Chapter 3

Turbulence Mixing

3.1 Introduction.

One of the most striking properties of turbulence is its ability to mix scalars (temperature, pollutants, chemical species) orders of magnitude faster than molecular diffusion alone. This enhanced mixing arises from the multi-scale structure of turbulent eddies. In nature, the dynamics of the atmosphere and oceans are strongly governed by scalar-mixing processes; similarly, phenomena ranging from stirring sugar into coffee, to industrial blending of fluids, to fuel–oxygen mixing in combustion chambers all rely on turbulent mixing.

Understanding mixing therefore requires more than the governing equations — it requires identifying the *characteristic scales and times* that describe how turbulent motions transport and dissipate scalar variance.

Turbulence is inherently multi-scale, no single length or time scale can capture its dynamics. In this chapter, we focus on the *physical interpretation* of these scales and their connection to mixing time-scales. Formal statistical definitions and derivations are presented in the following chapter Chapter 3; here, our aim is to build the intuition necessary for estimating how fast turbulence mixes scalars across a range of flows.

3.2 Why turbulent mixing matters.

1. **Climate and Atmosphere:** The Sun warms the tropics far more than the poles, creating a strong equator-to-pole temperature gradient. Without motion, this gradient would persist — tropics baking, poles freezing, mid-latitudes trapped in between. Instead, turbulent eddies and waves transport warm air poleward and cold air equatorward, blending them into the temperate conditions we know.

2. **Oceans:** Warm, light water overlies cold, dense water. Without turbulence, this stratification would persist indefinitely. Currents, winds, and tides inject turbulent motions that overturn layers, mixing heat, nutrients, and dissolved gases throughout the water column.

3. **Mixers** Pour cream into coffee and it will form slow, curling filaments; without stirring, they take minutes to blend. With a spoon, you inject turbulence — streaks stretch, fold, and fragment into ever finer scales until the drink is uniform.

4. **Combustion Systems:** Inject fuel into still air and it will form smooth plumes that mix slowly. Introduce turbulence — via swirl vanes, sharp velocity gradients, or sudden expansions — and the fuel sheet shreds into thin ribbons that wrap through the air, producing a nearly perfect mixture in milliseconds. Industrial burners, gas turbines, and refinery mixers use this principle deliberately: the faster and more uniformly fuel and oxidizer mix, the more efficient and cleaner the burn.

Whether in a coffee cup, a jet engine, or a chemical reactor, turbulence multiplies contact between components, accelerates the erosion of concentration gradients, and produces uniform mixtures in a fraction of the time molecular diffusion could achieve.

Whether driven by shear or buoyancy, the root cause of turbulence production is a *gradient* in velocity, temperature, or density that injects energy into the flow. In shear-driven turbulence, velocity gradients create layers sliding past each other, producing fluctuating momentum transport known as *Reynolds stresses*. In buoyancy-driven turbulence, temperature or composition gradients cause lighter fluid to rise and heavier fluid to sink, producing fluctuating vertical motions and associated turbulent stresses. In both cases, these turbulent stresses transfer energy from the mean flow (or mean density distribution) into chaotic fluctuations, dramatically increasing effective mixing rates.

Large eddies carry fluid parcels across many gradient lengths in a single motion. As these eddies break down into smaller ones, the *turbulent cascade* passes energy to ever-finer scales until molecular diffusion finally completes the homogenization.

3.3 Characteristics of Turbulent Flows

Turbulent flows are marked by irregular, three-dimensional motions with strong velocity fluctuations across a wide range of scales. They typically arise at high Reynolds numbers, where inertial forces dominate viscous forces, and are a property of the *flow* rather than of the fluid itself. In everyday and engineering contexts, turbulence is the rule rather than the exception—present in atmospheric winds, ocean currents, combustion chambers, industrial mixers, and even the air moving around you now.

Key characteristics of turbulent flows include:

- **Irregularity and Chaos** Observe smoke rising from a chimney or measure velocity in a turbulent pipe: no exact pattern repeats itself. This apparent randomness means deterministic prediction of the exact motion is impractical.

However, statistical quantities such as mean velocities, variances, and correlations are repeatable and form the basis for turbulence modeling and theory.

- **Enhanced Mixing** Turbulent motions transport momentum, heat, and scalar quantities far more efficiently than laminar flows. In laminar wind over water, the surface may ripple but particles remain largely in place; in turbulence, eddies carry particles over large distances, blending temperature, momentum, and chemical species.
- **Flow Property, Not Fluid Property** Turbulence is a state of motion, not a property of the substance.
- **Three-Dimensionality and Vorticity** Turbulent flows contain complex vortex structures. Sustained turbulence requires *vortex stretching*, a process unique to three-dimensional flows, where vortices are elongated and intensified, feeding energy into smaller scales.
- **Large Reynolds Number** Most turbulent flows have high Reynolds numbers, indicating that inertial effects dominate over viscous effects. Stability theory seeks to determine the critical Reynolds number above which laminar flow transitions to turbulence—one of the central challenges of fluid mechanics.
- **Dissipation and Energy Cascade** Turbulence rapidly converts large-scale kinetic energy into smaller-scale motion via the cascade process, ultimately dissipating energy as heat through molecular viscosity at the smallest scales. The dissipation rate is much higher than in laminar flows, so without continuous energy input, turbulence decays.
- **Intermittency** Turbulent energy is not evenly distributed in space or time. Large eddies can produce bursts of intense small-scale activity, and the cascade process itself can amplify these fluctuations. Intermittency occurs across the energy-containing, inertial, and dissipation ranges, making the local turbulence intensity highly variable even in statistically steady flows.

In summary, turbulence is a multi-scale, three-dimensional, energy-dissipating process that enhances transport far beyond molecular diffusion. Its apparent randomness conceals an underlying statistical structure, which is why turbulence theory often combines physical reasoning with statistical and spectral descriptions.

3.4 Turbulence categories

Two broad classes of turbulence dominate how mixing occurs:

Mechanical Generation of Turbulence with scalars

Mechanical generation of turbulence occurs when a mean velocity gradient, such as $\frac{dU}{dz}$, acts as a source of turbulent energy. The velocity shear extracts energy from the mean flow and converts it into velocity fluctuations, denoted by u', v', w'. These velocity fluctuations interact with scalar fluctuations, such as temperature or concentration, and give rise to turbulent scalar flux terms like $\overline{w'\phi'}$. The scalar fluxes transport scalars from regions of high mean value to regions of low mean value, thereby enhancing mixing and homogenization within the flow. In this process, the mean shear drives velocity perturbations, the perturbations correlate with scalar variations, and the resulting turbulent fluxes redistribute the scalar field. The mixing is therefore a direct outcome of the coupling between mechanically generated turbulence and scalar transport, where the source of turbulence is the velocity gradient and the effect is the redistribution and smoothing of scalar gradients across the flow.

Buoyant Generation of Turbulence with Scalars

Buoyant generation of turbulence arises when density variations in a gravitational field lead to convective motions. A mean scalar gradient, such as a vertical temperature gradient $\frac{dT}{dz}$, produces regions of lighter, warmer fluid below heavier, cooler fluid. This unstable stratification generates buoyancy-driven velocity fluctuations, represented by w', which in turn correlate with scalar fluctuations θ'. The resulting turbulent scalar flux, expressed as $\overline{w'\theta'}$, represents the vertical transport of heat or other scalars by turbulent eddies. Unlike mechanical generation, where turbulence extracts energy from mean shear, buoyant generation extracts energy from the available potential energy of the stratified fluid. The scalar fluxes transport scalars vertically, creating strong mixing and overturning that homogenizes the scalar field. Thus, buoyant turbulence is generated directly by the conversion of potential energy into kinetic energy, with scalar gradients serving as both the source of turbulence and the medium through which turbulent fluxes redistribute energy and matter.

3.5 Laminar mixing

Let us start the chapter by understanding what is mixing ? What are the differences between Laminar mixing and Turbulent mixing?

In laminar flows, the fluid moves in smooth, orderly layers. Molecules transfer mass, momentum, and energy through direct, orderly motion.

Fick's Second Law of Diffusion

The unsteady diffusion of a scalar (such as concentration or temperature) in one spatial dimension is governed by Fick's

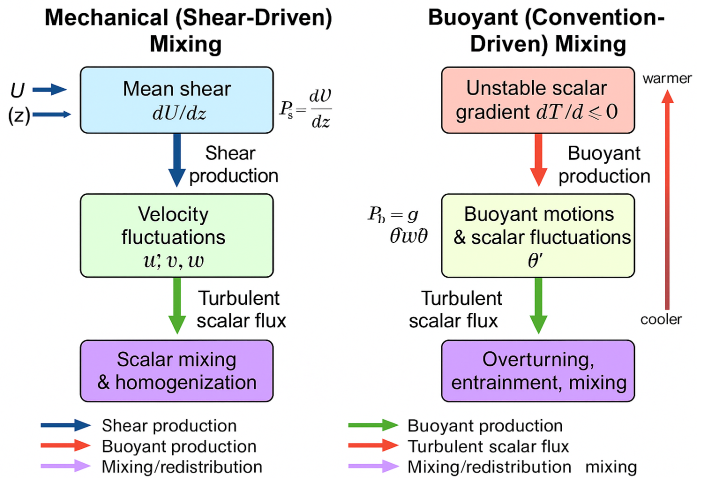

Figure 3.1: Mechanical and Buoyant generation of Turbulence with scalar.

second law:

$$\frac{\partial \phi}{\partial t} = \mu \frac{\partial^2 \phi}{\partial x^2} \tag{3.1}$$

Where, $\phi(x, t)$ is the scalar field (e.g., concentration),μ is the molecular diffusivity [units: m^2/s], x is the spatial coordinate and t is time.

Order-of-Magnitude Estimation

To estimate the characteristic time scale for diffusion over a length scale L, we apply dimensional (scaling) analysis to Eq. 3.2.

We define a time scale t_d and a length scale L, where we assume that the changes in ϕ occur. Let us assume that $t \sim t_d$, $x \sim L$

Then the terms in Eq. 3.2 scale as:

$$\underbrace{\frac{\partial \phi}{\partial t}}_{\sim \phi/t_d} = \mu \underbrace{\frac{\partial^2 \phi}{\partial x^2}}_{\sim \phi/L^2} \tag{3.2}$$

We obtain

$$\frac{\phi}{t_d} \sim \mu \frac{\phi}{L^2}.$$

Canceling ϕ (assuming it is non-zero) and rearranging gives

$$t_d \sim \frac{L^2}{\mu}. \tag{3.3}$$

Equation 3.3 provides the characteristic diffusion time over a distance L. The quadratic dependence on L demonstrates why diffusion is extremely slow across large distances.

Example: 3.1: Salt Dissolving in Still Water

Consider a small salt crystal is dropped into a quiescent tank of still water, the only mechanism available to mix the dissolved ions is through **molecular diffusion**. Because the surrounding fluid is initially at rest, there are no velocity gradients to generate shear or turbulence; therefore, the mass transfer process is *laminar* in the strictest sense.

The resulting mixing process is governed by Fick's second law; hence, we can assume that,

- Length scale: $L \sim 0.01$ m (1 cm),
- Diffusivity of salt in water: $\mu \sim 1 \times 10^{-9}$ m^2/s.

$$t_d \sim \frac{(0.01)^2}{10^{-9}} = 1 \times 10^5 \text{ s} \approx 28 \text{ hours}$$

This simple estimate shows that, without stirring or turbulence, even small-scale diffusion is slow, highlighting the inefficiency of pure molecular diffusion for mixing in quiescent fluids.

3.6 Turbulent Mixing

Turbulent scalar mixing begins with a gradient of velocity, temperature, or concentration. This gradient acts as a source of instability, initiating fluctuations that lead to mixing. The consequences of mixing are a homogenous mixture where the gradients of temperature, concentration, or any other scalar will diminish. In the limit $t \to \infty$ the mean scalar becomes a spatial constant, all gradients vanish, and further transport proceeds only by molecular mixing. In particular, turbulent kinetic energy (TKE) is produced by velocity shear in the mean flow; the process is known as *mechanical generation* or *shear production*. When adjacent fluid layers move at different velocities, velocity gradients (shear) stretch and tilt turbulent eddies, transferring energy from the mean flow into turbulent fluctuations. This process is distinct from buoyant production, where turbulence arises from density or temperature differences.

We introduce the concept of Reynolds decomposition first.

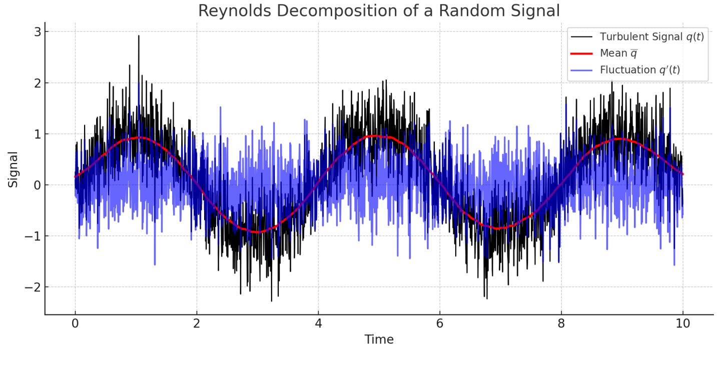

Figure 3.2: Reynolds decomposition of a random scalar signal $q(t)$. The curve $q(t)$ is split as $q(t) = \overline{q} + q'(t)$, where the running mean approximates the time average $\overline{q}$ and the fluctuation is $q'(t) = q(t) - \overline{q}$.

Let $q(t)$ be a turbulent signal (scalar in this example).

We define the long-time average over a long time-period T as,

$$\overline{q}(t) = \frac{1}{T}\int_t^{t+T} q(\tau)\, d\tau, \tag{3.4}$$

for averaging window T large compared to turbulent timescales.

We define the *ensemble average* of a random turbulent signal $q(t)$ as:

$$\langle q(t)\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} q^{(n)}(t), \tag{3.5}$$

where

- $q^{(n)}(t)$ is the n^{th} *realization* (or outcome) of the random process q,
- n is the index labeling each realization,
- N is the total number of realizations.

If we conduct 100 repeated wind-tunnel experiments under the same inlet and boundary conditions, then

$$q^{(1)}(t),\ q^{(2)}(t),\ \ldots,\ q^{(n)}(t)$$

denote the measured velocity signals from the $1^{\text{st}}, 2^{\text{nd}}, \ldots, n^{\text{th}}$ runs, respectively. The ensemble average at time t is then

$$\langle q(t)\rangle = \frac{1}{N}\sum_{n=1}^{N} q^{(n)}(t).$$

In practice, repeated experiments may not be feasible. Instead, we often replace the ensemble average by a *time average*:

$$\overline{q}(t) = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} q(\tau)\, d\tau.$$

The Reynolds averaging is defined as the process of decomposing any instantaneous turbulence quantity into a long-time averaged mean and a fluctuating component. This framework was introduced by Osborne Reynolds. The Reynolds Decomposition is defined as follows:

$$q(t) = \overline{q} + q'(t), \qquad \overline{q'} = 0. \tag{3.6}$$

The following are the properties of Reynolds Averaging. For any a, b and constant c,

$$\begin{aligned} \overline{a+b} &= \overline{a} + \overline{b}, \\ \overline{c\,a} &= c\,\overline{a}, \\ \overline{ab} &= \overline{a}\,\overline{b} + \overline{a'b'}. \end{aligned} \tag{3.7}$$

3.6.1 Reynolds Averaged Scalar Transport Equation

For a passive scalar $\phi(\mathbf{x}, t)$ advected by a mean flow with uniform velocity (U_1,0,0), the instantaneous scalar transport equation advected with mean velocity U_1 is:

$$\frac{\partial \phi}{\partial t} + U_1 \frac{\partial \phi}{\partial x_j} = \alpha\, \frac{\partial^2 \phi}{\partial x_j^2}, \tag{3.8}$$

where $\alpha = \mu/\rho$ is the molecular diffusivity of the scalar.

Reynolds Averaging: Applying Reynolds decomposition (Eq.3.6) for the ,

$$\phi = \overline{\phi} + \phi', \qquad U_1 = \overline{U_1} + u_1' \tag{3.9}$$

Here, $\overline{\phi}$ is the time-averaged mean concentration, and $\overline{U_1}$ is the mean velocity which is the same as the imposed uniform velocity U_1 in this problem; and ϕ' is the scalar fluctuations, and (u_1', u_2', u_3') are the velocity fluctuations. Note that as we are writing the equation for advection with the mean velocity U_1, the components u$_2'$, $u_3' donotappearintheequations.Itshouldbenotedthatwhenwe$

Plug in Eq.3.9 in Eq.3.8,

$$\frac{\partial(\overline{\phi}+\phi')}{\partial t}+(\overline{U_1}+u_1')\frac{\partial\,(\overline{\phi}+\phi')}{\partial x_j}=\alpha\,\frac{\partial^2(\overline{\phi}+\phi')}{\partial x_j^2}. \qquad (3.10)$$

,

We take a time-average of the above equation, and this is represented by the operator $\overline{()}$, resulting in,

$$\overline{\frac{\partial(\overline{\phi}+\phi')}{\partial t}}+\overline{(\overline{U_1}+u_1')\frac{\partial\,(\overline{\phi}+\phi')}{\partial x_j}}=\alpha\,\overline{\frac{\partial^2(\overline{\phi}+\phi')}{\partial x_j^2}}. \qquad (3.11)$$

,

Recognizing the following: $\overline{\frac{\partial(\phi')}{\partial t}}=0$, $\overline{\frac{\partial(\phi')}{\partial x_j}}=0$, $\overline{\frac{\partial(U_1\phi')}{\partial x_j}}=0$, uniform mean velocity it does not vary spatially,the resultant equation is,

$$\frac{\partial\overline{\phi}}{\partial t}+\overline{U}_1\frac{\partial\overline{\phi}}{\partial x_j}=\alpha\,\frac{\partial^2\overline{\phi}}{\partial x_j^2}-\frac{\partial\overline{u_1'\phi'}}{\partial x_j}. \qquad (3.12)$$

The additional term $\overline{u_1'\phi'}$ is the *turbulent scalar flux.*

3.7 Gradient-Diffusion Hypothesis (with a single local scale)

A *single* local turbulence scale (velocity u' and length ℓ) controls transport. The turbulent scalar flux is modeled analogously to Fick's law:

$$\overline{u_1'\phi'}=-\,K_\phi\,\frac{\partial\overline{\phi}}{\partial x_j}, \qquad (3.13)$$

where K_ϕ is the *eddy (turbulent) diffusivity* and κ is the molecular diffusivity.

Substituting (3.13) into (3.12) gives

$$\frac{\partial\overline{\phi}}{\partial t}+U_1\frac{\partial\overline{\phi}}{\partial x_j}=\frac{\partial}{\partial x_j}\left[(\kappa+K_\phi)\,\frac{\partial\overline{\phi}}{\partial x_j}\right], \qquad (3.14)$$

so the effective diffusivity is $\alpha_{\text{eff}} = \kappa + K_\phi$ with $K_\phi \gg \kappa$ in turbulence.

Scaling. Assuming $\phi' \sim \ell\,|\nabla\overline{\phi}|$ and a velocity scale u', we obtain

$$\boxed{K_\phi \;\sim\; C_\phi\, u'\,\ell}, \qquad C_\phi = \mathcal{O}(1), \tag{3.15}$$

and the mixing time over a layer of thickness L:

$$t_{\text{mix}} \;\sim\; \frac{L^2}{\kappa + K_\phi} \;\approx\; \frac{L^2}{u'\,\ell} \;\sim\; \frac{L}{u'} \;\text{ if } L \sim \ell. \tag{3.16}$$

3.8 Eddy Turnover time

Mixing in turbulent flows occurs over a wide range of time-scales. At one extreme, the integral scale represents the largest eddies in the flow. These large motions have sizes comparable to the characteristic dimension of the flow region (e.g., pipe diameter, boundary layer thickness, or stirred container size). The associated integral time-scale is

$$t_L \;\sim\; \frac{L}{u'}, \tag{3.17}$$

which is the time required for a large eddy to turn over. This sets the fastest effective stirring time, since these eddies sweep and fold scalar fields into smaller and smaller scales.

At the other extreme lies molecular diffusion, which operates at the smallest length scales where velocity fluctuations no longer dominate. The diffusive time-scale is

$$t_d \;\sim\; \frac{\ell^2}{\alpha}, \tag{3.18}$$

with ℓ the smallest scalar structures and α the molecular diffusivity. This represents the slowest possible mixing, since molecular transport alone is very inefficient compared to turbulent stirring.

Between these two extremes, turbulence provides a continuous range of scales: large eddies stretch and fold scalar fields into smaller scales until the scalar gradients become so sharp that diffusion acts rapidly. Thus, turbulent mixing time-scales span from the relatively short eddy-turnover time at the integral scale to the very long diffusive time in the absence of stirring.

The key idea is that turbulence accelerates mixing by replacing the very slow diffusive time with a much shorter eddy-turnover time, effectively bridging the two ends of the spectrum.

Schematic Representation

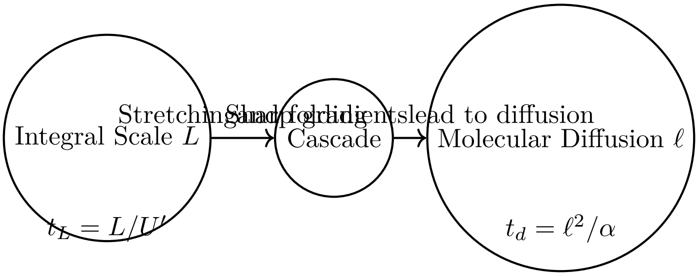

Figure 3.3: Schematic of turbulent mixing time-scales, with integral-scale eddy turnover time at one end and molecular diffusion time at the other. Turbulence bridges these extremes through a cascade of motions.

Eddy turnover time tells us how quickly large swirls can move fluid parcels around, fold them together, and promote mixing. Later, we will see how smaller and faster eddies emerge from these large motions, but for now the integral scale and its turnover time provide the fundamental reference for understanding turbulent mixing.

Key Concept: Turbulent Mixing Time-Scales

To understand turbulent mixing, it is essential to recognize the range of time-scales present in turbulent flows. At the upper end lies the **integral time-scale**, associated with the largest eddies that dominate stirring through vortex motion, stretching, and folding of fluid parcels. At the lower end are the **molecular diffusion time-scales**, where dissipation finally smooths out scalar gradients. Between these two extremes exists a continuum of eddy sizes, each with its own turnover time.

The central concept behind turbulent mixing is the **scalar turbulent flux term**, which represents the correlation between scalar fluctuations and velocity fluctuations. While each turbulent quantity:velocity, scalar, or pressure behaves as a seemingly random signal, their statistical correlations provide the organizing principle that governs mixing. These correlations are the defining characteristic of turbulence, enabling the transport and homogenization of momentum, heat, and scalars far more rapidly than molecular diffusion alone.

In the following chapters, we will analyze these correlations in greater depth and develop the statistical framework that explains how turbulent fluctuations combine to produce coherent fluxes and efficient mixing.

Example 3.3: Stirred Coffee Mixing Time

A typical mug has inner diameter $D \approx 9$ cm (radius $R \approx$ 4.5–5 cm) and liquid depth $\approx$ 9–10 cm. A teaspoon (or stirrer) has width $\approx$ 2–3 cm. A gentle stir produces RMS fluctuations $u' \approx 0.10\ \mathrm{m\,s^{-1}}$.

We need to estimate the *integral length scale* l, the *mixing length* L,turbulent fluctuation u', the turbulent scalar diffusivity

$\alpha_t \sim u'l$ to calculate the mixing time $t_{\text{mix}} \sim \dfrac{L^2}{\alpha_t}$ (from Eq.??.

The dominant energy–containing eddies are set by the shear layer around the spoon/vortex core. A reasonable choice is the spoon width $l \approx 2$. We can assume the mixing length $L \approx R \approx 5$ cm.

We can assume u' to be of 30% of the mean fluid speed generated by the stirring motion. If the spoon moves at roughly 0.3 m/s (U_s) and imparts similar speeds to the surrounding fluid, the kinetic energy per unit mass introduced by stirring is $KE \sim \frac{1}{2}U_s^2$. Assuming this energy is cascaded into turbulent eddies, the characteristic velocity of turbulent fluctuations u' is 30% of the stirring velocity,$u' = k^{\frac{1}{2}} \sim 0.3U_s$.

With $u' = 0.10\ \mathrm{m\,s^{-1}}$, $l = 0.02$ m, $L = 0.05$ m:

$$\alpha_t \approx u'l = (0.10)(0.02) = 2.0 \times 10^{-3}\ \mathrm{m^2\,s^{-1}},$$

$$t_{\text{mix}} \approx \frac{L^2}{\alpha_t} = \frac{(0.05)^2}{2.0 \times 10^{-3}} = \frac{2.5 \times 10^{-3}}{2.0 \times 10^{-3}} \approx 1.25\ \mathrm{s}.$$

Comparison to molecular diffusion. For heat in water, $\alpha \approx 1.4 \times 10^{-7}\ \mathrm{m^2\,s^{-1}}$:

$$t_d \approx \frac{L^2}{\alpha} = \frac{2.5 \times 10^{-3}}{1.4 \times 10^{-7}} \approx 1.8 \times 10^4\ \mathrm{s} \approx \boxed{5\ \mathrm{h}}.$$

Thus, turbulent stirring reduces the mixing time from hours to seconds.

Note that larger l (bigger spoon) or larger u' (faster stir) increases $\alpha_t = u'l$ and shortens t_{mix}.

Example 3.4 : Smoke Dispersal in Turbulent Air

Consider a turbulent atmospheric flow where the mean wind speed is approximately $U_s \sim 5$ m/s. Smoke plume is released into the atmosphere. The plume spread is characterized by a length scale of $L \sim 10$ m, while the integral length scale of the largest eddies is estimated as $l \sim 1$ m. Using these parameters, estimate the characteristic eddy turnover time, the ratio of large-eddy time scales to mean advection time, and discuss how turbulent mixing compares with advection by the mean flow.

In this example, we assume that the density of the smoke is only marginally different from that of the ambient air, hence the buoyancy forces are negligible. The dominant mechanism of mixing is by the turbulent velocity fluctuations.

Let: $U_s \sim 5m/s$ (Wind speed), $u' \sim 1.5$ m/s (turbulent fluctuation velocity is 30% of the wind), $L \sim 10$ m (plume spread), $l \sim 1$ m (integral length scale).

Then, using Eq.**??**

$$\alpha_t \sim u'l = 1.5 \times 1 = 1.5 \text{ m}^2/\text{s}$$
$$t_t \sim \frac{L^2}{\alpha_t} = \frac{100}{1.5} = 66 \text{ s} \approx 1.1 \text{ minutes}$$

Thus, turbulence reduces the mixing time from months to minutes. The mixing is greatly accelerated by the turbulent eddies. Instead of taking hours or days as would be expected with molecular diffusion alone, the scalar becomes uniformly distributed throughout the domain in approximately one minute. This example highlights the efficiency of turbulent mixing and the importance of eddy diffusivity in scalar transport. This is a very simplistic approximation, where we have assumed there is a single eddy that is formed, and the mixing is at the scale of this eddy. However, turbulence results in a cascade of eddies. Energy is injected at the large scale L, transported down to smaller scales via nonlinear interactions, until it reaches the *Kolmogorov scale*, where viscosity dominates and energy is dissipated as heat. The cascade ensures that large-scale stirring rapidly produces small-scale eddies, which are most effective at homogenizing scalars like temperature, dye, or concentration.

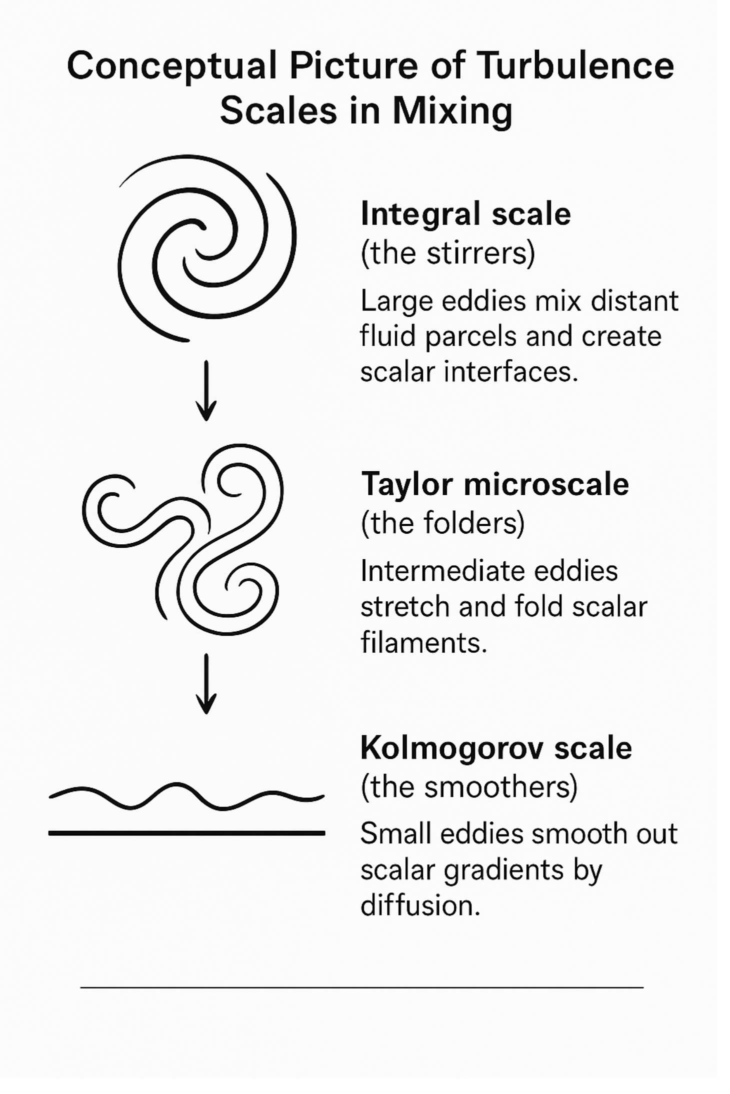

Figure 3.4: Conceptual picture of cascade model of turbulence in mixing: Large integral eddies (L), Intermediate eddies near the Taylor microscale (λ) and smallest Kolmogorov scale (η)

3.9 Buoyancy Generated Mixing

In addition to shear production, turbulence can also be generated through buoyancy. When the ground surface is heated, warmer and lighter air parcels rise while cooler parcels sink. This vertical motion couples temperature (or potential temperature θ) fluctuations directly to the turbulent velocity field.In the atmospheric boundary layer (ABL), turbulent exchange of heat and momentum is governed by surface fluxes and buoyancy generation. Unlike purely shear-driven turbulence, buoyancy couples scalar (temperature) fluctuations directly to turbulent kinetic energy (TKE) through vertical heat fluxes. Quantifying these fluxes and the associated similarity scales is essential for modeling surface–atmosphere exchange, atmospheric stability, and the relevant mixing time-scales.

3.9.1 Buoyancy-Driven Scalar Flux from a Heated Surface

Consider a uniformly heated horizontal plate that imposes a vertical mean thermal gradient. Let $\theta(x, y, z, t)$ denote potential temperature. Let the mean velocity be represented as u_j In a thin surface layer above the plate, horizontal variations in mean fields are weak compared to vertical variations, so we assume horizontal homogeneity: $\partial_x(\cdot) = \partial_y(\cdot) = 0$ for mean quantities, with Boussinesq density variations and $\overline{w} = 0$.

We start with the Instantaneous scalar equation for potential temperature,

$$\frac{\partial \theta}{\partial t} + u_j \frac{\partial \theta}{\partial x_j} = \kappa \nabla^2 \theta + S, \tag{3.19}$$

where κ is the molecular diffusivity and S represents sources/sinks (often negligible in the surface layer away from the wall). Using the Reynolds decomposition Eq.3.6, we now decompose velocity and temperature into mean and fluctuating parts:

$$u_i = \overline{U}_i + u_i', \qquad \theta = \overline{\theta} + \theta'. \tag{3.20}$$

With horizontal homogeneity, $\overline{\theta} = \overline{\theta}(z,t)$ and $\overline{\mathbf{U}} = (\overline{U}(z,t), 0, 0)$. Where, S represents the sources/sinks and κ is the molecular diffusivity The Reynolds-averaged scalar equation is,

$$\begin{aligned}\frac{\partial\overline{\theta}}{\partial t} &+ \underbrace{\overline{u}\,\frac{\partial\overline{\theta}}{\partial x}}_{\approx 0} + \underbrace{\overline{v}\,\frac{\partial\overline{\theta}}{\partial y}}_{\approx 0} + \overline{w}\,\frac{\partial\overline{\theta}}{\partial z} \\ &+ \underbrace{\frac{\partial}{\partial x}\overline{u'\theta'}}_{\ll \frac{\partial}{\partial z}\overline{w'\theta'}} + \underbrace{\frac{\partial}{\partial y}\overline{v'\theta'}}_{\ll \frac{\partial}{\partial z}\overline{w'\theta'}} + \frac{\partial}{\partial z}\overline{w'\theta'} = \kappa\,\nabla^2\overline{\theta} + \overline{S}.\end{aligned} \tag{3.21}$$

If we assume the mean subsidence $\overline{w} = 0$, the resultant equation is,

$$\frac{\partial\overline{\theta}}{\partial t} + \frac{\partial}{\partial z}\overline{w'\theta'} = \kappa\,\frac{\partial^2\overline{\theta}}{\partial z^2} + \overline{S}. \tag{3.22}$$

Dominant vertical-flux assumption. In buoyancy-driven surface layers (heated from below), vertical motions dominate near the wall. Assume

$$w' \gg u',\, v' \quad \text{(for transport)}, \qquad \partial_z\overline{\theta} \neq 0, \qquad \partial_x\overline{\theta} \approx \partial_y\overline{\theta} \approx 0.$$

Then the production of temperature fluctuations is primarily through the coupling term $w'(\partial\overline{\theta}/\partial z)$ in (**??**). Physically:

- Updrafts $w' > 0$ originating from the heated plate carry warmer-than-mean fluid ($\theta' > 0$).
- Downdrafts $w' < 0$ tend to carry cooler-than-mean fluid ($\theta' < 0$).

Both quadrants give a *positive* product $w'\theta'$, which builds a positive covariance $\overline{w'\theta'}$ above a heated surface.

The measurable, mean turbulent flux is

$$q_z^{(\theta)} \equiv \overline{w'\theta'}. \tag{3.23}$$

In gradient-diffusion (eddy-diffusivity) form this is modeled as

$$\overline{w'\theta'} \approx -K_\theta \frac{\partial \overline{\theta}}{\partial z}, \tag{3.24}$$

with K_θ the eddy diffusivity for heat. Over a heated plate, $\partial\overline{\theta}/\partial z < 0$ (temperature decreases with height), so (3.24) gives $\overline{w'\theta'} > 0$, consistent with the quadrant argument above.

Surface heat and buoyancy flux. At the wall $z = 0$, the turbulent flux equals the imposed surface flux (to leading order in the roughness sublayer):

$$Q_H = \rho_0 c_p \overline{w'\theta'}\Big|_{z=0}, \qquad B_0 = \frac{g}{\theta_0} \overline{w'\theta'}\Big|_{z=0}. \tag{3.25}$$

Here, Q_H is the sensible heat flux and B_0 is the surface buoyancy flux. Heating from below implies $Q_H > 0$ and $B_0 > 0$.

With w' dominating the turbulent transport, (3.22) reduces (steady, $\overline{S} \approx 0$) to

$$\frac{d}{dz} \overline{w'\theta'} \approx \kappa \frac{d^2\overline{\theta}}{dz^2}. \tag{3.26}$$

A vertical mean thermal gradient over a heated plate generates temperature fluctuations θ' that are *correlated* with vertical velocity fluctuations w'. Under $w' \gg u', v'$, this produces a dominant, positive turbulent heat flux $\overline{w'\theta'}$ that carries heat upward. This is the key mechanism by which buoyancy generates scalar flux and drives turbulent mixing in the atmospheric surface layer.

Thus, just as mechanical shear produces scalar fluxes through $\overline{u'\phi'}$, buoyancy couples vertical motion and temperature fluctuations to produce $\overline{w'\theta'}$. In the ABL, this process controls the growth of the convective boundary layer, sets characteristic velocity and temperature scales, and provides the essential input for similarity theory.

In later sections, we will use these fluxes to define convective scales such as the Deardorff velocity w_* and to establish the Monin-Obukhov framework for surface-layer similarity.

3.9.2 Convective Velocity and Temperature Scales

In a convective mixed layer of depth H (often denoted z_i), the Deardorff velocity scale is

$$w_* \;=\; (B_0 H)^{1/3}, \tag{3.27}$$

with a corresponding convective temperature scale

$$\theta_*^{(c)} \;=\; \frac{\overline{w'\theta'}|_0}{w_*}. \tag{3.28}$$

These give characteristic times for layer-scale transport:

$$\tau_* \sim \frac{H}{w_*}, \qquad \tau_T \approx \frac{H\,\Delta\theta}{|\overline{w'\theta'}|}, \tag{3.29}$$

where $\Delta\theta$ is a representative mean temperature contrast across H.

3.9.3 Stratification and Brunt-Väisälä Frequency

Stratification is set by the vertical gradient of potential temperature. When $d\theta/dz > 0$ (air aloft is warmer in potential temperature sense), a vertically displaced parcel feels a restoring buoyancy force and oscillates about its original level; this is *stable* stratification. The oscillation rate is the Brunt-Väisälä frequency

$$N = \sqrt{(g/\theta)\,d\theta/dz}, \qquad T_b = 2\pi/N \tag{3.30}$$

T_b is the time-period. For larger N means stronger resistance to vertical motion and suppressed turbulent mixing. When $d\theta/dz < 0$ (warmer near the surface), $N^2 < 0$ and N become imaginary: small vertical displacements grow rather than oscillate, signaling *unstable* stratification with buoyant convection, vigorous vertical transport, and typically positive $\overline{w'\theta'}$.
The *neutral* case has $d\theta/dz \approx 0$ (so $N \approx 0$), implying less buoyant effect and mixing governed primarily by shear.

Example 3.5: Density Currents

Lock Exchange Density Currents When two fluids of slightly different densities are initially separated and then allowed to interact (e.g., cold outdoor air entering a warm room through a gap under a door in winter), the denser fluid flows underneath the lighter fluid, with a well–defined gravity–current front. As the front advances, shear at the interface and buoyancy jointly produce turbulence, but the initial driving is from gravitational potential energy.

Problem (Convective room mixing). Consider a rectangular room of height $H = 2.5$ m in which a cold outdoor gravity current intrudes near the floor. From observations of the interior convective layer, the Deardorff velocity scale is estimated as $w_* \approx 0.33\ \mathrm{m\,s^{-1}}$.

Given:

- Room height: $H = 2.5$ m,
- Convective velocity: $w_* = 0.33\ \mathrm{m\,s^{-1}}$,
- Reference potential temperature: $\theta_0 = 300$ K,
- Air density: $\rho_0 = 1.2\ \mathrm{kg\,m^{-3}}$,
- Specific heat (constant pressure): $c_p = 1005\ \mathrm{J\,kg^{-1}\,K^{-1}}$.

From the Deardorff relation (Eq. 3.25),

$$B_0 = \frac{w_*^3}{H} = \frac{(0.33)^3}{2.5} \approx 1.44 \times 10^{-2}\ \mathrm{m^2\,s^{-3}}. \tag{3.31}$$

Using $B_0 = \dfrac{g}{\theta_0}\,\overline{w'\theta'}$ (Eq. **??**),

$$\overline{w'\theta'} \;=\; \frac{\theta_0}{g}\,B_0 \;\approx\; \frac{300}{9.81}\,(1.437\times10^{-2}) \;\approx\; 4.40\times10^{-1}\ \mathrm{K\,m\,s^{-1}}. \tag{3.32}$$

Define the convective temperature scale (Eq.3.28

$$\theta_*^{(c)} \;\equiv\; \frac{\overline{w'\theta'}}{w_*} \;\approx\; \frac{0.440}{0.33} \;\approx\; 1.33\ \mathrm{K}. \tag{3.33}$$

The (dynamic) sensible heat flux (Eq.3.25) is

$$Q_H \;=\; \rho_0 c_p\,\overline{w'T'} \;\approx\; \rho_0 c_p\,\overline{w'\theta'} \;\approx\; (1.2)(1005)(0.440) \;\approx\; 5.31\times10^{2}\ \mathrm{W\,m^{-2}}. \tag{3.34}$$

Convective (eddy-turnover) time (Eq. 3.27):

$$\tau_* \;=\; \frac{H}{w_*} \;=\; \frac{2.5}{0.33} \;\approx\; 7.6\ \mathrm{s}. \tag{3.35}$$

Example 3.7: Convective Atmospheric Boundary Layer

On a warm summer afternoon over flat terrain, surface sensible-heat fluxes can reach

$$Q_0 \;\approx\; 500\ \mathrm{W\,m^{-2}}.$$

For air at $T_0 \approx 300$ K and density $\rho \approx 1.2\ \mathrm{kg\,m^{-3}}$, the corresponding *buoyancy flux* (Eq.3.25 is

$$B \;=\; \frac{g}{T_0}\,\frac{Q_0}{\rho\,c_p} \;=\; \frac{9.81}{300}\,\frac{500}{(1.2)(1005)} \;\approx\; 1.35\times10^{-2}\ \mathrm{m^2\,s^{-3}}.$$

If the mixed-layer depth is $H \approx 1200$ m, the Deardorff convective velocity scale is

$$w_* \;=\; (B\,H)^{1/3} \;=\; \left[(1.35\times10^{-2})(1200)\right]^{1/3} \;\approx\; 2.6\ \mathrm{m\,s^{-1}}.$$

The turnover time of the largest eddies spanning the ABL depth is calculated using Eq.3.27

$$\tau_{\mathrm{turn}} = \frac{H}{w_*} \approx \frac{1200}{2.6} \approx 460 \text{ s } (\sim 8 \text{ min}).$$

This means that within about 8 minutes, a parcel can traverse the full ABL depth. Several such turnovers are typically needed for complete homogenization of heat, moisture, and tracers in the boundary layer.

In this convective ABL, turbulence is generated primarily by buoyancy due to surface heating. The energy-containing eddies span the entire mixed-layer depth, efficiently redistributing scalars. Small UAVs flying at 50–200 m altitude may encounter vertical gusts approaching $\pm w_*$, making stable flight control challenging.

Nomenclature

ϕ	Scalar quantity (e.g., temperature, concentration, salinity)
$\overline{\phi}$	Mean (time-averaged) scalar
$\mathbf{u}$	Instantaneous velocity vector
$\overline{\mathbf{u}}$	Mean velocity vector
u_i'	Fluctuating velocity component in the i direction
u'	Root-mean-square (RMS) turbulent velocity fluctuation
l	Integral length scale (size of the largest eddies)
L	Characteristic length scale of the mixing region
μ	Molecular diffusivity
μ_t	Eddy diffusivity (turbulent diffusivity)
k	Turbulent kinetic energy, $u' = \sqrt{k}$
t_{eddy}	Eddy turnover time
t_{mix}	Turbulent mixing time scale
t_d	Molecular-diffusion time scale
ε	Turbulent kinetic energy dissipation rate
u_l	Velocity fluctuation at scale l
t_l	Turnover time at scale l
g	Acceleration due to gravity
T'	Temperature fluctuation
w'	Vertical velocity fluctuation
T_0	Reference temperature
C	Scalar concentration (generic)
Q	Source strength or flux (when used in examples)
ρ	Density
Pr_t	Turbulent Prandtl number, $Pr_t = \mu_t/\alpha_t$
α_t	Turbulent thermal diffusivity

End-of-Chapter Problems

1. **Mixing-length baseline.** For a scalar with molecular diffusivity α in a turbulent region of size L, assume $\alpha_t \approx u'l$ with u' and l known. Show that $t_{\text{mix}} \sim L^2/(u'l)$. Evaluate t_{mix} and the ratio t_d/t_{mix} for $L = 0.05$ m, $u' = 0.10$ m s^{-1}, $l = 0.02$ m, and (i) heat in water $\alpha = 1.4 \times 10^{-7}$ m^2 s^{-1}, (ii) mass diffusion in water $D = 1 \times 10^{-9}$ m^2 s^{-1}.

2. **Stirred cup variants.** Using the stirred-coffee model, recompute t_{mix} for (a) a larger spoon $l = 0.03$ m, (b) faster stirring $u' = 0.2$ m s^{-1}, and (c) vertical mixing limited by depth $L = 0.10$ m with the baseline u', l. Comment on which change most reduces t_{mix}.

3. **Convective ABL scaling.** Given surface heat flux $Q_0 = 500$ W m^{-2}, air density $\rho = 1.2$ kg m^{-3}, $c_p = 1005$ J kg^{-1} K^{-1}, $T_0 = 300$ K, and mixed-layer depth $H = 1200$ m: (a) compute the buoyancy flux $B = (g/T_0)\,Q_0/(\rho c_p)$; (b) find the Deardorff velocity $w_* = (BH)^{1/3}$; (c) estimate the turnover time $\tau_{\text{turn}} = H/w_*$; (d) using $\mu_T \approx 0.1\,w_* H$, estimate $t_{\text{mix}} \approx H^2/(2\mu_T)$.

4. **Rayleigh–Bénard convection.** For water at 30°C with $\beta = 2.1 \times 10^{-4}$ K^{-1}, $\nu = 1.0 \times 10^{-6}$ m^2 s^{-1}, $\alpha = 1.4 \times 10^{-7}$ m^2 s^{-1}, layer depth $H = 0.10$ m, and $\Delta T = 10$ K: (a) compute Ra and state the regime; (b) estimate $w_* = (g\beta\Delta T H)^{1/3}$; (c) compute $\tau_{\text{turn}} = H/w_*$; (d) estimate $\mu_T \approx 0.1\,w_* H$ and $t_{\text{mix}} \approx H^2/(2\mu_T)$.

5. **Door-gap density current.** Cold air at $T_{\text{ext}} = 278$ K intrudes under a door into a room at $T_{\text{room}} = 298$ K. Gap depth $h = 0.02$ m. (a) Compute reduced gravity $g' = g\,\Delta T/T_0$ with $T_0 = 298$ K. (b) Estimate the front speed using $\text{Fr}_d \approx 1$: $U \approx \sqrt{g'h}$. (c) If the measured $w_* = 0.33$ m s^{-1} and $H = 2.5$ m, estimate

$\overline{w'T'} \approx w_*^3/[(g/T_0)H]$ and $P_b = (g/T_0)\overline{w'T'}$. (d) Compute $\tau_* = H/w_*$ and discuss mixing time compared with (b).

6. **Fuel–air premixing (jet in crossflow).** A jet of diameter $h = 0.02$ m injects fuel into air with $\Delta U = 40\ \mathrm{m\,s^{-1}}$, air density $\rho = 1.18\ \mathrm{kg\,m^{-3}}$, viscosity $\mu = 1.8 \times 10^{-5}$ Pa s, mass diffusivity $D_m = 7 \times 10^{-5}\ \mathrm{m^2\,s^{-1}}$, reduced gravity $g' = 8.41\ \mathrm{m\,s^{-2}}$. Compute $Re = \rho \Delta U h/\mu$, $Sc = \nu/D_m$, $\mathrm{Fr}_d = \Delta U/\sqrt{g'h}$, and the eddy time $t_e \sim h/\Delta U$. Comment on which mechanism dominates initial mixing.

7. **Integral time scale from autocorrelation.** A probe records $u'(t)$ at $\Delta t = 0.1$ s. The normalized autocorrelation at lags $m = 0, 1, 2, 3, 4, 5$ is $\rho = [1,\ 0.7,\ 0.45,\ 0.25,\ 0.10,\ 0.00]$. Estimate the Eulerian integral time scale $T_E \approx \Delta t \sum_{m=0}^{m_0} \rho(m)$ (to the first zero crossing m_0). With $u' = 0.3\ \mathrm{m\,s^{-1}}$, estimate $\alpha_t \sim u'^2 T_E$ and $l \sim u' T_E$.

8. **Room mixing via covariance.** In a room of height $H = 2.5$ m with measured $w_* = 0.33\ \mathrm{m\,s^{-1}}$ and $T_0 = 298$ K: (a) estimate $\overline{w'T'} \approx w_*^3/[(g/T_0)H]$; (b) compute $P_b = (g/T_0)\overline{w'T'}$; (c) find $\tau_* = H/w_*$; (d) for a bulk temperature difference $\Delta T = 5$ K, estimate $\tau_{\mathrm{mix}} \sim (gH/T_0)\,\Delta T/P_b$. Comment on the relation between τ_* and τ_{mix}.

Notation (extended)

ϕ	Generic scalar (temperature T, potential temperature θ, salinity S, concentration c); units problem-dependent.
$\mathbf{u}$, u_i	Velocity vector and components ($\mathrm{m\,s^{-1}}$).
κ	Molecular diffusivity of ϕ ($\mathrm{m^2\,s^{-1}}$).
K_ϕ	Eddy (turbulent) diffusivity of ϕ ($\mathrm{m^2\,s^{-1}}$).
K_m	Eddy viscosity ($\mathrm{m^2\,s^{-1}}$); $Pr_t = K_m/K_\phi$.
S_ϕ	Volumetric source/sink of ϕ (units of $\phi\,\mathrm{s^{-1}}$).
$q_i^{(\phi)}$	Turbulent scalar flux $\overline{u_i'\phi'}$ (units of $\phi\,\mathrm{m\,s^{-1}}$).
b	Buoyancy $= -g\rho'/\rho_0$ ($\mathrm{m\,s^{-2}}$); $\approx g\,\theta'/\theta_0$ (dry air).
$\overline{w'b'}$	Buoyancy flux ($\mathrm{m\,s^{-3}}$); $= \frac{g}{\theta_0}\overline{w'\theta'}$.
B_0	Surface buoyancy flux $\overline{w'b'}\vert_{z=0}$ ($\mathrm{m\,s^{-3}}$).
Q_H	Surface sensible heat flux $\rho_0 c_p\,\overline{w'T'}\vert_{z=0}$ ($\mathrm{W\,m^{-2}}$).
k	Turbulent kinetic energy $\frac{1}{2}\overline{u_i'u_i'}$ ($\mathrm{m^2\,s^{-2}}$).
P_s, ε	Shear production and dissipation of TKE ($\mathrm{m^2\,s^{-3}}$).
L, H	Vertical or characteristic mixing thickness/depth (m).
L_{int}	Turbulent integral length scale (m); u': rms velocity ($\mathrm{m\,s^{-1}}$).
α_t	Effective (eddy) diffusivity $\kappa + K_\phi$ ($\mathrm{m^2\,s^{-1}}$).
w_*	Convective velocity scale $(B_0 H)^{1/3}$ ($\mathrm{m\,s^{-1}}$).
τ_{diff}	Diffusive mixing time L^2/α_t (s).
τ_{flux}	Flux-controlled adjustment time $H\,\Delta\phi/\vert\overline{w'\phi'}\vert$ (s).
τ_e	Eddy-turnover time L_{int}/u' (s); $\tau_* = H/w_*$ (s).