E. M. SODANO
U. S. Army Engineer
(G. I. M. R. A. D. A.)
Fort Belvoir, Virginia

GENERAL NON-ITERATIVE SOLUTION OF THE INVERSE AND DIRECT GEODETIC PROBLEMS

Summary

Improved practical and theoretical formulas are presented for the calculation of geodetic distances, azimuths, and positions on a spheroid. The formulas are designed for use with either electronic computers or desk calculators. For the latter, the formulas lend themselves to the construction of useful interpolation tables.

The report includes convenient computation forms and auxiliary equations which assure a high degree of accuracy for any geodetic line, no matter how short or long (up to half or fully around the earth) and regardless of its orientation or location. Numerical examples illustrate the complete calculation procedure.

I. Introductory Background

Earlier, at the Army Map Service, the writer published a comprehensive study [1] for the rigorous non-iterative inverse solution of long geodesics, following its presentation at the XI General Assembly of the International Union of Geodesy and Geophysics. (At present, a translation by the German Geodetic Commission and an abstract by the U.S.S.R. Academy of Sciences, complete with final formulas, are also available). The method received favorable commentaries from several authoritative writers on the subject. A comparative evaluation study published in [2] indicated that the method was as accurate as contemporary solutions, yet simpler and shorter to compute. The procedure does not require any special geodetic tables and calls for relatively few trigonometric interpolations. Of greatest interest was the fact that it was the first rigorous non-iterative inverse solution to go not only beyong the first power of spheroidal flattening but through all the cubic terms of flattening. Yet, it was practical.

II. Theoretical Refinement of non-iterative inverse

Since [1] represented only the formative stage of development, the next phase consisted of making a thorough mathematical analysis of the formulas in order to determine whether there were any concealed intrinsic properties or relationships which could be used to obtain an optimum solution. This analysis resulted in the uncovering of three basic quantities, a, m, and ϕ, the exclusive substitution of which promised a concise, orderly pattern for the terms of the two main spheroidal power series, x and S, that were originally given in [1], at the top of page 18 and the bottom of page 19, respectively.

III. Development of corresponding direct

Since there were indications that a, m, and ϕ were a rather unique set of quantities (as will be shown later), an attempt was also made to introduce their equivalent into the formulation of a corresponding Direct solution, which so far was lacking. As if by design, two spheroidal power series resulted, again with a simple, orderly pattern of terms. Moreover, the format was identical to that of the Inverse. The three corresponding basic quantities were denoted as a_1, m_1, and ϕ_S. This Direct solution was derived from the formulas on pages 14 and 15 of [1]. Essentially, the power series of the quantity S (now appearing also in Appendix E) was used to solve for ϕ_0; then the expression of ϕ_0 was in turn placed into the λ power series. A critical factor in the mathematical determination of a_1 and m_1 was the proper choice of their common smaller order term. $5e'^2 \sin^2 \beta_1$, which is probably one in a series of others required for the orderly theoretical extension of the solution to higher degree.

IV. Notes on computation forms

The above efforts led to the Inverse and Direct computation forms now shown in Appendices A and B, respectively. The main Inverse spheroidal expressions are indicated by $(S \div b_0)$ and $(\lambda - L) \div c$, while the Direct by ϕ_0 and $(L - \lambda) \div \cos \beta_0$. It is to be noted that their outer coefficients are simple product combinations of a and m or a_1 and m_1, while their bracketed expressions are functions of only the variable ϕ or ϕ_S and powers of the spheroidal parameters f or e'^2. By tabulation of the functions of ϕ and ϕ_S so that they may be rapidly obtained by interpolation, it would then be very simple to multiply them by the easily determined outer coefficients. Since conventional trigonometric tables can be used to obtain the first-order term ϕ, and no tables are required to obtain the first-order term ϕ_S only short tables for the second- and third-order terms need be drawn up. The third-order terms, which are small, vary sufficiently slowly for visual interpolation.

In addition to their two pairs of main power series and the basic quantities necessary to numerically evaluate them, these Inverse and Direct computation forms generally contain one simple closed trigonometric expression for each required final quantity except when, for example, the trigonometric cofunction is given as an alternative for occasions when a weak determination or unlinear interpolation may otherwise result. The forms may appear cluttered with rules for choosing signs, trigonometric quadrants, and so forth. Actually, these do not entail any added calculations but simply define the problem without ambiguity. In addition, the choices provide for greater generality and more varied applications. For example, one may calculate either the shorter geodesic between two given positions, or the longer geodesic around the spheroid's back side. Also, the subscripts 1 or 2 may be assigned to either position without fear of ambiguity. For less accuracy, terms in f^2 and e'^4 may be omitted.

V. Formulas for very short and long geodesics -

From its inception, the non-iterative derivation was developed primarily for very long geodesics; therefore, the auxiliary trigonometric functions which were correspondingly designed provided the greatest simplicity at the expense of generality. Recently, however, this Agency as well as the Army Map Service placed a requirement for a single set of formulas applicable to very short as well as very long lines, since they were to be used also in the adjustment of triangulation and trilateration ground nets. It was felt that the basic long line formulas given in Appendices A and B appeared particularly convenient for the electronic computers which were to make the calculations. However, in order to obtain the same or even greater accuracy for very short geodesics, the alternate formulas presented in Appendix C were provided. The reason for the increased accuracy requirement is easily understood when one considers, for example, that if the length of a line is decreased a thousandfold, the positions of the new endpoints must be known a thousand times more accurately to maintain a constant azimuth accuracy. This means that the latitude and longitude will have a greater number of decimals and, therefore, additional significant digits. In order to avoid the carrying of too many fixed places, which are more apt to be affected by rounding errors if there is no spare digit capacity, Appendix C provides formulas whose terms are generally very small when a geodesic is very short, so the computations can be done conveniently by floating point for greater decimal accuracy. To obtain the full required accuracy, no additional terms need be added to the power series formulas, because (as shown in the second paragraph of Appendix F) they converge to more good decimal places for shorter geodesics. Since many of the small quantities in Appendix C consist of sines of small angles, their evaluation is especially adaptable to electronic computers, which by means of floating point can readily calculate trigonometric series that inherently converge to additional decimals for such small angles. Actually, the formulas in Appendix C are equally applicable to short and long lines, so only one set of equations need be programmed into an electronic computer. A floating point formula for a more accurate cosine of large absolute latitude is also included in Appendix C.

VI. Concluding remarks

Appendix D provides the complete numerical calculations for a very short and a very long geodesic--1 mile and 6,000 miles, respectively. In each case, the Direct solution provides a check on the Inverse. The discrepancies between the two types of solutions are given at the end of Appendix D. The better positional accuracy provided for the short line by the formulas of Appendix C is convincingly shown.

Appendix E provides a non-iterative inverse solution of higher order accuracy (that is, through f^3 and e'^6 terms) for use as a theoretical check on Direct or other Inverse formulas. In Appendix F, several interesting types of inter-relations of the terms of the power series are discussed and illustrated. These include relationships between numerical coefficients as well as algebraic terms. In Appendix G, meridional arc formulas are derived as special cases of the Inverse and Direct.

In conclusion, it should be noted that the Inverse case of almost antipodal positions, treated on pages 24 through 25 of [1], is omitted here because of its rare practical occurrence. Also, the elimination of β by substitution in terms of the given B is not undertaken because simple closed functions herein would become series expansions.

BIBLIOGRAPHY

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APPENDIX A

Inverse computation form

Given : B_1 , L_1 = Geodetic latitude and longitude of any point.

B_2 , L_2 = Latitute and longitude of any other point,

(South latitudes and west longitudes considered negative).

Required : α, S = Geodetic azimuths clockwise from north and distance.

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a_o , b_o = Semimajor and semiminor axes of spheroid.

f = Spheroidal flattening $= 1 - \dfrac{b_o}{a_o}$

$$L = (L_2 - L_1) \text{ or } (L_2 - L_1) + [\text{sign opposite of } (L_2 - L_1)](360^\circ)$$

Use whichever L has an absolute value < or > 180°, according to whether the shorter or the back-side's longer geodesic is intended. However, for meridional arcs ($|L| = 0^\circ$ or 180° or 360°), use either L but consider it (+) for the shorter and (-) for the longer.

$\tan\ \beta = (\tan B)\ (1 - f)$ when $|B| \leq 45^\circ$

or $\cot \beta = (\cot B) \div (1 - f)$ when $|B| > 45^\circ$

$a = \sin\beta_1 \sin\beta_2$; $b = \cos\beta_1 \cos\beta_2$; $\cos\phi = a + b\cos L$.

$$\sin\phi = \pm\sqrt{(\sin L \cos\beta_2)^2 + (\sin\beta_2 \cos\beta_1 - \sin\beta_1 \cos\beta_2 \cos L)^2}$$

The $\sin\phi$ is (+) for the shorter arc and (-) for the longer. Compute the radical entirely by floating decimals to prevent loss of digits, especially for very short geodesics.

ϕ = Positive radians in proper quadrant, reference angle being determined from $\sin\phi$ or $\cos\phi$, whichever has the smaller absolute value.

$c = (b \sin L) \div \sin\phi$; $\qquad m = 1 - c^2$.

$$\begin{aligned}
\frac{S}{b_o} &= [(1 + f + f^2)\phi] \\
&+ a[(f + f^2)\sin\phi - (\tfrac{f^2}{2})\phi^2 \csc\phi] \\
&+ m[-(\tfrac{f + f^2}{2})\phi - (\tfrac{f + f^2}{2})\sin\phi\cos\phi + (\tfrac{f^2}{2})\phi^2 \cot\phi] \\
&+ a^2[-(\tfrac{f^2}{2})\sin\phi\cos\phi] \\
&+ m^2[(\tfrac{f^2}{16})\phi + (\tfrac{f^2}{16})\sin\phi\cos\phi - (\tfrac{f^2}{2})\phi^2 \cot\phi - (\tfrac{f^2}{8})\sin\phi\cos^3\phi] \\
&+ am[(\tfrac{f^2}{2})\phi^2 \csc\phi + (\tfrac{f^2}{2})\sin\phi\cos^2\phi]
\end{aligned}$$

$$\frac{\lambda - L}{c} = [(f + f^2)\,\phi]$$
$$+ a[-(\frac{f^2}{2}) \sin\phi - (f^2)\,\phi^2 \csc\phi]$$
$$+ m[-(\frac{5f^2}{4})\,\phi + (\frac{f^2}{4}) \sin\phi \cos\phi + (f^2)\,\phi^2 \cot\phi] \text{ radians}$$

$$\cot\alpha_{1-2} = (\sin\beta_2 \cos\beta_1 - \cos\lambda \sin\beta_1 \cos\beta_2) \div \sin\lambda \cos\beta_2$$

$$\cot\alpha_{2-1} = (\sin\beta_2 \cos\beta_1 \cos\lambda - \sin\beta_1 \cos\beta_2) \div \sin\lambda \cos\beta_1$$

For meridional arcs, consider α as having $0°$ reference angle, and obtain only the signs of the cotangents by disregarding the denominators. For other geodesics, replace cotangent by tangent when $|\cot\alpha| > 1$, by taking the reciprocal of the quotient's value.

	Quadrant of α_{1-2}	Quadrant of α_{2-1}
If L is (+)	... and cot (tan) of α_{1-2} is (+) or (-), α_{1-2} is in quad I or II, respectively	... and cot (tan) of α_{2-1} is (+) or (-), α_{2-1} is in quad III or IV, respectively
If L is (-)	... and cot (tan) of α_{1-2} is (+) or (-), α_{1-2} is in quad III or IV, respectively	... and cot (tan) of α_{2-1} is (+) or (-), α_{2-1} is in quad I or II, respectively

APPENDIX B

Direct computation form

Given : B_1 , L_1 = Geodetic latitude and longitude of any point 1.

α_{1-2} S = Azimuth clockwise from north and distance to any point 2.

Required : Geodetic α_{2-1} , B_2 , and L_2 .

(South latitudes and west longitudes considered negative).

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a_o , b_o = Semimajor and semiminor axes of spheroid.

f = Spheroidal flattening $= 1 - \dfrac{b_o}{a_o}$

e'^2 = Second eccentricity squared $= (a_o^2 - b_o^2) \div b_o^2$

$$\tan\beta = (\tan B)(1 - f) \text{ when } |B| \leq 45°$$

$$\text{or } \cot\beta = (\cot B) \div (1 - f) \text{ when } |B| > 45°$$

$$\cos\beta_o = \cos\beta_1 \sin\alpha_{1-2} \; ; \quad g = \cos\beta_1 \cos\alpha_{1-2} \; ,$$

$$m_1 = \left(1 + \frac{e'^2}{2}\sin^2\beta_1\right)(1 - \cos^2\beta_o) \; ; \; \phi_S = (S \div b_o) \text{ radians;}$$

$$a_1 = \left(1 + \frac{e'^2}{2}\sin^2\beta_1\right)(\sin^2\beta_1 \cos\phi_S + g \sin\beta_1 \sin\phi_S).$$

$$\begin{aligned}
\phi_o &= [\phi_S] \\
&+ a_1\left[-\frac{e'^2}{2}\sin\phi_S\right] \\
&+ m_1\left[-\frac{e'^2}{4}\phi_S + \frac{e'^2}{4}\sin\phi_S\cos\phi_S\right] \\
&+ a_1^2\left[\frac{5e'^4}{8}\sin\phi_S\cos\phi_S\right] \\
&+ m_1^2\left[\frac{11e'^4}{64}\phi_S - \frac{13e'^4}{64}\sin\phi_S\cos\phi_S - \frac{e'^4}{8}\phi_S\cos^2\phi_S + \frac{5e'^4}{32}\sin\phi_S\cos^3\phi_S\right] \\
&+ a_1 m_1\left[\frac{3e'^4}{8}\sin\phi_S + \frac{e'^4}{4}\phi_S\cos\phi_S - \frac{5e'^4}{8}\sin\phi_S\cos^2\phi_S\right] \text{ radians}
\end{aligned}$$

$$\cot \alpha_{2\text{-}1} = (g \quad \cos \phi_0 \; - \sin \beta_1 \; \sin \phi_0) \div \cos \beta_0$$

For meridional arcs, consider $\alpha_{2\text{-}1}$ as having $0°$ reference angle, and obtain only the sign of the cotangent by disregarding the denominator. For other geodesics, replace cotangent by tangent when $|\cot \alpha_{2\text{-}1}| > 1$, by taking the reciprocal of the quotient's value.

	Quadrant of $\alpha_{2\text{-}1}$
If $(0° \leq \alpha_{1\text{-}2} \leq 180°$	.... and cot (tan) of $\alpha_{2\text{-}1}$ is (+) or (-), $\alpha_{2\text{-}1}$ is in quad III or IV, respectively.
If $(180° < \alpha_{1\text{-}2} < 360°)$	.... and cot (tan) of $\alpha_{2\text{-}1}$ is (+) or (-), $\alpha_{2\text{-}1}$ is in quad I or II, respectively.

$$\cot \lambda = (\cos \beta_1 \cos \phi_0 - \sin \beta_1 \; \sin \phi_0 \; \cos \; \alpha_{1\text{-}2}) \div \sin \phi_0 \; \sin \alpha_{1\text{-}2}$$

For meridional arcs, consider λ as having $0°$ reference angle, and obtain only the sign of the cotangent by disregarding $\sin \alpha_{1\text{-}2}$. For other geodesics, replace cotangent by tangent when $|\cot \lambda| > 1$, by taking the reciprocal of the quotient's value.

	Quadrant ang Sign of λ	
	When $0° < \phi_0 \leq 180°$ (sin ϕ_0 considered positive)	When $180° < \phi_0 \leq 360°$ (sin ϕ_0 considered negative)
and $(0° \leq \alpha_{1\text{-}2} \leq 180°)$	... then if cot (tan) of λ is (+) or (-), λ is in quad I or II, respectively.	... then if cot (tan) of λ is (+) or (-), λ is in quad III or IV, respectively.
and $(180° < \alpha_{1\text{-}2} < 360°)$	... then if cot (tan) of λ is (+) or (-), the associated angle is in quad III or IV, respectively, and λ is obtained by <u>subtracting</u> <u>360°</u>.	... then if cot (tan) of λ is (+) or (-), the associated angle is in quad I or II, respectively, and λ is obtained by <u>subtracting</u> <u>360°</u>.

$$\frac{L - \lambda}{\cos \beta_o} = [\, - f \phi_S \,]$$

$$+ \; a_1 \, [\, \frac{3 f^2}{2} \sin \phi_S \,]$$

$$+ \; m_1 \, [\, \frac{3 f^2}{4} \; \phi_S \; - \frac{3 f^2}{4} \sin \phi_S \cos \phi_S \,] \text{ radians}$$

$$L_2 = L_1 + L$$

[If $| L_2 | > 180°$, modify L_2 by adding <u>or</u> subtracting 360°, according to whether it is initially negative <u>or</u> positive.]

$$\sin \beta_2 = \sin \beta_1 \cos \phi_o + g \sin \phi_o$$

$$\cos \beta_2 = +\sqrt{(\cos \beta_o)^2 + (g \cos \phi_o - \sin \beta_1 \sin \phi_o)^2}$$

Compute the radical entirely by floating decimals to prevent loss of digits, especially for large absolute latitudes.

$$\tan \beta_2 = (\sin \beta_2 \div \cos \beta_2)$$

$$\text{or} \;\; \cot \beta_2 = (\cos \beta_2 \div \sin \beta_2)$$

Use whichever has the smaller absolute value.

Obtain tan (or cot) of B_2 from earlier defined relation of B to β.

Determine $(-90° \leq B_2 \leq 90°)$, applying sign of its tan (or cot).

APPENDIX C

Alternate inverse and direct formulas
(For very short as well as long geodesics)

The following alternate formulas for corresponding ones in Appendices A and B are designed to maintain or appropriately increase the accuracy of various elements of short geodesics, without decreasing the accuracy of long geodesics. The formulas specifically take advantage of inherently small quantities and of small differences of given large quantities, so as to provide--through the application of floating point calculations -- increased decimal place accuracy without requiring additional operational digits. The small angles involved are especially adaptable to electronic computers, which by means of floating point can readily obtain greater decimal accuracy inherent in trigonometric power series of such small angles.

1. For inverse solution :

$$\sin\phi = \pm\sqrt{(\sin L \cos\beta_2)^2 + [\sin(\beta_2 - \beta_1) + 2\cos\beta_2 \sin\beta_1 \sin^2 \frac{L}{2}]^2}$$

$$\cot\alpha_{1\text{-}2} = [\sin(\beta_2 - \beta_1) + 2\cos\beta_2 \sin\beta_1 \sin^2 \frac{\lambda}{2}] \div \cos\beta_2 \sin\lambda$$

$$\cot\alpha_{2\text{-}1} = [\sin(\beta_2 - \beta_1) - 2\cos\beta_1 \sin\beta_2 \sin^2 \frac{\lambda}{2}] \div \cos\beta_1 \sin\lambda$$

where

$$(\beta_2 - \beta_1) = (B_2 - B_1) + 2[\sin(B_2 - B_1)]\,[(n + n^2 + n^3)\,a - (n - n^2 + n^3)\,b]$$

2. For direct solution :

$$B_2 = B_1 + (\beta_2 - \beta_1) + 2[\sin(\beta_2 - \beta_1)]\,[(n + n^3)\cos(B_2 + B_1) + n^2\cos(B_2 - B_1)]$$

where $\sin(\beta_2 - \beta_1) = \sin\phi_0 \cos\alpha_{1\text{-}2} - 2\sin^2\frac{\lambda}{2}\sin\beta_1\cos\beta_2$

and the required approximate B_2 and $\cos\beta_2$ are obtained in Appendix B.

3. For inverse and direct at given absolute latitudes > 45° :

$$\cos\beta = \sin\{(90 \mp B) \pm 2[\sin(90 \mp B)]\;(n + n^2 + n^3)\sin\beta\}$$

the upper and lower signs of which are applied for the northern and southern hemispheres, respectively.

In the preceding three sets of formulas, $n = (a_0 - b_0) \div (a_0 + b_0)$. Some smaller coefficients of the almost negligible n^3 have been removed because they are unsymmetric, and because they become even smaller in Parts 1 and 2 for short geodesics and in Part 3 for large absolute latitudes. It should be noted that terms containing powers of n are in radians.

The accurate floating point calculations for short geodesics should be applied not only to the formulas of this appendix but, in turn, also to associated formulas in Appendices A and B, as illustrated numerically in Appendix D. The prescribed increase in decimal accuracy in the sine of a small angle, for example, can be obtained not only from the sine series, but also from trigonometric tables by taking the reciprocal of the large interpolated cosecant of the angle. However, in addition to sufficient significant digits, the table should have intervals small enough for accurate linear interpolation. Even better, of course, is a table of high decimal accuracy for the small sines themselves.

APPENDIX D

Numerical illustrations of inverse and direct

(Geodesics of approximately 1 and 6,000 miles for each)

The two extreme test distances noted above are chosen to illustrate by calculation not only the basic computation forms of Appendices A and B but also the alternate formulas of Appendix C. The degree of consistency of the answers has been determined below by checking each Inverse solution against the corresponding Direct. The resulting discrepancies, which for each geodesic are summarized at the end of this appendix, therefore represent the combined errors of the Inverse and Direct.

Inverse Solution	Long Geodesic	Short Geodesic
B_1	$+20^\circ$	$+45^\circ$
L_1	0°	$+12^\circ 11'18''$
B_2	$+45^\circ$	$+45^\circ 00'36''.5$
L_2	$+106^\circ$	$+12^\circ 12'09''.5$
a_0 (meters)	6 378 388.000	6 378 388.000
b_0 (meters)	6 356 911.946	6 356 911.946
f	.00 33670 03367	.00 33670 03367
n	- - -	.00 16863 40641
L	$+106^\circ$	$51''.5$
$\tan \beta_1$	.36274 47453	.99663 29966
$\tan \beta_2$	.99663 29966	.99698 57825
$\cos \beta_1$	.94006 23275	.70829 81969
$\cos \beta_2$	.70829 81969	.70817 32700
$\sin \beta_1$	.34100 26695	.70591 33545
$\sin \beta_2$	.70591 33545	.70603 86817

Inverse Solution	Long Geodesic	Short Geodesic
a	.24071 83383	.49840 21342
b	.66584 44515	.50159 78502
sin L	.96126 16959	.000 24967 90432
cos L	-.27563 73558	.99999 99688
cos ϕ	.05718 67343	.99999 99687
$\sin (B_2 - B_1)$	- - - - -	.000 17695 69927
$(\beta_2 - \beta_1)$ radians	- - - - -	.000 17695 60928
$\sin (\beta_2 - \beta_1)$	- - - - -	.000 17695 60919
$\sin^2 \frac{L}{2}$	- - - - -	.000 00001 55849
sin ϕ	.99836 34996	.000 25016 57049
ϕ (radians)	1.51357 83766	.000 25016 57075
c	.64109 99269	.50062 20631
m	.58899 08837	.74937 75499
$(S \div b_0)$ =	+1.51869 17590 + .00080 87665 - .00156 22320 - .00000 00188 + .00000 01279 + .00000 18468	+.000 25101 08523 +.000 00042 05152 -.000 00063 22699 -.000 00000 03522 -.000 00000 07962 +.000 00000 10592
S (meters)	9 649 412.505	1 594.307 213
$(\lambda - L) \div c$ =	+ .00511 33825 - .00000 76243 - .00001 16616	+.000 00084 51448 -.000 00000 21203 .000 00000 00000
λ (radians)	1.85331 48325	.000 25010 10825
sin λ	.96035 63877	.000 25010 10799
cos λ	-.27877 51927	.99999 99687
$\sin^2 \frac{\lambda}{2}$	- - - - -	.000 00001 56376

Inverse Solution	Long Geodesic	Short Geodesic
cot α_{1-2}	1.07455 96453	.99919 16383
cot α_{2-1}	-.47245 22960	.99883 88553
α_{1-2}	42°56'30".03503	45°01'23".40210
α_{2-1}	295°17'18".59981	225°01'59".82121

Direct Check	Long Geodesic	Short Geodesic
B_1	+20°	+45°
L_1	0°	+12°11'18"
α_{1-2}	42°56'30".03503	45°01'23".40210
S (meters)	9 649 412.505	1 594.307 213
a_0 (meters)	6 378 388.000	6 378 388.000
b_0 (meters)	6 356 911.946	6 356 911.946
f	.00 33670 03367	.00 33670 03367
e'^2	.00 67681 70197	.00 67681 70197
n	- - - - -	.00 16863 40641
tan β_1	.36274 47453	.99663 29966
cos β_1	.94006 23275	.70829 81969
sin β_1	.34100 26695	.70591 33545
sin α_{1-2}	.68125 35334	.70739 26381
cos α_{1-2}	.73204 75552	.70682 08083
cos β_0	.64042 07822	.50104 49301
g	.68817 03286	.50063 99041
m_1	.59009 33386	.25146 93699
ϕ_S (radians)	1.51794 02494	.000 25079 90085

Direct Check	Long Geodesic		Short Geodesic
$\sin \phi_S$	.99860 34425		.000 25079 90059
$\cos \phi_S$	.05283 14696		.99999 99685
a_1	.24057 82171		.49924 27565
ϕ_0 (radians) =	+1.51794 02494 - .00081 30002 - .00146 29306 + .00000 00874 + .00000 39825 + .00000 25543 +1.51567 09428		+.000 25079 90085 -.000 00042 37199 .000 00000 00000 +.000 00000 17897 .000 00000 00000 .000 00000 00000 +.000 25037 70783
$\sin \phi_0$	.99848 09807		.000 25037 70757
$\cos \phi_0$	.05509 74693		.99999 99687
$\cot \alpha_{2-1}$	-.47245 22450		.99883 88542
α_{2-1}	295°17'18".59121		225°01'59".82132
$\cot \lambda$ =	-.29028 29979		$\tan \lambda$ = .000 25010 10884
λ	106°11'13".61256		51".58705 146
$(L - \lambda) \div \cos \beta_0$ =	-.00511 09099 +.00000 40853 +.00000 73513		-.000 00084 44411 +.000 00000 21292 .000 00000 00000
L (radians)	1.85004 89647		.000 24967 90471
L_2	105°59'59".99117		12°12'09".50000 027
$\sin \beta_2$	.70591 33687	Preliminary values	.70603 86812
$\cos \beta_2$	.70829 81829		.70817 32700
$\tan \beta_2$	.99663 30364		.99698 57817
$\tan B_2$	1.00000 00399		1.00035 39769
B_2	45°00'00".00411		45°00'36".49992

Direct Check	Long Geodesic	Short Geodesic
$\sin^2 \frac{\lambda}{2}$	- - - - -	.000 00001 56376
$\sin (\beta_2 - \beta_1)$	- - - - -	.000 17695 60922
$(\beta_2 - \beta_1)$ radians	- - - - -	.000 17695 60931
$\cos (B_2 + B_1)$	- - - - -	-.00017 69566
$\cos (B_2 - B_1)$	- - - - -	.99999 99841
B_2 (improved value)	- - - - -	45°00'36".50000 005

Discrepancies between Inverse and Direct	Long Geodesic	Short Geodesic
ΔB_2	0".00411	0".00000 005
ΔL_2	0".00883	0".00000 027
$\Delta\alpha_{2-1}$	0".00860	0".00011

In addition, the preceding Inverse and Direct illustrative examples contain several common intermediate and secondary components whose values can be compared. Also, since the solutions of the long geodesic are illustrated by the same numerical problem that was used in reference [1] for the earlier form of the Inverse method, opportunities for other comparisons are available. It is apparent that the extremely high positional accuracy for the short geodesic is due to the use of alternate formulas given in Appendix C. The azimuth error is consistent with this positional error, in view of the line's shortness. Comparable accuracies are also obtainable at large absolute latitudes, but only if interpreted relative to the increasing convergence and closeness of the meridians in polar regions.

APPENDIX E

Theoretical formulas for higher accuracy

The results of the illustrative numerical examples given in Appendix D indicate that the formulas in Appendices A through C provide sufficient practical accuracy. For theoretical purposes, however, the formulas could be extended through f^3 and e'^6 terms or beyond. The outer coefficients of the formula for $(S \div b_0)$ in Appendix A would then include, for example, the higher order combinations a^3, m^3, $a^2 m$, and am^2. Similar orderly extensions should be expected for the $(\lambda - L) \div c$ formula in Appendix A and the ϕ_0 and $(L - \lambda) \div \cos \beta_0$ in Appendix B, except that in the case of the latter two their outer coefficients will bear the subscript 1, and their components a_1 and m_1 would have to be properly defined to higher powers of e'^2. If necessary, appropriate formulas in Appendix C can also be extended.

In the present appendix, only the $(\lambda - L) \div c$ power series of Appendix A will be given to the next higher order terms, since it provides a non-iterative rigorous solution for the quantity λ which is required in most of the classical methods for calculating the Inverse of long geodesics. The unique form of the extended $(\lambda - L) \div c$ power series given below has been derived from the top of page 18 of [1], by substitution in terms of a, m, ϕ, and f. The series is followed by accurate Inverse distance and azimuth formulas taken in large part from pages 14 and 15 of reference [1]. The resulting method of solution can be used for precise computation of Inverse problems, especially as a theoretical check on Direct or other Inverse formulas.

$$
\begin{aligned}
\frac{\lambda - L}{c} = {} & [\, (f + f^2 + f^3)\, \phi \,] \\
& + a\, [\, -(\frac{f^2}{2} + f^3) \sin \phi - (f^2 + 4f^3)\, \phi^2 \csc \phi \\
& \qquad + (\frac{3f^3}{2})\, \phi^3 \csc \phi \cot \phi \,] \\
& + m\, [\, -(\frac{5f^2}{4} + 3f^3)\, \phi + (\frac{f^2}{4} + \frac{f^3}{2}) \sin \phi \cos \phi \\
& \qquad + (f^2 + 4f^3)\, \phi^2 \cot \phi - (\frac{f^3}{2})\, \phi^3 \csc^2 \phi - (f^3)\, \phi^3 \cot^2 \phi \,] \\
& + a^2 [\, (f^3)\, \phi + (\frac{f^3}{2}) \sin \phi \cos \phi + (f^3)\, \phi^3 \csc^2 \phi \,] \\
& + m^2\, (\frac{31f^3}{16})\, \phi - (\frac{9f^3}{16}) \sin \phi \cos \phi + (\frac{f^3}{2})\, \phi \cos^2 \phi \\
& \qquad - (\frac{9f^3}{2})\, \phi^2 \cot \phi + (\frac{f^3}{8}) \sin \phi \cos^3 \phi \\
& \qquad + (\frac{f^3}{2})\, \phi^3 \csc^2 \phi + (2f^3)\, \phi^3 \cot^2 \phi \,]
\end{aligned}
$$

$$+ am\,[\,(f^3)\sin\phi - (\frac{3f^3}{2})\,\phi\cos\phi + (\frac{9f^3}{2})\,\phi^2\csc\phi$$

$$- (\frac{f^3}{2})\sin\phi\,\cos^2\phi - (\frac{7f^3}{2})\,\phi^3\csc\phi\,\cot\phi\,]\ \text{radians}\,.$$

where the component quantities are again defined in Appendix A, while some alternate definitions are found in Appendix C.

Next, ϕ_0 is obtained in the same manner as ϕ, except that the value of λ obtained from above is now to be used in place of L. Then continue as follows :

$$\cos\beta_0 = (b\sin\lambda) \div \sin\phi_0\,;\ \cos 2\sigma = (2a \div \sin^2\beta_0) - \cos\phi_0\,.$$

$$A_0 = 1 + \frac{e'^2}{4}\sin^2\beta_0 - \frac{3e'^4}{64}\sin^4\beta_0 + \frac{5e'^6}{256}\sin^6\beta_0$$

$$B_0 = \frac{e'^2}{4}\sin^2\beta_0 - \frac{e'^4}{16}\sin^4\beta_0 + \frac{15e'^6}{512}\sin^6\beta_0$$

$$C_0 = \frac{e'^4}{128}\sin^4\beta_0 - \frac{3e'^6}{512}\sin^6\beta_0$$

$$D_0 = \frac{e'^6}{1536}\sin^6\beta_0$$

$$S = b_0\,(A_0\,\phi_0 + B_0\sin\phi_0\cos 2\sigma - C_0\sin 2\phi_0\cos 4\sigma$$

$$+ D_0\sin 3\phi_0\cos 6\sigma)$$

To complement the above geodetic distance, S, the azimuths α_{1-2} and α_{2-1} are obtained from formulas given in Appendix A or C.

APPENDIX F

Inter-Relations of the terms of the power series

As noted earlier, the coefficients a and m in the $(S \div b_0)$ and $(\lambda - L) \div c$ Inverse power series in Appendices A and E display a unique set of product combinations. The identical simple pattern is also repeated in the two Direct power series in Appendix B, except that it occurs instead with the subscripted a_1 and m_1. Although not shown in this paper, even the higher degree combinations (such as a^2m, m^2a, a^3, and m^3) appear to enter in orderly fashion in the further extension of the power series. It is of significant importance that the a and m (or a_1 and m_1) combinations are completely factorable from the power series terms, since this permits the latter to be tabulated as a function of only the variable ϕ or ϕ_S and the parameter f or e'^2. Electronic computer programming and calculations also become simpler, whether for producing just the table or for calculating the entire Inverse or Direct.

Another interesting inter-relation of the terms of the series concerns the numerical coefficients of the powers of f and e'^2. It should be noted, for example, that in Appendix B the numerical coefficients related to the m_1^2 terms of the ϕ_0 power series are : $\frac{11}{64}$, $-\frac{13}{64}$, $-\frac{1}{8}$, $\frac{5}{32}$. The total of the above four numbers is found to be exactly zero. Upon closer inspection, it is found from the power series in Appendices A, B, and E that the zero sum occurs with all sets of terms having m or m_1 as one of the factors, even for the $(S \div b_0)$ series in Appendix A, if it is modified as shown later. When different powers of f are present, the sum is zero separately for the numerical coefficients of the f terms, f^2 terms, and so forth, such as in the $(\lambda - L) \div c$ series in Appendix E. In all instances described, the sum is zero by virtue of the fact that each term--which is a function of ϕ (or ϕ_S)-- is first put into a form which satisfies the following condition : The algebraic sum of the exponents of ϕ and sin ϕ (after all trigonometric functions of ϕ are converted to sines and cosines) is unity. Actually, the above condition can be (and has been) satisfied even for the non-m and the non-m_1 series terms. For very short geodesics (which of course have a small arc value ϕ and, therefore, sin ϕ approaches ϕ and cos ϕ approaches unity), the resulting unity exponent implies that every term is of the small order of ϕ, times its numerical coefficient and the proper power of f or e'^2. Since even the omitted terms of the series contain that small order of ϕ (or ϕ_S), the power series converge to a greater number of decimals for short geodesics. This is shown by the much better positional consistency obtained from the numerical example for the short geodesic in Appendix D. For terms which have m or m_1 as one of the coefficients, the convergency for short geodesics is even greater because (as noted above) the sum of the numerical coefficients is zero separately for each power of f or e'^2, and ϕ or ϕ_S is practically a common factor.

As for the $(S \div b_0)$ series mentioned in the preceding paragraph, the expression given in Appendix A can be reduced to the following form :

$$\begin{aligned}
\frac{S}{b_0} = \quad & [\,(1 + f + f^2)\,\phi\,] \\
& + (m \cos\phi - a)\,.[\,-(f + f^2)\sin\phi + (\frac{f^2}{2})\,\phi^2 \csc\phi\,] \\
& + \quad m \quad [\,-(\frac{f + f^2}{2})\,\phi + (\frac{f + f^2}{2})\sin\phi\cos\phi\,] \\
& + (m\cos\phi - a)^2\,[\,-(\frac{f^2}{2})\sin\phi\cos\phi\,] \\
& + \quad m^2 \quad (\frac{f^2}{16})\,\phi + (\frac{f^2}{16})\sin\phi\cos\phi - (\frac{f^2}{8})\sin\phi\cos^3\phi\,] \\
& + m\,(m\cos\phi - a)[\,(\frac{f^2}{2})\sin\phi\cos^2\phi - (\frac{f^2}{2})\,\phi^2\csc\phi\,]
\end{aligned}$$

The compound coefficient $(m \cos\phi - a)$ is an expression which appeared extensively in the course of the original derivation of the Inverse solution. As used above, it causes the numerical coefficients of the terms with the factor m to add to zero, just like the other power series. It is interesting to note that the next higher order extension of $(S \div b_0)$ continues to give the proper zero sum for the numerical coefficients of applicable terms, when the additional prescribed product combinations of the same $(m \cos\phi - a)$ and m are used.

In conclusion, it is worth noting that, of the four main power series given in Appendices A and B, only $(S \div b_0)$ does not lend itself to completely factoring out the ellipsoidal parameter from each series of terms. The capability of factoring for all four power series (at least to the extent of the number of terms given) may be important. It would mean, for example, that the total value of each series of terms could be tabulated independently of any specific spheroid flattening or eccentricity. (Of course, the parameters would then be made a part of the external coefficients instead). In the $(S \div b_0)$ formula given in the present appendix, only the terms whose coefficient is $(m \cos\phi - a)$ do not lend themselves to factoring out the function of flattening. Those terms, however, can be represented as in the following :

$$[\,(m\cos\phi - a)\,(1 - \frac{f\,\phi^2}{2\sin^2\phi})\,]\;[\,-(f + f^2)\sin\phi\,]$$

where the unwanted portion of flattening has been transferred to the external coefficient. This new compound coefficient may be used in place of the previous $(m \cos\phi - a)$ throughout the $(S \div b_0)$ expression for consistency, since the extraneous f^3 terms which are introduced are negligible.

APPENDIX G

Meridional arc as special case of non-iterative inverse and direct

An interesting indications of the simplicity and rapid convergence of the non-iterative inverse is to reduce it to the special case of meridional arc distances, for northern latitudes up to 90° from the equator. Since β_1 and L are then 0°, the following result :

$$a = 0, \quad m = 1, \quad \phi = \beta_2 \text{ radians.}$$

Therefore, such meridional distances, S_M, become :

$$S_M = b_o \left[\left(1 + \frac{f}{2} + \frac{9f^2}{16}\right) \beta_2 - \left(\frac{f}{2} + \frac{7f^2}{16}\right) \sin\beta_2 \cos\beta_2 - \left(\frac{f^2}{8}\right) \sin\beta_2 \cos^3 \beta_2 \right]$$

Similarly, β_2 can be derived for the corresponding S_M by letting β_1 and $\alpha_{1\text{-}2}$ equal 0° in the Direct solution, whence :

$$a_1 = 0, \quad m_1 = 1, \beta_2 = \phi_o$$

Then by substitution into the ϕ_o power series, there results :

$$\beta_2 = \left(1 - \frac{e'^2}{4} + \frac{11e'^4}{64}\right) \phi_M + \left(\frac{e'^2}{4} - \frac{13e'^4}{64}\right) \sin\phi_M \cos\phi_M + \left(\frac{5e'^4}{32}\right) \sin\phi_M \cos^3 \phi_M - \left(\frac{e'^4}{8}\right) \phi_M \cos^2 \phi_M \text{ radians,}$$

$$\text{where } \phi_M = (S_M \div b_o) \text{ radians.}$$

As a check (the complete details of which need not be shown), the above Inverse and Direct meridional arc solutions were compared mathematically and found to be fully consistent with each other. Essentially, dividing the Inverse meridional formula by b_o produced ϕ_M as a function of β_2, from which $\sin\phi_M$ and $\cos\phi_M$ were then obtained by expanding in series around $\sin\beta_2$ and $\cos\beta_2$, respectively. Substitution into the Direct meridional formula finally made the right side identically equal to the left side's β_2, up through all e'^4 terms.