CFD 100 Series

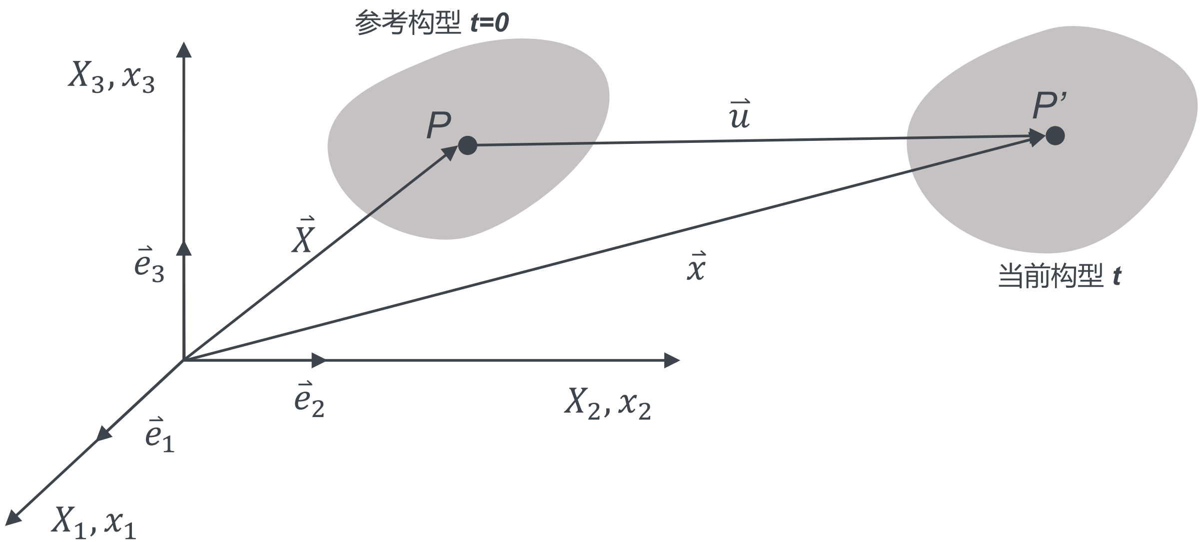
An introduction to Computational Fluid Dynamics

Lecture 01:

A brief derivation of the governing equations

1.1
描述运动的方法
参考构型和当前构型

- **参考构型**下***P***点的
  - 位置矢量 $\vec{\boldsymbol{X}} = X_1\vec{e}_1 + X_2\vec{e}_2 + X_3\vec{e}_3$
  - 物质坐标 $X_i = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$
- **当前构型**下***P***点的
  - 位置矢量 $\vec{\boldsymbol{x}} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$
  - 空间坐标 $x_i = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$



1.1
描述运动的方法
拉格朗日法和欧拉法

- 拉格朗日法（颗粒）
  - 颗粒*P*的运动方程用**拉格朗日变量**表示
  - $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}\left(\vec{\boldsymbol{X}}, t\right)$ 或
  - $x_i = x_i(X_1, X_2, X_3, t)$

相互转化

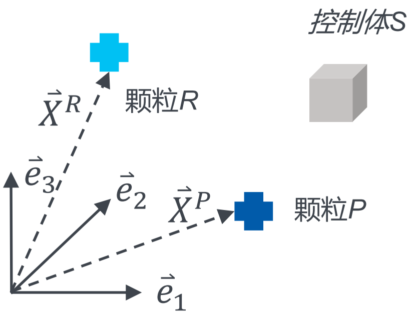
- 欧拉法（空间点）
  - 颗粒*P*的运动方程用**欧拉变量**表示
  - $\vec{\boldsymbol{X}} = \vec{\boldsymbol{X}}(\vec{\boldsymbol{x}}, t)$ 或
  - $X_i = X_i(x_1, x_2, x_3, t)$

- 拉格朗日法给出了在***t*=*0***时刻，即参考构型下坐标为(X_1, X_2, X_3)的**颗粒***P*，在*t*时刻位置坐标，着眼于追踪每个**质点**
- 欧拉法给出了在*t*时刻，占据坐标为(x_1, x_2, x_3)的**空间点**的**颗粒**，着眼于每个**空间点**
- 根据*Jacobian*行列式来判断运动方程是否合适：
  - $J = \left|\frac{\partial x_i}{\partial X_i}\right|$

1.1
描述运动的方法
拉格朗日法和欧拉法

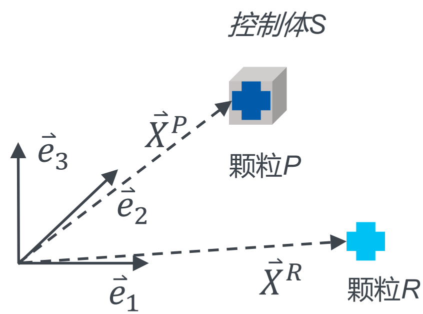
$$\vec{\boldsymbol{x}}\left(\vec{\boldsymbol{X}}^{P}, t_0\right) = \vec{\boldsymbol{x}}^{P}$$

参考构型 *t=0*



$$\vec{\boldsymbol{x}}\left(\vec{\boldsymbol{X}}^{P}, t_1\right) = \vec{\boldsymbol{x}}^{P}$$

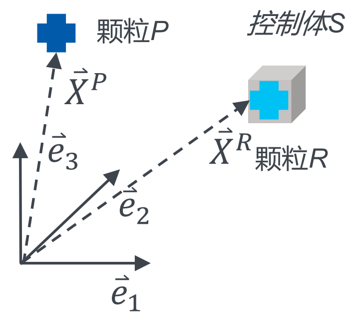
当前构型*t=t₁*



$$\vec{\boldsymbol{X}}\left(\boldsymbol{x}^{S}, t_1\right) = \vec{\boldsymbol{X}}^{P}$$

$$\vec{\boldsymbol{x}}\left(\vec{\boldsymbol{X}}^{P}, t_2\right) = \vec{\boldsymbol{x}}^{P}$$

当前构型 $t=t_2$



$$\vec{\boldsymbol{X}}\left(\boldsymbol{x}^{S}, t_2\right) = \vec{\boldsymbol{X}}^{R}$$

$$\begin{cases} x_1\left(\vec{\boldsymbol{X}}, t\right) = X_2 t^2 + X_1 \\ x_2\left(\vec{\boldsymbol{X}}, t\right) = X_3 t + X_2 \\ x_3\left(\vec{\boldsymbol{X}}, t\right) = X_3 \end{cases} \xleftrightarrow{\text{相互转化}} \begin{cases} X_1\left(\vec{\boldsymbol{x}}, t\right) = x_1 - t^2 x_1 + t^3 x_3 \\ X_2\left(\vec{\boldsymbol{x}}, t\right) = x_2 + t x_3 \\ X_3\left(\vec{\boldsymbol{x}}, t\right) = x_3 \end{cases}$$

1.1
描述运动的方法
物质导数：以温度为例

$$T=T(\vec{\boldsymbol{X}},t)=T\left(\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t),t\right)$$

$$\begin{aligned}\dot{T}=\frac{\mathrm{d}T\left(\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t),t\right)}{\mathrm{d}t}&=\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial t}+\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial x_1}\frac{\partial x_1(\vec{\boldsymbol{X}},t)}{\partial t}+\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial x_2}\frac{\partial x_2(\vec{\boldsymbol{X}},t)}{\partial t}+\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial x_3}\frac{\partial x_3(\vec{\boldsymbol{X}},t)}{\partial t}\\&=\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial t}+\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial x_1}v_1(\vec{\boldsymbol{X}},t)+\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial x_2}v_2(\vec{\boldsymbol{X}},t)+\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial x_3}v_3(\vec{\boldsymbol{X}},t)\\&=\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial t}+\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial x_1}v_1(\vec{\boldsymbol{x}},t)+\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial x_2}v_2(\vec{\boldsymbol{x}},t)+\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial x_3}v_3(\vec{\boldsymbol{x}},t)\\&=\underbrace{\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial t}}_{\text{当地导数}}+\underbrace{\frac{\partial T(\vec{\boldsymbol{x}},t)}{\partial x_k}v_k(\vec{\boldsymbol{x}},t)}_{\text{迁移导数}}\end{aligned}$$

$$\frac{\mathrm{d}\phi(\vec{\boldsymbol{x}},t)}{\mathrm{d}t}=\frac{\partial\phi(\vec{\boldsymbol{x}},t)}{\partial t}+\frac{\partial\phi(\vec{\boldsymbol{x}},t)}{\partial x_k}v_k(\vec{\boldsymbol{x}},t)$$

物质导数的一般形式，**温度**、**密度**、**速度**

1.1
描述运动的方法
物质导数：例子

$$\begin{cases} x_1 = X_1(1+t) \\ x_2 = X_2(1+t) \\ x_3 = X_3 \end{cases} \qquad T(\vec{x}) = {x_1}^2 + {x_2}^2$$

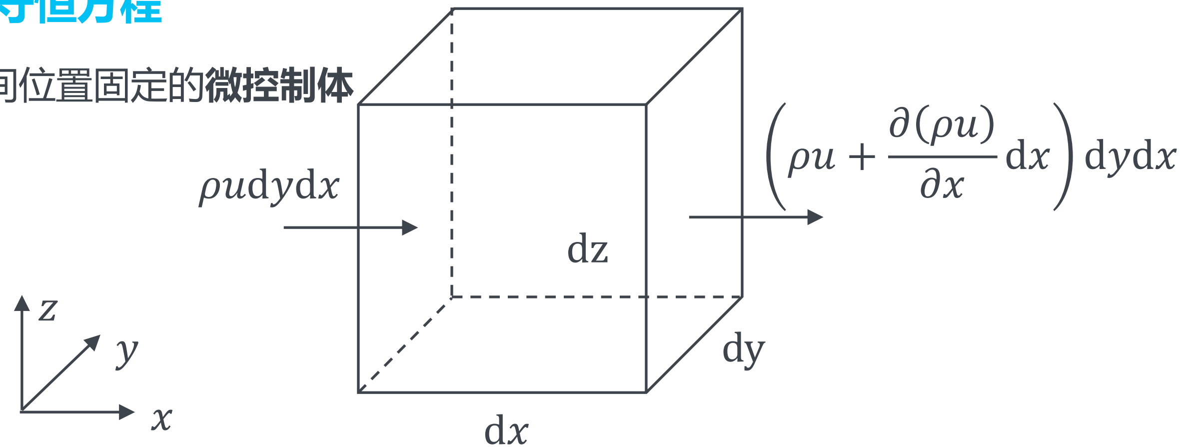
计算t=0时坐标为(X_1=3, X_2=1, X_3=0)的颗粒P，在t=1时温度的物质导数。
要求用**拉格朗日描述的物质导数**和**欧拉描述的物质导数**两种方法计算。

1.2

控制方程

质量守恒方程

取空间位置固定的**微控制体**



$$\Delta m_x=\left(\rho u+\frac{\partial(\rho u)}{\partial x}\mathrm{d}x\right)\mathrm{d}y\mathrm{d}x-\rho u\mathrm{d}y\mathrm{d}x=\frac{\partial(\rho u)}{\partial x}\mathrm{d}x\mathrm{d}y\mathrm{d}x$$

$$\Delta m_y=\left(\rho v+\frac{\partial(\rho v)}{\partial y}\mathrm{d}y\right)\mathrm{d}z\mathrm{d}x-\rho u\mathrm{d}z\mathrm{d}x=\frac{\partial(\rho v)}{\partial y}\mathrm{d}x\mathrm{d}y\mathrm{d}x$$

$$\Delta m_z=\left(\rho w+\frac{\partial(\rho w)}{\partial z}\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y-\rho w\mathrm{d}x\mathrm{d}y=\frac{\partial(\rho w)}{\partial z}\mathrm{d}x\mathrm{d}y\mathrm{d}x$$

1.2
控制方程
质量守恒方程

流出的总质量：
$$\begin{cases} \Delta m_x = \dfrac{\partial(\rho u)}{\partial x}\mathrm{d}x\mathrm{d}y\mathrm{d}x \\ \Delta m_y = \dfrac{\partial(\rho v)}{\partial y}\mathrm{d}x\mathrm{d}y\mathrm{d}x \\ \Delta m_z = \dfrac{\partial(\rho w)}{\partial z}\mathrm{d}x\mathrm{d}y\mathrm{d}x \end{cases}$$

质量的**减少量**：$-\dfrac{\partial \rho}{\partial t}\mathrm{d}x\mathrm{d}y\mathrm{d}z$

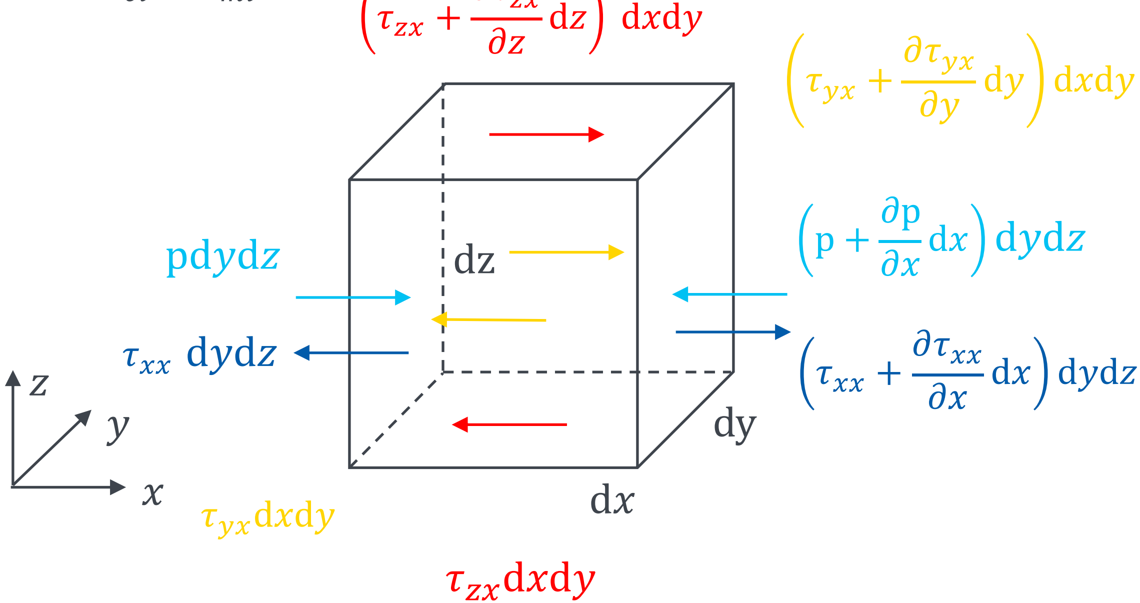
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho u)}{\partial x} = 0 \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \qquad \nabla \cdot (\rho \vec{u}) = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = \frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho(\nabla \cdot \vec{u}) = 0 \qquad \nabla \cdot \vec{u} = 0$$

1.2
控制方程
动量守恒方程

取随流体流动的**流体微团**，对其使用牛顿第二定律$\vec{F} = m\vec{a}$

取某一时刻，该**流体微团**在流动过程中占据了某**一微控制体**，则两者的受力是相同的，即$\vec{F}_{cv} = \vec{F}_{mv}$



1.2

控制方程

动量守恒方程

$$f_x = \left(\mathrm{p} - \left(\mathrm{p} + \frac{\partial \mathrm{p}}{\partial x}\mathrm{d}x\right)\right)\mathrm{d}y\mathrm{d}z + \left(\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}\mathrm{d}x\right) - \tau_{xx}\right)\mathrm{d}y\mathrm{d}z + \left(\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\mathrm{d}y\right) - \tau_{yx}\right)\mathrm{d}x\mathrm{d}z + \left(\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\mathrm{d}z\right) - \tau_{zx}\right)\mathrm{d}x\mathrm{d}y$$

$$F_{body,x} = f_x \rho \mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad m a_x = \rho \mathrm{d}x\mathrm{d}y\mathrm{d}z \frac{\mathrm{d}u}{\mathrm{d}t}$$

$$\begin{cases} \rho \dfrac{\mathrm{d}u}{\mathrm{d}t} = -\dfrac{\partial \mathrm{p}}{\partial x} + \dfrac{\partial \tau_{xx}}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} + f_x \rho \\ \rho \dfrac{\mathrm{d}v}{\mathrm{d}t} = -\dfrac{\partial \mathrm{p}}{\partial y} + \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \tau_{yy}}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} + f_y \rho \\ \rho \dfrac{\mathrm{d}w}{\mathrm{d}t} = -\dfrac{\partial \mathrm{p}}{\partial z} + \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \tau_{zz}}{\partial z} + f_z \rho \end{cases}$$

- 动量守恒方程的非守恒形式
  - 通过随流体运动的流体微团形式推导得到

1.2
控制方程
动量守恒方程

牛顿流体的本构方程：

$$\tau_{xx}=2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu(\nabla\cdot\vec{u})\qquad \tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$\tau_{yy}=2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu(\nabla\cdot\vec{u})\qquad \tau_{xz}=\tau_{zx}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$

$$\tau_{zz}=2\mu\frac{\partial w}{\partial z}-\frac{2}{3}\mu(\nabla\cdot\vec{u})\qquad \tau_{yz}=\tau_{zy}=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)$$

$$\begin{cases}\rho\dfrac{\mathrm{d}u}{\mathrm{d}t}=-\dfrac{\partial p}{\partial x}+\mu\dfrac{\partial^2 u}{\partial x^2}+f_x\rho\\[2ex]\rho\dfrac{\mathrm{d}v}{\mathrm{d}t}=-\dfrac{\partial p}{\partial y}+\mu\dfrac{\partial^2 v}{\partial y^2}+f_y\rho\\[2ex]\rho\dfrac{\mathrm{d}w}{\mathrm{d}t}=-\dfrac{\partial p}{\partial z}+\mu\dfrac{\partial^2 w}{\partial z^2}+f_z\rho\end{cases}$$

$$\frac{\partial(\rho\phi)}{\partial t}+\nabla\cdot(\rho\phi\vec{u})=\rho\left(\frac{\partial\phi}{\partial t}+\vec{u}\cdot\nabla\phi\right)+\phi\left(\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{u})\right)=\rho\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

$$\begin{cases}\dfrac{\partial(\rho u)}{\partial t}+\dfrac{\partial(\rho u\vec{u})}{\partial x}=-\dfrac{\partial p}{\partial x}+\mu\dfrac{\partial^2 u}{\partial x^2}+f_x\rho\\[2ex]\dfrac{\partial(\rho v)}{\partial t}+\dfrac{\partial(\rho v\vec{u})}{\partial y}=-\dfrac{\partial p}{\partial y}+\mu\dfrac{\partial^2 v}{\partial y^2}+f_y\rho\\[2ex]\dfrac{\partial(\rho w)}{\partial t}+\dfrac{\partial(\rho w\vec{u})}{\partial y}=-\dfrac{\partial p}{\partial z}+\mu\dfrac{\partial^2 w}{\partial z^2}+f_z\rho\end{cases}$$

- 动量守恒方程的守恒形式

1.3
控制方程
一般形式的守恒方程

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \\ \dfrac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u}\vec{u}) = -\nabla \mathrm{p} + \mu \nabla^2 \vec{u} + \vec{f}_b \end{cases}$$

- 守恒形式的：
  - 质量守恒方程
  - 动量守恒方程

$$\underbrace{\frac{\partial (\rho \phi)}{\partial t}}_{\text{非稳态项}} + \underbrace{\nabla \cdot (\rho \vec{u} \phi)}_{\text{对流项}} = \underbrace{\mu \nabla^2 \phi}_{\text{扩散项}} + \underbrace{Q^{\phi}}_{\text{源项}}$$

- 扩散是由于分子热运动所致，在不同方向的几率都是一样的，因此扩散过程会在**各个方向传递**
- 而对流是流体微团的定向流动过程，因此具有一定的**方向性**

1.4
参考资料

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- 王宏伟. *我所理解的流体力学*. 国防工业出版社, 2015. Page 56 - 74
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