3

A RIGOROUS NON-ITERATIVE PROCEDURE FOR RAPID INVERSE SOLUTION OF VERY LONG GEODESICS

EMANUEL M. SODANO

Army Map Service, Washington 25, D.C.

(Communication prèsentée à l'Assemblée Générale de Toronto)

I. INTRODUCTION

The purpose of this paper is to unify, under a single cover, the many theoretical and practical aspects of a rigorous, yet rapid, non-iterative procedure for the inverse solution of very long geodesics. The results represent the gradual extension and accumulated improvement of the original Army Map Service Technical Report No. 7.

The present version contains primarily a more stable formula for azimuths and an alternate formula for very short lines, whereby the solution becomes accurate and practical for any line in any position, yet requires no special purpose tables nor the carrying of excess digits. The final non-iterative formulas are convergent power series which have been extended through terms equivalent to the second, fourth and sixth powers of spheroid eccentricity, and may therefore be cut back term by term according to the requirements of time or accuracy.

A procedure of this kind is well adapted to the requirements of modern electronic computers as well as desk calculators. The solution can be expanded to even higher terms for purely theoretical considerations, although its accuracy is already beyond the tenth decimal place of radians for the azimuths as well as the arc of the distance. For the more practical computations to the second and fourth powers of eccentricity, the accuracy is to 7 and 9 places of radians respectively, even for distances circumscribing the earth. Yet the computations are significantly short and rapid, even without parametric latitude tables or other such aids.

This paper supplies the complete theoretical derivations starting with a rigorous modification of HELMERT's classical formulas. A variety of basic types of non-iterative solutions are developed, complete with an illustrative 6000 mile example and a procedure for antipodal points.

II. PRELIMINARY MODIFICATION OF HELMERT'S ITERATIVE SOLUTION

e = eccentricity of the spheroid.

e' = second eccentricity.

b_0 = semi-minor axis.

L = absolute difference of longitude on the spheroid, between the given endpoints of the geodesic.

β_1 and β_2 = parametric (or reduced) latitude of the westward and eastward endpoints respectively.

λ = difference of longitude (approximately L) on the reduced sphere, for which a progressively better value is found with each repetition of the following iteration process:

$$\cos \Phi_0 = \sin \beta_1 \sin \beta_2 + \cos \beta_1 \cos \beta_2 \cos \lambda$$

$$\sin \Phi_0 = (\text{sign of} \sin \lambda) \sqrt{1 - \cos^2 \Phi_0}$$

$$\Phi_0 = \text{positive radians}$$

$$\sin 2\Phi_0 = 2 \sin \Phi_0 \cos \Phi_0$$

$$\sin 3\Phi_0 = 3 \sin \Phi_0 - 4 \sin^3 \Phi_0$$

$$\cos \beta_0 = (\cos \beta_1 \cos \beta_2 \sin \lambda) \div \sin \Phi_0$$

$$\sin^2 \beta_0 = 1 - \cos^2 \beta_0$$

$$\cos 2\sigma = (2 \sin \beta_1 \sin \beta_2 \div \sin^2 \beta_0) - \cos \Phi_0$$

$$\cos 4\sigma = -1 + 2 \cos^2 2\sigma$$

$$\cos 6\sigma = 4 \cos^3 2\sigma - 3 \cos 2\sigma$$

$$A' = \frac{e^2 e'}{e' + e} - \frac{e^2 e'^2}{16} \sin^2 \beta_0 + \frac{3e^2 e'^4}{128} \sin^4 \beta_0$$

$$B' = \frac{e^2 e'^2}{16} \sin^2 \beta_0 - \frac{e^2 e'^4}{32} \sin^4 \beta_0$$

$$C' = \frac{e^2 e'^4}{256} \sin^4 \beta_0$$

$$\Upsilon = A'\Phi_0 - B' \sin \Phi_0 \cos 2\sigma + C' \sin 2\Phi_0 \cos 4\sigma$$

Next approximation to $\lambda = (L + \Upsilon \cos \beta_0)$ radians.

After a sufficiently accurate λ is found, and using the set of values from the last iteration, the geodetic distance (S) and azimuths (α) between the endpoints are obtained as follows:

$$A_0 = 1 + \frac{e'^2}{4}\sin^2\beta_0 - \frac{3e'^4}{64}\sin^4\beta_0 + \frac{5e'^6}{256}\sin^6\beta_0$$

$$B_0 = \frac{e'^2}{4}\sin^2\beta_0 - \frac{e'^4}{16}\sin^4\beta_0 + \frac{15e'^6}{512}\sin^6\beta_0$$

$$C_0 = \frac{e'^4}{128}\sin^4\beta_0 - \frac{3e'^6}{512}\sin^6\beta_0$$

$$D_0 = \frac{e'^6}{1536}\sin^6\beta_0$$

$$S = b_0\,(A_0\Phi_0 + B_0\sin\Phi_0\cos 2\sigma - C_0\sin 2\Phi_0\cos 4\sigma + D_0\sin 3\Phi_0\cos 6\sigma)$$

$$\cot\alpha_{1-2} = \frac{\tan\beta_2\cos\beta_1 - \cos\lambda\sin\beta_1}{\sin\lambda}$$

$$\cot\alpha_{2-1} = \frac{\sin\beta_2\cos\lambda - \cos\beta_2\tan\beta_1}{\sin\lambda}$$

where α_{1-2} and α_{2-1} range from 0° to 180° and 180° to 360°, respectively, clockwise from north.

III. REDUCTION OF THE HELMERT PROCEDURE TO POWER SERIES IN X

Let it be *assumed* that the *true value of* λ is known (that is, the value that would result from an *infinite number* of Helmert approximations) and let this true value be represented by the given absolute difference of longitude on the spheroid plus a quantity x which will be determined later.

Thus: $$\lambda = (L + x).$$

It will be evident, later, that x is a very small positive quantity of the order of e^2, and therefore well suited for setting up a convergent power series in x for each expression contained in the HELMERT procedure. For example, from the above assumed equation, the following is derived:

$$\begin{aligned}\cos\lambda &= \cos(L + x)\\ &= \cos L\cos x - \sin L\sin x\\ &= (\cos L)\left(1 - \frac{x^2}{2} + \ldots\right) - (\sin L)\,(x - \ldots)\end{aligned}$$

Therefore: $\cos\lambda = (\cos L) - (\sin L)\,x - \tfrac{1}{2}(\cos L)\,x^2 + \ldots$

There is thus available, at the outset, a series for the *true* $\cos\lambda$ with which to begin the Helmert solution and develop it in power series of x in its entirety. The process consists of substituting each new series into the succeeding Helmert expressions as required. For convenience, the following additional notation will be used:

$$\begin{aligned}
N &= e' \div (e' + e) \\
a &= \sin\beta_1 \sin\beta_2 \\
b &= \cos\beta_1 \cos\beta_2 \\
\cos\Phi &= a + b\cos L \\
c &= b \sin L \csc\Phi \\
m &= 1 - c^2 \\
h &= e'^2 m \\
P &= m\cot\Phi - a\csc\Phi \\
U_1 &= (\tan\beta_2\cos\beta_1 - \cos L\sin\beta_1) \div \sin L \\
U_2 &= (\sin\beta_2\cos L - \cos\beta_2\tan\beta_1) \div \sin L
\end{aligned}$$

Listed below, in the same sequence as the corresponding Helmert expressions, is the complete set of series through $\Upsilon \cos\beta_0$. The extent of the powers of x is such as to permit accuracies of the e^6 order in λ, for subsequent application to the distance and azimuths to the same degree of accuracy as the reference Helmert iteration form.

$$\begin{aligned}
\cos\Phi_0 &= (\cos\Phi) - (c\sin\Phi)\,x - \tfrac{1}{2}(c^2\cos\Phi + P\sin\Phi)\,x^2 \\
\sin\Phi_0 &= (\sin\Phi) + (c\cos\Phi)\,x - \tfrac{1}{2}(c^2\sin\Phi - P\cos\Phi)\,x^2 \\
\Phi_0 &= \Phi + (\Phi_0 - \Phi) = \Phi + \arcsin[\sin(\Phi_0 - \Phi)] \\
&= \Phi + \arcsin[\sin\Phi_0\cos\Phi - \cos\Phi_0\sin\Phi] \\
&= \Phi + (c)\,x + \tfrac{1}{2}(P)\,x^2 \\
\sin 2\Phi_0 &= 2\sin\Phi\cos\Phi \\
\sin\lambda &= (\sin L) + (\cos L)\,x - \tfrac{1}{2}(\sin L)\,x^2 \\
\cos\beta_0 &= (c) + (P)\,x - \tfrac{1}{2}(cm + 3c\,P\cot\Phi)\,x^2 \\
\sin^2\beta_0 &= m - (2cP)x \\
\cos 2\sigma &= \frac{1}{m}(m\cos\Phi - 2P\sin\Phi) \\
&\quad + \frac{1}{m^2}(cm^2\sin\Phi + 4cmP\cos\Phi - 4cP^2\sin\Phi)\,x \\
\cos 4\sigma &= \frac{1}{m^2}(m^2 - 2m^2\sin^2\Phi - 8mP\sin\Phi\cos\Phi + 8P^2\sin^2\Phi) \\
A' &= \frac{e^2}{128}(128N - 8h + 3h^2) + \frac{e^2}{8}(e'^2\,cP)\,x \\
B' &= \frac{e^2}{32}(2h - h^2) - \frac{e^2}{8}(e'^2\,cP)\,x
\end{aligned}$$

$$C' = \frac{e^2}{256}\,(h^2)$$

$$A'\Phi_0 = \frac{e^2}{128}\,(128N\Phi - 8h\Phi + 3h^2\Phi) + \frac{e^2}{16}\,(16Nc - hc + 2e'^2cP\Phi)x + \frac{e^2}{2}\,(NP)\,x^2$$

$$-B'\sin\Phi_0\cos 2\sigma = \frac{e^2}{128}\,(-8h\sin\Phi\cos\Phi + 16e'^2\,P\sin^2\Phi + 4h^2\sin\Phi\cos\Phi - 8e'^2\,hP\sin^2\Phi) - \frac{e^2}{16}\,(hc)\,x$$

$$C'\sin 2\Phi_0\cos 4\sigma = \frac{e^2}{128}\,(h^2\sin\Phi\cos\Phi - 2h^2\sin^3\Phi\cos\Phi - 8e'^2\,hP\sin^2\Phi\cos^2\Phi + 8e'^4P^2\sin^3\Phi\cos\Phi)$$

$$\begin{aligned}\Upsilon\cos\beta_0 = \frac{e^2}{128}\,(&128Nc\Phi - 8hc\Phi - 8hc\sin\Phi\cos\Phi + 16e'^2cP\sin^2\Phi \\ &+ 3h^2c\Phi + 5h^2c\sin\Phi\cos\Phi - 2h^2c\sin^3\Phi\cos\Phi \\ &- 8e'^2hcP\sin^2\Phi - 8e'^2hcP\sin^2\Phi\cos^2\Phi \\ &+ 8e'^4cP^2\sin^3\Phi\cos\Phi) + \frac{e^2}{16}\,(16Nc^2 + 16NP\Phi - 2hc^2 \\ &- hP\Phi + 2e'^2\,c^2\,P\Phi - hP\sin\Phi\cos\Phi + 2e'^2\,P^2\sin^2\Phi)x \\ &- \frac{e^2}{2}\,(Ncm\Phi - 3NcP + 3NcP\Phi\cot\Phi)x^2\end{aligned}$$

IV. DERIVATION OF THE UNKNOWN QUANTITY X

Since the substitution into the Helmert iteration began with an algebraic series representing the true λ, the next approximation to λ must of necessity be its equal; that is:

The next approximation to λ = the starting true λ

or $\quad L + \Upsilon\cos\beta_0 = L + x$

and therefore $\quad \Upsilon\cos\beta_0 = x.$

By replacing $\Upsilon\cos\beta_0$ with its corresponding power series, the above equation takes the following quadratic form:

$$Q_1 + Q_2x + Q_3x^2 = x$$

for which the required solution of x to the proper order is

$$x = Q_1(1 + Q_2 + Q_2^2 + Q_1Q_3).$$

Finally, substituting for Q_1, Q_2 and Q_3, produces the following end result:

$$
\begin{aligned}
x = \frac{e^2c}{128}\Big[& 128N\Phi + 128e^2N^2c^2\Phi - 8h\Phi - 8h\sin\Phi\cos\Phi + 128e^2N^2P\Phi^2 \\
& + 16e'^2P\sin^2\Phi + 128e^4N^3c^4\Phi - 24e^2Nhc^2\Phi + 3h^2\Phi \\
& - 8e^2Nhc^2\sin\Phi\cos\Phi + 5h^2\sin\Phi\cos\Phi - 64e^4N^3c^2m\Phi^3 \\
& - 2h^2\sin^3\Phi\cos\Phi + (16e^2e'^2N + 448e^4N^3)c^2P\Phi^2 \\
& - 16e^2NhP\Phi^2 + 16e^2e'^2Nc^2P\sin^2\Phi - 8e'^2hP\sin^2\Phi \\
& - 16e^2NhP\Phi\sin\Phi\cos\Phi - 192e^4N^3c^2P\Phi^3\cot\Phi \\
& - 8e'^2hP\sin^2\Phi\cos^2\Phi + 128e^4N^3P^2\Phi^3 \\
& + 32e^2e'^2NP^2\Phi\sin^2\Phi + 8e'^4P^2\sin^3\Phi\cos\Phi\Big]
\end{aligned}
$$

The above rigorously developed expression is completely non-iterative, since it requires only the *given* spheroidal longitude. It therefore permits a direct evaluation of the ultimately true λ (that is, $L + x$), extended in this case through terms equivalent to the e^2, e^4 and e^6 order consecutively, in accordance to the accuracy that may be desired. Furthermore, it represents the algebraic solution of the hitherto unknown quantity x used in the power series version of each of the intermediate Helmert expressions.

V. DETERMINATION OF GEODETIC DISTANCE AND AZIMUTHS

The non-iterative expression that has been developed for x suggests at once a numerical solution of distance and azimuths wherein, using the resulting true value of λ, only a single evaluation of Helmert's original formulas is necessary. An illustrative example by such a procedure is given in Sect. VII.

On the other hand, instead of reverting to functions of the true λ, the distance and azimuths themselves can be expanded non-iteratively into power series of x with coefficients in terms of the *given* spheroidal difference of longitude. This is accomplished below, but limited to the e^4 order of accuracy, since this manner of obtaining the distance and azimuths through e^6 would require each component series to one higher power of x than was necessary for λ. Again, the series are developed in the same sequence as the corresponding Helmert expressions.

$$A_0 = \frac{1}{64}(64 + 16h - 3h^2) - \frac{1}{2}(e'^2cP)x$$

$$B_0 = \frac{1}{16}(4h - h^2) - \frac{1}{2}(e'^2cP)x$$

$$C_0 = \frac{h^2}{128}$$

$$A_0\Phi_0 = \frac{1}{64}(64\Phi + 16h\Phi - 3h^2\Phi) + \frac{1}{4}(4c + hc - 2e'^2cP\Phi)x$$
$$+ \frac{1}{2}(P)\,x^2$$

$$B_0 \sin \Phi_0 \cos 2\sigma = \frac{1}{64}(16h \sin \Phi \cos \Phi - 32e'^2P \sin^2 \Phi - 4h^2 \sin \Phi \cos \Phi$$
$$+ 8e'^2hP \sin^2 \Phi) + \frac{1}{4}(hc)x$$

$$- C_0 \sin 2\Phi_0 \cos 4\sigma = \frac{1}{64}(- h^2 \sin \Phi \cos \Phi + 2h^2 \sin^3 \Phi \cos \Phi$$
$$+ 8e'^2hP \sin^2 \Phi \cos^2 \Phi - 8e'^4P^2 \sin^3 \Phi \cos \Phi)$$

$$S = \frac{b_0}{64}(64\Phi + 16h\Phi + 16h \sin \Phi \cos \Phi - 32e'^2P \sin^2 \Phi$$
$$- 3h^2\Phi - 5h^2 \sin \Phi \cos \Phi + 2h^2 \sin^3 \Phi \cos \Phi$$
$$+ 8e'^2hP \sin^2 \Phi + 8e'^2hP \sin^2 \Phi \cos^2 \Phi$$
$$- 8e'^4P^2 \sin^3 \Phi \cos \Phi) + \frac{b_0}{2}(2c + hc - e'^2cP\Phi)x$$
$$+ \frac{b_0}{2}(P)\,x^2$$

$$\cot \alpha_{1-2} = U_1 - \left(\frac{U_2 \cos \beta_1}{\sin L \cos \beta_2}\right) x + \left(\frac{U_1}{2 \sin^2 L} + \frac{U_2 \cos L \cos \beta_1}{2 \sin^2 L \cos \beta_2}\right) x^2$$

$$\cot \alpha_{2-1} = U_2 - \left(\frac{U_1 \cos \beta_2}{\sin L \cos \beta_1}\right) x + \left(\frac{U_2}{2 \sin^2 L} + \frac{U_1 \cos L \cos \beta_2}{2 \sin^2 L \cos \beta_1}\right) x^2$$

The x and x^2 for the above formulas of distance and azimuths can be substituted either numerically or algebraically using, in this case, only the first 6 terms of x for accuracies equivalent to the e^4 order. The *algebraic* substitution gives the following final expressions:

$$S = \frac{b_0}{64}\Big[64\Phi + 64e^2Nc^2\Phi + 16h\Phi + 16h \sin \Phi \cos \Phi$$
$$- 32e'^2P \sin^2 \Phi + 64\, e^4N^2c^4\Phi - 3h^2\Phi + (32e^2N - 4e^2)hc^2\Phi$$
$$- 4e^2hc^2 \sin \Phi \cos \Phi - 5h^2 \sin \Phi \cos \Phi + 2h^2 \sin^3 \Phi \cos \Phi$$
$$+ (96e^4N^2 - 32e^2e'^2N)c^2P\Phi^2 + 8e^2e'^2c^2P \sin^2 \Phi$$
$$+ 8e'^2hP \sin^2 \Phi + 8e'^2hP \sin^2 \Phi \cos^2 \Phi - 8e'^4P^2 \sin^3 \Phi \cos \Phi\Big]$$

$$\cot \alpha_{1-2} = U_1 - \frac{e^2 N c \Phi U_2 \cos \beta_1}{\sin L \cos \beta_2} - \frac{e^4 N^2 c^3 \Phi U_2 \cos \beta_1}{\sin L \cos \beta_2}$$

$$+ \frac{e^2 h c \Phi U_2 \cos \beta_1}{16 \sin L \cos \beta_2} + \frac{e^2 h c U_2 \sin \Phi \cos \Phi \cos \beta_1}{16 \sin L \cos \beta_2}$$

$$- \frac{e^4 N^2 c P \Phi^2 U_2 \cos \beta_1}{\sin L \cos \beta_2} - \frac{e^2 e'^2 c P U_2 \sin^2 \Phi \cos \beta_1}{8 \sin L \cos \beta_2}$$

$$+ \frac{e^4 N^2 c^2 \Phi^2 U_2 \cos L \cos \beta_1}{2 \sin^2 L \cos \beta_2} + \frac{e^4 N^2 c^2 \Phi^2 U_1}{2 \sin^2 L}$$

The corresponding $\cot \alpha_{2-1}$ is obtainable from the above by interchanging U_1 with U_2 and β_1 with β_2.

Thus, progressively, there have been developed three rigorous methods for determining geodetic distance and azimuths non-iteratively: as a function of the true λ, as a power series in x, and culminated by an *explicit* expression in essentially the given spheroidal latitude and longitude of the endpoints. For shorter lines, or for reduced accuracy on long lines, terms may be still further eliminated according to the next higher powers of e^2, e'^2, h and x, or equivalent combinations thereof.

VI. OTHER NON-ITERATIVE SOLUTIONS

The distance and azimuths by the original Helmert method are essentially functions of elements in the following *spherical* triangle:

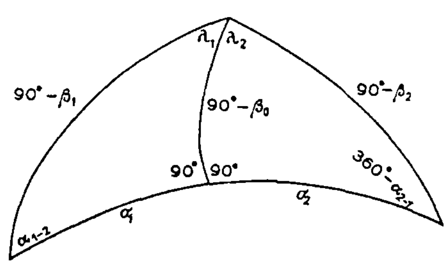

Fig. 1.

where
$$\lambda = \lambda_2 - \lambda_1$$
$$\Phi_0 = \sigma_2 - \sigma_1$$
$$2\sigma = \sigma_2 + \sigma_1$$

and λ_1, λ_2, σ_1 and σ_2 are regarded as negative or positive according to whether they are west or east of the perpendicular arc $90° - \beta_0$. (For this specific configuration, therefore, λ and Φ_0 actually represent the *sum* of the absolute components and 2σ the *difference.*)

Since Helmert's method of successive approximations can only determine λ first, the subsequent solution of the above spherical triangle would always begin with λ and the known β_1 and β_2. The present paper, however, has developed not only a non-iterative expression for λ, but also independent power series for the various elements of this spherical triangle or functions thereof. Therefore the combination of ways to compute quantities leading to the distance and azimuths is increased considerably. In addition, the x for such series can be substituted either numerically or algebraically, in the manner shown for the distance and azimuth series in Sect. V.

The above potentiality for increasing the number of non-iterative solutions is illustrated in AMS Technical Report No. 7, wherein the x and x^2 of the Φ_0 series are algebraically eliminated. The computed value of Φ_0 is then combined with β_1 and β_2 to obtain α's, followed by β_0, 2σ, A_0, B_0, C_0 and finally the geodetic distance. When adopting such varied procedures for solving the reference triangle, care should be taken to avoid formulations which lead to a weak determination of required quantities. These difficulties may most likely occur at extremes of latitude, longitude, or azimuth.

The non-iterative series, too, are functions of elements of a spherical triangle, but defined by β_1, β_2 and the *given* longitude L. This amounts simply to a substitution of L for λ, which results in a spherical triangle with parts corresponding as follows:

Series	β_1	β_2	L	Φ	c	U_1	U_2
Helmert	β_1	β_2	λ	Φ_0	$\cos\beta_0$	α_{1-2}	α_{2-1}

Starting with the given β_1, β_2 and L, the values of all quantities used in the non-iterative series may thus be solved trigonometrically in various orders.

It is also to be noted that in the relation $\lambda = (L + x)$, if x is assumed to be zero, L is considered to be equal to λ. Therefore in the various power series in x, the *constant term* can represent the true value of the series by simply replacing functions of L with λ. This is well illustrated by the A_0 series in x in Sect. V and its counterpart in Sect. II, to the proper order. The principle can well be incorporated in computation forms, such as the one in the next section as applied to H, I, etc.

VII. NUMERICAL ILLUSTRATION OF A SAMPLE SOLUTION

(International Spheroid)

Given $\begin{cases} L = \text{absolute difference of longitude} \\ B_1 = \text{latitude of westward point} \\ B_2 = \text{latitude of eastward point} \end{cases}$	106° 20° North 45° North
$\tan \beta_1 = 0{\cdot}99663\ 29966 \tan B_1$	0·36274 47453
$\tan \beta_2 = 0{\cdot}99663\ 29966 \tan B_2$	0·99663 29966
$\cos \beta_1 = 1 \div + \sqrt{1 + \tan^2 \beta_1}$	0·94006 23275
$\cos \beta_2 = 1 \div + \sqrt{1 + \tan^2 \beta_2}$	0·70829 81969
$\sin \beta_1 = \tan \beta_1 \cos \beta_1$	0·34100 26695
$\sin \beta_2 = \tan \beta_2 \cos \beta_2$	0·70591 33545
$a = \sin \beta_1 \sin \beta_2$	0·24071 83383
$b = \cos \beta_1 \cos \beta_2$	0·66584 44515
$\sin L$	0·96126 16959
$\cos L$	−0·27563 73558
$\cos \Phi = a + b \cos L$	0·05718 67343
$\sin \Phi = (\text{sign of} \sin L)\sqrt{1 - \cos^2 \Phi}$	0·99836 34996
$\Phi = \text{positive radians}$	1·51357 83766
$A = (b \sin L) \div \sin \Phi$	0·64109 99269
$B = A^2$	0·41100 91163
$C = [\cos \Phi - (\cos \Phi)B] \div 4{\cdot}9865\ 20649$	0·00675 47028
$D = -\,a(0{\cdot}40108\ 12630)$	−0·09654 76152
$E = -\,a(0{\cdot}79945\ 93686)$	−0·19244 45307
$F = (3{\cdot}9865\ 20649)C$	0·02692 77622
$G = \Phi^2 \div \sin \Phi$	2·29467 47388
$x_{rad} = \dfrac{\Phi(237{\cdot}2388918 + B) + \sin \Phi\,(C + D) + G(F + E)}{70519{\cdot}51145}.A$	0·00326 58167
$\lambda = L + x$	106° 11′ 13″·62305
$\sin \lambda$	0·96035 63902
$\cos \lambda$	−0·27877 51848
$\cos \Phi_0 = a + b \cos \lambda$	0·05509 74283
$\sin \Phi_0 = (\text{sign of} \sin \lambda)\sqrt{1 - \cos^2 \Phi_0}$	0·99848 09830
$\Phi_0 = \text{positive radians}$	1·51567 09835
$\sin 2\Phi_0 = (\sin \Phi_0 \cos \Phi_0) \div 0{\cdot}5$	0·11002 74687
$\cos \beta_0 = (b \sin \lambda) \div \sin \Phi_0$	0·64042 07839
$q = 1 - \cos^2 \beta_0$	0·58986 12195
$\cos 2\sigma = (2a - q \cos \Phi_0) \div q$	0·76108 89231
$\cos 4\sigma = (\cos^2 2\sigma - 0{\cdot}5) \div 0{\cdot}5$	0·15851 26977
$H = 6356911{\cdot}946 + 10756{\cdot}165q - 13{\cdot}650q^2$	6363251·841
$I = 10756{\cdot}165q - 18{\cdot}200q^2$	6338·312
$J = 2{\cdot}275q^2$	0·792
$\cot \alpha_{1-2} = (\tan \beta_2 \cos \beta_1 - \cos \lambda \sin \beta_1) \div \sin \lambda$	1·07455 96397
$\cot \alpha_{2-1} = (\sin \beta_2 \cos \lambda - \cos \beta_2 \tan \beta_1) \div \sin \lambda$	−0·47245 22891
α_{1-2} = Clockwise from North, in quad I or II for cot + or − respectively	42° 56′ 30″·03557
α_{2-1} = Clockwise from North, in quad III or IV for cot + or − respectively	295° 17′ 18″·59865
S meters $= H\Phi_0 + I \sin \Phi_0 \cos 2\sigma - J \sin 2\Phi_0 \cos 4\sigma$	9649412·854m.

VIII. NUMERICAL COEFFICIENTS FOR OTHER SPHEROIDS

The illustrative solution given in the preceding section contains fixed numerical coefficients which are functions solely of the size and shape of the International spheroid. The algebraic expressions of these coefficients, together with their values, are shown below in the order of appearance in the sample solution. For any other spheroid, these expressions can be quickly re-evaluated once and for all and substituted for the corresponding International values. (Note: $e^2\mathcal{N}$ = flattening.)

$$+\sqrt{1-e^2} = 0{\cdot}99663\ 29966$$

$$(16e^2\mathcal{N}^2 + e'^2) \div e'^2 = 4{\cdot}9865\ \ 20649$$

$$2e'^2 \div (16e^2\mathcal{N}^2 + e'^2) = 0{\cdot}40108\ 12630$$

$$16e^2\mathcal{N}^2 \div (16e^2\mathcal{N}^2 + e'^2) = 0{\cdot}79945\ 93686$$

$$16e^2\mathcal{N}^2 \div e'^2 = 3{\cdot}9865\ \ 20649$$

$$(16\mathcal{N} - e'^2) \div (16e^2\mathcal{N}^2 + e'^2) = 237{\cdot}23\ \ \ 88918$$

$$16 \div e^2(16e^2\mathcal{N}^2 + e'^2) = 70519{\cdot}51145$$

$$b_0 = 6356911{\cdot}946$$

$$b_0e'^2 \div 4 = 10756{\cdot}165$$

$$3b_0e'^4 \div 64 = 13{\cdot}650$$

$$b_0e'^2 \div 4 = 10756{\cdot}165$$

$$b_0e'^4 \div 16 = 18{\cdot}200$$

$$b_0e'^4 \div 128 = 2{\cdot}275$$

IX. ADDITIONAL NOTES ON COMPUTATIONAL PROCEDURES

(1) Although the illustrative solution given in Sect. VII is primarily intended for accuracy equivalent to the e^4 order, it easily lends itself to any required degree. This is accomplished simply by adding or subtracting appropriate terms of x, H, I, J, and S. The extended terms are given in the latter part of Sect. IV and II respectively. For short lines or reduced accuracy on long lines, x on the International spheroid becomes merely $(A\Phi \div 297)$ and all terms in q^2 are omitted, with the consequent elimination of many other supporting quantities. Similar savings are realized for other forms of solutions presented herein.

(2) For short lines, the resulting small Φ is computed more accurately from sin Φ obtained as follows:

$$\sin\frac{\Phi}{2} = +\sqrt{b\sin^2\frac{L}{2} + \sin^2\frac{\beta_1 - \beta_2}{2}}$$

$$\cos\frac{\Phi}{2} = (\text{sign of} \sin L)\sqrt{0{\cdot}5\,(1 + \cos\Phi)}$$

$$\sin\Phi = \left(\sin\frac{\Phi}{2}\cos\frac{\Phi}{2}\right) \div 0{\cdot}5$$

Similarly, $\sin\Phi_0$ is obtained as above by replacing Φ with Φ_0 and L with λ. In either case, squaring the small sines under the radical increases their significant decimal places.

(3) If the numerator of x is to be cumulated in a ten digit calculator, 9 decimal places should be allotted to Φ, $\sin\Phi$ and G, but only 7 decimals to their multipliers. However, when the value of G is 10 or greater, decrease its decimal places accordingly and increase those of F and E correspondingly. For a smaller calculator, reduce all decimals equally.

(4) Use co-function of $\tan\beta$ or $\cot\alpha$ when their values are too large.

$$\text{Thus}\quad \cot\beta_n = \frac{\cot B_n}{+\sqrt{1-e^2}} \quad \text{and} \quad \tan\alpha = \frac{1}{\cot\alpha}$$

(5) The accuracy of geodetic distances computed through the e^2, e^4 and e^6 order for *very long geodesics* is within a few meters, centimeters and tenths of millimeters respectively. Azimuths are good to tenths, thousandths, and hundred thousandths of a second. Further improvement of results occurs for shorter lines.

(6) Some of the terms in the sample solution of Sect. VII have been grouped for ease of computing by desk calculator. For electronic computers, however, the terms are best left in series form, thus being ideally suited to adding or removing them according to accuracy requirements.

X. ANTIPODAL POINTS

In the various series that have been presented, Φ represents a spherical arc distance which varies from 0° to 180° and even to 360° according to whether the geodetic line is very short, half around the earth or completely around it. At these specific instances, quantities such as $\csc\Phi$, $\cot\Phi$, and P approach infinity. For the case of the very short lines, this condition is equalized by the factors Φ and $\sin\Phi$ which approach zero. For the other two cases, however, the series gradually fail to converge due to the Φ becoming larger.

Closer inspection of the various series in x shows, nevertheless, that this condition of divergence never prevails in the constant terms, and for succeeding coefficients it is to no greater degree than the *power of the corresponding x.* Therefore, here too it could be equalized if x were *sufficiently small.*

The first equation of Sect. III relates x as follows:

$$\lambda = (L + x).$$

This true value of λ could have been represented, instead, by:

$$\lambda = (L_n + z)$$

where L_n is an arbitrary amount of longitude *more nearly* equal to λ and therefore z is correspondingly smaller than x. This new assumption leads to a set of power series in z which are identical to those in x, except that its coefficients will be a function of L_n instead of L. The obvious value to assign to L_n would be the slightly inaccurate result of solving an antipodal line in powers of the larger x.

The relation derived at the beginning of Sect. IV will accordingly change from:

$$\Upsilon \cos \beta_0 = x$$

to:

$$(L - L_n) + \Upsilon \cos \beta_0 = z$$

where, as noted, the substitution of the $\Upsilon \cos \beta_0$ series given at the end of Sect. III will now be in terms of L_n and z instead of L and x. Solving the above equation for z (this time through only the e^4 order of accuracy) gives:

$$z = \frac{16(L - L_n) + (16e^2 NcΦ - e^2hcΦ - e^2hc \sin Φ \cos Φ + 2e^2e'^2cP \sin^2 Φ)_n}{16(1 - e^2Nc^2 - e^2NPΦ)_n}$$

where the subscripts n to the parenthesis indicate that c, $Φ$, h, P, etc. are functions of L_n instead of L. This time, the denominator of the expression cannot be algebraically divided into the numerator, because the $e^2NPΦ$ term is relatively large for nearly antipodal lines.

With the above correction z to an arbitrary but sufficiently accurate value L_n, the true λ of antipodal lines is essentially obtained again non-iteratively, and therefore more rapidly than by numerous individual successive approximations. Thus, also, a previous 4″ longitude discrepancy noted by Mr. H. F. RAINSFORD[1] for a line of about 179° 46′ 18″ longitude would be resolved. In this connexion, this author would like to express his appreciation for the interest shown by Mr. RAINSFORD, resulting in profitable correspondence.

REFERENCE

1 RAINSFORD, H. F. *Bull. géod.* No. 37 (1955).