Final Project AE 569

IGM Guidance algorithm

Christian Engelbrecht

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1 Equations of motion; K1, K2

For a given set of final states of an ascent vehicle leaving the lunar surface, we can write the equations of motion simply as:

$$\begin{aligned}
\dot{x} &= V_x \\
\dot{y} &= V_y \\
\dot{V_x} &= a\cos\beta \\
\dot{V_y} &= -g + a\sin\beta
\end{aligned}$$

Where β is the flight angle of the vehicle, and

$$a = \frac{T}{m(t)} = \frac{T}{m_0 - \dot{m}(t - t_0)}$$

is the thrust acceleration of the vehicle,for a given thrust T.The vehicle ascends from some time 0 to a final time t_f, where the final condition of the vehicle is

$$\begin{aligned}
V_x(t_f) &= V_x^* \\
V_y(t_f) &= V_y^* \\
y(t_f) &= y_f^*
\end{aligned}$$

Using optimal control theory, we now aim to maximize the Hamiltonian H of the system. The Hamiltonian is

$$H = P_x V_x + P_y V_y + P_{V_x} a\cos(\beta) + P_{V_y}(-g + a\sin(\beta))$$

Where $\boldsymbol{P_r}$ is $\binom{P_x}{P_y}$, and $\boldsymbol{P_V}$ is $\binom{P_{V_x}}{P_{V_y}}$ are the costate vectors for position and velocity, respectively.
We also define a performance index J that will minimize the total flight time

$$J = t_f$$

The derivatives for the costate equations are

$$\begin{aligned}
\dot{P_x} - \frac{\partial H}{\partial x} &= 0 & \dot{P_y} - \frac{\partial H}{\partial y} &= 0 \\
\Rightarrow P_x &= c_1 & \Rightarrow P_y &= c_2 \\
\dot{P_{V_x}} = -\frac{\partial H}{\partial V_x} &= -P_x & \dot{P_{V_y}} - \frac{\partial H}{\partial V_y} &= -P_y \\
\Rightarrow P_{V_x} &= c_1 t + c3 & \Rightarrow P_{V_y} &= c_2 t + c4
\end{aligned}$$

Our control variable here is β, and since this is not a constrained control (β can take on any value), the optimility condtition $\frac{\partial H}{\partial \beta} = 0$ gives us an expression to relate β to the constants found above:

$$\begin{aligned}
\frac{\partial H}{\partial \beta} = (\frac{T}{M})(-P_{V_x}\sin\beta + P_{V_y}\cos\beta) = 0 \\
\Rightarrow -P_{V_x}\sin\beta + P_{V_y}\cos\beta = 0 \\
P_{V_x} tan \beta - P_{V_y} = 0 \\
\tan\beta = \frac{P_{V_y}}{P_{V_x}} \\
\tan\beta = \frac{c_2 t + c_4}{c_1 t + c_3}
\end{aligned}$$

This is the familiar bilinear tangent law. We can now use the transversaility conditions to trim down the number of unknowns. The transversality condition for P_{V_x} is

$$\begin{aligned}
P_x(t_f) = -\frac{\partial \Phi}{\partial x(t_f)} = -\frac{\partial t_f}{x(t_f)} = 0 \\
\Rightarrow P_x = c_1 = 0
\end{aligned}$$

Similarly, the transversality condition for P_y:

$$\begin{aligned}
P_y(t_f) = -\frac{\partial \Phi}{\partial y(t_f)} = -\frac{\partial t_f}{y(t_f)} = 0 \\
\Rightarrow P_y = c_2 = 0 \\
\boxed{\therefore \tan\beta = \frac{c_4}{c_3} = constant \equiv \tan\tilde{\chi}}
\end{aligned}$$

We can now integrate the equations of motion to find explicit closed form solutions for V_x, V_y, x, y. Integrating $V_x and V_y$:

$\mathrm{V}_y(t) =$