Final Project AE 569

IGM Guidance algorithm

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1 Equations of motion; K1, K2

For a given set of final states of an ascent vehicle leaving the lunar surface, we can write the equations of motion simply as:

$$\begin{aligned}\dot{x} &= V_x \\ \dot{y} &= V_y \\ \dot{V_x} &= a\cos\beta \\ \dot{V_y} &= -g + a\sin\beta\end{aligned}$$

Where β is the flight angle of the vehicle, and

$$a = \frac{T}{m(t)} = \frac{T}{m_0 - \dot{m}(t - t_0)}$$

is the thrust acceleration of the vehicle,for a given thrust T.The vehicle ascends from some time 0 to a final time t_f, where the final condition of the vehicle is

$$\begin{aligned}V_x(t_f) &= V_x^* \\ V_y(t_f) &= V_y^* \\ y(t_f) &= y_f^*\end{aligned}$$

Using optimal control theory, we now aim to maximize the Hamiltonian H of the system. The Hamiltonian is

$$H = P_x V_x + P_y V_y + P_{V_x} a\cos(\beta) + P_{V_y}(-g + a\sin(\beta))$$

Where $\boldsymbol{P_r}$ is $\binom{P_x}{P_y}$, and $\boldsymbol{P_V}$ is $\binom{P_{Vx}}{P_{V_y}}$ are the costate vectors for position and velocity, respectively.
We also define a performance index J that will minimize the total flight time

$$J = t_f$$

The derivatives for the costate equations are

$$\begin{aligned}\dot{P_x} - \frac{\partial H}{\partial x} &= 0 & \dot{P_y} - \frac{\partial H}{\partial y} &= 0 \\ \Rightarrow P_x &= c_1 & \Rightarrow P_y &= c_2 \\ \dot{P_{V_x}} = -\frac{\partial H}{\partial V_x} &= -P_x & \dot{P_{V_y}} - \frac{\partial H}{\partial V_y} &= -P_y \\ \Rightarrow P_{V_x} &= c_1 t + c3 & \Rightarrow P_{V_y} &= c_2 t + c4\end{aligned}$$

Our control variable here is β, and since this is not a constrained control (β can take on any value), the optimility condtition $\frac{\partial H}{\partial \beta} = 0$ gives us an expression to relate β to the constants found above:

$$
\begin{aligned}
\frac{\partial H}{\partial \beta} = (\frac{T}{M})(-P_{V_x} \sin\beta + P_{V_y} \cos\beta) = 0 \\
\Rightarrow -P_{V_x} \sin\beta + P_{V_y} \cos\beta = 0 \\
P_{V_x} tan\beta - P_{V_y} = 0 \\
\tan\beta = \frac{P_{V_y}}{P_{V_x}} \\
\tan\beta = \frac{c_2 t + c_4}{c_1 t + c_3}
\end{aligned}
$$

This is the familiar bilinear tangent law. We can now use the transversaility conditions to trim down the number of unknowns. The transversality condition for P_{V_x} is

$$
\begin{aligned}
P_x(t_f) = -\frac{\partial \Phi}{\partial x(t_f)} = -\frac{\partial t_f}{x(t_f)} = 0 \\
\Rightarrow P_x = c_1 = 0
\end{aligned}
$$

Similarly, the transversality condition for P_y, in a simplified situation where $y(t_f)$ is not specified:

$$
\begin{aligned}
P_y(t_f) = -\frac{\partial \Phi}{\partial y(t_f)} = -\frac{\partial t_f}{y(t_f)} = 0 \\
\Rightarrow P_y = c_2 = 0 \\
\boxed{\therefore \tan\beta = \frac{c_4}{c_3} = constant \equiv \tan\tilde{\chi}}
\end{aligned}
$$

We can now integrate the equations of motion to find explicit closed form solutions for V_x, V_y, x, y. Integrating V_x and V_y, recalling $\dot{m} = -\frac{T}{g_0 I_{sp}}$:

$$
\begin{aligned}
V_x(t) &= \cos\beta \int_{t_0}^{t} \frac{T d\tau}{m - \dot{m}(t_0 - \tau)} \\
&= T\cos\beta \cdot \frac{\ln(m_0 + \dot{m}(t_0 - \tau)}{\dot{m}} \Big|_{\tau=t_0}^{\tau=t} \\
&= -g_0 I_{sp} \cos\beta \cdot [\ln(m_0 + \dot{m}(t_0 + t)) - \ln(m_0)] + V_x(t_0) \\
&= g_0 I_{sp} \cos\beta \ln\left(\frac{m_0}{m_0 + \dot{m}(t_0 + t)}\right) + V_x(t_0)
\end{aligned}
$$

$$
\begin{aligned}
V_y(t) &= \sin\beta \int_{t_0}^{t} \frac{T d\tau}{m - \dot{m}(t_0 - \tau)} + \int_{t_0}^{t} g d\tau \\
&= T \sin\beta \left(\frac{\ln(m_0 + \dot{m}(t_0 - \tau)}{\dot{m}} \Big|_{\tau=t_0}^{\tau=t} + g\tau \Big|_{\tau=t_0}^{\tau=t} \right) \\
&= -g_0 I_{sp} \sin\beta [\ln(m_0 + \dot{m}(t_0 + t)) - \ln(m_0) + g(t - t_0)] + V_y(t_0) \\
&= g_0 I_{sp} \sin\beta \ln\left(\frac{m_0}{m_0 + \dot{m}(t_0 + t)} \right) - g(t - t_0) + V_y(t_0)
\end{aligned}
$$

We now use the end conditions $V_x(t_f) = V_x^*$ and $V_y(t_f) = V_y^*$. We define the temporary variable $\gamma \equiv g_0 I_{sp} \ln\left(\frac{m_0}{m_0 + \dot{m}(t_0+t)}\right)$

$$
\Rightarrow V_y^* + g(t_f - t_0) - V_y(t_0) = \gamma \sin\tilde{\chi} \tag{1}
$$

$$
V_x^* - V_x(t_0) = \gamma \cos\tilde{\chi} \tag{2}
$$

Then divide (1) by (2):

$$
\Rightarrow \tan\tilde{\chi} = \frac{V_y^* - V_y(t_0) + g(t_f - t_0)}{V_x^* - V_x(t_0)} \tag{3}
$$

This equation now only depends on $t_f - t_0$, which is called the Time to Go: t_{go}. We will use this equation to determine the value for $\tilde{\chi}$ given a value for t_{go} We now consider again the linear tangent law, with the full constraint on final vertical position (i.e. the transversality condition for P_y does not apply), the bilinear tangent law is now the linear tangent law, where the constants c_2, c_3 and c_4 are recast to $k1, k_0$:

$$
\tan\beta = \tag{4}
$$