Final Project AE 569

IGM Guidance algorithm

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1 Equations of motion; K1, K2

For a given set of final states of an ascent vehicle leaving the lunar surface, we can write the equations of motion simply as:

$$\begin{aligned}\dot{x} &= V_x \\ \dot{y} &= V_y \\ \dot{V_x} &= a\cos\beta \\ \dot{V_y} &= -g + a\sin\beta\end{aligned}$$

Where β is the flight angle of the vehicle, and

$$a = \frac{T}{m(t)} = \frac{T}{m_0 - \dot{m}(t - t_0)}$$

is the thrust acceleration of the vehicle,for a given thrust T.The vehicle ascends from some time 0 to a final time t_f, where the final condition of the vehicle is

$$\begin{aligned}V_x(t_f) &= V_x^* \\ V_y(t_f) &= V_y^* \\ y(t_f) &= y_f^*\end{aligned}$$

Using optimal control theory, we now aim to maximize the Hamiltonian H of the system. The Hamiltonian is

$$H = P_x V_x + P_y V_y + P_{V_x} a\cos(\beta) + P_{V_y}(-g + a\sin(\beta))$$

Where $\boldsymbol{P_r}$ is $\binom{P_x}{P_y}$, and $\boldsymbol{P_V}$ is $\binom{P_{V_x}}{P_{V_y}}$ are the costate vectors for position and velocity, respectively.
We also define a performance index J that will minimize the total flight time

$$J = t_f$$

The derivatives for the costate equations are

$$\begin{aligned}\dot{P_x} - \frac{\partial H}{\partial x} &= 0 & \dot{P_y} - \frac{\partial H}{\partial y} &= 0 \\ \Rightarrow P_x &= c_1 & \Rightarrow P_y &= c_2 \\ \dot{P_{V_x}} = -\frac{\partial H}{\partial V_x} &= -P_x & \dot{P_{V_y}} - \frac{\partial H}{\partial V_y} &= -P_y \\ \Rightarrow P_{V_x} &= c_1 t + c3 & \Rightarrow P_{V_y} &= c_2 t + c4\end{aligned}$$

Our control variable here is β, and since this is not a constrained control (β can take on any value), the optimility condtition $\frac{\partial H}{\partial \beta} = 0$ gives us an expression to relate β to the constants found above:

$$
\begin{aligned}
\frac{\partial H}{\partial \beta} = (\frac{T}{M})(-P_{V_x} \sin\beta + P_{V_y} \cos\beta) = 0 \\
\Rightarrow -P_{V_x} \sin\beta + P_{V_y} \cos\beta = 0 \\
P_{V_x} tan\beta - P_{V_y} = 0 \\
\tan\beta = \frac{P_{V_y}}{P_{V_x}} \\
\tan\beta = \frac{c_2 t + c_4}{c_1 t + c_3}
\end{aligned}
$$

This is the familiar bilinear tangent law. We can now use the transversaility conditions to trim down the number of unknowns. The transversality condition for P_{V_x} is

$$
\begin{aligned}
P_x(t_f) = -\frac{\partial \Phi}{\partial x(t_f)} = -\frac{\partial t_f}{x(t_f)} = 0 \\
\Rightarrow P_x = c_1 = 0
\end{aligned}
$$

Similarly, the transversality condition for P_y, in a simplified situation where $y(t_f)$ is not specified:

$$
\begin{aligned}
P_y(t_f) = -\frac{\partial \Phi}{\partial y(t_f)} = -\frac{\partial t_f}{y(t_f)} = 0 \\
\Rightarrow P_y = c_2 = 0 \\
\boxed{\therefore \tan\beta = \frac{c_4}{c_3} = constant \equiv \tan\tilde{\chi}}
\end{aligned}
$$

We can now integrate the equations of motion to find explicit closed form solutions for V_x, V_y, x, y. Integrating V_x and V_y, recalling $\dot{m} = -\frac{T}{g_0 I_{sp}}$:

$$
\begin{aligned}
V_x(t) &= \cos\beta \int_{t_0}^{t} \frac{T d\tau}{m - \dot{m}(t_0 - \tau)} \\
&= T\cos\beta \cdot \frac{\ln(m_0 + \dot{m}(t_0 - \tau))}{\dot{m}} \Big|_{\tau = t_0}^{\tau = t} \\
&= -g_0 I_{sp} \cos\beta \cdot [\ln(m_0 + \dot{m}(t_0 + t)) - \ln(m_0)] + V_x(t_0) \\
&= g_0 I_{sp} \cos\beta \ln\left(\frac{m_0}{m_0 + \dot{m}(t_0 + t)}\right) + V_x(t_0)
\end{aligned}
$$

$$
\begin{aligned}
V_y(t) &= \sin\beta \int_{t_0}^{t} \frac{T d\tau}{m - \dot{m}(t_0 - \tau)} + \int_{t_0}^{t} g d\tau \\
&= T \sin\beta \left(\frac{\ln(m_0 + \dot{m}(t_0 - \tau)}{\dot{m}} \Big|_{\tau = t_0}^{\tau = t} + g\tau \Big|_{\tau = t_0}^{\tau = t} \right) \\
&= -g_0 I_{sp} \sin\beta[\ln(m_0 + \dot{m}(t_0 + t)) - \ln(m_0) + g(t - t_0)] + V_y(t_0) \\
&= g_0 I_{sp} \sin\beta \ln\left(\frac{m_0}{m_0 + \dot{m}(t_0 + t)} \right) - g(t - t_0) + V_y(t_0)
\end{aligned}
$$

We now use the end conditions $V_x(t_f) = V_x^*$ and $V_y(t_f) = V_y^*$. We define the temporary variable $\gamma \equiv g_0 I_{sp} \ln\left(\frac{m_0}{m_0 + \dot{m}(t_0 + t)}\right)$

$$
\begin{aligned}
\Rightarrow V_y^* + g(t_f - t_0) - V_y(t_0) &= \gamma \sin\tilde{\chi} \qquad (1) \\
V_x^* - V_x(t_0) &= \gamma \cos\tilde{\chi} \qquad (2)
\end{aligned}
$$

Then divide (1) by (2):

$$
\Rightarrow \tan\tilde{\chi} = \frac{V_y^* - V_y(t_0) + g(t_f - t_0)}{V_x^* - V_x(t_0)} \qquad (3)
$$

This equation now only depends on $t_f - t_0$, which is called the Time to Go: t_{go}. We will use this equation to determine the value for $\tilde{\chi}$ given a value for t_{go} We now consider again the linear tangent law, with the full constraint on final vertical position (i.e. the transversality condition for P_y does not apply), the bilinear tangent law is now the linear tangent law, where the constants c_2, c_3 and c_4 are recast to $k1, k_0$. We consider the tangenet law not at a specific time t but at the time interval $t - t_0$

$$
\tan\beta = k_1(t - t_0) + k_0
$$

We discovered above that $\tan\beta$ is constant when some assumptions are taken into account, and equal to $\tilde{\chi}$ in those cases. Now, with those assumptions relaxed, we say that the linear tangent law still applies but there is a slight offset in $\tilde{\chi}$: that is, $k_0 = \tilde{\chi} + k_2$

$$
\tan\beta = k_1(t - t_0) + \tilde{\chi} + k_2
$$

If we further assume that offset is very small, i.e $|k_1(t - t_0) + k_2| << 1$, and β is very small, then we can use the small angle approximation

$$
\beta = k_1(t - t_0) + \tilde{\chi} + k_2 \qquad (4)
$$

This angle β is the angle necessary for integrating the equations of motion. We need in particular it's sine and cosine. Using the double angle identity, we can say

$$
\begin{aligned}
\sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B) \\
\cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B) \\
\Rightarrow \sin(\beta) &= \sin(\tilde{\chi} + (k_1(t-t_0)+k_2)) \\
&= \sin(\tilde{\chi})\cos(k_1(t-t_0)+k_2) + \cos(\tilde{\chi})\sin(k_1(t-t_0)+k_2) \\
&\approx \sin(\tilde{\chi}) + k_1(t-t_0)\cos(\tilde{\chi}) + k_2\cos(\tilde{\chi}) \\
\cos(\beta) &= \cos(\tilde{\chi})\cos(k_1(t-t_0)+k_2) - \sin(\tilde{\chi})\sin(k_1(t-t_0)+k_2) \\
&\approx cos(\tilde{\chi}) - k_2\sin(\tilde{\chi}) - k_1(t-t_0)\sin(\tilde{\chi})
\end{aligned}
$$

At this point the EOMs are entirely integrable, given a t_{go} To find this quantity, we use an iterative approach using the one dimensional rocket equation and the incremental velocity at a given time t_0. First, a time is chosen at random and a ΔV is calculated given the state of the velocity at that time (equation 5). Then the velocity is calculated using the Tsiolkovski rocket equation (equations 6 and 7): if the velocities match, the initial time chosen is a solution to the set of equations. If not, the time is incremented and the conditions is checked again.

$$
\Delta V = \sqrt{(V_x^* - V_x)^2 + (V_y^* - V_y + g(t_{go)}))} \tag{5}
$$

$$
\Delta V = g_0 I_{sp} \ln\frac{m_0}{m},\ m = m_0 - \dot{m} t_{go} \tag{6}
$$

$$
\Rightarrow t_{go} = \frac{-m_0}{\dot{m}}(e^{\frac{-\Delta V}{I_{sp}*g_0}} - 1) \tag{7}
$$

Now with all tools in hand, the equations of motion can properly be solved for each time step, and an iterative guidance program can be made. Starting by integrating V_y

$$
\begin{aligned}
\dot{V}_y &= -g + a\sin\beta \\
&= -g + \frac{T}{m_0 - \dot{m}(t-t_0)} * (\sin\tilde{\chi} + k_2\cos\tilde{\chi} + k_1\cos\tilde{\chi}(t-t_0)) \\
\therefore V_y(t) &= -g(t-t_0) + T\int_{t_0}^{t}\frac{\sin\tilde{\chi} + k_2\cos\tilde{\chi}}{m_0 - \dot{m}(\tau - t_0)}d\tau + T\int_{t_0}^{t}\frac{cos\tilde{\chi}k_1(\tau - t_0)}{m_0 - \dot{m}(\tau - t_0)}d\tau + V_y(t_0)
\end{aligned}
$$

Defining the constants a_{22}, a_{21} and a_{20} as the coeffiecients of k_2,k_1 and the constant coefficient, respectively:

$$
\begin{aligned}
a_{22} &= T\cos\tilde{\chi}\int_{t_0}^{t}\frac{d\tau}{m_0-\dot{m}(\tau-t_0)}\\
&= \frac{T\cos\tilde{\chi}}{\dot{m}}\ln\frac{m_0}{m_0-\dot{m}(t-t_0)}\\
a_{21} &= T\cos\tilde{\chi}\int_{t_0}^{t}\frac{(\tau-t_0)d\tau}{m_0-\dot{m}(\tau-t_0)}\\
&= T\cos\tilde{\chi}(\frac{m_0\ln(\frac{m_0}{m_0-\dot{m}(t_0-t)})}{\dot{m}^2}-\frac{(t+t_0)}{\dot{m}})\\
a_{20} &= V_y(t_0)-\int_{t_0}^{t}\frac{d\tau}{m_0-\dot{m}(\tau-t_0)}-g\\
&= \frac{T\sin\tilde{\chi}}{\dot{m}}\ln\frac{m_0}{m_0-\dot{m}(t_0-t)}-g(t-t_0)
\end{aligned}
$$

$$\Rightarrow V_y(t)=a_{22}k_2+a_{21}k_1+a_{20}$$

For the longer derivations of these and the following coefficients, please see the attached paper to this report. Now we can integrate this equation for $V_y(t)$ one more time to get $y(t)$

$$
\begin{aligned}
y(t) &= y(t_0)+\int_{t_0}^{t}V_y(\tau)d\tau\\
&= y(t_0)+\int_{t_0}^{t}a_{22}k_2dt+\int_{t_0}^{t}a_{21}k_1dt+\int_{t_0}^{t}+a_{20}dt\\
&= y(t_0)+a_{12}k_2+a_{11}k_1+a_{10}
\end{aligned}
$$

Defininig the coefficients a_{11}, a_{12} , a_{10}

$$
\begin{aligned}
a_{10} &= \int_{t_0}^{t}a_{20}dt\\
&= V_y(t_0)(t-t_0)-\frac{g}{2}(t-t_0)^2+\frac{T\sin\tilde{\chi}}{|\dot{m}|}\Big[\frac{(t-t_0)|\dot{m}|-(m_0|\dot{m}|(t-t_0)\ln\frac{m_0}{m_0-|\dot{m}|(t-t_0)}}{y}\Big]\\
a_{11} &= \int_{t_0}^{t}a_{21}k_1dt\\
&= T\cos\tilde{\chi}\Big[\frac{m_0}{\dot{m}^2}\Big(\frac{(t-t_0)|\dot{m}|-(m_0-|\dot{m}|(t-t_0)ln\frac{m_0}{m_0-|\dot{m}|(t-t_0)})}{|\dot{m}|}\Big)-\frac{(t-t_0)^2}{2|\dot{m}|}\Big]\\
a_{12} &= \int_{t_0}^{t}a_{22}k_1dt\\
&= T\cos\tilde{\chi}\Big[|\dot{m}|(t-t_0)-(m_0-|\dot{m}|(t-t_0)\ln\frac{m_0}{m_0-|\dot{m}|(t-t_0)}\Big]
\end{aligned}
$$

These derivations are also attached to this report, and simplified versions are implemented in the code
The coefficients of k_1 k_2 are now explicit functions of time, and the current state of the vehicle. We can use this, along with the explicit solution for V_y found above, to set up a system of equations to solve for the constants k_1 and k_2, given a t_{go} from the iterative method also found above.
That is, replacing β with $\tilde{\chi}$ in the explicit equation for V_y:

$$\begin{aligned} V_y(t_f) &= (a_{22})_{t_f} k_2 + (a_{21})_{t_f} k_1 + (a_{20})_{t_f} \\ &= g_0 I_{sp} \sin\tilde{\chi} \ln\left(\frac{m_0}{m_0 - |\dot{m}|(t_{go})}\right) - g t_{go} + V_y^* \end{aligned}$$

We also use the terminal constraint y^*

$$\begin{aligned} y(t_f) &= y_f^* \\ &= (a_{12})_{t_f} k_2 + (a_{11})_{t_f} k_1 + (a_{10})_{t_f} \end{aligned}$$

$$\therefore \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

For b_1,b_2:

$$\begin{aligned} b_1 &= g_0 I_{sp} \sin\tilde{\chi} \ln\left(\frac{m_0}{m_0 - |\dot{m}|(t_{go})}\right) - g t_{go} + V_y^* - a_{20} \\ b_2 &= y_f^* + V_y^* - t_{go} - a_{10} \end{aligned}$$

This is the linear system which is solved at every timestep, to find the required flight angle β from equation 4 above.

2 Mission 1 results

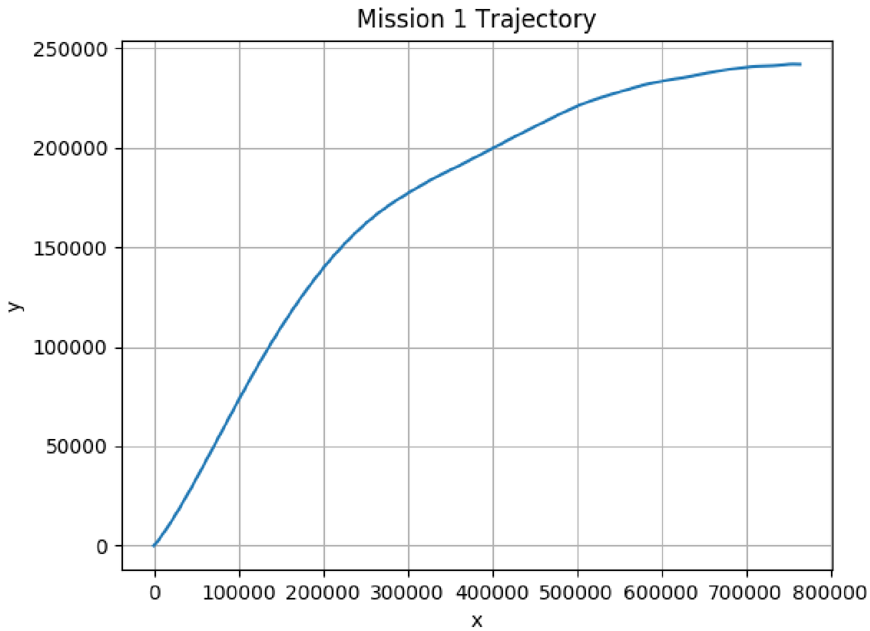
Unfortunately, my code did not work to implement the above routines. The possible sources of error include the derivation of the constants, as well as general coding bugs which were not squashed in time. The following plots and code are included only as a show of effort, and obviously do not reflect the actual accuracy of the real world IGM guidance.
The conditions for the first mission are:

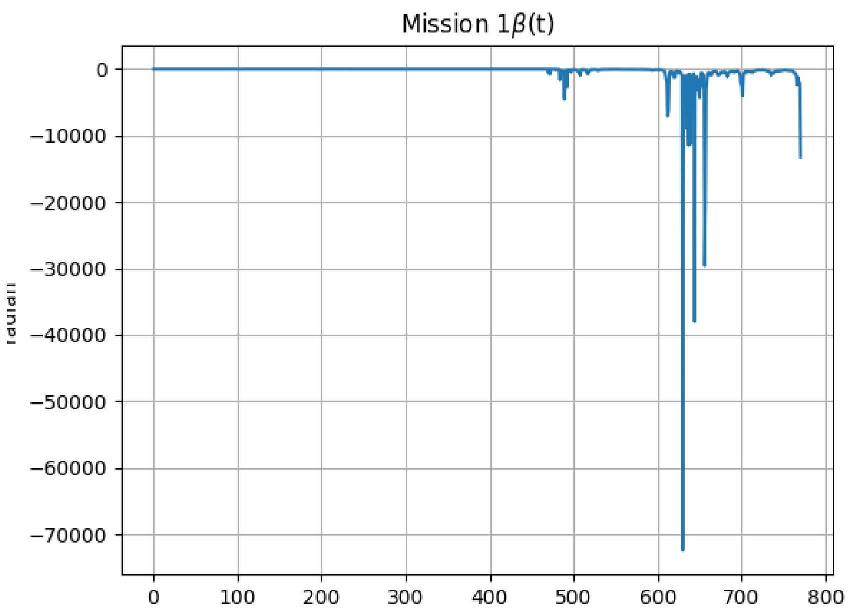
$$x(0) = y(0) = V_x(0) = V_y(0) = 0, m(0) = 17513kg$$

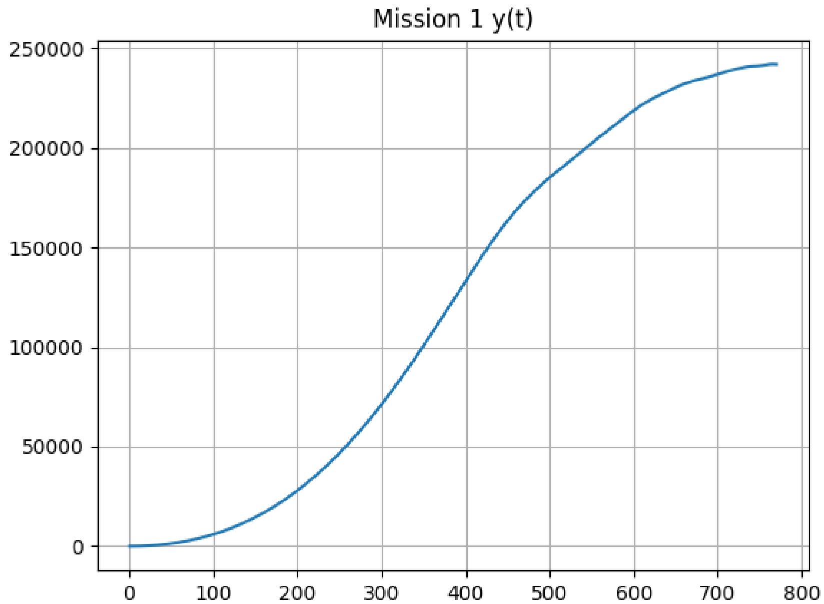
$$V_x^* = 1594.6\frac{m}{s}, V_y^* = 0\frac{m}{s}, y_f^* = 185000m$$

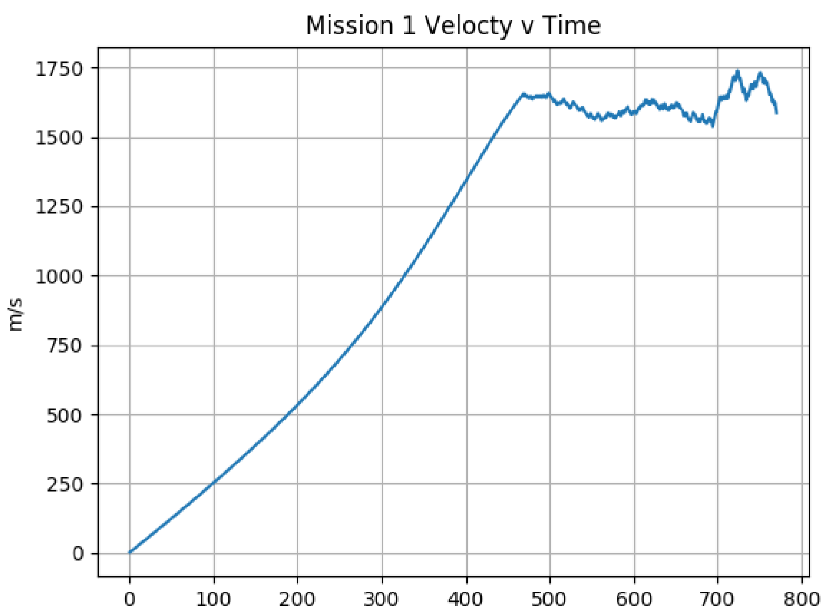
For both missions, the vehicle's characteristics were

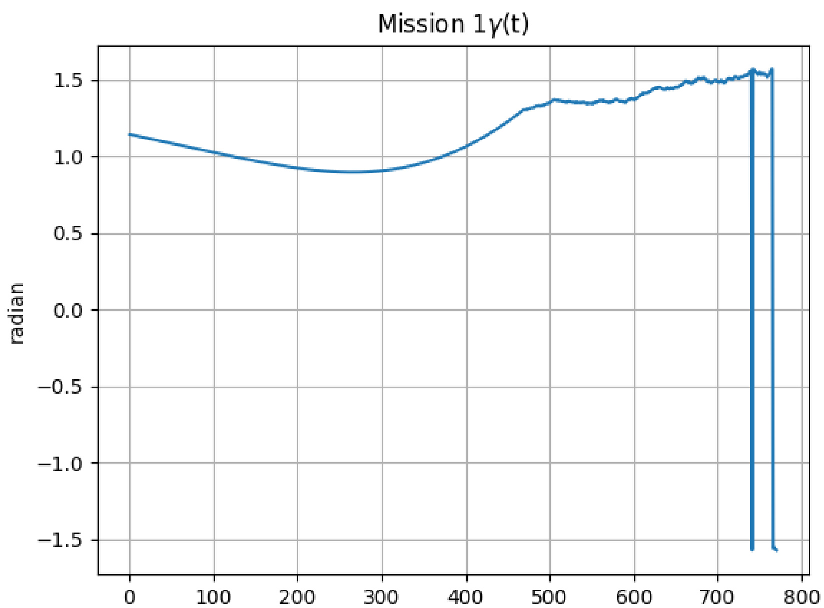
$$T = 60051.0, \dot{m} = -19.118 kg/s$$


Mission 1 Trajectory
250000
200000
150000
100000
50000
0
y
0
100000
200000
300000
400000
500000
600000
700000
800000
x


Mission 1β(t)
0
−10000
−20000
−30000
−40000
−50000
−60000
−70000
0
100
200
300
400
500
600
700
800


Mission 1 y(t)
250000
200000
150000
100000
50000
0
0
100
200
300
400
500
600
700
800


Mission 1 Velocty v Time
m/s
1750
1500
1250
1000
750
500
250
0
0
100
200
300
400
500
600
700
800

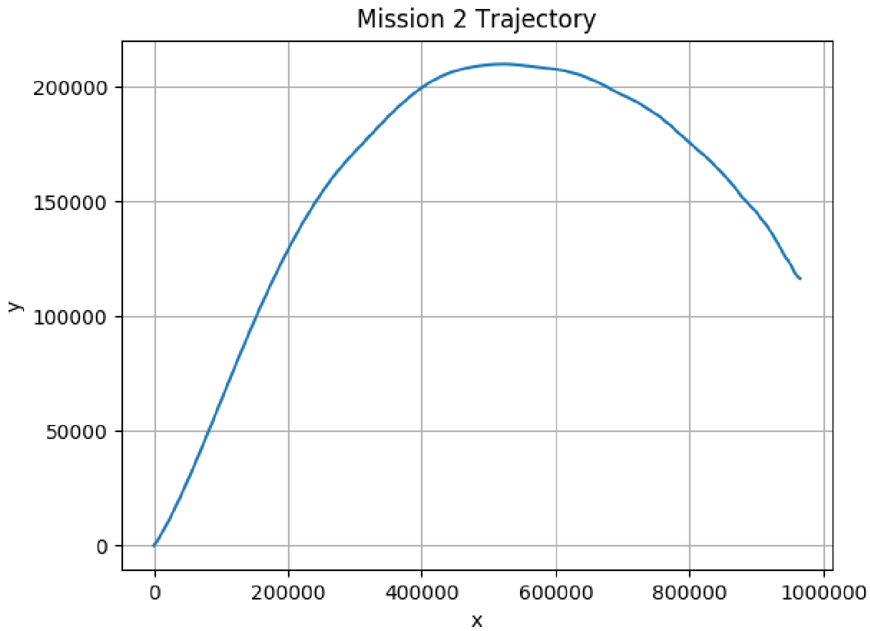


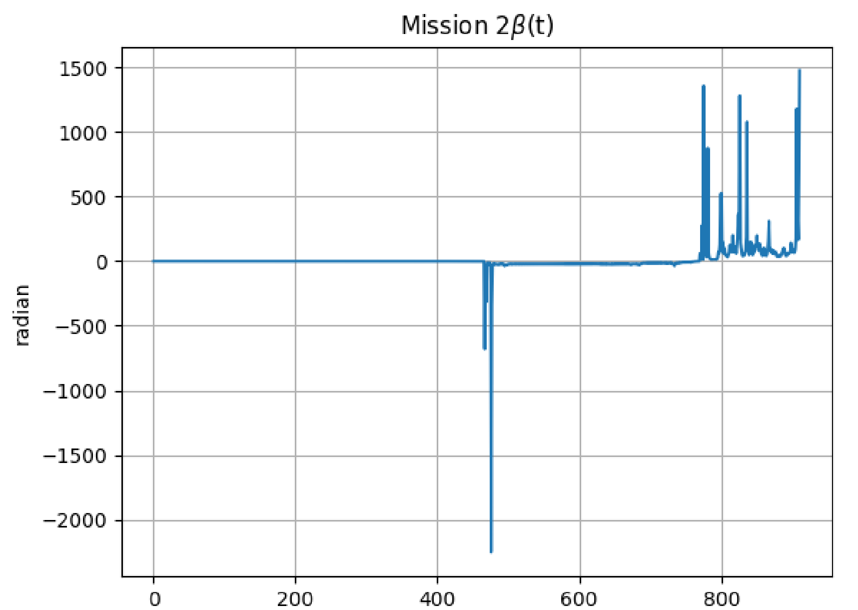
Mission 1γ(t)
radian
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
100
200
300
400
500
600
700
800

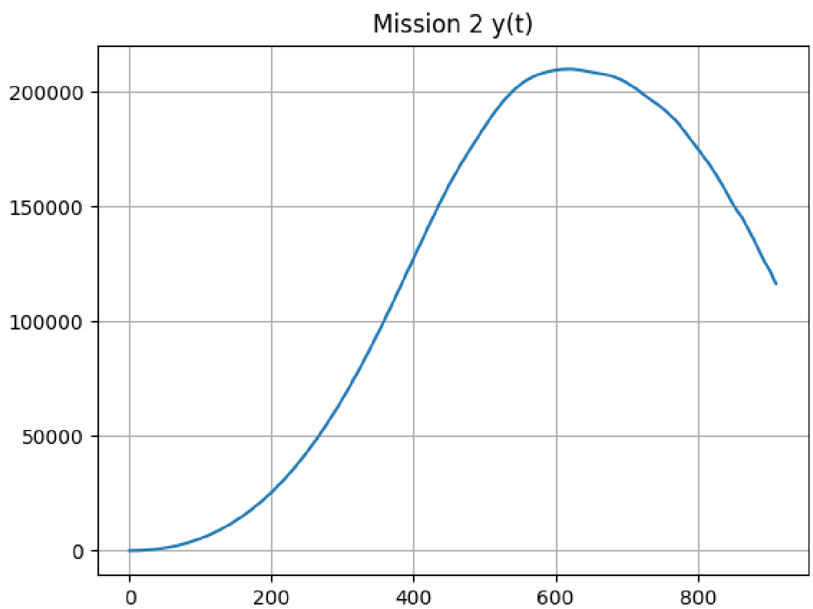
The plots for mission 2 do not look much better, unfortunately. The conditions for that mission are

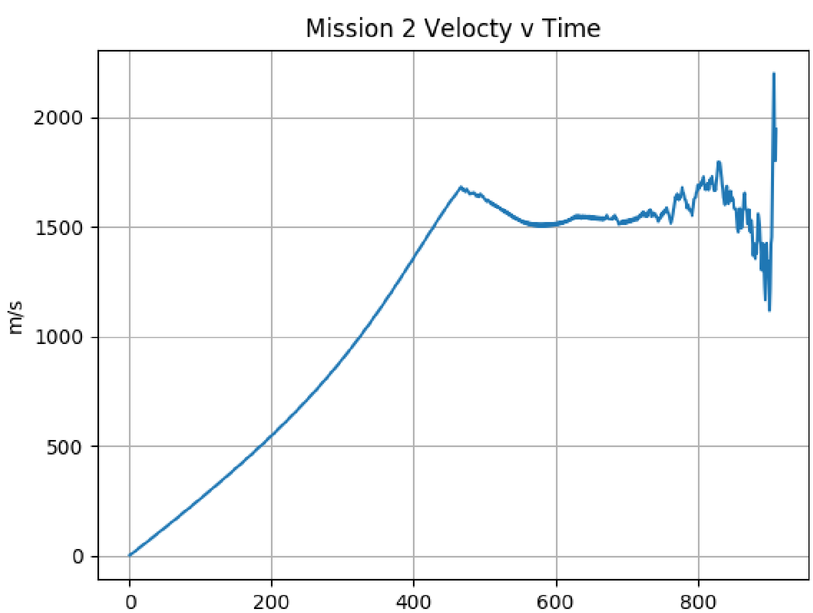
$$x(0) = y(0) = V_x(0) = V_y(0) = 0, m(0) = 17513kg$$

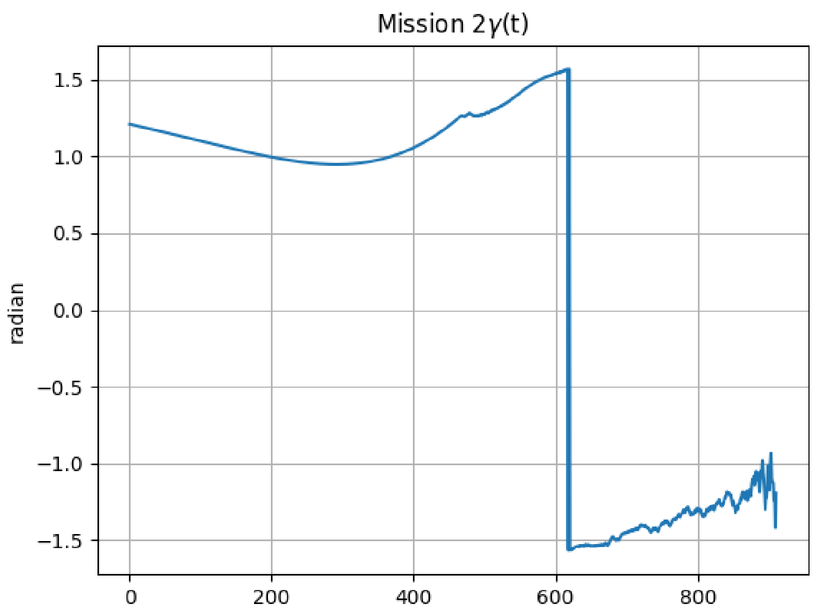
$$V_x^* = 1594.6\frac{m}{s}, V_y^* = 0\frac{m}{s}, y_f^* = 185000m$$


Mission 2 Trajectory
200000
150000
100000
50000
0
y
0
200000
400000
600000
800000
1000000
x


Mission 2β(t)
1500
1000
500
0
−500
−1000
−1500
−2000
radian
0
200
400
600
800


Mission 2 y(t)
200000
150000
100000
50000
0
0
200
400
600
800


Mission 2 Velocty v Time
m/s
2000
1500
1000
500
0
0
200
400
600
800


Mission 2γ(t)
radian
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
200
400
600
800