Module 2

Objectives:

- Write down the state space ~~& a~~ and probability matrix of a small DTMC
- Explain if a DTMC is irreducible and aperiodic or not
- ~~Explain if a DTMC is~~
- Derive a DTMC's stationary distribution (if it exists) using the global and local balance equations
- Derive properties of Geo/Geo/1 queue, Geo/Geo/1/B queue
- Be able to apply Little's Law to them

Concepts

a. Markov Chains → A mathematical system that ~~undergoes~~ goes transitions from one state to another according to certain probabilistic rules. **Note** that a key feature is the probability of transitioning to any particular state depends only on the current state, NOT on preceeding states

all numbers

Here $\{X_k, k \in \mathbb{N}\}$ ← state space

state space → state space #

X_k is a random variable

X_k is the state of the process at time k

Markov Chain → $Pr(X_{k+1} = j \mid X_k = i)$ Probability of the next state equals j ~~is~~ given that the current state equals i, independant of k

- **Only dependant on current state**

We can make a matrix of i (current states) and j (future next states), each part of the matrix showing the probability of the next state based on the current state (P_{ij})

The rows of the matrix equal to 1 (row stochastic)

$$\rightarrow P = \begin{array}{cc} & \text{Next State (column)} \\ \text{Current State (Row)} & \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \end{array}$$

$P_{11} + P_{12} + P_{13} = 1$

We can use this to make a diagram

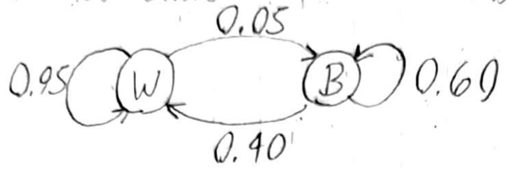
Example

$$P = \begin{array}{c|cc} & W & B \\ \hline W & 0.95 & 0.05 \\ B & 0.40 & 0.60 \end{array}$$

$\leftarrow$ Future state

Current state

Statement, if the machine is working today, there is a 0.05 probability it will be broken tomorrow.



$\leftarrow$ Transition diagram

$Pr(X_0 = \text{Working}) = 0.8$
$Pr(X_0 = \text{Broken}) = 0.2$ $\longrightarrow$ Probability of $Pr(X_i = \text{Working})$ $\leftarrow$ Working on Day 1

$Pr(X_i = W) = \underbrace{Pr(X_0 = W \cap X_i = W)}_{\text{Working union Working yesterday}} + Pr(X_0 = B \cap X_i = W)$ $\leftarrow$ Working union broken yesterday

Probability of working tomorrow $\rightarrow$ $Pr(X_0 = W) \cdot Pr(X_i = W \mid X_0 = W) + Pr(X_0 = B) \, Pr(X_i = W \mid X_0 = B)$

Given Above ... Given Above

$0.8 \, P_{WW} + 0.2 \, P_{BW}$

$0.8(0.95) + 0.2(0.4)$

$Pr(X_i = W) = 0.84$ $\leftarrow$ 84% chance of machine working tomorrow

Probability of a Future State

$$Pr(X_k = j) = \Sigma Pr(X_{k-1} = i) P_{ij}$$

Finding the probability of future states $\longrightarrow$ $P(X_k = j) = \underbrace{\Sigma Pr(X_{k-1} = i)}_{\text{Sum of previous states}} P_{ij}$ $\leftarrow$ current state

We can say

$$p[k] = \underbrace{p[k-1]}_{\text{previous state}} P \leftarrow \text{matrix}$$

Taking this to a specific step (ex $\rightarrow$ in step 3)

matrix $\rightarrow P^{n \leftarrow \text{step}}$ $\rightarrow$ $P_{ij}{}^{n} = Pr(X_n = j \mid X_0 = i)$

This means that to find the probability of a future state given the current, we can multiply the matrix.

N approaching ∞
Is there a limit

What fraction of time is the machine broken?

Try Stati.

Stationary Distribution

Capturing the long term of the system

Each matrix has a probability distribution matrix π

Things are stationary if

$\pi P = \pi$ → If π happens at $p[0]$ then step $p[k]$ will also be π

For this to work

1. We need to have π
2. $\lim_{n\to\infty} P^n$ needs to exist

Remeber DTMC must be

1. Irreducible
2. Aperoidic

π If π is

$\pi = 1$, then π has a unique stationary distribution

$\pi = \infty$, there is no stationary distribution, no limit

Finding distribution – Method #1 $\pi_j = \sum \pi_i P_{ij} \ \forall j$

Global Balance Equations

The amount of flows in equal the flow out

$$\begin{matrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{matrix} \qquad P = \begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/4 & 0 & 3/4 \\ 0 & 1/4 & 3/4 \end{pmatrix} \to \pi = P\pi$$

$$\begin{matrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{matrix} \qquad \pi_1 \ \pi_2 \ \pi_3 \begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/4 & 0 & 3/4 \\ 0 & 1/4 & 3/4 \end{pmatrix} \Rightarrow \begin{aligned} 1/4\pi_1 + 1/4\pi_2 + 0\pi_3 &= \pi_1 \\ 1/4\pi_1 + 0\pi_2 + 3/4\pi_3 &= \pi_2 \\ 0\pi_1 + 1/4\pi_2 + 3/4\pi_3 &= \pi_3 \end{aligned}$$

and

$\pi_1 + \pi_2 + \pi_3 = 1$

so → $\pi = (1/13, 3/13, 9/13)$

Local Balance Equations

Note These may not always work.

$\pi_j P_{ij} = \pi_i P_{ji}$ ←

$$\begin{aligned} \pi_1 P_{12} &= \pi_2 P_{21} & \pi_1 3/4 &= \pi_2 1/4 \\ \pi_2 P_{23} &= \pi_3 P_{32} \rightarrow & \pi_2 3/4 &= \pi_3 1/4 \\ \pi_1 P_{13} &= \pi_3 P_{31} & \pi_1 0 &= \pi_3 0 \end{aligned}$$

$\underline{\pi_1 + \pi_2 + \pi_3 = 1}$

Tells you if this works

Probability of time it is in this state

$\pi_1 \quad \pi_2 \quad \pi_2$

Limit / Theorem

The ~~system~~ system is equilibrium at only one point
As n step → ∞, you will arrive at one state

At step n, we know the probability is
P^n ← Exponent of the matrix

If this exists, this is the stationary distribution
$\pi P = \pi$ ✓

To find at the n^{th} step, we use
P^{12}

$$Ex \rightarrow P^{12} = \begin{pmatrix} 0.0770 & 0.2308 & 0.6923 \\ 0.0769 & 0.2308 & 0.6923 \\ 0.0769 & 0.2308 & 0.6923 \end{pmatrix}$$ ← Note, this is getting closer to a limit

If P was periodic, then $P(X_n = i) = 0$ for states I, then there would be no equilibrium

To ∞

What is the long run?
Assume $\pi = (1/13, 3/13, 9/13)$
Where p_1 is the 0 ← Bug Broken
p_2 is 0.25 ← Backup Broken
p_3 is 0.50 ← Working

$1/13(0) + 3/13(0.25) + 9/13(0.50) = 40.4\%$ ← Long run

@ $\pi_1\ \pi_2\ \pi_3$

π_1 tells us the amount of time we will be at state 1 in the long run

Geo/Geo/1 Assume an infinite buffer

\# of packets arrive at time slot k $\to a(k) \to$ Bernoulli independently and identically distributed

$s(k) \to$

↑

\# of packets served at k

$q(k) \to$ # of packets in line

In next time slot $k+1$

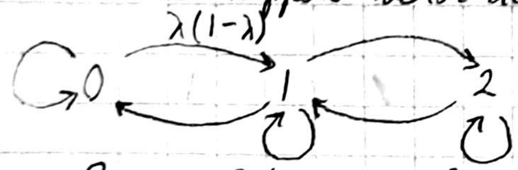
$q(k+1) = [q(k) + a(k) - s(k)]$

note

$s(k) \leq q(k)$

Arrival happens before departure

Imagine this



$\leftarrow$ A DTMC w/ no transient state

$P_{i,i+1} = \lambda(1-\mu) \to$ forward

$P_{i+1,i} = \mu(1-\lambda) \to$ backwards

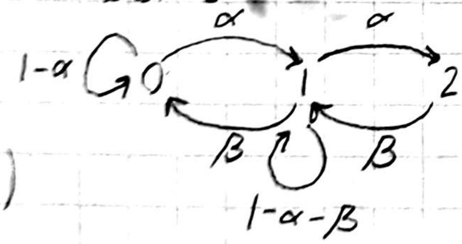
$P_{i,i} = \lambda\mu + (1-\lambda)(1-\mu)$

$P_{0,0} = 1-\lambda(1-\mu) \leftarrow$ Nothing in queue

$\alpha = \lambda(1-\mu) \leftarrow$ 1 arrive, doesn't leave

$\beta = \mu(1-\lambda) \leftarrow$ 0 arrivals, 1 departure

Rewrite



DTMC

$$P = \begin{bmatrix} 1-\alpha & \beta & 0 \\ \alpha & 1-\alpha-\beta & \beta \\ 0 & \alpha & 1-\alpha-\beta \end{bmatrix}$$

Irreducible

$P(X_{ij} = j \mid X_0 = 1)$

$=$

$P_{ij}^n > 0$

$P_r(X_{01} = 1 \mid X_0 = 0)$

$P_{01}^n > 0$

Aperiodic

State 0

$P_{00} > 0$

$\gcd(P_{00} > 0) = 1$

Local Balance

$\beta \pi_{i+1} = \alpha \pi_i$

$\frac{\cancel{\beta} \pi_{i+1}}{\cancel{\beta}} = \frac{\alpha}{\beta} \pi_i \to$ replace $\frac{\alpha}{\beta} = \rho$

$\pi_{i+1} = \rho \pi_i$

get to any state from 0

$\left.\begin{matrix} \pi_i = \rho^i \pi_0 \\ \text{sum of } \pi \text{ stationary distrib} \to \sum \pi_i = 1 \end{matrix}\right] \Rightarrow \sum_i \pi_i = \pi_0 \sum \rho^i = 1$

Note, if $\rho < 1$, $\sum \rho^i = 1/(1-\rho)$, $\pi_0 = 1-\rho$, $\pi_i = \rho^i(1-\rho)$

if $\rho \geq 1$, no stationary distribution

queue length infinite

Assume $\rho < 1$, $\pi_i = \rho^i(1-\rho)$

$\Sigma \rho^i = 1/1-\rho$

$\pi_0 = 1-\rho$

$\pi_i = \rho^i(1-\rho)$

$$E(q) = \Sigma\, i\pi^i$$
$$= \Sigma\, i\rho^i(1-\rho)$$
$$= (1-\rho)\rho \underbrace{\Sigma\, i\rho^{i-1}}$$

derivative of $\Sigma \frac{d\rho^i}{dt} = \frac{d\rho}{dt} \frac{1}{1-\rho}$

$$= (\cancel{1-\rho})\rho \frac{1}{1-\rho^2}$$

Long Term Average $\boxed{E(q) = \frac{\rho}{1-\rho}}$ $\Leftarrow$ Stationary Distr

Wait Time   Little Law

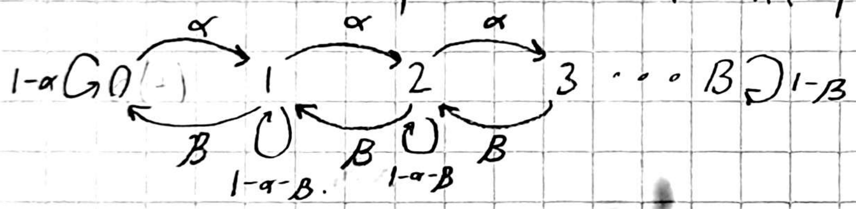
$$L = \lambda W$$

Long Term Average (L) = Mean arrival rate (λ) • Average Wait (W)

$L = E(q) = \frac{\rho}{1-\rho} \rightarrow W = \frac{L}{\lambda} = \frac{\rho}{\lambda(1-\rho)}$ $\rightarrow \rho \Rightarrow 1$ infinite

$\rho \Rightarrow 0 = W = 0$

Example



Irreducible and Aperiodic

$$\beta \pi_{1-i} = \alpha \pi_i \quad 0 < i < B-1$$

$$\rho = \frac{\alpha}{\beta} = \frac{\lambda(1-\mu)}{\mu(1-\lambda)}$$

$$\pi_i = \rho^i \pi_0 \rightarrow \pi_0 \Sigma_0^B \rho^i = 1$$

$$\rightarrow \pi_0 = \frac{1-\rho}{1-\rho^{B+1}}$$

$$\pi_i = \frac{(1-\rho)\rho^i}{1-\rho^{B+1}} \quad 0 > i > B$$

Packets dropped

$$P_d = P_r(q(t) = B \mid a(t) = 1) = P_r(q(t) = B) = \pi_B$$

t - time slot, $q(t)$ - queue length, $A(t)$, packet arrivals

$I(t)$ - Indicator of packet in queue

$I(t) \to 1$ arrival, in queue

$\quad\quad 0$ otherwise

$\lambda = \lim \lambda(T)$

$L = \lim L(T)$

$W = \lim W(n)$

Little's Law $\to$ λ - arrival rate

L - average queue length $\to$

W - average wait time of first n packets

Average Queue Length $\to E(q) = \pi_i = \alpha^i(1-\alpha) \Leftarrow$ Take π_i

$= \sum_{i=0}^{\infty} \alpha i(1-\alpha) \Leftarrow$ Find the sum

Results in a general expression

$$E(q) = \frac{\alpha}{1-\alpha}$$

$$\lambda = \frac{A(T)}{T} \qquad L(T) = \frac{\sum_1^T q(t)}{T} \qquad W = \frac{1}{n}\sum_{k=1}^{n} w_k$$

ave

$$L(T) = \sum_1^T \frac{x(t)/2}{T} = \frac{x(T)}{2T}$$

Averag