CFD 100 Series

An introduction to Computational Fluid Dynamics

Lecture 02:

Navier-Stokes Equations: Steady diffusion

1.1
有限差分法 Finite Difference Method
CFD模拟流程

- 前处理
  - 定义求解域，CAD几何建模
  - 离散求解域，生成网格
- CFD模拟
  - 离散控制方程，有限差分、有限体积、有限元
  - 网格点的离散控制方程
  - 组装、求解线性方程组
- 后处理
  - 流线图、速度矢量图
  - 升力、阻力

1.1
有限差分法 Finite Difference Method
离散控制方程

- 用适当的**代数差分**（有限差分）来代替控制方程中的**偏导数**
- 用**泰勒级数**展开可以推导出偏导数的有限差分形式

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \\ \dfrac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u}\vec{u}) = -\nabla \mathrm{p} + \mu \nabla^2 \vec{u} + \vec{f_b} \end{cases}$$

$$\frac{\partial (\rho \phi)}{\partial t} + \nabla \cdot (\rho \vec{u} \phi) = \mu \nabla^2 \phi + Q^{\phi}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u u}{\partial x} + \frac{\partial \rho v u}{\partial y} = -\frac{\partial \mathrm{p}}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial \rho u v}{\partial x} + \frac{\partial \rho v v}{\partial y} = -\frac{\partial \mathrm{p}}{\partial y} + \mu \frac{\partial^2 v}{\partial x^2} + \mu \frac{\partial^2 v}{\partial y^2}$$

1.1

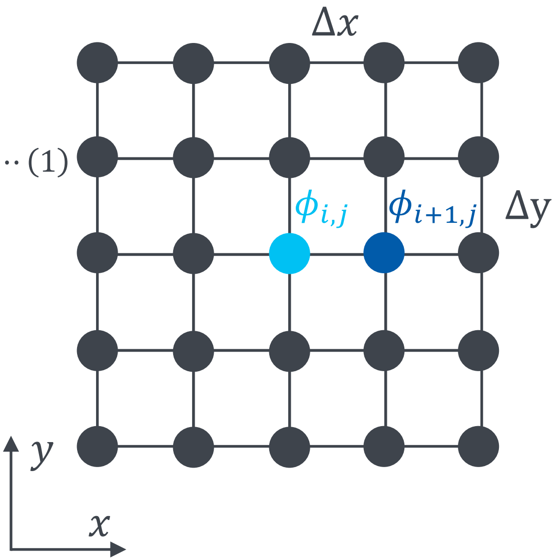
有限差分法 Finite Difference Method

离散控制方程： 一阶偏导数

$$\phi_{i+1,j} = \phi_{i,j} + \left(\frac{\partial\phi}{\partial x}\right)_{i,j}\Delta x + \left(\frac{\partial^2\phi}{\partial x^2}\right)_{i,j}\frac{(\Delta x)^2}{2!} + \left(\frac{\partial^3\phi}{\partial x^3}\right)_{i,j}\frac{(\Delta x)^3}{3!} + \cdots (1)$$

$$\left(\frac{\partial\phi}{\partial x}\right)_{i,j} = \underbrace{\frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x}}_{\text{差分表达式}} \underbrace{- \left(\frac{\partial^2\phi}{\partial x^2}\right)_{i,j}\frac{\Delta x}{2} - \left(\frac{\partial^3\phi}{\partial x^3}\right)_{i,j}\frac{(\Delta x)^2}{3!} + \cdots}_{\text{截断误差}} (2)$$

$$\left(\frac{\partial\phi}{\partial x}\right)_{i,j} \approx \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x} + \mathcal{O}(\Delta x) (3)$$



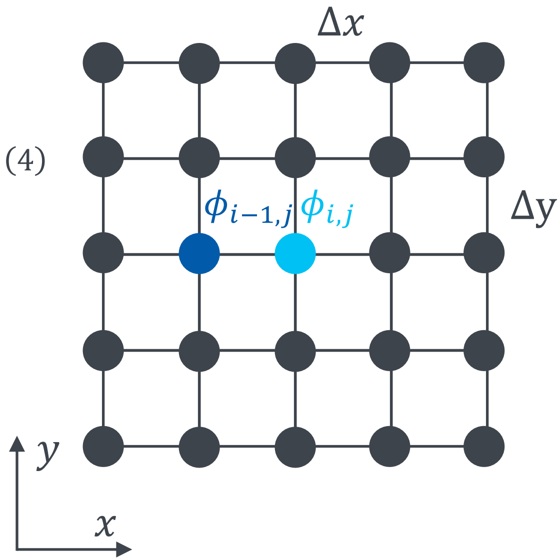
- 一阶精度
- 向前差分

1.1
有限差分法 Finite Difference Method
离散控制方程：一阶偏导数

$$\phi_{i-1,j} = \phi_{i,j} - \left(\frac{\partial \phi}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 \phi}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2!} - \left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i,j} \frac{\Delta x^3}{3!} + \cdots (4)$$

$$\left(\frac{\partial \phi}{\partial x}\right)_{i,j} = \underbrace{\frac{\phi_{i,j} - \phi_{i-1,j}}{\Delta x}}_{\text{差分表达式}} + \underbrace{\left(\frac{\partial^2 \phi}{\partial x^2}\right)_{i,j} \frac{\Delta x}{2} - \left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i,j} \frac{\Delta x^2}{3!}}_{\text{截断误差}} + \cdots (5)$$

$$\left(\frac{\partial \phi}{\partial x}\right)_{i,j} \approx \frac{\phi_{i,j} - \phi_{i-1,j}}{\Delta x} + \mathcal{O}(\Delta x) (6)$$



- 一阶精度
- 向后差分

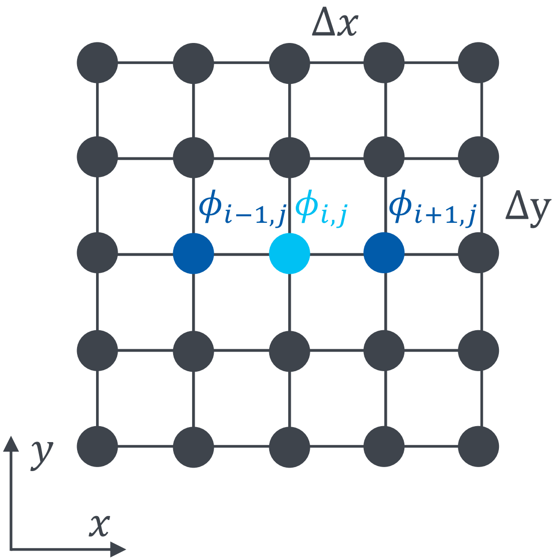
1.1
有限差分法 Finite Difference Method
离散控制方程：一阶偏导数

$$\phi_{i+1,j} - \phi_{i-1,j} = 2\left(\frac{\partial \phi}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{3!} + \cdots (7)$$

$$\left(\frac{\partial \phi}{\partial x}\right)_{i,j} = \underbrace{\frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x}}_{\text{差分表达式}} \underbrace{- \left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^2}{3} + \cdots}_{\text{截断误差}} (8)$$

$$\left(\frac{\partial \phi}{\partial x}\right)_{i,j} \approx \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x} + \mathcal{O}(\Delta x^2) (3)$$

- 二阶精度
- 中心差分



1.1

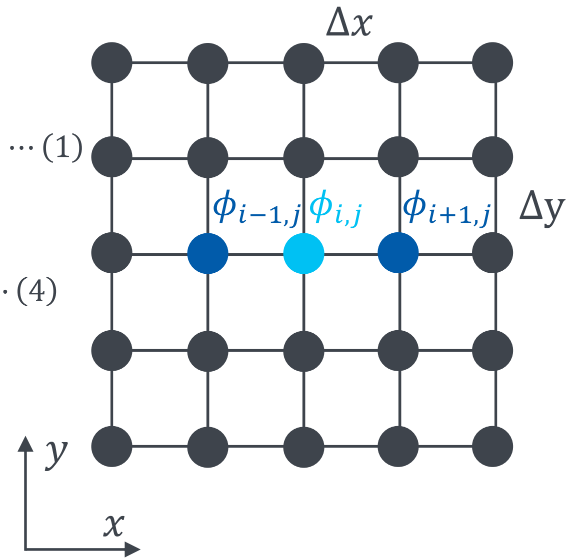
有限差分法 Finite Difference Method

离散控制方程：二阶偏导数

$$\phi_{i+1,j} = \phi_{i,j} + \left(\frac{\partial \phi}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 \phi}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2!} + \left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{3!} + \cdots (1)$$

$$\phi_{i-1,j} = \phi_{i,j} - \left(\frac{\partial \phi}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 \phi}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2!} - \left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i,j} \frac{\Delta x^3}{3!} + \cdots (4)$$

$$\left(\frac{\partial^2 \phi}{\partial x^2}\right)_{i,j} = \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} + \mathcal{O}(\Delta x^2) (9)$$



1.2
有限差分法 Finite Difference Method
一维稳态导热问题

环境温度T_∞

$x = x_L$ $x = x_R$

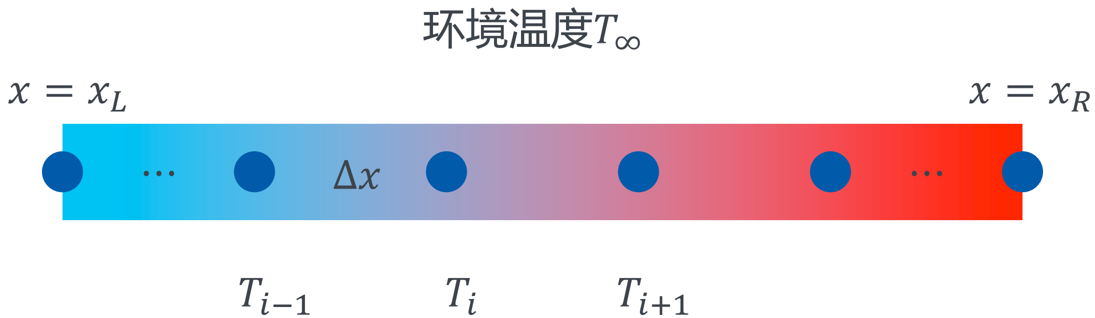
$T = T_L$ $T = T_R$

$$\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} = \frac{hP}{\lambda A_c}(T - T_\infty) \to \frac{\mathrm{d}^2 T}{\mathrm{d}x^2} = H(T - T_\infty)$$

1.2

有限差分法 Finite Difference Method

一维稳态导热问题



$$\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} = H(T - T_\infty) \rightarrow \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} = H(T_i - H_\infty)$$

$$T_{i+1} - (2 + H\Delta x^2)T_i + T_{i-1} = -H\Delta x^2 T_\infty$$

1.2
有限差分法 Finite Difference Method
一维稳态导热问题

$$T_{i+1} - (2 + H\Delta x^2)T_i + T_{i-1} = -H\Delta x^2 T_\infty$$

$$T = T_L, T = T_R$$

$$T_1 = T_L$$

$$T_3 - (2 + H\Delta x^2)T_2 + T_1 = -H\Delta x^2 T_\infty$$

$$T_4 - (2 + H\Delta x^2)T_3 + T_3 = -H\Delta x^2 T_\infty$$

$$\vdots$$

$$T_n - (2 + H\Delta x^2)T_{n-1} + T_{n-2} = -H\Delta x^2 T_\infty$$

$$T_n = T_R$$

1.2
有限差分法 Finite Difference Method
一维稳态导热问题

$$T_{i+1} - (2 + H\Delta x^2)T_i + T_{i-1} = -H\Delta x^2 T_\infty$$

$$T = T_L, T = T_R$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -(2 + H\Delta x^2) & 1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 1 & -(2 + H\Delta x^2) & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_{n-1} \\ T_n \end{pmatrix} = \begin{pmatrix} T_L \\ -H\Delta x^2 T_\infty \\ \vdots \\ -H\Delta x^2 T_\infty \\ T_R \end{pmatrix}$$