Least Square Monte Carlo for Pricing American Put Option and Variance Reductions

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1 Introduction

Option pricing is an important topic in the world of Mathematical Finance, and has attracted many researchers in Mathematics, Statistics, Computer Science, and Operations Research to use quantitative methods to work on it. For example, the Black-Schole-Merton model provides analytical prices of European options. However, European options are just the simplest form of options, and there are much more on the market, such as American put options, Asian options, Barrier options, and Israeli options, have much more complicated payment structure, and it is exceedingly hard to derive their theoretical prices. Such options are sometimes called 'exotic options', and are usually priced using Monte Carlo techniques in the industry.

Because of the complex payoff structures of these options, it's usually not sufficient to just generate a lot of paths and take the average of the corresponding payoff. In this project, we will focus on a regression technique equipped with Monte Carlo simulation, the Least Square Monte Carlo algorithm (Longstaff &Schwartz, 2001), to develop a program for pricing American put options. In the LSMC method, the choice of regressor in the regression step is influential to the result. We are going to compare the effect of different regressor choices. Variance reduction techniques (e.g. Control Variates), and Greek estimations (e.g. Finite Difference) will also be integrated into the project.

2 Problem Description

The American put option gives the owner of a put the right, but not the obligation, to sell an asset (the underlying), at a specified price (the strike), by a predetermined date (the expiry or maturity) to a given party (the seller of the put), where buyer can exercise the put for a payout of $K - S(t)$ any time until the option's maturity time T. Mathematically speaking, let T stands for the maturity of the option, $\tau \in [0, T]$ be any time before the maturity, K as the strike price,

S as the price of underlying stock, and r as the risk-free interest rate. Let $P_{A,P}$ be the price of American put option, we have:

$$P_{A,P} = e^{-r\tau} \sup_{\tau}(\max(K - S(\tau)), 0) \tag{1}$$

And the LSMC algorithm, which is what we are going to implement in this project, provides a convergent estimator, $\overline{P_{A,P}}$, of the above price.

In Monte Carlo simulation, one of our concerns is the accuracy of the estimator (because of randomness), which is measured by the variance. The easiest way to decrease variance is to increase the Monte Carlo sample size. However, there are many intelligent ways to get an estimator with smaller variance, under a particular sample size. Such tricks are called variance deduction techniques. In this project, we are going to integrate a few such techniques in our simulation, and compare their performance.

3 Model Description

3.1 The Theoretical Option Price

The theoretical price of American put options with finite maturity is not mathematically derived by researchers yet. In this project, we will use online option price calculators as reference of their real prices. Such calculators usually use binomial trees. When the time interval of the binomial trees get smaller and smaller, the price will also converge to the real value, just like the LSMC algorithm. The link of a few well functioning online option price calculators are listed as references.

Another way to have an idea about the real price, is to notice that the American put options always have higher value that the corresponding European put options, since the option holders have more flexibility in American put options. If our simulated price are slightly higher than the price of corresponding European put option (which can be easily obtained), we can say that the estimation we get is not too biased.

3.2 The LSMC Algorithm

In the LSMC algorithm, we firstly generate a lot of price paths, and compute the cash flow of each path, and compute the present value of each cash flow. The average of these present values would be our Monte Carlo estimation.

The main challenge is how to derive the cash flow reflecting the payoff structure of American put options. On the day of maturity, the payoff is simply $\max(0, K - S(T))$. Now consider the payoff on day n is given. For day $n-1$, the cash flow is simply 0 if the stock price is above the strike price. If the stock price is lower than the strike price, we need to consider whether it's better to exercise, or hold it for higher payoff in the future. What we will do here is to build a regression model, where the stock prices at time $n-1$ serve as independent variable(s), and the payoff at day n (which we already calculated) serves as the dependent variable. With this linear model, we predict the future payoff using stock prices. For a particular path, if the predicted future payoff is higher than the payoff of immediate exercise, then we are going to hold the option. Otherwise, we exercise the option and get the payoff at time $n-1$, and the payoff at time n becomes 0 in the cash flow.

A mathematical expression of the above process could be:

- Simulate m stock price paths with length n. Construct a path matrix P_{mn}.
- Define cash flow matrix C_{mn}.
- $C_{in} = \max(0, K - P_{in})$ for $i \in 1, ..., m$
- For $j : P_{j,n-1} >= K, C_{j,n-1} = 0$. For $\{i : P_{i,n-1} < K\}$, build a linear model where $d * C_{i,n}$ is the dependent variable, and the independent variables are functions of $P_{i,n-1}$.
- Use the above model to predict the expected future payoff $F_{i,n}$ for each $P_{i,n-1}$. Then, $C_{i,n-1} = 0$ if $F_{i,n} > \max(0, K - P_{i,n-1})$, and $C_{i,n-1} = \max(0, K - P_{i,n-1})$, $C_{i,n} = 0$ if $F_{i,n} <= \max(0, K - P_{i,n-1})$.
- Repeat the above until we get all the elements of C.
- Each row of C should have at most 1 non-zero element. Discount it to today and take their average.

Where d is the discount factor for a time interval in the simulation.

Various choices are possible for the linear model in the algorithm described above. In the paper by Longstaff and Schwartz, the authors suggested to use the Laguerre polynomials. In the project, we will test the following 3 linear models:

- LM1: $y \sim x + x^2$

- LM2: The first two Laguerre polynomials, $y \sim (-x+1) + (\frac{1}{2}(x^2 - 4x + 2))$

- LM3: The first three Legendre polynomials, $y \sim x + (\frac{1}{2}(3x^2 - 1)) + (\frac{1}{2}(5x^3 - 3x))$

Example: Suppose we have a stock priced 100 today. An American put option has payoff 110 and matures two years later. Time is considered discrete and the length of an interval of time is 1 year. We assume that the risk-free rate is 0. Suppose we generate 3 stock price paths: $\{100, 120, 111\}, \{100, 95, 85\}, \{100, 99, 108\}$. Firstly looking at $t = 2$, we determine that the cash flows are $\{?, 0\}, \{?, 25\}, \{?, 2\}$. Now, since path 1 is above payoff at $t = 1$, we know that the cash flow is $\{0, 0\}, \{?, 25\}, \{?, 2\}$. For the two ? left here, we build a regression model $y \sim x$ where $y = [25, 2]$ and $x = [85, 99]$. Such regression model is $y \sim -1.643x + 164.643$. The predicted future payoff for 85, 99 are relatively 25 and 2. The profits of immediate exercise are relatively 15 and 11. Therefore, in path 2 it's better to hold the option at $t = 1$, which gives the cash flows $\{0, 0\}, \{0, 25\}, \{?, 2\}$. On the other hand, in path 3 it's better to exercise at $t = 1$ to get a payoff of 11, so our cash flow becomes $\{0, 0\}, \{0, 25\}, \{11, 0\}$. Therefore, the price of the option is estimated as $\frac{25+11}{2} = 18$.

3.3 Variance Reduction Techniques

3.3.1 Control Variates

Since the estimator proposed in the LSMC algorithm is proven to be convergent, we can assume that it is approximately unbiased when the Monte Carlo sample size is large. Denote the LSMC estimate $\hat{\theta}_{LSMC}$. Then $\hat{\theta}_{LSMC} - c(\hat{\omega} - \omega)$ will be also approximately unbiased if $\hat{\omega}$ is another estimator on something with mean ω, and has less variance if c is optimized. We can easily derive that the optimal c is $c* = -\frac{Cov(\theta,\omega)}{Var(\omega)}$. Therefore, the reasonable choices of ω should have a close relationship with θ. In this project, we will try the following three control variate:

- CV1: The price of the call option with same simulated paths

- CV2: The price of the European put option with same simulated paths

The process of our computation will be the following:

- Obtain the list of discount cash flows of each simulated path, $\hat{\theta} = \{\theta_1, ..., \theta_n\}$ (Their average is the LSMC estimator).

- Obtain the list of prices of the option serving as control variate for each path, $\hat{\omega} = \{\omega_1, ..., \omega_n\}$. Get ω using Black-Scholes formula.

- Estimate $Cov(\omega, \theta)$ and $Var(\omega)$ and obtain $c*$.

- Compute the estimator.

3.3.2 Antithetic Variates

The simulation inputs of this project are standard normal random numbers. The estimator can be considered as a function of these normal random numbers, $f_1 = f(z_1, ..., z_n)$. Consider a standard normal random variable Z. We can easily show that $-Z$ is also a standard normal (for example, via Moment Generating Functions), and has negative covariance with Z. Our hope is that $f(z_1, ..., z_n)$ and $f_2 = f(-z_1, ..., -z_n)$ also have negative covariance. If they have, then from

$$Var(\frac{f_1 + f_2}{2}) = \frac{Var(f_1) + Var(f_2) + 2Cov(f_1, f_2)}{4} \tag{2}$$

We know that doing $\frac{f_1+f_2}{2}$ will reduce the variance.

The process of our computation will be the following:

- When use $z_1, ..., z_n$ to generate a stock path, we use $-z_1, ..., -z_n$ to generate an antithetic stock path. Once the stock path generations complete, we have a stock price matrix, and an antithetic stock price matrix.

- Run the LSMC algorithm with two set of stock paths, separately.Obtain two lists of discounted cash flows.

- Average the above two lists pairwisely and obtain a new list. The mean of this list will be our antithetic estimation.

4 Simulation Results

In our simulation, we generate 100 stock paths, and each path has 100 steps. This will limit the preciseness of the estimation. However, simulations with more paths/steps than that turned to be quite time consuming. Since we have to carry out a lot of comparisons, we have to limit at the above levels.

4.1 Choice of Linear Models

First of all, we'd like to take a look on the performance of the above three choices of linear models. We use the following three options maturing in 1 year for checking:

- Option 1: $S_0 = 100, K = 92.5, r = 0.03, \sigma = 0.4$
- Option 2: $S_0 = 100, K = 98, r = 0.03, \sigma = 0.25$
- Option 3: $S_0 = 100, K = 93, r = 0.03, \sigma = 0.25$

The results turned to be:

- Option 1: theoretical price 10.7713, LM1 price 5.446, LM2 price 10.905, LM3 price 4.117
- Option 2: theoretical price 7.7146, LM1 price 2.882, LM2 price 8.075, LM3 price 2.742
- Option 3: theoretical price 5.5614, LM1 price 3.228, LM2 price 5.898, LM3 price 1.864

It was very obvious that linear model 2, which was the one suggested in the original paper, was the only one that was giving close estimates. Therefore, we will be using linear model 2 in the following parts of the project.

4.2 Choice of Variance Reduction

Table 1: Option Price Estimates

Option	No Variance Reduction	CV1	CV2	Antithetic	Theoretical
1	10.905	11.23	10.09	9.46	10.7713
2	8.075	8.277	7.444	7.45	7.7146
3	5.898	6.11	5.36	5.24	5.5614

Option 1, 2, and 3 and their theoretical prices refer to above.

First of all, we can say that the above estimations with variance reduction are still unbiased. (Since we only simulated 100 stock paths, and each path has only 100 steps, we cannot expect the estimations to be very precise).

For control variates, it turns out that using the price of European/American call options won't work. It actually increases the variance of the estimation. However, using the prices of European put options will significantly reduce the variance.

Table 2: Standard Deviations of Estimates

Option	No Variance Reduction	CV1	CV2	Antithetic
1	14.7	24.8	1.8	7.01
2	11.33	16.7	0.11	5.64
3	9.56	15.5	0.1	5.38

For the antithetic variate we proposed, it turned out that using the negations of standard normal random numbers to build antithetic paths is working. It approximately reduce the variance to a half.

Conclusion: Using the first few Laguerre polynomials as regressors in the regression step in LSMC, the algorithm is working. Using corresponding prices of European put options as a control variate is a very good way to reduce variance of the estimation. The project is basically successful!

5 Future Works

Due to time constraints, we cannot put everything we can think of into the project. Here we will elaborate a few features that we could work on in the future.

5.1 Greeks

Greeks are derivatives of option prices with respect to its parameters. For example, the 'Delta' of an option is the rate of the change of the option price, with respect to the change of the original price of the underlying stock. Such Greeks (derivatives) are usually estimated with the Finite Difference Method. For example, using $P(S)$ to denote the price of the option when the initial price of the underlying stock is S, we have

$$\Delta = \lim_{t \to 0} \frac{P(S+t) - P(S)}{t} \tag{3}$$

Therefore, setting t to be a small number, the above ratio will be an estimation of the Delta at price S and other give parameters.

5.2 LSMC and Other Options

We can think about applying the Least Square Monte Carlo method to options that are more complicated.

For example, the Israeli options, or Game options, are American options that the issuer can cancel the option at any time with a penalty δ. We can think about how to apply the LSMC method to this kind of option. A possible approach is to set up a cash flow matrix to the option issuer as well. We can induce this matrix from the maturity, just like how we worked with the cash flow matrix with American put options. When the option issuer predicts that the future payoff is larger than the penalty he/she pays, the option is cancelled. Integrated the simulation for holder and issuer together will probably give us an estimation of the option.

6 Reference

- Valuing American Options by Simulation: A Simple Least-Squares Approach, Francis A. Longstaff & Eduardo S. Schwartz.
- Robert's Online Option Pricer, $http://www.intrepid.com/robertl/option-pricer1.html?q = robertl/option-pricer1.html$
- Cboe - IVolatility Services, Option Price Calculator: The link cannot be pasted, but searching the name on Google it will come out.
- $http://www.option-price.com/$