



# AS-AD in the Standard Dynamic Neoclassical Model: Business Cycles and Growth Trends

by

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## **Abstract**

The paper shows how a dynamic neoclassical AS-AD can be derived and used to describe business cycles and growth trends to undergraduates. Derived within the Ramsey-Cass-Koopmans (RCK) model, the AS-AD is the stationary equilibrium of the deterministic dynamic general equilibrium framework. Allowing Solow exogenous growth, the AS-AD is derived along the balanced growth path equilibrium. The derivation first builds consumption demand, aggregate demand, and then aggregate supply through the equilibrium conditions and a closed form solution for the capital stock. Through a comparative static change in goods sector productivity, the paper shows the basic failing of the standard RBC model. Allowing a second comparative static change in the consumer's time endowment, this captures a change in the "external margin" of labor supply. These comparative statics enable explanation of the business cycle, and "Solow-plus" growth trends including education time and working time. In extension of RCK, the paper shows beyond the undergraduate level, how to derive AS-AD when including human capital and endogenous growth. This allows an endogenous change in the time endowment for work and leisure through a change in human capital productivity, with a similar but more fundamental AS-AD story of business cycles and growth trends.

Key words: Ramsey-Cass-Koopmans, supply, demand, state variables

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# 1 Introduction

Colander (1995) famously clarifies how the aggregate supply and aggregate demand ( $AS - AD$ ) analysis is derived in part from the Keynesian model and is "incorrectly specified". It lacks internal consistency in that it combines a Keynesian demand which implicitly assumes an unknown supply condition, and then adds a classical supply curve, thereby mixing analyses. At the heart of the Keynesian demand is the Keynesian "cross" that goes back at least to Samuelson's (1951) seminal formulation. Samuelson starts with a "fitted" consumption expenditure schedule with a slope less than one graphed around a 45 degree line (p. 266). This becomes the basis of the standard derivation of the Keynesian demand. In contrast, Friedman (1957) puts forth that the Keynesian assumption about the nature of the consumption function is not only important but also wrong, a debate that ensues today. Friedman offers the alternative of instead deriving a "classical" consumption demand that is a fraction of permanent income, using Fisher's intertemporal analysis.

Colander (1995) suggests making the Keynesian  $AD$  consistent with a Keynesian  $AS$ , perhaps through a focus on how the price level is determined. Originally, Fisher derived the price level as a function of the money supply in his quantity theory. Keynes (1923) embraced this Fisher derivation in his *Tract*, but Keynes (1930) radically departs from its price level determination in his *Treatise*. Instead Keynes follows his teacher Marshall and writes the aggregate price as the aggregate average cost plus aggregate profit. However Keynes (1930) assumes that profit equals investment minus savings, which he defends in footnotes to his 1936 *General Theory*, but which has no basis in Marshallian theory.<sup>1</sup>

This paper could be said to formulate  $AS - AD$  by following in part both Keynes's (1930) fascinating Marshall-based attempt and Friedman's Fisher-based consumption theory. It derives consumption demand as in the

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<sup>1</sup>Gillman (2002) argues that the Keynesian cross analysis is based on the incorrect assumption that profit equals investment (I) minus savings (S), as in Keynes's (1930) *Treatise on Money*. The *Treatise* presents a "cross" and business cycle whereby  $I > S$  causes profit that leads to expansion, while  $I < S$  leads to contraction, with a "stable" equilibrium at  $I = S$  where the cross intersection occurs. In modern micro-founded real business cycle theory, the representative agent finds  $I = S$  in equilibrium, profit is defined residually as revenue minus cost, and such a dynamic basis for a business cycle through I and S is not explicitly postulated.

permanent income hypothesis of consumption, but as an outcome of utility optimization rather than as any sort of assumption. Aggregate demand then adds the consumption demand to investment, while aggregate supply is the firm's normalized upward sloping marginal cost curve. The elusive aggregate price of the  $AS - AD$  analysis is simply a relative price as in all micro theory, in this case of goods to leisure (or labor if one prefers). Its "balanced growth path" ( $BGP$ ) time-less solution bypasses the time dimension that Colander notes exists in the context of classic  $AS$  supply curves that are described as short, medium or long run. Instead the  $AS - AD$  presented here is the "stationary equilibrium" that we (in misnomer?) often call the long run. As was the aim of Keynes (1930), the (relative) price of output is not tied up in any way with monetary theory, the quantity theory of money in particular, or with any money stock at all. Rather the price is simply the inverse of the real wage rate given that the goods price is normalized to one. In turn, in the labor market the relative price is the real wage rate divided by one. Monetary theory can be added within an  $AS - AD$  extension (Gillman, 2011), but is beyond the scope of this initial presentation of the "neoclassical"  $AS - AD$ .

The precise  $AS - AD$  presented is simply that of the Ramsey (1928)-Cass (1965)-Koopmans (1965) [ $RCK$ ] model. And while Colander (1995) rightly focuses on the intricacies of the dynamics involved in telling the  $AS - AD$  stories of Keynesian origin, also discussed in Gillman (2002), here the model is fully dynamic but only the  $BGP$  equilibrium is presented. Transition dynamics are not investigated. Comparative statics show the new  $BGP$  equilibrium when an exogenous parameter is changed. Only two comparative static changes are presented here. They provide analysis of the central focus of modern macroeconomics: the business cycle and growth theory. And while rigorous, only partial derivatives and algebraically solving systems of equations is required. The  $RCK$  derived  $AS - AD$  is an equilibrium of a dynamic model and so naturally has complexity. But arguably it is within reach of advanced undergraduates and, in my pure speculation, is the future for undergraduate teaching as mathematical description become increasingly accepted over time as in any science.

Surprisingly it appears that never before has  $AS - AD$  been derived rigorously within our standard dynamic general equilibrium framework. Such derivation is the paper's main contribution. Making it accessible to under-

graduates is a second contribution. And its other contribution is to use only two comparative static changes to explain qualitative business cycles and growth trends. It does this with no assumptions beyond standard parameter calibration for utility and production within the standard *RCK* model of neoclassical exogenous growth. Allowing for positive Solow exogenous technological progress,  $AS - AD$  is shown to be equally well-derived with sustained growth. Extension to endogenous growth follows rather seamlessly and improves key aspects of the "macroeconomic story we tell"; as this gets beyond undergraduate reach, but helps motivate the comparative statics used in the *RCK* model, this extension is briefly sketched as well.

Motivation for the paper is clear. It bridges a daunting gap for students of microeconomics embarking upon the study of macroeconomics. In modern macroeconomic teaching, microeconomic derived supply and demand analysis with comparative statics is left behind. In advanced macro research, it is replaced by numerical simulation of the equilibrium with impulse responses to shocks. The advantage of graphical market analysis has been lost even though it lurks just below the surface of our foundation modern macro research models. The paper's  $AS - AD$  fills the micro to macro void that has befuddled us since the dawn of modern macro general equilibrium theory.

## 2 Modeling Overview

Widespread adoption of the recursive methodology, stimulated for example by Stokey, Lucas and Prescott (1989), seemingly has pushed  $AS - AD$  even farther from the underlying markets based on supply and demand. Ironically it is the recursive methodology itself that holds a key to a "restart" of teaching modern macro with the supply and demand of micro. This key is the so-called state variable. This is the stock variable of the system which has its accumulation stated in discrete time as a first-order difference equation. And for typical timing conventions, the current period stock is known in the current time period while the choice of the stock for next period is the decision variable. The *RCK* known state variable at time  $t$  is the capital stock at time  $t$ , denoted here by  $k_t$ . The first trick to forming  $AS - AD$  is to exploit exactly this fact that the equilibrium  $k_t$  is known at time  $t$ .

The second trick is to identify  $AS - AD$  as we would in micro: at the

"stationary" equilibrium. In the dynamic world of *RCK* with capital accumulation and Solow (1956) growth, this equilibrium is the *BGP* equilibrium with either zero or some positive rate of exogenous growth. The formulation of  $AS - AD$  makes supply and demand a function of  $k_t$ . This is similar to how partial equilibrium demand typically is a function of exogenously given income, except that  $k_t$  is an endogenously determined equilibrium value. The capital stock can be solved as a closed-form solution of exogenous parameters by using market clearing to set  $AS$  equal to  $AD$  at the equilibrium relative price (or the real wage); it then in addition requires using the exogenous growth aspect of *RCK* to solve uniquely for the real wage in terms of exogenous parameters. This leaves the goods market clearing condition as one equation in  $k_t$ , with an explicit  $k_t$  solution..

Deriving  $AS - AD$  in the *RCK* model, as a function of  $k_t$ , involves using the *RCK* equilibrium conditions to solve the same two margins we already teach and derive in textbook-style to varying degrees: the intratemporal marginal rate of substitution between goods and leisure and the intertemporal margin between consumption this period and next period. From these margins, the permanent income hypothesis of consumption demand is formed. Construction of the  $AS - AD$  follows readily, along with the supply and demand for labor, also contingent on  $k_t$ .

A change in any exogenous parameter implies a new  $k_t$  in the *BGP* equilibrium, and a new set of  $AS - AD$  curves. In addition to changing the capital stock, an exogenous parameter change directly cause shifts in the supply and demand functions. So parameter changes cause changes in the equilibrium  $k_t$ , which shifts the  $AS - AD$ , and also the parameters directly enter the  $AS - AD$  functions and cause additional shifts.

The main comparative static exercise of neoclassical growth and real business cycles (*RBC*) is a change in the goods sector productivity parameter. Doing this within the  $AS - AD$  framework can illustrate the stark but well-known result that shows the central failure of the standard *RBC* model to explain business cycles. Using the homothetic utility and production functions, employment does not change when goods sector productivity changes, in contrast to actual business cycle experience. This gives a deterministic rendering of *RBC* theory whereby the income and substitution effects of the wage rate change offset each other and leave employment unchanged. Using

log-utility and Cobb-Douglas production throughout the paper, this comparative static also illustrates an equally well-known result in growth theory. A trend up in productivity over time results in hours worked staying constant rather than trending downwards as in data. The downward trend has been explained in a variety of ways. Call this a "Solow-plus" trend that we would prefer to capture if we could while still explaining the other Solow growth facts.<sup>2</sup>

Illustrating for the student central failings of the *RCK* model clears the way to embark upon a neat solution. To see a way to breathe new life into the *RCK* model, consider what a change in the goods sector productivity (TFP) actually does. Given the production technology, an increase in TFP raises the consumer's endowment of goods. Our neoclassical growth and business cycle analysis rests on this one side of our endowed world. The other dimension of the consumer endowment, Beckerian (1965) time, is ignored.

The solution is to allow not just the goods endowment to change, but rather both goods and time endowments to change. Allowing both endowments exogenously to rise and fall in comparative static fashion shows a business cycle in the  $AS - AD$  of the goods market and in the labor market. Allowing for certain exogenous trends in both goods and time endowment gives the standard Solow facts plus the trend down in hours worked per week.

Fortunately, the additional change in the time endowment not only balances out our treatment of considering changes in total endowment, goods and time, but is also consistent with how research has extended the standard model to better explain both business cycles and growth. Exogenously changing the time endowment given for work and leisure within the *RCK* model is similar to changing the "external" labor margin, which is sometimes thought of as the "labor force participation rate". Adding an external labor margin has been found to be a key to explaining cycles with an *RBC* approach.<sup>3</sup> In extension to endogenous growth, the time left for work and leisure becomes endogenous when there is a second sector using time to produce human capital investment. A change in the productivity of the human

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<sup>2</sup>These four Solow facts are a constant real interest rate and output to capital ratio, and a rising real wage and output to labor ratio.

<sup>3</sup>See Hansen (1985) and Rogerson (1988), along with the closely related two sector extensions of Benhabib et al (1991) and Greenwood and Hercowitz (1991).

capital sector, in general, causes a change in the time allocated for work and leisure.<sup>4</sup>

### 3 Analysis

Let the representative agent act as both firm and consumer. The firm rents capital  $k_t$  from the consumer at the real competitive rate  $r_t$  and pays wages for labor time  $l_t$  at the competitive rate  $w_t$ . The firm's production technology for goods output  $y_t$  is Cobb-Douglas with  $A_G \geq 0$  and  $\gamma \in [0, 1]$  :

$$y_t = A_G (l_t)^\gamma (k_t)^{1-\gamma}.$$

The firm profit  $\Pi_t$  maximization yields that the wage rate and capital rental rate equal their respective marginal products:

$$\begin{aligned} \underset{l_t, k_t}{Max} \Pi_t &= A_G (l_t)^\gamma (k_t)^{1-\gamma} - w_t l_t - r_t k_t; \\ w_t &= \gamma A_G (l_t)^{\gamma-1} (k_t)^{1-\gamma}, \\ r_t &= (1 - \gamma) A_G (l_t)^\gamma (k_t)^{-\gamma}. \end{aligned}$$

The consumer's period  $t$  utility is of log form in goods  $c_t$  and in leisure  $x_t$ , such that with the leisure preference parameter  $\alpha \geq 0$ ,

$$u(c_t, x_t) = \ln c_t + \alpha \ln x_t.$$

The consumer spends time working for the firm,  $l_t$ , and time in leisure, such that the total time endowment is equal to  $T$  :

$$T = l_t + x_t.$$

The consumer's goods budget constraint sets expenditure on consumption  $c_t$  equal to income from wages  $w_t l_t$  and capital rental  $r_t k_t$  minus investment in capital  $i_t$ . With  $\delta_K \in [0, 1]$  the depreciation rate on capital stock,  $i_t$  is given by

$$i_t = k_{t+1} - k_t (1 - \delta_K).$$

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<sup>4</sup>This literature dates to Uzawa's (1965) two sector model with human capital, which was modernized by *RCK*-style utility maximization in Lucas (1988).



The goods constraint is then

$$c_t = w_t l_t + r_t k_t - k_{t+1} + k_t (1 - \delta_K).$$

The consumer maximization problem is over the discounted infinite horizon, with  $\beta \in (0, 1) \equiv \frac{1}{1+\rho}$  being the discount factor, with  $\lambda_t$  the Lagrangian multiplier on the goods constraint, with  $\mu_t$  the Lagrangian multiplier on the time constraint, and with the choice being with respect to consumption of goods  $c_t$ , leisure  $x_t$ , and next period's capital stock  $k_{t+1}$  :

$$\begin{aligned} \text{Max}_{c_t, x_t, l_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \{ & \ln c_t + \alpha \ln x_t \\ & + \lambda_t [w_t l_t + r_t k_t - k_{t+1} + k_t (1 - \delta_K) - c_t] \\ & + \mu_t [T - l_t - x_t] \}. \end{aligned}$$

Or eliminating the constraints, the problem is equally stated as

$$\text{Max}_{l_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \{ \ln [w_t l_t + r_t k_t - k_{t+1} + k_t (1 - \delta_K)] + \alpha \ln (T - l_t) \}.$$

When taking the first-order conditions, the problem requires writing out the constrained utility in two adjacent time periods:  $t$  and  $t + 1$ . This is because next period's capital stock  $k_{t+1}$  appears in the goods resource constraint at time  $t$  and at time  $t + 1$ . The equilibrium conditions give the consumer's intratemporal marginal rate of substitution between goods and leisure and the intertemporal marginal rate of substitution between consumption at time  $t$  and at time  $t + 1$ .

The nature of the infinite horizon problem that requires a focus on only two periods,  $t$  and  $t + 1$ , immediately lends itself to the simpler recursive framework that indeed uses only those two periods. In particular, through infinite substitution the present discounted value of the infinitely discounted utility can be written simply over the two periods  $t$  and  $t + 1$ . Call the maximized Lagrangian  $V(k_t)$ . Then the problem rewrites in recursive form as

$$\begin{aligned} V(k_t) = \text{Max}_{c_t, x_t, l_t, k_{t+1}} \{ & \ln c_t + \alpha \ln x_t + \beta V(k_{t+1}) \\ & + \lambda_t [w_t l_t + r_t k_t - k_{t+1} + k_t (1 - \delta_K) - c_t] \\ & + \mu_t [T - l_t - x_t] \}. \end{aligned}$$

With constraints substituted in,

$$V(k_t) = \underset{l_t, k_{t+1}}{Max} \ln [w_t l_t + r_t k_t - k_{t+1} + k_t (1 - \delta_K)] + \alpha \ln (T - l_t) + \beta V(k_{t+1}).$$

In addition to the two first-order conditions, equilibrium requires the envelope condition of the derivative with respect to the state variable  $k_t$ .

$$\begin{aligned} \frac{w_t}{c_t} - \frac{\alpha}{x_t} &= 0, \\ \frac{-1}{c_t} + \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}} &= 0, \\ \frac{\partial V(k_t)}{\partial k_t} &= \frac{1 + r_t - \delta_k}{c_t}. \end{aligned}$$

Eliminating  $\frac{\partial V(k_t^s)}{\partial k_t^s}$  and  $\frac{\partial V(k_{t+1}^s)}{\partial k_{t+1}^s}$  and defining the rate of time preference  $\rho$  as  $\frac{1}{1+\rho} \equiv \beta$ , the equilibrium conditions reduce to the goods and time constraints plus the intratemporal and intertemporal margins that are

$$\frac{\frac{\alpha}{x_t}}{\frac{1}{c_t}} = w_t, \quad \frac{c_{t+1}}{c_t} = \frac{1 + r_t - \delta_k}{1 + \rho}.$$

### 3.1 Aggregate Demand: AD

Use the allocation of time constraint to solve for labor supply  $l_t^s = T - x_t$ . Substitute in for  $x_t = \frac{\alpha c_t}{w_t}$  from the intratemporal margin so that  $l_t^s = T - \frac{\alpha c_t}{w_t}$ . Then substitute in for  $l_t^s$  in the budget constraint:  $c_t = w_t \left( T - \frac{\alpha c_t}{w_t} \right) - k_{t+1} + k_t (1 + r_t - \delta_K)$ . Solving for consumption demand,  $c_t = \frac{w_t T - k_{t+1} + k_t (1 + r_t - \delta_K)}{1 + \alpha}$ . Bringing together terms such that  $c_t = \frac{w_t T + k_t \left( -\frac{k_{t+1}}{k_t} + 1 + r_t - \delta_k \right)}{1 + \alpha}$ , the *BGP* solution sees all non-stationary variables growing at the same rate, say  $g$ . Then consumption demand is

$$c_t = \frac{w_t T + k_t (r_t - \delta_k - g)}{1 + \alpha}$$

The intertemporal margin along the *BGP* implies that  $1 + g = \frac{c_{t+1}}{c_t} = \frac{1 + r_t - \delta_k}{1 + \rho}$ , so that  $r_t - \delta_k - g = \rho (1 + g)$ , and  $c_t = \frac{w_t T + k_t \rho (1 + g)}{1 + \alpha}$ .

In the baseline case, set the exogenous growth rate to zero, so that  $g = 0$ ,  $r = \rho + \delta_k$ , and  $c_t = \frac{w_t T + k_t \rho}{1 + \alpha}$ . Consumption is a fraction of permanent income  $y_{pt} \equiv w_t T + \rho k_t$ :

$$c_t = \frac{1}{1 + \alpha} (wT + \rho k) \equiv \frac{y_{pt}}{1 + \alpha}.$$

Consumption is a fraction of the flow of the full value of time plus the interest flow of capital, with the fraction a function of how much leisure is preferred. This is the permanent income hypothesis of consumption within the deterministic *RCK* model.

Adding stationary investment demand means adding the maintenance of capital or  $\delta_k k$  to consumption demand. This give a *BGP* aggregate demand of

$$AD : y^d = \frac{1}{1 + \alpha} (wT + \rho k) + \delta_k k.$$

The relative price of goods to leisure can be solved for so that a typical demand graph can ensue in price-quantity space:

$$\frac{1}{w} = \frac{T}{y(1 + \alpha) - k[\rho + (1 + \alpha)\delta_k]}.$$

Given  $k$  and the parameter values, a downward sloping demand (hyperbola) results. The solution to  $k$  is found by setting  $AD$  equal to  $AS$ .

### 3.2 Aggregate Supply: AS

Aggregate supply of goods is derived from the firm's equilibrium conditions. From the marginal product of labor equilibrium condition to the goods producer problem, labor demand is  $l_t^d = \left(\frac{\gamma A_G}{w_t}\right)^{\frac{1}{1-\gamma}} k_t$ . Substituting this  $l_t^d$  into the firm's production function  $y_t^s = A_G (l_t^d)^\gamma (k_t)^{1-\gamma}$  gives the aggregate supply  $AS$  as a function of the relative price  $\frac{1}{w_t}$  and the capital stock  $k_t$ .

$$AS : y_t^s = A_G^{\frac{1}{1-\gamma}} \left(\frac{\gamma}{w_t}\right)^{\frac{\gamma}{1-\gamma}} k_t.$$

Solving for the relative price, and along the *BGP* with zero growth, so that time subscripts can be dropped,

$$\frac{1}{w} = \frac{1}{\gamma A_G^{\frac{1}{1-\gamma}}} \left(\frac{y^s}{k}\right)^{\frac{1-\gamma}{\gamma}}.$$

Given  $k$  and the parameter values, the supply slopes upward with convexity if  $\gamma < 0.5$ , a linear function if  $\gamma = 0.5$  and with concavity if  $\gamma > 0.5$ .

### 3.3 Marginal Cost of Output

Here the price of goods is indeed the marginal cost of output as in microeconomic theory. Let total cost be denoted by  $TC_t$ , which in the model is equal to the firm's input costs, or

$$TC_t = w_t l_t + r_t k_t.$$

Solve for labor from the production function of output,  $l_t^d = \left(\frac{y_t}{A_G}\right)^{\frac{1}{\gamma}} (k_t)^{\frac{\gamma-1}{\gamma}}$ , and use the *BGP* facts that  $r = \rho + \delta_k$ , and that  $k$  and  $w$  are known equilibrium values. Then total cost can be written in terms of output  $y$  :

$$TC = \frac{w}{(A_G)^{\frac{1}{\gamma}}} (y)^{\frac{1}{\gamma}} (k)^{\frac{\gamma-1}{\gamma}} + (\rho + \delta_k) k.$$

Taking the partial derivative with respect to  $y$  defines marginal cost ( $MC$ ) in a typical way as

$$MC = \frac{\partial (TC)}{\partial y} = \frac{w}{\gamma A_G^{\frac{1}{\gamma}} (k)^{\frac{1-\gamma}{\gamma}}} y^{\frac{1-\gamma}{\gamma}}.$$

Normalizing the goods price to unity, and making the price a relative one by dividing by the real wage, then we again get the *AS* curve:  $\frac{1}{w_t} = \frac{y^{\frac{1-\gamma}{\gamma}}}{\gamma A_G^{\frac{1}{\gamma}} (k)^{\frac{1-\gamma}{\gamma}}}$ .

Varian (1978, p. 22) calls the same marginal cost function the short run marginal cost for a fixed capital stock  $k$ . Here  $k$  is the stationary solution, or "long run" solution, not a fixed factor per se. In general equilibrium, a "short run" ad hoc can be derived if the firm's time  $t$  capital stock is held constant when taking the equilibrium conditions; but that would violate the envelope condition and so is not an equilibrium in the *RCK* neoclassical model.

### 3.4 Solution for the Capital Stock

The *AS - AD* framework is convenient not only for its explanatory power through graphs but also as a solution methodology. With market clearing in the goods market, let the total quantity of goods demanded equal the total quantity of goods supplied. Denoting excess demand by the function  $Y(w_t, k_t)$ , this gives that

$$Y(w_t, k_t) \equiv y_t^d - y_t^s = \frac{w_t T + k_t [\rho + (1 + \alpha) \delta_k]}{1 + \alpha} - A_G^{\frac{1}{1-\gamma}} \left(\frac{\gamma}{w_t}\right)^{\frac{\gamma}{1-\gamma}} k_t.$$

Since excess demand is zero in equilibrium, and dropping time subscripts along the zero growth *BGP*,

$$0 = \frac{wT + k[\rho + (1 + \alpha)\delta_k]}{1 + \alpha} - A_G^{\frac{1}{1-\gamma}} \left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}} k.$$

Eliminate the wage rate  $w$  and solve for  $k$  by using the consumer fact that  $r = \rho + \delta_k$  and the firm side fact that the marginal product of capital is  $r = (1 - \gamma) A_G \left(\frac{l_t}{k_t}\right)^\gamma$ . This gives the equilibrium input ratio  $\frac{l_t}{k_t} = \left[\frac{\rho + \delta_k}{(1 - \gamma) A_G}\right]^{\frac{1}{\gamma}}$ , which can be substituted back into the firm's marginal product of labor

$$w = \gamma A_G \left(\frac{l}{k}\right)^{\gamma-1} = \gamma A_G \left[\frac{\rho + \delta_k}{(1 - \gamma) A_G}\right]^{\frac{\gamma-1}{\gamma}}.$$

Substituting in the above solution for  $w$  into  $Y(w, k) = 0$  gives the explicit closed form solution for the capital stock.

$$k = \frac{T \gamma A_G^{\frac{1}{\gamma}} \left[\frac{(1-\gamma)}{\rho + \delta_k}\right]^{\frac{1}{\gamma}}}{\gamma + \alpha - \alpha \delta_k \left(\frac{(1-\gamma)}{\rho + \delta_k}\right)}.$$

It is independent of time and the initial capital stock at time 0. From this solution  $AS - AD$  can be graphed given any calibration for the parameters, and comparative statics can be conducted. With zero leisure preference,  $\alpha = 0$ , consumption equals permanent income and the capital stock is simply  $k = T \left[\frac{A_G(1-\gamma)}{\rho + \delta_k}\right]^{\frac{1}{\gamma}} = T \left(\frac{w}{\gamma A_G}\right)^{\frac{1}{1-\gamma}}$ .

### 3.5 Labor Market

The supply and demand for labor follows directly. With consumption demand given as  $c_t = \frac{1}{1+\alpha} (w_t T + \rho k_t)$ , and the intratemporal margin as  $x_t = \frac{\alpha c_t}{w_t}$ , then using the allocation of time constraint:

$$l_t^s = T - x_t = T - \frac{\alpha c_t^d}{w_t} = T - \frac{\alpha}{1 + \alpha} \left[ T + \left(\frac{\rho}{w_t}\right) k_t \right].$$

To graph this in a price-quantity supply and demand space, solve for the relative price of labor (leisure) to goods, which is the real wage:

$$w_t = \frac{\alpha \rho k_t}{T - (1 + \alpha) l_t^s}.$$

It is clear that this will be an upward sloping supply of labor curve as  $l_t^s$  enters the righthand-side positively.

Labor demand is from the firm's marginal product of labor condition:

$$l_t^d = \left( \frac{\gamma A_G}{w_t} \right)^{\frac{1}{1-\gamma}} k_t,$$

which inversely is

$$w_t = \gamma A_G \left( \frac{k_t}{l_t^d} \right)^{1-\gamma}.$$

This demand curve hyperbolically slopes downward as the quantity of labor demanded enters the righthand-side negatively.

## 4 Calibrated AS-AD with Business Cycle

With a baseline calibration the *AS* and *AD* can be graphed and comparative statics conducted. In particular, increasing the goods endowment for a given production function involves simply increasing the goods productivity parameter  $A_G$ , the key parameter change in the *RBC* revolution ushered in by Kydland and Prescott (1982). Here this causes output, the real wage, and the capital stock to rise but has no effect on the employment of labor as the income and substitution effects exactly offset each other. An increase in the time endowment  $T$  causes output, the capital stock and the employment to all rise, while the real wage stays constant. Combining such an increase in both goods and time endowments causes a business cycle type increase in output, the capital stock, the real wage and employment; this captures basic elements by which we describe an expansion in the business cycle. Decreasing both endowments mimics what we think of as a contraction, or downturn, in the business cycle.

### 4.1 Example Calibration

Assume as in Gillman (2011) that  $\gamma = \frac{1}{3}$ ,  $\alpha = 0.5$ ,  $\rho = 0.03$ ,  $T = 1$ ,  $\delta_k = 0.03$  and  $A_G = 0.15$ . Then the equilibrium capital stock will equal  $k =$

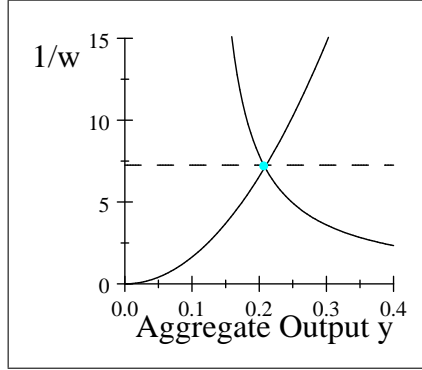


Figure 1: RCK  $AS - AD$  Equilibrium

$\frac{(1)(\frac{1}{3})(0.15)^3(\frac{2}{3(0.06)})^3}{(\frac{1}{3}+0.5)-0.5(0.03)(\frac{2}{3(0.06)})} = 2.3148$ . The  $AD$  and  $AS$  are given respectively as

$$\frac{1}{w} = \frac{1}{y^d (1 + 0.5) - 2.3148 [0.03 + (1.5) 0.03]},$$

$$\frac{1}{w} = \frac{(y^s)^2}{\frac{1}{3} (0.15)^3 (2.3148)^2}.$$

Figure 1 graphs these equations with the equilibrium wage of  $w = 0.138$ .

Similarly the labor market equations are given by

$$w = \frac{0.5 (0.03) 2.3148}{1 - (1.5) l^s},$$

$$w = \frac{1}{3} (0.15) \left( \frac{2.3148}{l^d} \right)^{\frac{2}{3}}.$$

Figure 2 graphs these with an employment rate of 0.5.

## 4.2 Comparative Statics: Goods, Time Endowments

Now let each  $A_G$  and  $T$  rise respectively by 5%, and then let's combine the two changes at one time for an expansion, and similarly for a contraction. First let  $A_G$  rise from 0.15 to 0.1575, with no other parameter changes. The capital stock rises to  $k = \frac{(1)(\frac{1}{3})(0.1575)^3(\frac{2}{3(0.06)})^3}{(\frac{1}{3}+0.5)-0.5(0.03)(\frac{2}{3(0.06)})} = 2.68$ . The supply and

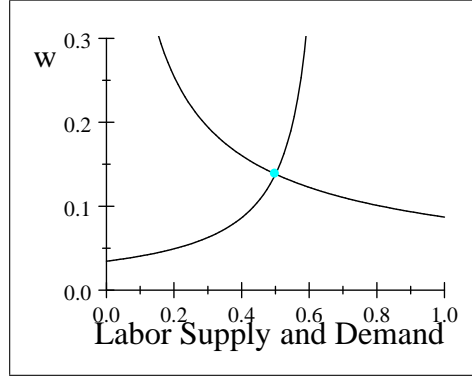


Figure 2: RCK Labor Market.

demand equations in the goods and labor markets become:

$$\begin{aligned}
 AD & : \quad \frac{1}{w} = \frac{1}{y^d (1 + 0.5) - 2.68 [0.03 + (1.5) 0.03]}, \\
 AS & : \quad \frac{1}{w} = \frac{(y^s)^2}{\frac{1}{3} (0.1575)^3 (2.68)^2}; \\
 w_t & = \frac{(0.03) 2.68}{1 - (1.5) l_t^s}, \quad w_t = \frac{1}{3} (0.1575) \left( \frac{2.68}{l^d} \right)^{\frac{2}{3}}.
 \end{aligned}$$

Figure 3 shows that both  $AD$  and  $AS$  shift outwards from the red baseline to the new black curves. The rise in  $k$  shifts out both curves directly while the increase in  $A_G$  acts to diminish the degree to which the  $AS$  shifts out. The  $AS$  shifts out by more than the  $AD$  curve and the relative price of goods  $\frac{1}{w}$  falls as Harberger (1988) stated we should expect with such Solow type productivity increases.<sup>5</sup>

Figure 4 shows with the black curves that labor supply shifts back due to the income effect of a higher  $k$  while labor demand expands due to both a higher  $k$  and a higher  $A_G$ . The wage rate rises as in Solow growth facts while the income and substitution effects offset each other so as to leave employment unchanged. The unchanged employment, resulting as well with more general homothetic utility and production functions, is a central problem with the  $RKC$  model's ability to provide a qualitative business cycle.

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<sup>5</sup>This graph is as in Harberger's (1988) AEA Presidential address explaining how the average cost of output falls as Solow technological progress ensues.



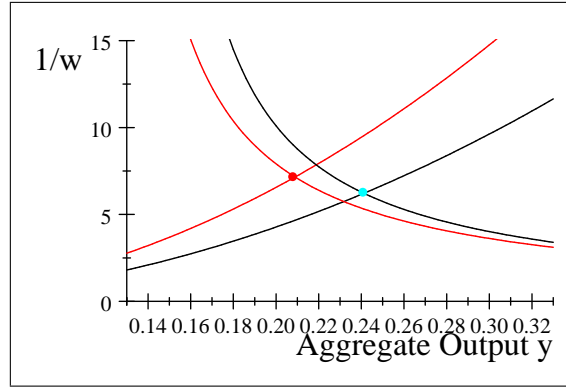


Figure 3: RCK  $AS - AD$  : Increase in Goods Endowment Through 5% increase in  $A_G$  (in black) Compared to Baseline (in red).

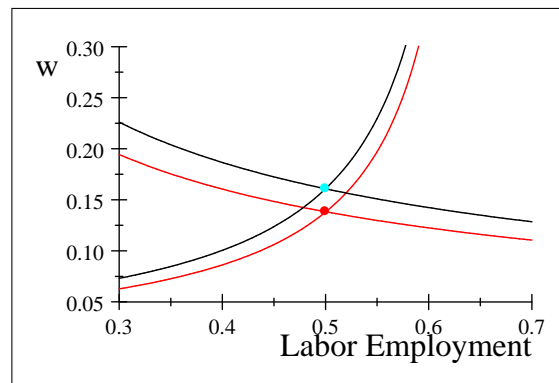


Figure 4: RCK Labor Market: Increase in Goods Endowment Through 5% increase in  $A_G$  (in black) Compared to Baseline (in red).

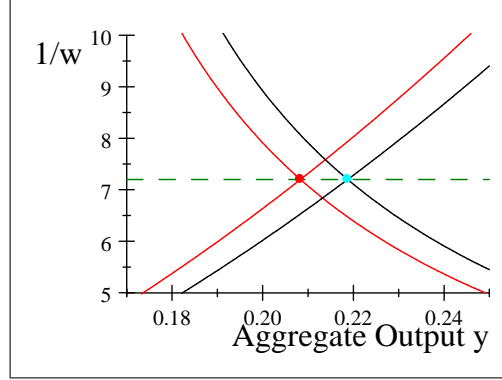


Figure 5: RCK  $AS - AD$  : Time Endowment Increase to 1.05 (in black) Compared to Baseline (in red).

Now consider the baseline parameters, including  $A_G = 0.15$ , but with a 5% increase in time endowment from 1 to 1.05. Then the capital stock rises to  $k = \frac{(1.05)(\frac{1}{3})(0.15)^3(\frac{2}{3(0.06)})^3}{(\frac{1}{3}+0.5)-0.5(0.03)(\frac{2}{3(0.06)})} = 2.431$ . The goods market and labor markets equations become:

$$\begin{aligned}
 AD & : \quad \frac{1}{w} = \frac{1.05}{y^d (1 + 0.5) - 2.431 [0.03 + (1.5) 0.03]}, \\
 AS & : \quad \frac{1}{w} = \frac{(y^s)^2}{\frac{1}{3} (0.15)^3 (2.431)^2}; \\
 w_t & = \frac{(0.03) 2.431}{1.05 - (1.5) l_t^s}, \quad w_t = \frac{1}{3} (0.15) \left( \frac{2.431}{l^d} \right)^{\frac{2}{3}}.
 \end{aligned}$$

Figure 5 shows how  $AS$  and  $AD$  shift out (to the black curves) with a higher  $k$  and with the increased time endowment directly increasing  $AD$  as well. This causes output to rise while the relative price  $1/w$  is unchanged. Figure 6 displays the labor market changes from the red to black curves. The wage rate is unchanged while now the employment rises.

### 4.3 Business Cycle: Expansion and Contraction

Combining the two comparative static increases in goods and time endowments results in both the real wage and employment increasing, along with output increasing. Taking a 5% increase simultaneously in both goods and

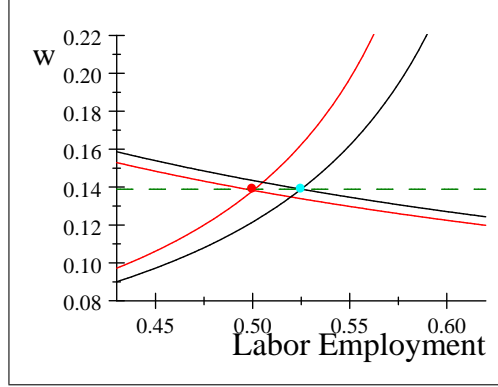


Figure 6: RCK Labor Market: Increase in Time Endowment to 1.05 (in black) Compared to Baseline (in red).

time endowments, through an increase in both  $A_G$  and  $T$ , shows that a typical expansion results qualitatively. The capital stock now is  $k = \frac{(1.05)(\frac{1}{3})(0.1575)^3(\frac{2}{3(0.06)})^3}{(\frac{1}{3}+0.5)-0.5(0.03)(\frac{2}{3(0.06)})} = 2.81$ . The new goods and labor market equations are each affected by both parameters directly and through the increased  $k$ .

$$\begin{aligned}
 AD &: \frac{1}{w} = \frac{1.05}{y^d(1+0.5) - 2.81[0.03 + (1.5)0.03]}, \\
 AS &: \frac{1}{w} = \frac{(y^s)^2}{\frac{1}{3}(0.1575)^3(2.81)^2}; \\
 w_t &= \frac{0.5(0.03)2.81}{1.05 - (1.5)l^s}, \quad w_t = \frac{1}{3}(0.1575)\left(\frac{2.81}{l^d}\right)^{\frac{2}{3}}.
 \end{aligned}$$

Figures 7 and 8 show, in the movement from the baseline red curves to the new black curves, the type of typical expansion in output, fall in the relative price of output, rise in employment and in the real wage that is consistent with a business cycle.<sup>6</sup> This much is established qualitatively with only an increase in  $A_G$ . Therefore the big difference comes in Figure 8 with the pivoting out of the labor supply curve and the subsequent rise in employment from 0.5 to 0.52, a 4% rise in the employment rate. The combined endowment

<sup>6</sup>Den Haan (2004) presents evidence that comovement of aggregate prices is significantly negative in the long run during the postwar period for G7 countries, using VAR forecast errors and frequency domain filters.

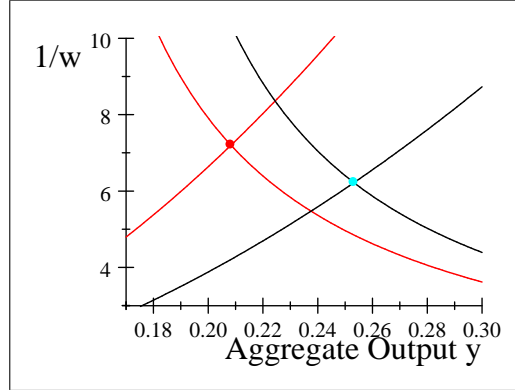


Figure 7: RCK  $AS - AD$  Expansion: Both Goods and Time Endowment 5% Increase (in black) Compared to Baseline (in red).

increases cause the quantity of labor supplied to appear to "move up" along the labor supply curve, although actually it is pivoting outwards.

Figure 9 shows for completeness that a similarly generated contraction occurs in the labor market when the goods and time endowment fall by 5% back down to the baseline parameter values, starting at the red curves and going to the black curves. The demand for labor shift down while the supply for labor twists up somewhat, leaving employment to fall back down to 0.5.

## 5 RCK Output, Input Markets

Another far-reaching advantage of deriving goods and labor markets separately, in an  $AS - AD$  version of the  $RCK$  model, is that the general equilibrium nature allows for exactly drawn production possibility curve output diagrams and isoquant-isocost input diagrams. And the comparative statics can be shown here as well, allowing for a business cycle explanation in output and input spaces using the same two comparative static changes in goods and time endowments.

### 5.1 Indifference and Production Possibility Curves

Consumption can be written in terms of the production function and the utility level, and graphed in  $(c, l)$  space. Consider that consumption is  $c_t^d =$

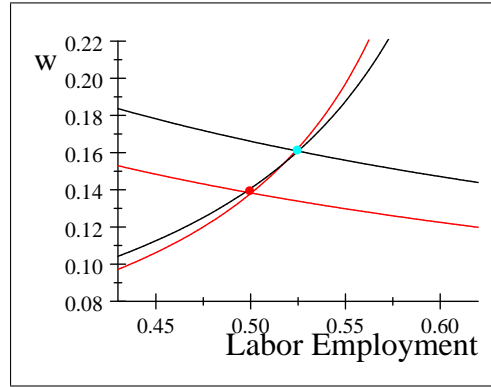


Figure 8: RCK Labor Market Expansion: Both Goods and Time Endowment 5% Increase (in black) Compared to Baseline (in red).

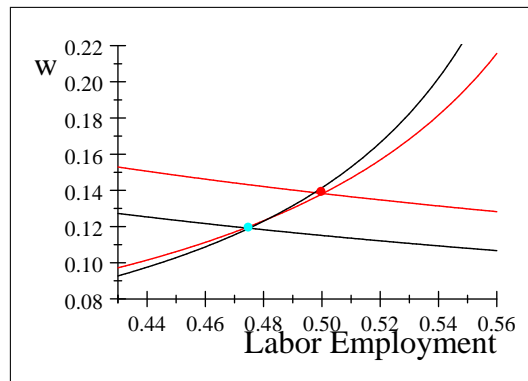


Figure 9: RCK Labor Market Contraction: Both Goods and Time Endowment 5% Decrease (in black) relative to the Expansion Case (in red).

$y_t^s - i_t = A_G (l_t^d)^\gamma (k_t)^{1-\gamma} - \delta_k k_t$ . Substituting in the baseline parameters,  $k = 2.315$ , and  $A_G = 0.15$ , this gives a "production possibility curve" (PPC) of

$$c_t^d = 0.262 (l_t^d)^{\frac{1}{3}} - 0.069.$$

Similarly, the utility level can be found in the baseline as  $\ln c_t + \alpha \ln (1 - l_t) = \ln 0.13889 + 0.5 \ln 0.5 = -2.32$ . Solving for  $c_t$ , the indifference curve at the new equilibrium after the  $A_G$  increase is

$$c_t = \frac{e^{-2.32}}{(1 - l_t)^{0.5}}.$$

The budget line in equilibrium is

$$c_t^d = w_t l_t^s + \rho k_t = c_t^d = (0.13889) l_t^s + (0.03) (2.3148).$$

It is worth pointing out the symmetry of the budget line between both consumer and firm. In fact, the consumer's budget line here is the same line as the firm's profit line in the *BGP* equilibrium. For the firm  $y_t = w_t l_t + r_t k_t$ , and since  $r_t = \rho + \delta_k$ , then  $y_t = w_t l_t + (\rho + \delta_k) k_t$ . Given also that output equals consumption plus investment,  $y_t = c_t + i_t$ , and that *BGP* investment equals the maintenance for capital depreciation, or  $i_t = \delta_k k_t$ , then consumption as solved for from the firm's profit definition is  $c_t = y_t - i_t = w_t l_t^d + (\rho + \delta_k) k_t - \delta_k k_t$ , or  $c_t = w_t l_t^d + \rho k_t$ . With labor supply equal to labor demand, one can see that the consumer's budget line and the firm's profit line when expressed in terms of  $c$  are indeed the same "separating hyperplane".

Figure 10 draws the baseline calibration plus with  $A_G$  increased to 0.1575 in terms of the production possibility curves, the indifference curves and the budget lines, with the upwards-shifted set of curves representing the new equilibrium after the  $A_G$  increase. Employment stays the same as consumption rises and the slope of the indifference curve rises in reflection of a higher wage rate.

Similarly, with the time endowment rising from  $T = 1$  to  $T = 1.05$  Figure 11 shows that the tangency point between the PPC and indifference curves shifts upwards again but in such a way that employment increases, while

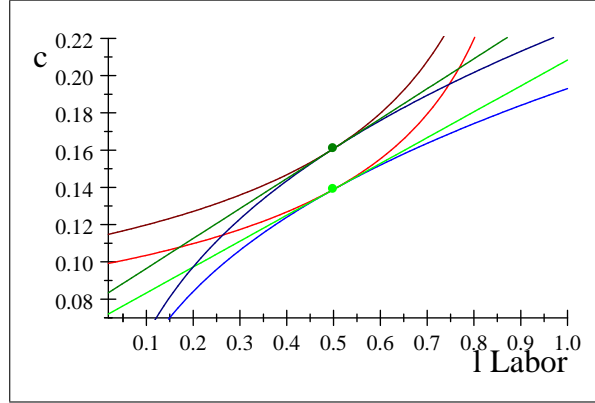


Figure 10: PPC and Indifference Curve with Goods Productivity Increase as Compared to Baseline

the slope of the budget line remains the same since the wage rate does not change.<sup>7</sup>

## 5.2 Input Space: Isoquants and Isocosts

The isoquant curve is based on output, which is  $y_t^s = A_G (l_t^d)^\gamma (k_t)^{1-\gamma}$  in general. The baseline output can be written as  $0.208 = 0.15 (l_t^d)^{\frac{1}{3}} (k_t)^{\frac{2}{3}}$ , giving an isoquant of

$$k_t = \frac{1.64}{(l_t^d)^{\frac{1}{2}}}.$$

The isocost line is  $y_t = w_t l_t + r_t k_t$ , which in the example is  $0.208 = (0.13889) l_t + (0.06) k_t$ , giving an isocost line of

$$k_t = 3.472 - 2.315 l_t.$$

The factor input ratio is

$$\frac{k_t}{l_t} = \frac{2.315}{0.50} = 4.63.$$

Drawing the baselin set of input market curves, along with those for when  $A_G$  rises by 5%, Figure 12 shows from the input market perspective how the

<sup>7</sup>Note that utility curves can "cross" when graphed in the  $(c, l)$  space because of the investment. Alternatively in  $(y, l)$  space, utility curves would not cross. The choice of  $(c, l)$  space here is so that the consumer's utility items,  $c$  and  $x = 1 - l$ , are the goods graphed as is done in a typical microeconomic PPC curve of any two "goods".

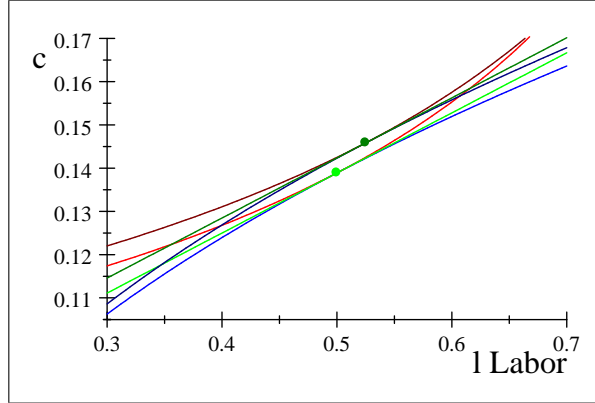


Figure 11: PPC and Indifference Curves with Time Endowment Increase as Compared to Baseline.

increase in  $A_G$  causes the equilibrium isoquant to shift up, employment to remain the same and the input ratio  $k/l$  to pivot up; a steeper isocost shows the higher real wage.

For the increase in the time endowment to  $T = 1.05$ , Figure 13 reflects that the factor input ratio remains the same (an identical input ratio for any  $T$ ) as the equilibrium isoquant shifts up and employment rises, as compared to the baseline. The same sloped isocost reflects that factor prices are unchanged.

Simultaneously increasing the goods and time endowments as in a business cycle expansion would result in similar output and input market graphs. The "production possibility curve" would pivot upwards with an increased slope of the consumer-budget/firm-profit line at the tangency with the new higher indifference curve, as the real wage, output and labor employment rise. And in the input market the isoquant would shift out and the isocost pivot upwards such that again a higher real wage, output level, and employment would be shown.

## 6 Exogenous Growth Theory

Allowing the goods sector productivity factor  $A_G$  to rise at a constant rate of growth gives all of Solow's growth facts in the  $RCK$  model. And it is easily illustrated in both the goods market, using  $AS - AD$  analysis, and in the



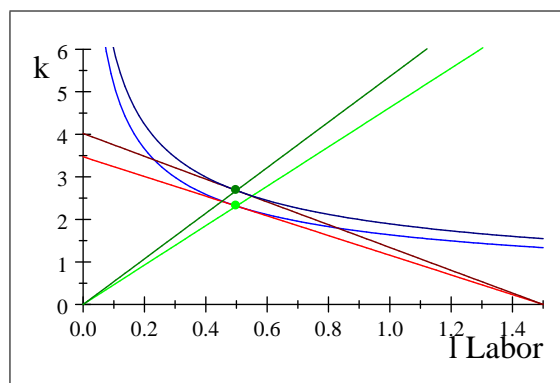


Figure 12: Isoquants and Isocosts with a Goods Productivity Increase as Compared to Baseline.

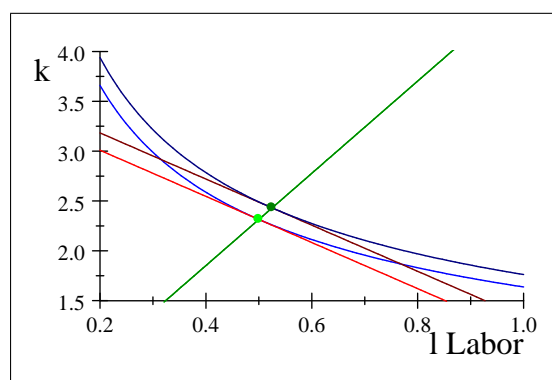


Figure 13: Isoquants and Isocosts with a Time Endowment Increase as Compared to Baseline.

labor market. Letting  $A_G$  grow at a constant rate  $\mu$ ,

$$A_{Gt+1} = A_{Gt}(1 + \mu).$$

In the baseline calibration, except now with a *BGP* growth rate  $g$  of 2%, or  $g = 0.02$ , then  $\mu = 0.0067$ .

Foregoing all of the equations, Figures 14 and 15 graph the resulting goods and labor markets over four periods of time, with the equilibria shifting over time from red to purple to blue to black. Solow's (1965) stylistic growth facts that output and the real wage rise are seen in Figure 14, along with Harberger's (1988) stylistic growth fact of a falling relative price of output. Figure 15 shows in the labor market again that the wage rate rises. The third and fourth Solow stylistic facts are that the real interest rate and the output-capital ratio ( $y/k$ ) remain constant. These hold also since over time it remains true that  $r = \rho + \delta_k$ , which is constant. And the marginal product of capital,  $r$ , is just a fixed fraction of the output-capital ratio in a Cobb-Douglas production function, so  $y/k$  likewise is constant.

However notice the *RBC* problem again reflected in Figure 15: employment is unchanged over time. Taken in the context of typical textbooks, this is deemed inconsistent with a falling hours worked per week over the long historical trend in industrial countries such as the US. It is the growth theory "flip-side" puzzle to the *RBC* puzzle of the inability of goods productivity changes to cause changes in employment over the business cycle.

Now consider a slightly different angle on this standard analysis. Consistent with the earlier comparative static exercises, again allow the time endowment  $T$  to trend instead of the goods endowment factor  $A_G$ . Of course this is only of interest if it allows additional long run stylistic trends to be explained that are in addition to the Solow facts. In particular, the decline in the hours worked per week has been a centuries old phenomenon. Does it make sense to describe this by allowing less time to be devoted to work and leisure by having  $T$  trend down, and employment likewise trend down? I think it does, as this indeed is the result of such a trend over time.

Even more interesting for illustrative purposes, consider a combination of trends in goods and time productivity in a fashion similar to the business cycle explanation. Let there be a combination of  $A_G$  trending up and  $T$

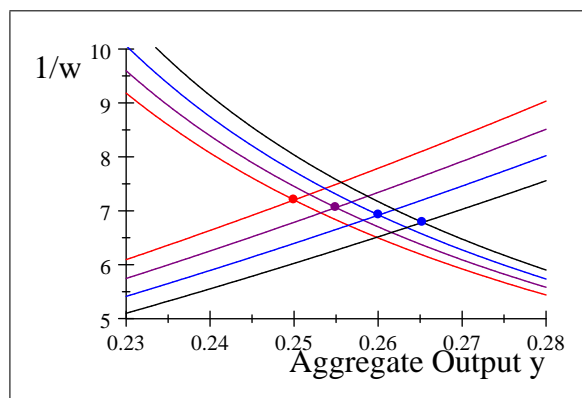


Figure 14:  $AS - AD$  Equilibria Over Four Periods with a  $BGP$  Exogenous Growth Rate of  $g = 0.02$ .

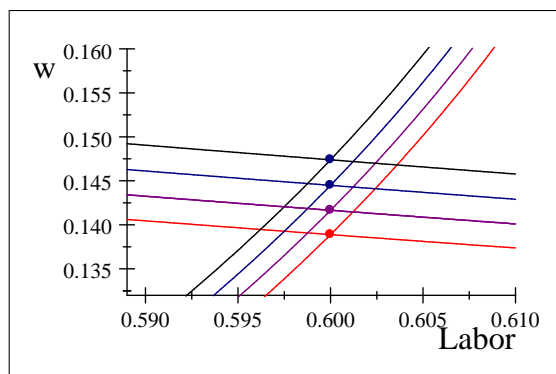


Figure 15: Labor Market with 2% Exogenous Growth.

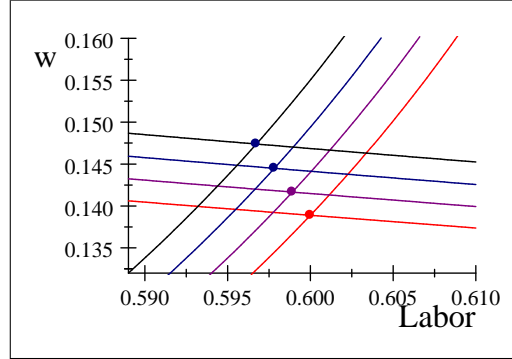


Figure 16: Labor Market with  $A_G$  Trending Up and  $T$  Trending Down.

trending down slightly, as presented in both the goods and labor markets.<sup>8</sup> The goods market qualitatively looks like Figure 14 and so will not be reproduced;  $1/w$  falls as output rises. However in Figure 16 the labor market is shown to reproduce a rising wage rate along with a very gradually declining trend in time spent working, as the equilibria over time again go from the red to purple to blue and to the black in the most recent time period. All of these trending equilibria can also be shown in the PPC-indifference curve space and the factor input space, although those graphs are foregone here for simplicity of presentation.

The idea is to capture the trend downwards in the work week by accounting for the trend upwards in time spent in education, over the centuries. In other words, it is plausible that our lifetime allocation of time for work and leisure has fallen over time as our education time has gradually trended up. For example, looking at the time since the industrial revolution began in the 1700s, average education levels have gone from no primary school to some years in primary, to some in high school, to a high school diploma average, to some university years, to a bachelors degree, etc.

Capturing a fall in  $T$  is here a representation of how education time has increased, although this is not explained explicitly within the *RCK* model or one with Solow exogenous growth. However an increase in education time and a consequent fall in  $T$  can be made endogenous by extending the Solow

<sup>8</sup>The trend down in  $T$  is 0.00182 per year, based on evidence reporting a 12% decrease in US labor time from 1965 to 2005, in Aguiar and Hurst (2009).

model as did Lucas (1988) through an additional sector for investment in human capital.

## 7 Endogenous Growth and Business Cycles

Thus far the modelling approach, themes of business cycles and growth, and mathematical sophistication is within reach of advanced undergraduates, or those even at an intermediate level if they have a good grasp of partial derivatives, solving systems of equations, and economic intuition. All the solutions have been closed-form analytic ones. Going one step further, to endogenize growth through human capital investment, can still be done using  $AS - AD$ , only partial derivatives and solving systems of equations, and all within a closed form solution. However now the solution involves is a quadratic which has a closed form explicit solution, but obviously is more complex looking and may be accessible for only the most mathematically inclined undergraduates. At this point, the material arguably becomes more of a masters level, or first PhD course level.

Here I will present the extension briefly, as it is relatively simple and it provides the key to how  $T$  is changed endogenously, rather than exogenously in the work up until now. This is important because the only additional comparative static beyond the typical goods productivity has been this change in  $T$ . In the *RBC* and Solow-growth fashion of changing productivity parameters, when the productivity parameter of the human capital sector is changed, the time that the representative agent devotes to education changes and this in turn endogenously changes the time left-over for goods and leisure, or  $T$ . This provides understanding that the change in  $T$  in the *RCK* model is not an arbitrary artifice by which to describe cycles and growth. The change in  $T$  with endogenous growth results from a sectoral productivity change in the education sector. In different terms, this is like changing the productivity of the "non-market" sector in the two sector business cycle theory begun in particular with Benhabib et al. (1991) and Greenwood and Hercowitz (1991), after evolving from Hansen (1985) and Rogerson (1988). But now the additional non-market sector is the sector producing human capital investment in a costly Lucas (1988) fashion.

Besides consistency with adding greater labor volatility in the *RBC* lit-

erature by adding an external labor margin, the extension in turn explains an important additional trend explicitly: the rise in lifetime education time through a trend up in human capital sectoral productivity. This also is a rather simple way to explain how the *BGP* growth rate can rise over time, as has indeed been witnessed in the last 250 years at the same time as education time has trended upwards. Therefore these two additional trends in education and the growth rate are explained in a very simple fashion through a trend up in the productivity parameter in the Lucas (1988) human capital investment sector.

## 7.1 The Human Capital Extension

With  $h_t$  denoting human capital, the goods sector production function now is

$$y_t^s = A_G (l_t^d h_t)^\gamma (k_t)^{1-\gamma},$$

so that the change relative to the *RCK* with Solow growth is to think of  $A_{Gt}$  now being defined instead as an  $\hat{A}_{Gt} \equiv A_G (h_t)^\gamma$ . Instead of exogenous technological change in  $A_G$ , there is endogenous change in  $\hat{A}_{Gt}$  as a result of the consumer's choice of human capital.

The new sector for producing human capital investment is given by

$$h_{t+1} = h_t (1 - \delta_h) + i_{ht},$$

where  $\delta_h \in [0, 1]$  is the depreciation rate of human capital and  $i_{ht}$  is the sectoral investment function given by a simple linear (Lucas, 1988) production function:

$$i_{ht} = A_H l_{Ht} h_t;$$

in which  $A_H$  is the sectoral productivity parameter, and  $l_{Ht}$  the fraction of time spent accumulating human capital. In short,  $h_{t+1} = h_t (1 + A_H l_{Ht} - \delta_H)$ . Time allocation now includes not just labor and leisure, which will still be denoted in sum as  $T$ , but also the time  $l_{ht}$  devoted to "education":

$$1 = T + l_{ht} = l_t^s + x_t + l_{ht}.$$

The time  $T$  becomes endogenous as more or less time goes into producing human capital.

The consumer problem becomes

$$V(k_t, h_t) = \underset{k_{t+1}, h_{t+1}, l_t^s, l_{Ht}, x_t}{Max} : \ln[w_t l_t^s h_t + r_t k_t - k_{t+1} + k_t - \delta_k k_t] + \alpha \ln(x_t) + \beta V(k_{t+1}, h_{t+1}),$$

subject to  $h_{t+1} = h_t(1 - \delta_h) + A_H l_{Ht} h_t$  and  $1 = l_t^s + x_t + l_{Ht}$ , where  $c_t = w_t l_t^s h_t + r_t k_t - k_{t+1} + k_t - \delta_k k_t$ . Note that the only difference in the goods budget constraint is that wages are indexed by the human capital of the consumer, being equal to  $w_t l_t^s h_t$  instead of  $w_t l_t^s$ . Substituting in the constraints, the consumer problem is<sup>9</sup>

$$V(k_t, h_t) = \underset{k_{t+1}, l_t^s, l_{Ht}}{Max} : \ln[w_t l_t^s h_t + r_t k_t - k_{t+1} + k_t - \delta_k k_t] + \alpha \ln(1 - l_{Ht} - l_t^s) + \beta V[k_{t+1}, h_t(1 - \delta_h) + A_H l_{Ht} h_t].$$

Along the *BGP* the intratemporal goods-leisure margin ( $MRS_{c,x} : \frac{\alpha c_t^d}{x_t} = w_t h_t$ ), and the physical capital intertemporal margin ( $1 + g_t = \frac{1+r_t-\delta_k}{1+\rho}$ ) are joined by the human capital intertemporal margin:  $1 + g_t = \frac{1+A_H(1-x_t)-\delta_h}{1+\rho}$ . The growth rate become endogenous by virtue of the consumer's choice of leisure  $x_t$ . More leisure means a lower "human capital capacity utilization rate" of  $1 - x_t$ , and a lower return on human capital. Since the *BGP* sees equivalence of capital returns,  $r_t - \delta_k = A_H(1 - x_t) - \delta_h$ . If leisure increases and so the productively employed time ( $l_t^s + l_{Ht}$ ) decreases, then also it follows that  $r_t$  falls. This happens by increasing physical capital relative to human

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<sup>9</sup>The equilibrium and envelope conditions are

$$\begin{aligned} k_{t+1} &: \frac{1}{c_t^d}(-1) + \beta \frac{\partial V(k_{t+1}, h_{t+1})}{\partial k_{t+1}} = 0, \\ l_t^s &: \frac{1}{c_t^d}(w_t h_t) + \frac{\alpha}{x_t}(-1) = 0, \\ l_{Ht} &: \frac{\alpha}{x_t}(-1) + \beta \frac{\partial V(k_{t+1}, h_{t+1})}{\partial h_{t+1}}(A_H h_t) = 0; \\ k_t &: \frac{\partial V(k_t, h_t)}{\partial k_t} = \frac{1}{c_t^d}(1 + r_t - \delta_k), \\ h_t &: \frac{\partial V(k_t, h_t)}{\partial h_t} = \frac{1}{c_t^d}(w_t l_t) + \beta \frac{\partial V(k_{t+1}, h_{t+1})}{\partial h_{t+1}}(1 + A_H l_{Ht} - \delta_H). \\ h_t &: \frac{\partial V(k_t, h_t)}{\partial h_t} = \frac{1}{c_t^d}(w_t l_t) + \beta \frac{\partial V(k_{t+1}, h_{t+1})}{\partial h_{t+1}}(1 + A_H l_{Ht} - \delta_H). \end{aligned}$$

capital, since the return to human capital has fallen. In this environment, the new "state" variable along the *BGP* is indeed the ratio  $k/h$ . A decrease in  $A_H$  causes less  $l_{Ht}$ , more leisure, and a higher  $k/h$  as the consumer substitutes from human to physical capital.

## 7.2 Extended AS-AD

The consumer demand function can now be solved with the addition that time in human capital accumulation enters it. Following similar steps to the above derivation of consumer demand, now

$$c_t^d = \frac{1}{1+\alpha} [w_t h_t (1 - l_{Ht}) + k_t (r_t - \delta_k - g)].$$

Using the human capital investment and its *BGP* condition that  $1 + g = \frac{h_{t+1}}{h_t} = 1 + A_H l_{Ht} - \delta_H$ , we can write human capital time as  $l_{Ht} = \frac{g + \delta_H}{A_H}$ . Stationary consumption is written as normalized by human capital  $h_t$ , where  $\frac{k_t}{h_t}$  is constant in the *BGP* equilibrium:

$$\frac{c_t^d}{h_t} = \frac{1}{1+\alpha} \left[ w_t \left( 1 - \frac{g + \delta_H}{A_H} \right) + \frac{k_t}{h_t} \rho (1 + g) \right].$$

In terms of the permanent income hypothesis of consumption, now  $c_t^d = \left( \frac{1}{1+\alpha} \right) y_{Pt}$  where  $y_{Pt}$  can be written as  $y_{Pt} = w_t h_t T_t + k_t \rho (1 + g)$ , similar to the *RCK* model with a positive  $g$  except for the indexing of wage income by  $h_t$ .

Investment with a positive growth rate  $g$  is  $(g + \delta_k) k_t$ . Adding this to consumer demand and normalizing by  $h_t$  gives aggregate demand.

$$\begin{aligned} AD : \quad \frac{y_t^d}{h_t} &= \frac{1}{1+\alpha} \left[ w_t \left( 1 - \frac{g + \delta_H}{A_H} \right) + \frac{k_t}{h_t} [\rho (1 + g) + (g + \delta_k) (1 + \alpha)] \right]; \\ \frac{1}{w_t} &= \frac{1 - \frac{g + \delta_H}{A_H}}{\frac{y_t^d}{h_t} (1 + \alpha) - \frac{k_t}{h_t} [\rho (1 + g) + (g + \delta_k) (1 + \alpha)]}. \end{aligned}$$

Aggregate supply ends up completely unchanged as  $y_t^s = A_G \left( \frac{\gamma A_G}{w_t} \right)^{\frac{\gamma}{1-\gamma}} k_t$ , which can be made stationary through normalization by  $h_t$ :

$$AS : \frac{y_t^s}{h_t} = A_G \left( \frac{\gamma A_G}{w_t} \right)^{\frac{\gamma}{1-\gamma}} \frac{k_t}{h_t}, \quad \frac{1}{w_t} = \frac{1}{\gamma A_G} \left( \frac{y_t^s}{A_G k_t} \right)^{\frac{1-\gamma}{\gamma}}.$$



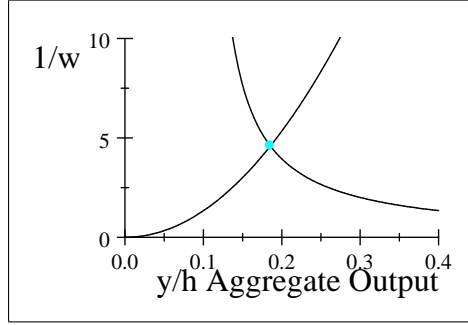


Figure 17:  $AS - AD$  with Human Capital Extension.

Using a calibration that gives a  $BGP$  growth rate  $g$  of 2%, or  $g = 0.02$ ,<sup>10</sup> Figure 17 graphs  $AS - AD$ . Comparative static changes in  $A_G$  and  $A_H$  can be done to show business cycles in a fashion very similar to the  $RCK$  model above. And growth trends also follow. To show this in brief, consider that the Figure 17 is drawn for the stationary relative price  $1/w$ . In the Solow exogenous growth  $RCK$  model,  $w$  is rising over time. With endogenous growth,  $w$  is the stationary rate of "raw" labor. What is rising is the consumer wage income of  $wh_t$ , where  $h_t$  grows at the  $BGP$  growth rate of  $g = 0.02$  in this example. Drawing in Figure 18 the  $AS - AD$  using  $1/wh_t$  as the relative price and  $y_t$  as the other axis gives the comparable  $AS - AD$  shifting up during growth, going from red to purple to blue to black curves at time passes. Figure 19 shows the similar graph for the labor market. Notice that human capital and endogenous growth does not solve the issue that employment stays stationary over time, rather than slightly declining as in the evidence. Further, as in  $RCK$ , a comparative static increase in  $A_G$ , or a trending upwards increase over time, leaves the employment time unchanged.

In contrast consider a slight upward trend in human capital productivity,  $A_H$ . Then indeed the time  $T$  trends down along with labor time  $l^s$ . To illustrate this most simply, let there be a comparative static 5% increase in  $A_H$  to 0.20. Figure 20 shows the labor market shifts from the red to the

<sup>10</sup>Also as in Gillman (2011),  $\gamma = \frac{1}{3}$ ,  $\alpha = 1$ ,  $A_H = 0.189$ ,  $\delta_k = 0.05$ ,  $\delta_h = 0.015$ ,  $\beta = \frac{1}{1+\rho} = 0.95$ ,  $\rho = \frac{1}{0.95} - 1 = 0.0526$ ,  $A_G = 0.282$ . The equilibrium here includes that  $l_H = 0.185$ ,  $l = 0.284$ ,  $1 - x = 0.185 + 0.284 = 0.469$ ; leisure is 0.531 as in the related two sector model of Gomme and Rupert (2007) for postwar US data. Also  $r = 0.1237$ ,  $w = 0.21772$  ( $\frac{1}{w_t} = 4.59$ ), and  $\frac{k_t}{h_t} = 1.00$ .

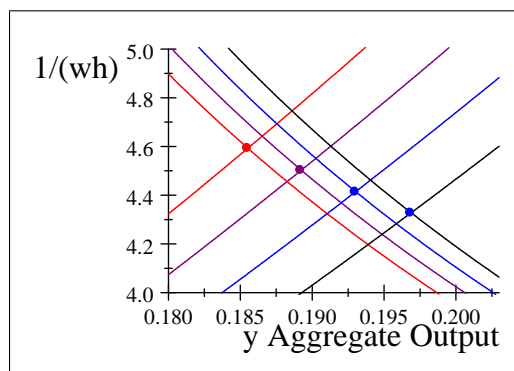


Figure 18:  $AS - AD$  Shift up from Time  $t$  to  $t + 3$ ; All Parameters and  $w$  are Stationary.

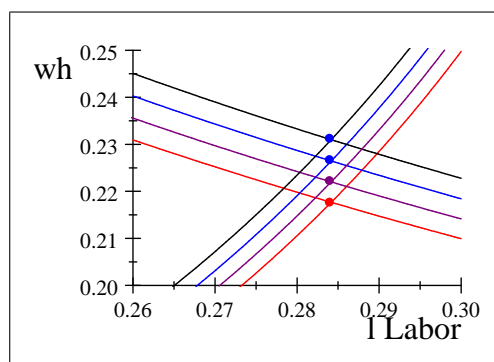


Figure 19: Labor Market with Human Capital Extension.

black supply and demand curves, with employment falling. Meanwhile the time  $l_H$  in the education sector rises, human capital rises, and output rises (not shown). The "countercyclical" increase in education time is consistent with US evidence as in Harris and Koubi (2003). The idea is that education time increases in the downturn, while (exogenously) the education sector is more productive, and labor shifts from the market to the non-market sector; conversely education time decreases when  $A_H$  falls and labor shifts back to the goods sector. This shows an endogenous change in  $T$ , time for goods and leisure, that compares to the exogenous changes in  $T$  in the *RCK* model. In *RCK*, the time  $T$  rose (exogenously) in the upturn and fell in the downturn; here when  $A_H$  falls,  $T$  rises (since  $l_H$  falls) and when  $A_H$  rises,  $T$  falls (since  $l_H$  rises). Extended over time by allowing for a very slight (exogenous) trend up in  $A_H$ , both the falling trend in working time and the rising trend in education time is directly explained. In addition, the *BGP* growth rate  $g$  would rise gradually as well, also broadly consistent with the trend since the industrial revolution.

Such a slight trend in  $A_H$  is not a far reach given that our entire neo-classical growth and business cycle is built upon exogenously changing  $A_G$ . Here we consider not just the goods endowment, which is increased given the production function when  $A_G$  increases, but also consider the time endowment. An exogenous change in  $A_H$  causes an endogenous change in  $T$ , and does so in such a way as to describe a business cycle, Solow growth trends, and Solow-plus trends in working time, education time, and the growth rate. Albeit, the explanation is a simple, but arguably fundamental, one.

### 7.3 Output, Input General Equilibrium

The production possibility curve representation as well as the isoquant and isocost graphs follow through with human capital and endogenous growth. Again foregoing the equations, which follow in a fashion as described for the *RCK* model, Figures 21 and 22 present these graphs for the same example calibration and with normalization by  $h_t$ .

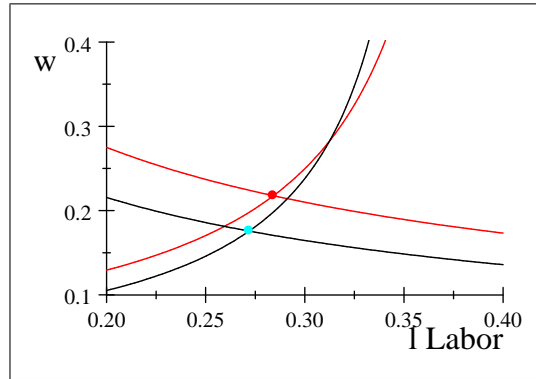


Figure 20: Labor Market with  $A_H$  Increase.

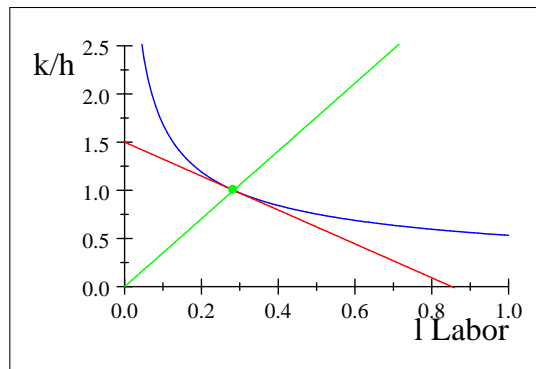


Figure 21: Factor Market with Human Capital Extension of  $RCK$ .

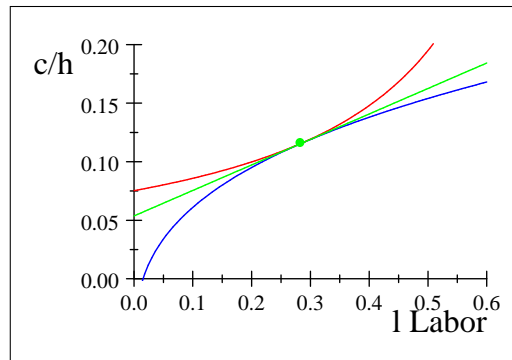


Figure 22: Goods and Labor with Human Capital Extension of  $RCK$ .

## 8 Qualification and Conclusion

The most easy criticism of the approach here is that *BGP* equilibrium does not occur over the course of a business cycle and so cannot be used to explain business cycles. Is this a fair criticism that can be used to argue that *AS – AD* cannot be done in the dynamic recursive *RCK* model? I do not think so. Economies are constantly being shocked and tending back towards their new equilibrium. Transition dynamics can be the most important story in some cases. But over the business cycle can we really say that the economy over some 4 or more years in an upturn, and sometimes as long in a downturn (although usually shorter) really have nothing to do with the equilibrium as in the stationary state? Consider that the shock processes in the *RBC* model is always with a persistence parameter near 0.95 whereas 1.0 is the persistence parameter for a comparative static change as in the deterministic analysis here. It is not so different, and "medium term" business cycle analysis as in Comin and Gertler (2006) uses band-pass filters that include the low frequency. Given the ability to teach textbook style business cycle and growth with key *RBC* comparative statics changes using *AS – AD*, as presented in this paper, it would seem to over-ride heuristic arguments that we cannot describe macroeconomic phenomena without transition dynamics.

The paper shows that the *AS – AD* can be solved in the *RCK* dynamic model by deriving the intratemporal and intertemporal equilibrium margins, finding the closed-form solution for the capital stock, and graphing the supply and demand in terms of the relative price of goods to leisure  $1/w$  and output  $y$ . The supply and demand are functions of the "state" variable, the capital stock  $k$ , which is given (or shall we say "known") in equilibrium as of any given time  $t$ . Changes in goods and time endowments through changes in goods productivity  $A_G$  and time for work and leisure  $T$  can be used to show a business cycle, along with "Solow-plus" growth-trend stylistic facts. Going beyond undergraduates, human capital can be introduced with the result that the growth rate is endogenous. Changing human capital sectoral productivity  $A_H$  causes an endogenous change in time for work and leisure  $T$ , such that a similar but perhaps more fundamental explanation of business cycle and Solow-plus growth facts result.

The aim is to reinvigorate macroeconomics teaching so that the vast

chasm between graduate and undergraduate study is seamlessly bridged. Neoclassical  $AS - AD$  is a way to do this. Certainly it can be extended to have any degree of price rigidity, monopoly power, or other Neo-Keynesian elements that one thinks should be there, with examples of fixed prices given in Gillman (2011).

The gradual evolution of macroeconomic thought has brought us from the conceptions of aggregate demand in Keynes and in Samuelson's (1951) Keynesian cross to where we have found ourselves today, rooted in an eclectic  $AS - AD$  static theory. Colander (1995) rightly points out the mix does not add up to internal consistency. Rather than righting the analysis through the Keynesian side as he suggested might be done, here I have righted the analysis through the Marshallian classical theory, something Keynes (1930) himself also explicitly set out to accomplish. But instead of defining profit as investment minus savings as did Keynes, I have followed Keynes's brilliant student Ramsey (1928). Using a case of his exact model,  $AS - AD$  analysis from a Marshallian foundation is presented with zero  $BGP$  growth. Extension to exogenous growth follows directly. Comparative statics describing a business cycle then become possible using Beckerian (1965) insights on time endowment. Adding one more sector, human capital, and the Uzawa (1965) -Lucas (1988) extension of  $RCK$  also emerges in  $AS - AD$  terms. The story of cycles and growth trends becomes enhanced by using changes only in the two sectoral productivities

It may appear surprising that this  $RCK$  analysis with  $AS - AD$  seems not to have been presented before. For example Obstfeld and Rogoff's (1996) beautiful methodologically consistent development of macroeconomics could have had it embedded. Perhaps the early and ongoing controversy on consumption demand theory in terms of Friedman's (1957) permanent income hypothesis versus the consumption function in Samuelson (1951) has kept the consumption demand within  $RCK$  under wraps and unexploited for deriving  $AD$  in the classical dynamic model. Such speculation on economic history goes beyond the scope of this paper.

Similarly the question of why endogenous growth of Uzawa-Lucas has not become more mainstream remains a puzzle. The roots of disregarding it certainly start as far back as Cass (1965), who in a footnote comments directly about Uzawa's approach that he does not incorporate production of

investment in a separate sector since "no further insight is gained" (footnote 2, p. 233). Certainly the success of Solow's (1956) growth facts and Kydland and Prescott's (1982) *RBC* facts may indicate that further delving is unnecessary. However the benefits of human capital investment, the resulting nature of endogenous growth, and its potential enhancement of the *RCK AS – AD* analysis have been suggested in this paper as being of fundamental interest for students of macroeconomics.<sup>11</sup>

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<sup>11</sup>In a stochastic setting, Dang et al. (2011) suggest a solution to the fundamental *RBC* consumption-output, growth persistence, and labor puzzles through a two-sector endogenous growth model with an economy-wide shock to the productivity of both goods and human capital sectors.

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