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APPROXIMATE SOLUTIONS TO STOCHASTIC DYNAMIC PROGRAMS

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This paper examines the properties of various approximation methods for solving stochastic dynamic programs in structural estimation problems. The problem addressed is evaluating the expected value of the maximum of available choices. The paper shows that approximating this by the maximum of expected values frequently has poor properties. It also shows that choosing a convenient distributional assumptions for the errors and then solving exactly conditional on the distributional assumption leads to small approximation errors even if the distribution is misspecified.

1. INTRODUCTION

Over the last few years, there has been much research on estimating stochastic dynamic programming models of rational behavior.[1] Some papers, including those by Berkovec and Stern (1991), Miller (1984), Pakes (1987), Rust (1989), and Wolpin (1984, 1987), solve the dynamic programming problem explicitly. In particular, they make assumptions so that they can compute the expected value of the maximum of the values of choices available next period, called Emax in this paper. Papers such as those by Stock and Wise (1990) and Manski (1988) use the maximum of expected values, called maxE, to approximate Emax. Lumsdaine, Stock, and Wise (1991) showed that, for a particular example, their approximation has the same predictive power as the exact solution and is much better than a model that ignores dynamics completely. Ausink and Wise (1993) provided similar results for another application.

In this paper, I examine the properties of maxE as an approximation for Emax in dynamic programming models and the properties of Emax when the distribution of the errors is misspecified. The results suggest that maxE is a good approximation for Emax only in limited situations but that Emax performs well even when error distributions are misspecified.

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2. THE MODEL

I use a simplified version of Berkovec and Stern (1991). An individual can either be in state 0 or state 1 at age t. When in state i, the individual receives a flow that has a nonrandom component $\omega_i(t)$ and a stationary, stochastic component ϵ_{ti}. For a work-retirement model, $\omega_i(t)$ would be the deterministic component of wages, pensions, leisure, and other nonpecuniary flows, and ϵ_{ti} would be the stochastic component. Throughout, I assume that $\epsilon_{ti} \sim$ i.i.d. G for $i = 0,1$, where G does not vary over i. Furthermore, I assume $E\epsilon_{ti} = 0$, Var $\epsilon_{ti} = \sigma^2 < \infty$.

It is assumed that the individual observes ϵ_{t0} and ϵ_{t1} before making a decision so that he or she contributes randomness to observed choices. This is equivalent in the retirement model to observing one's wages and working conditions before deciding whether or not to retire. If the individual just entered state i, he must pay a start-up cost c_i. In a retirement model, there may be significant costs associated with starting a new job such as moving, learning the job, or buying new clothes or equipment (see Berkovec and Stern, 1991, for an estimate of the large magnitude of these costs).

The value of being in state i at time t is

$$V_e(i,t,s) = \omega_i(t) + \epsilon_{ti} - c_i s + \beta E \max_i [V_e(i,t+1,0), V_e(1-i,t+1,1)], \tag{2.1}$$

where s is an indicator of whether state i was just entered ($s = 1$ if just entered, and $s = 0$ if not), β is a discount factor, and $1 - i$ is the state other than i. The value function is indexed with "e" to denote that it is the exact value function.

In general, the difficult problem is to evaluate the expected value of the maximum of the two available choices.[2] Let $\bar{V}_{ti} = V(i,t,s) - \epsilon_{ti}$. Rust (1989) and Berkovec and Stern (1991) have assumed $\epsilon_{ti} \sim$ i.i.d. Extreme Value, implying

$$E \max[V(i,t,s), V(1-i,t,1-s)] = \theta \ln \sum_i \exp\{\bar{V}_{ti}/\theta\}, \tag{2.2}$$

where θ is a dispersion parameter. The use of (2.2) makes it easy to evaluate the value functions backwards recursively. For the remainder of this paper, I will refer to using the extreme value assumption in solving the value function problem as the Emax method even though it is the expected value of the maximum only if G is extreme value.

Stock and Wise (1990) approximated the value function as

$$V_a(i,t,s) = \omega_i(t) + \epsilon_{ti} - c_i s + \beta \max_i [EV_a(i,t+1,0), EV_a(1-i,t+1,1)]. \tag{2.3}$$

For the remainder of the paper, this approximation will be called the maxE approximation. This approximation differs from the Emax function in that the expected value of the maximum is replaced by the maximum of expected values. This ignores Jensen's inequality and most of the selection bias literature such as that by Heckman (1979). However, if it causes only small approximation errors, then it may be a reasonable alternative to exact computation. In some sense, exact computation is also an approximation because it depends on knowing the distributional assumption G. Some of the motivation in Manski (1988, 1991) and Ahn and Manski (1993) comes from not wanting to impose an arbitrary assumption on G; however, they must assume that ϵ_t is observed only after the choice at age t is made. Given this assumption, the maxE approximation is exact.

In fact, $V_a(.,.,.)$ converges uniformly to $V_e(.,.,.)$, and the hazard rates[3] between states converge uniformly as σ, the standard deviation of ϵ_{ti}, approaches zero. These results are proven in the Appendix. However, a model with a very small σ is problematic in that (a) it will not be able to explain variation in individual choices and (b) it will imply a hazard rate function between states that is zero almost everywhere with a spike at one time t (when it is optimal for almost all individuals to switch states). In contrast, when σ is relatively large and either of the start-up costs, c_0 or c_1, are large, the approximation behaves poorly (Stern 1994). Both Berkovec and Stern (1991) and Stock and Wise (1990) estimated large values of σ.

I consider the following special case for the remainder of the paper. Replace (2.1) by

$$V(F,t,s) = w + \epsilon_{tF} - c_F s + \beta E \max[V(F,t+1,0), V(R,t+1,1)] \quad \textbf{(2.4)}$$

and

$$V(R,t,s) = r(t) + \epsilon_{tR} + \beta E \max[V(R,t+1,0), V(F,t+1,1)], \quad \textbf{(2.5)}$$

where the two states are full-time work F or retirement R. Assume that wage is constant without loss of generality because only $w - r(t)$ is relevant. Assume that $c_R = 0$ following Berkovec and Stern (1991). Note that individuals can return to work after retiring, as is consistent with, for example, Berkovec and Stern (1991) or Rust (1989). When it is not confusing, I use the notation Emax (V_F, V_R) to denote the last term in (2.4) and (2.5). Section 3 reports Monte Carlo results for the specification in (2.4) and (2.5) concerning the precision of various approximations for the value functions.

3. HAZARD RATE RESULTS

Using the model described in (2.4) and (2.5) with i.i.d. Extreme Value errors, I examine the precision of four different methods of approximating the value functions[4]: (a) Emax, (b) maxE, (c) ignore dynamics by setting Emax $(V_F, V_R) = 0$, and (d) exact solution assuming G is normal with variance equal to the Extreme Value variance $(\pi^2\theta^2/6)$. Throughout, I assume

that $\beta = 0.9$, $w = 4.0$, and $r(t) = a\exp\{b(t-1)/100\}$, where $a = 3.0$ and $b = 0.2$.

Construction of the hazard rate requires computing probabilities that depend on G and the expected value of the value functions. I focus on hazard rates because they are essentially what are used to estimate dynamic programming models of this type. The model parameter estimates will be consistent only if the hazard rates are correctly specified given true parameters.

In the first example, $c_F = 0.2$ and $\theta = 1.0$. The hazard rates are displayed in Panel A of Figure 1. It is clear that, for this example, no method is very different from any other. In all cases, hazard rates are increasing because $r(t)$ is increasing relative to w. The shape of the hazard rate using the maxE approximation is a little different than the other three but probably not in a particularly significant way.

In the second example, displayed in Panel B, $c_F = 2.0$ and $\theta = 1.0$. These parameter values are reasonable values given the estimates in Berkovec and Stern (1991). The results here vary significantly over methods. The maxE approximation is very biased whether G is extreme value or normal. In fact, it is not clear in this example that the maxE approximation is better than just ignoring dynamics. However, the choice of G seems to be relatively unimpor-

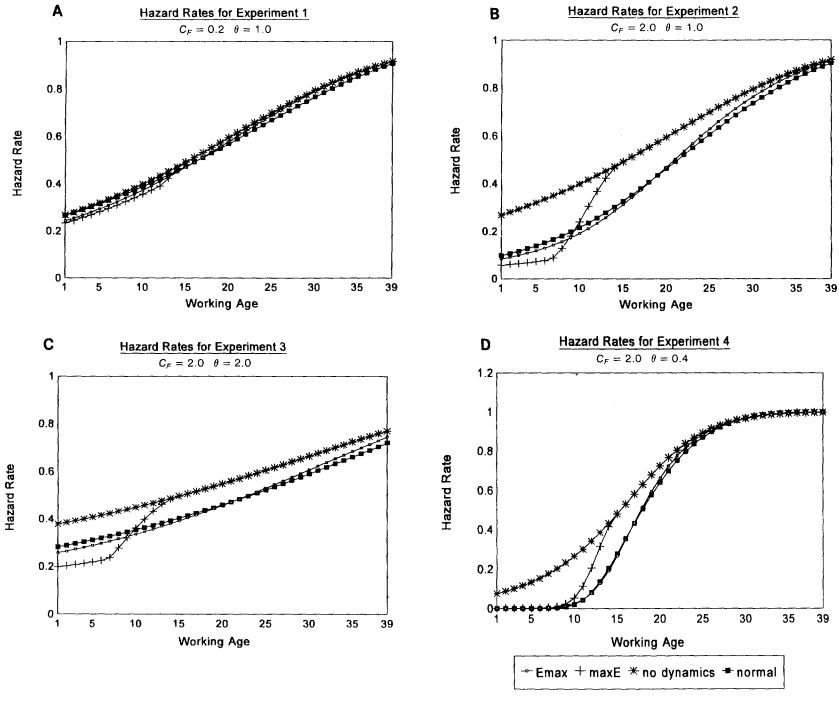

FIGURE 1. Hazard rates for experiments 1–4.

tant given the proximity of the hazard rates for the Emax and exact normal methods.

One should expect the maxE approximation to perform poorly here because, with a large c_F, the option value of working is significant relative to the option value of retiring. The maxE approximation misses the part of the option value due to the conditional expected value of the error (CEVE),

$$\sum_{i=0}^{1} E[\epsilon_{ti} \mid V(i,t,s) \geq V(1-i,t,1-s)] \Pr[V(i,t,s) > V(1-i,t,1-s)], \quad \textbf{(3.1)}$$

which is closely related to the last term in equation (2.1).

This is confirmed in panel C, where $c_F = 2.0$ and $\theta = 2.0$. Increasing θ makes the CEVE more important. As can be seen, the maxE approximation performs even worse than in Panel B. Yet the Emax and exact normal solutions are still close to each other.

Panel D sets $c_F = 2.0$ and $\theta = 0.4$. In this case, the variance of ϵ_{ti} is small. Thus, the CEVE should be relatively less important. In fact, for those ages where it is unimportant, the exit probability is near zero or one because one choice clearly dominates the other. At those ages, the methods allowing for dynamics are close; however, for those ages where $\overline{V}_{tF}$ and $\overline{V}_{tR}$ are close enough to each other so that the exit probability is not near zero or one, the maxE approximation hazard rate again deviates from the Emax and exact normal solution hazard rates. Yet the Emax and exact normal solution hazard rates are almost indistinguishable.

The maxE approximation hazard rate is less smooth than the Emax approximation for panels B–D. In particular, it starts off looking like the Emax approximation hazard rates and then, at some point, increases quickly to look like the hazard rate for the no dynamics solution. At the beginning, retiring is such a bad option that the maxE individual knows she will want to work again next period (and therefore will incur the start-up cost c_F). Thus, she behaves like an individual solving the exact solution. At some point, when the two expected flows are relatively close, returning to work is no longer as good an option, especially when not properly accounting for the CEVE. So the hazard rate rises faster than it would if one accounted for the CEVE. Eventually, when the expected retirement flow is greater than the expected work flow, the individual assumes she will not return to work. This is equivalent to what the individual ignoring dynamics would do.

It may be that, even though the maxE approximation performs poorly in terms of the hazard rate, it results in only small losses in utility. Table 1 displays the loss in utility when any of the three approximations is used. In the first experiment (low c_F, intermediate θ), the maxE utility loss is small, but it is larger than the loss associated with ignoring dynamics. Essentially, dynamics do not affect choices much in this example. In the next three examples, the loss is equal to about 0.4% of lifetime utility. This is much smaller

TABLE 1. Initial utility levels

	$c_F = 0.2$ $\theta = 1.0$	$c_F = 2.0$ $\theta = 1.0$	$c_F = 2.0$ $\theta = 2.0$	$c_F = 2.0$ $\theta = 0.4$
Emax	45.508	43.537	48.406	42.040
Loss	(0.000)	(0.000)	(0.000)	(0.000)
maxE	45.428	43.310	48.201	41.876
Loss	(0.080)	(0.227)	(0.205)	(0.164)
No dynamics	45.456	42.587	47.477	41.153
Loss	(0.052)	(0.950)	(0.929)	(0.887)
Normal	45.493	43.517	48.385	42.034
Loss	(0.015)	(0.020)	(0.021)	(0.006)

Notes: Numbers in each cell are utility of having an old full-time job in the first period when the policy rule corresponds to the left column even though reality implies the best rule is the exact rule. Loss numbers are utility differences between the exact rule and the rule used to make decisions. Figures are based on 10,000 simulations.

than the loss associated with ignoring dynamics (2% of lifetime utility), but it is larger than the loss when the normal approximation is used (0.04% of lifetime utility). Thus, the maxE losses are only moderate in size, but they still lead to significantly different decision rules.

4. THE ROBUSTNESS OF Emax

One might still argue that there are many distributions for which the $\text{Emax}(V_F, V_R)$ is very different than if G is normal or extreme value. The Williams-Daly-Zachary Theorem (McFadden, 1981) describes restrictions on $\text{Emax}(V_F, V_R)$. This section simulates $\text{Emax}(V_F, V_R)$ for a large class of distributions.

I consider a class of distributions with common mean and variance but different degrees of kurtosis so that I can measure the effect of varying tail thickness on $\text{Emax}(V_F, V_R)$. Let $X \sim N(0,1)$, and let

$$Y = \text{sign}\, X \cdot |X|^{\alpha}. \tag{4.1}$$

Let $\epsilon_i = Y/\sqrt{\text{Var}\, Y}$ for $i = F, R$. Then, $E\epsilon_i = 0$, $\text{Var}\, \epsilon_i = 1$, and $E\epsilon_i^4$ depends on α, as can be shown in Table 2. I will call this distribution the exponential normal distribution. The fourth moments in Table 2 are simulated using antithetic acceleration (Geweke, 1988). The standard normal has a fourth moment of 3 (simulated to be 3.099), and the extreme value has a fourth moment of 5.4 (Johnson and Kotz, 1970, p. 278). Thus, this class allows for a wide range of kurtosis. This class should reasonably approxi-

TABLE 2. Simulated fourth moments of errors

Exponent (α)	Fourth moment	$\Pr(\lvert\epsilon\rvert > 3)$
0.50	1.576	$8.16e-13$
0.75	2.181	$9.98e-5$
1.00	3.099	0.0026
1.20	4.233	0.0075
1.60	9.517	0.0168
2.00	31.999	0.0216
2.50	204.081	0.0220

mate the range of $\text{Emax}(V_F, V_R)$ because that should depend mostly on higher moments of the data.

Figure 2 graphs $\text{Emax}(V_F, V_R)$ for those values of α in Table 2 less than 2.5. This allows for kurtosis between 1.576 and 31.999. The value of

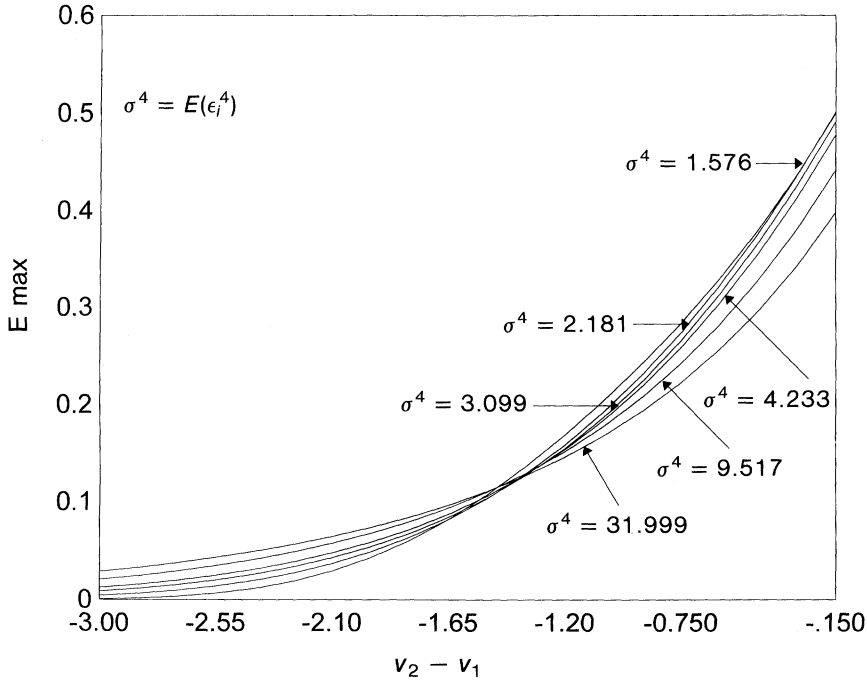

FIGURE 2. Expected value of the maximum.

$V(F,t+1,s)$ is 0, and the value of $V(R,t+1,1-s)$ is graphed on the horizontal axis. Only negative values of $V(R,t+1,1-s)$ are considered because one always can standardize $\text{Emax}(V_F, V_R)$ by subtracting $\max[EV_F, EV_R]$.

Figure 2 shows that Emax is quite robust to the choice of distribution G. Furthermore, it is very precise relative to the maxE approximation, $\max(EV_F, EV_R) = 0$.

5. ESTIMATION RESULTS

Perhaps the best measure of how well different approximations to $\text{Emax}(V_F, V_R)$ perform is to use them to estimate some of the parameters in (2.4) and (2.5). Table 3 and Figure 3 provide results from such an experiment. The parameters to be estimated are c_F, a, and b. The true values are

TABLE 3. Parameter bias estimates

Method	Distribution	Cost	A	B	Time
Emax	Extreme Value	0.00063	0.00015	0.00075	206
		(0.0539)	(0.0404)	(0.1093)	
		[0.0539]	[0.0404]	[0.1093]	
maxE	Extreme Value	−0.00458*	0.56433*	−1.51805*	348
		(0.0537)	(0.0335)	(0.0734)	
		[0.0539]	[0.5653]	[1.5198]	
Logit	Extreme Value	0.00159	−1.36384*	2.88996*	217
		(0.0541)	(0.0447)	(0.1887)	
		[0.0541]	[1.3646]	[2.8961]	
Emax	Expl Nrml ($\alpha = 1.0$)	−0.15823*	0.05557*	−0.12191	214
	Kurtosis = 2.988	(0.0497)	(0.0383)	(0.1037)	
		[0.1659]	[0.0675]	[0.1600]	
Emax	Expl Nrml ($\alpha = 1.25$)	−0.01190*	0.02398*	−0.06060*	213
	Kurtosis = 4.113	(0.0490)	(0.0411)	(0.1127)	
		[0.0504]	[0.0476]	[0.1280]	
Emax	Expl Nrml ($\alpha = 1.5$)	0.13443*	−0.00761	0.00071	212
	Kurtosis = 5.623	(0.0490)	(0.0411)	(0.1127)	
		[0.1431]	[0.0418]	[0.1127]	

Notes: 1000 Monte Carlo experiments were run for the first three experiments, and 100 for the last three experiments. Each experiment has a sample of 500. Reported numbers are mean bias for the experiments. Numbers in parentheses are standard deviations of the bias. To find the standard error of each bias estimate, divide the reported standard deviations by $\sqrt{1{,}000}$ or $\sqrt{100}$ depending on the number of Monte Carlo experiments. Numbers in brackets are root mean squared errors. Asterisks indicate that items are asymptotically significantly different from zero. Reported times are average number of CPU seconds on an IBM RS6000 to estimate the model once.

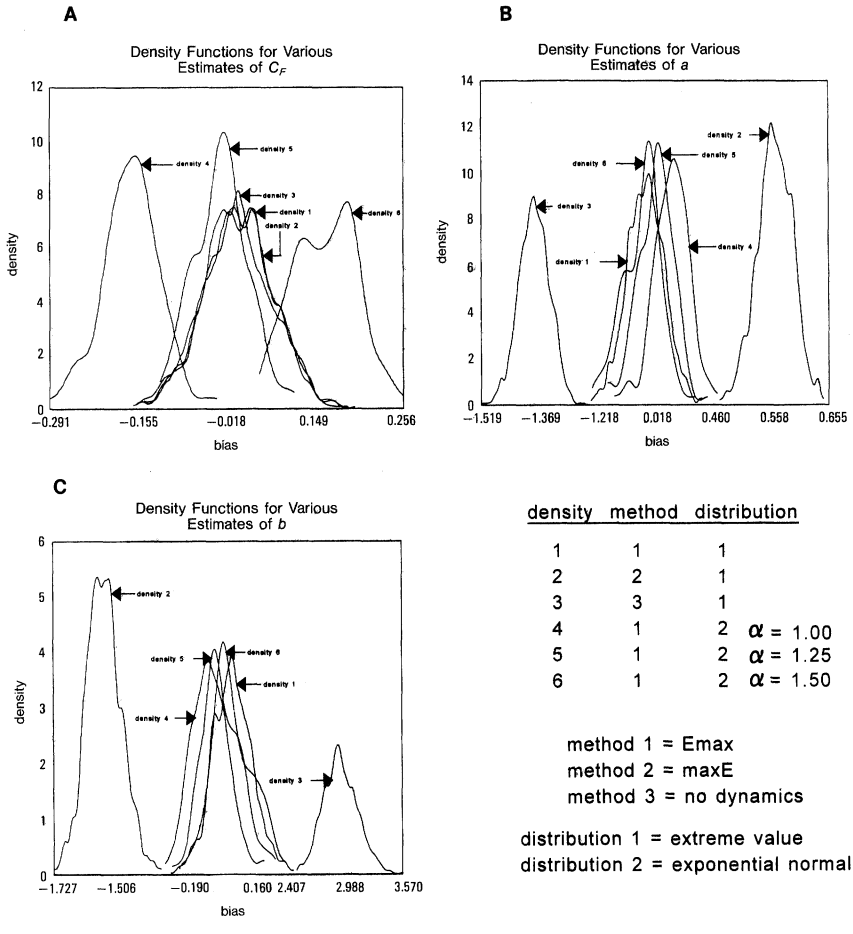

FIGURE 3. Density functions of various estimates.

those used in Panel B of Figure 1; $c_F = 2.0$, $\theta = 1.0$, $a = 3.0$, and $b = 2.0$. The model is estimated using the method of moments with optimal instruments. This is equivalent to maximum likelihood estimation for this example.

First, 500 observations are generated assuming G is Extreme Value. These 500 observations are used to estimate a, b, and c_F using the Emax approximation, the maxE approximation, and ignoring $\text{Emax}(V_F, V_R)$. Biases are reported for 1,000 Monte Carlo experiments. It is clear that the Emax approximation provides consistent estimates and that the other two approximations cause large biases. One can see this in the small biases, standard errors, and root mean squared errors of the Emax approximation relative to the much larger statistics for the maxE approximation and logit method (this

corresponds to ignoring dynamics). One can see the same thing by observing the location and height of the density functions of bias for methods 1–3 in Panels A–C of Figure 3. Furthermore, while the maxE approximation can be evaluated faster than the Emax approximation,[5] the Emax approximation takes less total time to estimate the model (reported in the last column of Table 3) because it converges in fewer iterations than the other two methods.[6] For example, for the first three rows of Table 3, the mean number of iterations per estimation procedure is 3.8 for the Emax approximation, 9.7 for the maxE approximation, and 6.0 for the logit method.

Next, 500 observations are generated assuming G is the exponential normal distribution described in (4.1) for $\alpha = 1$, 1.25, and 1.5. I estimate these three models using the Emax approximation. The estimates, reported in Table 3, and the bias densities for methods 4–6 in Figure 3 show bias. But the bias is relatively small, especially compared to the bias using the other two methods.[7] Furthermore, because the cost of computing the true $\text{Emax}(V_F, V_R)$ is large when G is exponential normal, the exact method provides a feasible, reasonable approximation.

Serially correlated errors provide a significant problem for the Emax approximation while they provide no problem for the maxE approximation, but it is not clear that the maxE approximation is accurate in that case. Furthermore, Berkovec and Stern (1991), Christensen (1990), Engen (1992), Hotz, Miller, Sanders, and Smith (1994), Hubbard, Skinner, and Zeldes (1993), Miller (1984), and Palumbo (1991) all allow for particular types of serial correlation and still solve the exact value functions. Stern (1994) provided an encouraging example of the model in this paper with serial correlation.

While the results in Stern (1994) show that the maxE approximation performs poorly relative to the exact method even when there is serial correlation, Lumsdaine, Stock, and Wise (1991) found that the maxE approximation and exact method perform similarly, and Ausink and Wise (1993) found that the maxE approximation performs better than the exact method. However, in both papers, the authors assumed that the Emax approximation could not feasibly allow for serial correlation. Thus, they allowed for serial correlation while using the maxE approximation and restricted the correlation parameter to be zero while using the Emax approximation.

6. CONCLUSIONS

This paper shows that approximating the expected value of the maximum by its functional form assuming i.i.d. Extreme Value errors provides a relatively precise approximation even when the error distribution is misspecified. This method is significantly better than ignoring dynamics or using the maximum of expected values.

One might argue that my results lose validity if the number of choices the agent has increases. While I cannot claim that the normal and extreme value distribution hazard rates will remain close, other parts of the argument become stronger. As the number of choices increases, the CEVE becomes greater and the relative computational ease of using the extreme value distribution increases.[8]

The usual assumption made in economic models is that people behave as if they were solving optimization problems. In many models where the value of an option is not very important, the maxE approximation is probably a good approximation. However, there are many empirical examples where the maxE approximation would ignore all of the interesting behavior in the model. For example, an option pricing model using maxE would imply that options are worthless, and a search model using maxE would imply that the distribution of wage offers affects a person's reservation wage only through its mean. Of course, no one would use the maxE approximation in such an application. But, to the degree that valuing options is important in other, less obvious applications, the effect is similar if not as overwhelming as in the stock option and search examples. Thus, the choice of approximation depends on the problem and should be chosen carefully when it is an important part of the model.

NOTES

1. See Eckstein and Wolpin (1989) for a survey.
2. Hotz, Miller, Sanders, and Smith (1994) suggest an alternative solution to this problem.
3. The hazard rate is the one-period transition probability.
4. These numbers provide hazard rates similar to those observed in Berkovec and Stern (1991) in that they generally are rising without significant spikes at age 62 or 65. Otherwise, both the parameter values and the associated hazard rates have no particular significance. Spikes could be generated easily by adding pensions or Social Security with significant events at 62 or 65.
5. Evaluation of log-likelihood function requires 35.8 CPU seconds on average using maxE and 54.3 using Emax.
6. All Monte Carlo experiments use the true values as starting values. However, because the estimation algorithm takes very efficient steps, the starting value is not important. This was confirmed in some smaller experiments.
7. The cost term has a larger bias than for the other two methods, but a and b have much smaller biases.
8. The computational cost increases linearly in the number of choices for the extreme value distribution, and the cost increases exponentially for the normal distribution.

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APPENDIX

Consider the model defined in (2.1) and its approximation in (2.3).

THEOREM 1. $|V_e(i,t,s) - V_a(i,t,s)| \to 0$ *uniformly as* $\sigma \to 0$ *for all* i, t, *and* s.

Before proving Theorem 1, two helpful lemmas are proven. Define

$$\tilde{V}_e(i,t,s) = EV_e(1-i,t,1-s) - EV_e(i,t,s). \tag{A.1}$$

LEMMA 1. *If* $V_e(i,t,s)$ *is defined by* (2.1), *then*

$$E \max_i [V_e(i,t,s), V_e(1-i,t,1-s)]$$
$$= \max_i [EV_e(i,t,s), EV_e(1-i,t,1-s)]$$
$$+ \sigma \int_{-\infty}^{\infty} \int_{[\tilde{V}_e(i,t,s)+\sigma u_{1-i}]/\sigma}^{\infty} u_i h(u_i) h(u_{1-i})\, du_i\, du_{1-i}, \tag{A.2}$$

where $h(u) = g(\sigma u)$ *and* g *is the density associated with* G.

Proof. The left-hand side of (A.2) can be written as

$$V_e(1-i,t,1-s) + \sigma \int_{-\infty}^{\infty} \int_{[\tilde{V}_e(i,t,s)+\sigma u_{1-i}]/\sigma}^{\infty} (1 - H(u_i))\, du_i\, h(u_{1-i})\, du_{1-i}, \tag{A.3}$$

where $u_i = \epsilon_i/\sigma$ and $H(u_i) = \int_{-\infty}^{u_i} h(u)\, du$. Using integration by parts, (A.3) can be written as the right-hand side of (A.2). ■

Next, define

$$D = \sigma \int_{-\infty}^{\infty} \int_{[\bar{V}+\sigma u_{1-i}]/\sigma}^{\infty} u_i g(u_i) g(u_{1-i})\, du_i\, du_{1-i}$$
$$= \sigma E\left[u_i \middle| u_i > \frac{\bar{V} + \sigma u_{1-i}}{\sigma}\right] \Pr\left[u_i > \frac{\bar{V} + \sigma u_{1-i}}{\sigma}\right]. \tag{A.4}$$

LEMMA 2. $D \to 0$ *uniformly as* $\sigma \to 0$ *for all* $\bar{V}$.

Proof. The assumption that $E\epsilon_{ti} = 0$ and $\operatorname{Var} E\epsilon_{ti} = \sigma^2 < \infty$ implies

$$\int_{-\infty}^{\infty} |\epsilon_{it}|\, g(\epsilon_{ti})\, d\epsilon_{ti} < 0. \tag{A.5}$$

The result is a straightforward conclusion from this assumption. ■

Proof of Theorem 1. Let

$$V_d(i,t,s) = V_e(i,t,s) - V_a(i,t,s) \tag{A.6}$$

be the difference between the exact and approximate value functions. I want to show that $|V_d(i,t,s)| \to 0$ uniformly as $\sigma \to 0$ for all i, t, and s. Using (2.1), (2.3), and Lemma 1,

$$V_d(i,t,s) = \beta D(i,t,s) + \beta \max[EV_e(i,t+1,0), EV_e(1-i,t+1,1)]$$
$$- \beta \max[EV_a(i,t+1,0), EV_a(1-i,t+1,1)], \tag{A.7}$$

where $D(i,t,s)$ is defined in (A.4) with $\bar{V} = \tilde{V}_e(i,t,s)$. Then,

$$|V_d(i,t,s)| \le \beta |D(i,t,s)| + \beta |\max[EV_e(i,t+1,0), EV_e(1-i,t+1,1)]$$
$$- \max[EV_a(i,t+1,0), EV_a\,(1-i,t+1,1)]|$$
$$\le \beta |D(i,t,s)| + \beta \max[|EV_e(i,t+1,0) - EV_a(i,t+1,0)|,$$
$$|EV_e(1-i,t+1,1) - EV_a(1-i,t+1,1)|]. \tag{A.8}$$

The last inequality is straightforward to verify by considering all 4! orderings of the four terms in the max's. Then,

$$|V_d(i,t,s)| \leq \beta|D(i,t,s)| + \beta|\max[EV_d(i,t+1,0), EV_d(1-i,t+1,1)]|. \quad \textbf{(A.9)}$$

Define

$$V_d^*(i,t,s) = \beta D(i,t,s) + \beta \max[V_d^*(i,t+1,0), V_d^*(1-i,t+1,1)]. \quad \textbf{(A.10)}$$

This is a value function, and its unique solution is

$$V_d^*(i,t,s) = \beta \sum_{\tau=t}^{\infty} \beta^{\tau-t} D(i_\tau, \tau, s_\tau), \quad \textbf{(A.11)}$$

where i_τ is the state one should be in at time τ. By Lemma 2, $D(i,t,s) \to 0$ uniformly as $\sigma \to 0$ for all i, τ, and s. Thus, $V_d^*(i,t,s) \to 0$ uniformly as $\sigma \to 0$ for all i, t, and s. Therefore, because $V_d^*(i,t,s)$ and $V_d(i,t,s)$ are the same value functions, $|V_d(i,t,s)| \to 0$ uniformly as $\sigma \to 0$. ■

Remark 1. Define $i_e(t, i_{t-1})$ to be the policy function for the exact value function at t conditional on being in state i_{t-1} at time $t-1$:

$$i_e(t, i_{t-1}) = \underset{j \in \{0,1\}}{\operatorname{argmax}} [V_e(j,t,|j - i_{t-1}|), V_e(1-j,t,|l-j-i_{t-1}|)]. \quad \textbf{(A.12)}$$

The policy function is the optimal choice of i at time t. Define $i_a(t, i_{t-1})$ analogously for the approximate value function. Then, even though the difference in value functions converges to zero uniformly, the difference in policy functions does not. In particular, convergence of the policy function requires that $V_e(i,t,s) - V_e(1-i,t,1-s) > 0$ if and only if $V_a(i,t,s) - V_a(1-i,t,1-s) > 0$. However, if $|V_e(i,t,s) - V_e(1-i,t,1-s)| \to 0$, then this condition is not necessarily guaranteed.

Despite this problem, the exact and approximate hazard rates between states converge uniformly. The exact hazard rate from state $1-i$ to i is

$$\Pr_e[s=1] = \Pr[V_e(i,t,1) > V_e(1-i,t,0)], \quad \textbf{(A.13)}$$

and the approximation is

$$\Pr_a[s=1] = \Pr[V_a(i,t,1) > V_a(1-i,t,0)].$$

THEOREM 2.

$|\Pr_e[s=1] - \Pr_a[s=1]| \to 0$ *uniformly as* $\sigma \to 0$.

Proof.

$$\begin{aligned}\Pr_e[s=1] &= \Pr[V_e(i,t,1) > V_e(1-i,t,0)] \\ &= \Pr[\varepsilon_{it} - \epsilon_{1-i,t} > EV_e(1-i,t,0) - EV_e(i,t,1)]. \quad \textbf{(A.14)}\end{aligned}$$

But $[EV_e(1-i,t,0) - EV_e(i,t,1)] - [EV_a(1-i,t,0) - EV_a(i,t,1)]$ converges uniformly to 0 as $\sigma \to 0$ by Theorem 1. Because $\Pr[\cdot]$ is bounded, the result follows. The same analysis applies for hazard rates between i and $1-i$. ■