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# Solutions of Inverse Geodetic Problem in Navigational Applications

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ABSTRACT: Solutions of such navigational problems as an orthodromic navigation (courses, distances and intermediate points), maximum latitude and a composite navigation with limited latitude as well as, for comparison, a loxodromic navigation (courses, distances) without any simplifications for a sphere, by an application of solutions of the inverse geodetic problem are presented. An exemplary rigorous, rapid, non-iterative solution of the inverse geodetic problem according to Sodano, for any length of geodesic, is attached.

## 1 INTRODUCTION

The traditional method of computing the shortest path between two points on the Earth known as Great Circle method approximate the Earth as the sphere.

These simplifications have been necessary to reduce the number of calculations and justified in times of manual mechanical or electronic calculators, but are completely unnecessary and unjustified in times of computer calculations. Therefore we will directly apply the solution of the problem known in geodesy as the inverse geodetic problem.

In the solution of the inverse geodetic problem (Fig. 1) from the given coordinates $\varphi_1$, $\lambda_1$ at the start of geodesic $P_1$ and coordinates $\varphi_2$, $\lambda_2$ of the endpoint $P_2$ are calculated the length S, the azimuth $\alpha_{1\text{-}2}$ and the reversed azimuth $\alpha_{2\text{-}1}$, on any reference ellipsoid.

E. M. Sodano (Sodano 1958, 1965, 1967) from Helmert's classical iterative formulas derived a rigorous non-iterative procedure, for any length of geodesic and for any required accuracy, which is attached (as exemplary) in Appendix A. This procedure (or any other solution of the inverse geodetic problem) will be used in this paper in the formal notation

$$S = \mathrm{IGP}\,(\varphi_1, \lambda_1, \varphi_2, \lambda_2) \tag{1}$$

$$\alpha_{1\text{-}2}, \alpha_{2\text{-}1} = \mathrm{IGP}\,(\varphi_1, \lambda_1, \varphi_2, \lambda_2) \tag{2}$$

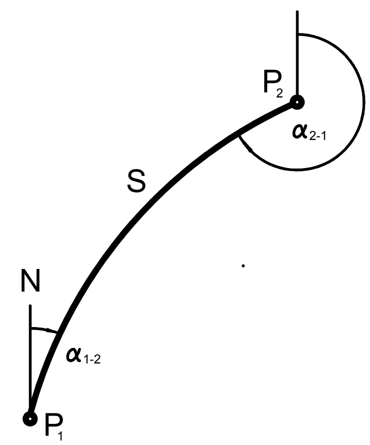


Figure 1. The direct and the inverse geodetic problem.

## 2 ORTHODROMIC DISTANCE AND COURSES

We define the orthodrome as the shortest path on any surface and not only the Great Circle distance on the sphere as commonly is used.

The geodesic is (locally - not long way round) the shortest path between two points on an ellipsoid of revolution. Therefore we can obtain orthodromic distance and courses directly from Equations 1 and 2 with navigational substitutions

$$C_{gs} = \alpha_{1\text{-}2} \tag{3}$$

$$C_{ge} = \alpha_{2\text{-}1} - 180° \tag{4}$$

where $C_{gs}$ = the course over ground at the start of the orthodrome; and $C_{ge}$ = the course over ground at the end of the orthodrome.

East longitudes and north latitudes are considered positive and west longitudes and south latitudes are considered negative.

## 3 ACCURACY OF THE SOLUTION OF THE INVERSE GEODETIC PROBLEM

"The accuracy of geodetic distances computed through the $e^2$, $e^4$, $e^6$ order for very long geodesics is within a few meters, centimeters and tenth of millimeters respectively. Azimuths are good to tenth, thousandths and hundreds thousandths of a second. Further improvement of results occurs for shorter lines" (Sodano 1958).

This accuracy can be easy tested in the case of equatorial orthodrome. Substitution $\varphi_1 = \varphi_2 = 0$ to Equations A 2 to A 10 yields

$$S = b_0(1 + f + f^2)|L| \tag{5}$$

whereas the correct value is given by the equation

$$S^* = a_0|L| = \frac{b_0}{1-f}|L| \tag{6}$$

therefore the relative error is

$$\Delta S = \frac{S - S^*}{S^*} \approx -38 \cdot 10^{-9} \approx -38\,\text{cm}/10\,000\,\text{km} \tag{7}$$

## 4 ERRORS OF CALCULATIONS ON THE SPHERE

According to Euler's theorem for an ellipsoid of revolution the radius of curvature in meridian is the smallest and the radius of curvature in the prime vertical is the largest at a point. These radii are given respectively by the equations

$$R_M = \frac{a_0(1-e^2)}{\sqrt{(1-e^2\sin^2\varphi)^3}} \tag{8}$$

$$R_N = \frac{a_0}{\sqrt{1-e^2\sin^2\varphi}} \tag{9}$$

The widest span has the radius of curvature in meridian since

$$R_{M\,min} = R_M(\varphi = 0°) = \frac{b_0^{\,2}}{a_0} \tag{10}$$

$$R_{M\,max} = R_M(\varphi = 90°) = R_N(\varphi = 90°) = \frac{a_0^{\,2}}{b_0} \tag{11}$$

The substitute radius of curvature of any orthodrome will be within these limits. The minimum absolute value of deviation gives an assumption that the substitute radius of sphere is given by the equation (for a global range of latitudes)

$$R_S = \frac{R_{M\,max} + R_{M\,min}}{2} \tag{12}$$

Then the maximal relative error of calculation on such a sphere, instead of an ellipsoid, gives the equation

$$\Delta S_S = \frac{\pm(R_{M\,max} - R_{M\,min})}{R_{M\,max} + R_{M\,min}} \approx \pm 0.5\% \approx \pm 50\ \text{km}/10\ 000\ \text{km} \tag{13}$$

These results are similar to obtained by Earle (2006) with much more complicated methods.

## 5 INTERMEDIATE POINTS ON THE ORTHODROME

For calculating intermediate points on the orthodrome we can use, as exemplary, the solution of the direct geodetic problem presented in Lenart (2011), also according to Sodano, having similar accuracy.

In the solution of the direct geodetic problem (Fig. 1) from the given coordinates $\varphi_1$, $\lambda_1$ and azimuth $\alpha_{1\text{-}2}$ at the start of geodesic $P_1$ and their length S are calculated coordinates $\varphi_2$, $\lambda_2$ of the endpoint $P_2$ and the reversed azimuth $\alpha_{2\text{-}1}$, on any reference ellipsoid.

This procedure (or any other solution of the direct geodetic problem) will be used in this paper in the formal notation

$$\varphi_2, \lambda_2 = \text{DGP}(\varphi_1, \lambda_1, \alpha_{1\text{-}2}, S) \tag{14}$$

$$\alpha_{2\text{-}1} = \text{DGP}(\varphi_1, \lambda_1, \alpha_{1\text{-}2}, S) \tag{15}$$

The orthodrome of the length S we will divide for n suborthodromes (Fig. 2) of any length $S_i$ such as

$$\sum_{i=1}^{n} S_i = S \quad \text{(from Equation 1)} \tag{16}$$

and intermediate points are calculated in n iterations:

FOR i=1 to n

IF i=1 THEN

$$(\varphi_2, \lambda_2)_{i-1} = \varphi_1, \lambda_1 \tag{17}$$

$$(\alpha_{2-1})_{i-1} - 180° = \alpha_{2-1} \text{ (from Equation 2)} \tag{18}$$

ENDIF

$$(\varphi_2, \lambda_2)_i = \text{DGP}\left((\varphi_2, \lambda_2)_{i-1}, (\alpha_{2-1})_{i-1} - 180°, S_i\right) \tag{19}$$

$$(\alpha_{2-1})_i = \text{DGP}\left((\varphi_2, \lambda_2)_{i-1}, (\alpha_{2-1})_{i-1} - 180°, S_i\right) \tag{20}$$

NEXT i

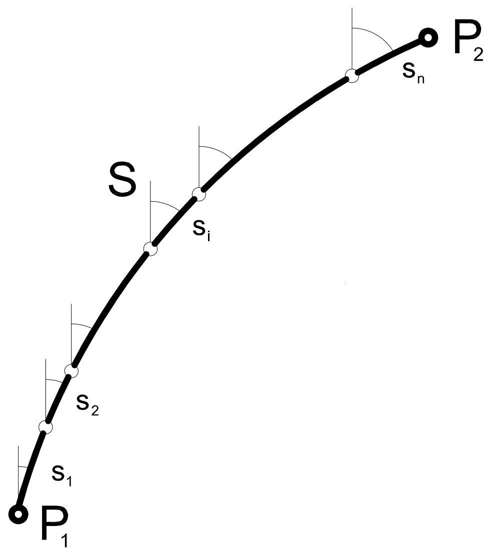


Figure 2. Intermediate points on the orthodrome.

or even n-1 iterations because the last iteration is for verification only that

$$(\varphi_2, \lambda_2)_n = P_2(\varphi_2, \lambda_2) \tag{21}$$

In traditional navigation intermediate points are calculated during planning the voyage to navigate between them along a loxodrome. If we have programmed procedure on the bridge during the voyage then the situation can be quite different. In this case the intermediate points are needed e. g. for the verification of the path on the map only. During the voyage if we enter as $P_1$ the current position and $P_2$ is constantly the endpoint then the procedure gives the course over ground for the orthodrome to the endpoint, even if we e. g. due set and drift are out of track, and not to an intermediate point. We can calculate a new course for the orthodrome after each position fixing and to navigate always along the current orthodrome without these calculated intermediate points.

## 6 MAXIMUM LATITUDE

According to Clairaut's relation for a geodesic on a surface of revolution

$$r \sin \alpha = \text{const.} = C \tag{22}$$

where r = the radius of parallel.

For an ellipsoid of revolution

$$r = R_N \cos \varphi \tag{23}$$

At $\varphi_{max}$

$$|\sin \alpha| = 1 \tag{24}$$

therefore

$$\left| \frac{a_0 \cos \varphi_{max}}{\sqrt{1 - e^2 \sin \varphi_{max}}} \right| = |C| \tag{25}$$

and finally

$$|\sin \varphi_{max}| = \sqrt{\frac{a_0^2 - C^2}{a_0^2 - e^2 C^2}} \tag{26}$$

where C e.g. is

$$C = r(\varphi_1) \sin \alpha_{1-2} \tag{27}$$

The above equations are valid if

$$|\sin \alpha| = 1 \tag{28}$$

exists on our orthodrome i.e when

$$(C_{gs} - 90°)(C_{ge} - 90°) < 0 \tag{29}$$

or

$$(C_{gs} - 270°)(C_{ge} - 270°) < 0 \tag{30}$$

or else

$$|\varphi_{max}| = \max(|\varphi_1|, |\varphi_2|) \tag{31}$$

## 7 COMPOSITE NAVIGATION

If for any reason $\varphi_{max}$ is limited to $\varphi_{lim}$ then (Fig. 3) we have:

1 The orthodome I (Ort I) – from φ1, λ1 to φlim, λ2OI.
2 The loxodrome (Lx) – at φlim from λ2OI to λ1OII.
3 The orthodome II (Ort II) – from φlim, λ1OII to φ2, λ2.

We will obtain $\lambda_{2OI}$ and $\lambda_{1OII}$ in iterative procedures:

$C_{ge}$ = IGP ($\varphi_1$, $\lambda_1$, $\varphi_{lim}$, $\lambda_{2OI}$ = var) (32)

where $\lambda_{2OI}$ is adjusted by any small increments until $C_{ge}$ = 90° if L > 0 or 270° if L < 0

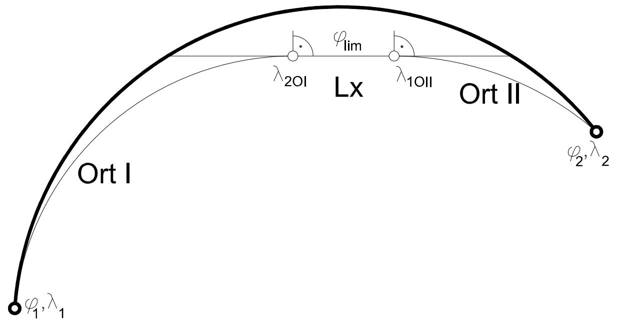


Figure 3. Composite navigation.

and

$C_{gs}$ = IGP ($\varphi_{lim}$, $\lambda_{1OII}$=var, $\varphi_2$, $\lambda_2$) (33)

where $\lambda_{1OII}$ is adjusted by any small increments until $C_{gs}$ = 90° if L > 0 or 270° if L< 0

This iterative process, although looks as very complicated, is very fast and simple with using e.g. the Solver in Microsoft Excel.

## 8 LOXODROMIC DISTANCE AND COURSE

The ortodromic navigation is for shorter distances then in the loxodromic navigation therefore to calculate this difference we will calculate, for comparison, the loxodromic distance and course on an ellipsoid.

$$S_{lx} = \left|\frac{S_M(\varphi_1,\varphi_2)}{\cos\alpha_{lx}}\right| \quad (34)$$

where $S_M(\varphi_1, \varphi_2)$ = the meridian distance between $\varphi_1$ and $\varphi_2$; and $\alpha_{lx}$ = the course over ground for loxodrome.

$$\tan\alpha_{lx} = \frac{L}{\Delta\varphi} \quad (35)$$

where

$$\Delta\varphi = \ln(tg(\pi/4+\varphi_2/2)) - \ln(tg(\pi/4+\varphi_1/2)) + \\ + \frac{e}{2}(\ln\frac{1-e\sin\varphi_2}{1+e\sin\varphi_2} - \ln\frac{1-e\sin\varphi_1}{1+e\sin\varphi_1}) \quad (36)$$

In our case

$S_M(\varphi_1, \varphi_2)$ = IGP ($\varphi_1$, $\lambda_1$, $\varphi_2$, $\lambda_1$) (37)

Equation 33 is valid if $\varphi_1 \neq \varphi_2$ or else

$$S_{lx} = r\,(\varphi_1 = \varphi_2)\cdot|L| \quad (38)$$

## 9 INVERSE COMPUTATION FORM SIMPLIFIED

For shorter distances (the very long geodesic in paragraph 3 means even 20 000 km) or lower required accuracies we can use equations from Appendix A reduced to f order (having the accurate solution for reference in errors calculations). Therefore Equation A 10 becomes to

$$S = b_0[(1+f+f^2)\Phi - (f+f^2)m\Phi/2 + \\ +(f+f^2)(2a - m\cos\Phi)\sin\Phi/2] \quad (39)$$

and Equation A 11 becomes to

$$\gamma = (f+f^2)\Phi c + L \quad (40)$$

or

$$S = A_S\Phi + B_S[c^2\Phi + (2a - m\cos\Phi)\sin\Phi] \quad (41)$$

where

$$A_S = [1 + \frac{1}{2}(f+f^2)]b_0 \quad (42)$$

$$B_S = [\frac{1}{2}(f+f^2)]b_0 \quad (43)$$

It is evident that f + $f^2$ are from reduced higher order elements from series

$$f + f^2 + f^3 + \cdots = \frac{1}{1-f} - 1 \quad (44)$$

Noting that

$$\frac{b_0}{1-f} = a_0 \quad (45)$$

Equations 42 and 43 are thus

$$A_S = (a_0 + b_0)/2 \quad (46)$$

$$B_S = (a_0 - b_0)/2 \quad (47)$$

This simplified computation form gives errors in the range of meters (and has no errors for equatorial orthodromes.

## 10 CIRCULAR FUNCTIONS

The angles $\alpha_{1\text{-}2}$, $\alpha_{2\text{-}1}$ from Equations A 12, A 13 and $\alpha_{lx}$ from Equation 35 have to be calculated with the circular function $\tan^{-1}()$, but this function gives solutions in the range (-90°, 90°). For full range (0°, 360°) retrieving tables of quadrants are used in Sodano (1965).

For computer calculations a special procedure should be used to retrieve the full range (0°, 360°) from the signs of the numerator N and the denominator D and to detect and support a division by zero case e.g.:

For

$$\text{ANGLE} = \text{TAN}^{-1}\frac{N}{D}$$

IF D≠0 THEN

ANGLE=ATN(N/D)

IF D<0 THEN ANGLE=ANGLE+180°: END IF

ELSE

ANGLE=(2-SGN(N))*ABS(SGN(N))*90°

END IF

IF ANGLE<0 THEN ANGLE=ANGLE+360°: END IF

## 11 CONCLUSIONS

The set of presented procedures are quite general and universal. They can be used with any solutions of the inverse (and direct) geodetic problems as well during the voyage planning as during the voyage in real time, for "full" orthodrome navigation.

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## APPENDIX A

**Inverse computation form** (Sodano 1965, 1967)

Given: $\varphi_1$, $\lambda_1$, $\varphi_2$, $\lambda_2$

Required: $\alpha_{1\text{-}2}$, $\alpha_{2\text{-}1}$, S

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Reference ellipsoid: $a_0$, $b_0$ = semi-major and semi-minor axes

Flattening

$$f = 1 - \frac{b_0}{a_0} \tag{A 1}$$

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$$\tan\beta_1 = (1-f)\tan\varphi_1 \tag{A 2}$$

$$\tan\beta_2 = (1-f)\tan\varphi_2 \tag{A 3}$$

$$a = \sin\beta_1 \sin\beta_2 \tag{A 4}$$

$$b = \cos\beta_1 \cos\beta_2 \tag{A 5}$$

$$L = \lambda_2 - \lambda_1 \tag{A 6}$$

$$\cos\Phi = a + b\cos L \tag{A 7}$$

$$c = \frac{b\sin L}{\sin\Phi} \tag{A 8}$$

$$m = 1 - c^2 \tag{A 9}$$

$$\begin{aligned} S = b_0[&(1+f+f^2)\Phi \\ &- \frac{(f+f^2)m\Phi}{2} \\ &+ \frac{(f+f^2)(2a - m\cos\Phi)\sin\Phi}{2} \\ &+ \frac{f^2m^2(\Phi + \sin\Phi\cos\Phi)}{16} \\ &- \frac{f^2(2a - m\cos\Phi)^2\sin\Phi\cos\Phi}{8} \\ &- \frac{f^2(1-m)(a - m\cos\Phi)\Phi^2\csc\Phi}{2}] \end{aligned} \tag{A 10}$$

$$\begin{aligned} \gamma = \Big\{ &(f+f^2)\Phi - [\frac{f^2\sin\Phi}{2} + f^2\Phi^2\csc\Phi]a \\ &+ [-\frac{5f^2\Phi}{4} + \frac{f^2\sin\Phi\cos\Phi}{4} + f^2\Phi^2\cot\Phi]m \Big\}c + L \end{aligned} \tag{A 11}$$

$$\tan\alpha_{1-2} = \frac{\cos\beta_2\sin\gamma}{\sin\beta_2\cos\beta_1 - \sin\beta_1\cos\beta_2\cos\gamma} \tag{A 12}$$

$$\tan\alpha_{2-1} = \frac{-\cos\beta_1\sin\gamma}{(\sin\beta_1\cos\beta_2 - \sin\beta_2\cos\beta_1\cos\gamma)} \tag{A 13}$$