\mu\mu'] &= \sigma_\mu^2 I_{N^{(1)}N^{(2)}N^{(3)}}\otimes J_T+\rho_{(3)}\left(I_{N^{(1)}N^{(2)}}\otimes J_{N^{(3)}T}-I_{N^{(1)}N^{(2)}N^{(3)}}\otimes J_T\right)+\\
&\quad+\rho_{(2)}\left(I_{N^{(1)}}\otimes\left(J_{N^{(2)}}-I_{N^{(2)}}\right)\otimes\left(I_{N^{(3)}}\otimes J_T\right)\right)+\rho_{(1)}\left(J_{N^{(1)}}-I_{N^{(1)}}\right)\otimes\left(I_{N^{(2)}N^{(3)}}\otimes J_T\right)
\end{aligned}$$

Thus, the covariance matrix of model (2) takes the form

$$\begin{aligned}
E[uu'] &= \sigma_\mu^2 I_{N^{(1)}N^{(2)}N^{(3)}}\otimes J_T+\rho_{(3)}\left(I_{N^{(1)}N^{(2)}}\otimes J_{N^{(3)}T}-I_{N^{(1)}N^{(2)}N^{(3)}}\otimes J_T\right)+\\
&\quad+\rho_{(2)}\left(I_{N^{(1)}}\otimes\left(J_{N^{(2)}}-I_{N^{(2)}}\right)\otimes\left(I_{N^{(3)}}\otimes J_T\right)\right)+\rho_{(1)}\left(J_{N^{(1)}}-I_{N^{(1)}}\right)\otimes\left(I_{N^{(2)}N^{(3)}}\otimes J_T\right)+\\
&\quad+\sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}
\end{aligned}$$

the covariance matrix of model (3) looks like

$$
\begin{aligned}
E[uu'] &= \sigma_\mu^2 I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T + \rho_{(3)} \left(I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} - I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T\right) + \\
&+ \rho_{(2)} \left(I_{N^{(1)}} \otimes \left(J_{N^{(2)}} - I_{N^{(2)}}\right) \otimes \left(I_{N^{(3)}} \otimes J_T\right)\right) + \rho_{(1)} \left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes \left(I_{N^{(2)}N^{(3)}} \otimes J_T\right) + \\
&+ \sigma_\lambda^2 J_{N^{(1)}N^{(2)}N^{(3)}} \otimes I_T + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}
\end{aligned}
$$

and finally, the covariance matrix of model (6) is

$$
\begin{aligned}
E[uu'] &= \sigma_\mu^2 I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T + \rho_{(3)} \left(I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} - I_{N^{(1)}N^{(2)}N^{(3)}} \otimes J_T\right) + \\
&+ \rho_{(2)} \left(I_{N^{(1)}} \otimes \left(J_{N^{(2)}} - I_{N^{(2)}}\right) \otimes \left(I_{N^{(3)}} \otimes J_T\right)\right) + \rho_{(1)} \left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes \left(I_{N^{(2)}N^{(3)}} \otimes J_T\right) + \\
&+ \sigma_v^{(1)^2} I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}} \otimes I_T + \sigma_v^{(2)^2} J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}} \otimes I_T + \\
&+ \sigma_v^{(3)^2} J_{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}T} + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}
\end{aligned}
$$

Models (4), (5) and (7)

The covariance matrixes of models (4), (5) and (7) are slightly more complicated as there more variance components to take into account. The covariance matrix of model (4) now is

$$
\begin{aligned}
E[uu'] &= \sigma_\mu^{(1)^2} I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} + \rho_{(2)}^{(1)} \left(I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}T} - I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T}\right) + \\
&+ \rho_{(1)}^{(1)} \left(\left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes \left(I_{N^{(2)}} \otimes J_{N^{(3)}T}\right)\right) + \sigma_\mu^{(2)^2} I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T + \\
&+ \rho_{(2)}^{(2)} \left(I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}T} - I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T\right) + \\
&+ \rho_{(1)}^{(2)} \left(\left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T\right) + \sigma_\mu^{(3)^2} J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T + \\
&+ \rho_{(2)}^{(3)} \left(J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}T} - J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T\right) + \\
&+ \rho_{(1)}^{(3)} \left(J_{N^{(1)}} \otimes \left(J_{N^{(2)}} - I_{N^{(2)}}\right) \otimes I_{N^{(3)}} \otimes J_T\right) + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}
\end{aligned}
$$

For model (5) we get

$$
\begin{aligned}
E[uu'] &= \sigma_\mu^{(1)^2} I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} + \rho_{(2)}^{(1)} \left(I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}T} - I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T}\right) + \\
&+ \rho_{(1)}^{(1)} \left(\left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes \left(I_{N^{(2)}} \otimes J_{N^{(3)}T}\right)\right) + \sigma_\mu^{(2)^2} I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T + \\
&+ \rho_{(2)}^{(2)} \left(I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}T} - I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T\right) + \\
&+ \rho_{(1)}^{(2)} \left(\left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T\right) + \\
&+ \sigma_\mu^{(3)^2} J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T + \rho_{(2)}^{(3)} \left(J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}T} - J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T\right) + \\
&+ \rho_{(1)}^{(3)} \left(J_{N^{(1)}} \otimes \left(J_{N^{(2)}} - I_{N^{(2)}}\right) \otimes I_{N^{(3)}} \otimes J_T\right) + \sigma_\lambda^2 J_{N^{(1)}N^{(2)}N^{(3)}} \otimes I_T + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}
\end{aligned}
$$

And finally, for model (7) we get the covariance matrix

$$
\begin{aligned}
E[uu'] &= \sigma_\mu^{(1)^2} I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T} + \rho_{(2)}^{(1)} \left(I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}T} - I_{N^{(1)}N^{(2)}} \otimes J_{N^{(3)}T}\right) + \\
&+ \rho_{(1)}^{(1)} \left(\left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes \left(I_{N^{(2)}} \otimes J_{N^{(3)}T}\right)\right) + \sigma_\mu^{(2)^2} I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T + \\
&+ \rho_{(2)}^{(2)} \left(I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}T} - I_{N^{(1)}} \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T\right) + \\
&+ \rho_{(1)}^{(2)} \left(\left(J_{N^{(1)}} - I_{N^{(1)}}\right) \otimes J_{N^{(2)}} \otimes I_{N^{(3)}} \otimes J_T\right) + \sigma_\mu^{(3)^2} J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T + \\
&+ \rho_{(2)}^{(3)} \left(J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}T} - J_{N^{(1)}} \otimes I_{N^{(2)}N^{(3)}} \otimes J_T\right) + \\
&+ \rho_{(1)}^{(3)} \left(J_{N^{(1)}} \otimes \left(J_{N^{(2)}} - I_{N^{(2)}}\right) \otimes I_{N^{(3)}} \otimes J_T\right) + \sigma_v^{(1)^2} I_{N^{(1)}} \otimes J_{N^{(2)}N^{(3)}} \otimes I_T + \\
&+ \sigma_v^{(2)^2} J_{N^{(1)}} \otimes I_{N^{(2)}} \otimes J_{N^{(3)}} \otimes I_T + \sigma_v^{(3)^2} J_{N^{(1)}N^{(2)}} \otimes I_{N^{(3)}T} + \sigma_\epsilon^2 I_{N^{(1)}N^{(2)}N^{(3)}T}
\end{aligned}
$$

3.4 Estimation of the Variance Components and Cross Correlations

Models (2) and (3)

The estimation of the variance components for models (2) and (3) does not change in this case, but of course the cross correlation coefficients need to be estimated. For model (2) the identifying equation are

$$
E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(1)}}\sigma_\mu^2 + \frac{1}{N^{(1)}}\sigma_\epsilon^2 + \frac{N^{(1)}-1}{N^{(1)}}\rho_{(1)}
$$

$$
E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(2)}}\sigma_\mu^2 + \frac{1}{N^{(2)}}\sigma_\epsilon^2 + \frac{N^{(2)}-1}{N^{(2)}}\rho_{(2)}
$$

$$
E\left[\left(\frac{1}{N^{(3)}}\sum_{s=1}^{N^{(3)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(3)}}\sigma_\mu^2 + \frac{1}{N^{(3)}}\sigma_\epsilon^2 + \frac{N^{(3)}-1}{N^{(3)}}\rho_{(3)}
$$

So we get

$$
\hat{\rho}_{(1)} = \frac{1}{\left(N^{(1)}-1\right) N^{(1)}N^{(2)}N^{(3)}T}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{i=1}^{N^{(1)}} \hat{u}_{ijst}\right)^2 - \frac{1}{N^{(1)}-1}\hat{\sigma}_\mu^2 - \frac{1}{N^{(1)}-1}\hat{\sigma}_\epsilon^2
$$

$$
\hat{\rho}_{(2)} = \frac{1}{\left(N^{(2)}-1\right) N^{(1)}N^{(2)}N^{(3)}T}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{j=1}^{N^{(2)}} \hat{u}_{ijst}\right)^2 - \frac{1}{N^{(2)}-1}\hat{\sigma}_\mu^2 - \frac{1}{N^{(2)}-1}\hat{\sigma}_\epsilon^2
$$

$$
\hat{\rho}_{(3)} = \frac{1}{\left(N^{(3)}-1\right) N^{(1)}N^{(2)}N^{(3)}T}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T}\left(\sum_{s=1}^{N^{(3)}} \hat{u}_{ijst}\right)^2 - \frac{1}{N^{(3)}-1}\hat{\sigma}_\mu^2 - \frac{1}{N^{(3)}-1}\hat{\sigma}_\epsilon^2
$$

Turning our attention to model (3) now

$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}}\sigma_\mu^2+\sigma_\lambda^2+\frac{1}{N^{(1)}}\sigma_\epsilon^2+\frac{N^{(1)}-1}{N^{(1)}}\rho_{(1)}$$
$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(2)}}\sigma_\mu^2+\sigma_\lambda^2+\frac{1}{N^{(2)}}\sigma_\epsilon^2+\frac{N^{(2)}-1}{N^{(2)}}\rho_{(2)}$$
$$E\left[\left(\frac{1}{N^{(3)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(3)}}\sigma_\mu^2+\sigma_\lambda^2+\frac{1}{N^{(3)}}\sigma_\epsilon^2+\frac{N^{(3)}-1}{N^{(3)}}\rho_{(3)}$$

and so

$$\hat{\rho}_{(1)}=\frac{1}{\left(N^{(1)}-1\right)N^{(1)}N^{(2)}N^{(3)}T}\times$$
$$\times\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{i=1}^{N^{(1)}}\hat{u}_{ijst}\right)^2-\frac{1}{N^{(1)}-1}\hat{\sigma}_\mu^2-\frac{N^{(1)}}{N^{(1)}-1}\hat{\sigma}_\lambda^2-\frac{1}{N^{(1)}-1}\hat{\sigma}_\epsilon^2$$
$$\hat{\rho}_{(2)}=\frac{1}{\left(N^{(2)}-1\right)N^{(1)}N^{(2)}N^{(3)}T}\times$$
$$\times\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{j=1}^{N^{(2)}}\hat{u}_{ijst}\right)^2-\frac{1}{N^{(2)}-1}\hat{\sigma}_\mu^2-\frac{N^{(2)}}{N^{(2)}-1}\hat{\sigma}_\lambda^2-\frac{1}{N^{(2)}-1}\hat{\sigma}_\epsilon^2$$
$$\hat{\rho}_{(3)}=\frac{1}{\left(N^{(3)}-1\right)N^{(1)}N^{(2)}N^{(3)}T}\times$$
$$\times\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T}\left(\sum_{s=1}^{N^{(3)}}\hat{u}_{ijst}\right)^2-\frac{1}{N^{(3)}-1}\hat{\sigma}_\mu^2-\frac{N^{(3)}}{N^{(3)}-1}\hat{\sigma}_\lambda^2-\frac{1}{N^{(3)}-1}\hat{\sigma}_\epsilon^2$$

Models (6)

In the case of model (6) the estimation of the variance components of ϵ and μ remain unchanged (i.e., are as in the case of the model without cross correlation), otherwise

$$E\left[u_{ijst}^2\right] = \sigma_\mu^2 + \sigma_v^{(1)^2} + \sigma_v^{(2)^2} + \sigma_v^{(3)^2} + \sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(1)}}\sigma_\mu^2 + \frac{1}{N^{(1)}}\sigma_v^{(1)^2} + \sigma_v^{(2)^2} + \sigma_v^{(3)^2} + \frac{1}{N^{(1)}}\sigma_\epsilon^2 + \frac{N^{(1)}-1}{N^{(1)}}\rho_{(1)}$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(2)}}\sigma_\mu^2 + \sigma_v^{(1)^2} + \frac{1}{N^{(2)}}\sigma_v^{(2)^2} + \sigma_v^{(3)^2} + \frac{1}{N^{(2)}}\sigma_\epsilon^2 + \frac{N^{(2)}-1}{N^{(2)}}\rho_{(2)}$$

$$E\left[\left(\frac{1}{N^{(3)}}\sum_{s=1}^{N^{(3)}} u_{ijst}\right)^2\right] = \frac{1}{N^{(3)}}\sigma_\mu^2 + \sigma_v^{(1)^2} + \sigma_v^{(2)^2} + \frac{1}{N^{(3)}}\sigma_v^{(3)^2} + \frac{1}{N^{(3)}}\sigma_\epsilon^2 + \frac{N^{(3)}-1}{N^{(3)}}\rho_{(3)}$$

$$E\left[\left(\frac{1}{N^{(1)}T}\sum_{i=1}^{N^{(1)}}\sum_{t=1}^{T} u_{ijst}\right)^2\right] = \frac{1}{N^{(1)}}\sigma_\mu^2 + \frac{1}{N^{(1)}T}\sigma_v^{(1)^2} + \frac{1}{T}\sigma_v^{(2)^2} + \frac{1}{T}\sigma_v^{(3)^2} +$$
$$+ \frac{1}{N^{(1)}T}\sigma_\epsilon^2 + \frac{N^{(1)}-1}{N^{(1)}}\rho_{(1)}$$

$$E\left[\left(\frac{1}{N^{(2)}T}\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T} u_{ijst}\right)^2\right] = \frac{1}{N^{(2)}}\sigma_\mu^2 + \frac{1}{T}\sigma_v^{(1)^2} + \frac{1}{N^{(2)}T}\sigma_v^{(2)^2} + \frac{1}{T}\sigma_v^{(3)^2} + \frac{1}{N^{(2)}T}\sigma_\epsilon^2 +$$
$$+ \frac{N^{(2)}-1}{N^{(2)}}\rho_{(2)}$$

So we get for the estimation of the cross correlations and the variance components

$$\hat{\rho}_{(1)} = \frac{1}{(T-1)\left(N^{(1)}-1\right)N^{(1)}N^{(2)}N^{(3)}T}\times$$
$$\times \sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\left[\left(\sum_{i=1}^{N^{(1)}}\sum_{t=1}^{T}\hat{u}_{ijst}\right)^2 - \sum_{t=1}^{T}\left(\sum_{i=1}^{N^{(1)}}\hat{u}_{ijst}\right)^2\right] - \frac{1}{N^{(1)}-1}\hat{\sigma}_\mu^2$$

$$\hat{\rho}_{(2)} = \frac{1}{(T-1)\left(N^{(2)}-1\right)N^{(1)}N^{(2)}N^{(3)}T}\times$$
$$\times \sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\left[\left(\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T}\hat{u}_{ijst}\right)^2 - \sum_{t=1}^{T}\left(\sum_{j=1}^{N^{(2)}}\hat{u}_{ijst}\right)^2\right] - \frac{1}{N^{(2)}-1}\hat{\sigma}_\mu^2$$

$$\hat{\sigma}_v^{(1)^2} = \frac{1}{\left(N^{(1)}-1\right)N^{(1)}N^{(2)}N^{(3)}T}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t}^{T}\left(N^{(1)}\sum_{i=1}^{N^{(1)}}\hat{u}_{ijst}^2-\left(\sum_{i=1}^{N^{(1)}}\hat{u}_{ijst}\right)^2\right)+\hat{\rho}_{(1)}$$

$$\hat{\sigma}_v^{(2)^2} = \frac{1}{\left(N^{(2)}-1\right)N^{(1)}N^{(2)}N^{(3)}T}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\sum_{t}^{T}\left(N^{(2)}\sum_{j=1}^{N^{(2)}}\hat{u}_{ijst}^2-\left(\sum_{j=1}^{N^{(2)}}\hat{u}_{ijst}\right)^2\right)+\hat{\rho}_{(2)}$$

$$\hat{\sigma}_v^{(3)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\hat{u}_{ijst}^2-\hat{\sigma}_\mu^2-\hat{\sigma}_v^{(1)^2}-\hat{\sigma}_v^{(2)^2}-\hat{\sigma}_\epsilon^2$$

$$\hat{\rho_{(3)}} = \frac{1}{\left(N^{(3)}-1\right)N^{(1)}N^{(2)}N^{(3)}T}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{t}^{T}\left(\sum_{s=1}^{N^{(3)}}\hat{u}_{ijst}\right)^2-\frac{1}{N^{(3)}-1}\hat{\sigma}_\mu^2-\frac{N^{(3)}}{N^{(3)}-1}\hat{\sigma}_v^{(1)^2}-$$
$$-\frac{N^{(3)}}{N^{(3)}-1}\hat{\sigma}_v^{(2)^2}-\frac{1}{N^{(3)}-1}\hat{\sigma}_v^{(3)^2}-\frac{1}{N^{(3)}-1}\hat{\sigma}_\epsilon^2$$

Models (4)

For model (4) the Within transformation remains as for the model without cross correlation, so the estimation of the variance of ϵ is exactly as in section 3.2. Overall, the following identifying equations can be derived

$$E\left[u_{ijst}^2\right]=\sigma_\mu^{(1)^2}+\sigma_\mu^{(2)^2}+\sigma_\mu^{(3)^2}+\sigma_\epsilon^2$$
$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(1)}}\sigma_\mu^{(2)^2}+\sigma_\mu^{(3)^2}+\frac{1}{N^{(1)}}\sigma_\epsilon^2+$$
$$+\frac{N^{(1)}-1}{N^{(1)}}\rho_{(1)}^{(1)}+\frac{N^{(1)}-1}{N^{(1)}}\rho_{(1)}^{(2)}$$
$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(2)}}\sigma_\mu^{(1)^2}+\sigma_\mu^{(2)^2}+\frac{1}{N^{(2)}}\sigma_\mu^{(3)^2}+\frac{1}{N^{(2)}}\sigma_\epsilon^2+$$
$$+\frac{N^{(2)}-1}{N^{(2)}}\rho_{(2)}^{(1)}+\frac{N^{(2)}-1}{N^{(2)}}\rho_{(1)}^{(3)}$$
$$E\left[\left(\frac{1}{N^{(3)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\sigma_\mu^{(1)^2}+\frac{1}{N^{(3)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(3)}}\sigma_\mu^{(3)^2}+\frac{1}{N^{(3)}}\sigma_\epsilon^2+$$
$$+\frac{N^{(3)}-1}{N^{(3)}}\rho_{(2)}^{(2)}+\frac{N^{(3)}-1}{N^{(3)}}\rho_{(2)}^{(3)}$$

$$E\left[\left(\frac{1}{N^{(1)}N^{(2)}}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}N^{(2)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(1)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(2)}}\sigma_\mu^{(3)^2}+\frac{1}{N^{(1)}N^{(2)}}\sigma_\epsilon^2+$$

$$+\frac{N^{(1)}-1}{N^{(1)}N^{(2)}}\rho_{(1)}^{(1)}+\frac{N^{(1)}-1}{N^{(1)}N^{(2)}}\rho_{(1)}^{(2)}+\frac{N^{(2)}-1}{N^{(2)}N^{(1)}}\rho_{(2)}^{(1)}+\frac{N^{(2)}-1}{N^{(2)}N^{(1)}}\rho_{(1)}^{(3)}$$

$$E\left[\left(\frac{1}{N^{(1)}N^{(3)}}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(1)}N^{(3)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(3)}}\sigma_\mu^{(3)^2}+\frac{1}{N^{(1)}N^{(3)}}\sigma_\epsilon^2+$$

$$+\frac{N^{(1)}-1}{N^{(1)}N^{(3)}}\rho_{(1)}^{(1)}+\frac{N^{(1)}-1}{N^{(1)}N^{(3)}}\rho_{(1)}^{(2)}+\frac{N^{(3)}-1}{N^{(1)}N^{(3)}}\rho_{(2)}^{(2)}+\frac{N^{(3)}-1}{N^{(1)}N^{(3)}}\rho_{(2)}^{(3)}$$

$$E\left[\left(\frac{1}{N^{(2)}N^{(3)}}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(2)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(3)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(2)}N^{(3)}}\sigma_\mu^{(3)^2}+\frac{1}{N^{(2)}N^{(3)}}\sigma_\epsilon^2+$$

$$+\frac{N^{(2)}-1}{N^{(2)}N^{(3)}}\rho_{(2)}^{(1)}+\frac{N^{(2)}-1}{N^{(2)}N^{(3)}}\rho_{(1)}^{(3)}+\frac{N^{(3)}-1}{N^{(2)}N^{(3)}}\rho_{(2)}^{(2)}+\frac{N^{(3)}-1}{N^{(2)}N^{(3)}}\rho_{(2)}^{(3)}$$

$$E\left[\left(\frac{1}{N^{(1)}N^{(2)}N^{(3)}}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\frac{1}{N^{(1)}N^{(2)}}\sigma_\mu^{(1)^2}+\frac{1}{N^{(1)}N^{(3)}}\sigma_\mu^{(2)^2}+\frac{1}{N^{(2)}N^{(3)}}\sigma_\mu^{(3)^2}+$$

$$+\frac{1}{N^{(1)}N^{(2)}N^{(3)}}\sigma_\epsilon^2+\frac{N^{(1)}-1}{N^{(1)}N^{(2)}N^{(3)}}\rho_{(1)}^{(1)}+\frac{N^{(1)}-1}{N^{(1)}N^{(2)}N^{(3)}}\rho_{(1)}^{(2)}+\frac{N^{(2)}-1}{N^{(1)}N^{(2)}N^{(3)}}\rho_{(2)}^{(1)}+$$

$$+\frac{N^{(2)}-1}{N^{(1)}N^{(2)}N^{(3)}}\rho_{(1)}^{(3)}+\frac{N^{(3)}-1}{N^{(1)}N^{(2)}N^{(3)}}\rho_{(2)}^{(2)}+\frac{N^{(3)}-1}{N^{(1)}N^{(2)}N^{(3)}}\rho_{(2)}^{(3)}$$

Altogether we have 8 identifying equations but unfortunately 9 unknown variance components and correlation coefficients. These cannot be estimated without further restrictions on the parameters. Let us impose the additional assumption that $\sigma_\mu^{(1)^2}=\sigma_\mu^{(2)^2}=\sigma_\mu^{(3)^2}=\sigma_\mu^2$. Under this assumption we need to estimate only 7 unknown parameters. From the first identifying equation

$$\hat\sigma_\mu^2=\frac{1}{3N^{(1)}N^{(2)}N^{(3)}T}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\hat u_{ijst}^2-\frac{1}{3}\hat\sigma_\epsilon^2$$

Notice however, that the above identifying equations 5 – 8 are, unfortunately, linear combinations of the equations 2 – 5. This means that we need to impose further restrictions on the model. Let us assume, in addition, that

$$E[\mu_{ij}^{(1)}\mu_{i'j'}^{(1)}]=\begin{cases}\sigma_\mu^2 & i=i' \text{ and } j=j'\\ \rho_{(1)} & i=i' \text{ and } j\neq j'\\ \rho_{(2)} & i\neq i' \text{ and } j=j'\\ 0 & i\neq i' \text{ and } j\neq j'\end{cases}$$

$$E[\mu_{is}^{(2)}\mu_{i's'}^{(2)}] = \begin{cases} \sigma_\mu^2 & i = i' \text{ and } s = s' \\ \rho_{(1)} & i = i' \text{ and } s \neq s' \\ \rho_{(3)} & i \neq i' \text{ and } s = s' \\ 0 & i \neq i' \text{ and } j \neq j' \end{cases}$$

$$E[\mu_{js}^{(3)}\mu_{j's'}^{(3)}] = \begin{cases} \sigma_\mu^2 & j = j' \text{ and } s = s' \\ \rho_{(2)} & j = j' \text{ and } s \neq s' \\ \rho_{(3)} & j \neq j' \text{ and } s = s' \\ 0 & i \neq i' \text{ and } j \neq j' \end{cases}$$

Now, the identifying equations take the form

$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}} u_{ijst}\right)^2\right] = \frac{2+N^{(1)}}{N^{(1)}}\sigma_\mu^2 + \frac{1}{N^{(1)}}\sigma_\epsilon^2 + \frac{N^{(1)}-1}{N^{(1)}}\rho_{(2)} + \frac{N^{(1)}-1}{N^{(1)}}\rho_{(3)}$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}} u_{ijst}\right)^2\right] = \frac{2+N^{(2)}}{N^{(2)}}\sigma_\mu^2 + \frac{1}{N^{(2)}}\sigma_\epsilon^2 + \frac{N^{(2)}-1}{N^{(2)}}\rho_{(1)} + \frac{N^{(2)}-1}{N^{(2)}}\rho_{(3)}$$

$$E\left[\left(\frac{1}{N^{(3)}}\sum_{s=1}^{N^{(3)}} u_{ijst}\right)^2\right] = \frac{2+N^{(3)}}{N^{(3)}}\sigma_\mu^2 + \frac{1}{N^{(3)}}\sigma_\epsilon^2 + \frac{N^{(3)}-1}{N^{(3)}}\rho_{(1)} + \frac{N^{(3)}-1}{N^{(3)}}\rho_{(2)}$$

So we get the following estimators for the cross correlations

$$\begin{aligned}
\hat{\rho}_{(1)} = & \frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{j=1}^{N^{(2)}}\hat{u}_{ijst}\right)^2 + \\
& + \frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(3)}-1\right)}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T}\left(\sum_{s=1}^{N^{(3)}}\hat{u}_{ijst}\right)^2 - \\
& - \frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{i=1}^{N^{(1)}}\hat{u}_{ijst}\right)^2 - \\
& - \hat{\sigma}_\mu^2\frac{1}{2}\left(\frac{N^{(3)}+2}{N^{(3)}-1} + \frac{N^{(2)}+2}{N^{(2)}-1} - \frac{N^{(1)}+2}{N^{(1)}-1}\right) - \\
& - \hat{\sigma}_\epsilon^2\frac{1}{2}\left(\frac{1}{N^{(3)}-1} + \frac{1}{N^{(2)}-1} - \frac{1}{N^{(1)}-1}\right)
\end{aligned}$$

$$\hat{\rho}_{(2)} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(3)}-1\right)}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T}\left(\sum_{s=1}^{N^{(3)}}\hat{u}_{ijst}\right)^2 - \hat{\sigma}_\mu^2\frac{N^{(3)}+2}{N^{(3)}-1} - \hat{\sigma}_\epsilon^2\frac{1}{N^{(3)}-1} - \hat{\rho}_{(1)}$$

$$\hat{\rho}_{(3)} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{j=1}^{N^{(2)}}\hat{u}_{ijst}\right)^2 - \hat{\sigma}_\mu^2\frac{N^{(2)}+2}{N^{(2)}-1} - \hat{\sigma}_\epsilon^2\frac{1}{N^{(2)}-1} - \hat{\rho}_{(1)}$$

Models (5)

Let us continue with model (5). Like for the previous model, the Within transformation is still as for the model without cross correlation, so the estimation of the variance of ϵ and that of λ remains as in section 3.2. Making the same assumption as above for model (4), the identifying equations now are

$$E[u_{ijst}^2] = 3\sigma_\mu^2 + \sigma_\lambda^2 + \sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}}u_{ijst}\right)^2\right] = \frac{2+N^{(1)}}{N^{(1)}}\sigma_\mu^2 + \sigma_\lambda^2 + \frac{1}{N^{(1)}}\sigma_\epsilon^2 + \frac{N^{(1)}-1}{N^{(1)}}\rho_{(2)} + \frac{N^{(1)}-1}{N^{(1)}}\rho_{(3)}$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right] = \frac{2+N^{(2)}}{N^{(2)}}\sigma_\mu^2 + \sigma_\lambda^2 + \frac{1}{N^{(2)}}\sigma_\epsilon^2 + \frac{N^{(2)}-1}{N^{(2)}}\rho_{(1)} + \frac{N^{(2)}-1}{N^{(2)}}\rho_{(3)}$$

$$E\left[\left(\frac{1}{N^{(3)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right] = \frac{2+N^{(3)}}{N^{(3)}}\sigma_\mu^2 + \sigma_\lambda^2 + \frac{1}{N^{(3)}}\sigma_\epsilon^2 + \frac{N^{(3)}-1}{N^{(3)}}\rho_{(1)} + \frac{N^{(3)}-1}{N^{(3)}}\rho_{(2)}$$

And so this leads to

$$\hat{\sigma}_\mu^2 = \frac{1}{3N^{(1)}N^{(2)}N^{(3)}T}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\hat{u}_{ijst}^2 - \frac{1}{3}\hat{\sigma}_\lambda^2 - \frac{1}{3}\hat{\sigma}_\epsilon^2$$

$$\hat{\rho}_{(1)} = \frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)} \sum_{i=1}^{N^{(1)}} \sum_{s=1}^{N^{(3)}} \sum_{t=1}^{T} \left(\sum_{j=1}^{N^{(2)}} \hat{u}_{ijst} \right)^2 +$$

$$+ \frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(3)}-1\right)} \sum_{i=1}^{N^{(1)}} \sum_{j=1}^{N^{(2)}} \sum_{t=1}^{T} \left(\sum_{s=1}^{N^{(3)}} \hat{u}_{ijst} \right)^2 -$$

$$- \frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)} \sum_{j=1}^{N^{(2)}} \sum_{s=1}^{N^{(3)}} \sum_{t=1}^{T} \left(\sum_{i=1}^{N^{(1)}} \hat{u}_{ijst} \right)^2 -$$

$$- \hat{\sigma}_{\mu}^2 \frac{1}{2} \left(\frac{N^{(3)}+2}{N^{(3)}-1} + \frac{N^{(2)}+2}{N^{(2)}-1} - \frac{N^{(1)}+2}{N^{(1)}-1} \right) - \hat{\sigma}_{\lambda}^2 \frac{1}{2} \left(\frac{N^{(3)}}{N^{(3)}-1} + \frac{N^{(2)}}{N^{(2)}-1} - \frac{N^{(1)}}{N^{(1)}-1} \right) -$$

$$- \hat{\sigma}_{\epsilon}^2 \frac{1}{2} \left(\frac{1}{N^{(3)}-1} + \frac{1}{N^{(2)}-1} - \frac{1}{N^{(1)}-1} \right)$$

$$\hat{\rho}_{(2)} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(3)}-1\right)} \sum_{i=1}^{N^{(1)}} \sum_{j=1}^{N^{(2)}} \sum_{t=1}^{T} \left(\sum_{s=1}^{N^{(3)}} \hat{u}_{ijst} \right)^2 - \hat{\sigma}_{\mu}^2 \frac{N^{(3)}+2}{N^{(3)}-1} - \hat{\sigma}_{\lambda}^2 \frac{N^{(3)}}{N^{(3)}-1} -$$

$$- \hat{\sigma}_{\epsilon}^2 \frac{1}{N^{(3)}-1} - \hat{\rho}_{(1)}$$

$$\hat{\rho}_{(3)} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)} \sum_{i=1}^{N^{(1)}} \sum_{s=1}^{N^{(3)}} \sum_{t=1}^{T} \left(\sum_{j=1}^{N^{(2)}} \hat{u}_{ijst} \right)^2 - \hat{\sigma}_{\mu}^2 \frac{N^{(2)}+2}{N^{(2)}-1} - \hat{\sigma}_{\lambda}^2 \frac{N^{(2)}}{N^{(2)}-1} -$$

$$- \hat{\sigma}_{\epsilon}^2 \frac{1}{N^{(2)}-1} - \hat{\rho}_{(1)}$$

Models (7)

Finally, for model (7), making the same additional parameter restrictions as for models (4) and (5) we get the following identifying equations

$$E[u_{ijst}^2] = 3\sigma_{\mu}^2 + \sigma_v^{(1)^2} + \sigma_v^{(2)^2} + \sigma_v^{(3)^2} + \sigma_{\epsilon}^2$$

$$E\left[\left(\frac{1}{T}\sum_{t=1}^{T}u_{ijst}\right)^2\right]=3\sigma_\mu^2+\frac{1}{T}\sigma_v^{(1)^2}+\frac{1}{T}\sigma_v^{(2)^2}+\frac{1}{T}\sigma_v^{(3)^2}+\frac{1}{T}\sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{N^{(1)}}\sum_{i=1}^{N^{(1)}}u_{ijst}\right)^2\right]=\frac{2+N^{(1)}}{N^{(1)}}\sigma_\mu^2+\frac{1}{N^{(1)}}\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+\frac{1}{N^{(1)}}\sigma_\epsilon^2+$$

$$+\frac{N^{(1)}-1}{N^{(1)}}\rho_{(2)}+\frac{N^{(1)}-1}{N^{(1)}}\rho_{(3)}$$

$$E\left[\left(\frac{1}{N^{(2)}}\sum_{j=1}^{N^{(2)}}u_{ijst}\right)^2\right]=\frac{2+N^{(2)}}{N^{(2)}}\sigma_\mu^2+\sigma_v^{(1)^2}+\frac{1}{N^{(2)}}\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+\frac{1}{N^{(2)}}\sigma_\epsilon^2+$$

$$+\frac{N^{(2)}-1}{N^{(2)}}\rho_{(1)}+\frac{N^{(2)}-1}{N^{(2)}}\rho_{(3)}$$

$$E\left[\left(\frac{1}{N^{(3)}}\sum_{s=1}^{N^{(3)}}u_{ijst}\right)^2\right]=\frac{2+N^{(3)}}{N^{(3)}}\sigma_\mu^2+\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+\frac{1}{N^{(3)}}\sigma_v^{(3)^2}+\frac{1}{N^{(3)}}\sigma_\epsilon^2+$$

$$+\frac{N^{(3)}-1}{N^{(3)}}\rho_{(1)}+\frac{N^{(3)}-1}{N^{(3)}}\rho_{(2)}$$

$$E\left[\left(\frac{1}{N^{(1)}T}\sum_{i=1}^{N^{(1)}}\sum_{t=1}^{T}u_{ijst}\right)^2\right]=\frac{2+N^{(1)}}{N^{(1)}}\sigma_\mu^2+\frac{1}{N^{(1)}T}\sigma_v^{(1)^2}+\frac{1}{T}\sigma_v^{(2)^2}+\frac{1}{T}\sigma_v^{(3)^2}+$$

$$+\frac{1}{N^{(1)}T}\sigma_\epsilon^2+\frac{N^{(1)}-1}{N^{(1)}}\rho_{(2)}+\frac{N^{(1)}-1}{N^{(1)}}\rho_{(3)}$$

$$E\left[\left(\frac{1}{N^{(2)}T}\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T}u_{ijst}\right)^2\right]=\frac{2+N^{(2)}}{N^{(2)}}\sigma_\mu^2+\frac{1}{T}\sigma_v^{(1)^2}+\frac{1}{N^{(2)}T}\sigma_v^{(2)^2}+\frac{1}{T}\sigma_v^{(3)^2}+$$

$$+\frac{1}{N^{(2)}T}\sigma_\epsilon^2+\frac{N^{(2)}-1}{N^{(2)}}\rho_{(1)}+\frac{N^{(2)}-1}{N^{(2)}}\rho_{(3)}$$

which lead to the following estimators

$$\hat{\sigma}_\mu^2 = \frac{1}{3N^{(1)}N^{(2)}N^{(3)}T(T-1)} \sum_{i=1}^{N^{(1)}} \sum_{j=1}^{N^{(2)}} \sum_{s=1}^{N^{(3)}} \left(\left(\sum_{t=1}^{T} \hat{u}_{ijst} \right)^2 - \sum_{t=1}^{T} \hat{u}_{ijst}^2 \right)$$

$$\hat{\sigma}_v^{(1)^2} = \frac{1}{\left(N^{(1)}-1\right) N^{(2)}N^{(3)}T} \sum_{i=1}^{N^{(1)}} \sum_{j=1}^{N^{(2)}} \sum_{s=1}^{N^{(3)}} \sum_{t=1}^{T} \hat{u}_{ijst}^2 -$$

$$- \frac{1}{N^{(1)}\left(N^{(1)}-1\right) N^{(2)}N^{(3)}(T-1)} \sum_{j=1}^{N^{(2)}} \sum_{s=1}^{N^{(3)}} \sum_{t=1}^{T} \left(\sum_{i=1}^{N^{(1)}} \hat{u}_{ijst} \right)^2 +$$

$$+ \frac{1}{N^{(1)}\left(N^{(1)}-1\right) N^{(2)}N^{(3)}T(T-1)} \sum_{j=1}^{N^{(2)}} \sum_{s=1}^{N^{(3)}} \left(\sum_{i=1}^{N^{(1)}} \sum_{t=1}^{T} \hat{u}_{ijst} \right)^2 - \frac{3N^{(1)}}{N^{(1)}-1} \hat{\sigma}_\mu^2 - \hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_v^{(2)^2} = \frac{1}{N^{(1)}\left(N^{(2)}-1\right) N^{(3)}T} \sum_{i=1}^{N^{(1)}} \sum_{j=1}^{N^{(2)}} \sum_{s=1}^{N^{(3)}} \sum_{t=1}^{T} \hat{u}_{ijst}^2 -$$

$$- \frac{1}{N^{(1)}N^{(2)}\left(N^{(2)}-1\right) N^{(3)}(T-1)} \sum_{i=1}^{N^{(1)}} \sum_{s=1}^{N^{(3)}} \sum_{t=1}^{T} \left(\sum_{j=1}^{N^{(2)}} \hat{u}_{ijst} \right)^2 +$$

$$+ \frac{1}{N^{(1)}N^{(2)}\left(N^{(2)}-1\right) N^{(3)}T(T-1)} \sum_{i=1}^{N^{(1)}} \sum_{s=1}^{N^{(3)}} \left(\sum_{j=1}^{N^{(2)}} \sum_{t=1}^{T} \hat{u}_{ijst} \right)^2 - \frac{3N^{(2)}}{N^{(2)}-1} \hat{\sigma}_\mu^2 - \hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_v^{(3)^2} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T} \sum_{i=1}^{N^{(1)}} \sum_{j=1}^{N^{(2)}} \sum_{s=1}^{N^{(3)}} \sum_{t=1}^{T} \hat{u}_{ijst}^2 - 3\hat{\sigma}_\mu^2 - \hat{\sigma}_v^{(1)^2} - \hat{\sigma}_v^{(2)^2} - \hat{\sigma}_\epsilon^2$$

$$\hat{\rho}_{(1)} = \frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)} \sum_{i=1}^{N^{(1)}} \sum_{s=1}^{N^{(3)}} \sum_{t=1}^{T} \left(\sum_{j=1}^{N^{(2)}} \hat{u}_{ijst} \right)^2 +$$

$$+ \frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(3)}-1\right)} \sum_{i=1}^{N^{(1)}} \sum_{j=1}^{N^{(2)}} \sum_{t=1}^{T} \left(\sum_{s=1}^{N^{(3)}} \hat{u}_{ijst} \right)^2 -$$

$$- \frac{1}{2N^{(1)}N^{(2)}N^{(3)}T\left(N^{(1)}-1\right)} \sum_{j=1}^{N^{(2)}} \sum_{s=1}^{N^{(3)}} \sum_{t=1}^{T} \left(\sum_{i=1}^{N^{(1)}} \hat{u}_{ijst} \right)^2 -$$

$$- \hat{\sigma}_\mu^2 \frac{1}{2} \left(\frac{N^{(3)}+2}{N^{(3)}-1} + \frac{N^{(2)}+2}{N^{(2)}-1} - \frac{N^{(1)}+2}{N^{(1)}-1} \right) - \hat{\sigma}_\epsilon^2 \frac{1}{2} \left(\frac{1}{N^{(3)}-1} + \frac{1}{N^{(2)}-1} - \frac{1}{N^{(1)}-1} \right) -$$

$$- \hat{\sigma}_v^{(1)^2} \frac{2N^{(1)}-1}{2N^{(1)}} - \hat{\sigma}_v^{(2)^2} \frac{1}{2N^{(2)}} - \hat{\sigma}_v^{(3)^2} \frac{1}{2N^{(3)}}$$

$$\hat{\rho}_{(2)} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(3)}-1\right)}\sum_{i=1}^{N^{(1)}}\sum_{j=1}^{N^{(2)}}\sum_{t=1}^{T}\left(\sum_{s=1}^{N^{(3)}}\hat{u}_{ijst}\right)^2 - \hat{\sigma}^2_\mu\frac{N^{(3)}+2}{N^{(3)}-1} -$$

$$-\hat{\sigma}^2_\epsilon\frac{1}{N^{(3)}-1} - \hat{\sigma}_v^{(1)^2} - \hat{\sigma}_v^{(2)^2} - \frac{1}{N^{(3)}}\hat{\sigma}_v^{(3)^2} - \hat{\rho}_{(1)}$$

$$\hat{\rho}_{(3)} = \frac{1}{N^{(1)}N^{(2)}N^{(3)}T\left(N^{(2)}-1\right)}\sum_{i=1}^{N^{(1)}}\sum_{s=1}^{N^{(3)}}\sum_{t=1}^{T}\left(\sum_{j=1}^{N^{(2)}}\hat{u}_{ijst}\right)^2 - \hat{\sigma}^2_\mu\frac{N^{(2)}+2}{N^{(2)}-1} -$$

$$-\hat{\sigma}^2_\epsilon\frac{1}{N^{(2)}-1} - \hat{\sigma}_v^{(1)^2} - \frac{1}{N^{(2)}}\hat{\sigma}_v^{(2)^2} - \hat{\sigma}_v^{(3)^2} - \hat{\rho}_{(1)}$$

4. Unbalanced Data

In order to be able formalize the nature of the data here, some new notations need to be introduced. Let be:

Z_{ijs} the set of time periods when firm i sells product j to country s, with T_{ijs} now being the number of elements in Z_{ijs}, $Z_{ijs} = \left\{z^1_{ijs}, ..., z^{T_{ijs}}_{ijs}\right\}$;

$Z^{(1)}_{js}$ the set of time periods when any firm sells product j to country s, with $T^{(1)}_{js}$ being the number of elements in $Z^{(1)}_{js}$;

$Z^{(2)}_{is}$ the set of time periods when firm i sells any product to country s, with $T^{(2)}_{is}$ being the number of elements in $Z^{(2)}_{is}$;

$Z^{(3)}_{ij}$ the set of time periods when firm i sells product j to any country, with $T^{(3)}_{ij}$ being the number of elements in $Z^{(3)}_{ij}$;

$Z^{(1)}_{i}$ the set of time periods when firm i sells any product to any country, with $T^{(1)}_{i}$ being the number of elements in $Z^{(1)}_{i}$;

$Z^{(2)}_{j}$ the set of time periods when any firm sells product j to any country, with $T^{(2)}_{j}$ being the number of elements in $Z^{(2)}_{j}$;

$Z^{(3)}_{s}$ the set of time periods when any firm sells any product to country s, with $T^{(3)}_{s}$ being number of elements in $Z^{(3)}_{s}$;

$Q^{(1)}_{jst}$ the set of firms that sell product j to country s at period of time t, with $N^{(1)}_{jst}$ being the number of elements in $Q^{(1)}_{jst}$;

$Q^{(2)}_{ist}$ the set of products that firm i sells to country s at period t, with $N^{(2)}_{ist}$ being the number of elements in $Q^{(2)}_{ist}$;

$Q^{(3)}_{ijt}$ the set of countries to which firm i sells product j at time t, with $N^{(3)}_{ijt}$ being the number of elements in $Q^{(3)}_{ijt}$;

$Q_{ij}^{(3)}$ the set of countries to which firm i sells product j at any time, with $N_{ij}^{(3)}$ being the number of elements in $Q_{ij}^{(3)}$;

$Q_{st}^{(1)}$ the set of firms that sell any product to country s at period of time t, with $N_{st}^{(1)}$ being the number of elements in $Q_{st}^{(1)}$;

$Q_{jt}^{(1)}$ the set of firms that sell product j to any country at period of time t, with $N_{jt}^{(1)}$ being the number of elements in $Q_{jt}^{(1)}$;

$Q_{it}^{(2)}$ the set of products that firm i sells to any country at period t, with $N_{it}^{(2)}$ being the number of elements in $Q_{it}^{(2)}$;

$Q_{i}^{(2)}$ the set of products that firm i sells to any country at any time, with $N_{i}^{(2)}$ being the number of elements in $Q_{i}^{(2)}$;

$Q_{i}^{(3)}$ the set of countries to which firm i sells any product at any time, with $N_{i}^{(3)}$ being the number of elements in $Q_{i}^{(3)}$;

$Q_{j}^{(3)}$ the set of countries to which any firm sells product j at any time, with $N_{j}^{(3)}$ being the number of elements in $Q_{j}^{(3)}$;

$Q^{(1)}$ the set of firms that at least sell a product to any country at any times, with $N^{(1)}$ being the number of elements in $Q^{(1)}$;

$Q^{(2)}$ the set of products being sold by any firm to any country at any times, with $N^{(2)}$ being the number of elements in $Q^{(2)}$;

$Q^{(3)}$ the set of countries to which any product has been sold by any firm at any times, with $N^{(3)}$ being the number of elements in $Q^{(3)}$.

4.1 Covariance Matrixes of the Different Models

Models (1)

For this model we have

$$u_{ijst} = \mu_i + \gamma_j + \alpha_s + \lambda_t + \epsilon_{ijst}$$

So we can build up the covariance matrix in the following way

$$\begin{aligned}
u_{ijs} &= \mu_i \otimes l_{T_{ijs}} + \gamma_j \otimes l_{T_{ijs}} + \alpha_s \otimes l_{T_{ijs}} + \lambda_{Z_{ijs}} + \epsilon_{ijs} \\
E\left[u_{ijs} {u'}_{ijs}\right] &= \sigma_\mu^2 J_{T_{ijs}} + \sigma_\gamma^2 J_{T_{ijs}} + \sigma_\alpha^2 J_{T_{ijs}} + \sigma_\lambda^2 I_{T_{ijs}} + \sigma_\epsilon^2 I_{T_{ijs}} \\
u_{ij} &= \mu_i \otimes l_{\sum_s T_{ijs}} + \gamma_j \otimes l_{\sum_s T_{ijs}} + \tilde{\alpha}_{ij} + \tilde{\lambda}_{ij} + \epsilon_{ij} \\
E\left[u_{ij} {u'}_{ij}\right] &= \sigma_\mu^2 J_{\sum_s T_{ijs}} + \sigma_\gamma^2 J_{\sum_s T_{ijs}} + \sigma_\alpha^2 A_{ij} + \sigma_\lambda^2 D_{ij} + \sigma_\epsilon^2 I_{\sum_s T_{ijs}} \\
u_i &= \mu_i \otimes l_{\sum_j \sum_s T_{ijs}} + \tilde{\gamma}_i + \tilde{\alpha}_i + \tilde{\lambda}_i + \epsilon_i \\
E\left[u_i {u'}_i\right] &= \sigma_\mu^2 J_{\sum_j \sum_s T_{ijs}} + \sigma_\gamma^2 B_i + \sigma_\alpha^2 F_{i,i} + \sigma_\lambda^2 D_i + \sigma_\epsilon^2 I_{\sum_j \sum_s T_{ijs}} \\
u &= \tilde{\mu} + \tilde{\gamma} + \tilde{\alpha} + \tilde{\lambda} + \epsilon \\
E\left[uu'\right] &= \sigma_\mu^2 C + \sigma_\gamma^2 B + \sigma_\alpha^2 F + \sigma_\lambda^2 D + \sigma_\epsilon^2 I_{\sum_i \sum_j \sum_s T_{ijs}}
\end{aligned}$$

where

$$\tilde{\mu}' = \left(\underbrace{\mu_1 \dots \mu_1}_{\sum_j \sum_s T_{1js} \text{ times}} \quad \underbrace{\mu_2 \dots \mu_2}_{\sum_j \sum_s T_{2js} \text{ times}} \quad \cdots \quad \underbrace{\mu_N^{(1)} \dots \mu_N^{(1)}}_{\sum_j \sum_s T_{N^{(1)}js} \text{ times}} \right)$$

$$C = \begin{pmatrix} J_{\sum_j \sum_s T_{1js}} & 0 & \cdots & 0 \\ 0 & J_{\sum_j \sum_s T_{2js}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{\sum_j \sum_s T_{N^{(1)}js}} \end{pmatrix}$$

$$\begin{aligned}
\tilde{\gamma}_i' &= \left(\underbrace{\gamma_1 \cdots \gamma_1}_{\sum_s T_{i1s} \text{ times}} \quad \underbrace{\gamma_2 \cdots \gamma_2}_{\sum_s T_{i2s} \text{ times}} \quad \cdots \quad \underbrace{\gamma_N^{(2)} \cdots \gamma_N^{(2)}}_{\sum_s T_{iN^{(2)}s} \text{ times}} \right) \\
\tilde{\gamma}' &= \left(\tilde{\gamma}_1', \tilde{\gamma}_2', ..., \tilde{\gamma}_{N^{(1)}}' \right)
\end{aligned}$$

$$B_i = \begin{pmatrix} J_{\sum_s T_{i1s}} & 0 & \cdots & 0 \\ 0 & J_{\sum_s T_{i2s}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{\sum_s T_{iN^{(2)}s}} \end{pmatrix}, \quad B = \begin{pmatrix} P_{1,1} & P_{1,2} & \cdots & P_{1,N^{(1)}} \\ P_{2,1} & P_{2,2} & \cdots & P_{2,N^{(1)}} \\ \vdots & \vdots & \ddots & \vdots \\ P_{N^{(1)},1} & P_{N^{(1)},2} & \cdots & P_{N^{(1)},N^{(1)}} \end{pmatrix}$$

where

$$P_{i,p} = \begin{pmatrix} J_{\left(\sum_s T_{i1s} \times \sum_s T_{p1s}\right)} & 0 & \cdots & 0 \\ 0 & J_{\left(\sum_s T_{i2s} \times \sum_s T_{p2s}\right)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{\left(\sum_s T_{iN^{(2)}s} \times \sum_s T_{pN^{(2)}s}\right)} \end{pmatrix}$$

$$\tilde{\alpha}'_{ij} = \left(\underbrace{\alpha_1 \dots \alpha_1}_{T_{ij1}\text{ times}} \quad \underbrace{\alpha_2 \dots \alpha_2}_{T_{ji2}\text{ times}} \quad \cdots \quad \underbrace{\alpha_{N^{(3)}} \dots \alpha_{N^{(3)}}}_{T_{ijN^{(3)}}\text{ times}} \right)$$

$$\tilde{\alpha}'_i = \left(\tilde{\alpha}'_{i1}, \tilde{\alpha}'_{i2}, ..., \tilde{\alpha}'_{iN^{(2)}}\right), \quad \tilde{\alpha}' = \left(\tilde{\alpha}'_1, \tilde{\alpha}'_2, ..., \tilde{\alpha}'_{N^{(1)}}\right)$$

$$A_{ij} = \begin{pmatrix} J_{T_{ij1}} & 0 & \dots & 0 \\ 0 & J_{T_{ij2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{T_{ijN^{(3)}}} \end{pmatrix}$$

$$K^{j,l}_{i,p} = \begin{pmatrix} J_{(T_{ij1} \times T_{pl1})} & 0 & \dots & 0 \\ 0 & J_{(T_{ij2} \times T_{pl2})} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{\left(T_{ijN^{(3)}} \times T_{plN^{(3)}}\right)} \end{pmatrix}$$

$$F_{i,p} = \begin{pmatrix} K^{1,1}_{i,p} & K^{1,2}_{i,p} & \dots & K^{1,N^{(2)}}_{i,p} \\ K^{2,1}_{i,p} & K^{2,2}_{i,p} & \dots & K^{2,N^{(2)}}_{i,p} \\ \vdots & \vdots & \ddots & \vdots \\ K^{N^{(2)},1}_{i,p} & K^{N^{(2)},2}_{i,p} & \dots & K^{N^{(2)},N^{(2)}}_{i,p} \end{pmatrix}, \quad F = \begin{pmatrix} F_{1,1} & F_{1,2} & \dots & F_{1,N^{(1)}} \\ F_{2,1} & F_{2,2} & \dots & F_{2,N^{(1)}} \\ \vdots & \vdots & \ddots & \vdots \\ F_{N^{(1)},1} & F_{N^{(1)},2} & \dots & F_{N^{(1)},N^{(1)}} \end{pmatrix}$$

Let us denote by $\lambda_{Z_{ijs}}$ the vector of length T_{ijs} of time effects associated with time periods from the set Z_{ijs}. Then $\tilde{\lambda}'_{ij} = \left(\lambda_{Z_{ij1}}, ..., \lambda_{Z_{ijN^{(3)}}}\right)$ and

$$\tilde{\lambda}'_i = \left(\tilde{\lambda}'_{i1}\tilde{\lambda}'_{i2}, ..., \tilde{\lambda}'_{iN^{(2)}}\right)$$
$$\tilde{\lambda}' = \left(\tilde{\lambda}'_1, \tilde{\lambda}'_2, ..., \tilde{\lambda}'_{N^{(1)}}\right)$$

Now about the $M_{T_{ijs} \times T_{lpr}}$ matrix of size $(T_{ijs} \times T_{lpr})$. Element of n-th row and k-th column is

$$\{m\}_{nk} = \begin{cases} 1 & \text{if } z^n_{ijs} = z^k_{lpr} \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

Further,

$$E_s = \begin{pmatrix} M_{T_{ij1} \times T_{ijs}} \\ M_{T_{ij2} \times T_{ijs}} \\ \dots \\ M_{T_{ijN^{(3)}} \times T_{ijs}} \end{pmatrix}, \quad D_{ij} = (E_1, E_2, ..., E_{N^{(3)}})$$

$$E_{js} = \begin{pmatrix} M_{T_{i11} \times T_{ijs}} \\ M_{T_{i12} \times T_{ijs}} \\ \dots \\ M_{T_{iN^{(2)}N^{(3)}} \times T_{ijs}} \end{pmatrix}, \quad D_i = (E_{11}, E_{12}, ..., E_{N^{(2)}N^{(3)}})$$

$$E_{ijs} = \begin{pmatrix} M_{T_{111} \times T_{ijs}} \\ M_{T_{112} \times T_{ijs}} \\ \dots \\ M_{T_{N^{(1)}N^{(2)}N^{(3)}} \times T_{ijs}} \end{pmatrix}, \quad D = (E_{111}, E_{112}, ..., E_{N^{(1)}N^{(2)}N^{(3)}})$$

Models (2)

This is a slightly simpler case than model (1) above. We have

$$u_{ijst} = \mu_{ijs} + \epsilon_{ijst}$$

So we can build up the covariance matrix in the usual way

$$\begin{aligned}
u_{ijs} &= \mu_{ijs} \otimes l_{T_{ijs}} + \epsilon_{ijst} \\
E\left[u_{ijs}u'_{ijs}\right] &= \sigma^2_\mu J_{T_{ijs}} + \sigma^2_\epsilon I_{T_{ijs}} \\
u_{ij} &= \tilde{\mu}_{ij} + \epsilon_{ij} \\
E\left[u_{ij}u'_{ij}\right] &= \sigma^2_\mu A_{ij} + \sigma^2_\epsilon I_{\sum_s T_{ijs}} \\
u_i &= \tilde{\mu}_i + \epsilon_i \\
E\left[u_i u'_i\right] &= \sigma^2_\mu A_i + \sigma^2_\epsilon I_{\sum_j \sum_s T_{ijs}} \\
u &= \tilde{\mu} + \epsilon \\
E\left[uu'\right] &= \sigma^2_\mu A + \sigma^2_\epsilon I_{\sum_i \sum_j \sum_s T_{ijs}}
\end{aligned}$$

where

$$\tilde{\mu}'_{ij} = \left(\underbrace{\mu_{ij1} \dots \mu_{ij1}}_{T_{ij1} \text{ times}} \quad \underbrace{\mu_{ij2} \dots \mu_{ij2}}_{T_{ij2} \text{ times}} \quad \dots \quad \underbrace{\mu_{ijN^{(3)}} \dots \mu_{ijN^{(3)}}}_{T_{ijN^{(3)}} \text{ times}} \right)$$

$$\tilde{\mu}'_i = (\tilde{\mu}'_{i1}, \tilde{\mu}'_{i2}, ..., \tilde{\mu}'_{i3}), \quad \tilde{\mu}' = (\tilde{\mu}'_1, \tilde{\mu}'_2, ..., \tilde{\mu}'_3)$$

$$A_i = \begin{pmatrix} A_{i1} & 0 & \dots & 0 \\ 0 & A_{i2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{iN^{(2)}} \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & 0 & \dots & 0 \\ 0 & A_{12} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{N^{(1)}N^{(2)}} \end{pmatrix}$$

Model (3)

Now we have

$$u_{ijst} = \mu_{ijs} + \lambda_t + \epsilon_{ijst}$$

We have already derived the covariance matrix of μ_{ijs} in model (2) and λ_t in model (1). Thus for model (3) covariance matrix takes the form

$$E\left[uu'\right] = \sigma_\mu^2 A + \sigma_\lambda^2 D + \sigma_\epsilon^2 I_{\sum_i \sum_j \sum_s T_{ijs}}$$

Model (4)

The composition of the disturbance term now is

$$u_{ijst} = \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + \epsilon_{ijst}$$

So we have

$$\begin{aligned}
u_{ijs} &= \mu_{ij}^{(1)} \otimes l_{T_{ijs}} + \mu_{is}^{(2)} \otimes l_{T_{ijs}} + \mu_{js}^{(3)} \otimes l_{T_{ijs}} + \epsilon_{ijs} \\
E\left[u_{ijs}u_{ijs}'\right] &= \sigma_\mu^{(1)^2} J_{T_{ijs}} + \sigma_\mu^{(2)^2} J_{T_{ijs}} + \sigma_\mu^{(3)^2} J_{T_{ijs}} + \sigma_\epsilon^2 I_{T_{ijs}} \\
u_{ij} &= \mu_{ij}^{(1)} \otimes l_{\sum_s T_{ijs}} + \tilde{\mu}_{ij}^{(2)} + \tilde{\mu}_{ij}^{(3)} + \epsilon_{ij} \\
E\left[u_{ij}u_{ij}'\right] &= \sigma_\mu^{(1)^2} J_{\sum_s T_{ijs}} + \sigma_\mu^{(2)^2} A_{ij} + \sigma_\mu^{(3)^2} A_{ij} + \sigma_\epsilon^2 I_{\sum_s T_{ijs}} \\
u_i &= \tilde{\mu}_i^{(1)} + \tilde{\mu}_i^{(2)} + \tilde{\mu}_i^{(3)} + \epsilon_i \\
E\left[u_i u_i'\right] &= \sigma_\mu^{(1)^2} B_i + \sigma_\mu^{(2)^2} F_{i,i} + \sigma_\mu^{(3)^2} A_i + \sigma_\epsilon^2 I_{\sum_j \sum_s T_{ijs}} \\
u &= \tilde{\mu}^{(1)} + \tilde{\mu}^{(2)} + \tilde{\mu}^{(3)} + \epsilon \\
E\left[uu'\right] &= \sigma_\mu^{(1)^2} G + \sigma_\mu^{(2)^2} L + \sigma_\mu^{(3)^2} M + \sigma_\epsilon^2 I_{\sum_i \sum_j \sum_s T_{ijs}}
\end{aligned}$$

with

$$\begin{aligned}
\tilde{\mu}_{ij}^{(2)} &= \left(\underbrace{\mu_{i1}^{(2)} \dots \mu_{i1}^{(2)}}_{T_{ij1}\text{ times}} \quad \underbrace{\mu_{i2}^{(2)} \dots \mu_{i2}^{(2)}}_{T_{ij2}\text{ times}} \quad \dots \quad \underbrace{\mu_{iN^{(3)}}^{(2)} \dots \mu_{iN^{(3)}}^{(2)}}_{T_{ijN^{(3)}}\text{ times}} \right) \\
\tilde{\mu}_i^{(2)} &= \left(\tilde{\mu}_{i1}^{(2)}, \tilde{\mu}_{i2}^{(2)}, \dots, \tilde{\mu}_{iN^{(2)}}^{(2)} \right) \\
\tilde{\mu}^{(2)} &= \left(\tilde{\mu}_1^{(2)}, \tilde{\mu}_2^{(2)}, \dots, \tilde{\mu}_{N^{(1)}}^{(2)} \right)
\end{aligned}$$

$$
\begin{aligned}
\tilde{\mu}_{ij}^{(3)} &= \left(\underbrace{\mu_{j1}^{(3)} \dots \mu_{j1}^{(3)}}_{T_{ij1} \text{ times}} \quad \underbrace{\mu_{j2}^{(3)} \dots \mu_{j2}^{(3)}}_{T_{ij2} \text{ times}} \quad \dots \quad \underbrace{\mu_{jN^{(3)}}^{(3)} \dots \mu_{jN^{(3)}}^{(3)}}_{T_{ijN^{(3)}} \text{ times}} \right) \\
\tilde{\mu}_{i}^{(3)} &= \left(\tilde{\mu}_{i1}^{(3)}, \tilde{\mu}_{i2}^{(3)}, ..., \tilde{\mu}_{iN^{(2)}}^{(3)} \right) \\
\tilde{\mu}^{(3)} &= \left(\tilde{\mu}_{1}^{(3)}, \tilde{\mu}_{2}^{(3)}, ..., \tilde{\mu}_{N^{(1)}}^{(3)} \right) \\
\tilde{\mu}_{i}^{(1)} &= \left(\underbrace{\mu_{i1}^{(1)} \dots \mu_{i1}^{(1)}}_{\sum_s T_{i1s} \text{ times}} \quad \underbrace{\mu_{i2}^{(1)} \dots \mu_{i2}^{(1)}}_{\sum_s T_{i2s} \text{ times}} \quad \dots \quad \underbrace{\mu_{iN^{(2)}}^{(1)} \dots \mu_{iN^{(2)}}^{(1)}}_{\sum_s T_{iN^{(2)}s} \text{ times}} \right) \\
\tilde{\mu}^{(1)} &= \left(\tilde{\mu}_{1}^{(1)}, \tilde{\mu}_{2}^{(1)}, ..., \tilde{\mu}_{N^{(1)}}^{(1)} \right)
\end{aligned}
$$

and

$$
G = \begin{pmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{N^{(1)}} \end{pmatrix}, \quad L = \begin{pmatrix} F_{1,1} & 0 & \dots & 0 \\ 0 & F_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F_{N^{(1)},N^{(1)}} \end{pmatrix}
$$

$$
Z_{i,p} = \begin{pmatrix} J_{(T_{i11} \times T_{p11})} & 0 & \dots & 0 \\ 0 & J_{(T_{i12} \times T_{p12})} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{\left(T_{iN^{(2)}N^{(3)}} \times T_{pN^{(2)}N^{(3)}}\right)} \end{pmatrix}
$$

$$
M = \begin{pmatrix} Z_{1,1} & Z_{1,2} & \dots & Z_{1,N^{(1)}} \\ Z_{2,1} & Z_{2,2} & \dots & Z_{2,N^{(1)}} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{N^{(1)},1} & Z_{N^{(1)},2} & \dots & Z_{N^{(1)},N^{(1)}} \end{pmatrix}
$$

Model (5)

The disturbance term is now structured as

$$
u_{ijst} = \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + \lambda_t + \epsilon_{ijst}
$$

Given that we have already derived the covariance matrix of $\mu_{ij}^{(1)}$, $\mu_{is}^{(2)}$, and $\mu_{js}^{(3)}$ in model (4) and λ_t in model (1), for covariance matrix of model (5) we get

$$
E\left[uu'\right] = \sigma_{\mu}^{(1)^2} G + \sigma_{\mu}^{(2)^2} L + \sigma_{\mu}^{(3)^2} M + \sigma_{\lambda}^2 D + \sigma_{\epsilon}^2 I_{\sum_i \sum_j \sum_s T_{ijs}}
$$

Model (6)

The disturbance term now is

$$u_{ijst} = \mu_{ijs} + v_{it}^{(1)} + v_{jt}^{(2)} + v_{st}^{(3)} + \epsilon_{ijst}$$

so we get for the covariance matrix

$$\begin{aligned}
u_{ijs} &= \mu_{ijs} \otimes l_{T_{ijs}} + v_{iZ_{ijs}}^{(1)} + v_{jZ_{ijs}}^{(2)} + v_{sZ_{ijs}}^{(3)} + \epsilon_{ijs} \\
E\left[u_{ijs}u_{ijs}'\right] &= \sigma_\mu^2 J_{T_{ijs}} + {\sigma_v^{(1)}}^2 I_{T_{ijs}} + {\sigma_v^{(2)}}^2 I_{T_{ijs}} + {\sigma_v^{(3)}}^2 I_{T_{ijs}} + \sigma_\epsilon^2 I_{T_{ijs}} \\
u_{ij} &= \tilde{\mu}_{ij} + \tilde{v}_{ij}^{(1)} + \tilde{v}_{ij}^{(2)} + \tilde{v}_{ij}^{(3)} + \epsilon_{ij} \\
E\left[u_{ij}u_{ij}'\right] &= \sigma_\mu^2 A_{ij} + {\sigma_v^{(1)}}^2 D_{ij} + {\sigma_v^{(2)}}^2 D_{ij} + {\sigma_v^{(3)}}^2 I_{\sum_s T_{ijs}} + \sigma_\epsilon^2 I_{\sum_s T_{ijs}} \\
u_i &= \tilde{\mu}_i + \tilde{v}_i^{(1)} + \tilde{v}_i^{(2)} + \tilde{v}_i^{(3)} + \epsilon_i \\
E\left[u_i u_i'\right] &= \sigma_\mu^2 A_i + {\sigma_v^{(1)}}^2 D_i + {\sigma_v^{(2)}}^2 R_i^1 + {\sigma_v^{(3)}}^2 N_{i,i} + \sigma_\epsilon^2 I_{\sum_j \sum_s T_{ijs}} \\
u &= \tilde{\mu} + \tilde{v}^{(1)} + \tilde{v}^{(2)} + \tilde{v}^{(3)} + \epsilon \\
E\left[uu'\right] &= \sigma_\mu^2 A + {\sigma_v^{(1)}}^2 H + {\sigma_v^{(2)}}^2 Q + {\sigma_v^{(3)}}^2 N + \sigma_\epsilon^2 I_{\sum_i \sum_j \sum_s T_{ijs}}
\end{aligned}$$

Denoting by
$v_{iZ_{ijs}}^{(1)}$ the vector of length T_{ijs} of the individual-time varying effects associated with the firm i and time periods from the set Z_{ijs},
$v_{jZ_{ijs}}^{(2)}$ the vector of length T_{ijs} of the individual-time varying effects associated with the product j and time periods from the set Z_{ijs}, and
$v_{sZ_{ijs}}^{(3)}$ the vector of length T_{ijs} of the individual-time varying effects associated with the country s and time periods from the set Z_{ijs}
we get

$$\begin{aligned}
\tilde{v}_{ij}^{(1)} &= \left(v_{iZ_{ij1}}^{(1)}, ..., v_{iZ_{ijN^{(3)}}}^{(1)}\right), \;\; \tilde{v}_i^{(1)} = \left(\tilde{v}_{i1}^{(1)}, \tilde{v}_{i2}^{(1)}, ..., \tilde{v}_{iN^{(2)}}^{(1)}\right) \\
\tilde{v}^{(1)} &= \left(\tilde{v}_1^{(1)}, \tilde{v}_2^{(1)}, ..., \tilde{v}_{N^{(1)}}^{(1)}\right) \\
\tilde{v}_{ij}^{(2)} &= \left(v_{jZ_{ij1}}^{(2)}, ..., v_{jZ_{ijN^{(3)}}}^{(2)}\right), \;\; \tilde{v}_i^{(2)} = \left(\tilde{v}_{i1}^{(2)}, \tilde{v}_{i2}^{(2)}, ..., \tilde{v}_{iN^{(2)}}^{(2)}\right) \\
\tilde{v}^{(2)} &= \left(\tilde{v}_1^{(2)}, \tilde{v}_2^{(2)}, ..., \tilde{v}_{N^{(1)}}^{(2)}\right) \\
\tilde{v}_{ij}^{(3)} &= \left(v_{1Z_{ij1}}^{(3)}, ..., v_{N^{(3)}Z_{ijN^{(3)}}}^{(3)}\right), \;\; \tilde{v}_i^{(3)} = \left(\tilde{v}_{i1}^{(3)}, \tilde{v}_{i2}^{(3)}, ..., \tilde{v}_{iN^{(2)}}^{(3)}\right) \\
\tilde{v}^{(3)} &= \left(\tilde{v}_1^{(3)}, \tilde{v}_2^{(3)}, ..., \tilde{v}_{N^{(1)}}^{(3)}\right)
\end{aligned}$$

and

$$H = \begin{pmatrix} D_l & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_N^{(1)} \end{pmatrix}$$

$$E_s^l = \begin{pmatrix} M_{T_{ij1} \times T_{ljs}} \\ M_{T_{ij2} \times T_{ljs}} \\ \dots \\ M_{T_{ijN^{(3)}} \times T_{ljs}} \end{pmatrix}, \quad D_{ij}^l = \left(E_1^l, E_2^l, ..., E_{N^{(3)}}^l\right)$$

$$R_i^l = \begin{pmatrix} D_{i1}^l & 0 & \dots & 0 \\ 0 & D_{i2}^l & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_{iN^{(2)}}^l \end{pmatrix}, \quad Q = \begin{pmatrix} R_1^1 & R_1^2 & \dots & R_1^{N^{(1)}} \\ R_2^1 & R_2^2 & \dots & R_2^{N^{(1)}} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N^{(1)}}^1 & R_{N^{(1)}}^2 & \dots & R_{N^{(1)}}^{N^{(1)}} \end{pmatrix}$$

$$S_{i,s}^{j,l} = \begin{pmatrix} M_{T_{ij1} \times T_{sl1}} & 0 & \dots & 0 \\ 0 & M_{T_{ij2} \times T_{sl2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_{T_{ijN^{(3)}} \times T_{slN^{(3)}}} \end{pmatrix}$$

$$N_{i,s} = \begin{pmatrix} S_{i,s}^{1,1} & S_{i,s}^{1,2} & \dots & S_{i,s}^{1,N^{(2)}} \\ S_{i,s}^{2,1} & S_{i,s}^{2,2} & \dots & S_{i,s}^{2,N^{(2)}} \\ \vdots & \vdots & \ddots & \vdots \\ S_{i,s}^{N^{(2)},1} & S_{i,s}^{N^{(2)},2} & \dots & S_{i,s}^{N^{(2)},N^{(2)}} \end{pmatrix}$$

$$N = \begin{pmatrix} N_{1,1} & N_{1,2} & \dots & N_{1,N^{(1)}} \\ N_{2,1} & N_{2,2} & \dots & N_{2,N^{(1)}} \\ \vdots & \vdots & \ddots & \vdots \\ N_{N^{(1)},1} & N_{N^{(1)},2} & \dots & N_{N^{(1)},N^{(1)}} \end{pmatrix}$$

Model (7)

The disturbance term now is

$$u_{ijst} = \mu_{ij}^{(1)} + \mu_{is}^{(2)} + \mu_{js}^{(3)} + v_{it}^{(1)} + v_{jt}^{(2)} + v_{st}^{(3)} + \epsilon_{ijst}$$

Fortunately, we have already derived the covariance matrix of $\mu_{ij}^{(1)}$, $\mu_{is}^{(2)}$, $\mu_{js}^{(3)}$, $v_{it}^{(1)}$, $v_{jt}^{(2)}$, and $v_{st}^{(3)}$ previously, thus for model (7) the covariance matrix takes the form

$$E\left[uu'\right] = {\sigma_\mu^{(1)}}^2 G + {\sigma_\mu^{(2)}}^2 L + {\sigma_\mu^{(3)}}^2 M + {\sigma_v^{(1)}}^2 H + {\sigma_v^{(2)}}^2 Q + {\sigma_v^{(3)}}^2 N + \sigma_\epsilon^2 I_{\sum_i \sum_j \sum_s T_{ijs}}$$

4.2 Estimation of the Variance Components

Just like in the balanced case, the estimation of the variance components is carried out in two steps. First, some identifying equations are presented, then, based on these, estimators for the different variance components are derived. But first, let us introduce some additional notation. Let denote

$$\tilde{A} = \frac{1}{T} \sum_{i \in Q^{(1)}} \sum_{j \in Q_i^{(2)}} \sum_{s \in Q_{ij}^{(3)}} \sum_{t \in Z_{ijs}} \hat{u}_{ijst}^2$$

$$\tilde{B} = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{N_j^{(3)}} \sum_{s \in Q_j^{(3)}} \frac{1}{T_{js}^{(1)}} \sum_{t \in Z_{js}^{(1)}} \left(\frac{1}{N_{jst}^{(1)}} \sum_{i \in Q_{jst}^{(1)}} \hat{u}_{ijst} \right)^2$$

$$\tilde{C} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(3)}} \sum_{s \in Q_i^{(3)}} \frac{1}{T_{is}^{(2)}} \sum_{t \in Z_{is}^{(2)}} \left(\frac{1}{N_{ist}^{(2)}} \sum_{j \in Q_{ist}^{(2)}} \hat{u}_{ijst} \right)^2$$

$$\tilde{D} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j \in Q_i^{(2)}} \frac{1}{T_{ij}^{(3)}} \sum_{t \in Z_{ij}^{(3)}} \left(\frac{1}{N_{ijt}^{(3)}} \sum_{s \in Q_{ijt}^{(3)}} \hat{u}_{ijst} \right)^2$$

$$\tilde{E} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j \in Q_i^{(2)}} \frac{1}{N_{ij}^{(3)}} \sum_{s \in Q_{ij}^{(3)}} \left(\frac{1}{T_{ijs}} \sum_{t \in Z_{ijs}} \hat{u}_{ijst} \right)^2$$

and

$$b = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{N_j^{(3)}} \sum_{s \in Q_j^{(3)}} \frac{1}{T_{js}^{(1)}} \sum_{t \in Z_{js}^{(1)}} \frac{1}{N_{jst}^{(1)}}$$

$$c = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(3)}} \sum_{s \in Q_i^{(3)}} \frac{1}{T_{is}^{(2)}} \sum_{t \in Z_{is}^{(2)}} \frac{1}{N_{ist}^{(2)}}$$

$$d = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j \in Q_i^{(2)}} \frac{1}{T_{ij}^{(3)}} \sum_{t \in Z_{ij}^{(3)}} \frac{1}{N_{ijt}^{(3)}}$$

$$e = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j \in Q_i^{(2)}} \frac{1}{N_{ij}^{(3)}} \sum_{s \in Q_{ij}^{(3)}} \frac{1}{T_{ijs}}$$

Model (1)

The identifying equations are the following

$$E\left[u_{ijst}^2\right] = \sigma_\mu^2 + \sigma_\gamma^2 + \sigma_\alpha^2 + \sigma_\lambda^2 + \sigma_\epsilon^2$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right] = b\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\gamma^2+\sigma_\alpha^2+\sigma_\lambda^2$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right] = c\left(\sigma_\gamma^2+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_\alpha^2+\sigma_\lambda^2$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right] = d\left(\sigma_\alpha^2+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_\gamma^2+\sigma_\lambda^2$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{N_{ij}^{(3)}}\sum_{s\in Q_{ij}^{(3)}}\left(\frac{1}{T_{ijs}}\sum_{t\in Z_{ijs}}u_{ijst}\right)^2\right] = e\left(\sigma_\lambda^2+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_\gamma^2+\sigma_\alpha^2$$

These leads to the following estimators

$$\hat\sigma_\epsilon^2 = \frac{1}{3}\left(\frac{\tilde B}{b-1}+\frac{\tilde C}{c-1}+\frac{\tilde D}{d-1}+\frac{\tilde E}{e-1}\right)-\frac{\tilde A}{3}\left(\frac{1}{b-1}+\frac{1}{c-1}+\frac{1}{d-1}+\frac{1}{e-1}+1\right)$$

$$\hat\sigma_\lambda^2 = \frac{\tilde A-\tilde E}{1-e}-\hat\sigma_\epsilon^2$$

$$\hat\sigma_\alpha^2 = \frac{\tilde A-\tilde D}{1-d}-\hat\sigma_\epsilon^2$$

$$\hat\sigma_\gamma^2 = \frac{\tilde A-\tilde C}{1-c}-\hat\sigma_\epsilon^2$$

$$\hat\sigma_\mu^2 = \frac{\tilde A-\tilde B}{1-b}-\hat\sigma_\epsilon^2$$

Model (2)

Using the identifying equations

$$E\left[u_{ijst}^2\right] = \sigma_\mu^2 + \sigma_\epsilon^2$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{N_{ij}^{(3)}}\sum_{s\in Q_{ij}^{(3)}}\left(\frac{1}{T_{ijs}}\sum_{t\in Z_{ijs}}u_{ijst}\right)^2\right] = \sigma_\mu^2+e\sigma_\epsilon^2$$

the estimators of variance components are

$$\hat{\sigma}^2_\mu = \frac{\tilde{E} - e\tilde{A}}{1-e}$$
$$\hat{\sigma}^2_\epsilon = \tilde{A} - \hat{\sigma}^2_\mu$$

Model (3)

The identifying equations are

$$E\left[u^2_{ijst}\right] = \sigma^2_\mu + \sigma^2_\lambda + \sigma^2_\epsilon$$
$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N^{(3)}_j}\sum_{s\in Q^{(3)}_j}\frac{1}{T^{(1)}_{js}}\sum_{t\in Z^{(1)}_{js}}\left(\frac{1}{N^{(1)}_{jst}}\sum_{i\in Q^{(1)}_{jst}}u_{ijst}\right)^2\right] = b\left(\sigma^2_\mu + \sigma^2_\epsilon\right) + \sigma^2_\lambda$$
$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N^{(2)}_i}\sum_{j\in Q^{(2)}_i}\frac{1}{N^{(3)}_{ij}}\sum_{s\in Q^{(3)}_{ij}}\left(\frac{1}{T_{ijs}}\sum_{t\in Z_{ijs}}u_{ijst}\right)^2\right] = e\left(\sigma^2_\lambda + \sigma^2_\epsilon\right) + \sigma^2_\mu$$

And so the estimators are

$$\hat{\sigma}^2_\mu = \frac{\tilde{E} - e\tilde{A}}{1-e}$$
$$\hat{\sigma}^2_\lambda = \frac{\tilde{B} - b\tilde{A}}{1-b}$$
$$\hat{\sigma}^2_\epsilon = \tilde{A} - \hat{\sigma}^2_\mu - \hat{\sigma}^2_\lambda$$

Model (4)

The identifying equations in this case are simply

$$E\left[u^2_{ijst}\right] = \sigma^{(1)^2}_\mu + \sigma^{(2)^2}_\mu + \sigma^{(3)^2}_\mu + \sigma^2_\epsilon$$
$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N^{(3)}_j}\sum_{s\in Q^{(3)}_j}\frac{1}{T^{(1)}_{js}}\sum_{t\in Z^{(1)}_{js}}\left(\frac{1}{N^{(1)}_{jst}}\sum_{i\in Q^{(1)}_{jst}}u_{ijst}\right)^2\right] = b\left(\sigma^{(1)^2}_\mu + \sigma^{(2)^2}_\mu + \sigma^2_\epsilon\right) + \sigma^{(3)^2}_\mu$$
$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N^{(3)}_i}\sum_{s\in Q^{(3)}_i}\frac{1}{T^{(2)}_{is}}\sum_{t\in Z^{(2)}_{is}}\left(\frac{1}{N^{(2)}_{ist}}\sum_{j\in Q^{(2)}_{ist}}u_{ijst}\right)^2\right] = c\left(\sigma^{(1)^2}_\mu + \sigma^{(3)^2}_\mu + \sigma^2_\epsilon\right) + \sigma^{(2)^2}_\mu$$
$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N^{(2)}_i}\sum_{j\in Q^{(2)}_i}\frac{1}{T^{(3)}_{ij}}\sum_{t\in Z^{(3)}_{ij}}\left(\frac{1}{N^{(3)}_{ijt}}\sum_{s\in Q^{(3)}_{ijt}}u_{ijst}\right)^2\right] = d\left(\sigma^{(2)^2}_\mu + \sigma^{(3)^2}_\mu + \sigma^2_\epsilon\right) + \sigma^{(1)^2}_\mu$$

And the estimators are

$$\hat{\sigma}_{\mu}^{(1)^2} = \frac{\tilde{D} - d\tilde{A}}{1-d}$$
$$\hat{\sigma}_{\mu}^{(2)^2} = \frac{\tilde{C} - c\tilde{A}}{1-c}$$
$$\hat{\sigma}_{\mu}^{(3)^2} = \frac{\tilde{B} - b\tilde{A}}{1-b}$$
$$\hat{\sigma}_{\epsilon}^{2} = \tilde{A} - \hat{\sigma}_{\mu}^{(1)^2} - \hat{\sigma}_{\mu}^{(2)^2} - \hat{\sigma}_{\mu}^{(3)^2}$$

Model (5)

The identifying equations now are

$$E\left[u_{ijst}^2\right] = \sigma_{\mu}^{(1)^2} + \sigma_{\mu}^{(2)^2} + \sigma_{\mu}^{(3)^2} + \sigma_{\lambda}^2 + \sigma_{\epsilon}^2$$

$$E\left[\frac{1}{N^{(2)}} \sum_{j\in Q^{(2)}} \frac{1}{N_j^{(3)}} \sum_{s\in Q_j^{(3)}} \frac{1}{T_{js}^{(1)}} \sum_{t\in Z_{js}^{(1)}} \left(\frac{1}{N_{jst}^{(1)}} \sum_{i\in Q_{jst}^{(1)}} u_{ijst}\right)^2\right] =$$
$$= b\left(\sigma_{\mu}^{(1)^2} + \sigma_{\mu}^{(2)^2} + \sigma_{\epsilon}^2\right) + \sigma_{\mu}^{(3)^2} + \sigma_{\lambda}^2$$

$$E\left[\frac{1}{N^{(1)}} \sum_{i\in Q^{(1)}} \frac{1}{N_i^{(3)}} \sum_{s\in Q_i^{(3)}} \frac{1}{T_{is}^{(2)}} \sum_{t\in Z_{is}^{(2)}} \left(\frac{1}{N_{ist}^{(2)}} \sum_{j\in Q_{ist}^{(2)}} u_{ijst}\right)^2\right] =$$
$$= c\left(\sigma_{\mu}^{(1)^2} + \sigma_{\mu}^{(3)^2} + \sigma_{\epsilon}^2\right) + \sigma_{\mu}^{(2)^2} + \sigma_{\lambda}^2$$

$$E\left[\frac{1}{N^{(1)}} \sum_{i\in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j\in Q_i^{(2)}} \frac{1}{T_{ij}^{(3)}} \sum_{t\in Z_{ij}^{(3)}} \left(\frac{1}{N_{ijt}^{(3)}} \sum_{s\in Q_{ijt}^{(3)}} u_{ijst}\right)^2\right] =$$
$$= d\left(\sigma_{\mu}^{(2)^2} + \sigma_{\mu}^{(3)^2} + \sigma_{\epsilon}^2\right) + \sigma_{\mu}^{(1)^2} + \sigma_{\lambda}^2$$

$$E\left[\frac{1}{N^{(1)}} \sum_{i\in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j\in Q_i^{(2)}} \frac{1}{N_{ij}^{(3)}} \sum_{s\in Q_{ij}^{(3)}} \left(\frac{1}{T_{ijs}} \sum_{t\in Z_{ijs}} u_{ijst}\right)^2\right] =$$
$$= e\left(\sigma_{\lambda}^2 + \sigma_{\epsilon}^2\right) + \sigma_{\mu}^{(1)^2} + \sigma_{\mu}^{(2)^2} + \sigma_{\mu}^{(3)^2}$$

The variance components' estimators now are

$$\hat{\sigma}_\lambda^2 = \frac{1}{3}\left(\frac{\tilde{B}}{1-b} + \frac{\tilde{C}}{1-c} + \frac{\tilde{D}}{1-d} - \frac{\tilde{E}}{1-e}\right) - \frac{\tilde{A}}{3}\left(\frac{b}{1-b} + \frac{c}{1-c} + \frac{d}{1-d} - \frac{e}{1-e}\right)$$

$$\hat{\sigma}_\mu^{(1)^2} = \frac{\tilde{D} - d\tilde{A}}{1-d} - \hat{\sigma}_\lambda^2$$

$$\hat{\sigma}_\mu^{(2)^2} = \frac{\tilde{C} - c\tilde{A}}{1-c} - \hat{\sigma}_\lambda^2$$

$$\hat{\sigma}_\mu^{(3)^2} = \frac{\tilde{B} - b\tilde{A}}{1-b} - \hat{\sigma}_\lambda^2$$

$$\hat{\sigma}_\epsilon^2 = \tilde{A} - \hat{\sigma}_\mu^{(1)^2} - \hat{\sigma}_\mu^{(2)^2} - \hat{\sigma}_\mu^{(3)^2} - \hat{\sigma}_\lambda^2$$

Model (6)

The identifying equations for this model are

$$E\left[u_{ijst}^2\right] = \sigma_\mu^2 + \sigma_v^{(1)^2} + \sigma_v^{(2)^2} + \sigma_v^{(3)^2} + \sigma_\epsilon^2$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right] =$$

$$= b\left(\sigma_\mu^2 + \sigma_v^{(1)^2} + \sigma_\epsilon^2\right) + \sigma_v^{(2)^2} + \sigma_v^{(3)^2}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right] =$$

$$= c\left(\sigma_\mu^2 + \sigma_v^{(2)^2} + \sigma_\epsilon^2\right) + \sigma_v^{(1)^2} + \sigma_v^{(3)^2}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right] =$$

$$= d\left(\sigma_\mu^2 + \sigma_v^{(3)^2} + \sigma_\epsilon^2\right) + \sigma_v^{(1)^2} + \sigma_v^{(2)^2}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{N_{ij}^{(3)}}\sum_{s\in Q_{ij}^{(3)}}\left(\frac{1}{T_{ijs}}\sum_{t\in Z_{ijs}}u_{ijst}\right)^2\right] =$$

$$= e\left(\sigma_v^{(1)^2} + \sigma_v^{(2)^2} + \sigma_v^{(3)^2} + \sigma_\epsilon^2\right) + \sigma_\mu^2$$

And the estimators are

$$\hat{\sigma}_\mu^2 = \frac{\tilde{E} - e\tilde{A}}{1-e}$$

$$\hat{\sigma}_\epsilon^2 = \tilde{A}\left(1 + \frac{b}{2(1-b)} + \frac{c}{2(1-c)} + \frac{d}{2(1-d)}\right) - \frac{1}{2}\left(\frac{\tilde{B}}{1-b} + \frac{\tilde{C}}{1-c} + \frac{\tilde{D}}{1-d}\right) - \hat{\sigma}_\mu^2$$

$$\hat{\sigma}_v^{(1)^2} = \frac{\tilde{A} - \tilde{B}}{1-b} - \hat{\sigma}_\mu^2 - \hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_v^{(2)^2} = \frac{\tilde{A} - \tilde{C}}{1-c} - \hat{\sigma}_\mu^2 - \hat{\sigma}_\epsilon^2$$

$$\hat{\sigma}_v^{(3)^2} = \tilde{A} - \hat{\sigma}_\mu^2 - \hat{\sigma}_v^{(1)^2} - \hat{\sigma}_v^{(2)^2} - \hat{\sigma}_\epsilon^2$$

Model (7)

For this last model we need to introduce some additional notations:

$$\tilde{F} = \frac{1}{N^{(3)}} \sum_{s \in Q^{(3)}} \frac{1}{T_s^{(3)}} \sum_{t \in Z_s^{(3)}} \left(\frac{1}{N_{st}^{(1)}} \sum_{i \in Q_{st}^{(1)}} \frac{1}{N_{ist}^{(2)}} \sum_{j \in Q_{ist}^{(2)}} \hat{u}_{ijst} \right)^2$$

$$\tilde{G} = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{T_j^{(2)}} \sum_{t \in Z_j^{(2)}} \left(\frac{1}{N_{jt}^{(1)}} \sum_{i \in Q_{jt}^{(1)}} \frac{1}{N_{ijt}^{(3)}} \sum_{s \in Q_{ijt}^{(3)}} \hat{u}_{ijst} \right)^2$$

$$f_1 = \frac{1}{N^{(3)}} \sum_{s \in Q^{(3)}} \frac{1}{T_s^{(3)}} \sum_{t \in Z_s^{(3)}} \frac{1}{N_{st}^{(1)^2}} \sum_{i \in Q_{st}^{(1)}} \frac{1}{N_{ist}^{(2)}}$$

$$f_2 = \frac{1}{N^{(3)}} \sum_{s \in Q^{(3)}} \frac{1}{T_s^{(3)}} \sum_{t \in Z_s^{(3)}} \frac{1}{N_{st}^{(1)}}$$

$$f_3 = \frac{1}{N^{(3)}} \sum_{s \in Q^{(3)}} \frac{1}{T_s^{(3)}} \sum_{t \in Z_s^{(3)}} \frac{1}{N_{st}^{(1)^2}} \left(\sum_{i \in Q_{st}^{(1)}} \frac{1}{N_{ist}^{(2)}} \left(1 + \sum_{l \in Q_{st}^{(1)}, l \neq i} \frac{1}{N_{lst}^{(2)}} I_{il}^{st} \right) \right)$$

$$f = \frac{1 - f_2 - f_3}{f_1}$$

$$g_1 = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{T_j^{(2)}} \sum_{t \in Z_j^{(2)}} \frac{1}{N_{jt}^{(1)^2}} \sum_{i \in Q_{jt}^{(1)}} \frac{1}{N_{ijt}^{(3)}}$$

$$g_2 = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{T_j^{(2)}} \sum_{t \in Z_j^{(2)}} \frac{1}{N_{jt}^{(1)^2}} \left(\sum_{i \in Q_{jt}^{(1)}} \frac{1}{N_{ijt}^{(3)}} \left(1 + \sum_{p \in Q_{jt}^{(1)}} \frac{1}{N_{pjt}^{(3)}} J_{ip}^{jt} \right) \right)$$

$$g_3 = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{T_j^{(2)}} \sum_{t \in Z_j^{(2)}} \frac{1}{N_{jt}^{(1)}}$$

$$g = \frac{1 - g_2 - g_3}{g_1}$$

where I_{il}^{st} is the number of common products that firm i and l sell to country s at time t and J_{ip}^{jt} is the number of common countries to which firms i and p sell product j at time t.

Turning now our attention to the identifying equations

$$E\left[u_{ijst}^2\right] = \sigma_\mu^{(1)^2} + \sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2} + \sigma_v^{(1)^2} + \sigma_v^{(2)^2} + \sigma_v^{(3)^2} + \sigma_\epsilon^2$$

$$E\left[\frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{N_j^{(3)}} \sum_{s \in Q_j^{(3)}} \frac{1}{T_{js}^{(1)}} \sum_{t \in Z_{js}^{(1)}} \left(\frac{1}{N_{jst}^{(1)}} \sum_{i \in Q_{jst}^{(1)}} u_{ijst} \right)^2 \right] =$$
$$= b\left(\sigma_\mu^{(1)^2} + \sigma_\mu^{(2)^2} + \sigma_v^{(1)^2} + \sigma_\epsilon^2\right) + \sigma_\mu^{(3)^2} + \sigma_v^{(2)^2} + \sigma_v^{(3)^2}$$

$$E\left[\frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(3)}} \sum_{s \in Q_i^{(3)}} \frac{1}{T_{is}^{(2)}} \sum_{t \in Z_{is}^{(2)}} \left(\frac{1}{N_{ist}^{(2)}} \sum_{j \in Q_{ist}^{(2)}} u_{ijst} \right)^2 \right] =$$
$$= c\left(\sigma_\mu^{(1)^2} + \sigma_\mu^{(3)^2} + \sigma_v^{(2)^2} + \sigma_\epsilon^2\right) + \sigma_\mu^{(2)^2} + \sigma_v^{(1)^2} + \sigma_v^{(3)^2}$$

$$E\left[\frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j \in Q_i^{(2)}} \frac{1}{T_{ij}^{(3)}} \sum_{t \in Z_{ij}^{(3)}} \left(\frac{1}{N_{ijt}^{(3)}} \sum_{s \in Q_{ijt}^{(3)}} u_{ijst} \right)^2 \right] =$$
$$= d\left(\sigma_\mu^{(2)^2} + \sigma_\mu^{(3)^2} + \sigma_v^{(3)^2} + \sigma_\epsilon^2\right) + \sigma_\mu^{(1)^2} + \sigma_v^{(1)^2} + \sigma_v^{(2)^2}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{N_{ij}^{(3)}}\sum_{s\in Q_{ij}^{(3)}}\left(\frac{1}{T_{ijs}}\sum_{t\in Z_{ijs}}u_{ijst}\right)^2\right]=$$
$$=e\left(\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+\sigma_\epsilon^2\right)+\sigma_\mu^{(1)^2}+\sigma_\mu^{(2)^2}+\sigma_\mu^{(3)^2}$$
$$E\left[\frac{1}{N^{(3)}}\sum_{s\in Q^{(3)}}\frac{1}{T_s^{(3)}}\sum_{t\in Z_s^{(3)}}\left(\frac{1}{N_{st}^{(1)}}\sum_{i\in Q_{st}^{(1)}}\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=$$
$$=f_1\left(\sigma_\mu^{(1)^2}+\sigma_\epsilon^2\right)+f_2\left(\sigma_\mu^{(2)^2}+\sigma_v^{(1)^2}\right)+f_3\left(\sigma_\mu^{(3)^2}+\sigma_v^{(2)^2}\right)+\sigma_v^{(3)^2}$$
$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{T_j^{(2)}}\sum_{t\in Z_j^{(2)}}\left(\frac{1}{N_{jt}^{(1)}}\sum_{i\in Q_{jt}^{(1)}}\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$
$$=g_1\left(\sigma_\mu^{(2)^2}+\sigma_\epsilon^2\right)+g_2\left(\sigma_\mu^{(3)^2}+\sigma_v^{(3)^2}\right)+g_3\left(\sigma_\mu^{(1)^2}+\sigma_v^{(1)^2}\right)+\sigma_v^{(2)^2}$$

So the estimators of variance components of the last model are

$$\hat\sigma_\epsilon^2=\frac{\tilde A}{2}\left(1+\frac{b}{1-b}+\frac{c}{1-c}+\frac{d}{1-d}-\frac{e}{1-e}\right)-\frac{1}{2}\left(\frac{\tilde B}{1-b}+\frac{\tilde C}{1-c}+\frac{\tilde D}{1-d}-\frac{\tilde E}{1-e}\right)$$
$$\hat\sigma_v^{(3)^2}=\frac{1}{1+f}\left(\frac{\tilde E-e\tilde A}{1-e}-\frac{\tilde D-d\tilde A}{1-d}-\frac{f_2}{f_1}\cdot\frac{\tilde C-c\tilde A}{1-c}-\frac{f_3}{f_1}\cdot\frac{\tilde B-b\tilde A}{1-b}+\frac{\tilde F}{f_1}-2\hat\sigma_\epsilon^2\right)$$
$$\hat\sigma_v^{(2)^2}=\frac{1}{1+g}\left(\frac{\tilde E-e\tilde A}{1-e}-\frac{\tilde C-c\tilde A}{1-c}-\frac{g_2}{g_1}\cdot\frac{\tilde B-b\tilde A}{1-b}-\frac{g_3}{g_1}\cdot\frac{\tilde D-d\tilde A}{1-d}+\frac{\tilde G}{g_1}-2\hat\sigma_\epsilon^2\right)$$

$$\hat\sigma_v^{(1)^2}=\frac{\tilde E-e\tilde A}{1-e}-\hat\sigma_\epsilon^2-\hat\sigma_v^{(3)^2}-\hat\sigma_v^{(2)^2}$$
$$\hat\sigma_\mu^{(1)^2}=\frac{\tilde D-d\tilde A}{1-d}-\hat\sigma_v^{(1)^2}-\hat\sigma_v^{(2)^2}$$
$$\hat\sigma_\mu^{(2)^2}=\frac{\tilde C-c\tilde A}{1-c}-\hat\sigma_v^{(1)^2}-\hat\sigma_v^{(3)^2}$$
$$\hat\sigma_\mu^{(3)^2}=\tilde A-\hat\sigma_\mu^{(1)^2}-\hat\sigma_\mu^{(2)^2}-\hat\sigma_v^{(1)^2}-\hat\sigma_v^{(2)^2}-\hat\sigma_v^{(3)^2}-\hat\sigma_\epsilon^2$$

4.3 Covariance Matrixes of the Models with Cross Correlation

Let us turn now our attention to the models with cross correlations. For models (2), (3) and (6) we have

$$\begin{aligned} E[\mu_{ij}\mu_{ij}'] &= \sigma_\mu^2 A_{ij} + \rho_{(3)}\left(J_{N^{(3)}T} - A_{ij}\right) \\ E[\mu_i\mu_i'] &= \sigma_\mu^2 A_i + \rho_{(3)}\left(B_i - A_i\right) + \rho_{(2)}\left(F_{i,i} - A_i\right) \\ E[\mu\mu'] &= \sigma_\mu^2 A + \rho_{(3)}\left(G - A\right) + \rho_{(2)}\left(L - A\right) + \rho_{(1)}\left(M - A\right) \end{aligned}$$

Thus the covariance matrix of model (2) takes the form

$$E[uu'] = \sigma_\mu^2 A + \rho_{(3)}\left(G - A\right) + \rho_{(2)}\left(L - A\right) + \rho_{(1)}\left(M - A\right) + \sigma_\epsilon^2 I_{\sum_i\sum_j\sum_s T_{ijs}}$$

the covariance matrix of model (3) looks like

$$E\left[uu'\right] = \sigma_\mu^2 A + \rho_{(3)}\left(G - A\right) + \rho_{(2)}\left(L - A\right) + \rho_{(1)}\left(M - A\right) + \sigma_\lambda^2 D + \sigma_\epsilon^2 I_{\sum_i\sum_j\sum_s T_{ijs}}$$

and, finally, the covariance matrix of model (6) is

$$\begin{aligned} E\left[uu'\right] = \sigma_\mu^2 A + \rho_{(3)}\left(G - A\right) + \rho_{(2)}\left(L - A\right) + \rho_{(1)}\left(M - A\right) + \\ + \sigma_v^{(1)^2} H + \sigma_v^{(2)^2} Q + \sigma_v^{(3)^2} N + \sigma_\epsilon^2 I_{\sum_i\sum_j\sum_s T_{ijs}} \end{aligned}$$

In the case of models (4), (5), and (7) we have for $\mu_{ij}^{(1)}$

$$\begin{aligned} E[\tilde{\mu}_i^{(1)}\tilde{\mu}_i^{(1)'}] &= \sigma_\mu^{(1)^2} B_i + \rho_{(2)}^{(1)}\left(J_{\sum_j\sum_s T_{ijs}} - B_i\right) \\ E[\tilde{\mu}^{(1)}\tilde{\mu}^{(1)'}] &= \sigma_\mu^{(1)^2} G + \rho_{(2)}^{(1)}\left(C - G\right) + \rho_{(1)}^{(1)}\left(B - G\right) \end{aligned}$$

for $\mu_{is}^{(2)}$

$$\begin{aligned} E[\tilde{\mu}_i^{(2)}\tilde{\mu}_i^{(2)'}] &= \sigma_\mu^{(2)^2} F_{i,i} + \rho_{(2)}^{(2)}\left(J_{\sum_j\sum_s T_{ijs}} - F_{i,i}\right) \\ E[\tilde{\mu}^{(2)}\tilde{\mu}^{(2)'}] &= \sigma_\mu^{(2)^2} L + \rho_{(2)}^{(2)}\left(C - L\right) + \rho_{(1)}^{(2)}\left(F - L\right) \end{aligned}$$

and for $\mu_{js}^{(3)}$

$$\begin{aligned} E[\tilde{\mu}_{ij}^{(3)}\tilde{\mu}_{ij}^{(3)'}] &= \sigma_\mu^{(3)^2} A_{ij} + \rho_{(2)}^{(3)}\left(J_{\sum_s T_{ijs}} - A_{ij}\right) \\ E[\tilde{\mu}_i^{(3)}\tilde{\mu}_i^{(3)'}] &= \sigma_\mu^{(3)^2} A_i + \rho_{(2)}^{(3)}\left(B_i - A_i\right) + \rho_{(1)}^{(3)}\left(F_{i,i} - A_i\right) \\ E[\tilde{\mu}^{(3)}\tilde{\mu}^{(3)'}] &= \sigma_\mu^{(3)^2} M + \rho_{(2)}^{(3)}\left(B - M\right) + \rho_{(1)}^{(3)}\left(F - M\right) \end{aligned}$$

So the covariance matrix of model (4) now is

$$E[uu'] = {\sigma_{\mu}^{(1)}}^2 G + \rho_{(2)}^{(1)} (C - G) + \rho_{(1)}^{(1)} (B - G) + {\sigma_{\mu}^{(2)}}^2 L + \rho_{(2)}^{(2)} (C - L) + \rho_{(1)}^{(2)} (F - L) + \\ + {\sigma_{\mu}^{(3)}}^2 M \rho_{(2)}^{(3)} (B - M) + \rho_{(1)}^{(3)} (F - M) + \sigma_{\epsilon}^2 I_{\sum_i \sum_j \sum_s T_{ijs}}$$

for model (5) we get

$$E[uu'] = {\sigma_{\mu}^{(1)}}^2 G + \rho_{(2)}^{(1)} (C - G) + \rho_{(1)}^{(1)} (B - G) + {\sigma_{\mu}^{(2)}}^2 L + \rho_{(2)}^{(2)} (C - L) + \rho_{(1)}^{(2)} (F - L) + \\ + {\sigma_{\mu}^{(3)}}^2 M + \rho_{(2)}^{(3)} (B - M) + \rho_{(1)}^{(3)} (F - M) + \sigma_{\lambda}^2 D + \sigma_{\epsilon}^2 I_{\sum_i \sum_j \sum_s T_{ijs}}$$

and, finally, for model (7) we get the following covariance matrix

$$E[uu'] = {\sigma_{\mu}^{(1)}}^2 G + \rho_{(2)}^{(1)} (C - G) + \rho_{(1)}^{(1)} (B - G) + {\sigma_{\mu}^{(2)}}^2 L + \rho_{(2)}^{(2)} (C - L) + \rho_{(1)}^{(2)} (F - L) + \\ + {\sigma_{\mu}^{(3)}}^2 M + \rho_{(2)}^{(3)} (B - M) + \rho_{(1)}^{(3)} (F - M) + {\sigma_{v}^{(1)}}^2 H + {\sigma_{v}^{(2)}}^2 Q + {\sigma_{v}^{(3)}}^2 N + \\ + \sigma_{\epsilon}^2 I_{\sum_i \sum_j \sum_s T_{ijs}}$$

4.4 Estimation of the Variance Components and Cross Correlations

Again, some additional notations need to be introduced:

$$\tilde{K} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{T_i^{(1)}} \sum_{t \in Z_i^{(1)}} \left(\frac{1}{N_{it}^{(2)}} \sum_{j \in Q_{it}^{(2)}} \frac{1}{N_{ijt}^{(3)}} \sum_{s \in Q_{ijt}^{(3)}} \hat{u}_{ijst} \right)^2$$

$$k_1 = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{T_i^{(1)}} \sum_{t \in Z_i^{(1)}} \frac{1}{{N_{it}^{(2)}}^2} \sum_{j \in Q_{it}^{(2)}} \frac{1}{N_{ijt}^{(3)}}$$

$$k_2 = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{T_i^{(1)}} \sum_{t \in Z_i^{(1)}} \frac{1}{N_{it}^{(2)}}$$

$$k_3 = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{T_i^{(1)}} \sum_{t \in Z_i^{(1)}} \frac{1}{{N_{it}^{(2)}}^2} \left(\sum_{j \in Q_{it}^{(2)}} \frac{1}{N_{ijt}^{(3)}} \left(1 + \sum_{k \in Q_{it}^{(2)}, k \neq j} \frac{1}{N_{ikt}^{(3)}} S_{jk}^{it} \right) \right)$$

$$b^{(1)} = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{N_j^{(3)}} \sum_{s \in Q_j^{(3)}} \frac{1}{T_{js}^{(1)}} \sum_{t \in Z_{js}^{(1)}} \frac{N_{jst}^{(1)} - 1}{N_{jst}^{(1)}}$$

$$c^{(1)} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(3)}} \sum_{s \in Q_i^{(3)}} \frac{1}{T_{is}^{(2)}} \sum_{t \in Z_{is}^{(2)}} \frac{N_{ist}^{(2)} - 1}{N_{ist}^{(2)}}$$

$$d^{(1)} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{N_i^{(2)}} \sum_{j \in Q_i^{(2)}} \frac{1}{T_{ij}^{(3)}} \sum_{t \in Z_{ij}^{(3)}} \frac{N_{ijt}^{(3)} - 1}{N_{ijt}^{(3)}}$$

$$f^{(1)} = \frac{1}{N^{(3)}} \sum_{s \in Q^{(3)}} \frac{1}{T_s^{(3)}} \sum_{t \in Z_s^{(3)}} \frac{1}{N_{st}^{(1)^2}} \sum_{i \in Q_{st}^{(1)}} \sum_{l \in Q_{st}^{(1)}, l \neq i} \frac{1}{N_{ist}^{(2)}} \frac{1}{N_{lst}^{(2)}} I_{il}^{st}$$

$$f^{(2)} = \frac{1}{N^{(3)}} \sum_{s \in Q^{(3)}} \frac{1}{T_s^{(3)}} \sum_{t \in Z_s^{(3)}} \frac{1}{N_{st}^{(1)^2}} \sum_{i \in Q_{st}^{(1)}} \frac{N_{ist}^{(2)} - 1}{N_{ist}^{(2)}}$$

$$g^{(1)} = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{T_j^{(2)}} \sum_{t \in Z_j^{(2)}} \frac{1}{N_{jt}^{(1)^2}} \sum_{i \in Q_{jt}^{(1)}} \sum_{p \in Q_{jt}^{(1)}, p \neq i} \frac{1}{N_{ijt}^{(3)}} \frac{1}{N_{pjt}^{(3)}} J_{ip}^{jt}$$

$$g^{(3)} = \frac{1}{N^{(2)}} \sum_{j \in Q^{(2)}} \frac{1}{T_j^{(2)}} \sum_{t \in Z_j^{(2)}} \frac{1}{N_{jt}^{(1)^2}} \sum_{i \in Q_{jt}^{(1)}} \frac{N_{ijt}^{(3)} - 1}{N_{ijt}^{(3)}}$$

$$k^{(2)} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{T_i^{(1)}} \sum_{t \in Z_i^{(1)}} \frac{1}{N_{it}^{(2)^2}} \sum_{j \in Q_{it}^{(2)}} \sum_{k \in Q_{it}^{(2)}, k \neq j} \frac{1}{N_{ijt}^{(3)}} \frac{1}{N_{ikt}^{(3)}} S_{jk}^{it}$$

$$k^{(3)} = \frac{1}{N^{(1)}} \sum_{i \in Q^{(1)}} \frac{1}{T_i^{(1)}} \sum_{t \in Z_i^{(1)}} \frac{1}{N_{it}^{(2)^2}} \sum_{j \in Q_{it}^{(2)}} \frac{N_{ijt}^{(3)} - 1}{N_{ijt}^{(3)}}$$

where S_{jk}^{it} is the number of countries to which firm i sells both products j and k at time t.

Model (2)

The estimation of variance components of ϵ and μ remain as in the model without cross correlations, but the cross correlation coefficients themselves need to be estimated.

In this case the identifying equations are

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right]=b\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+b^{(1)}\rho_{(1)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=c\left(\sigma_\gamma^2+\sigma_\epsilon^2\right)+c^{(1)}\rho_{(2)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=d\left(\sigma_\alpha^2+\sigma_\epsilon^2\right)+d^{(1)}\rho_{(3)}$$

So we get

$$\hat\rho_{(1)}=\frac{1}{b^{(1)}}\left[\tilde B-b\left(\hat\sigma_\mu^2+\hat\sigma_\epsilon^2\right)\right]$$
$$\hat\rho_{(2)}=\frac{1}{c^{(1)}}\left[\tilde C-c\left(\hat\sigma_\mu^2+\hat\sigma_\epsilon^2\right)\right]$$
$$\hat\rho_{(3)}=\frac{1}{d^{(1)}}\left[\tilde D-d\left(\hat\sigma_\mu^2+\hat\sigma_\epsilon^2\right)\right]$$

Model (3)

The estimation of the variance of μ remains the same, as above, however it changes for other variance components. Now the identifying equations are

$$E\left[u_{ijst}^2\right]=\sigma_\mu^2+\sigma_\lambda^2+\sigma_\epsilon^2$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right]=b\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\lambda^2+b^{(1)}\rho_{(1)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=c\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\lambda^2+c^{(1)}\rho_{(2)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=d\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\lambda^2+d^{(1)}\rho_{(3)}$$

$$E\left[\frac{1}{N^{(3)}}\sum_{s\in Q^{(3)}}\frac{1}{T_s^{(3)}}\sum_{t\in Z_s^{(3)}}\left(\frac{1}{N_{st}^{(1)}}\sum_{i\in Q_{st}^{(1)}}\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=$$
$$=f_1\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\lambda^2+f^{(1)}\rho_{(1)}+f^{(2)}\rho_{(2)}$$

and thus

$$
\begin{aligned}
\hat{\sigma}_\epsilon^2 &= \frac{1}{f_1 - 1 - (b-1)\frac{f^{(1)}}{b^{(1)}} - (c-1)\frac{f^{(2)}}{c^{(1)}}} \left[\tilde{A}\left(\frac{f^{(1)}}{b^{(1)}} + \frac{f^{(2)}}{c^{(1)}} - 1\right) - \frac{f^{(1)}}{b^{(1)}}\tilde{B} - \frac{f^{(2)}}{c^{(1)}}\tilde{C} + \tilde{F}\right] - \hat{\sigma}_\mu^2 \\
\hat{\sigma}_\lambda^2 &= \tilde{A} - \hat{\sigma}_\mu^2 - \hat{\sigma}_\epsilon^2 \\
\hat{\rho}_{(1)} &= \frac{1}{b^{(1)}}\left[\tilde{B} - b\left(\hat{\sigma}_\mu^2 + \hat{\sigma}_\epsilon^2\right) - \hat{\sigma}_\lambda^2\right] \\
\hat{\rho}_{(2)} &= \frac{1}{c^{(1)}}\left[\tilde{C} - c\left(\hat{\sigma}_\mu^2 + \hat{\sigma}_\epsilon^2\right) - \hat{\sigma}_\lambda^2\right] \\
\hat{\rho}_{(3)} &= \frac{1}{d^{(1)}}\left[\tilde{D} - d\left(\hat{\sigma}_\mu^2 + \hat{\sigma}_\epsilon^2\right) - \hat{\sigma}_\lambda^2\right]
\end{aligned}
$$

Model (6)

Again the estimation of the variance of μ remains unchanged, but we need, of course, to estimate all the remaining variance components and cross-correlations. The identifying equations now are

$$E\left[u_{ijst}^2\right] = \sigma_\mu^2 + {\sigma_v^{(1)}}^2 + {\sigma_v^{(2)}}^2 + {\sigma_v^{(3)}}^2 + \sigma_\epsilon^2$$

$$
\begin{aligned}
&E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right] = \\
&\qquad = b\left(\sigma_\mu^2 + {\sigma_v^{(1)}}^2 + \sigma_\epsilon^2\right) + {\sigma_v^{(2)}}^2 + {\sigma_v^{(3)}}^2 + b^{(1)}\rho_{(1)} \\
&E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right] = \\
&\qquad = c\left(\sigma_\mu^2 + {\sigma_v^{(2)}}^2 + \sigma_\epsilon^2\right) + {\sigma_v^{(1)}}^2 + {\sigma_v^{(3)}}^2 + c^{(1)}\rho_{(2)} \\
&E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right] = \\
&\qquad = d\left(\sigma_\mu^2 + {\sigma_v^{(3)}}^2 + \sigma_\epsilon^2\right) + {\sigma_v^{(1)}}^2 + {\sigma_v^{(2)}}^2 + d^{(1)}\rho_{(3)}
\end{aligned}
$$

$$E\left[\frac{1}{N^{(3)}}\sum_{s\in Q^{(3)}}\frac{1}{T_s^{(3)}}\sum_{t\in Z_s^{(3)}}\left(\frac{1}{N_{st}^{(1)}}\sum_{i\in Q_{st}^{(1)}}\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=$$
$$=f_1\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+f_2\sigma_v^{(1)^2}+f_3\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+f^{(1)}\rho_{(1)}+f^{(2)}\rho_{(2)}$$
$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{T_j^{(2)}}\sum_{t\in Z_j^{(2)}}\left(\frac{1}{N_{jt}^{(1)}}\sum_{i\in Q_{jt}^{(1)}}\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$
$$=g_1\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+g_2\sigma_v^{(3)^2}+g_3\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+g^{(1)}\rho_{(1)}+g^{(3)}\rho_{(3)}$$
$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{T_i^{(1)}}\sum_{t\in Z_i^{(1)}}\left(\frac{1}{N_{it}^{(2)}}\sum_{j\in Q_{it}^{(2)}}\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$
$$=k_1\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+k_2\sigma_v^{(2)^2}+k_3\sigma_v^{(3)^2}+\sigma_v^{(1)^2}+k^{(2)}\rho_{(2)}+k^{(3)}\rho_{(3)}$$

In order to get estimators of variance components and cross-correlations one needs to solve the following system of 7 equations for 7 unknowns: $\hat{\sigma}_v^{(1)^2}$, $\hat{\sigma}_v^{(2)^2}$, $\hat{\sigma}^2_{v^{(3)}}$, $\hat{\sigma}^2_\epsilon$, $\hat{\rho}_{(1)}$, $\hat{\rho}_{(2)}$, $\hat{\rho}_{(3)}$

$$\begin{aligned}
\hat{\sigma}_v^{(1)^2}+\hat{\sigma}_v^{(2)^2}+\hat{\sigma}^2_{v^{(3)}}+\hat{\sigma}^2_\epsilon &= \tilde{A}-\hat{\sigma}^2_\mu\\
b\left(\hat{\sigma}_v^{(1)^2}+\hat{\sigma}^2_\epsilon\right)+\hat{\sigma}_v^{(2)^2}+\hat{\sigma}^2_{v^{(3)}}+b^{(1)}\hat{\rho}_{(1)} &= \tilde{B}-b\hat{\sigma}^2_\mu\\
c\left(\hat{\sigma}_v^{(2)^2}+\hat{\sigma}^2_\epsilon\right)+\hat{\sigma}_v^{(1)^2}+\hat{\sigma}^2_{v^{(3)}}+c^{(1)}\hat{\rho}_{(2)} &= \tilde{C}-c\hat{\sigma}^2_\mu\\
d\left(\hat{\sigma}^2_{v^{(3)}}+\hat{\sigma}^2_\epsilon\right)+\hat{\sigma}_v^{(1)^2}+\hat{\sigma}_v^{(2)^2}+d^{(1)}\hat{\rho}_{(3)} &= \tilde{D}-d\hat{\sigma}^2_\mu\\
f_1\hat{\sigma}^2_\epsilon+f_2\hat{\sigma}_v^{(1)^2}+f_3\hat{\sigma}_v^{(2)^2}+\hat{\sigma}^2_{v^{(3)}}+f^{(1)}\hat{\rho}_{(1)}+f^{(2)}\hat{\rho}_{(2)} &= \tilde{F}-f_1\hat{\sigma}^2_\mu\\
g_1\hat{\sigma}^2_\epsilon+g_2\hat{\sigma}^2_{v^{(3)}}+g_3\hat{\sigma}_v^{(1)^2}+\hat{\sigma}_v^{(2)^2}+g^{(1)}\hat{\rho}_{(1)}+g^{(3)}\hat{\rho}_{(3)} &= \tilde{G}-g_1\hat{\sigma}^2_\mu\\
k_1\hat{\sigma}^2_\epsilon+k_2\hat{\sigma}_v^{(2)^2}+k_3\hat{\sigma}^2_{v^{(3)}}+\hat{\sigma}_v^{(1)^2}+k^{(2)}\hat{\rho}_{(2)}+k^{(3)}\hat{\rho}_{(3)} &= \tilde{K}-k_1\hat{\sigma}^2_\mu
\end{aligned}$$

It is hard to solve this system in its general form, however, it can easily be solved for given data set.

Model (4)

For models (4), (5) and (7) we make the same assumptions for the variance components of $\mu_{ij}^{(1)}$, $\mu_{is}^{(2)}$, $\mu_{js}^{(3)}$ and for covariance parameters as in the balanced case. We now have the following identifying equations

$$E\left[u^2_{ijst}\right]=3\sigma^2_\mu+\sigma^2_\epsilon$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right]=b\left(2\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\mu^2+b^{(1)}\rho_{(1)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=c\left(2\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\mu^2+c^{(1)}\rho_{(2)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=d\left(2\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\mu^2+d^{(1)}\rho_{(3)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{N_{ij}^{(3)}}\sum_{s\in Q_{ij}^{(3)}}\left(\frac{1}{T_{ijs}}\sum_{t\in Z_{ijs}}u_{ijst}\right)^2\right]=3\sigma_\mu^2+e\sigma_\epsilon^2$$

Thus

$$\hat{\sigma}_\epsilon^2=\frac{\tilde{A}-\tilde{E}}{1-e}$$
$$\hat{\sigma}_\mu^2=\frac{1}{3}\left(\tilde{A}-\hat{\sigma}_\epsilon^2\right)$$
$$\hat{\rho}_{(1)}=\frac{1}{b^{(1)}}\left[\tilde{B}-(2b+1)\,\hat{\sigma}_\mu^2-b\hat{\sigma}_\epsilon^2\right]$$
$$\hat{\rho}_{(2)}=\frac{1}{c^{(1)}}\left[\tilde{C}-(2c+1)\,\hat{\sigma}_\mu^2-c\hat{\sigma}_\epsilon^2\right]$$
$$\hat{\rho}_{(3)}=\frac{1}{d^{(1)}}\left[\tilde{D}-(2d+1)\,\hat{\sigma}_\mu^2-d\hat{\sigma}_\epsilon^2\right]$$

Model (5)

From the identifying equations

$$E\left[u_{ijst}^2\right]=3\sigma_\mu^2+\sigma_\lambda^2+\sigma_\epsilon^2$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right]=$$
$$=b\left(2\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_\lambda^2+b^{(1)}\rho_{(1)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=$$
$$=c\left(2\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_\lambda^2+c^{(1)}\rho_{(2)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$
$$=d\left(2\sigma_\mu^2+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_\lambda^2+d^{(1)}\rho_{(3)}$$
$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{N_{ij}^{(3)}}\sum_{s\in Q_{ij}^{(3)}}\left(\frac{1}{T_{ijs}}\sum_{t\in Z_{ijs}}u_{ijst}\right)^2\right]=$$
$$=3\sigma_\mu^2+e\left(\sigma_\lambda^2+\sigma_\epsilon^2\right)$$

we get

$$\hat\sigma_\mu^2=\frac{1}{3}\cdot\frac{\tilde E-e\tilde A}{1-e}$$

$$\begin{aligned}\hat\sigma_\epsilon^2=&\frac{1}{f_1-1-f_1^{(1)}\frac{b-1}{b^{(1)}}-f_1^{(2)}\frac{c-1}{c^{(1)}}-f_1^{(3)}\frac{d-1}{d^{(1)}}}\Bigg[\tilde A\left(\frac{f_1^{(1)}}{b^{(1)}}+\frac{f_1^{(2)}}{c^{(1)}}+\frac{f_1^{(3)}}{d^{(1)}}-1\right)-\frac{f_1^{(1)}}{b^{(1)}}\tilde B-\\&-\frac{f_1^{(2)}}{c^{(1)}}\tilde C-\frac{f_1^{(3)}}{d^{(1)}}\tilde D+\tilde F-\hat\sigma_\mu^2\Bigg((f_1+f_2+f_3-3+2(b-1)\frac{f_1^{(1)}}{b^{(1)}}+2(c-1)\frac{f_1^{(c)}}{c^{(1)}}+\\&+2(d-1)\frac{f_1^{(3)}}{d^{(1)}}\Bigg)\Bigg]\end{aligned}$$

$$\hat\sigma_\lambda^2=\tilde A-3\hat\sigma_\mu^2-\hat\sigma_\epsilon^2$$
$$\hat\rho_{(1)}=\frac{1}{b^{(1)}}\left(\tilde B-(2b+1)\hat\sigma_\mu^2-\hat\sigma_\lambda^2-b\hat\sigma_\epsilon^2\right)$$
$$\hat\rho_{(2)}=\frac{1}{c^{(1)}}\left(\tilde C-(2c+1)\hat\sigma_\mu^2-\hat\sigma_\lambda^2-c\hat\sigma_\epsilon^2\right)$$
$$\hat\rho_{(3)}=\frac{1}{d^{(1)}}\left(\tilde D-(2d+1)\hat\sigma_\mu^2-\hat\sigma_\lambda^2-d\hat\sigma_\epsilon^2\right)$$

Model (7)

As above, from

$$E\left[u_{ijst}^2\right]=2\sigma_\mu^2+\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+\sigma_\epsilon^2$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{N_j^{(3)}}\sum_{s\in Q_j^{(3)}}\frac{1}{T_{js}^{(1)}}\sum_{t\in Z_{js}^{(1)}}\left(\frac{1}{N_{jst}^{(1)}}\sum_{i\in Q_{jst}^{(1)}}u_{ijst}\right)^2\right]=$$
$$=b\left(2\sigma_\mu^2+\sigma_v^{(1)^2}+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_v^{(2)^2}+\sigma_v^{(3)^2}+b^{(1)}\rho_{(1)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(3)}}\sum_{s\in Q_i^{(3)}}\frac{1}{T_{is}^{(2)}}\sum_{t\in Z_{is}^{(2)}}\left(\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=$$

$$=c\left(2\sigma_\mu^2+\sigma_v^{(2)^2}+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_v^{(1)^2}+\sigma_v^{(3)^2}+c^{(1)}\rho_{(2)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{N_i^{(2)}}\sum_{j\in Q_i^{(2)}}\frac{1}{T_{ij}^{(3)}}\sum_{t\in Z_{ij}^{(3)}}\left(\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$

$$=d\left(2\sigma_\mu^2+\sigma_v^{(3)^2}+\sigma_\epsilon^2\right)+\sigma_\mu^2+\sigma_v^{(1)^2}+\sigma_v^{(2)^2}+d^{(1)}\rho_{(3)}$$

$$E\left[\frac{1}{N^{(3)}}\sum_{s\in Q^{(3)}}\frac{1}{T_s^{(3)}}\sum_{t\in Z_s^{(3)}}\left(\frac{1}{N_{st}^{(1)}}\sum_{i\in Q_{st}^{(1)}}\frac{1}{N_{ist}^{(2)}}\sum_{j\in Q_{ist}^{(2)}}u_{ijst}\right)^2\right]=$$

$$=f_1\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+f_2\left(\sigma_\mu^2+\sigma_v^{(1)^2}\right)+f_3\left(\sigma_\mu^2+\sigma_v^{(2)^2}\right)+\sigma_v^{(3)^2}+$$

$$+f_1^{(1)}\rho_{(1)}+f_1^{(2)}\rho_{(2)}+f_1^{(3)}\rho_{(3)}$$

$$E\left[\frac{1}{N^{(2)}}\sum_{j\in Q^{(2)}}\frac{1}{T_j^{(2)}}\sum_{t\in Z_j^{(2)}}\left(\frac{1}{N_{jt}^{(1)}}\sum_{i\in Q_{jt}^{(1)}}\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$

$$=g_1\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+g_2\left(\sigma_\mu^2+\sigma_v^{(3)^2}\right)+g_3\left(\sigma_\mu^2+\sigma_v^{(1)^2}\right)+\sigma_v^{(2)^2}+$$

$$+g_1^{(1)}\rho_{(1)}+g_1^{(2)}\rho_{(2)}+g_1^{(3)}\rho_{(3)}$$

$$E\left[\frac{1}{N^{(1)}}\sum_{i\in Q^{(1)}}\frac{1}{T_i^{(1)}}\sum_{t\in Z_i^{(1)}}\left(\frac{1}{N_{it}^{(2)}}\sum_{j\in Q_{it}^{(2)}}\frac{1}{N_{ijt}^{(3)}}\sum_{s\in Q_{ijt}^{(3)}}u_{ijst}\right)^2\right]=$$

$$=k_1\left(\sigma_\mu^2+\sigma_\epsilon^2\right)+k_2\left(\sigma_\mu^2+\sigma_v^{(2)^2}\right)+k_3\left(\sigma_\mu^2+\sigma_v^{(3)^2}\right)+\sigma_v^{(1)^2}+$$

$$+k_1^{(1)}\rho_{(1)}+k_1^{(2)}\rho_{(2)}+k_1^{(3)}\rho_{(3)}$$

Here again as in the case of the previous model, it is hard to get estimators of variance components in general form. However, it can easily be done for a given data set. One needs to solve the following system:

$$\begin{aligned}
\hat{\sigma}_v^{(1)^2} + \hat{\sigma}_v^{(2)^2} + \hat{\sigma}_{v(3)}^2 + \hat{\sigma}_\epsilon^2 &= \tilde{A} - 2\hat{\sigma}_\mu^2 \\
b\left(\hat{\sigma}_v^{(1)^2} + \hat{\sigma}_\epsilon^2\right) + \hat{\sigma}_v^{(2)^2} + \hat{\sigma}_{v(3)}^2 + b^{(1)}\hat{\rho}_{(1)} &= \tilde{B} - (2b+1)\hat{\sigma}_\mu^2 \\
c\left(\hat{\sigma}_v^{(2)^2} + \hat{\sigma}_\epsilon^2\right) + \hat{\sigma}_v^{(1)^2} + \hat{\sigma}_{v(3)}^2 + c^{(1)}\hat{\rho}_{(2)} &= \tilde{C} - (2c+1)\hat{\sigma}_\mu^2 \\
d\left(\hat{\sigma}_{v(3)}^2 + \hat{\sigma}_\epsilon^2\right) + \hat{\sigma}_v^{(1)^2} + \hat{\sigma}_v^{(2)^2} + d^{(1)}\hat{\rho}_{(3)} &= \tilde{D} - (2d+1)\hat{\sigma}_\mu^2 \\
f_1\hat{\sigma}_\epsilon^2 + f_2\hat{\sigma}_v^{(1)^2} + f_3\hat{\sigma}_v^{(2)^2} + \hat{\sigma}_{v(3)}^2 + f_1^{(1)}\hat{\rho}_{(1)} + f_1^{(2)}\hat{\rho}_{(2)} + f_1^{(3)}\hat{\rho}_{(3)} &= \tilde{F} - (f_1 + f_2 + f_3)\,\hat{\sigma}_\mu^2 \\
g_1\hat{\sigma}_\epsilon^2 + g_2\hat{\sigma}_{v(3)}^2 + g_3\hat{\sigma}_v^{(1)^2} + \hat{\sigma}_v^{(2)^2} + g_1^{(1)}\hat{\rho}_{(1)} + g_1^{(2)}\hat{\rho}_{(2)} + g_1^{(3)}\hat{\rho}_{(3)} &= \tilde{G} - (g_1 + g_2 + g_3)\,\hat{\sigma}_\mu^2 \\
k_1\hat{\sigma}_\epsilon^2 + k_2\hat{\sigma}_v^{(2)^2} + k_3\hat{\sigma}_{v(3)}^2 + \hat{\sigma}_v^{(1)^2} + k_1^{(1)}\hat{\rho}_{(1)} + k_1^{(2)}\hat{\rho}_{(2)} + k_1^{(3)}\hat{\rho}_{(3)} &= \tilde{K} - (k_1 + k_2 + k_3)\,\hat{\sigma}_\mu^2
\end{aligned}$$

5. Conclusion

In this paper we derived several four-dimensional random effects panel data models, suited to deal with economic flow type data like trade or FDI. All necessary tools have been introduced in order to be able to implement these models using standard econometric/statistical software packages like Stata or Matlab. Higher dimensional panel data sets, however, can become very large, very quickly. Even for smaller data sets, the number of observations can be in the magnitude of $10^4 - 10^5$, but can easily reach 10^6. This is translated into huge computational resource requirements, both in hard drive capacity and CPU time. Typically, the models and methods presented for the balanced case (when closed form spectral decompositions are available) can be used when the data is of the size 10^5. However, when the data set becomes bigger or unbalanced methods and models need to be used, resource requirements can be forbidding. The main difficulty is that in the case of the GLS estimator the covariance matrix of the model, which has the size of the overall number of observations, needs to be inverted. If we run out of computing power the solution can be to use OLS instead of GLS. The OLS estimator is still consistent, although not optimal, for the models introduced in this paper. But then the covariance matrix of the OLS estimator needs to be properly adjusted using the covariance matrix of each model and the estimated variance components derived above, in order to get the appropriate standard errors of the estimated parameters. Another way to ease computing power requirements is to use lower level programming languages like C+, etc., but this requires serious code writing skills and additional resources.

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