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THE SOLUTION AND ESTIMATION OF DISCRETE CHOICE DYNAMIC PROGRAMMING MODELS BY SIMULATION AND INTERPOLATION: MONTE CARLO EVIDENCE

Michael P. Keane and Kenneth I. Wolpin*

Abstract—Over the past decade, a substantial literature on methods for the estimation of discrete choice dynamic programming (DDP) models of behavior has developed. However, the implementation of these methods can impose major computational burdens because solving for agents' decision rules often involves high dimensional integrations that must be performed at each point in the state space.

In this paper we develop an approximate solution method that consists of: 1) using Monte Carlo integration to stimulate the required multiple integrals at a subset of the state points, and 2) interpolating the non-simulated values using a regression function. The overall performance of this approximation method appears to be excellent.

Introduction

OVER the last decade there has been growing a substantial literature on the structural estimation of dynamic discrete choice models of behavior. The reason for this growth is that many behavioral economic models are described naturally as sequential discrete choice optimization problems constrained by resource limitations and imperfect information about future events. Such models have found application in industrial organization, labor economics, development economics, health economics, public finance, and economic demography. Recent surveys by Eckstein and Wolpin (1989a) and by Rust (1992, forthcoming) provide a good introduction to this literature.

A major impediment to the application of this approach is computational. As in static discrete choice models when the choice set is large, it is typical that high dimensional integrations must be performed to calculate the choice probabilities needed for estimation. However, in the dynamic setting, it is typically the case that high dimensional integrations also must be performed to solve the dynamic optimization problem itself.[1] Moreover, those integrations must be performed at all values of the state space (discrete, or discretized if continuous) upon which the evaluation of choices is conditioned, which is what Bellman (1957) called the "curse of dimensionality."

The inherent computational problem of this approach has been accommodated in the literature in several ways. In many applications the dimensionality of the problem, both in terms of the number of choices and the size of the state space, has been kept small. A significant part of the literature has been restricted to problems of only two alternatives. Among the earliest contributions of this type were studies of the following dichotomous decisions: to re-enlist in the air force or not (Gotz and McCall, 1984), to remain in an occupation or choose a different occupation (Miller, 1984), to renew a patent or let it expire (Pakes, 1986), to replace a bus engine or not (Rust, 1987), to have a child or not (Wolpin, 1984), to accept employment or continue to search (Wolpin, 1987).

A number of alternatives to reducing the size of the choice set and/or the state space have been developed and implemented. They can be classified as methods that rely on the full solution of the dynamic programming model but take advantage of particular structures, functional forms or distributional assumptions (Miller, 1984; Pakes, 1987; Rust, 1987), or methods that circumvent having to solve completely the optimization problem (Hotz and Miller, 1991; Manski, 1991; Hotz,

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[1] In random utility models (McFadden, 1973), it is usually assumed that the additive random errors attached to each choice, while possibly correlated, have some unique component. However, a statistically non-degenerate model, one in which all mutually exclusive choices have non-zero probability, can be obtained with as few errors as the number of distinct (non-mutually exclusive) decision variables. Thus, the relevant dimensionality of the numerical integrations need increase only linearly with the number of decision variables.

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Miller, Sanders and Smith, 1992). The advantages and drawbacks of these methods will be addressed in a later section.

Computational complexity will always be limiting if exact full solutions to optimization problems are desired. The purpose of this paper is to explore the performance of approximate solutions obtained by simulation and interpolation of the integral values that must be computed within full solution methods. In section I we present the general problem and the specific example of occupational choice we use throughout the paper to illustrate alternative solution methods. In the second section, we present a brief overview of methods that have been developed in the literature to ameliorate the computational complexities inherent in the estimation of dynamic choice models.

Section III begins with the presentation of our approximation method, which consists of simulating the multiple integrations embedded in the optimal solution by Monte Carlo integration for a subset of the state space elements and interpolating the non-simulated values using a specific regression function approximation that we develop.[2] This procedure can potentially greatly ameliorate the "curse of dimensionality" problem. We next present the results of using the approximation, first in terms of the degree to which it mimics the optimal choice sequence, and second in terms of the parameter estimates it generates and in its resulting predictive accuracy. Evaluations are made for three different sets of structural parameter values that generate different behavioral choice patterns. The overall performance of the approximations is excellent, although not universally across the three data sets and not with respect to all performance criteria. The method is sufficiently promising, in our view, to consider its implementation on real data as a serious alternative to other methods of estimation.

The particular interpolating function we develop has the disadvantage that it may become computationally infeasible when the state space is extremely large, as occurs, for example, when the underlying unobservables are serially correlated. We therefore present several alternatives where the computational burden does not necessarily grow with the dimension of the state space and assess their relative performance. An important advantage of the approximation methods explored in this paper, in our view, is that their performance will improve naturally as computational power continues to increase.

I. The General Problem

A. *The Choice-Theoretic Framework*

We consider a general model in which an individual decides among K possible alternatives in each of T (finite) discrete periods of time. Alternatives are defined to be mutually exclusive so that if $d_k(t) = 1$ indicates that alternative k is chosen at time t and $d_k(t) = 0$ indicates otherwise, then $\sum_{k=1}^{K} d_k(t) = 1$. Associated with each choice at time t is a current period reward, $R_k(t)$, that is known to the individual at time t but that is random from the perspective of periods prior to t.

The objective of the individual at any time $t = 0, \dots, T$, is to maximize

$$E\left[\sum_{\tau=t}^{T} \delta^{\tau-t} \sum_{k\in K} R_k(\tau) d_k(\tau) | S(t)\right], \tag{1}$$

where $\delta > 0$ is the individual's discount factor, $E(\cdot)$ is the mathematical expectations operator, and $S(t)$ is the state space at time t. The state space consists of all factors, known to the individual, that affect current rewards or the probability distribution of any of the future rewards.

Maximization of (1) is accomplished by choice of the optimal sequence of control variables $\{d_k(t)\}_{k\in K}$ for $t = 0, \dots, T$. Define the maximal expected value of the discounted lifetime reward at t as

$$V(S(t), t) = \max_{\{d_k(t)\}_{k\in K}} E\left[\sum_{\tau=t}^{T} \delta^{\tau-t} \sum_{k=1}^{K} R_k(\tau) \times d_k(\tau) | S(t)\right]. \tag{2}$$

The value function $V(S(t), t)$ depends on the state space at t and on t itself (due to the finiteness of the horizon or the direct effect of age on rewards), and can be written as

$$V(S(t), t) = \max_{k\in K}\{V_k(S(t), t)\}, \tag{3}$$

[2] Bellman, Kalaba and Kotkin (1963) proposed using polynomials in the state space to approximate value functions in a continuous choice setting. As will become apparent, our approach to approximation, while related, differs in important ways.

where $V_k(S(t), t)$, the alternative-specific expected lifetime reward or value function, obeys the Bellman equation (Bellman, 1957)

$$\begin{aligned} V_k(S(t),t) &= R_k(S(t),t) \\ &\quad +\delta E[V(S(t+1),t+1) \\ &\quad |S(t), d_k(t)=1], \\ t \le T-1, V_k(S(T),T) &= R_k(S(T),T). \end{aligned} \tag{4}$$

Notice that the dependence of the current period reward on the state space (or at least a subset of it) is made explicit in (4). As seen in (4), the alternative-specific value function assumes that future choices are optimally made for any given current period decision. The expectation in (4) is taken over the distribution of $S(t+1)$ conditional on $S(t)$ and $d_k(t)=1$, with the conditional density denoted by

$$p_{kt}(S(t+1)|S(t), d_k(t)=1). \tag{5}$$

The randomness in rewards arises from the existence of state variables at time $t+1$ observable to the agent at $t+1$, but unobservable at t or before. The formulation in (5) allows for contemporaneously and serially correlated rewards.

B. *A Model of Occupational Choice*

To understand the difficulties in solving and estimating dynamic discrete choice problems, we consider as a concrete example a model of occupational choice. This example is the basis for the subsequent work in this paper on the simulation method we propose as a means of solving and estimating the general model. We chose a specific example because it is difficult to conceive a generic specification of the current reward functions. We chose an occupational choice model because its structure accommodates a wide range of complications that are illustrative of the general problem and because the economics literature on occupational choice has been stagnant for some time (Miller (1984) discussed below, is an important exception).[3] Nevertheless, the reader should understand that the approximation method we propose is applicable to discrete choice dynamic programming models generally.

Assume that in each period an individual chooses to work in either of two occupations, to attend school, or to remain at home (consume leisure or engage in home production).[4] There are thus four mutually exclusive alternatives: occupation one, occupation two, schooling, and home. The per-period reward functions are given by

$$\begin{aligned} R_1(t) &= w_{1t} = \tilde{w}_1(s_t, x_{1t}, x_{2t}; \alpha_1)e^{\epsilon_{1t}} \\ R_2(t) &= w_{2t} = \tilde{w}_2(s_t, x_{1t}, x_{2t}; \alpha_2)e^{\epsilon_{2t}} \\ R_3(t) &= \beta_0 - \beta_1 I(s_t \ge 12) \\ &\quad - \beta_2(1-d_3(t-1)) + \epsilon_{3t}, \\ R_4(t) &= \gamma_0 + \epsilon_{4t}. \end{aligned} \tag{6}$$

In (6), s_t is the number of periods of schooling obtained by the beginning of period t, x_{1t} is the number of periods that the individual worked in occupation one (experience) by the beginning of period t, x_{2t} is the analogously defined level of experience in occupation two, α_1 and α_2 are parameter vectors associated with the wage functions, β_0 is the consumption value of schooling, β_1 is the post-secondary tuition cost of schooling, with I an indicator function equal to one if $s_i \ge 12$ (the individual has completed high school) and zero otherwise, β_2 is an adjustment cost associated with returning to school (if $d_3(t-1) = 0$), and γ_0 is the (mean) value of the non-market alternative. The ϵ_{kt}'s are alternative-specific shocks, to skill levels ($k = 1,2$), to the consumption value of schooling ($k=3$), and to the value of non-market time ($k=4$). Note that these shocks appear multiplicatively in the wage, and thus reward, functions for the two occupations, but additively for the schooling and home alternatives. Wage offers are always non-negative, but consumption values of school and home may be of either sign.[5]

C. *Solving the Optimization Problem*

Case: ϵ_k's serially uncorrelated.

It is convenient to consider first the case where the ϵ_k's are jointly serially uncorrelated. In that case, the joint distribution of the ϵ_k's is

[3] We are also currently implementing a version of this model on actual data.

[4] See Keane and Wolpin (1994) for a more complete specification of the model.

[5] Given that the model represents the optimization problem of a single individual, incorporating individual (permanent) heterogeneity in occupation-specific skill endowments and school and home preferences does not change anything of substance. The same is true of the opposite extreme in which there are aggregate shocks affecting all individuals in the same way.

$\pi_{t=0}^{T} f(\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}; \eta)$ where η is the vector of distribution parameters and f is the marginal (joint) distribution of time t errors. Further, denote the full parameter vector of the model as $\theta = \{\alpha_1, \alpha_2, \beta, \gamma, \eta\}$ where the elements of θ are the appropriate vectors in (6). The individual knows θ and must solve for the sequence $\{d_k, (t)\}$ for $t = 0, \ldots, T$, that maximizes (1) subject to the evolution of the state space (5).

The state space for this problem at time t is $S(t) = \{s_t, x_{1t}, x_{2t}, d_3(t-1), \epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}\}$. It is convenient to denote the deterministic (more accurately, predetermined) elements of the state space, $s_t, x_{1t}, x_{2t}, d_3(t-1)$, as $\bar{s}(t)$. The elements of the state space evolve according to

$$\begin{aligned} x_{1,t+1} &= x_{1t} + d_1(t), \\ x_{2,t+1} &= x_{2t} + d_2(t), \\ s_{t+1} &= s_t + d_3(t), \\ f(\epsilon_{t+1}|S(t), d_k(t)) &= f(\epsilon_{t+1}|\bar{S}(t), d_k(t)). \end{aligned} \tag{7}$$

The last equation in (7) reflects the fact that the ϵ_k's are serially independent. Initial conditions are $x_{10} = x_{20} = 0, s_0 = 10$. Notice that the deterministic elements of the state space take on only discrete values.

Consider an individual entering the last decision period, T, with particular values of the deterministic state space elements $S(T)$. At T the individual receives a draw from the joint distribution of the ϵ_{kT}'s. The individual would then calculate each of the T period reward functions (equation (6) at $t = T$) and choose the alternative with the largest realized reward.

Suppose the individual were instead at period $T-1$, again with particular values of the deterministic state space elements, $\bar{S}(T-1)$. In order to calculate the alternative-specific value functions, (4) at $T-1$, the individual must calculate

$$\begin{aligned} &\operatorname{Emax}\big(R_1(T), R_2(T), R_3(T), R_4(T) \\ &\qquad |\bar{S}(T-1), d_k(T-1)\big) \\ &= \int_{\epsilon_{4T}}\int_{\epsilon_{3T}}\int_{\epsilon_{2T}}\int_{\epsilon_{1T}} \max\big(R_1(T), R_2(T), \\ &\qquad R_3(T), R_4(T)|\bar{S}(T-1), d_k(T-1)\big) \\ &\quad \times f(\epsilon_{1T}, \epsilon_{2T}, \epsilon_{3T}, \epsilon_{4T})\, d\epsilon_{1T}\, d\epsilon_{2T}\, d\epsilon_{3T}\, d\epsilon_{4T}, \end{aligned} \tag{8}$$

for each $k = 1, \ldots, 4$. It is important to note that the Emax$(\cdot)$ function is in general a multivariate integral even when the ϵ_{kT}'s are stochastically independent. Further, because each choice $k = 1, \ldots, 4$ leads to a different state at time T, Emax$(\cdot)$ must be calculated at each of these four time T state points for each element of $\bar{S}(T-1)$.

Having calculated (8), the value functions at $T-1$, $V_k(S(T-1), T-1)$, are known up to the random draws of the $\epsilon_{k,T-1}$'s. The individual receives a set of draws at $T-1$ and chooses the alternative with the highest value.

Moving backwards in time, the individual must compute, analogously to (8), the expected maximum of the alternative-specific value functions at every $t = 0, \ldots, T$. These take the form

$$\begin{aligned} &\operatorname{Emax}\big[V_1(S(t+1), t+1), \\ &\qquad V_2(S(t+1), t+1) \\ &\qquad V_3(S(t+1), t+1), \\ &\qquad V_4(S(t+1), t+1)|\bar{S}(t), d_k(t)\big]. \end{aligned} \tag{9}$$

As in (8), (9) is a four-variate multiple integral over the joint $\epsilon_{k,t+1}$ distribution. Moreover, in order to calculate (9) the alternative-specific value functions at (t) must have been calculated for all of the possible state space values at $t+1$, $\bar{S}(t+1)$, that may arise given $\bar{S}(t)$ and $d_k(t)$. This implies that at $t+2, t+3, \ldots, T$, the alternative-specific value functions at those times must have been calculated at all of the feasible state points that could have arisen at those times given $\bar{S}(t)$ and $d_k(t)$. Thus, in order to solve for the $t = 0$ alternative-specific value functions, it is necessary to have calculated the alternative-specific value functions at each future date at all feasible state points. At time T, this means calculating (8) for every feasible combination of $\bar{S}(T-1)$ and $d_k(T-1)$, i.e., for every possible point in $\bar{S}(T)$. In the case of $T = 40$, $\bar{S}(T)$ has 13,150 elements.[6]

Case II: ϵ_k's serially correlated.

In this case the joint distribution of the ϵ's is $f(\epsilon_{10}, \ldots, \epsilon_{40}, \ldots, \epsilon_{1T}, \ldots, \epsilon_{4T}; \eta)$. The state space at t now must include all of the ϵ's that are known at t and that affect the distribution of

[6] This number reflects the appropriate constraint that the sum of the occupation-specific work experiences and schooling cannot exceed T. We also assumed that schooling could not exceed ten additional years.

ϵ_{t+1}. Thus, the state space at t is $S(t) = \{\bar{S}(t), \epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}, \ldots, \epsilon_{10}, \epsilon_{20}, \epsilon_{30}, \epsilon_{40}\}$, which is composed of both discrete and continuous elements.

With serial dependence, (8) becomes

$$\begin{aligned}
&\mathrm{Emax}(R_1(T), R_2(T), R_3(T),\\
&\qquad R_4(T)|S(T-1), d_k(T-1))\\
&= \int_{\epsilon_{4T}}\int_{\epsilon_{3T}}\int_{\epsilon_{2T}}\int_{\epsilon_{1T}} \max(R_1(T), R_2(T),\\
&\qquad R_3(T), R_4(T)|S(T-1), d_k(T-1))\\
&\qquad \times f(\epsilon_{1T}, \epsilon_{2T}, \epsilon_{3T}, \epsilon_{4T}|S(T-1),\\
&\qquad d_k(T-1))d\epsilon_{1T}, d\epsilon_{2T}, d\epsilon_{3T}, d\epsilon_{4T}. \qquad (10)
\end{aligned}$$

The expected maximum of the period T rewards now depends on the prior ϵ_{kt} draws as well as on the state space elements $\bar{S}(T-1)$. This carries over to all periods of the Emax functions; $S(t)$ replaces $\bar{S}(t)$ in (9). Clearly, the computational effort is greatly magnified when there is serial dependence because the state space becomes infinitely large. Lacking analytical solutions for (10), backwards solution of the dynamic programming problem requires discretization of the ϵ_{kt}'s. With four distinct shocks, the increase in the number of state space elements can be substantial.

C. Estimation

The computational complexity that arises in providing an exact solution to the optimization problem is that, as the model is specified, the expected maximum function entails a multiple integration of dimension K and that function must be calculated for each element of the state space. However, the reason for solving the optimization problem is that the solution serves as the input into estimating the parameters of the model given data on choices and possibly some of the rewards. Any inexactness in the solution presumably translates into estimation bias.

To understand the connection between the solution of the model and estimation, consider having data on a sample of homogeneous individuals from the same birth cohort who are assumed to be solving the occupational choice model previously described and for whom choices are observed in each of the periods $t = 0, \ldots, T$. In addition, as is usual, wages are assumed to be observed only in the periods in which market work is chosen and only for the occupation that is chosen.

To simplify the presentation, consider the case where the random elements of the reward functions are serially independent. Then, the joint probability of choosing occupation one at time t and its corresponding (accepted) wage is

$$\begin{aligned}
\Pr(d_1(t) = 1, w_{1t}|\bar{S}(t)) &= \Pr\big(w_{1t} + \delta\,\mathrm{Emax}(V(S(t+1), t+1)|\bar{S}(t), d_1(t) = 1)\\
&\qquad \geq \tilde{w}_{2t}(\bar{S}(t))e^{\epsilon_{2t}} + \delta\,\mathrm{Emax}(V(S(t+1), t+1)|\bar{S}(t), d_2(t) = 1),\\
&W_{1t} + \delta\,\mathrm{Emax}(V(S(t+1), t+1)|\bar{S}(t), d_1(t) = 1)\\
&\qquad \geq \beta_0 - \beta_1 I(s_t \geq 12) - \beta_2(1 - d_3(t-1))\\
&\qquad\quad + \epsilon_{3t} + \delta\,\mathrm{Emax}(V(S(t+1), t+1)|\bar{S}(t), d_3(t) = 1),\\
&w_{1t} + \delta\,\mathrm{Emax}(V(S(t+1), t+1)|\bar{S}(t), d_1(t) = 1)\\
&\qquad \geq \gamma_0 + \epsilon_{4t}\delta\,\mathrm{Emax}(V(S(t+1), t+1)|\bar{S}(t), d_4(t) = 1), w_{1t}\big), \qquad (11)
\end{aligned}$$

namely, the probability that the alternative one value function exceeds the other three and that the wage that is accepted is the observed wage. An exactly analogous probability statement holds for occupation two, with the difference between those for the occupations and the probability statements for schooling and home being that the current period rewards are not observed for the latter.

The likelihood function for the sample is the product of these probability statements over time and people. Maximizing the sample likelihood with respect to θ would yield consistent and asymptotically normal estimates. Evaluation of the likelihood itself requires the calculation of four-variate integrals. In the context of serial dependence in the stochastic elements of the reward functions, maximum likelihood estimation would require the calculation of a KT variate integral rather than a sequence of K variate integrals.

If the Emax($\cdot$) functions could be calculated without error, it would be possible to construct unbiased simulators of the choice probabilities and form an MSM (method of simulated moments) estimator of the type developed by McFadden (1989) and Pakes and Pollard (1989) for static discrete choice models.[7] The computational gain over maximum likelihood would be no different in kind for the dynamic discrete choice model. The MSM estimator is linear in the choice probabilities and is therefore a consistent estimator for a fixed number of simulation draws used to simulate choice probabilities. However, it is important to understand that the MSM estimator combined with a simulated estimate of the Emax($\cdot$) functions is not consistent. As seen in (11) the choice probabilities are nonlinear functions of the Emax($\cdot$) functions, which implies that the simulated choice probabilities using simulated Emax($\cdot$) functions will be biased. Both the likelihood estimator and the MSM estimator, which depend on simulated Emax($\cdot$) functions, are inconsistent in the context of nonlinear measurement error.

II. A Brief Review of Existing Solution and Estimation Methods

In this section, we review existing methods in the literature for dealing with the computational problems described in the previous section that arise in the full solution and estimation of discrete choice dynamic programming models. Notice that there are no conceptual problems in implementing models with large choice sets, large state spaces, and serial dependencies in unobservables. The problem is in implementing interesting economic models that are computationally tractable. A comprehensive review is beyond the scope of this paper, particularly in light of the several surveys previously mentioned that have appeared in the literature. Our purpose is to give the reader a context within which to place the approximation method explored in this paper.

Full-Solution Methods

Computational simplifications for handling large choice sets and/or large state spaces (which includes the case of serially correlated unobservables), while remaining within the full-solution paradigm, have involved finding convenient forms for the reward functions and error distributions.[8] Miller (1984), Pakes (1986), and Rust (1987) are the leading examples of this approach. The approach in Rust has been more widely adopted in the economics literature, perhaps because of its structural similarity with the static random utility model.[9] That formulation has also served as the basis for the implementation of several of the non-full solution methods discussed below.

Rust makes the following assumptions: (i) the reward functions are additively separable in the unobservables, with each unobservable associated with a mutually exclusive choice, (ii) the unobservables are conditionally independent, i.e., conditional on observable state variables, the unobservables are serially independent, and (iii) the unobservables are distributed as multivariate extreme value. There are two very appealing consequences of these assumptions for solution and

[7] Berkovec and Stern (1991) use the MSM estimator for a dynamic programming optimization problem for which there are analytical solutions.

[8] The earliest examples of obtaining tractability essentially through small choice sets and state spaces are those of Gotz and McCall (1984) and Wolpin (1984).

[9] Miller formulates an occupational choice model as a multi-armed bandit problem. The method he develops for tractably solving (employing the Gittens index) and estimating that model, accommodating as it does a large choice set and serial correlation in wages, is generalizable to problems with the same structure. Its main drawback is that the assumption in such problems of independence across arms may be too restrictive over a broad range of economic problems. For example, it would be inapplicable to the occupational choice model we consider if work experience in one occupation affects productivity in another occupation. Pakes considers an optimal stopping problem (whether or not to renew a patent) in which serially correlated unobservables enter the reward function additively. While the distributional assumptions that make the solution of the dynamic programming problem tractable are specific to the particular problem, and thus not generally transportable to other stopping problems, Pakes demonstrated the feasibility of incorporating serial correlation into the estimation of discrete choice dynamic programming models.

estimation:

1. The Emax$(\cdot)$ function (the expected value of (3) appearing in (4)) has the closed form solution

$$E[V(S(t),t)] = \gamma + \tau \ln\left\{ \sum_{k=1}^{K} \frac{\exp(\bar{V}_k(\bar{S}(t),t))}{\tau} \right\}, \tag{12}$$

where $\bar{V}_k$ is the expectation of the alternative-specific value functions, the expectation of (4), and γ is Euler's constant. Thus, multivariate numerical integrations are avoided in solving the dynamic programming problem. It should be noted that (conditional) independence of the alternative-specific errors is not sufficient to obtain an analytical form; additive separability and the extreme value assumptions are crucial.

2. The choice probabilities are multinomial logit, i.e., with τ normalized to unity,

$$\Pr(d_k(t) = 1)|\bar{S}(t)) = \frac{\exp(\bar{V}_k(\bar{S}(t),t))}{\sum_{j \in K} \exp(V_j(S(t),t))}. \tag{13}$$

Therefore, multivariate numerical integrations are also avoided in likelihood estimation. However, as in the static logit model, only limited forms of correlation among the alternative-specific errors can be accommodated.[10]

In addition to simplification achieved through functional form assumptions, there are several examples in the published literature of what can be viewed as a simplification achieved through an approximation to the full solution. Stock and Wise (1990) estimate a model of retirement which they call an "option value"' model, but which is equivalent to substituting the maximum of the expected alternative-specific value functions for the expected maximum of the alternative-specific value functions. Lumsdaine, Stock, and Wise (1990) evaluated the performance of this approximation vs. the exact solution in predicting the effect of the pension window plan studied by Stock and Wise, and concluded that the fit was about the same. Stern (1991), analyzing a different model of retirement concluded from simulation evidence that while the approximation did predict well the large impact of a pension window, it did not predict well other dynamic aspects of the model.

Wolpin (1992) estimates a structural model of labor force dynamics in which agents are assumed to optimize over longer discrete periods the further are the periods into the future. This simplification has the effect of reducing the size of the state space. More recently, Geweke, Slonim and Zarkin (1992) have proposed a solution and estimation method based on approximating the agent's decision rules that still recover structural parameters. This latter paper is the closest in spirit to the approximation method proposed and analyzed in this paper.

Non-Full-Solution Methods

The non-full-solution methods that have appeared in the literature, Hotz and Miller (1993), Hotz, Miller, Sanders, and Smith (1994), and Manski (1988), use alternative representations of the future component of the choice-specific valuations that do not depend explicitly on the structural parameters of the model. The Emax$(\cdot)$ functions are estimated from data on future choice or reward probabilities. Because the choice-specific valuation functions are specified in the first two papers as they are presented in section I, it is relatively easy to describe their methodologies. However, the representation in Manski's path utility framework is sufficiently different that we will forgo discussion of it. Manski's approach does share many of the same advantages and drawbacks.[11]

The insight of Hotz and Miller can be most easily illustrated under Rust's assumptions. Using (12) and (13), it can be shown that

$$E[V(S(t),t)] = \sum_{k=1}^{K} \Pr(d_k(t) = 1|\bar{S}(t)) \times \left[\gamma + \bar{V}_k(\bar{S}(t),t) - \ln(\Pr(d_k(t) = 1|\bar{S}(t))\right]. \tag{14}$$

[10] There is a direct analogy to nested logit, but without its usual implied sequential decision-making interpretation. Even in the non-nested case, the independence of irrelevant alternatives axiom does not hold in the dynamic setting because augmenting the choice set must affect the valuation attached to all choices.

[11] For a brief description of Manski's approach see Eckstein and Wolpin (1989).

Successive forward substitution for $\bar{V}_k(\bar{S}(t), t)$, recognizing that it contains future expected maximum functions, implies that the expected maximum function at any t can be written as a function of the future conditional choice probabilities. Hotz and Miller show that this result is not dependent on the extreme value distribution assumption, but generalizes to any distribution. The extreme value distribution is appealing because the representation has a closed form. Empirical implementation uses data on observed future choices to obtain the conditional choice probabilities that are needed for calculating alternative-specific value functions. Because choice probabilities are obtained non-parametrically from the data and are state-specific, implementation may require very large observation sets, particularly when the state space is large. Structural parameters are recovered from the contemporaneous reward functions, which are the only places they enter in this formulation. The estimation therefore does not take into account all of the parameter restrictions contained in the theory.

Although this method is significantly more tractable than a full solution method, an inherent limitation of this approach is that it cannot admit to the existence of individual-specific unobservables as a component of the state space, generally ruling out forms of serial correlation including permanent unobserved heterogeneity. Furthermore the Emax$(\cdot)$ values calculated from the data are obviously not policy invariant as they depend on the structural parameters of the model. Thus, full solution would be necessary after estimation in order to conduct policy experiments.

The methodology in Hotz, Miller, Sanders, and Smith also uses the Hotz and Miller representation theorem. However, rather than computing the alternative-specific valuation functions by considering all feasible future paths as in Hotz and Miller, they simulate future paths in calculating the expected maximum functions, using (14) in the extreme value case. Noting that in the extreme value case, for example, the expected values of the alternative-specific value functions (normalized against one of the alternatives) are just the log-odds of the choice probabilities, data on choice probabilities are sufficient to estimate (non-parametrically) the normalized value functions. Parameter estimates are obtained by comparing the data to the simulated value functions using a weighted distance estimator. Because the estimator is linear in the simulated value functions, analogous to the MSM estimator, only one future path needs to be simulated to obtain consistent estimates. While this method is even more tractable computationally than Hotz and Miller, its limitations are not different.

III. Approximating the Solutions by Simulation and Interpolation

A. *The Method*

In this section, we present an approximation method based on simulation and interpolation. As illustrated by (11), the choice probabilities that enter the likelihood function are nonlinear functions of the expected maximum function (EMAX). Consistent estimation of θ (T fixed as N grows) requires that EMAX be precisely calculated for all elements of the state space. Approximations to EMAX lead to inconsistent estimates and presumably to greater finite sample bias. The extent of this bias as the accuracy of the EMAX calculation varies is unknown a priori and is the subject of the rest of the paper. The method we propose can accommodate both iid and serially correlated unobservables.

We propose to approximate EMAX by crude Monte Carlo integration.[12] That is, we take D draws from the joint ϵ distribution, calculate the maximum of the value functions over the four choices for the given ϵ draw, and average the maximum over the D draws. Clearly the simulated EMAX is unbiased, and is consistent as D becomes large. It is possible to simulate EMAX in this way at each point in a finite state space and a continuous state space can be discretized. However, for problems of the size we would like to consider, such a pure simulation strategy is not computationally practicable.

As a method for coping with the "curse of dimensionality," we propose to simulate EMAX by the above Monte Carlo integration for only a subset of the possible state points and to interpolate EMAX at the remaining state points. There are several possible ways to do the interpolations. One method would be to use a neighborhood

[12] We do not employ acceleration techniques, e.g., antithetical variates, in order to keep the method simple. Using such methods might increase accuracy for given computational burden.

criterion, interpolating EMAX values from those states that are nearby. However, there is no obvious metric of proximity among state points in a multi-dimensional setting. A second approach, the one we adopt, is to search for a function that relates EMAX (as in (8) or (9)) to its arguments and that approximates EMAX well. Those arguments depend only on the state space elements, so that potential interpolating functions are functions of the state space elements themselves or functions of transformations of the state points, such as expected rewards or expected alternative-specific value functions. After considerable experimentation, we found that the following general class of functions works well. Denoting, as before, $\bar{V}_k(S(t),t)$ as the expected value of $V_k(S(t),t)$, and $\mathrm{MAXE}(S(t),t)$ as $\max_k(\bar{V}_k(S(t))$, the $\mathrm{Emax}(V(S(t),t))$ function is approximated by

$$\mathrm{EMAX}(S(t),t) \approx \mathrm{MAXE}(S(t),t) + g\big(\mathrm{MAXE}(S(t),t) - \bar{V}_k(S(t),t),t\big), \tag{15}$$

where the term inside the $g(\cdot)$ function is a vector of elements over $k \in K$ and where $g(\cdot) > 0$. The intuition for this form is that the difference between EMAX and MAXE, which must be positive, will depend on how far apart are the expected alternative-specific value functions as captured by the individual differences between MAXE and the $\bar{V}_k$'s. The inclusion of MAXE in g, along with the alternative-specific value functions, captures the notion of distance between the value functions. The parameters of the $g(\cdot)$ function depend on properties of the joint ϵ distribution, i.e., on its higher-order moments, and on t itself.[13] Below, we compare the accuracy of several alternative interpolating functions ($g(\cdot)$).[14]

[13] As can be seen by manipulating (12), this representation is exact, i.e., it is an equality, in the case of the multivariate extreme value distribution.

[14] For the case with additive errors, Stern (1991) shows that the EMAX function must satisfy the following derivative conditions with respect to $\bar{V}_k$ (Stern, 1991): (1) the first-partial derivatives are positive, (2) the first-partial derivatives sum (over k) to one, (3) the first-partial derivatives are less than one, (4) the second own-partial derivatives are positive, and (5) the second cross-partial derivatives are negative. When the errors are multiplicative, or a mixture of multiplicative and additive errors as in the occupational choice model, conditions (1), (4) and (5) still hold.

Consider how a backwards recursive solution would be obtained using our simulation-interpolation method. At T we perform the EMAX simulations as described above for a subset of state points, $S^*(T)$. These simulated EMAX values along with the alternative-specific expected value functions, $\bar{V}_k(S^*(T),T)$, are used to estimate the interpolating function by a regression equation. Moving backwards to $T-1$ we wish to perform the same calculation for a subset $S^*(T-1)$. To calculate the simulated EMAX's as well as the arguments for the interpolating function, as seen from equation (4), requires that we calculate EMAX at T at every point in the state space that can be reached from a point in $S^*(T-1)$. Those EMAX values that are in $S^*(T)$ are available directly, while those not in $S^*(T)$ are calculated from the interpolating function, $g(T)$. These steps are repeated as the backwards solution proceeds.

It is important to note that the backwards solution using the interpolating function (15) requires that the value of EMAX either be simulated or interpolated at every point in the state space. To see this, denote by $S^{**}(t)$ the set of points at which we either simulate or interpolate EMAX at time t. Clearly, if K alternatives are available in each state in each time period, then $S^{**}(T)$ must be K times larger than $S^{**}(T-1)$, which must be K times larger than $S^{**}(T-2)$, etc. Thus, $S^{**}(t)$ must contain the entire state space regardless of the number of state points contained in $S^*(t)$.

The computational advantage of this approximate solution method is that, for $S^*(t)$ a small subset of $S(t)$, most of the time-consuming multiple integrations necessary to construct the EMAX functions are replaced by fast interpolations using a regression estimate of (15). If the interpolating function provides a close approximation to the $\mathrm{EMAX}(\cdot)$ function, this method has the potential to ameliorate the "curse of dimensionality" problem. However, it is important to determine how rapidly the number of state space points used to estimate the interpolating function must increase with the size of the state space in order to maintain accuracy. Moreover, even though the interpolations are considerably faster than the multiple integrations that would otherwise be necessary (regardless of the numerical integration technique), if the size of the state space becomes sufficiently large, the computational burden of calculating the fitted values of the non-simulated state points from the regression may become excessive. Later, in section III.D, we analyze the performance of alternative interpolating functions that permit $S^{**}(t)$ to be less than the full state space at time t, $S(t)$.

An important point about our approximate solution method is that we use the interpolated values of the Emax($\cdot$) functions only at those points in the state space needed for the backwards solution for which we did not simulate the Emax($\cdot$) function. We always use the simulated EMAX values when they occur in the backwards solution. Thus, as the number of state points at which we calculate simulated EMAX values increases and the number of draws used in the simulation become large, our approximation approaches the exact solution.

As already noted, the interpolating function based on (15) may be implemented in the serially dependent case using discretized values of the $\epsilon_k(t)$'s. If they take on M possible values at each t and the serial dependence is first-order Markov, then the state space will be $M * K'$ (K' is the number of disturbances, which in our example is the same as the number of alternatives, K) times larger than in the iid case. If this state space is too large, given a "reasonable" value for M and given the size of $\bar{S}$, for the interpolation using (15) to be computationally feasible, one of the alternative interpolating functions described in section III.D below can be used. These interpolating functions do not require discretization of the shocks.

B. *Simulated Data*

To ascertain the performance of the approximate solution method described above, we adopt the following specifications of the four-alternative occupational choice model

$$\begin{aligned}
\tilde{w}_{1t} &= \alpha_{10} + \alpha_{11}s_t + \alpha_{12}x_{1t} - \alpha_{13}x_{1t}^2 + \alpha_{14}x_{2t} \\
&\quad - \alpha_{15}x_{2t}^2, \\
\tilde{w}_{2t} &= \alpha_{20} + \alpha_{21}s_t + \alpha_{22}x_{1t} - \alpha_{23}x_{1t}^2 + \alpha_{24}x_{2t} \\
&\quad - \alpha_{25}x_{2t}^2, \\
& f\big(\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t} | S(t-1), d_k(t-1)\big) \\
&\quad = f\big(\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t} | \bar{S}(t-1), d_k(t-1)\big) \\
&\quad \sim N(0, \Sigma). \qquad (16)
\end{aligned}$$

The other aspects of the model are as previously described.

Table 1 reports parameter values for three different data sets that we use for assessing performance.[15] In all of the data sets, occupation two is more skill-intensive in the sense that schooling has a higher return (except in data set three where they are equal), and "own" experience has a higher return. In addition, experience in occupation one increases skills (and thus the wage) in occupation two while the reverse is never true. Thus, both schooling and occupation one experience provide general skills useful in both occupations, while occupation two experience provides skills specific only to occupation two.

Data set two differs from data set one in several aspects: (i) occupation two has a lower mean wage at $t = 0$ in the second data set than in the first, (ii) schooling has positive consumption value, there is a non-zero tuition cost for college and a larger adjustment cost associated with returning to school in data set two, (iii) the value of nonmarket time is lower in data set two, and (iv) the standard deviations of the (ln) wage errors are twice as large, and those of the schooling value and home value errors four times as large, in data set two.

Data set three differs from one and two in that (i) the (ln) wage functions in both occupations are linear in experience, rather than concave, in data set three, (ii) the schooling adjustment cost is larger in data set three, (iii) the value of home time is larger in data set three, (iv) the standard deviations of the (ln) wage errors are at least twice as big in data set three as in two and at least four times as big as in one, and the standard deviations of the schooling and home time value errors are slightly bigger in three than in two, and (v) the ln wage errors of the two occupations have a contemporaneous correlation of 0.5 and the schooling and home time value errors have a correlation of -0.5 in data set three, while all contemporaneous correlations are zero in data sets one and two.

These particular parameter values were chosen for two reasons: because they give substantively different life-cycle choice patterns and because

[15] The discount factor is set to 0.95 throughout the analysis.

TABLE 1.—PARAMETER VALUES

Parameters	Data Set One	Data Set Two	Data Set Three
α_{10}	9.21	9.21	8.00
α_{11}	0.038	0.04	0.07
α_{12}	0.033	0.033	0.055
α_{13}	0.0005	0.0005	0.0
α_{14}	0.0	0.0	0.0
α_{15}	0.0	0.0	0.0
α_{20}	8.48	8.20	7.90
α_{21}	0.07	0.08	0.07
α_{22}	0.067	0.067	0.06
α_{23}	0.001	0.001	0.0
α_{24}	0.022	0.022	0.055
α_{25}	0.0005	0.0005	0.0
β_0	0.	5000.	5000.
β_1	0.	5000.	5000.
β_2	4000.	15000.	20000.
γ_0	17750.	14500.	21500.
$(\sigma_{11})^{1/2}$	0.2	0.4	1.0
σ_{12}	0.0	0.0	0.5
σ_{13}	0.0	0.0	0.0
σ_{14}	0.0	0.0	0.0
$(\sigma_{22})^{1/2}$	0.25	0.5	1.0
σ_{23}	0.0	0.0	0.0
σ_{24}	0.0	0.0	0.0
$(\sigma_{33})^{1/3}$	1500.	6000.	7000.
σ_{34}	0.0	0.0	-2.975×10^7
$(\sigma_{44})^{1/2}$	1500.	6000.	8500.

$$R_1(t) = w_{1t} = \exp\left(\alpha_{10} + \alpha_{11}s_t + \alpha_{12}x_{1t} - \alpha_{13}x_{1t}^2 + \alpha_{1x2t} - \alpha_{15}x_{2t}^2 + \epsilon_{1t}\right)$$
$$R_2(t) = w_{2t} = \exp\left(\alpha_{20} + \alpha_{21}s_t + \alpha_{22}x_{2t} - \alpha_{23}x_{2t}^2 + \alpha_{24}x_{1t} - \alpha_{25}x_{1t}^2 + \epsilon_{2t}\right)$$
$$R_3(t) = \beta_0 - \beta_1 I(s_t \geq 13) - \beta_2(1 - d_3(t-1)) + \epsilon_{3t}$$
$$R_4(t) = \gamma_0 + \epsilon_{4t}$$
$$\Sigma = (\sigma_{ij})$$

they have increasingly larger error-variances. The latter characteristic should, by itself, reduce the accuracy of the EMAX simulations. Using the parameter values in table 1, we generated three artificial data sets of 1000 persons each based on an "exact" solution of the optimization problem that uses 100,000 draws for EMAX computed for all state space elements.[16] The individuals are initially identical; their different life cycle paths are generated solely by their different sequences of lifetime error draws.

The choice distributions generated by each set of parameter values are reported in Keane and Wolpin (1994). Here we describe their main features. In data set one, the proportion of individuals in occupation one begins at 0.39, rises to a peak in period 4 of 0.46 and gradually declines throughout the rest of the life cycle to 0.23 in period 40. In data set two, the pattern is similar although the initial rise in participation in occupation one is greater, from 0.34 in period 1 to 0.66 in period 7, and the fall more gradual to 0.55 in period 40. In data set three the rise is even more pronounced, from 0.17 in period 1 to 0.80 in period 12, as the skills acquired in occupation one are more general, and the decline more pronounced, to only 0.27 in period 40, as the returns are obtained by switching to occupation two. Participation in occupation two increases continuously over the life cycle in all three data sets, although more steeply in data sets one and three.

The proportion of individuals in school declines rapidly with age in all three data sets as schooling is essentially only an investment good. Schooling has its highest overall return in data set three. Home-time is lowest in data set one and constant over the life cycle, has an inverted u-shape in data set two reaching a peak of 0.09 in period 7, and has a u-shape in data set three rising to 0.13 in period 40. Thus, data set three generates some degree of voluntary retirement as individuals leave occupation one because its investment opportunities are less valuable as the end of the horizon is approached.

C. *Results: Assessing Performance*

In this section, we report on the performance of the approximation method using several criteria: (1) how well the approximate solution corresponds to the true solution at the given parameter values, (2) the extent of the bias in estimated parameters resulting from using the approximate solution method, (3) the extent of the bias in out-of-sample predictions using the approximate solution method, and (4) the extent of the bias in predicting the impact of the policy experiment of introducing a college tuition subsidy using the approximate solution method.

In order to implement the method we need to specify a functional form for the interpolating function (15). The specification that worked best

[16] We did not create a data set with serially correlated unobservables because obtaining an exact solution and conducting an analysis of the performance of our approximation method would have been computationally prohibitive. Except for the errors caused by the discretization of the disturbances, there is no reason to believe that, for a given dimension of the state space, the performance of the approximation method would be different with serially correlated disturbances.

(evidence is presented below) is of the form:

$$\text{EMAX} - \text{MAXE} = \pi_0 + \sum_{j=1}^{4} \pi_{1j}\left(\text{MAXE} - \bar{V}_j\right) + \sum_{j=1}^{4} \pi_{2j}\left(\text{MAXE} - \bar{V}_j\right)^{1/2}. \tag{17}$$

The π's are freely time-varying and are estimated by ordinary least squares.[17] The subset of state points used to estimate the interpolating function, for a fixed number of elements in the subset, is chosen randomly.[18]

1. Approximate Solutions at True Parameter Values: The first set of results considers the performance of approximate solutions in terms of how closely they correspond to the optimal solutions at the true parameter values. Simulating EMAX at all of the state points, we vary the number of EMAX simulation draws using 2000, 1000, and 250 draws. Then, fixing the draws at 2000 we simulated EMAX values at randomly selected state points and used these points to estimate the interpolating function (17). The EMAX values are interpolated at the remaining state points. We report results of this interpolation method using 2000 and 500 state points. In contrast to the simulation-interpolation approximation we also solve the optimization problem using MAXE instead of EMAX.[19] For each approximation specification we generate a sample of 1000 people using the same ϵ draws. Thus, difference between approximate solutions and the optimal solution are due solely to the approximation error. Further, because estimation in the case of serial independence is based on one-step ahead forecasts rather than full forecasts that are based only on initial conditions, we also report one-step ahead forecasts for the case with 2000 draws and 500 points.[20]

The top panels in tables 2.1–2.3 report the proportion of times an approximation specification yields an optimal choice at ten period intervals and over all 40 periods for each data set, respectively. In all three data sets, using 2000 simulation draws at all state points almost always produces optimal solutions: 98.5% of the time in data set one, 99.4% of the time in data set two, and 99.1% of the time in data set three. While performance deteriorates as the number of draws is reduced and all state points are used, even with only 250 draws over 96% of the choices are correctly matched in all of the data sets. Performance also deteriorates only mildly as the number of state points used is reduced. Simulating EMAX at 2000 state points (as opposed to all of the state points), and applying our interpretation method reduces the overall correct matches to no worse than 96.6%. Using only 500 state points (3.8% of the maximum number in period 40) reduces the percentage of correct matches to 96.8% in data set one, to 92.3% in data set two, and to 94.2% in data set three. The comparable one-step ahead forecasts do considerably better, particularly in data set two for which the full forecast was least effective. The use of MAXE as an approximation to EMAX does much worse than any of the other approximations, achieving a 34% overall success rate in data set one, a 74% success rate in data set two, and a 50% success rate in data set three.

The previous set of results refers only to cross-sectional matches. The bottom panels of tables 2.1–2.3 evaluate performance of the same set of

[17] Not all of the derivative properties of the EMAX function are consistent with (17); specifically, some of the cross-partial derivatives in (17) are identically zero. However, adding interaction terms generally led to worse out-of-sample EMAX predictions. And, imposing the set of restrictions when the interpolating function is only an approximation will not necessarily improve the predictions. For example, using the form for EMAX obtained in the extreme value case (12) performed worse than our interpolating function, even though it obviously satisfies the derivative restrictions. The one restriction we did impose on the interpolated values was to set EMAX equal to MAXE if the predicted value for EMAX was below MAXE.

[18] We were not able to find a systematic method of selecting the state points that improved the EMAX out-of-sample predictions. This is clearly an area for further research.

[19] Even ignoring the existing literature on the use of MAXE, the fact that MAXE is a lower bound to EMAX and that they move together makes it a natural choice as a comparison approximation. Moreover, EMAX approaches MAXE as the dispersion in the disturbances go to zero.

[20] Calculating the exact solution using 100,000 draws and all of the state points took approximately 50 minutes of cpu time on a CRAY 2. Using 2000 draws and all state points took 47 seconds, using 2000 draws and 2000 state points took 20 seconds, and using 2000 draws and 500 state points took 10 seconds. Cutting the number of draws in half, to 1000, using 2000 state point values reduced the cpu time to 13 seconds and using 500 state points to 6 seconds. Comparable times on an IBM RISC 6000 Model 350 are 7–8 times greater.

TABLE 2.1.—PROPORTION CORRECT CHOICES FOR ALTERNATIVE APPROXIMATIONS DATA SET ONE[a]

No. EMAX Draws[b]	2000	1000	250	2000	2000	2000[c]	MAXE[d]
No. States	All	All	All	2000	500	500	All
				At Selected Periods			
Period							
1	0.979	0.977	0.974	0.984	0.971	0.971	0.401
10	0.982	0.972	0.969	0.983	0.963	0.992	0.386
20	0.986	0.977	0.970	0.983	0.967	0.994	0.312
30	0.985	0.979	0.972	0.986	0.971	1.00	0.279
40	0.985	0.977	0.971	0.988	0.973	1.00	0.264
Total	0.985	0.977	0.970	0.984	0.968	0.994	0.338
				Number of Periods Over the Lifetime			
Periods							
1–10	1.3	2.1	2.6	1.0	2.4	0.0	71.0
11–35	0.7	0.7	1.5	1.0	2.2	0.0	4.4
36–38	2.4	3.1	3.1	2.8	5.2	1.8	1.6
39	1.4	2.4	3.0	9.5	10.6	18.6	1.9
40	94.2	91.7	89.8	85.7	79.6	79.6	21.1
Average Number of Periods	39.4	39.1	38.8	39.4	38.7	39.8	13.6

[a] Based on a simulated sample of 1000 people.
[b] EMAX = $E\max(V^1, V^2, V^3, V^4)$.
[c] One-step ahead forecast.
[d] MAXE = $\max(EV^1, EV^2, EV^3, EV^4)$.

TABLE 2.2.—PROPORTION CORRECT CHOICES FOR ALTERNATIVE APPROXIMATIONS DATA SET TWO[a]

No. EMAX Draws[b]	2000	1000	250	2000	2000	2000[c]	MAXE[d]
No. States	All	All	All	2000	500	500	All
				At Selected Periods			
Period							
1	0.997	0.980	0.986	0.987	0.976	0.976	0.742
10	0.995	0.975	0.957	0.969	0.898	0.966	0.769
20	0.996	0.974	0.962	0.956	0.907	0.965	0.728
30	0.996	0.978	0.960	0.961	0.927	0.994	0.721
40	0.996	0.984	0.970	0.981	0.930	1.00	0.734
Total	0.994	0.975	0.962	0.967	0.923	0.978	0.740
				Number of Periods Over the Lifetime			
Periods							
1–10	0.0	0.1	0.2	0.0	0.4	0.0	3.4
11–35	1.5	6.3	9.8	7.7	19.1	0.4	58.4
36–38	2.4	4.9	8.3	12.3	18.0	23.1	17.7
39	1.9	3.4	5.5	16.8	19.7	33.7	8.6
40	94.2	85.3	76.2	63.2	42.8	42.8	11.9
Average Number of Periods	39.7	39.0	38.5	38.7	36.9	39.1	29.6

[a] Based on a simulated sample of 1000 people.
[b] EMAX = $E\max(V^1, V^2, V^3, V^4)$.
[c] One-step ahead forecast.
[d] MAXE = $\max(EV^1, EV^2, EV^3, EV^4)$.

TABLE 2.3.—PROPORTION CORRECT CHOICES FOR ALTERNATIVE APPROXIMATIONS DATA SET THREE[a]

No. EMAX Draws[b]	2000	1000	250	2000	2000	2000[c]	MAXE[d]
No. States	All	All	All	2000	500	500	All
			At Selected Periods				
Period							
1	0.995	0.995	0.992	0.993	0.969	0.969	0.709
10	0.989	0.992	0.956	0.970	0.898	0.941	0.427
20	0.994	0.997	0.987	0.970	0.924	0.961	0.421
30	0.989	0.997	0.990	0.959	0.960	0.974	0.514
40	0.996	0.998	0.990	0.982	0.975	1.00	0.765
Total	0.991	0.994	0.982	0.966	0.942	0.963	0.508
			Number of Periods Over the Lifetime				
Periods							
1–10	0.0	0.0	0.0	0.0	0.0	0.0	1.4
11–35	1.6	0.7	3.4	6.9	15.3	2.7	97.8
36–38	6.4	5.7	13.1	25.8	35.6	38.9	0.6
39	7.4	5.3	11.6	21.2	23.9	33.2	0.3
40	84.7	88.3	71.9	46.1	25.2	25.2	0.0
Average Number of Periods	39.7	39.8	39.3	38.7	37.7	38.5	20.3

[a] Based on a simulated sample of 1000 people.
[b] EMAX = $E\max(V^1, V^2, V^3, V^4)$.
[c] One-step ahead forecast.
[d] MAXE = $\max(EV^1, EV^2, EV^3, EV^4)$.

alternative approximation specifications on a longitudinal basis by reporting the distribution of the number of periods over the 40 period lifetime that choices were optimal and the average of the total number of periods in which the optimal choice is made. Using this criterion, the approximations other than MAXE perform very well in all three data sets. Although the fraction of individuals with all 40 choices correctly predicted does diminish when fewer draws are used and also when interpolation is used, the fraction of individuals with at least 30 periods correctly matched never is below 90%. And even in the worst case, over 50% of individuals are matched correctly in 39 or 40 periods. In contrast, for the MAXE approximation only in data set two are even 50% of the individuals matched correctly for 30 or more periods.[21]

Overall, the evidence presented above implies that (i) when EMAX is approximated by Monte Carlo integration at all state points, in most cases even 250 simulation draws may be adequate to approximate the solution of the optimization problem, and (ii) the approximation solution, while more sensitive to the number of state points for which EMAX must be interpolated than was the EMAX approximation using all state points to the number of draws, is by most criteria close to the true solution.

It is possible that the deterioration in the performance of the approximations as fewer state points are used to estimate the interpolating function occurs because the interpolating function that we have specified doesn't predict EMAX very well. Table 3, however, provides limited evidence that the function we use provides a quite accurate prediction. Table 6 reports the correlation between the actual EMAX in period 40 and the predicted EMAX for the three data sets, when all state points are used and when only 200 state points are used, for four different specifications of the interpolating function. The different

[21] We also performed period-by-period chi-square fit tests, based on the same simulated sample of 1000 people, of the choice distribution and the state variables, accumulated schooling and occupation-specific work experience, for each approximation specification. The results are reported in Keane and Wolpin (1994). To summarize, for data set one all of the approximations except for MAXE performed extremely well in this dimension. For data set two, the fit was only rejected consistently for the 2000 points-500 draws approximation and for the MAXE approximation. For data set three, the schooling distribution was not well fit even for the 2000/2000 and 250/ALL cases, while the choice distribution fit well even for the 2000/500 approximation and MAXE continued to perform poorly.

TABLE 3.—CORRELATION OF PREDICTED EMAX USING 200 STATE POINTS AND ALL (13150) STATE POINTS IN PERIOD 40 FOR ALTERNATIVE INTERPOLATING FUNCTIONS

	Linear			Square Root			Logarithmic			Linear and Square Root		
		200 Points			200 Points			200 Points			200 Points	
Data Set	All Pts	In Sample	Out of Sample	All Pts	In Sample	Out of Sample	All Pts	In Sample	Out of Sample	All Pts	In Sample	Out of Sample
One	0.874	0.870	0.870	0.931	0.916	0.930	0.873	0.874	0.870	0.980	0.975	0.973
Two	0.978	0.980	0.978	0.982	0.986	0.982	0.849	0.918	0.950	0.994	0.996	0.994
Three	0.974	0.979	0.974	0.941	0.941	0.938	0.724	0.764	0.721	0.989	0.990	0.989

specifications refer to different functional forms for $g(\cdot)$ in (15). The forms we report are linear, square root, logarithmic, and both linear and square root. Recall that the linear plus square root specification is the function used in producing the previous tables. The combined linear-square root specification yields the highest out-of-sample correlation among the different specifications, 0.973 for data set one, 0.994 for data set two, and 0.989 for data set three.

Figures 1.1–1.3 depict the actual and predicted dependent variable, EMAX − MAXE, as $\bar{V}_2$ varies over a representative part of the state space at fixed values of $\bar{V}_1$ and $\bar{V}_3$ ($\bar{V}_4$ is always fixed) for each of the three data sets. When $\bar{V}_2$ is below MAXE, i.e., $\bar{V}_2$ is not the maximum of $\bar{V}_1$ through $\bar{V}_4$, EMAX − MAXE rises with $\bar{V}_2$ as MAXE is constant. EMAX − MAXE reaches a maximum when $\bar{V}_2$ equals MAXE and then declines as $\bar{V}_2$ increases; each unit increase in $\bar{V}_2$ and thus MAXE increases EMAX by less than one unit and by smaller amounts the greater is $\bar{V}_2$ (and thus MAXE). As is evident from the figures, and consistent with table 3, the predictions based on the interpolating function (17) are usually very close.

Figures 2.1–2.3 replace EMAX − MAXE with EMAX itself, which is what enters the dynamic programming solution. The figures also include the MAXE approximation which is below $\bar{V}_2$ until $\bar{V}_2$ = MAXE and is thereafter equal to $\bar{V}_2$. While the interpolating function fits quite well, as is already known from the previous set of figures, MAXE differs from EMAX by as much as 40%.[22] In spite of the evidence from table 2 and these figures, it is nevertheless true that the errors induced by the simulation-interpolation of EMAX must be sufficiently large in some cases to produce the false predictions reported in table 2.

2. *Estimation Based on Approximate Solutions.* Given the fit-test results reported in the preceding tables, it is unclear to what extent the approximations would generate biases in parameters estimated from data obtained from the exact solution. To ascertain how large estimation biases would be, we performed a Monte Carlo experiment of estimating the parameters of the occupational choice model for 40 different 100 person data sets generated from the exact solution. Estimation was performed by simulated maximum likelihood (Albright et al., 1977) using 200 draws to form kernel-smoothed simulators of the probabilities, as given by (11), that enter the likelihood function.[23] For these experiments, we used a 500-200 approximation specification, 500 draws to simulated EMAX and 200 state points for the interpolating function, considerably smaller than the draw-state point combinations reported in tables 3–5.[24]

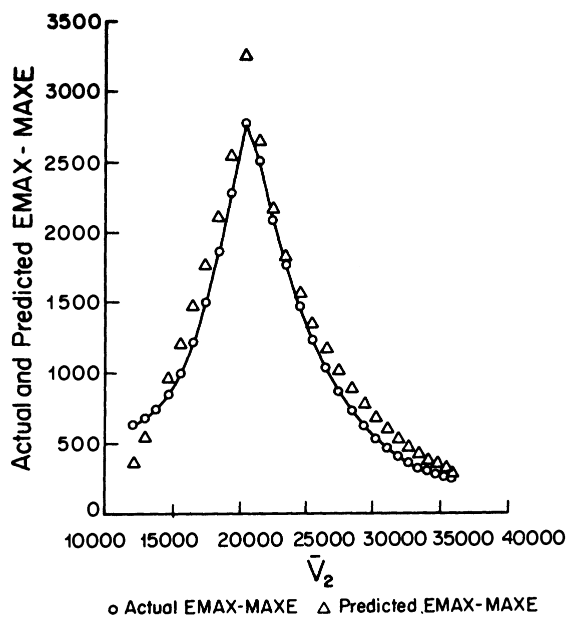
Tables 4.1–4.3 report the results of this Monte Carlo experiment for the three data sets. The parameters are as previously defined except for the a_{ij}'s which are the Cholesky decomposition parameters used to generate the joint ϵ_k error covariance matrix. The third column in the tables

[22] Interestingly, while the MAXE function is convex in $\bar{V}_2$, as must be the Emax function, the best fitting approximation function is not.

[23] The smoothing function is necessary because with only 200 draws there are cells that have no simulated observations and because it enables the use of gradient methods for estimation. The smoothing function used was the kernel smoothing function described in McFadden (1989) with a window parameter of 500.

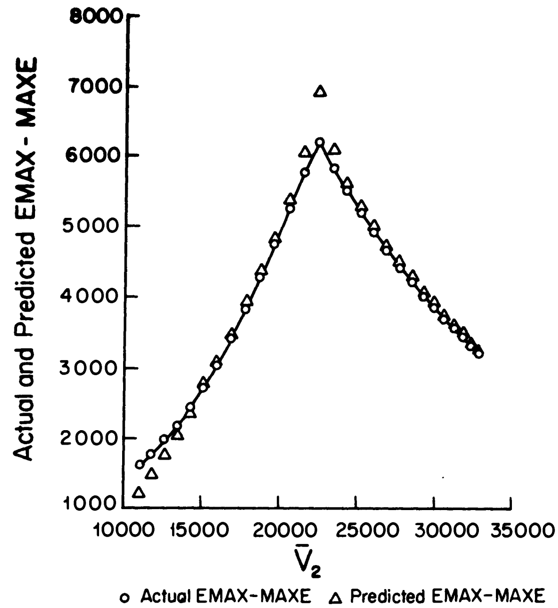
[24] The simulated ML estimator is consistent as the number of state points, the number of EMAX simulation draws, and the number of choice probability simulation draws all become large.

FIGURE 1.1.—ACTUAL VS. PREDICTED VALUE OF EMAX − MAXE: DATA SET ONE



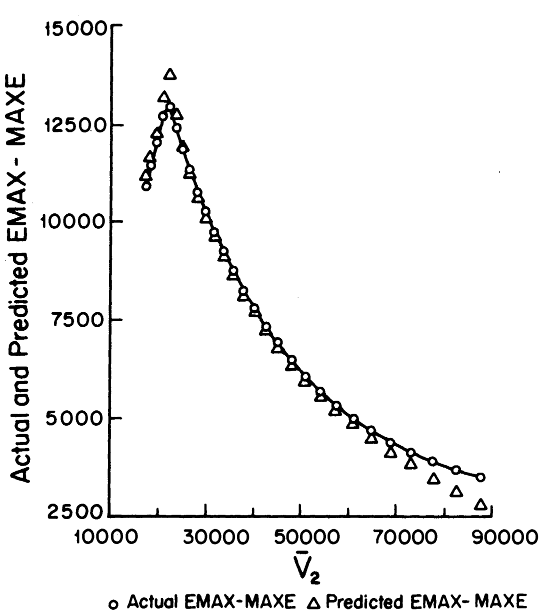
Note: Dataset one, points 415 through 442 in the state space in period 40. At these points, $\bar{V}_1 = 20{,}619.65$, $\bar{V}_3 = -4000.0$ and $\bar{V}_4 = 17{,}750.0$.

FIGURE 1.2.—ACTUAL VS. PREDICTED VALUE OF EMAX − MAXE: DATA SET TWO



Note: Dataset two, points 415 through 442 in the state space in period 40. At these points, $\bar{V}_1 = 22{,}337.01$, $\bar{V}_3 = -10{,}000.0$ and $\bar{V}_4 = 14{,}500.0$.

FIGURE 1.3.—ACTUAL VS. PREDICTED VALUE OF EMAX − MAXE: DATA SET THREE

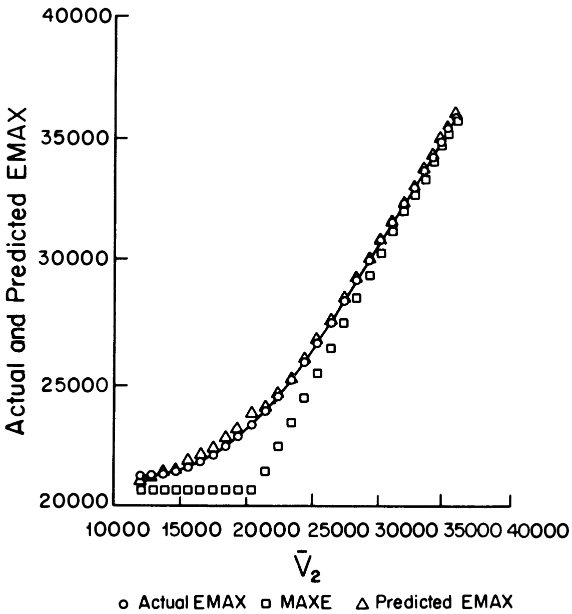


Note: Dataset three, points 415 through 442 in the state space in period 40. At these points, $\bar{V}_1 = 19{,}148.89$, $\bar{V}_3 = -15{,}000.0$ and $\bar{V}_4 = 21{,}500.0$.

reports the estimated bias, that is, the average deviation of the estimated parameter from the true parameter over the 40 experiments. The t-statistic for that bias, reported in the next column, is obtained from the standard deviation of the estimated parameters over the 40 experiments, which is shown in column five. The last column shows the average estimated standard error of the parameter estimate obtained from the (simulated) first derivative outer product approximation of the Hessian matrix of the likelihood.

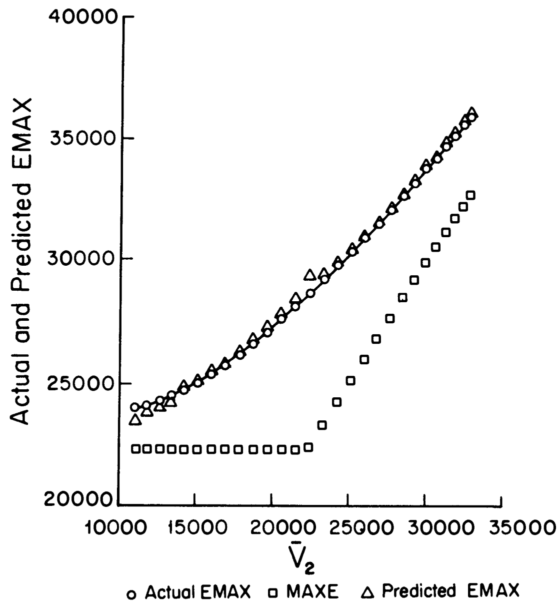
Consider the results for each of the data sets. For data set one, the biases overall seem to be very small and precisely estimated. The only parameters for which the bias seems at all substantively significant are the Cholesky parameters a_{33}, a_{42}, and a_{44}. Given that we observe the current period rewards for occupations, namely (accepted) wages, but not for schooling or home, it might be expected that pinning down error variances and covariances involving those choices would be most difficult. Based on the empirical standard deviations, the model parameters (true value + bias) are themselves in most cases also

FIGURE 2.1.—ACTUAL VS. PREDICTED EMAX: DATA SET ONE



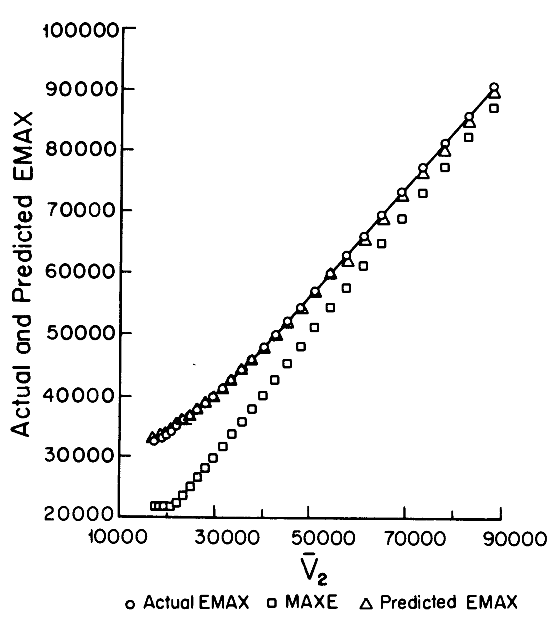
Note: Dataset one, points 415 through 442 in the state space in period 40. At these points, $\bar{V}_1$ = 20,619.65, $\bar{V}_3$ = −4000.0 and $\bar{V}_4$ = 17,750.0.

FIGURE 2.2.—ACTUAL VS. PREDICTED EMAX: DATA SET TWO



Note: Dataset two, points 415 through 442 in the state space in period 40. At these points, $\bar{V}_1$ = 22,337.01, $\bar{V}_3$ = −10,000.0 and $\bar{V}_4$ = 14,500.0.

FIGURE 2.3.—ACTUAL VS. PREDICTED EMAX: DATA SET THREE



Note: Dataset three, points 415 through 442 in the state space in period 40. At these points, $\bar{V}_1$ = 19,148.89, $\bar{V}_3$ = −15,000.0 and $\bar{V}_4$ = 21,500.0.

precisely estimated; t-statistics over 100 are not unusual. However, standard errors based on the simulated approximation to the Hessian matrix are significantly overstated, often by an order of magnitude.

The biases are also generally small for data sets two and three, although less precisely estimated. Again, the only substantively important biases seem to be for the Cholesky decomposition parameters associated with schooling and home, a_{31} and a_{32} for data set two and a_{31} and a_{42} for data set three. Also, as with data set one, parameters are estimated precisely, but standard errors using the simulated outer product approximation to the Hessian, while generally overstated, seem to be less severely overstated than was the case with data set one.

While the biases appear small in some sense, it is unclear what is an appropriate metric. If one is interested in the estimates of particular parameters in themselves, then it probably makes little difference if the "return" to schooling is thought to be 3.8% in occupation one or 3.822% as is the case for data set one. However, it is likely that the parameter values themselves are not of pri-

TABLE 4.1.—MONTE CARLO ESTIMATION RESULTS
DATA SET ONE[a]

Parameter	True Value	Bias[b]	t-statistic Bias[c]	Standard Deviation of Estimated Parameter[d]	Mean of Estimated Standard Error[e]
α_{10}	9.21	0.0025	9.92	0.0016	0.014
α_{11}	0.038	0.00022	8.73	0.00016	0.0015
α_{12}	0.033	0.00037	16.0	0.00014	0.00079
α_{13}	0.0005	−0.000035	−19.5	0.000011	0.000019
α_{14}	0.0	−0.00062	−3.52	0.0011	0.0024
α_{15}	0.0	0.0000009	0.10	0.000058	0.000096
α_{20}	8.48	0.0023	7.59	0.0019	0.0123
α_{21}	0.07	0.000007	0.49	0.000095	0.00096
α_{22}	0.067	0.00031	13.6	0.00014	0.0010
α_{23}	0.001	−0.000029	−23.1	0.000008	0.000030
α_{24}	0.022	−0.00040	−6.24	0.00040	0.00090
α_{25}	0.0005	−0.000035	−5.46	0.000041	0.000070
β_0	0.0	−67	4.24	100	459
β_1	0.0	147	6.42	145	410
β_2	4000	−207	−4.15	317	660
γ_0[f]	17750	−111	−4.40	159	1442
a_{11}	0.2	−0.0014	−3.10	0.0030	0.0056
a_{21}	0.0	−0.0017	−0.70	0.016	0.023
a_{22}	0.25	−0.00029	−0.41	0.0044	0.0046
a_{31}	0.0	0.270	6.56	0.261	0.413
a_{32}	0.0	−0.037	−1.05	0.224	0.379
a_{33}	1500	−424	−12.3	218	350
a_{41}	0.0	0.042	2.19	0.123	0.911
a_{42}	0.0	0.210	3.27	0.406	0.624
a_{43}	0.0	−0.103	−1.10	−0.588	0.870
a_{44}	1500	−467	−8.20	360	786

[a] Based on 40 sets of 100 individuals. EMAX simulation uses 500 draws, interpolation uses 200 points, and likelihood simulation uses 200 draws.

[b] $\hat{\theta} - \theta,\ \hat{\theta} = \frac{1}{40}\sum_{j=1}^{40} \hat{\theta}_j.$

[c] $\left(\frac{\hat{\theta} - \theta}{\sigma_{\hat{\theta}}}\right)\sqrt{40}.$

[d] $\sigma_{\hat{\theta}} = \left[\frac{1}{39}\sum_{j=1}^{40}\left(\hat{\theta}_j - \hat{\theta}\right)^2\right]^{1/2}.$

[e] $\frac{1}{40}\sum_{j=1}^{40} \hat{\sigma}_{\hat{\theta}_j}.$

[f] $\epsilon_{1t} = a_{11}\eta_{1t}$
$\epsilon_{2t} = a_{21}\eta_{1t} + a_{22}\eta_{2t}$
$\epsilon_{3t} = a_{31}\eta_{1t} + a_{32}\eta_{2t} + a_{33}\eta_{3t}$
$\epsilon_{4t} = a_{41}\eta_{1t} + a_{42}\eta_{2t} + a_{43}\eta_{3t} + a_{44}\eta_{4t}$
$\eta_{kt} \sim N(0, 1),\ k = 1, \ldots, 4.$

mary interest, but rather that the model would be used for some forecasting purpose. It is therefore important to consider the performance of the approximation when the estimated models are used for out-of-sample prediction. Table 5 provides such information. To obtain out-of-sample predictions, a new set of 40 samples of 100 persons each was drawn and matched to each of the 40 sets of estimated parameter vectors obtained from the original 40 samples of 100 persons. From each of those 40 new samples, we calculated mean schooling attainment and work experience in each occupation at the end of life based on the estimated parameters and using the approximate solution method. The same calculations were performed at the true parameters using the exact solution method.

For data set one, table 5 shows that the approximate solution overstates schooling attainment by 0.06 periods, understates work experience in occupation one by 0.36 periods, and overstates work experience in occupation two by 0.32 periods. The residual home time, not reported in the table, is overstated by 0.02 periods. In percentage

TABLE 4.2.—MONTE CARLO ESTIMATION RESULTS DATA SET TWO[a]

Parameter	True Value	Bias[b]	t-statistic Bias[c]	Standard Deviation of Estimated Parameter[d]	Mean of Estimated Standard Error[e]
α_{10}	9.21	0.00076	2.58	0.0019	0.0041
α_{11}	0.40	0.000079	2.63	0.00019	0.00051
α_{12}	0.33	0.000059	2.43	0.00015	0.00030
α_{13}	0.0005	−0.000022	−10.0	0.000014	0.000012
α_{14}	0.0	−0.00017	−3.29	0.00033	0.00050
α_{15}	0.0	−0.000032	−4.55	0.000044	0.000036
α_{20}	8.20	0.00023	0.70	0.0021	0.0054
α_{21}	0.08	−0.000058	−1.87	0.00020	0.00060
α_{22}	0.067	0.000017	0.61	0.00017	0.00060
α_{23}	0.001	−0.000036	−7.53	0.000030	0.000026
α_{24}	0.022	0.000039	1.62	0.00015	0.00040
α_{25}	0.0005	−0.000023	−5.77	0.000025	0.000017
β_0	5000	−223	−2.31	610	906
β_1	5000	218	1.89	732	999
β_2	15000	−114	−0.67	1064	2565
γ_0	14500	−392	−5.03	493	1601
a_{11}[f]	0.4	−0.00028	−0.40	0.0044	0.0057
a_{21}	0.0	0.0051	1.55	0.021	0.022
a_{22}	0.5	−0.00039	−0.62	0.0040	0.0076
a_{31}	0.0	−0.394	−3.20	0.778	0.971
a_{32}	0.0	0.421	2.71	0.982	0.793
a_{33}	6000	106	1.24	541	1034
a_{41}	0.0	−0.065	−0.059	0.699	1.02
a_{42}	0.0	0.070	0.49	0.903	0.641
a_{43}	0.0	0.117	0.59	1.26	0.658
a_{44}	6000	89	−0.85	660	955

[a] Based on 40 sets of 100 individuals. EMAX simulation uses 500 draws, interpolation uses 200 points, and likelihood simulation uses 200 draws.

[b] $\hat{\theta} - \theta, \hat{\theta} = \frac{1}{40}\sum_{j=1}^{40} \hat{\theta}_j$.

[c] $\left(\frac{\hat{\theta} - \theta}{\sigma_{\hat{\theta}}}\right)\sqrt{40}$.

[d] $\sigma_{\hat{\theta}} = \left[\frac{1}{39}\sum_{j=1}^{40} \left(\hat{\theta}_j - \hat{\theta}\right)^2\right]^{1/2}$.

[e] $\frac{1}{40}\sum_{j=1}^{40} \hat{\sigma}_{\hat{\theta}_j}$.

[f] $\epsilon_{1t} = a_{11}\eta_{1t}$
$\epsilon_{2t} = a_{21}\eta_{1t} + a_{22}\eta_{2t}$
$\epsilon_{3t} = a_{31}\eta_{1t} + a_{32}\eta_{2t} + a_{33}\eta_{3t}$
$\epsilon_{4t} = a_{41}\eta_{1t} + a_{42}\eta_{2t} + a_{43}\eta_{3t} + a_{44}\eta_{4t}$
$\eta_{kt} = \sim N(0,1),\ k = 1,\ldots,4.$

terms, prediction errors are all below 3%. While the standard errors of the prediction error estimates generally exceed their point estimates, based on 95% confidence intervals one would probably judge these differences to be substantively small. The prediction biases for data set two are larger than those obtained for data set one, and they are also much more precisely estimated. However, 95% confidence intervals span still reasonably small biases. Data set three has the largest schooling prediction bias, 15%, but small experience biases. The 95% confidence interval for the schooling bias includes what might be judged an economically significant error, at the outer bound 0.75 periods (years) given actual additional attainment of 3.78 periods or a 20% error.

An important reason for obtaining estimates of the structural choice model is to calculate the effects of counterfactual policy interventions on decisions. Table 6 compares the estimates of a college tuition subsidy on end-of-life schooling and occupation-specific experience using the approximate and exact solutions based on the same

TABLE 4.3.—MONTE CARLO ESTIMATION RESULTS
DATA SET THREE[a]

Parameter	True Value	Bias[b]	t-statistic Bias[c]	Standard Deviation of Estimated Parameter[d]	Mean of Estimated Standard Error[e]
α_{10}	8.00	0.00032	2.72	0.00075	0.012
α_{11}	0.070	−0.000047	−2.47	0.00012	0.00038
α_{12}	0.055	0.000020	1.29	0.000097	0.00023
α_{13}	0.0	−0.0000018	−0.69	0.000011	0.0000079
α_{14}	0.0	−0.00034	−2.60	0.00083	0.0016
α_{15}	0.0	0.00014	2.95	0.00031	0.00021
α_{20}	7.90	0.00021	1.74	0.00076	0.0068
α_{21}	0.070	0.000026	1.07	0.00016	0.00037
α_{22}	0.06	−0.00013	−3.00	0.00027	0.00049
α_{23}	0.0	−0.000064	−4.93	0.000082	0.000032
α_{24}	0.55	0.0000078	0.64	0.000077	0.00026
α_{25}	0.0	−0.0000086	−3.66	0.000015	0.000011
β_0	5000	71.0	1.18	381	1073
β_1	5000	489	4.45	695	1208
β_2	20000	243	1.59	962	2004
γ_0	21500	−0.78	−0.26	19.2	25.3
a_{11}[f]	1.0	−0.000033	−0.25	0.00082	0.011
a_{21}	0.5	0.00051	1.95	0.0016	0.0059
a_{22}	0.866	0.000086	0.40	0.0013	0.0082
a_{31}	0.0	−0.321	−4.37	0.465	0.756
a_{32}	0.0	−0.110	−0.75	0.927	0.630
a_{33}	7000	182	2.69	426	725
a_{41}	0.0	0.055	1.00	0.346	0.518
a_{42}	0.0	0.179	1.87	0.604	0.494
a_{43}	−4250	12.3	0.11	735	579
a_{44}	7361	−1.11	−0.01	695	519

[a] Based on 40 sets of 100 individuals. EMAX simulation uses 500 draws, interpolation uses 200 points, and likelihood simulation uses 200 draws.

[b] $\hat{\theta} - \theta$, $\hat{\theta} = \frac{1}{40}\sum_{j=1}^{40} \hat{\theta}_j$.

[c] $\left(\frac{\hat{\theta} - \theta}{\sigma_{\hat{\theta}}}\right)\sqrt{40}$.

[d] $\sigma_{\hat{\theta}} = \left[\frac{1}{39}\sum_{j=1}^{40}\left(\hat{\theta}_j - \hat{\theta}\right)^2\right]^{1/2}$

[e] $\frac{1}{40}\sum_{j=1}^{40} \hat{\sigma}_{\hat{\theta}_j}$.

[f] $\epsilon_{1t} = a_{11}\eta_{1t}$
$\epsilon_{2t} = a_{21}\eta_{1t} + a_{22}\eta_{2t}$
$\epsilon_{3t} = a_{31}\eta_{1t} + a_{32}\eta_{2t} + a_{33}\eta_{3t}$
$\epsilon_{4t} = a_{41}\eta_{1t} + a_{42}\eta_{2t} + a_{43}\eta_{3t} + a_{44}\eta_{4t}$
$\eta_{kt} \sim N(0,1)$, $k = 1,\ldots,4$.

method of constructing out-of-sample data used in table 5 described above. The amount of the tuition subsidy varies across the data sets in order to keep the effects of similar magnitude. As table 6 indicates, a 500 dollar per-period subsidy assuming data set one parameters increases schooling by 1.44 periods, a 1000 dollar per-period subsidy assuming data set two parameters increasing schooling by 1.12 periods, and a 2000 dollar per-period subsidy assuming data set three parameters increases schooling by 1.67 periods. The corresponding tuition effects based on the parameter estimates obtained from the approximate solution are 1.72, 1.08, and 1.14 periods. Using the approximate solution would overstate the cost of the program by 140 dollars per cohort member for data set one (860 dollars vs. 720 dollars per cohort member). It would understate the cost by only 35 dollars per cohort member for data set two (1082 vs 1117), but would understate the cost by 1060 dollars per cohort member for data set three (2280 vs. 3340). Confidence intervals on all of these estimates are quite narrow. Whether an understatement of 30% of the total

TABLE 5.—OUT-OF-SAMPLE FIT OF MEAN OF STATE VARIABLES AFTER PERIOD 40 FOR APPROXIMATE SOLUTION METHOD (standard deviations in parentheses)

	Data Set One			Data Set Two			Data Set Three		
		Experience			Experience			Experience	
	Schooling	Occ. 1	Occ. 2	Schooling	Occ. 1	Occ. 2	Schooling	Occ. 1	Occ. 2
Exact Solution[a]	12.75 (0.25)	12.73 (1.40)	23.90 (1.31)	12.30 (0.23)	23.81 (0.78)	11.36 (0.75)	13.78 (0.27)	24.65 (0.49)	10.58 (0.42)
Approximate Solution[b]	12.81 (0.38)	12.37 (2.33)	24.22 (2.20)	12.54 (0.39)	22.64 (1.28)	12.38 (1.22)	13.21 (0.48)	24.72 (0.82)	10.89 (0.63)
Absolute Prediction Error	0.06	−0.36	0.32	0.24	−1.17	1.02	−0.57	0.07	0.31
Percent Prediction Error	2.2[c]	2.8	1.3	10.4	−4.9	9.0	−15.1	0.3	2.9
0.95 Confidence Interval for Absolute Prediction Error	−0.082, 0.202	−1.18, 0.462	−0.480, 1.12	0.106, 0.374	−0.746, −1.59	0.590, 1.46	−0.370, −0.750	−0.228, 0.368	0.070, 0.550

[a] Based on 40 samples of 100 persons using true parameter values.
[b] Based on 40 samples of 100 persons using estimated parameter values for each of the 40 samples as in tables 4.1–4.3.
[c] Relative to schooling −10.0.

TABLE 6.—EFFECT OF COLLEGE TUITION SUBSIDY ON STATE VARIABLES AFTER PERIOD 40 FOR APPROXIMATE VS. EXACT SOLUTION

	Data Set One[a]			Data Set Two[b]			Data Set Three[c]		
		Experience			Experience			Experience	
	Schooling	Occ. 1	Occ. 2	Schooling	Occ. 1	Occ. 2	Schooling	Occ. 1	Occ. 2
Exact Solution[d]	1.44 (0.18)	−3.43 (0.94)	2.19 (0.89)	1.12 (0.22)	−2.71 (0.53)	2.08 (0.43)	1.67 (0.20)	−1.27 (0.18)	−0.236 (0.10)
Approximate Solution[e]	1.72 (0.27)	−4.36 (1.10)	2.90 (1.06)	1.08 (0.30)	−2.70 (0.75)	2.12 (0.60)	1.14 (0.23)	−0.812 (0.27)	−0.154 (0.20)
Absolute Prediction Error	0.28	−0.93	0.71	−0.035	0.006	0.039	−0.529	0.455	0.083
0.95 Confidence Interval for Difference	0.178, 0.382	−1.47, −0.390	0.190, 1.27	−0.159 0.089	−0.304, 0.316	−0.207 0.285	−0.641 −0.417	0.339, 0.569	0.015, 0.151

[a] 500 dollar tuition subsidy.
[b] 1000 dollar tuition subsidy.
[c] 2000 dollar tuition subsidy.
[d] Based on 40 samples of 100 persons using true parameter values.
[e] Based on 40 samples of 100 persons using estimated parameter values for each of the 40 samples as in tables 4.1–4.3.

cost of the intervention, as for data set three, is large depends on the accuracy of alternative forecasts.

D. The Case of Extremely Large State Spaces: Alternative Interpolating Functions

1. The Problem: As discussed in the previous section, the interpolating function based on (15) would become infeasible to implement if the state space is sufficiently large. Recall that the approximate solution method based on (15) requires that the value of EMAX be either simulated or interpolated at every point in the state space. The reason is that the arguments in the EMAX interpolating function (15), the $\bar{V}_k$'s, are themselves functions of the next period EMAX function, as can be seen by taking the expectation of (4). Even very fast interpolations will become too computationally burdensome when the state space, and therefore the number of fitted EMAX's, is sufficiently large. Computer memory limitations will be reached eventually as well.

Serial correlation in unobservable state variables is a special case of the general problem because, as noted, it has the effect of increasing the size of the state space. If each disturbance is first-order Markov, calculating $\text{EMAX}(S(t+1), t+1|\bar{S}(t), \epsilon(t), d_k(t)=1)$ requires integrating over the distribution of $\epsilon(t+1)$ conditional on

$\epsilon(t)$. Backwards solution using (15) involves constructing and saving in memory a value (simulated or interpolated) of $\text{EMAX}(S(t+1), t+1|\bar{S}(t), \epsilon(t), d_k(t)=1)$ for every $(\bar{S}(t+1), \epsilon(t))$ point in the state space.

The general problem of an extremely large state space can be avoided by using interpolating functions for EMAX that do not include next period EMAX values as arguments. Examples would be interpolating functions for EMAX at time t whose arguments were the state variable at time t themselves, or the expected reward functions at time t. In these cases, the fitted EMAX values can be constructed as they are needed in the backwards solution from the estimated interpolating function parameters, and none of the interpolated EMAX values need to be saved in memory. Thus, the curse of dimensionality is essentially circumvented provided the interpolating function provides accurate predictions based on a computationally feasible number of simulated state points.

In the case of serially correlated unobservables, it is only necessary to treat them symmetrically with the other state variables. This approach has the additional advantage that the disturbances need not be discretized. In terms of estimation, simulated maximum likelihood is substantially complicated when there is serial correlation. However, the recursive simulation method developed by Keane (1994) applies directly to this case.

2. Results Using Alternative Interpolating Functions: In this section, we provide evidence on the performance of three alternative interpolating functions that are computationally tractable even when the state space is sufficiently large that it becomes infeasible to fill in all of the EMAX values by interpolation. The first alternative we consider is to interpolate EMAX from a quadratic approximation in the deterministic state space elements. Recall that the deterministic part of the state space for the dynamic program consists of the number of periods of experience in each occupation, the number of periods of schooling, and a dichotomous indicator of whether the individual attended school in the previous period. The second alternative uses a quadratic approximation in the contemporary payoffs evaluated at the means of their stochastic components; the contemporary payoffs are given by the deterministic components of the reward functions shown in (6).[25] The third specification of the interpolating function modifies the second by adding the maximum of the deterministic components of the value functions, MAXE (0), as a regressor. This last formulation is still feasible in the case of serial correlation because, unlike the $\bar{V}_k$ functions, this deterministic function does not depend on $\epsilon(t-1)$. However, this form is not feasible if the deterministic component of the state space is extremely large, because MAX(0) must be constructed at all points of the deterministic part of the state space.

The performance of these interpolating functions is compared in tables 7 and 8. We use the same three data sets as in tables 1–6. Unfortunately, it is not feasible to compare the approximations to the exact solutions for models where serial correlation is actually present or where the state space is many times larger than that we have already considered, because obtaining an exact solution of the model is computationally too burdensome.

Table 7 reports the proportion of correct choices, paralleling the top panels of tables 2.1–2.3, and the average number of periods correct, as in the bottom panels. The dynamic program is solved using 2000 simulation draws for the EMAX calculations and 500 state points for the interpolating regression. The results indicate that the third specification is the most consistent, performing slightly better than our preferred interpolating function in data set three, slightly worse in data set two, and fairly significantly worse in data set one. The third specification dominates the more parismonious second specification in all three data sets. The first specification, using the state space elements directly for the interpolation, is almost identical to the third specification in data three, is somewhat better in data set two, but is very significantly worse in data set one.

Table 8 reports the performance of the three alternative interpolating functions in estimation. Monte Carlo experiments were performed using the same design as those in table 4. However, given the computational burden of this exercise,

[25] Note that the deterministic reward in the non-market sector does not vary with the state space and is not used in the interpolating function.

TABLE 7.—PROPORTION CORRECT CHOICES FOR ALTERNATIVE INTERPOLATING FUNCTIONS AT SELECTED PERIODS USING 2000 DRAWS AND 500 STATE POINTS

	Data Set One Interpolating Function			Data Set Two Interpolating Function			Data Set Three Interpolating Function		
Period	1[a]	2[b]	3[c]	1[a]	2[b]	3[c]	1[a]	2[b]	3[c]
1	0.608	0.954	0.983	0.993	0.908	0.933	0.992	0.952	0.977
5	0.493	0.796	0.888	0.934	0.844	0.930	0.950	0.826	0.933
10	0.578	0.753	0.875	0.918	0.820	0.889	0.958	0.843	0.941
15	0.645	0.774	0.891	0.925	0.827	0.891	0.972	0.940	0.970
20	0.702	0.775	0.910	0.936	0.850	0.889	0.971	0.955	0.977
25	0.719	0.773	0.916	0.928	0.859	0.909	0.974	0.963	0.977
30	0.744	0.768	0.909	0.938	0.836	0.907	0.955	0.963	0.965
35	0.758	0.775	0.912	0.943	0.851	0.902	0.958	0.961	0.983
40	0.760	0.768	0.910	0.938	0.847	0.924	0.979	0.974	0.984
Total	0.665	0.778	0.904	0.938	0.837	0.901	0.966	0.931	0.968
Average No. Periods Correct	26.6	31.1	36.1	37.2	33.5	36.0	38.7	37.2	38.7

[a] Quadratic in state space.
[b] Quadratic in contemporaneous payoffs evaluated at mean errors.
[c] Same as [b] with maximum of future value functions evaluated at mean errors as additional regressor.

we use only data set one for the evaluation. Unlike the results in table 4.1, the quadratic in state variables approximation shown in table 8 reveals biases of substantial economic magnitude in several parameters, in particular, the blue-collar wage function intercept, the cost of returning to school, the value of home-time, and the concavity of the experience effects. Biases in the covariance matrix of the disturbances are also considerably larger than in table 4.1. Approximations based on the second alternative interpolating function, quadratics in the rewards evaluated

TABLE 8.—MONTE CARLO ESTIMATION RESULTS: ALTERNATIVE APPROXIMATION FUNCTIONS

	Quadratics in State Space		Quadratics in Rewards at Mean Errors		Quadratic in Rewards Plus MAX(0)	
Parameter	Bias	S.D.[a]	Bias	S.D.[a]	Bias	S.D.[a]
α_{10}	−0.028	0.011	−0.00050	0.0064	0.0045	0.0021
α_{11}	0.0061	0.00085	0.0014	0.00063	0.00024	0.00025
α_{12}	0.0024	0.0010	0.00069	0.00039	0.00076	0.00028
α_{13}	−0.00010	0.000034	−0.000049	0.000018	−0.000045	0.000017
α_{14}	−0.0067	0.0027	0.0065	0.0017	−0.0016	0.0015
α_{15}	0.00022	0.000092	−0.00021	0.000073	0.000064	0.000070
α_{20}	0.00049	0.0069	−0.00099	0.0050	0.0055	0.0030
α_{21}	−0.0022	0.00053	−0.00020	0.00036	0.000094	0.00013
α_{22}	0.0030	0.00063	0.00085	0.00068	0.00078	0.00026
α_{23}	−0.00014	0.000026	−0.00013	0.000019	−0.000055	0.000015
α_{24}	0.0016	0.0016	0.0041	0.0011	−0.00081	0.00069
α_{25}	−0.00017	0.00013	−0.00024	0.000078	−0.000035	0.000053
β_0	0.66	0.703	−78	32	66	151
β_1	0.71	0.795	−31	42	35	197
β_2	3556	1100	1184	960	−228	370
γ_0	−3141	2261	−2425	1486	−273	291
a_{11}	−0.0068	0.0056	−0.0045	0.0036	−0.0026	0.0035
a_{21}	0.0041	0.026	0.022	0.019	0.011	0.016
a_{22}	0.0046	0.0053	0.0060	0.0058	−0.0016	0.0045
a_{31}	−1.61	0.85	−1.70	0.46	0.267	0.31
a_{32}	−1.83	0.623	0.11	0.55	−0.073	0.26
a_{33}	950	698	45	454	−458	241
a_{41}	−0.73	2.04	−1.5	1.01	−0.024	0.21
a_{42}	2.29	1.27	3.00	0.91	0.420	0.60
a_{43}	1.05	2.92	65	1.23	−0.272	0.75
a_{44}	714	1773	−1058	1159	−575	706

[a] See note *d* in table 4.1.

at the mean disturbance, generally have smaller biases as seen in table 8. However, the bias in the estimated cost of returning to school and in the value of home-time remains large, as do those of some of the Cholesky parameters. As table 8 indicates, the third interpolating function causes, on balance, the smallest biases. Although larger than those in table 4.1, they are usually of the same order of magnitude.

The evidence indicates that the interpolating function based directly on the state variables can perform poorly. Taking into account how the state variables enter into the reward functions seems to matter. Experiments with alternative functional forms in these sets of arguments is needed before strong generalizations should be drawn.

IV. Conclusion

In this paper we have proposed a new method for approximately solving discrete choice dynamic programming problems. The method is based on simulation and interpolation. It requires that one simulate the expected maxima of the value functions only at a subset of the state points. Then, these simulated expected maxima are used to fit an interpolating regression that provides fitted values for the expected maxima at the other points, which are needed in the backwards solution process. Thus, our approximation method ameliorates Bellman's "curse of dimensionality" problem, obtaining approximate solutions for problems with otherwise intractably large state spaces.

The method we propose requires choosing a particular interpolating function. For the fairly general type of problem we consider, we find, in a number of Monte Carlo experiments, that a function that includes the expected alternative-specific value functions performs quite well in several dimensions. First, it produces optimal decision paths very similar to those produced by the true solution. Second, estimation based on this approximation method embedded in a simulated ML procedure produces parameter estimates with biases that seem negligible from a substantive economic point of view. Third, the estimates so obtained, used in conjunction with the same approximate solution method, do exceedingly well at out-of-sample prediction, and produce simulated policy effects very similar in most cases to those predicted by the true parameter values in conjunction with the exact solution method.

A drawback to the interpolating function that uses the expected value functions as regressors is that it requires that the simulated plus interpolated EMAX values span the entire state space. As the state space grows, the number of interpolated values can themselves become intractably large in terms of computational capacity and memory. Alternative interpolating functions that do not require spanning the whole state space were presented and evaluated. We find that approximations based on quadratics in the state space elements perform rather poorly at least with respect to one of the simulated data sets, while those based on quadratics in the expected rewards perform reasonably well. None perform as well overall as the interpolating function based on the expected value functions.

In conclusion, it is our view that the approximate solution method proposed in this paper is a promising way to greatly increase the complexity of the optimization models that are feasible to solve and estimate. However, much additional work, with expanded state spaces and choice sets, needs to be done to determine the method's general applicability.

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