Optimal Replacement Policies for Rejuvenated Assets

John W. McClelland, Michael E. Wetzstein, and Richard K. Noles

The Faustmann-Samuelson solution for optimal asset rotation is extended to consider both asset rejuvenation and nonconstant prices. A dynamic theoretical model is developed in terms of an optimal control problem with discontinuous control variables, and numerical solution procedures are outlined. The model is applied to layer hen replacement where hen rejuvenation by forced molting is a common industrial practice. Results indicate a sensitivity between the decision to rejuvenate or replace a hen and the egg price cycle, suggesting a mixed rotation strategy.

Key words: asset replacement, dynamic programming, eggs, forced molting.

Faustmann, Samuelson (1937), Prienreich, Hirschleifer, Jorgenson, and Bellman provide a basic structure for the examination of asset replacement. Within this structure is the Faustmann-Samuelson (F-S) replacement criterion, predicated on maximizing the present discounted value of net returns from an infinite chain of assets for a competitive firm with constant output prices (Samuelson 1976). This value of present and future assets is established in the context of depreciation and the time value of assets (Prienreich, Hirschleifer, Jorgenson). Bellman provides additional mathematical structure and proposes tractable solution methods for replacement problems.

Research on capital asset replacement in agriculture proliferated once appropriate modifications to the theory were correctly established by Chisholm, Burt, and Perrin. This research has extended the basic theory to include taxes, inflation, and technical change (Bates, Rayner, and Custance; Bradford and Reid; Kay and Rister; Short and McNeill); alternative remaining value functions and nonidentical replacements (Reid and Bradford 1983); and nonconstant prices (Bentley and Shumway, Trapp). A holistic firm perspective and opportunity costs in analyzing replacement decisions were addressed by Reid and Bradford (1987) and Trapp. The techniques of optimal control which refine and extend the earlier theoretical replacement framework have also been applied (Chavas, Kliebenstein, and Crenshaw; Karp, Sadeh, and Griffin). However, a specific asset replacement methodology which incorporates recycled or rejuvenated assets is lacking.

Decisions to rejuvenate versus replace assets frequently occur in agricultural production. The decision in layer flocks is whether to force molt (rejuvenate) or replace a flock, and farm equipment may be either modified or replaced. Generally, rejuvenation is only indirectly considered in the context of repair costs or in the residual market for used equipment (Rust, Reid and Bradford 1983). However, rejuvenation is a major consideration in the formulation of replacement decisions, particularly in the presence of varying prices. In layer flock replacement decisions, an annual egg price cycle directly influences replacement decisions. Thus, an original aspect of the research reported in this paper is the specification of a generalized optimal control model incorporating the possibility of cyclical price patterns and rejuvenation.

Major objectives of the paper are, first, to develop a dynamic theoretical model for the replacement of a recyclible asset under a cyclical price pattern. Theoretical results de-

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velop the F-S criterion for a replacement/rejuvenation (R/R) model, and turnpike properties of the dynamic system are established as an outcome of cyclical prices. The theoretical model is then applied to layer flock replacement where the problem centers on decisions to rejuvenate a flock by force molting. Optimal R/R policies for a representative layer flock over an infinite planning horizon are identified by estimating the theoretical model numerically.

The Conventional Replacement Model

Consider an asset producing marketable products in a controlled production and competitive market environment. In this case it is possible to operate in a continuously optimal state with replacement decisions given by a scalar-valued transversality condition when prices are constant. Karp, Sadeh, and Griffin include a discussion of the replacement problem in a controlled environment; a similar approach is employed here to accentuate aspects of the problem salient to rejuvenation. An uncontrolled environment provides a more general model but also requires more extensive assumptions about the relationships among inputs, outputs, and the environment than is required for this analysis. Instead, the model is generalized on nonconstant prices to allow for variability in initial R/R decisions associated with an infinite planning horizon.

A decision policy is a sequence of R/R decisions, and an optimal policy denotes a decision policy which maximizes net present value of future income streams from a R/R sequence.[1] Maximizing net present value results in the following F-S replacement criterion for a conventional replacement decision with no possibility of rejuvenation and constant prices,

$$V(T) = k(TPV + S), \tag{1}$$

where $V(t)$, a concave functional in t, denotes the net returns from an asset at time t; S denotes the fixed salvage value at the time of replacement; TPV denotes the sum of discounted present values from each asset in the decision policy over an infinite time horizon; and k denotes a fixed discount rate. Optimal rotation time for a single asset, T, is determined by solving (1) for T.

Price Cycle Replacement Model

This conventional F-S criterion can be generalized by assuming input and output prices are represented by a periodic stationary process with period length Z in time domain t.[2] This implies a cyclical price pattern repeating indefinitely in t, requiring a notation for mapping a function with an infinite domain into a finite range. Denote T_i, $i = 1, 2, \ldots \infty$, as the cumulative time duration of the decision policy after the ith asset is replaced. Partition the unit circle into Z discrete sections denoted $z = 1, 2, \ldots, Z$.[3] Define $z_i = z(T_i)$ as a wrapping function, where z_i denotes the starting point of an asset.[4] By convention, let $T_o = 0$ establish the mapping $z_i = (z_o + T_i)\,\mathrm{mod}(Z)$, where z_o is a point of orientation on the circle, the initial starting point in the price cycle.[5] Thus, each point in the time domain measuring the decision policy duration is associated with a point on the unit circle giving the policy duration a relative position in the infinitely repeating price cycle. This notational device allows movements in the price cycle to be reflected in replacement decisions.

Let $V(z_i, t)$ be a concave functional in t denoting the net returns in time t from an asset placed in production at z_i in the price cycle. Explicitly, the revenue functional is

$$V(z_i, t) = p(z_i, t) \cdot Y(z_i, t - T_i) - w(z_i, t) \cdot X(z_i, t - T_i),$$

where $p(z_i, t)$ and $w(z_i, t)$ denote output and input price vectors, respectively, and $Y(z_i, t - T_i)$ and $X(z_i, t - T_i)$ denote vectors of output

[1] Employing a net present value criterion assumes no capital rationing and no production occurring in a lesser developed region. In these two cases, an internal rate of return and maximum sustainable yield criterion, respectively, may be more appropriate (Burt, Meyer, Tisdell and DeSilva).

[2] A function $f(X)$ is periodic with period Z if the domain of f contains $(X + Z)$ whenever it contains X, and if $f(X + Z) = f(X)$ for all X and all integer multiples of X.

[3] Periodic functions can be defined using the unit circle with equation $X^2 + Y^2 = 1$.

[4] Wrapping function is defined by letting C be a unit circle centered at the origin of a cartesian axis. The wrapping function z is a function with domain Z which maps the real numbers onto C. Specifically, for the real number 0, there corresponds a point $a = (1, 0)$ on C such that $z(0) = a$. Likewise, for the real number π there is a point $b = (-1, 0)$ on C such that $z(\pi) = b$ and $z(2\pi) = a$. Letting j be any real number, then there is an integer k such that $2\pi k \le j < 2\pi(k + 1)$, and, thus, $0 \le j - 2\pi k < 2\pi$ and $z(j) = z(j - 2\pi k)$. Therefore, if any two real numbers differ by an integer multiple of π, then z maps these numbers onto the same point on C.

[5] Congruence modulo (*mod*) is defined by $a = b\ mod(Z)$ if $(a - b)$ is divisible by Z, where Z is a fixed integer. Thus, a is the remainder after dividing b by Z.

and input levels, respectively, for an asset placed in production at z_i. Assuming a constant salvage value, S, and fixed cost of replacement, F, the calculus of variations problem for a conventional replacement decision with nonconstant prices is[6]

$$(2)\quad \text{Max } TPV(T_1, T_2, \ldots) = -F + \int_0^{T_1} V(z_o, t)e^{-kt}dt + (S - F)e^{-kT_1} + \int_{T_1}^{T_2} V(z_1, t)e^{-kt}dt + (S - F)e^{-kT_2} + \ldots .$$

The objective functional (2) is periodic on the decision set with exponential damping; thus, the solution is in the form of trajectories converging to a stable limit cycle (Boyce and Deprima). A limit cycle is a closed trajectory where all nonclosed trajectories converge after the transitory effects of initial conditions have dissipated. A periodic solution to (2) represents a steady-state limit cycle toward which all neighboring solutions tend. This is analogous to the turnpike property, which establishes that any optimal trajectory will almost always be in the neighborhood of an extremal stationary trajectory called the von Neumann path or the golden rule path (Haurie). The turnpike property and asymptotic behavior of optimal trajectories provide the necessary conditions for existence of a solution to (2).

Limit cycles are determined by letting the number of assets in a decision policy tend to infinity; they represent a perpetual return to an investment after the effects of initial conditions die out. Thus, evaluation of asset replacement with periodic prices constitutes an evaluation of the present value of a trajectory with m assets converging to a perpetuity (limit cycle) consisting of n assets. The number of assets in the limit cycle, n, is fixed; whereas, m, the number of assets in the trajectory leading to the limit cycle, depends on the initial starting point in the price cycle. An initial starting point, z_i, may be further away from the turnpike than other starting points. However, once on the turnpike the distance traveled is the same. This is completely consistent with asset replacement theory developed under the constant price assumption but broadens the theoretical base for replacement problem analysis.

Theoretically, let the present value of the ith asset PV_i in a decision policy be given by

$$(3)\quad PV_i = -F + \int_{T_{i-1}}^{T_i} V(z_{i-1}, t)e^{-k(t-T_{i-1})}dt + Se^{-k(T_i-T_{i-1})}.$$

Then denoting T_c as the duration of the limit cycle, determined by the duration of its n member assets, the limit cycle's perpetuity, P_c, is

$$(4)\quad P_c = \frac{\sum_{i=1}^{n} PV_i e^{-kT_{i-1}}/PVIFA(k, T_c)}{k},$$

where $PVIFA$ is the present value interest factor of an annuity

$$PVIFA(k, T_c) = (1 - e^{-kT_c})/k.$$

The total present value (TPV) of a decision policy beginning at z_o is the present value of a nonclosed trajectory leading to a limit cycle plus the discounted present value of the perpetuity for that limit cycle:

$$(5)\quad TPV(z_o) = \sum_{i=1}^{m} PV_i(z_o)e^{-kT_{i-1}} + P_c(z_o)e^{-kT_m},$$

where $PV_i(z_o)$ denotes the present value of the ith asset in the nonclosed trajectory of m assets started at z_o, $P_c(z_o)$ denotes the present value of the stream of returns from the limit cycle associated with a nonclosed trajectory beginning at z_o, and T_m denotes the total duration of the nonclosed trajectory. A decision policy exists for every initial condition, z_i, on the unit circle; and, thus, a trajectory and limit cycle exists for each z_i. However, the trajectories at some z_i may be nested in the trajectories of others.

For the maximum of (5), Euler's equation calls for optimizing the integrands at each T_i (Kamien and Schwartz). This results in the following transversality conditions:

[6] The assumptions of fixed salvage values and replacement costs are maintained throughout the theoretical model. Although these assumptions are restrictive, they are consistent with the empirical application of the model to layer replacement. Salvage values and replacement costs are significant factors in replacement decisions characterized by the R/R model. However, in the data set obtained for the empirical application, salvage values and replacement costs tend to exhibit a stationary process through time.

$$(6) \quad \frac{\partial TPV}{\partial T_i} = V(z_{i-1}, T_i)e^{-kT_i} - k(S - F)e^{-kT_i} + \int_{T_i}^{T_{i+1}} \frac{\partial V(z_i, t)}{\partial T_i} e^{-kt}dt = 0, \quad i = 1, \ldots, \infty.$$

The first and second terms on the right-hand side of (6) are the net returns from the last point in the production period and the marginal opportunity cost from selling the asset, respectively. The last term is the marginal net returns from a delay in starting the next asset. Thus, net returns from the last point in the production period must equal the marginal opportunity cost from selling the asset minus the discounted stream of marginal net returns from a delay in starting the next asset. This marginal net return is the incremental change in present value associated with a postponement in starting the subsequent asset. The initial condition of the asset with respect to its relative position in the price cycle determines the sign of this partial derivative.

A major difference between (6) and the F-S criterion (1) is the multiple equations associated with nonconstant prices compared to the single-equation criterion under constant prices. Multiple equations result because prices change from one period to the next, and, thus, net returns at the last point in the production period are not unique. The unique asset duration associated with constant prices results in an expression, $k(TPV)$, for the opportunity cost of postponement. However, for nonconstant prices this opportunity cost varies with the price pattern requiring a simultaneous solution of the transversality conditions (6) for determining optimal values of T_i.

The R/R Model

Equations (6) are appropriate for a wide range of problems. Trapp recently applied a similar approach in investigating beef-breeding herds. However, many problems are also characterized by rejuvenation. An asset can enter a production process at any time and will remain in production until its value declines to a threshold level, at which time it is either replaced or removed from production, rejuvenated, and then returned to the production process. This possibility of rejuvenation expands the decision by including the option of either rejuvenating or replacing an asset.

Without loss of generality, assume an asset can be rejuvenated only once and when sold is immediately replaced by a similar but new asset (pure replacement). The production process is then characterized by two production stages, the initial production stage associated with the introduction of a net asset and the recycled stage associated with production after rejuvenation. Denote $V'(z_i', t)$ as net returns at t associated with an asset rejuvenated at z_i' where

$$z_i' = (z_o + T_i' + L)\ mod(Z).$$

Variables T_i' and L denote the end of an initial production stage and a fixed duration of rejuvenation, respectively. Cost of rejuvenation may be explicitly incorporated by denoting C as a constant incremental rejuvenation cost. Considering rejuvenation the calculus of variations problem (2) is transformed into the following optimal control problem

$$(7) \quad \text{Max } TPV(T_1', T_2', \ldots, T_1, T_2, \ldots) = -F + \int_0^{T_1'} V(z_o, t)e^{-kt}dt - U_1\left[\int_{T_1'}^{T_1'+L} Ce^{-kt}dt - \int_{T_1'+L}^{T_1} V'(z_1', t)e^{-kt}dt\right] + (S - F)e^{-kT_1} + \int_{T_1}^{T_2'} V(z_1, t)e^{-kt}dt - U_2\left[\int_{T_2'}^{T_2'+L} Ce^{-kt}dt - \int_{T_2'+L}^{T_2} V'(z_2', t)e^{-kt}dt\right] + (S - F)e^{-kT_2} + \ldots,$$

subject to the following characteristic function:

$$U_i = \begin{cases} 1 \text{ replacement with rejuvenation } (M) \\ 0 \text{ otherwise } (R). \end{cases}$$

Solution to the optimal R/R policy is derived from the solution to the price cycle replacement model (2) with the following modification. The present value of the ith asset, PV_i, in (3) is modified to consider the R/R problem as follows:

$$(8) \quad PV_i = -F + \int_{T_{i-1}}^{T_i'} V(z_{i-1}, t)e^{-k(t-T_{i-1})}dt - U_i\left[\int_{T_i'}^{T_i'+L} Ce^{-k(t-T_{i-1})}dt\right.$$

$$- \int_{T_i'+L}^{T_i} V'(z_i', t)e^{-k(t-T_{i-1})}dt\Big] + Se^{-K(T_i-T_{i-1})}.$$

Replacing (3) with (8) in equations (4) and (5) results in a *TPV* of an R/R policy beginning at z_o.

To maximize this *TPV*, optimal levels of initial and recycled production durations and the discrete control variables must be determined. Initially, treating the control variables as parameters transforms the optimal control problem (7) into a calculus of variations problem. This results in the following transversality conditions (Kamien and Schwartz):

$$\text{(9a)} \quad \frac{\partial TPV}{\partial T_i'} = V(z_{i-1}, T_i')e^{-kT_i'} - U_i\Big[Ce^{-k(T_i'+L)} - Ce^{-kT_i'} - \int_{T_i'+L}^{T_i} \frac{\partial V'(z_i', t)}{\partial T_i'} e^{-kt}dt\Big] = 0,$$

$$\text{(9b)} \quad \frac{\partial TPV}{\partial T_i} = U_i \cdot V'(z_i', T_i)e^{-kT_i} - k(S - F)e^{-kT_i} + \int_{T_i}^{T_{i+1}'} \frac{\partial V(z_i, t)}{\partial T_i} e^{-kt}dt = 0, \quad i = 1, \ldots, \infty.$$

Assuming $U_i = 1$ for all i, (9a) states that discounted net returns from the final point of the initial production stage must equal marginal costs of recycling minus the discounted stream of marginal net returns from a delay in recycling. Equation (9b) reflects the interdependence between the present production stage and the production stage following rejuvenation. The equation states that net returns from the last point in a recycled production stage must equal the marginal opportunity cost from selling the asset minus the discounted stream of marginal net returns from the next asset. The optimal duration of the initial production stage is only indirectly affected by future returns through T_{i+1}'. If the marginal net returns in (9b) are negative, the duration of the recycled period will be shortened, resulting in a longer initial duration. Alternatively, if $U_i = 0$ for all i, then the problem reduces to the price cycle replacement decision (2), indicating $T_i' = T_i$, and resulting in the classical transversality conditions (6).

Solution Procedure

Initial estimates for T_i' and T_i in (6) and (9) are estimated numerically. Equation (5) is then maximized conditional on the estimates for T_i' and T_i by varying the sequence of control variables, indicating a possible R/R policy consisting of mixed policies, by a dynamic programming process. The R/R policy is then employed to reestimate T_i' and T_i. This iterative solution process continues until T_i', T_i, and the R/R policy stablize. The result is an optimal solution set to (5).

Specifically, the method of central differences is employed to estimate the series of partial derivatives, and numerical integration is performed by applying the trapezoid rule to (6) and (9) (Cheney and Kincaid). Initial estimates for T_i' and T_i are determined by first noting that z_i' and z_i are defined on the unit circle, which implies $|z_i' - z_{i+1}'|$ and $|z_i - z_{i+1}|$ are also on the unit circle, where

$$|z_i - z_{i+1}| = |T_i - T_{i+1}| \; mod(Z), \text{ and}$$
$$|z_i' - z_{i+1}'| = |T_i' - T_{i+1}'| \; mod(Z).$$

This implies a finite set of Z initial and recycled durations, given the seasonal nature of prices. Furthermore, denoting $T_i'|z_o$ and $T_i|z_o$, respectively, as the points in time at the end of an initial and recycled stage in a R/R policy started at z_o, then

$$\text{(10a)} \quad T_{i+1}'|z_o - T_i|z_o = T_1'|z_i, \text{ and}$$

$$\text{(10b)} \quad T_{i+1}|z_o - (T_{i+1}' + L)|z_o = T_1|z_{i+1} - (T_1' + L)|z_{i+1}.$$

Equation (10a) states that the duration of the initial production stage for the $(i + 1)$ asset in a decision policy started at z_o, on the left-hand side of (10a), is equal to the duration of the initial production stage of the first asset in decision policy started at $z_i = (z_o + T_i)$ $mod(Z)$, the right-hand side of (10a). Equation (10b) has a similar interpretation. Equations (10) indicate that the infinite sequence of T_i' and T_i can be represented by a finite set of initial durations, T_1', and total life of the first asset, T_1, for each z_i on the unit circle. Thus, (6) and (9) are reduced from an infinite to a finite set of equations.

Optimal solutions for T_1' and T_1, when $U_i = 1$ for all i, (9), are determined by minimizing

$$\text{(11)} \quad G(T_1', T_1) = (\partial TPV/\partial T_1')^2 + (\partial TPV/\partial T_1)^2$$

for each of the finite z_i's simultaneously.[7] The iterative procedure for numerically solving (9) is the following:

(*a*) Estimate values for T_1' and T_1 from (9), given (11) for the initial and recycled production stages assuming no replacement $T_1' = T_1$ in (9b) for each z_i.

(*b*) Estimate T_2' and T_2 for each z_i replacement policy by (10).

(*c*) Reestimate T_1' and T_1, given estimates for T_2' and T_2 for each z_i.

(*d*) Repeat (*b*) and (*c*) until T_1' and T_1 stabilize for all z_i.

Following this iterative procedure, estimates for T_1' and T_1 are generated for each of the finite number of z_i's. Information from all future assets in an infinite decision policy is considered by this iterative procedure. There is a finite number of z_i's, thus subsequent assets in the infinite sequence of replacements begin at points in the seasonal price cycle which are themselves the first asset at an alternative point in the seasonal price cycle. Specifically, T_2', T_2 and all subsequent stages associated with z_0 are T_1', T_1 for an alternative point in the seasonal price cycle. This condition is employed to reevaluate the first iteration estimates, steps (*b*) through (*d*), until all estimates stabilize. When $U_i = 0$ for all i, (6), indicating no rejuvenation, a similar procedure for numerically solving (9) is employed for solving (6).

Determining the optimal value of the control variables U_i at each z_i on the unit circle constitutes an optimal R/R policy. Approximation in policy space, described by Bellman and Wagner, is employed for determining this optimal policy. This solution method exploits the natural duality between value space of functions and the discontinuous policy space. Iteration in policy space is related specifically to the solution of (5) and the value of the control variables U_i. Simultaneously, considering all Z initial conditions implies a $(Z \times 1)$ policy vector. A null policy vector indicates a perpetual replacement policy without rejuvenation for each z_i on the unit circle; whereas, a unit policy vector indicates a perpetual replacement policy with rejuvenation. Mixed policies are characterized by zero and one elements in the policy vector.

Denote the policy vector for the jth iteration in policy space as $I_j, j = 0, 1, \ldots, J$ for a finite number of policy iterations, where I_o is the initial policy vector. For every policy vector I_j, there exists a stable limit cycle. A limit cycle occurs when the z_i of an asset in a R/R policy is the same as the z_i of a previous asset in the decision policy. The z_i's belonging to a limit cycle may be referred to as the nodes of the cycle. A nonclosed trajectory is a sequence of z_i's in an R/R policy which are not nodes of a limit cycle but which converge to a limit cycle. Thus, each z_i is either a node of a limit cycle or part of a nonclosed trajectory leading to the limit cycle.

Iteration in policy space is performed by first selecting an initial policy vector I_o. According to Bellman's principle of optimality, this selection is independent of the final optimal policy decision and, thus, is unconstrained in terms of policy mix. Limit cycles and nonclosed trajectories for each z_i are determined given initial estimates of T_1' and T_1. The present values of perpetual returns (5) for each z_i, $TPV(I_o, z_i)$, are then calculated as follows. Denote $R(z_i, z_r)$ as the present value of a nonrejuvenated asset started at point z_i in the price policies and replaced at point z_r, and define $M(z_i, z_v)$ to be the present value of an asset started at point z_i, rejuvenated and replaced at point z_v. The TPV's of perpetual R/R policies, started at z_i with initial policy vector I_o are

$$TPV^R(I_o, z_i) = R(z_i, z_r) + TPV(I_o, z_r), \text{ and}$$

$$TPV^M(I_o, z_i) = M(z_i, z_v) + TPV(I_o, z_v), i = 1, \ldots, Z,$$

where the superscripts R and M indicate nonrejuvenated and rejuvenated first assets, respectively, and $TPV(I_o, z_r)$ and $TPV(I_o, z_v)$ are the present values, discounted to the initial starting point z_i, of the perpetual R/R policy started at points z_r and z_v, respectively.

The subsequent policy vector I_1 is determined by

$$\text{(12)} \quad \mathrm{TPV}(I_1, z_i) = \max[TPV^R(I_o, z_i), TPV^M(I_o, z_i)].$$

Equation (12) states that the policy vector I_1 is determined by comparing two alternative estimates of the future and choosing the action which maximizes future returns. In general, the jth policy vector is determined by

$$\text{(13)} \quad \mathrm{TPV}(I_j, z_i) = \max[TPV^R(I_{j-1}, z_i), TPV^M(I_{j-1}, z_i)].$$

[7] Terms on the right-hand side of (11) are squared to avoid possible problems associated with signs.

Thus, each iteration in policy space is a simultaneous reestimation of perpetual future returns for each z_i. The iterations continue until

$$TPV(I_j, z_i) = TPV(I_{j-1}, z_i).$$

Note that at every z_i, the nonrejuvenated and rejuvenated policies are evaluated to maximize returns in each interaction, and the R/R policy for each z_i is based on the decision that maximizes (13). No policy is ever selected that reduces the present value of an R/R policy at any z_i, so $TPV(I_j, z_i)$ is monotone increasing in I. Algorithm (13) will, therefore, converge monotonely to a unique maximum for given values of T_1' and T_1. This solution will be a unique maximum for all T_1' and T_1 if the estimated values of T_1' and T_1 are robust to small changes in the limits of integration for subsequent assets in the R/R policy. This robustness can be tested by applying the optimal policy vector conditional on estimates for T_1' and T_1 to (11) and determining the stability of T_1' and T_1. If necessary reestimated values for T_1' and T_1 can be employed in maximizing (5). This iterative solution process continues until T_1', T_1 and the policy vector stabilize.

Optimal Layer Hen R/R Policies

As an application of the preceding theoretical model and solution method, the flock replacement problem of egg producing firms is analyzed. Rejuvenation by forced molting of layer hens is a decision variable. Relatively short asset durations result in small changes in a firm's economic environment which impact on these decisions. Previous research on the layer replacement problem indicates that replacement costs are a significant portion of total production costs (North). Research has established the feasibility of whole flock replacement over incremental replacement and the use of forced molting as a method of improving flock productivity and extending a flock's length of lay (Noles, Long, and Fortson). In general the layer hen R/R problem is dynamic in nature and dependent on seasonal price patterns. However, previous research fails in developing a dynamic general equilibrium framework, as outlined above, which determines duration of the flock laying cycles and R/R strategies endogenously.

Data

All of the data used in this study were supplied by egg-producing firms in northeast Georgia with the exception of feed prices. Specifically, price data on eggs, replacement pullets, spent hens, and molting were provided by the producers. Production data were obtained from the producers covering years 1975 to 1984. Weekly observations from 153 flocks of commercial white leghorn laying hens were collected, consisting of a flock's production of thirteen different egg grade categories, pounds of feed fed, hen mortality, and if the week is within a molt production stage. Summary information was also collected on date of hatch and replacement of each flock. The data set consists of 7,659 observations for pullet production and 4,675 observations for molt production.

Egg price data for years 1976 to 1984 provided by a producer listed the weekly prices received for the egg categories. These prices were converted to price per egg and adjusted \$0.06 per dozen for payment to a contract grower. The mean value of egg prices by grade for each week was employed to represent the egg price cycles. Seasonal variation for feed ingredients were obtained from average monthly U.S. Department of Agriculture (USDA) statistics for number two shelled corn in Chicago and 44% soybean meal, Decatur, Illinois. Additional per bird costs provided by the producing firms are \$0.185 molting costs, \$0.40 salvage value, and \$2.00 replacement costs. Egg and feed prices were pooled over the time series by week for each egg grade and feed type providing a seasonal price pattern. Feed fed and each egg size were pooled across flocks over the time series by week for the pullet and molt production stages separately.

Pullets were placed into production at twenty weeks of age and a five-week forced molting period was assumed, $L = 5$. Sensitivity of the model to changes in input prices, discount rate, and replacement costs were addressed. The monthly average input price series were adjusted to vary over the price cycle about mean values of \$0.07, \$0.075, \$0.08, and \$0.085 per pound. These prices reflect fixed costs from \$140.00 to \$170.00 per ton and are consistent with yearly average feed prices paid in Georgia from 1976 to 1984 (Georgia Agricultural Facts). Discount rates of 6%, 10%, and 25% and replacement costs of \$2.00 and \$2.40 were also analyzed.

Results

One set of results is sufficient to illustrate the above theoretical model and solution approach. For this application the numerical estimation of (6) and (9) was calculated for input prices at a mean value of \$0.075, a 10% discount rate, and a \$2.00 replacement cost. (For a discussion of model sensitivity to changes in input prices, discount rate, and replacement costs refer to McClelland.) Results illustrating the optimal R/R policy solution are presented in table 1.

The table is structured to present results for each week of the year as a starting week or z_i in the interative procedure. Column one, labeled start week, represents 52 weeks in the price cycle $Z = 52$. The next three columns under the heading final duration provide information on the durations of both pullet T_1', and molt stages $T_1 - (T_1' + L)$ and the total duration T_1, of a flock for the final (optimal) policy at each week. A flock which is not force molted has a pullet duration of 57 weeks, $T_1' = T_1 = 57$, and no molt duration. The column labeled mod week indicates the week in the price cycle, z_i, that the next flock in the R/R policy starts. Mod week is determined by taking $mod(52)$ of T_1 in each week. Columns indicating the initial and final policy decisions for each starting week denote a nonmolted flock as R and a forced molted flock as M, corresponding to the values of the characteristics function, U_i. The final policy represents the convergence of the dynamic programming algorithm and indicates the policy to be taken for each z_i. Calculated initial and final policy present values, based on perpetuities per bird (5) for the R/R policy started in the corresponding starting week z_i, are presented in the last two columns.

A significant characteristic of these results is the dominance of the final policy over the initial policy. This occurs as a result of the optimal final policy transforming the initial policy of conventional replacement to a mixed policy with eight starting weeks, z_i, indicating force molting. Table 1 illustrates this mixed policy for each starting week z_i. The final policy decision associated with nonmolted flocks results in a total asset duration of 57 weeks. The nodes of the unique and stable limit cycle for the final policy are (11_R, 16_R, 21_R, 26_R, 31_R, 36_R, 41_M, 17_R, 22_R, 27_R, 32_M, 11_R), where the subscripts R and M refer to nonmolted and forced molted starting weeks, respectively. All other R/R policies starting in alternative weeks converge to this limit cycle. For example, the nonclosed trajectory associated with week one, z_1, is (1_R, 6_R, 11_R).

These trajectories and the limit cycle result from the general U-shaped nature of the egg price cycle, with relatively low prices between weeks 20 and 30, end of May to end of July, and high prices toward the end of the cycle, December. The turnpike for this problem is associated with the juxtaposition of peak production with peak prices. Forced molting maintains the limit cycle within a neighborhood of this turnpike. The limit cycle indicates a forced molt at starting weeks 41 and 32 in the cycle, which will allow continuance of relatively high production levels through the peak price interval. These results empirically support the ideas of Swanson and Bell and Noles, Long, and Fortson in their discussions of optimal replacement policies and are in contrast to the idea that forced molting should be employed as a means of foregoing replacement costs during periods of depressed output prices (Chen, McNaughton, and Malone; Parlour and Halter; Zeelan).

Summary and Conclusions

An extension to the literature on asset replacement is presented in this paper, where the basic methodology for analysis of recyclable assets is considered in a replacement problem with seasonal price patterns. A theoretical model for this problem is developed in terms of an optimal control problem with discontinuous control variables, and numerical solution procedures are outlined. The theoretical model rests on the major assumption of a controlled environment, which implies that cyclical patterns in asset replacement are predominantly governed by price cycles. Poultry production provides an excellent application for illustrating the theoretical model. However, caution is required in applying this model to uncontrolled environments without explicitly considering possible stochastic environmental constraints.

The optimal limit cycle and nonclosed trajectories for the layer hen problem are discussed. Results indicate that a mixed policy of forced molting and nonmolting is optimal, given the seasonal nature of egg prices. Such a mixed policy is associated with the juxtaposition of peak production and prices. With re-

Table 1. Initial and Final Policy Iterations for $0.075 per Pound Feed Costs, 10% Discount Rate, and $2.00 Replacement Cost

	Final Duration				Decision[c]		Present Value	
Start Week	Pullet	Molt	Total[a]	Mod Week[b]	Initial	Final	Initial Policy	Final Policy
							------ ($) ------	
1	57		57	6	*R*	*R*	1.43	1.89
2	57		57	7	*R*	*R*	1.43	1.60
3	57		57	8	*R*	*R*	1.44	1.60
4	54	20	79	31	*R*	*M*	1.45	1.94
5	57		57	10	*R*	*R*	1.45	1.52
6	57		57	11	*R*	*R*	1.47	1.99
7	51	15	71	26	*R*	*M*	1.49	1.70
8	57		57	13	*R*	*R*	1.52	1.70
9	57		57	14	*R*	*R*	1.55	1.69
10	57		57	15	*R*	*R*	1.59	1.65
11	57		57	16	*R*	*R*	1.60	2.19
12	57		57	17	*R*	*R*	1.64	1.85
13	57		57	18	*R*	*R*	1.67	1.88
14	57		57	19	*R*	*R*	1.71	1.87
15	57		57	20	*R*	*R*	1.74	1.82
16	57		57	21	*R*	*R*	1.76	2.41
17	57		57	22	*R*	*R*	1.80	2.02
18	57		57	23	*R*	*R*	1.83	2.05
19	57		57	24	*R*	*R*	1.87	2.03
20	57		57	25	*R*	*R*	1.89	1.97
21	57		57	26	*R*	*R*	1.91	2.62
22	57		57	27	*R*	*R*	1.94	2.18
23	57		57	28	*R*	*R*	1.97	2.19
24	57		57	29	*R*	*R*	2.00	2.16
25	57		57	30	*R*	*R*	2.02	2.08
26	57		57	31	*R*	*R*	2.03	2.78
27	57		57	32	*R*	*R*	2.03	2.27
28	57		57	33	*R*	*R*	2.04	2.27
29	57		57	34	*R*	*R*	2.04	2.21
30	57		57	35	*R*	*R*	2.04	2.09
31	57		57	36	*R*	*R*	2.02	2.85
32	58	20	83	11	*R*	*M*	2.00	2.26
33	57		57	38	*R*	*R*	2.00	2.24
34	57		57	39	*R*	*R*	1.98	2.16
35	57		57	40	*R*	*R*	1.96	2.01
36	57		57	41	*R*	*R*	1.92	2.84
37	57		57	42	*R*	*R*	1.88	2.00
38	57		57	43	*R*	*R*	1.84	2.13
39	58	19	82	17	*R*	*M*	1.81	2.02
40	57		57	45	*R*	*R*	1.76	1.84
41	54	21	80	17	*R*	*M*	1.71	2.76
42	54	22	81	19	*R*	*M*	1.66	1.90
43	57		57	48	*R*	*R*	1.62	1.96
44	54	20	79	19	*R*	*M*	1.59	1.75
45	54	19	78	19	*R*	*M*	1.56	1.67
46	57		57	51	*R*	*R*	1.52	1.89
47	57		57	52	*R*	*R*	1.50	1.52
48	57		57	1	*R*	*R*	1.48	1.88
49	57		57	2	*R*	*R*	1.47	1.61
50	57		57	3	*R*	*R*	1.45	1.57
51	57		57	4	*R*	*R*	1.44	1.87
52	57		57	5	*R*	*R*	1.43	1.48

[a] Total is the sum of pullet and molt durations plus a five-week molting period.
[b] Mod week is the starting week for the next flock in the replacement chain.
[c] *R* and *M* denote a no-molt and molt flock, respectively.

spect to future research, the methodology could be extended to include time-series analysis of both production and prices. For layer hens, this type of analysis would extend the work of Hartman, provide empirical evidence on the nature of egg prices and production over time, and provide the basis for analyses under uncertainty. The methodology could also be extended and modified for the evaluation of individual flocks and specific firms operating under various economic circumstances. This extension holds the most promise for the layer industry as a whole.

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