Stan Functions Reference

Version 2.24

Stan Development Team

Contents

Overview

This is the reference for the functions defined in the Stan math library and available in the Stan programming language.

The Stan project comprises a domain-specific language for probabilistic programming, a differentiable mathematics and probability library, algorithms for Bayesian posterior inference and posterior analysis, along with interfaces and analysis tools in all of the popular data analysis languages.

In addition to this reference manual, there is a user's guide and a language reference manual for the Stan language and algorithms. The *Stan User's Guide* provides example models and programming techniques for coding statistical models in Stan. The *Stan Reference Manual* specifies the Stan programming language and inference algorithms.

There is also a separate installation and getting started guide for each of the Stan interfaces (R, Python, Julia, Stata, MATLAB, Mathematica, and command line).

Interfaces and Platforms

Stan runs under Windows, Mac OS X, and Linux.

Stan uses a domain-specific programming language that is portable across data analysis languages. Stan has interfaces for R, Python, Julia, MATLAB, Mathematica, Stata, and the command line, as well as an alternative language interface in Scala. See the web site https://mc-stan.org for interface-specific links and getting started instructions

Web Site

The official resource for all things related to Stan is the web site:

https://mc-stan.org

The web site links to all of the packages comprising Stan for both users and developers. This is the place to get started with Stan. Find the interface in the language you want to use and follow the download, installation, and getting started instructions.

GitHub Organization

Stan's source code and much of the developer process is hosted on GitHub. Stan's organization is:

https://github.com/stan-dev

Each package has its own repository within the `stan-dev` organization. The web site is also hosted and managed through GitHub. This is the place to peruse the source code, request features, and report bugs. Much of the ongoing design discussion is hosted on the GitHub Wiki.

Forums

Stan hosts message boards for discussing all things related to Stan.

https://discourse.mc-stan.org

This is the place to ask questions about Stan, including modeling, programming, and installation.

Licensing

- *Computer code:* BSD 3-clause license

The core C++ code underlying Stan, including the math library, language, and inference algorithms, is licensed under the BSD 3-clause licensed as detailed in each repository and on the web site along with the distribution links.

- *Logo:* Stan logo usage guidelines

Acknowledgements

The Stan project could not exist without the generous grant funding of many grant agencies to the participants in the project. For more details of direct funding for the project, see the web site and project pages of the Stan developers.

The Stan project could also not exist without the generous contributions of its users in reporting and in many cases fixing bugs in the code and its documentation. We used to try to list all of those who contributed patches and bug reports for the manual here, but when that number passed into the hundreds, it became too difficult to manage reliably. Instead, we will defer to GitHub (link above), where all contributions to the project are made and tracked.

Finally, we should all thank the Stan developers, without whom this project could not exist. We used to try and list the developers here, but like the bug reporters, once the list grew into the dozens, it became difficult to track. Instead, we will defer to the Stan web page and GitHub itself for a list of core developers and all developer contributions respectively.

Built-In Functions

1. Void Functions

Stan does not technically support functions that do not return values. It does support two types of statements, one printing and one for rejecting outputs.

Although `print` and `reject` appear to have the syntax of functions, they are actually special kinds of statements with slightly different form and behavior than other functions. First, they are the constructs that allow a variable number of arguments. Second, they are the the only constructs to accept string literals (e.g., `"hello world"`) as arguments. Third, they have no effect on the log density function and operate solely through side effects.

The special keyword `void` is used for their return type because they behave like variadic functions with void return type, even though they are special kinds of statements.

1.1. Print Statement

Printing has no effect on the model's log probability function. Its sole purpose is the side effect (i.e., an effect not represented in a return value) of arguments being printed to whatever the standard output stream is connected to (e.g., the terminal in command-line Stan or the R console in RStan).

`void` **`print`**`(T1 x1,..., TN xN)`
Print the values denoted by the arguments x1 through xN on the output message stream. There are no spaces between items in the print, but a line feed (LF; Unicode U+000A; C++ literal `'\n'`) is inserted at the end of the printed line. The types `T1` through `TN` can be any of Stan's built-in numerical types or double quoted strings of ASCII characters.

1.2. Reject Statement

The reject statement has the same syntax as the print statement, accepting an arbitrary number of arguments of any type (including string literals). The effect of executing a reject statement is to throw an exception internally that terminates the current iteration with a rejection (the behavior of which will depend on the algorithmic context in which it occurs).

`void` **`reject`**`(T1 x1,..., TN xN)`
Reject the current iteration and print the values denoted by the arguments x1 through xN on the output message stream. There are no spaces between items in the print, but a line feed (LF; Unicode U+000A; C++ literal `'\n'`) is inserted at the end of the printed

line. The types `T1` through `TN` can be any of Stan's built-in numerical types or double quoted strings of ASCII characters.

2. Integer-Valued Basic Functions

This chapter describes Stan's built-in function that take various types of arguments and return results of type integer.

2.1. Integer-Valued Arithmetic Operators

Stan's arithmetic is based on standard double-precision C++ integer and floating-point arithmetic. If the arguments to an arithmetic operator are both integers, as in `2 + 2`, integer arithmetic is used. If one argument is an integer and the other a floating-point value, as in `2.0 + 2` and `2 + 2.0`, then the integer is promoted to a floating point value and floating-point arithmetic is used.

Integer arithmetic behaves slightly differently than floating point arithmetic. The first difference is how overflow is treated. If the sum or product of two integers overflows the maximum integer representable, the result is an undesirable wraparound behavior at the bit level. If the integers were first promoted to real numbers, they would not overflow a floating-point representation. There are no extra checks in Stan to flag overflows, so it is up to the user to make sure it does not occur.

Secondly, because the set of integers is not closed under division and there is no special infinite value for integers, integer division implicitly rounds the result. If both arguments are positive, the result is rounded down. For example, `1 / 2` evaluates to 0 and `5 / 3` evaluates to 1.

If one of the integer arguments to division is negative, the latest C++ specification (C++11), requires rounding toward zero. This would have `1 / 2` and `-1 / 2` evaluate to 0, `-7 / 2` evaluate to -3, and `7 / 2` evaluate to 3. Before the C++11 specification, the behavior was platform dependent, allowing rounding up or down. All compilers recent enough to be able to deal with Stan's templating should follow the C++11 specification, but it may be worth testing if you are not sure and plan to use integer division with negative values.

Unlike floating point division, where `1.0 / 0.0` produces the special positive infinite value, integer division by zero, as in `1 / 0`, has undefined behavior in the C++ standard. For example, the clang++ compiler on Mac OS X returns 3764, whereas the g++ compiler throws an exception and aborts the program with a warning. As with overflow, it is up to the user to make sure integer divide-by-zero does not occur.

Binary Infix Operators

Operators are described using the C++ syntax. For instance, the binary operator of addition, written `X + Y`, would have the Stan signature `int operator+(int,int)` indicating it takes two real arguments and returns a real value. As noted previously, the value of integer division is platform-dependent when rounding is platform dependent before C++11; the descriptions below provide the C++11 definition.

`int` **`operator+`**`(int x, int y)`
The sum of the addends x and y

$$\text{operator+}(x, y) = (x + y)$$

`int` **`operator-`**`(int x, int y)`
The difference between the minuend x and subtrahend y

$$\text{operator-}(x, y) = (x - y)$$

`int` **`operator*`**`(int x, int y)`
The product of the factors x and y

$$\text{operator*}(x, y) = (x \times y)$$

`int` **`operator/`**`(int x, int y)`
The integer quotient of the dividend x and divisor y

$$\text{operator/}(x, y) = \begin{cases} \lfloor x/y \rfloor & \text{if } x/y \geq 0 \\ -\lfloor \text{floor}(-x/y) \rfloor & \text{if } x/y < 0. \end{cases}$$

`int` **`operator%`**`(int x, int y)`
x modulo y, which is the positive remainder after dividing x by y. If both x and y are non-negative, so is the result; otherwise, the sign of the result is platform dependent.

$$\text{operator\%}(x, y) \;=\; x \bmod y \;=\; x - y * \lfloor x/y \rfloor$$

Unary Prefix Operators

`int` **`operator-`**`(int x)`
The negation of the subtrahend x [operator-(x) = -x

`int` **`operator+`**`(int x)`
This is a no-op.

$$\text{operator+}(x) = x$$

2.2. Absolute Functions

`R` **`abs`**`(T x)`
absolute value of x

`int` **`int_step`**`(int x)`

`int` **`int_step`**`(real x)`
Return the step function of x as an integer,

$$\text{int\_step}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \text{ or } x \text{ is } NaN \end{cases}$$

Warning: `int_step(0)` and `int_step(NaN)` return 0 whereas `step(0)` and `step(NaN)` return 1.

See the warning in section step functions about the dangers of step functions applied to anything other than data.

2.3. Bound Functions

`int` **`min`**`(int x, int y)`
Return the minimum of x and y.

$$\min(x, y) = \begin{cases} x & \text{if } x < y \\ y & \text{otherwise} \end{cases}$$

`int` **`max`**`(int x, int y)`
Return the maximum of x and y.

$$\max(x, y) = \begin{cases} x & \text{if } x > y \\ y & \text{otherwise} \end{cases}$$

3. Real-Valued Basic Functions

This chapter describes built-in functions that take zero or more real or integer arguments and return real values.

3.1. Vectorization of Real-Valued Functions

Although listed in this chapter, many of Stan's built-in functions are vectorized so that they may be applied to any argument type. The vectorized form of these functions is not any faster than writing an explicit loop that iterates over the elements applying the function—it's just easier to read and write and less error prone.

Unary Function Vectorization

Many of Stan's unary functions can be applied to any argument type. For example, the exponential function, `exp`, can be applied to `real` arguments or arrays of `real` arguments. Other than for integer arguments, the result type is the same as the argument type, including dimensionality and size. Integer arguments are first promoted to real values, but the result will still have the same dimensionality and size as the argument.

Real and real array arguments

When applied to a simple real value, the result is a real value. When applied to arrays, vectorized functions like `exp()` are defined elementwise. For example,

```
// declare some variables for arguments
real x0;
real x1[5];
real x2[4, 7];
...
// declare some variables for results
real y0;
real y1[5];
real y2[4, 7];
...
// calculate and assign results
y0 = exp(x0);
y1 = exp(x1);
y2 = exp(x2);
```

When `exp` is applied to an array, it applies elementwise. For example, the statement

above,

```
y2 = exp(x2);
```

produces the same result for y2 as the explicit loop

```
for (i in 1:4)
  for (j in 1:7)
    y2[i, j] = exp(x2[i, j]);
```

Vector and matrix arguments

Vectorized functions also apply elementwise to vectors and matrices. For example,

```
vector[5] xv;
row_vector[7] xrv;
matrix[10, 20] xm;

vector[5] yv;
row_vector[7] yrv;
matrix[10, 20] ym;

yv = exp(xv);
yrv = exp(xrv);
ym = exp(xm);
```

Arrays of vectors and matrices work the same way. For example,

```
matrix[17, 93] u[12];

matrix[17, 93] z[12];

z = exp(u);
```

After this has been executed, `z[i, j, k]` will be equal to `exp(u[i, j, k])`.

Integer and integer array arguments

Integer arguments are promoted to real values in vectorized unary functions. Thus if `n` is of type `int`, `exp(n)` is of type `real`. Arrays work the same way, so that if `n2` is a one dimensional array of integers, then `exp(n2)` will be a one-dimensional array of reals with the same number of elements as `n2`. For example,

```
int n1[23];
```

```
real z1[23];
z1 = exp(n1);
```

It would be illegal to try to assign `exp(n1)` to an array of integers; the return type is a real array.

3.2. Mathematical Constants

Constants are represented as functions with no arguments and must be called as such. For instance, the mathematical constant π must be written in a Stan program as `pi()`.

`real` **`pi`**`()`
π, the ratio of a circle's circumference to its diameter

`real` **`e`**`()`
e, the base of the natural logarithm

`real` **`sqrt2`**`()`
The square root of 2

`real` **`log2`**`()`
The natural logarithm of 2

`real` **`log10`**`()`
The natural logarithm of 10

3.3. Special Values

`real` **`not_a_number`**`()`
Not-a-number, a special non-finite real value returned to signal an error

`real` **`positive_infinity`**`()`
Positive infinity, a special non-finite real value larger than all finite numbers

`real` **`negative_infinity`**`()`
Negative infinity, a special non-finite real value smaller than all finite numbers

`real` **`machine_precision`**`()`
The smallest number x such that $(x + 1) \neq 1$ in floating-point arithmetic on the current hardware platform

3.4. Log Probability Function

The basic purpose of a Stan program is to compute a log probability function and its derivatives. The log probability function in a Stan model outputs the log density on the unconstrained scale. A log probability accumulator starts at zero and is then incremented in various ways by a Stan program. The variables are first transformed from unconstrained to constrained, and the log Jacobian determinant added to the

log probability accumulator. Then the model block is executed on the constrained parameters, with each sampling statement (~) and log probability increment statement (`increment_log_prob`) adding to the accumulator. At the end of the model block execution, the value of the log probability accumulator is the log probability value returned by the Stan program.

Stan provides a special built-in function `target()` that takes no arguments and returns the current value of the log probability accumulator.[1] This function is primarily useful for debugging purposes, where for instance, it may be used with a print statement to display the log probability accumulator at various stages of execution to see where it becomes ill defined.

`real` **`target`**`()`
Return the current value of the log probability accumulator.

`real` **`get_lp`**`()`
Return the current value of the log probability accumulator; **deprecated;** - use `target()` instead.

Both `target` and the deprecated `get_lp` act like other functions ending in `_lp`, meaning that they may only may only be used in the model block.

3.5. Logical Functions

Like C++, BUGS, and R, Stan uses 0 to encode false, and 1 to encode true. Stan supports the usual boolean comparison operations and boolean operators. These all have the same syntax and precedence as in C++; for the full list of operators and precedences, see the reference manual.

Comparison Operators

All comparison operators return boolean values, either 0 or 1. Each operator has two signatures, one for integer comparisons and one for floating-point comparisons. Comparing an integer and real value is carried out by first promoting the integer value.

`int` **`operator<`**`(int x, int y)`

`int` **`operator<`**`(real x, real y)`
Return 1 if x is less than y and 0 otherwise.

$$\text{operator<}(x, y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$$

[1] This function used to be called `get_lp()`, but that name has been deprecated; using it will print a warning. The function `get_lp()` will be removed in a future release.

`int` **`operator<=`**`(int x, int y)`

`int` **`operator<=`**`(real x, real y)`
Return 1 if x is less than or equal y and 0 otherwise.

$$\text{operator<=}(x,y) = \begin{cases} 1 & \text{if } x \le y \\ 0 & \text{otherwise} \end{cases}$$

`int` **`operator>`**`(int x, int y)`

`int` **`operator>`**`(real x, real y)`
Return 1 if x is greater than y and 0 otherwise.

$$\text{operator>} = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{otherwise} \end{cases}$$

`int` **`operator>=`**`(int x, int y)`

`int` **`operator>=`**`(real x, real y)`
Return 1 if x is greater than or equal to y and 0 otherwise.

$$\text{operator>=} = \begin{cases} 1 & \text{if } x \ge y \\ 0 & \text{otherwise} \end{cases}$$

`int` **`operator==`**`(int x, int y)`

`int` **`operator==`**`(real x, real y)`
Return 1 if x is equal to y and 0 otherwise.

$$\text{operator==}(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

`int` **`operator!=`**`(int x, int y)`

`int` **`operator!=`**`(real x, real y)`
Return 1 if x is not equal to y and 0 otherwise.

$$\text{operator!=}(x,y) = \begin{cases} 1 & \text{if } x \ne y \\ 0 & \text{otherwise} \end{cases}$$

Boolean Operators

Boolean operators return either 0 for false or 1 for true. Inputs may be any real or integer values, with non-zero values being treated as true and zero values treated as false. These operators have the usual precedences, with negation (not) binding the most tightly, conjunction the next and disjunction the weakest; all of the operators bind more tightly than the comparisons. Thus an expression such as `!a && b` is interpreted as `(!a) && b`, and `a < b || c >= d && e != f` as `(a < b) || (((c >= d) && (e != f)))`.

`int` **`operator!`**`(int x)`

`int` **`operator!`**`(real x)`

Return 1 if x is zero and 0 otherwise.

$$\text{operator!}(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

`int` **`operator&&`**`(int x, int y)`

`int` **`operator&&`**`(real x, real y)`

Return 1 if x is unequal to 0 and y is unequal to 0.

$$\text{operator\&\&}(x, y) = \begin{cases} 1 & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

`int` **`operator||`**`(int x, int y)`

`int` **`operator||`**`(real x, real y)`

Return 1 if x is unequal to 0 or y is unequal to 0.

$$\text{operator||}(x, y) = \begin{cases} 1 & \text{if } x \neq 0 \text{ or } y \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Boolean Operator Short Circuiting

Like in C++, the boolean operators && and || and are implemented to short circuit directly to a return value after evaluating the first argument if it is sufficient to resolve the result. In evaluating `a || b`, if `a` evaluates to a value other than zero, the expression returns the value 1 without evaluating the expression `b`. Similarly, evaluating `a && b` first evaluates `a`, and if the result is zero, returns 0 without evaluating `b`.

Logical Functions
The logical functions introduce conditional behavior functionally and are primarily provided for compatibility with BUGS and JAGS.

`real` **`step`**`(real x)`
Return 1 if x is positive and 0 otherwise.

$$\text{step}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases}$$

Warning: `int_step(0)` and `int_step(NaN)` return 0 whereas `step(0)` and `step(NaN)` return 1.

The step function is often used in BUGS to perform conditional operations. For instance, `step(a-b)` evaluates to 1 if `a` is greater than `b` and evaluates to 0 otherwise. `step` is a step-like functions; see the warning in section step functions applied to expressions dependent on parameters.

`int` **`is_inf`**`(real x)`
Return 1 if x is infinite (positive or negative) and 0 otherwise.

`int` **`is_nan`**`(real x)`
Return 1 if x is NaN and 0 otherwise.

Care must be taken because both of these indicator functions are step-like and thus can cause discontinuities in gradients when applied to parameters; see section step-like functions for details.

3.6. Real-Valued Arithmetic Operators

The arithmetic operators are presented using C++ notation. For instance `operator+(x,y)` refers to the binary addition operator and `operator-(x)` to the unary negation operator. In Stan programs, these are written using the usual infix and prefix notations as `x + y` and `-x`, respectively.

Binary Infix Operators
`real` **`operator+`**`(real x, real y)`
Return the sum of x and y.

$$(x + y) = \text{operator+}(x, y) = x + y$$

`real` **`operator-`**`(real x, real y)`
Return the difference between x and y.

$$(x - y) = \text{operator-}(x, y) = x - y$$

`real` **`operator*`**`(real x, real y)`
Return the product of x and y.

$$(x * y) = \text{operator*}(x, y) = xy$$

`real` **`operator/`**`(real x, real y)`
Return the quotient of x and y.

$$(x/y) = \text{operator/}(x, y) = \frac{x}{y}$$

`real` **`operator^`**`(real x, real y)`
Return x raised to the power of y.

$$(x \hat{} y) = \text{operator}\hat{}(x, y) = x^y$$

Unary Prefix Operators
`real` **`operator-`**`(real x)`
Return the negation of the subtrahend x.

$$\text{operator-}(x) = (-x)$$

`real` **`operator+`**`(real x)`
Return the value of x.

$$\text{operator+}(x) = x$$

3.7. Step-like Functions

Warning: *These functions can seriously hinder sampling and optimization efficiency for gradient-based methods (e.g., NUTS, HMC, BFGS) if applied to parameters (including transformed parameters and local variables in the transformed parameters or model block). The problem is that they break gradients due to discontinuities coupled with zero gradients elsewhere. They do not hinder sampling when used in the data, transformed data, or generated quantities blocks.*

Absolute Value Functions
`R` **`fabs`**`(T x)`
absolute value of x

`real` **`fdim`**`(real x, real y)`
Return the positive difference between x and y, which is x - y if x is greater than y and 0 otherwise; see warning above.

$$\text{fdim}(x, y) = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

Bounds Functions

`real` **`fmin`**`(real x, real y)`

Return the minimum of x and y; see warning above.

$$\text{fmin}(x, y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

`real` **`fmax`**`(real x, real y)`

Return the maximum of x and y; see warning above.

$$\text{fmax}(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{otherwise} \end{cases}$$

Arithmetic Functions

`real` **`fmod`**`(real x, real y)`

Return the real value remainder after dividing x by y; see warning above.

$$\text{fmod}(x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor y$$

The operator $\lfloor u \rfloor$ is the floor operation; see below.

Rounding Functions

Warning: Rounding functions convert real values to integers. Because the output is an integer, any gradient information resulting from functions applied to the integer is not passed to the real value it was derived from. With MCMC sampling using HMC or NUTS, the MCMC acceptance procedure will correct for any error due to poor gradient calculations, but the result is likely to be reduced acceptance probabilities and less efficient sampling.

The rounding functions cannot be used as indices to arrays because they return real values. Stan may introduce integer-valued versions of these in the future, but as of now, there is no good workaround.

R **`floor`**`(T x)`

floor of x, which is the largest integer less than or equal to x, converted to a real value; see warning at start of section step-like functions

R **`ceil`**`(T x)`

ceiling of x, which is the smallest integer greater than or equal to x, converted to a real value; see warning at start of section step-like functions

R **round**(T x)
nearest integer to x, converted to a real value; see warning at start of section step-like functions

R **trunc**(T x)
integer nearest to but no larger in magnitude than x, converted to a double value; see warning at start of section step-like functions

3.8. Power and Logarithm Functions

R **sqrt**(T x)
square root of x

R **cbrt**(T x)
cube root of x

R **square**(T x)
square of x

R **exp**(T x)
natural exponential of x

R **exp2**(T x)
base-2 exponential of x

R **log**(T x)
natural logarithm of x

R **log2**(T x)
base-2 logarithm of x

R **log10**(T x)
base-10 logarithm of x

real **pow**(real x, real y)
Return x raised to the power of y.

$$\mathrm{pow}(x, y) = x^y$$

R **inv**(T x)
inverse of x

R **inv_sqrt**(T x)
inverse of the square root of x

R **inv_square**(T x)
inverse of the square of x

3.9. Trigonometric Functions

`real` **`hypot`**`(real x, real y)`
Return the length of the hypotenuse of a right triangle with sides of length x and y.

$$\text{hypot}(x, y) = \begin{cases} \sqrt{x^2 + y^2} & \text{if } x, y \geq 0 \\ \text{NaN} & \text{otherwise} \end{cases}$$

`R` **`cos`**`(T x)`
cosine of the angle x (in radians)

`R` **`sin`**`(T x)`
sine of the angle x (in radians)

`R` **`tan`**`(T x)`
tangent of the angle x (in radians)

`R` **`acos`**`(T x)`
principal arc (inverse) cosine (in radians) of x

`R` **`asin`**`(T x)`
principal arc (inverse) sine (in radians) of x

`R` **`atan`**`(T x)`
principal arc (inverse) tangent (in radians) of x, with values from $-\pi$ to π

`real` **`atan2`**`(real y, real x)`
Return the principal arc (inverse) tangent (in radians) of y divided by x,

$$\text{atan2}(y, x) = \arctan\left(\frac{y}{x}\right)$$

3.10. Hyperbolic Trigonometric Functions

`R` **`cosh`**`(T x)`
hyperbolic cosine of x (in radians)

`R` **`sinh`**`(T x)`
hyperbolic sine of x (in radians)

`R` **`tanh`**`(T x)`
hyperbolic tangent of x (in radians)

`R` **`acosh`**`(T x)`
inverse hyperbolic cosine (in radians)

R **asinh**(T x)
inverse hyperbolic cosine (in radians)

R **atanh**(T x)
inverse hyperbolic tangent (in radians) of x

3.11. Link Functions

The following functions are commonly used as link functions in generalized linear models. The function Φ is also commonly used as a link function (see section probability-related functions).

R **logit**(T x)
log odds, or logit, function applied to x

R **inv_logit**(T x)
logistic sigmoid function applied to x

R **inv_cloglog**(T x)
inverse of the complementary log-log function applied to x

3.12. Probability-Related Functions

Normal Cumulative Distribution Functions

The error function `erf` is related to the standard normal cumulative distribution function Φ by scaling. See section normal distribution for the general normal cumulative distribution function (and its complement).

R **erf**(T x)
error function, also known as the Gauss error function, of x

R **erfc**(T x)
complementary error function of x

R **Phi**(T x)
standard normal cumulative distribution function of x

R **inv_Phi**(T x)
standard normal inverse cumulative distribution function of p, otherwise known as the quantile function

R **Phi_approx**(T x)
fast approximation of the unit (may replace `Phi` for probit regression with maximum absolute error of 0.00014, see (Bowling et al. 2009) for details)

Other Probability-Related Functions

`real` **`binary_log_loss`**`(int y, real y_hat)`
Return the log loss function for for predicting $\hat{y} \in [0,1]$ for boolean outcome $y \in \{0,1\}$.

$$\text{binary\_log\_loss}(y, \hat{y}) = \begin{cases} -\log \hat{y} & \text{if } y = 0 \\ -\log(1 - \hat{y}) & \text{otherwise} \end{cases}$$

`real` **`owens_t`**`(real h, real a)`
Return the Owen's T function for the probability of the event $X > h$ and $0 < Y < aX$ where X and Y are independent standard normal random variables.

$$\text{owens\_t}(h, a) = \frac{1}{2\pi} \int_0^a \frac{\exp(-\frac{1}{2}h^2(1 + x^2))}{1 + x^2} dx$$

3.13. Combinatorial Functions

`real` **`inc_beta`**`(real alpha, real beta, real x)`
Return the regularized incomplete beta function up to x applied to alpha and beta. See section appendix for a definition.

`real` **`lbeta`**`(real alpha, real beta)`
Return the natural logarithm of the beta function applied to alpha and beta. The beta function, $\mathrm{B}(\alpha, \beta)$, computes the normalizing constant for the beta distribution, and is defined for $\alpha > 0$ and $\beta > 0$.

$$\text{lbeta}(\alpha, \beta) = \log \Gamma(a) + \log \Gamma(b) - \log \Gamma(a + b)$$

See section appendix for definition of $\mathrm{B}(\alpha, \beta)$.

`R` **`tgamma`**`(T x)`
gamma function applied to x. The gamma function is the generalization of the factorial function to continuous variables, defined so that $\Gamma(n + 1) = n!$. See for a full definition of $\Gamma(x)$. The function is defined for positive numbers and non-integral negative numbers,

`R` **`lgamma`**`(T x)`
natural logarithm of the gamma function applied to x,

`R` **`digamma`**`(T x)`
digamma function applied to x. The digamma function is the derivative of the natural logarithm of the Gamma function. The function is defined for positive numbers and non-integral negative numbers

`R` **`trigamma`**`(T x)`
trigamma function applied to x. The trigamma function is the second derivative of the natural logarithm of the Gamma function

`real` **`lmgamma`**`(int n, real x)`
Return the natural logarithm of the multivariate gamma function Γ_n with n dimensions applied to x.

$$\text{lmgamma}(n,x) = \begin{cases} \frac{n(n-1)}{4}\log\pi + \sum_{j=1}^{n}\log\Gamma\left(x+\frac{1-j}{2}\right) & \text{if } x \notin \{\ldots,-3,-2,-1,0\} \\ \text{error} & \text{otherwise} \end{cases}$$

`real` **`gamma_p`**`(real a, real z)`
Return the normalized lower incomplete gamma function of a and z defined for positive a and nonnegative z.

$$\text{gamma\_p}(a,z) = \begin{cases} \frac{1}{\Gamma(a)}\int_0^z t^{a-1}e^{-t}dt & \text{if } a>0, z\geq 0 \\ \text{error} & \text{otherwise} \end{cases}$$

`real` **`gamma_q`**`(real a, real z)`
Return the normalized upper incomplete gamma function of a and z defined for positive a and nonnegative z.

$$\text{gamma\_q}(a,z) = \begin{cases} \frac{1}{\Gamma(a)}\int_z^\infty t^{a-1}e^{-t}dt & \text{if } a>0, z\geq 0 \\ \text{error} & \text{otherwise} \end{cases}$$

`real` **`binomial_coefficient_log`**`(real x, real y)`
Warning: This function is deprecated and should be replaced with `lchoose`. Return the natural logarithm of the binomial coefficient of x and y. For non-negative integer inputs, the binomial coefficient function is written as $\binom{x}{y}$ and pronounced "x choose y." This function generalizes to real numbers using the gamma function. For $0 \leq y \leq x$,

$$\text{binomial\_coefficient\_log}(x,y) = \log\Gamma(x+1) - \log\Gamma(y+1) - \log\Gamma(x-y+1).$$

`int` **`choose`**`(int x, int y)`
Return the binomial coefficient of x and y. For non-negative integer inputs, the binomial coefficient function is written as $\binom{x}{y}$ and pronounced "x choose y." In its the antilog of the `lchoose` function but returns an integer rather than a real number with no non-zero decimal places. For $0 \leq y \leq x$, the binomial coefficient function can be defined via the factorial function

$$\text{choose}(x,y) = \frac{x!}{(y!)\,(x-y)!}.$$

`real` **`bessel_first_kind`**`(int v, real x)`
Return the Bessel function of the first kind with order v applied to x.

$$\text{bessel\_first\_kind}(\nu, x) = J_\nu(x),$$

where

$$J_\nu(x) = \left(\frac{1}{2}x\right)^\nu \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4}x^2\right)^k}{k!\,\Gamma(\nu + k + 1)}$$

`real` **`bessel_second_kind`**`(int v, real x)`
Return the Bessel function of the second kind with order v applied to x defined for positive x and v. For $x, \nu > 0$,

$$\text{bessel\_second\_kind}(\nu, x) = \begin{cases} Y_\nu(x) & \text{if } x > 0 \\ \text{error} & \text{otherwise} \end{cases}$$

where

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

`real` **`modified_bessel_first_kind`**`(int v, real z)`
Return the modified Bessel function of the first kind with order v applied to z defined for all z and v.

$$\text{modified\_bessel\_first\_kind}(\nu, z) = I_\nu(z)$$

where

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\,\Gamma(\nu + k + 1)}$$

`real` **`modified_bessel_second_kind`**`(int v, real z)`
Return the modified Bessel function of the second kind with order v applied to z defined for positive z and v.

$$\text{modified\_bessel\_second\_kind}(\nu, z) = \begin{cases} K_\nu(z) & \text{if } z > 0 \\ \text{error} & \text{if } z \le 0 \end{cases}$$

where

$$K_\nu(z) = \frac{\pi}{2} \cdot \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}$$

`real` **`falling_factorial`**`(real x, real n)`
Return the falling factorial of x with power n defined for positive x and real n.

$$\text{falling\_factorial}(x, n) = \begin{cases} (x)_n & \text{if } x > 0 \\ \text{error} & \text{if } x \leq 0 \end{cases}$$

where

$$(x)_n = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}$$

`real` **`lchoose`**`(real x, real y)`
Return the natural logarithm of the generalized binomial coefficient of x and y. For non-negative integer inputs, the binomial coefficient function is written as $\binom{x}{y}$ and pronounced "x choose y." This function generalizes to real numbers using the gamma function. For $0 \leq y \leq x$,

$$\text{binomial\_coefficient\_log}(x, y) = \log\Gamma(x+1) - \log\Gamma(y+1) - \log\Gamma(x-y+1).$$

`real` **`log_falling_factorial`**`(real x, real n)`
Return the log of the falling factorial of x with power n defined for positive x and real n.

$$\text{log\_falling\_factorial}(x, n) = \begin{cases} \log(x)_n & \text{if } x > 0 \\ \text{error} & \text{if } x \leq 0 \end{cases}$$

`real` **`rising_factorial`**`(real x, int n)`
Return the rising factorial of x with power n defined for positive x and integer n.

$$\text{rising\_factorial}(x, n) = \begin{cases} x^{(n)} & \text{if } x > 0 \\ \text{error} & \text{if } x \leq 0 \end{cases}$$

where

$$x^{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)}$$

`real` **`log_rising_factorial`**`(real x, real n)`
Return the log of the rising factorial of x with power n defined for positive x and real n.

$$\text{log\_rising\_factorial}(x, n) = \begin{cases} \log x^{(n)} & \text{if } x > 0 \\ \text{error} & \text{if } x \leq 0 \end{cases}$$

3.14. Composed Functions

The functions in this section are equivalent in theory to combinations of other functions. In practice, they are implemented to be more efficient and more numerically stable than defining them directly using more basic Stan functions.

`R` **`expm1`**`(T x)`
natural exponential of x minus 1

`real` **`fma`**`(real x, real y, real z)`
Return z plus the result of x multiplied by y.

$$\text{fma}(x, y, z) = (x \times y) + z$$

`real` **`multiply_log`**`(real x, real y)`
Warning: This function is deprecated and should be replaced with `lmultiply`. Return the product of x and the natural logarithm of y.

$$\text{multiply\_log}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ x \log y & \text{if } x, y \neq 0 \\ \text{NaN} & \text{otherwise} \end{cases}$$

`real` **`lmultiply`**`(real x, real y)`
Return the product of x and the natural logarithm of y.

$$\text{lmultiply}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ x \log y & \text{if } x, y \neq 0 \\ \text{NaN} & \text{otherwise} \end{cases}$$

`R` **`log1p`**`(T x)`
natural logarithm of 1 plus x

`R` **`log1m`**`(T x)`
natural logarithm of 1 minus x

`R` **`log1p_exp`**`(T x)`
natural logarithm of one plus the natural exponentiation of x

`R` **`log1m_exp`**`(T x)`
logarithm of one minus the natural exponentiation of x

`real` **`log_diff_exp`**`(real x, real y)`
Return the natural logarithm of the difference of the natural exponentiation of x and

the natural exponentiation of y.

$$\text{log\_diff\_exp}(x, y) = \begin{cases} \log(\exp(x) - \exp(y)) & \text{if } x > y \\ \text{NaN} & \text{otherwise} \end{cases}$$

`real` **`log_mix`**`(real theta, real lp1, real lp2)`
Return the log mixture of the log densities lp1 and lp2 with mixing proportion theta, defined by

$$\begin{aligned} \text{log\_mix}(\theta, \lambda_1, \lambda_2) &= \log(\theta \exp(\lambda_1) + (1 - \theta) \exp(\lambda_2)) \\ &= \text{log\_sum\_exp}(\log(\theta) + \lambda_1,\ \log(1 - \theta) + \lambda_2)\,. \end{aligned}$$

`real` **`log_sum_exp`**`(real x, real y)`
Return the natural logarithm of the sum of the natural exponentiation of x and the natural exponentiation of y.

$$\text{log\_sum\_exp}(x, y) = \log(\exp(x) + \exp(y))$$

`R` **`log_inv_logit`**`(T x)`
natural logarithm of the inverse logit function of x

`R` **`log1m_inv_logit`**`(T x)`
natural logarithm of 1 minus the inverse logit function of x

4. Array Operations

4.1. Reductions

The following operations take arrays as input and produce single output values. The boundary values for size 0 arrays are the unit with respect to the combination operation (min, max, sum, or product).

Minimum and Maximum

`real` **`min`**`(real[] x)`
The minimum value in x, or $+\infty$ if x is size 0.

`int` **`min`**`(int[] x)`
The minimum value in x, or error if x is size 0.

`real` **`max`**`(real[] x)`
The maximum value in x, or $-\infty$ if x is size 0.

`int` **`max`**`(int[] x)`
The maximum value in x, or error if x is size 0.

Sum, Product, and Log Sum of Exp

`int` **`sum`**`(int[] x)`
The sum of the elements in x, defined for x of size N by

$$\text{sum}(x) = \begin{cases} \sum_{n=1}^{N} x_n & \text{if} N > 0 \\ 0 & \text{if} N = 0 \end{cases}$$

`real` **`sum`**`(real[] x)`
The sum of the elements in x; see definition above.

`real` **`prod`**`(real[] x)`
The product of the elements in x, or 1 if x is size 0.

`real` **`prod`**`(int[] x)`
The product of the elements in x,

$$\text{product}(x) = \begin{cases} \prod_{n=1}^{N} x_n & \text{if} N > 0 \\ 1 & \text{if} N = 0 \end{cases}$$

`real` **`log_sum_exp`**`(real[] x)`
The natural logarithm of the sum of the exponentials of the elements in x, or $-\infty$ if the array is empty.

Sample Mean, Variance, and Standard Deviation
The sample mean, variance, and standard deviation are calculated in the usual way. For i.i.d. draws from a distribution of finite mean, the sample mean is an unbiased estimate of the mean of the distribution. Similarly, for i.i.d. draws from a distribution of finite variance, the sample variance is an unbiased estimate of the variance.[1] The sample deviation is defined as the square root of the sample deviation, but is not unbiased.

`real` **`mean`**`(real[] x)`
The sample mean of the elements in x. For an array x of size $N > 0$,

$$\text{mean}(x) \;=\; \bar{x} \;=\; \frac{1}{N}\sum_{n=1}^{N} x_n.$$

It is an error to the call the mean function with an array of size 0.

`real` **`variance`**`(real[] x)`
The sample variance of the elements in x. For $N > 0$,

$$\text{variance}(x) \;=\; \begin{cases} \frac{1}{N-1}\sum_{n=1}^{N}(x_n - \bar{x})^2 & \text{if } N > 1 \\ 0 & \text{if } N = 1 \end{cases}$$

It is an error to call the `variance` function with an array of size 0.

`real` **`sd`**`(real[] x)`
The sample standard deviation of elements in x.

$$\text{sd}(x) = \begin{cases} \sqrt{\text{variance}(x)} & \text{if } N > 1 \\ 0 & \text{if } N = 0 \end{cases}$$

It is an error to call the `sd` function with an array of size 0.

Euclidean Distance and Squared Distance
`real` **`distance`**`(vector x, vector y)`
The Euclidean distance between x and y, defined by

$$\text{distance}(x, y) \;=\; \sqrt{\sum_{n=1}^{N}(x_n - y_n)^2}$$

[1] Dividing by N rather than $(N-1)$ produces a maximum likelihood estimate of variance, which is biased to underestimate variance.

where N is the size of x and y. It is an error to call `distance` with arguments of unequal size.

`real` **`distance`**`(vector x, row_vector y)`
The Euclidean distance between x and y

`real` **`distance`**`(row_vector x, vector y)`
The Euclidean distance between x and y

`real` **`distance`**`(row_vector x, row_vector y)`
The Euclidean distance between x and y

`real` **`squared_distance`**`(vector x, vector y)`
The squared Euclidean distance between x and y, defined by

$$\text{squared\_distance}(x, y) \;=\; \text{distance}(x, y)^2 \;=\; \sum_{n=1}^{N} (x_n - y_n)^2,$$

where N is the size of x and y. It is an error to call `squared_distance` with arguments of unequal size.

`real` **`squared_distance`**`(vector x, row_vector [] y)`
The squared Euclidean distance between x and y

`real` **`squared_distance`**`(row_vector x, vector [] y)`
The squared Euclidean distance between x and y

`real` **`squared_distance`**`(row_vector x, row_vector[] y)`
The Euclidean distance between x and y

4.2. Array size and dimension function

The size of an array or matrix can be obtained using the `dims()` function. The `dims()` function is defined to take an argument consisting of any variable with up to 8 array dimensions (and up to 2 additional matrix dimensions) and returns an array of integers with the dimensions. For example, if two variables are declared as follows,

```
real x[7,8,9];
matrix[8,9] y[7];
```

then calling `dims(x)` or `dims(y)` returns an integer array of size 3 containing the elements 7, 8, and 9 in that order.

The `size()` function extracts the number of elements in an array. This is just the top-level elements, so if the array is declared as

```
real a[M,N];
```

the size of `a` is `M`.

The function `num_elements`, on the other hand, measures all of the elements, so that the array `a` above has $M \times N$ elements.

The specialized functions `rows()` and `cols()` should be used to extract the dimensions of vectors and matrices.

`int[] dims(T x)`
Return an integer array containing the dimensions of x; the type of the argument T can be any Stan type with up to 8 array dimensions.

`int num_elements(T[] x)`
Return the total number of elements in the array x including all elements in contained arrays, vectors, and matrices. T can be any array type. For example, if `x` is of type `real[4,3]` then `num_elements(x)` is 12, and if `y` is declared as `matrix[3,4] y[5]`, then `size(y)` evaluates to 60.

`int size(T[] x)`
Return the number of elements in the array x; the type of the array T can be any type, but the size is just the size of the top level array, not the total number of elements contained. For example, if `x` is of type `real[4,3]` then `size(x)` is 4.

4.3. Array broadcasting

The following operations create arrays by repeating elements to fill an array of a specified size. These operations work for all input types T, including reals, integers, vectors, row vectors, matrices, or arrays.

`T[] rep_array(T x, int n)`
Return the n array with every entry assigned to x.

`T[,] rep_array(T x, int m, int n)`
Return the m by n array with every entry assigned to x.

`T[,,] rep_array(T x, int k, int m, int n)`
Return the k by m by n array with every entry assigned to x.

For example, `rep_array(1.0,5)` produces a real array (type `real[]`) of size 5 with all values set to 1.0. On the other hand, `rep_array(1,5)` produces an integer array (type `int[]`) of size 5 with all values set to 1. This distinction is important because it is not possible to assign an integer array to a real array. For example, the following example contrasts legal with illegal array creation and assignment

```
real y[5];
int x[5];
```

```
x = rep_array(1,5);     // ok
y = rep_array(1.0,5);   // ok

x = rep_array(1.0,5);   // illegal
y = rep_array(1,5);     // illegal

x = y;                  // illegal
y = x;                  // illegal
```

If the value being repeated v is a vector (i.e., T is vector), then rep_array(v,27) is a size 27 array consisting of 27 copies of the vector v.

```
vector[5] v;
vector[5] a[3];

a = rep_array(v,3);  // fill a with copies of v
a[2,4] = 9.0;        // v[4], a[1,4], a[2,4] unchanged
```

If the type T of x is itself an array type, then the result will be an array with one, two, or three added dimensions, depending on which of the rep_array functions is called. For instance, consider the following legal code snippet.

```
real a[5,6];
real b[3,4,5,6];

b = rep_array(a,3,4); //  make (3 x 4) copies of a
b[1,1,1,1] = 27.9;    //  a[1,1] unchanged
```

After the assignment to b, the value for b[j,k,m,n] is equal to a[m,n] where it is defined, for j in 1:3, k in 1:4, m in 1:5, and n in 1:6.

4.4. Array concatenation

T **append_array**(T x, T y)

Return the concatenation of two arrays in the order of the arguments. T must be an N-dimensional array of any Stan type (with a maximum N of 7). All dimensions but the first must match.

For example, the following code appends two three dimensional arrays of matrices together. Note that all dimensions except the first match. Any mismatches will cause an error to be thrown.

```
matrix[4, 6] x1[2, 1, 7];
```

```
matrix[4, 6] x2[3, 1, 7];
matrix[4, 6] x3[5, 1, 7];

x3 = append_array(x1, x2);
```

4.5. Sorting functions

Sorting can be used to sort values or the indices of those values in either ascending or descending order. For example, if v is declared as a real array of size 3, with values

$$\mathrm{v} = (1, -10.3, 20.987),$$

then the various sort routines produce

$$\begin{aligned} \text{sort\_asc}(\mathrm{v}) &= (-10.3, 1, 20.987) \\ \text{sort\_desc}(\mathrm{v}) &= (20.987, 1, -10.3) \\ \text{sort\_indices\_asc}(\mathrm{v}) &= (2, 1, 3) \\ \text{sort\_indices\_desc}(\mathrm{v}) &= (3, 1, 2) \end{aligned}$$

`real[] sort_asc(real[] v)`
Sort the elements of v in ascending order

`int[] sort_asc(int[] v)`
Sort the elements of v in ascending order

`real[] sort_desc(real[] v)`
Sort the elements of v in descending order

`int[] sort_desc(int[] v)`
Sort the elements of v in descending order

`int[] sort_indices_asc(real[] v)`
Return an array of indices between 1 and the size of v, sorted to index v in ascending order.

`int[] sort_indices_asc(int[] v)`
Return an array of indices between 1 and the size of v, sorted to index v in ascending order.

`int[] sort_indices_desc(real[] v)`
Return an array of indices between 1 and the size of v, sorted to index v in descending order.

`int[] sort_indices_desc(int[] v)`
Return an array of indices between 1 and the size of v, sorted to index v in descending order.

`int rank(real[] v, int s)`
Number of components of v less than v[s]

`int rank(int[] v, int s)`
Number of components of v less than v[s]

4.6. Reversing functions

Stan provides functions to create a new array by reversing the order of elements in an existing array. For example, if `v` is declared as a real array of size 3, with values

$$\mathrm{v} = (1, -10.3, 20.987),$$

then

$$\mathrm{reverse}(\mathrm{v}) = (20.987, -10.3, 1).$$

`T[] reverse(T[] v)`
Return a new array containing the elements of the argument in reverse order.

5. Matrix Operations

5.1. Integer-Valued Matrix Size Functions

`int` **`num_elements`**`(vector x)`
The total number of elements in the vector x (same as function `rows`)

`int` **`num_elements`**`(row_vector x)`
The total number of elements in the vector x (same as function `cols`)

`int` **`num_elements`**`(matrix x)`
The total number of elements in the matrix x. For example, if `x` is a 5×3 matrix, then `num_elements(x)` is 15

`int` **`rows`**`(vector x)`
The number of rows in the vector x

`int` **`rows`**`(row_vector x)`
The number of rows in the row vector x, namely 1

`int` **`rows`**`(matrix x)`
The number of rows in the matrix x

`int` **`cols`**`(vector x)`
The number of columns in the vector x, namely 1

`int` **`cols`**`(row_vector x)`
The number of columns in the row vector x

`int` **`cols`**`(matrix x)`
The number of columns in the matrix x

5.2. Matrix Arithmetic Operators

Stan supports the basic matrix operations using infix, prefix and postfix operations. This section lists the operations supported by Stan along with their argument and result types.

Negation Prefix Operators
`vector` **`operator-`**`(vector x)`
The negation of the vector x.

`row_vector` **`operator-`**`(row_vector x)`
The negation of the row vector x.

`matrix` **`operator-`**`(matrix x)`
The negation of the matrix x.

Infix Matrix Operators

`vector` **`operator+`**`(vector x, vector y)`
The sum of the vectors x and y.

`row_vector` **`operator+`**`(row_vector x, row_vector y)`
The sum of the row vectors x and y.

`matrix` **`operator+`**`(matrix x, matrix y)`
The sum of the matrices x and y

`vector` **`operator-`**`(vector x, vector y)`
The difference between the vectors x and y.

`row_vector` **`operator-`**`(row_vector x, row_vector y)`
The difference between the row vectors x and y

`matrix` **`operator-`**`(matrix x, matrix y)`
The difference between the matrices x and y

`vector` **`operator*`**`(real x, vector y)`
The product of the scalar x and vector y

`row_vector` **`operator*`**`(real x, row_vector y)`
The product of the scalar x and the row vector y

`matrix` **`operator*`**`(real x, matrix y)`
The product of the scalar x and the matrix y

`vector` **`operator*`**`(vector x, real y)`
The product of the scalar y and vector x

`matrix` **`operator*`**`(vector x, row_vector y)`
The product of the vector x and row vector y

`row_vector` **`operator*`**`(row_vector x, real y)`
The product of the scalar y and row vector x

`real` **`operator*`**`(row_vector x, vector y)`
The product of the row vector x and vector y

`row_vector` **`operator*`**`(row_vector x, matrix y)`
The product of the row vector x and matrix y

`matrix` **`operator`**`*(matrix x, real y)`
The product of the scalar y and matrix x

`vector` **`operator`**`*(matrix x, vector y)`
The product of the matrix x and vector y

`matrix` **`operator`**`*(matrix x, matrix y)`
The product of the matrices x and y

Broadcast Infix Operators
`vector` **`operator`**`+(vector x, real y)`
The result of adding y to every entry in the vector x

`vector` **`operator`**`+(real x, vector y)`
The result of adding x to every entry in the vector y

`row_vector` **`operator`**`+(row_vector x, real y)`
The result of adding y to every entry in the row vector x

`row_vector` **`operator`**`+(real x, row_vector y)`
The result of adding x to every entry in the row vector y

`matrix` **`operator`**`+(matrix x, real y)`
The result of adding y to every entry in the matrix x

`matrix` **`operator`**`+(real x, matrix y)`
The result of adding x to every entry in the matrix y

`vector` **`operator`**`-(vector x, real y)`
The result of subtracting y from every entry in the vector x

`vector` **`operator`**`-(real x, vector y)`
The result of adding x to every entry in the negation of the vector y

`row_vector` **`operator`**`-(row_vector x, real y)`
The result of subtracting y from every entry in the row vector x

`row_vector` **`operator`**`-(real x, row_vector y)`
The result of adding x to every entry in the negation of the row vector y

`matrix` **`operator`**`-(matrix x, real y)`
The result of subtracting y from every entry in the matrix x

`matrix` **`operator`**`-(real x, matrix y)`
The result of adding x to every entry in negation of the matrix y

`vector` **`operator/`**`(vector x, real y)`
The result of dividing each entry in the vector x by y

`row_vector` **`operator/`**`(row_vector x, real y)`
The result of dividing each entry in the row vector x by y

`matrix` **`operator/`**`(matrix x, real y)`
The result of dividing each entry in the matrix x by y

Elementwise Arithmetic Operations
`vector` **`operator.*`**`(vector x, vector y)`
The elementwise product of y and x

`row_vector` **`operator.*`**`(row_vector x, row_vector y)`
The elementwise product of y and x

`matrix` **`operator.*`**`(matrix x, matrix y)`
The elementwise product of y and x

`vector` **`operator./`**`(vector x, vector y)`
The elementwise quotient of y and x

`vector` **`operator./`**`(vector x, real y)`
The elementwise quotient of y and x

`vector` **`operator./`**`(real x, vector y)`
The elementwise quotient of y and x

`row_vector` **`operator./`**`(row_vector x, row_vector y)`
The elementwise quotient of y and x

`row_vector` **`operator./`**`(row_vector x, real y)`
The elementwise quotient of y and x

`row_vector` **`operator./`**`(real x, row_vector y)`
The elementwise quotient of y and x

`matrix` **`operator./`**`(matrix x, matrix y)`
The elementwise quotient of y and x

`matrix` **`operator./`**`(matrix x, real y)`
The elementwise quotient of y and x

`matrix` **`operator./`**`(real x, matrix y)`
The elementwise quotient of y and x

`vector` **`operator.^`**`(vector x, vector y)`
The elementwise power of y and x

`vector` **`operator.^`**`(vector x, real y)`
The elementwise power of y and x

`vector` **`operator.^`**`(real x, vector y)`
The elementwise power of y and x

`row_vector` **`operator.^`**`(row_vector x, row_vector y)`
The elementwise power of y and x

`row_vector` **`operator.^`**`(row_vector x, real y)`
The elementwise power of y and x

`row_vector` **`operator.^`**`(real x, row_vector y)`
The elementwise power of y and x

`matrix` **`operator.^`**`(matrix x, matrix y)`
The elementwise power of y and x

`matrix` **`operator.^`**`(matrix x, real y)`
The elementwise power of y and x

`matrix` **`operator.^`**`(real x, matrix y)`
The elementwise power of y and x

5.3. Transposition Operator

Matrix transposition is represented using a postfix operator.

`matrix` **`operator'`**`(matrix x)`
The transpose of the matrix x, written as `x'`

`row_vector` **`operator'`**`(vector x)`
The transpose of the vector x, written as `x'`

`vector` **`operator'`**`(row_vector x)`
The transpose of the row vector x, written as `x'`

5.4. Elementwise Functions

Elementwise functions apply a function to each element of a vector or matrix, returning a result of the same shape as the argument. There are many functions that are vectorized in addition to the ad hoc cases listed in this section; see section function vectorizationfor the general cases.

5.5. Dot Products and Specialized Products

`real` **`dot_product`**`(vector x, vector y)`
The dot product of x and y

`real` **`dot_product`**`(vector x, row_vector y)`
The dot product of x and y

`real` **`dot_product`**`(row_vector x, vector y)`
The dot product of x and y

`real` **`dot_product`**`(row_vector x, row_vector y)`
The dot product of x and y

`row_vector` **`columns_dot_product`**`(vector x, vector y)`
The dot product of the columns of x and y

`row_vector` **`columns_dot_product`**`(row_vector x, row_vector y)`
The dot product of the columns of x and y

`row_vector` **`columns_dot_product`**`(matrix x, matrix y)`
The dot product of the columns of x and y

`vector` **`rows_dot_product`**`(vector x, vector y)`
The dot product of the rows of x and y

`vector` **`rows_dot_product`**`(row_vector x, row_vector y)`
The dot product of the rows of x and y

`vector` **`rows_dot_product`**`(matrix x, matrix y)`
The dot product of the rows of x and y

`real` **`dot_self`**`(vector x)`
The dot product of the vector x with itself

`real` **`dot_self`**`(row_vector x)`
The dot product of the row vector x with itself

`row_vector` **`columns_dot_self`**`(vector x)`
The dot product of the columns of x with themselves

`row_vector` **`columns_dot_self`**`(row_vector x)`
The dot product of the columns of x with themselves

`row_vector` **`columns_dot_self`**`(matrix x)`
The dot product of the columns of x with themselves

`vector` **`rows_dot_self`**`(vector x)`
The dot product of the rows of x with themselves

`vector` **`rows_dot_self`**`(row_vector x)`
The dot product of the rows of x with themselves

`vector` **`rows_dot_self`**`(matrix x)`
The dot product of the rows of x with themselves

Specialized Products
`matrix` **`tcrossprod`**`(matrix x)`
The product of x postmultiplied by its own transpose, similar to the tcrossprod(x) function in R. The result is a symmetric matrix $xx^\top$.

`matrix` **`crossprod`**`(matrix x)`
The product of x premultiplied by its own transpose, similar to the crossprod(x) function in R. The result is a symmetric matrix $x^\top x$.

The following functions all provide shorthand forms for common expressions, which are also much more efficient.

`matrix` **`quad_form`**`(matrix A, matrix B)`
The quadratic form, i.e., `B' * A * B`.

`real` **`quad_form`**`(matrix A, vector B)`
The quadratic form, i.e., `B' * A * B`.

`matrix` **`quad_form_diag`**`(matrix m, vector v)`
The quadratic form using the column vector v as a diagonal matrix, i.e., `diag_matrix(v) * m * diag_matrix(v)`.

`matrix` **`quad_form_diag`**`(matrix m, row_vector rv)`
The quadratic form using the row vector rv as a diagonal matrix, i.e., `diag_matrix(rv) * m * diag_matrix(rv)`.

`matrix` **`quad_form_sym`**`(matrix A, matrix B)`
Similarly to quad_form, gives `B' * A * B`, but additionally checks if A is symmetric and ensures that the result is also symmetric.

`real` **`quad_form_sym`**`(matrix A, vector B)`
Similarly to quad_form, gives `B' * A * B`, but additionally checks if A is symmetric and ensures that the result is also symmetric.

`real` **`trace_quad_form`**`(matrix A, matrix B)`
The trace of the quadratic form, i.e., `trace(B' * A * B)`.

`real` **`trace_gen_quad_form`**`(matrix D,matrix A, matrix B)`
The trace of a generalized quadratic form, i.e., `trace(D * B' * A * B)`.

`matrix` **`multiply_lower_tri_self_transpose`**`(matrix x)`
The product of the lower triangular portion of x (including the diagonal) times its own transpose; that is, if `L` is a matrix of the same dimensions as x with `L(m,n)` equal to `x(m,n)` for $n \leq m$ and `L(m,n)` equal to 0 if $n > m$, the result is the symmetric matrix $LL^\top$. This is a specialization of tcrossprod(x) for lower-triangular matrices. The input matrix does not need to be square.

`matrix` **`diag_pre_multiply`**`(vector v, matrix m)`
Return the product of the diagonal matrix formed from the vector v and the matrix m, i.e., `diag_matrix(v) * m`.

`matrix` **`diag_pre_multiply`**`(row_vector rv, matrix m)`
Return the product of the diagonal matrix formed from the vector rv and the matrix m, i.e., `diag_matrix(rv) * m`.

`matrix` **`diag_post_multiply`**`(matrix m, vector v)`
Return the product of the matrix m and the diagonal matrix formed from the vector v, i.e., `m * diag_matrix(v)`.

`matrix` **`diag_post_multiply`**`(matrix m, row_vector rv)`
Return the product of the matrix `m` and the diagonal matrix formed from the the row vector `rv`, i.e., `m * diag_matrix(rv)`.

5.6. Reductions

Log Sum of Exponents

`real` **`log_sum_exp`**`(vector x)`
The natural logarithm of the sum of the exponentials of the elements in x

`real` **`log_sum_exp`**`(row_vector x)`
The natural logarithm of the sum of the exponentials of the elements in x

`real` **`log_sum_exp`**`(matrix x)`
The natural logarithm of the sum of the exponentials of the elements in x

Minimum and Maximum

`real` **`min`**`(vector x)`
The minimum value in x, or $+\infty$ if x is empty

`real` **`min`**`(row_vector x)`
The minimum value in x, or $+\infty$ if x is empty

`real` **`min`**`(matrix x)`
The minimum value in x, or $+\infty$ if x is empty

`real` **`max`**`(vector x)`
The maximum value in x, or $-\infty$ if x is empty

`real` **`max`**`(row_vector x)`
The maximum value in x, or $-\infty$ if x is empty

`real` **`max`**`(matrix x)`
The maximum value in x, or $-\infty$ if x is empty

Sums and Products
`real` **`sum`**`(vector x)`
The sum of the values in x, or 0 if x is empty

`real` **`sum`**`(row_vector x)`
The sum of the values in x, or 0 if x is empty

`real` **`sum`**`(matrix x)`
The sum of the values in x, or 0 if x is empty

`real` **`prod`**`(vector x)`
The product of the values in x, or 1 if x is empty

`real` **`prod`**`(row_vector x)`
The product of the values in x, or 1 if x is empty

`real` **`prod`**`(matrix x)`
The product of the values in x, or 1 if x is empty

Sample Moments
Full definitions are provided for sample moments in section array reductions.

`real` **`mean`**`(vector x)`
The sample mean of the values in x; see section array reductions for details.

`real` **`mean`**`(row_vector x)`
The sample mean of the values in x; see section array reductions for details.

`real` **`mean`**`(matrix x)`
The sample mean of the values in x; see section array reductions for details.

`real` **`variance`**`(vector x)`
The sample variance of the values in x; see section array reductions for details.

`real` **`variance`**`(row_vector x)`
The sample variance of the values in x; see section array reductions for details.

`real` **`variance`**`(matrix x)`
The sample variance of the values in x; see section array reductions for details.

`real` **`sd`**`(vector x)`
The sample standard deviation of the values in x; see section array reductions for details.

`real` **`sd`**`(row_vector x)`
The sample standard deviation of the values in x; see section array reductions for details.

`real` **`sd`**`(matrix x)`
The sample standard deviation of the values in x; see section array reductions for details.

5.7. Broadcast Functions

The following broadcast functions allow vectors, row vectors and matrices to be created by copying a single element into all of their cells. Matrices may also be created by stacking copies of row vectors vertically or stacking copies of column vectors horizontally.

`vector` **`rep_vector`**`(real x, int m)`
Return the size m (column) vector consisting of copies of x.

`row_vector` **`rep_row_vector`**`(real x, int n)`
Return the size n row vector consisting of copies of x.

`matrix` **`rep_matrix`**`(real x, int m, int n)`
Return the m by n matrix consisting of copies of x.

`matrix` **`rep_matrix`**`(vector v, int n)`
Return the m by n matrix consisting of n copies of the (column) vector v of size m.

`matrix` **`rep_matrix`**`(row_vector rv, int m)`
Return the m by n matrix consisting of m copies of the row vector rv of size n.

Unlike the situation with array broadcasting (see section array broadcasting), where there is a distinction between integer and real arguments, the following two statements produce the same result for vector broadcasting; row vector and matrix broadcasting behave similarly.

```
vector[3] x;
```

```
x = rep_vector(1, 3);
x = rep_vector(1.0, 3);
```

There are no integer vector or matrix types, so integer values are automatically promoted.

5.8. Diagonal Matrix Functions

`matrix` **`add_diag`**`(matrix m, row_vector d)`
Add row_vector `d` to the diagonal of matrix `m`.

`matrix` **`add_diag`**`(matrix m, vector d)`
Add vector `d` to the diagonal of matrix `m`.

`matrix` **`add_diag`**`(matrix m, real d)`
Add scalar `d` to every diagonal element of matrix `m`.

`vector` **`diagonal`**`(matrix x)`
The diagonal of the matrix x

`matrix` **`diag_matrix`**`(vector x)`
The diagonal matrix with diagonal x

Although the `diag_matrix` function is available, it is unlikely to ever show up in an efficient Stan program. For example, rather than converting a diagonal to a full matrix for use as a covariance matrix,

```
y ~ multi_normal(mu, diag_matrix(square(sigma)));
```

it is much more efficient to just use a univariate normal, which produces the same density,

```
y ~ normal(mu, sigma);
```

Rather than writing `m * diag_matrix(v)` where `m` is a matrix and `v` is a vector, it is much more efficient to write `diag_post_multiply(m, v)` (and similarly for pre-multiplication). By the same token, it is better to use `quad_form_diag(m, v)` rather than `quad_form(m, diag_matrix(v))`.

5.9. Slicing and Blocking Functions

Stan provides several functions for generating slices or blocks or diagonal entries for matrices.

Columns and Rows

`vector` **`col`**`(matrix x, int n)`
The n-th column of matrix x

`row_vector` **`row`**`(matrix x, int m)`
The m-th row of matrix x

The `row` function is special in that it may be used as an lvalue in an assignment statement (i.e., something to which a value may be assigned). The row function is also special in that the indexing notation `x[m]` is just an alternative way of writing `row(x,m)`. The `col` function may **not**, be used as an lvalue, nor is there an indexing based shorthand for it.

Block Operations
Matrix Slicing Operations

Block operations may be used to extract a sub-block of a matrix.

`matrix` **`block`**`(matrix x, int i, int j, int n_rows, int n_cols)`
Return the submatrix of x that starts at row i and column j and extends n_rows rows and n_cols columns.

The sub-row and sub-column operations may be used to extract a slice of row or column from a matrix

`vector` **`sub_col`**`(matrix x, int i, int j, int n_rows)`
Return the sub-column of x that starts at row i and column j and extends n_rows rows and 1 column.

`row_vector` **`sub_row`**`(matrix x, int i, int j, int n_cols)`
Return the sub-row of x that starts at row i and column j and extends 1 row and n_cols columns.

Vector and Array Slicing Operations

The head operation extracts the first *n* elements of a vector and the tail operation the last. The segment operation extracts an arbitrary subvector.

`vector` **`head`**`(vector v, int n)`
Return the vector consisting of the first n elements of v.

`row_vector` **`head`**`(row_vector rv, int n)`
Return the row vector consisting of the first n elements of rv.

`T[]` **`head`**`(T[] sv, int n)`
Return the array consisting of the first n elements of sv; applies to up to three-dimensional arrays containing any type of elements `T`.

`vector` **`tail`**`(vector v, int n)`
Return the vector consisting of the last n elements of v.

`row_vector` **`tail`**`(row_vector rv, int n)`
Return the row vector consisting of the last n elements of rv.

`T[]` **`tail`**`(T[] sv, int n)`
Return the array consisting of the last n elements of sv; applies to up to three-dimensional arrays containing any type of elements `T`.

`vector` **`segment`**`(vector v, int i, int n)`
Return the vector consisting of the n elements of v starting at i; i.e., elements i through through i + n - 1.

`row_vector` **`segment`**`(row_vector rv, int i, int n)`
Return the row vector consisting of the n elements of rv starting at i; i.e., elements i through through i + n - 1.

`T[]` **`segment`**`(T[] sv, int i, int n)`
Return the array consisting of the n elements of sv starting at i; i.e., elements i through through i + n - 1. Applies to up to three-dimensional arrays containing any type of elements `T`.

5.10. Matrix Concatenation

Stan's matrix concatenation operations `append_col` and `append_row` are like the operations `cbind` and `rbind` in R.

Horizontal concatenation

`matrix` **`append_col`**`(matrix x, matrix y)`
Combine matrices x and y by columns. The matrices must have the same number of rows.

`matrix` **`append_col`**`(matrix x, vector y)`
Combine matrix x and vector y by columns. The matrix and the vector must have the same number of rows.

`matrix` **`append_col`**`(vector x, matrix y)`
Combine vector x and matrix y by columns. The vector and the matrix must have the same number of rows.

`matrix` **`append_col`**`(vector x, vector y)`
Combine vectors x and y by columns. The vectors must have the same number of rows.

`row_vector` **`append_col`**`(row_vector x, row_vector y)`
Combine row vectors x and y of any size into another row vector.

`row_vector` **`append_col`**`(real x, row_vector y)`
Append x to the front of y, returning another row vector.

`row_vector` **`append_col`**`(row_vector x, real y)`
Append y to the end of x, returning another row vector.

Vertical concatenation

`matrix` **`append_row`**`(matrix x, matrix y)`
Combine matrices x and y by rows. The matrices must have the same number of columns.

`matrix` **`append_row`**`(matrix x, row_vector y)`
Combine matrix x and row vector y by rows. The matrix and the row vector must have the same number of columns.

`matrix` **`append_row`**`(row_vector x, matrix y)`
Combine row vector x and matrix y by rows. The row vector and the matrix must have the same number of columns.

`matrix` **`append_row`**`(row_vector x, row_vector y)`
Combine row vectors x and y by row. The row vectors must have the same number of columns.

`vector` **`append_row`**`(vector x, vector y)`
Concatenate vectors x and y of any size into another vector.

`vector` **`append_row`**`(real x, vector y)`
Append x to the top of y, returning another vector.

`vector` **`append_row`**`(vector x, real y)`
Append y to the bottom of x, returning another vector.

5.11. Special Matrix Functions

Softmax

The softmax function maps[1] $y \in \mathbb{R}^K$ to the K-simplex by

$$\text{softmax}(y) = \frac{\exp(y)}{\sum_{k=1}^{K} \exp(y_k)},$$

[1]The softmax function is so called because in the limit as $y_n \to \infty$ with y_m for $m \neq n$ held constant, the result tends toward the "one-hot" vector θ with $\theta_n = 1$ and $\theta_m = 0$ for $m \neq n$, thus providing a "soft" version of the maximum function.

where $\exp(y)$ is the componentwise exponentiation of y. Softmax is usually calculated on the log scale,

$$\begin{aligned}\log \text{softmax}(y) &= y - \log \sum_{k=1}^{K} \exp(y_k) \\ &= y - \text{log\_sum\_exp}(y).\end{aligned}$$

where the vector y minus the scalar $\text{log\_sum\_exp}(y)$ subtracts the scalar from each component of y.

Stan provides the following functions for softmax and its log.

`vector` **`softmax`**`(vector x)`
The softmax of x

`vector` **`log_softmax`**`(vector x)`
The natural logarithm of the softmax of x

Cumulative Sums
The cumulative sum of a sequence $x_1, \ldots, x_N$ is the sequence $y_1, \ldots, y_N$, where

$$y_n = \sum_{m=1}^{n} x_m.$$

`real[]` **`cumulative_sum`**`(real[] x)`
The cumulative sum of x

`vector` **`cumulative_sum`**`(vector v)`
The cumulative sum of v

`row_vector` **`cumulative_sum`**`(row_vector rv)`
The cumulative sum of rv

5.12. Covariance Functions

Exponentiated quadratic covariance function
The exponentiated quadratic kernel defines the covariance between $f(x_i)$ and $f(x_j)$ where $f : \mathbb{R}^D \mapsto \mathbb{R}$ as a function of the squared Euclidian distance between $x_i \in \mathbb{R}^D$ and $x_j \in \mathbb{R}^D$:

$$\text{cov}(f(x_i), f(x_j)) = k(x_i, x_j) = \alpha^2 \exp\left(-\frac{1}{2\rho^2} \sum_{d=1}^{D} (x_{i,d} - x_{j,d})^2\right)$$

with α and ρ constrained to be positive.

There are two variants of the exponentiated quadratic covariance function in Stan. One builds a covariance matrix, $K \in \mathbb{R}^{N \times N}$ for $x_1, \ldots, x_N$, where $K_{i,j} = k(x_i, x_j)$, which is necessarily symmetric and positive semidefinite by construction. There is a second variant of the exponentiated quadratic covariance function that builds a $K \in \mathbb{R}^{N \times M}$ covariance matrix for $x_1, \ldots, x_N$ and $x'_1, \ldots, x'_M$, where $x_i \in \mathbb{R}^D$ and $x'_i \in \mathbb{R}^D$ and $K_{i,j} = k(x_i, x'_j)$.

`matrix` **`cov_exp_quad`**`(row_vectors x, real alpha, real rho)`
The covariance matrix with an exponentiated quadratic kernel of x.

`matrix` **`cov_exp_quad`**`(vectors x, real alpha, real rho)`
The covariance matrix with an exponentiated quadratic kernel of x.

`matrix` **`cov_exp_quad`**`(real[] x, real alpha, real rho)`
The covariance matrix with an exponentiated quadratic kernel of x.

`matrix` **`cov_exp_quad`**`(row_vectors x1, row_vectors x2, real alpha, real rho)`
The covariance matrix with an exponentiated quadratic kernel of x1 and x2.

`matrix` **`cov_exp_quad`**`(vectors x1, vectors x2, real alpha, real rho)`
The covariance matrix with an exponentiated quadratic kernel of x1 and x2.

`matrix` **`cov_exp_quad`**`(real[] x1, real[] x2, real alpha, real rho)`
The covariance matrix with an exponentiated quadratic kernel of x1 and x2.

5.13. Linear Algebra Functions and Solvers

Matrix Division Operators and Functions

In general, it is much more efficient and also more arithmetically stable to use matrix division than to multiply by an inverse. There are specialized forms for lower triangular matrices and for symmetric, positive-definite matrices.

Matrix division operators

`row_vector` **`operator/`**`(row_vector b, matrix A)`
The right division of b by A; equivalently `b * inverse(A)`

`matrix` **`operator/`**`(matrix B, matrix A)`
The right division of B by A; equivalently `B * inverse(A)`

`vector` **`operator\`**`(matrix A, vector b)`
The left division of A by b; equivalently `inverse(A) * b`

`matrix` **`operator\`**`(matrix A, matrix B)`
The left division of A by B; equivalently `inverse(A) * B`

Lower-triangular matrix division functions

There are four division functions which use lower triangular views of a matrix. The lower triangular view of a matrix $\mathrm{tri}(A)$ is used in the definitions and defined by

$$\mathrm{tri}(A)[m,n] = \begin{cases} A[m,n] & \text{if } m \geq n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

When a lower triangular view of a matrix is used, the elements above the diagonal are ignored.

`vector` **`mdivide_left_tri_low`**`(matrix A, vector b)`
The left division of b by a lower-triangular view of A; algebraically equivalent to the less efficient and stable form `inverse(tri(A)) * b`, where `tri(A)` is the lower-triangular portion of A with the above-diagonal entries set to zero.

`matrix` **`mdivide_left_tri_low`**`(matrix A, matrix B)`
The left division of B by a triangular view of A; algebraically equivalent to the less efficient and stable form `inverse(tri(A)) * B`, where `tri(A)` is the lower-triangular portion of A with the above-diagonal entries set to zero.

`row_vector` **`mdivide_right_tri_low`**`(row_vector b, matrix A)`
The right division of b by a triangular view of A; algebraically equivalent to the less efficient and stable form `b * inverse(tri(A))`, where `tri(A)` is the lower-triangular portion of A with the above-diagonal entries set to zero.

`matrix` **`mdivide_right_tri_low`**`(matrix B, matrix A)`
The right division of B by a triangular view of A; algebraically equivalent to the less efficient and stable form `B * inverse(tri(A))`, where `tri(A)` is the lower-triangular portion of A with the above-diagonal entries set to zero.

Symmetric positive-definite matrix division functions
There are four division functions which are specialized for efficiency and stability for symmetric positive-definite matrix dividends. If the matrix dividend argument is not symmetric and positive definite, these will reject and print warnings.

`matrix` **`mdivide_left_spd`**`(matrix A, vector b)`
The left division of b by the symmetric, positive-definite matrix A; algebraically equivalent to the less efficient and stable form `inverse(A) * b`.

`vector` **`mdivide_left_spd`**`(matrix A, matrix B)`
The left division of B by the symmetric, positive-definite matrix A; algebraically equivalent to the less efficient and stable form `inverse(A) * B`.

`row_vector` **`mdivide_right_spd`**`(row_vector b, matrix A)`
The right division of b by the symmetric, positive-definite matrix A; algebraically equivalent to the less efficient and stable form `b * inverse(A)`.

`matrix` **`mdivide_right_spd`**`(matrix B, matrix A)`
The right division of B by the symmetric, positive-definite matrix A; algebraically equivalent to the less efficient and stable form `B * inverse(A)`.

Matrix Exponential
The exponential of the matrix A is formally defined by the convergent power series:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

`matrix` **`matrix_exp`**`(matrix A)`
The matrix exponential of A

`matrix` **`matrix_exp_multiply`**`(matrix A, matrix B)`
The multiplication of matrix exponential of A and matrix B; algebraically equivalent to the less efficient form `matrix_exp(A) * B`.

`matrix` **`scale_matrix_exp_multiply`**`(real t, matrix A, matrix B)`
The multiplication of matrix exponential of tA and matrix B; algebraically equivalent to the less efficient form `matrix_exp(t * A) * B`.

Matrix Power
Returns the nth power of the specific matrix:

$$M^n = M_1 * ... * M_n$$

`matrix` **`matrix_power`**`(matrix A, int B)`
Matrix A raised to the power B.

Linear Algebra Functions
Trace

`real` **`trace`**`(matrix A)`
The trace of A, or 0 if A is empty; A is not required to be diagonal

Determinants

`real` **`determinant`**`(matrix A)`
The determinant of A

`real` **`log_determinant`**`(matrix A)`
The log of the absolute value of the determinant of A

Inverses

It is almost never a good idea to use matrix inverses directly because they are both inefficient and arithmetically unstable compared to the alternatives. Rather than inverting a matrix `m` and post-multiplying by a vector or matrix `a`, as in `inverse(m) * a`, it is better to code this using matrix division, as in `m \ a`. The pre-multiplication case is similar, with `b * inverse(m)` being more efficiently coded as as `b / m`. There are also useful special cases for triangular and symmetric, positive-definite matrices that use more efficient solvers.

Warning: The function `inv(m)` is the elementwise inverse function, which returns `1 / m[i, j]` for each element.

`matrix` **`inverse`**`(matrix A)`
The inverse of A

`matrix` **`inverse_spd`**`(matrix A)`
The inverse of A where A is symmetric, positive definite. This version is faster and more arithmetically stable when the input is symmetric and positive definite.

Eigendecomposition

`vector` **`eigenvalues_sym`**`(matrix A)`
The vector of eigenvalues of a symmetric matrix A in ascending order

`matrix` **`eigenvectors_sym`**`(matrix A)`
The matrix with the (column) eigenvectors of symmetric matrix A in the same order as returned by the function `eigenvalues_sym`

Because multiplying an eigenvector by -1 results in an eigenvector, eigenvectors returned by a decomposition are only identified up to a sign change. In order to compare the eigenvectors produced by Stan's eigendecomposition to others, signs may need to be normalized in some way, such as by fixing the sign of a component, or doing comparisons allowing a multiplication by -1.

The condition number of a symmetric matrix is defined to be the ratio of the largest eigenvalue to the smallest eigenvalue. Large condition numbers lead to difficulty in numerical algorithms such as computing inverses, and thus known as "ill conditioned." The ratio can even be infinite in the case of singular matrices (i.e., those with eigenvalues of 0).

QR Decomposition

`matrix` **`qr_thin_Q`**`(matrix A)`
The orthogonal matrix in the thin QR decomposition of A, which implies that the resulting matrix has the same dimensions as A

`matrix` **`qr_thin_R`**`(matrix A)`
The upper triangular matrix in the thin QR decomposition of A, which implies that the resulting matrix is square with the same number of columns as A

`matrix` **`qr_Q`**`(matrix A)`
The orthogonal matrix in the fat QR decomposition of A, which implies that the resulting matrix is square with the same number of rows as A

`matrix` **`qr_R`**`(matrix A)`
The upper trapezoidal matrix in the fat QR decomposition of A, which implies that the resulting matrix will be rectangular with the same dimensions as A

The thin QR decomposition is always preferable because it will consume much less memory when the input matrix is large than will the fat QR decomposition. Both versions of the decomposition represent the input matrix as

$$A = Q\,R.$$

Multiplying a column of an orthogonal matrix by -1 still results in an orthogonal matrix, and you can multiply the corresponding row of the upper trapezoidal matrix by -1 without changing the product. Thus, Stan adopts the normalization that the diagonal elements of the upper trapezoidal matrix are strictly positive and the columns of the orthogonal matrix are reflected if necessary. Also, these QR decomposition algorithms do not utilize pivoting and thus may be numerically unstable on input matrices that have less than full rank.

Cholesky Decomposition

Every symmetric, positive-definite matrix (such as a correlation or covariance matrix) has a Cholesky decomposition. If Σ is a symmetric, positive-definite matrix, its Cholesky decomposition is the lower-triangular vector L such that

$$\Sigma = L\,L^{\top}.$$

`matrix` **`cholesky_decompose`**`(matrix A)`
The lower-triangular Cholesky factor of the symmetric positive-definite matrix A

Singular Value Decomposition

Stan only provides functions for the singular values, not for the singular vectors involved in a singular value decomposition (SVD).

`vector` **`singular_values`**`(matrix A)`
The singular values of A in descending order

5.14. Sort Functions

See the sorting functions section for examples of how the functions work.

`vector` **`sort_asc`**`(vector v)`
Sort the elements of v in ascending order

`row_vector` **`sort_asc`**`(row_vector v)`
Sort the elements of v in ascending order

`vector` **`sort_desc`**`(vector v)`
Sort the elements of v in descending order

`row_vector` **`sort_desc`**`(row_vector v)`
Sort the elements of v in descending order

`int[]` **`sort_indices_asc`**`(vector v)`
Return an array of indices between 1 and the size of v, sorted to index v in ascending order.

`int[]` **`sort_indices_asc`**`(row_vector v)`
Return an array of indices between 1 and the size of v, sorted to index v in ascending order.

`int[]` **`sort_indices_desc`**`(vector v)`
Return an array of indices between 1 and the size of v, sorted to index v in descending order.

`int[]` **`sort_indices_desc`**`(row_vector v)`
Return an array of indices between 1 and the size of v, sorted to index v in descending order.

`int` **`rank`**`(vector v, int s)`
Number of components of v less than v[s]

`int` **`rank`**`(row_vector v, int s)`
Number of components of v less than v[s]

5.15. Reverse Functions

`vector` **`reverse`**`(vector v)`
Return a new vector containing the elements of the argument in reverse order.

`row_vector` **`reverse`**`(row_vector v)`
Return a new row vector containing the elements of the argument in reverse order.

6. Sparse Matrix Operations

For sparse matrices, for which many elements are zero, it is more efficient to use specialized representations to save memory and speed up matrix arithmetic (including derivative calculations). Given Stan's implementation, there is substantial space (memory) savings by using sparse matrices. Because of the ease of optimizing dense matrix operations, speed improvements only arise at 90% or even greater sparsity; below that level, dense matrices are faster but use more memory.

Because of this speedup and space savings, it may even be useful to read in a dense matrix and convert it to a sparse matrix before multiplying it by a vector. This chapter covers a very specific form of sparsity consisting of a sparse matrix multiplied by a dense vector.

6.1. Compressed Row Storage

Sparse matrices are represented in Stan using compressed row storage (CSR). For example, the matrix

$$A = \begin{bmatrix} 19 & 27 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 52 \\ 81 & 0 & 95 & 33 \end{bmatrix}$$

is translated into a vector of the non-zero real values, read by row from the matrix A,

$$w(A) = \begin{bmatrix} 19 & 27 & 52 & 81 & 95 & 33 \end{bmatrix}^{\top},$$

an array of integer column indices for the values,

$$v(A) = \begin{bmatrix} 1 & 2 & 4 & 1 & 3 & 4 \end{bmatrix},$$

and an array of integer indices indicating where in $w(A)$ a given row's values start,

$$u(A) = \begin{bmatrix} 1 & 3 & 3 & 4 & 7 \end{bmatrix},$$

with a padded value at the end to guarantee that

$$u(A)[n+1] - u(A)[n]$$

is the number of non-zero elements in row n of the matrix (here 2, 0, 1, and 3). Note that because the second row has no non-zero elements both the second and third

elements of $u(A)$ correspond to the third element of $w(A)$, which is 52. The values $(w(A), v(A), u(A))$ are sufficient to reconstruct A.

The values are structured so that there is a real value and integer column index for each non-zero entry in the array, plus one integer for each row of the matrix, plus one for padding. There is also underlying storage for internal container pointers and sizes. The total memory usage is roughly $12K + M$ bytes plus a small constant overhead, which is often considerably fewer bytes than the $M \times N$ required to store a dense matrix. Even more importantly, zero values do not introduce derivatives under multiplication or addition, so many storage and evaluation steps are saved when sparse matrices are multiplied.

6.2. Conversion Functions

Conversion functions between dense and sparse matrices are provided.

Dense to Sparse Conversion

Converting a dense matrix m to a sparse representation produces a vector w and two integer arrays, u and v.

`vector` **`csr_extract_w`**`(matrix a)`
Return non-zero values in matrix a; see section compressed row storage.

`int[]` **`csr_extract_v`**`(matrix a)`
Return column indices for values in `csr_extract_w(a)`; see compressed row storage.

`int[]` **`csr_extract_u`**`(matrix a)`
Return array of row starting indices for entries in `csr_extract_w(a)` followed by the size of `csr_extract_w(a)` plus one; see section compressed row storage.

Sparse to Dense Conversion

To convert a sparse matrix representation to a dense matrix, there is a single function.

`matrix` **`csr_to_dense_matrix`**`(int m, int n, vector w, int[] v, int[] u)`
Return dense m $\times$ n matrix with non-zero matrix entries w, column indices v, and row starting indices u; the vector w and array v must be the same size (corresponding to the total number of nonzero entries in the matrix), array v must have index values bounded by m, array u must have length equal to m + 1 and contain index values bounded by the number of nonzeros (except for the last entry, which must be equal to the number of nonzeros plus one). See section compressed row storage for more details.

6.3. Sparse Matrix Arithmetic

Sparse Matrix Multiplication

The only supported operation is the multiplication of a sparse matrix A and a dense vector b to produce a dense vector Ab. Multiplying a dense row vector b and a sparse matrix A can be coded using transposition as

$$bA = (A^\top b^\top)^\top,$$

but care must be taken to represent $A^\top$ rather than A as a sparse matrix.

`vector` **`csr_matrix_times_vector`**`(int m, int n, vector w, int[] v, int[] u, vector b)`
Multiply the m × n matrix represented by values w, column indices v, and row start indices u by the vector b; see compressed row storage.

7. Mixed Operations

These functions perform conversions between Stan containers matrix, vector, row vector and arrays.

`matrix` **`to_matrix`**`(matrix m)`
Return the matrix m itself.

`matrix` **`to_matrix`**`(vector v)`
Convert the column vector v to a `size(v)` by 1 matrix.

`matrix` **`to_matrix`**`(row_vector v)`
Convert the row vector v to a 1 by `size(v)` matrix.

`matrix` **`to_matrix`**`(matrix m, int m, int n)`
Convert a matrix m to a matrix with m rows and n columns filled in column-major order.

`matrix` **`to_matrix`**`(vector v, int m, int n)`
Convert a vector v to a matrix with m rows and n columns filled in column-major order.

`matrix` **`to_matrix`**`(row_vector v, int m, int n)`
Convert a row_vector a to a matrix with m rows and n columns filled in column-major order.

`matrix` **`to_matrix`**`(matrix m, int m, int n, int col_major)`
Convert a matrix m to a matrix with m rows and n columns filled in row-major order if col_major equals 0 (otherwise, they get filled in column-major order).

`matrix` **`to_matrix`**`(vector v, int m, int n, int col_major)`
Convert a vector v to a matrix with m rows and n columns filled in row-major order if col_major equals 0 (otherwise, they get filled in column-major order).

`matrix` **`to_matrix`**`(row_vector v, int m, int n, int col_major)`
Convert a row_vector a to a matrix with m rows and n columns filled in row-major order if col_major equals 0 (otherwise, they get filled in column-major order).

`matrix` **`to_matrix`**`(real[] a, int m, int n)`
Convert a one-dimensional array a to a matrix with m rows and n columns filled in column-major order.

`matrix` **`to_matrix`**`(int[] a, int m, int n)`
Convert a one-dimensional array a to a matrix with m rows and n columns filled in column-major order.

`matrix` **`to_matrix`**`(real[] a, int m, int n, int col_major)`
Convert a one-dimensional array a to a matrix with m rows and n columns filled in row-major order if col_major equals 0 (otherwise, they get filled in column-major order).

`matrix` **`to_matrix`**`(int[] a, int m, int n, int col_major)`
Convert a one-dimensional array a to a matrix with m rows and n columns filled in row-major order if col_major equals 0 (otherwise, they get filled in column-major order).

`matrix` **`to_matrix`**`(real[,] a)`
Convert the two dimensional array a to a matrix with the same dimensions and indexing order.

`matrix` **`to_matrix`**`(int[,] a)`
Convert the two dimensional array a to a matrix with the same dimensions and indexing order. If any of the dimensions of a are zero, the result will be a 0×0 matrix.

`vector` **`to_vector`**`(matrix m)`
Convert the matrix m to a column vector in column-major order.

`vector` **`to_vector`**`(vector v)`
Return the column vector v itself.

`vector` **`to_vector`**`(row_vector v)`
Convert the row vector v to a column vector.

`vector` **`to_vector`**`(real[] a)`
Convert the one-dimensional array a to a column vector.

`vector` **`to_vector`**`(int[] a)`
Convert the one-dimensional integer array a to a column vector.

`row_vector` **`to_row_vector`**`(matrix m)`
Convert the matrix m to a row vector in column-major order.

`row_vector` **`to_row_vector`**`(vector v)`
Convert the column vector v to a row vector.

`row_vector` **`to_row_vector`**`(row_vector v)`
Return the row vector v itself.

`row_vector` **`to_row_vector`**`(real[] a)`
Convert the one-dimensional array a to a row vector.

`row_vector` **`to_row_vector`**`(int[] a)`
Convert the one-dimensional array a to a row vector.

`real[,]` **`to_array_2d`**`(matrix m)`
Convert the matrix m to a two dimensional array with the same dimensions and indexing order.

`real[]` **`to_array_1d`**`(vector v)`
Convert the column vector v to a one-dimensional array.

`real[]` **`to_array_1d`**`(row_vector v)`
Convert the row vector v to a one-dimensional array.

`real[]` **`to_array_1d`**`(matrix m)`
Convert the matrix m to a one-dimensional array in column-major order.

`real[]` **`to_array_1d`**`(real[...] a)`
Convert the array a (of any dimension up to 10) to a one-dimensional array in row-major order.

`int[]` **`to_array_1d`**`(int[...] a)`
Convert the array a (of any dimension up to 10) to a one-dimensional array in row-major order.

8. Compound Arithmetic and Assignment

Compound arithmetic and assignment statements combine an arithmetic operation and assignment,

```
x = x op y;
```

replacing them with the compound form

```
x op= y;
```

For example, `x = x + 1` may be replaced with `x += 1`.

The signatures of the supported compound arithmetic and assignment operations are as follows.

8.1. Compound Addition and Assignment

`void` **`operator+=`**`(int x, int y)`
`x += y` is equivalent to `x = x + y`.

`void` **`operator+=`**`(real x, real y)`
`x += y` is equivalent to `x = x + y`.

`void` **`operator+=`**`(vector x, real y)`
`x += y` is equivalent to `x = x + y`.

`void` **`operator+=`**`(row_vector x, real y)`
`x += y` is equivalent to `x = x + y`.

`void` **`operator+=`**`(matrix x, real y)`
`x += y` is equivalent to `x = x + y`.

`void` **`operator+=`**`(vector x, vector y)`
`x += y` is equivalent to `x = x + y`.

`void` **`operator+=`**`(row_vector x, row_vector y)`
`x += y` is equivalent to `x = x + y`.

`void` **`operator+=`**`(matrix x, matrix y)`
`x += y` is equivalent to `x = x + y`.

8.2. Compound Subtraction and Assignment

`void` **`operator-=`**`(int x, int y)`
`x -= y` is equivalent to `x = x - y`.

`void` **`operator-=`**`(real x, real y)`
`x -= y` is equivalent to `x = x - y`.

`void` **`operator-=`**`(vector x, real y)`
`x -= y` is equivalent to `x = x - y`.

`void` **`operator-=`**`(row_vector x, real y)`
`x -= y` is equivalent to `x = x - y`.

`void` **`operator-=`**`(matrix x, real y)`
`x -= y` is equivalent to `x = x - y`.

`void` **`operator-=`**`(vector x, vector y)`
`x -= y` is equivalent to `x = x - y`.

`void` **`operator-=`**`(row_vector x, row_vector y)`
`x -= y` is equivalent to `x = x - y`.

`void` **`operator-=`**`(matrix x, matrix y)`
`x -= y` is equivalent to `x = x - y`.

8.3. Compound Multiplication and Assignment

`void` **`operator*=`**`(int x, int y)`
`x *= y` is equivalent to `x = x * y`.

`void` **`operator*=`**`(real x, real y)`
`x *= y` is equivalent to `x = x * y`.

`void` **`operator*=`**`(vector x, real y)`
`x *= y` is equivalent to `x = x * y`.

`void` **`operator*=`**`(row_vector x, real y)`
`x *= y` is equivalent to `x = x * y`.

`void` **`operator*=`**`(matrix x, real y)`
`x *= y` is equivalent to `x = x * y`.

`void` **`operator*=`**`(row_vector x, matrix y)`
`x *= y` is equivalent to `x = x * y`.

`void` **`operator*=`**`(matrix x, matrix y)`
`x *= y` is equivalent to `x = x * y`.

8.4. Compound Division and Assignment

`void` **`operator/=`**`(int x, int y)`
`x /= y` is equivalent to `x = x / y`.

`void` **`operator/=`**`(real x, real y)`
`x /= y` is equivalent to `x = x / y`.

`void` **`operator/=`**`(vector x, real y)`
`x /= y` is equivalent to `x = x / y`.

`void` **`operator/=`**`(row_vector x, real y)`
`x /= y` is equivalent to `x = x / y`.

`void` **`operator/=`**`(matrix x, real y)`
`x /= y` is equivalent to `x = x / y`.

8.5. Compound Elementwise Multiplication and Assignment

`void` **`operator.*=`**`(vector x, vector y)`
`x .*= y` is equivalent to `x = x .* y`.

`void` **`operator.*=`**`(row_vector x, row_vector y)`
`x .*= y` is equivalent to `x = x .* y`.

`void` **`operator.*=`**`(matrix x, matrix y)`
`x .*= y` is equivalent to `x = x .* y`.

8.6. Compound Elementwise Division and Assignment

`void` **`operator./=`**`(vector x, vector y)`
`x ./= y` is equivalent to `x = x ./ y`.

`void` **`operator./=`**`(row_vector x, row_vector y)`
`x ./= y` is equivalent to `x = x ./ y`.

`void` **`operator./=`**`(matrix x, matrix y)`
`x ./= y` is equivalent to `x = x ./ y`.

`void` **`operator./=`**`(vector x, real y)`
`x ./= y` is equivalent to `x = x ./ y`.

`void` **`operator./=`**`(row_vector x, real y)`
`x ./= y` is equivalent to `x = x ./ y`.

`void` **`operator./=`**`(matrix x, real y)`
`x ./= y` is equivalent to `x = x ./ y`.

9. Higher-Order Functions

Stan provides a few higher-order functions that act on other functions. In all cases, the function arguments to the higher-order functions are defined as functions within the Stan language and passed by name to the higher-order functions.

9.1. Algebraic Equation Solver

Stan provides two built-in algebraic equation solvers, respectively based on Powell's and Newton's methods. The Newton method constitutes a more recent addition to Stan; its use is recommended for most problems. Although they look like other function applications, algebraic solvers are special in two ways.

First, an algebraic solver is a higher-order function, i.e. it takes another function as one of its arguments. Other functions in Stan which share this feature are the ordinary differential equation solvers (see section Ordinary Differential Equation (ODE) Solvers). Ordinary Stan functions do not allow functions as arguments.

Second, some of the arguments of the algebraic solvers are restricted to data only expressions. These expressions must not contain variables other than those declared in the data or transformed data blocks. Ordinary Stan functions place no restriction on the origin of variables in their argument expressions.

Specifying an Algebraic Equation as a Function

An algebraic system is specified as an ordinary function in Stan within the function block. The algebraic system function must have this signature:

```
vector algebra_system(vector y, vector theta,
                      real[] x_r, int[] x_i)
```

The algebraic system function should return the value of the algebraic function which goes to 0, when we plug in the solution to the algebraic system.

The argument of this function are:

- *y*, the unknowns we wish to solve for
- *theta*, parameter values used to evaluate the algebraic system
- *x_r*, data values used to evaluate the algebraic system
- *x_i*, integer data used to evaluate the algebraic system

The algebraic system function separates parameter values, *theta*, from data values, *x_r*, for efficiency in propagating the derivatives through the algebraic system.

Call to the Algebraic Solver

`vector` **`algebra_solver`**`(function algebra_system, vector y_guess, vector theta, real[] x_r, int[] x_i)`
Solves the algebraic system, given an initial guess, using the Powell hybrid algorithm.

`vector` **`algebra_solver`**`(function algebra_system, vector y_guess, vector theta, real[] x_r, int[] x_i, real rel_tol, real f_tol, int max_steps)`
Solves the algebraic system, given an initial guess, using the Powell hybrid algorithm with additional control parameters for the solver.

Note: In future releases, the function `algebra_solver` will be deprecated and replaced with `algebra_solver_powell`.

`vector` **`algebra_solver_newton`**`(function algebra_system, vector y_guess, vector theta, real[] x_r, int[] x_i)`
Solves the algebraic system, given an initial guess, using Newton's method.

`vector` **`algebra_solver_newton`**`(function algebra_system, vector y_guess, vector theta, real[] x_r, int[] x_i, real rel_tol, real f_tol, int max_steps)`
Solves the algebraic system, given an initial guess, using Newton's method with additional control parameters for the solver.

Arguments to the Algebraic Solver

The arguments to the algebraic solvers are as follows:

- *algebra_system*: function literal referring to a function specifying the system of algebraic equations with signature `(vector, vector, real[], int[]):vector`. The arguments represent (1) unknowns, (2) parameters, (3) real data, and (4) integer data, and the return value contains the value of the algebraic function, which goes to 0 when we plug in the solution to the algebraic system,
- *y_guess*: initial guess for the solution, type `vector`,
- *theta*: parameters only, type `vector`,
- *x_r*: real data only, type `real[]`, and
- *x_i*: integer data only, type `int[]`.

For more fine-grained control of the algebraic solver, these parameters can also be provided:

- *rel_tol*: relative tolerance for the algebraic solver, type `real`, data only,
- *function_tol*: function tolerance for the algebraic solver, type `real`, data only,
- *max_num_steps*: maximum number of steps to take in the algebraic solver, type `int`, data only.

Return value

The return value for the algebraic solver is an object of type `vector`, with values which, when plugged in as `y` make the algebraic function go to 0.

Sizes and parallel arrays

Certain sizes have to be consistent. The initial guess, return value of the solver, and return value of the algebraic function must all be the same size.

The parameters, real data, and integer data will be passed from the solver directly to the system function.

Algorithmic details

Stan offers two algebraic solvers: `algebra_solver` and `algebra_solver_newton`. `algebra_solver` is baed on the Powell hybrid method (Powell 1970), which in turn uses first-order derivatives. The Stan code builds on the implementation of the hybrid solver in the unsupported module for nonlinear optimization problems of the Eigen library (Guennebaud, Jacob, and others 2010). This solver is in turn based on the algorithm developed for the package MINPACK-1 (Jorge J. More 1980).

`algebra_solver_newton`, uses Newton's method, also a first-order derivative based numerical solver. The Stan code builds on the implementation in KINSOL from the SUNDIALS suite (Hindmarsh et al. 2005). For many problems, we find that `algebra_solver_newton` is faster than Powell's method. If however Newton's method performs poorly, either failing to or requiring an excessively long time to converge, the user should be prepared to switch to `algebra_solver`.

For both solvers, the Jacobian of the solution with respect to auxiliary parameters is computed using the implicit function theorem. Intermediate Jacobians (of the algebraic function's output with respect to the unknowns y and with respect to the auxiliary parameters theta) are computed using Stan's automatic differentiation.

9.2. Ordinary Differential Equation (ODE) Solvers

Stan provides several higher order functions for solving initial value problems specified as Ordinary Differential Equations (ODEs).

Solving an initial value ODE means given a set of differential equations $y'(t, \theta) = f(t, y, \theta)$ and initial conditions $y(t_0, \theta)$, solving for y at a sequence of times $t_0 < t_1 \leq t_2, \cdots \leq t_n$. $f(t, y, \theta)$ is referred to here as the ODE system function.

$f(t, y, \theta)$ will be defined as a function with a certain signature and provided along with the initial conditions and output times to one of the ODE solver functions.

To make it easier to write ODEs, the solve functions take extra arguments that are passed along unmodified to the user-supplied system function. Because there can be any number of these arguments and they can be of different types, they are denoted below as `...`. The types of the arguments represented by `...` in the ODE solve function call must match the types of the arguments represented by `...` in the user-supplied system function.

Non-Stiff Solver

`vector[] ode_rk45(function ode, vector initial_state, real initial_time, real[] times, ...)`
Solves the ODE system for the times provided using the Dormand-Prince algorithm, a 4th/5th order Runge-Kutta method.

`vector[] ode_rk45_tol(function ode, vector initial_state, real initial_time, real[] times, real rel_tol, real abs_tol, int max_num_steps, ...)`
Solves the ODE system for the times provided using the Dormand-Prince algorithm, a 4th/5th order Runge-Kutta method with additional control parameters for the solver.

`vector[] ode_adams(function ode, vector initial_state, real initial_time, real[] times, ...)`
Solves the ODE system for the times provided using the Adams-Moulton method.

`vector[] ode_adams_tol(function ode, vector initial_state, real initial_time, real[] times, data real rel_tol, data real abs_tol, data int max_num_steps, ...)`
Solves the ODE system for the times provided using the Adams-Moulton method with additional control parameters for the solver.

Stiff Solver

`vector[] ode_bdf(function ode, vector initial_state, real initial_time, real[] times, ...)`

Solves the ODE system for the times provided using the backward differentiation formula (BDF) method.

`vector[] ode_bdf_tol(function ode, vector initial_state, real initial_time, real[] times, data real rel_tol, data real abs_tol, data int max_num_steps, ...)`
Solves the ODE system for the times provided using the backward differentiation formula (BDF) method with additional control parameters for the solver.

ODE System Function
The first argument to one of the ODE solvers is always the ODE system function. The ODE system function must have a `vector` return type, and the first two arguments must be a `real` and `vector` in that order. These two arguments are followed by the variadic arguments that are passed through from the ODE solve function call:

```
vector ode(real time, vector state, ...)
```

The ODE system function should return the derivative of the state with respect to time at the time and state provided. The length of the returned vector must match the length of the state input into the function.

The arguments to this function are:

- *time*, the time to evaluate the ODE system
- *state*, the state of the ODE system at the time specified
- `...`, sequence of arguments passed unmodified from the ODE solve function call. The types here must match the types in the `...` arguments of the ODE solve function call.

Arguments to the ODE solvers
The arguments to the ODE solvers in both the stiff and non-stiff solvers are the same.

- *ode*: ODE system function,
- *initial_state*: initial state, type `vector`,
- *initial_time*: initial time, type `int` or `real`,
- *times*: solution times, type `real[]`,
- `...`: sequence of arguments that will be passed through unmodified to the ODE system function. The types here must match the types in the `...` arguments of the ODE system function.

For the versions of the ode solver functions ending in `_tol`, these three parameters must be provided after `times` and before the `...` arguments:

- `data` *`rel_tol`*: relative tolerance for the ODE solver, type `real`, data only,
- `data` *`abs_tol`*: absolute tolerance for the ODE solver, type `real`, data only, and
- `data` *`max_num_steps`*: maximum number of steps to take between output times in the ODE solver, type `int`, data only.

Because these are all `data` arguments, they must be defined in either the data or transformed data blocks. They cannot be parameters, transformed parameters or functions of parameters or transformed parameters.

Return values

The return value for the ODE solvers is an array of vectors (type `vector[]`), one vector representing the state of the system at every time in specified in the `times` argument.

Array and vector sizes

The sizes must match, and in particular, the following groups are of the same size:

- state variables passed into the system function, derivatives returned by the system function, initial state passed into the solver, and length of each vector in the output,
- number of solution times and number of vectors in the output,

9.3. 1D Integrator

Stan provides a built-in mechanism to perform 1D integration of a function via quadrature methods.

It operates similarly to the algebraic solver and the ordinary differential equations solver in that it allows as an argument a function.

Like both of those utilities, some of the arguments are limited to data only expressions. These expressions must not contain variables other than those declared in the data or transformed data blocks.

Specifying an Integrand as a Function

Performing a 1D integration requires the integrand to be specified somehow. This is done by defining a function in the Stan functions block with the special signature:

```
real integrand(real x, real xc, real[] theta,
                       real[] x_r, int[] x_i)
```

The function should return the value of the integrand evaluated at the point x.

The argument of this function are:

- *x*, the independent variable being integrated over
- *xc*, a high precision version of the distance from x to the nearest endpoint in a definite integral (for more into see section Precision Loss).
- *theta*, parameter values used to evaluate the integral
- *x_r*, data values used to evaluate the integral
- *x_i*, integer data used to evaluate the integral

Like algebraic solver and the differential equations solver, the 1D integrator separates parameter values, `theta`, from data values, `x_r`.

Call to the 1D Integrator

`real` **`integrate_1d`** `(function integrand, real a, real b, real[] theta, real[] x_r, int[] x_i)`
Integrates the integrand from a to b.

`real` **`integrate_1d`** `(function integrand, real a, real b, real[] theta, real[] x_r, int[] x_i, real relative_tolerance)`
Integrates the integrand from a to b with the given relative tolerance.

Arguments to the 1D Integrator

The arguments to the 1D integrator are as follows:

- *integrand*: function literal referring to a function specifying the integrand with signature `(real, real, real[], real[], int[]):real` The arguments represent
  - (1) where integrand is evaluated,
  - (2) distance from evaluation point to integration limit for definite integrals,
  - (3) parameters,
  - (4) real data
  - (5) integer data, and the return value is the integrand evaluated at the given point,
- *a*: left limit of integration, may be negative infinity, type `real`,
- *b*: right limit of integration, may be positive infinity, type `real`,
- *theta*: parameters only, type `real[]`,
- *x_r*: real data only, type `real[]`,
- *x_i*: integer data only, type `int[]`.

A `relative_tolerance` argument can optionally be provided for more control over the algorithm:

- *`relative_tolerance`*: relative tolerance for the 1d integrator, type `real`, data only.

Return value

The return value for the 1D integrator is a `real`, the value of the integral.

Zero-crossing integrals

For numeric stability, integrals on the (possibly infinite) interval (a, b) that cross zero are split into two integrals, one from $(a, 0)$ and one from $(0, b)$. Each integral is separately integrated to the given `relative_tolerance`.

Precision loss near limits of integration in definite integrals

When integrating certain definite integrals, there can be significant precision loss in evaluating the integrand near the endpoints. This has to do with the breakdown in precision of double precision floating point values when adding or subtracting a small number from a number much larger than it in magnitude (for instance, `1.0 - x`). `xc` (as passed to the integrand) is a high-precision version of the distance between `x` and the definite integral endpoints and can be used to address this issue. More information (and an example where this is useful) is given in the User's Guide. For zero crossing integrals, `xc` will be a high precision version of the distance to the endpoints of the two smaller integrals. For any integral with an endpoint at negative infinity or positive infinity, `xc` is set to `NaN`.

Algorithmic details

Internally the 1D integrator uses the double-exponential methods in the Boost 1D quadrature library. Boost in turn makes use of quadrature methods developed in (Takahasi and Mori 1974), (Mori 1978), (Bailey, Jeyabalan, and Li 2005), and (Tanaka et al. 2009).

The gradients of the integral are computed in accordance with the Leibniz integral rule. Gradients of the integrand are computed internally with Stan's automatic differentiation.

9.4. Reduce-Sum Function

Stan provides a higher-order reduce function for summation. A function which returns a scalar `g: U -> real` is mapped to every element of a list of type `U[]`, `{ x1, x2, ... }` and all the results are accumulated,

```
g(x1) + g(x2) + ...
```

For efficiency reasons the reduce function doesn't work with the element-wise evaluated function `g` itself, but instead works through evaluating partial sums, `f: U[] -> real`, where:

```
f({ x1 }) = g(x1)
f({ x1, x2 }) = g(x1) + g(x2)
f({ x1, x2, ... }) = g(x1) + g(x2) + ...
```

Mathematically the summation reduction is associative and forming arbitrary partial sums in an aribitrary order will not change the result. However, floating point numerics on computers only have a limited precision such that associativity does not hold exactly. This implies that the order of summation determines the exact numerical result. For this reason, the higher-order reduce function is available in two variants:

- `reduce_sum`: Automatically choose partial sums partitioning based on a dynamic scheduling algorithm.
- `reduce_sum_static`: Compute the same sum as `reduce_sum`, but partition the input in the same way for given data set (in `reduce_sum` this partitioning might change depending on computer load). This should result in stable numerical evaluations.

Specifying the Reduce-sum Function

The higher-order reduce function takes a partial sum function `f`, an array argument `x` (with one array element for each term in the sum), a recommended `grainsize`, and a set of shared arguments. This representation allows parallelization of the resultant sum.

```
real reduce_sum(F f, T[] x, int grainsize, T1 s1, T2 s2, ...)
real reduce_sum_static(F f, T[] x, int grainsize, T1 s1, T2 s2, ...)
```

Returns the equivalent of `f(x, 1, size(x), s1, s2, ...)`, but computes the result in parallel by breaking the array `x` into independent partial sums. `s1, s2, ...` are shared between all terms in the sum.

- *`f`*: function literal referring to a function specifying the partial sum operation. Refer to the partial sum function.

- *x*: array of T, one for each term of the reduction, T can be any type,
- *grainsize*: For reduce_sum, grainsize is the recommended size of the partial sum (grainsize = 1 means pick totally automatically). For reduce_sum_static, grainsize determines the maximum size of the partial sums, type int,
- *s1*: first (optional) shared argument, type T1, where T1 can be any type
- *s2*: second (optional) shared argument, type T2, where T2 can be any type,
- ...: remainder of shared arguments, each of which can be any type.

The Partial sum Function

The partial sum function must have the following signature where the type T, and the types of all the shared arguments (T1, T2, ...) match those of the original reduce_sum (reduce_sum_static) call.

```
(T[] x_subset, int start, int end, T1 s1, T2 s2, ...):real
```

The partial sum function returns the sum of the start to end terms (inclusive) of the overall calculations. The arguments to the partial sum function are:

- *x_subset*, the subset of x a given partial sum is responsible for computing, type T[], where T matches the type of x in reduce_sum (reduce_sum_static)
- *start*, the index of the first term of the partial sum, type int
- *end*, the index of the last term of the partial sum (inclusive), type int
- *s1*, first shared argument, type T1, matching type of s1 in reduce_sum (reduce_sum_static)
- *s2*, second shared argument, type T2, matching type of s2 in reduce_sum (reduce_sum_static)
- ..., remainder of shared arguments, with types matching those in reduce_sum (reduce_sum_static)

9.5. Map-Rect Function

Stan provides a higher-order map function. This allows map-reduce functionality to be coded in Stan as described in the user's guide.

Specifying the Mapped Function

The function being mapped must have a signature identical to that of the function f in the following declaration.

```
vector f(vector phi, vector theta,
         data real[] x_r, data int[] x_i);
```

The map function returns the sequence of results for the particular shard being evaluated. The arguments to the mapped function are:

- *phi*, the sequence of parameters shared across shards
- *theta*, the sequence of parameters specific to this shard
- *x_r*, sequence of real-valued data
- *x_i*, sequence of integer data

All input for the mapped function must be packed into these sequences and all output from the mapped function must be packed into a single vector. The vector of output from each mapped function is concatenated into the final result.

Rectangular Map
The rectangular map function operates on rectangular (not ragged) data structures, with parallel data structures for job-specific parameters, job-specific real data, and job-specific integer data.

`vector` **`map_rect`**`(F f, vector phi, vector[] theta, data real[,] x_r, data int[,] x_i)`
Return the concatenation of the results of applying the function f, of type `(vector, vector, real[], int[]):vector` elementwise, i.e., `f(phi, theta[n], x_r[n], x_i[n])` for each `n` in `1:N`, where `N` is the size of the parallel arrays of job-specific/local parameters `theta`, real data `x_r`, and integer data `x_r`. The shared/global parameters `phi` are passed to each invocation of `f`.

10. Deprecated Functions

This appendix lists currently deprecated functionality along with how to replace it. These deprecated features are likely to be removed in the future.

10.1. integrate_ode_rk45, integrate_ode_adams, integrate_ode_bdf ODE Integrators

These ODE integrator functions have been replaced by those described in:

Specifying an Ordinary Differential Equation as a Function
A system of ODEs is specified as an ordinary function in Stan within the functions block. The ODE system function must have this function signature:

```
real[] ode(real time, real[] state, real[] theta,
         real[] x_r, int[] x_i)
```

The ODE system function should return the derivative of the state with respect to time at the time provided. The length of the returned real array must match the length of the state input into the function.

The arguments to this function are:

- *time*, the time to evaluate the ODE system
- *state*, the state of the ODE system at the time specified
- *theta*, parameter values used to evaluate the ODE system
- *x_r*, data values used to evaluate the ODE system
- *x_i*, integer data values used to evaluate the ODE system.

The ODE system function separates parameter values, *theta*, from data values, *x_r*, for efficiency in computing the gradients of the ODE.

Non-Stiff Solver
`real[ , ] integrate_ode_rk45(function ode, real[] initial_state, real initial_time, real[] times, real[] theta, real[] x_r, int[] x_i)`
Solves the ODE system for the times provided using the Dormand-Prince algorithm, a 4th/5th order Runge-Kutta method.

`real[ , ] integrate_ode_rk45(function ode, real[] initial_state, real initial_time, real[] times, real[] theta, real[] x_r, int[] x_i,`

real rel_tol, real abs_tol, int max_num_steps)
Solves the ODE system for the times provided using the Dormand-Prince algorithm, a 4th/5th order Runge-Kutta method with additional control parameters for the solver.

real[,] **integrate_ode**(function ode, real[] initial_state, real initial_time, real[] times, real[] theta, real[] x_r, int[] x_i)
Solves the ODE system for the times provided using the Dormand-Prince algorithm, a 4th/5th order Runge-Kutta method.

real[,] **integrate_ode_adams**(function ode, real[] initial_state, real initial_time, real[] times, real[] theta, data real[] x_r, data int[] x_i)
Solves the ODE system for the times provided using the Adams-Moulton method.

real[,] **integrate_ode_adams**(function ode, real[] initial_state, real initial_time, real[] times, real[] theta, data real[] x_r, data int[] x_i, data real rel_tol, data real abs_tol, data int max_num_steps)
Solves the ODE system for the times provided using the Adams-Moulton method with additional control parameters for the solver.

Stiff Solver
real[,] **integrate_ode_bdf**(function ode, real[] initial_state, real initial_time, real[] times, real[] theta, data real[] x_r, data int[] x_i)
Solves the ODE system for the times provided using the backward differentiation formula (BDF) method.

real[,] **integrate_ode_bdf**(function ode, real[] initial_state, real initial_time, real[] times, real[] theta, data real[] x_r, data int[] x_i, data real rel_tol, data real abs_tol, data int max_num_steps)
Solves the ODE system for the times provided using the backward differentiation formula (BDF) method with additional control parameters for the solver.

Arguments to the ODE solvers
The arguments to the ODE solvers in both the stiff and non-stiff cases are as follows.

- *ode*: function literal referring to a function specifying the system of differential equations with signature:

(real, real[], real[], data real[], data int[]):real[]

The arguments represent (1) time, (2) system state, (3) parameters, (4) real data, and

(5) integer data, and the return value contains the derivatives with respect to time of the state,

- *initial_state*: initial state, type `real[]`,
- *initial_time*: initial time, type `int` or `real`,
- *times*: solution times, type `real[]`,
- *theta*: parameters, type `real[]`,
- `data` *x_r*: real data, type `real[]`, data only, and
- `data` *x_i*: integer data, type `int[]`, data only.

For more fine-grained control of the ODE solvers, these parameters can also be provided:

- `data` *rel_tol*: relative tolerance for the ODE solver, type `real`, data only,
- `data` *abs_tol*: absolute tolerance for the ODE solver, type `real`, data only, and
- `data` *max_num_steps*: maximum number of steps to take in the ODE solver, type `int`, data only.

Return values

The return value for the ODE solvers is an array of type `real[,]`, with values consisting of solutions at the specified times.

Sizes and parallel arrays

The sizes must match, and in particular, the following groups are of the same size:

- state variables passed into the system function, derivatives returned by the system function, initial state passed into the solver, and rows of the return value of the solver,
- solution times and number of rows of the return value of the solver,
- parameters, real data and integer data passed to the solver will be passed to the system function

Discrete Distributions

11. Conventions for Probability Functions

Functions associated with distributions are set up to follow the same naming conventions for both built-in distributions and for user-defined distributions.

11.1. Suffix Marks Type of Function

The suffix is determined by the type of function according to the following table.

function	outcome	suffix
log probability mass function	discrete	`_lpmf`
log probability density function	continuous	`_lpdf`
log cumulative distribution function	any	`_lcdf`
log complementary cumulative distribution function	any	`_lccdf`
random number generator	any	`_rng`

For example, `normal_lpdf` is the log of the normal probability density function (pdf) and `bernoulli_lpmf` is the log of the bernoulli probability mass function (pmf). The log of the corresponding cumulative distribution functions (cdf) use the same suffix, `normal_lcdf` and `bernoulli_lcdf`.

11.2. Argument Order and the Vertical Bar

Each probability function has a specific outcome value and a number of parameters. Following conditional probability notation, probability density and mass functions use a vertical bar to separate the outcome from the parameters of the distribution. For example, `normal_lpdf(y | mu, sigma)` returns the value of mathematical formula $\log \text{Normal}(y \mid \mu, \sigma)$. Cumulative distribution functions separate the outcome from the parameters in the same way (e.g., `normal_lcdf(y_low | mu, sigma)`

11.3. Sampling Notation

The notation

```
y ~ normal(mu, sigma);
```

provides the same (proportional) contribution to the model log density as the explicit target density increment,

```
target += normal_lpdf(y | mu, sigma);
```

In both cases, the effect is to add terms to the target log density. The only difference is that the example with the sampling (~) notation drops all additive constants in the log density; the constants are not necessary for any of Stan's sampling, approximation, or optimization algorithms.

11.4. Finite Inputs

All of the distribution functions are configured to throw exceptions (effectively rejecting samples or optimization steps) when they are supplied with non-finite arguments. The two cases of non-finite arguments are the infinite values and not-a-number value—these are standard in floating-point arithmetic.

11.5. Boundary Conditions

Many distributions are defined with support or constraints on parameters forming an open interval. For example, the normal density function accepts a scale parameter $\sigma > 0$. If $\sigma = 0$, the probability function will throw an exception.

This is true even for (complementary) cumulative distribution functions, which will throw exceptions when given input that is out of the support.

11.6. Pseudorandom Number Generators

For most of the probability functions, there is a matching pseudorandom number generator (PRNG) with the suffix `_rng`. For example, the function `normal_rng(real, real)` accepts two real arguments, an unconstrained location μ and positive scale $\sigma > 0$, and returns an unconstrained pseudorandom value drawn from Normal(μ, σ). There are also vectorized forms of random number generators which return more than one random variate at a time.

Restricted to Transformed Data and Generated Quantities

Unlike regular functions, the PRNG functions may only be used in the transformed data or generated quantities blocks.

Limited Vectorization

Unlike the probability functions, only some of the PRNG functions are vectorized.

11.7. Cumulative Distribution Functions

For most of the univariate probability functions, there is a corresponding cumulative distribution function, log cumulative distribution function, and log complementary cumulative distribution function.

For a univariate random variable Y with probability function $p_Y(y \mid \theta)$, the cumulative distribution function (CDF) F_Y is defined by

$$F_Y(y) \;=\; \Pr[Y \leq y] \;=\; \int_{-\infty}^{y} p(y \mid \theta)\, \mathrm{d}y.$$

The complementary cumulative distribution function (CCDF) is defined as

$$\Pr[Y > y] \;=\; 1 - F_Y(y).$$

The reason to use CCDFs instead of CDFs in floating-point arithmetic is that it is possible to represent numbers very close to 0 (the closest you can get is roughly 10^{-300}), but not numbers very close to 1 (the closest you can get is roughly $1 - 10^{-15}$).

In Stan, there is a cumulative distribution function for each probability function. For instance, `normal_cdf(y, mu, sigma)` is defined by

$$\int_{-\infty}^{y} \text{Normal}(y \mid \mu, \sigma) \; \mathrm{d}y.$$

There are also log forms of the CDF and CCDF for most univariate distributions. For example, `normal_lcdf(y | mu, sigma)` is defined by

$$\log \left(\int_{-\infty}^{y} \text{Normal}(y \mid \mu, \sigma) \; \mathrm{d}y \right)$$

and `normal_lccdf(y | mu, sigma)` is defined by

$$\log \left(1 - \int_{-\infty}^{y} \text{Normal}(y \mid \mu, \sigma) \; \mathrm{d}y \right).$$

11.8. Vectorization

Stan's univariate log probability functions, including the log density functions, log mass functions, log CDFs, and log CCDFs, all support vectorized function application, with results defined to be the sum of the elementwise application of the function. Some of the PRNG functions support vectorization, see section vectorized PRNG functions for more details.

In all cases, matrix operations are at least as fast and usually faster than loops and vectorized log probability functions are faster than their equivalent form defined with loops. This isn't because loops are slow in Stan, but because more efficient automatic differentiation can be used. The efficiency comes from the fact that a vectorized log probability function only introduces one new node into the expression graph, thus reducing the number of virtual function calls required to compute gradients in C++, as well as from allowing caching of repeated computations.

Stan also overloads the multivariate normal distribution, including the Cholesky-factor form, allowing arrays of row vectors or vectors for the variate and location parameter. This is a huge savings in speed because the work required to solve the linear system for the covariance matrix is only done once.

Stan also overloads some scalar functions, such as `log` and `exp`, to apply to vectors (arrays) and return vectors (arrays). These vectorizations are defined elementwise and unlike the probability functions, provide only minimal efficiency speedups over repeated application and assignment in a loop.

Vectorized Function Signatures

Vectorized Scalar Arguments

The normal probability function is specified with the signature

```
normal_lpdf(reals | reals, reals);
```

The pseudotype `reals` is used to indicate that an argument position may be vectorized. Argument positions declared as `reals` may be filled with a real, a one-dimensional array, a vector, or a row-vector. If there is more than one array or vector argument, their types can be anything but their size must match. For instance, it is legal to use `normal_lpdf(row_vector | vector, real)` as long as the vector and row vector have the same size.

Vectorized Vector and Row Vector Arguments

The multivariate normal distribution accepting vector or array of vector arguments is written as

```
multi_normal_lpdf(vectors | vectors, matrix);
```

These arguments may be row vectors, column vectors, or arrays of row vectors or column vectors.

Vectorized Integer Arguments

The pseudotype `ints` is used for vectorized integer arguments. Where it appears either an integer or array of integers may be used.

Evaluating Vectorized Log Probability Functions

The result of a vectorized log probability function is equivalent to the sum of the evaluations on each element. Any non-vector argument, namely `real` or `int`, is repeated. For instance, if `y` is a vector of size `N`, `mu` is a vector of size `N`, and `sigma` is a scalar, then

```
ll = normal_lpdf(y | mu, sigma);
```

is just a more efficient way to write

```
ll = 0;
```

```
for (n in 1:N)
  ll = ll + normal_lpdf(y[n] | mu[n], sigma);
```

With the same arguments, the vectorized sampling statement

```
y ~ normal(mu, sigma);
```

has the same effect on the total log probability as

```
for (n in 1:N)
  y[n] ~ normal(mu[n], sigma);
```

Evaluating Vectorized PRNG Functions
Some PRNG functions accept sequences as well as scalars as arguments. Such functions are indicated by argument pseudotypes `reals` or `ints`. In cases of sequence arguments, the output will also be a sequence. For example, the following is allowed in the transformed data and generated quantities blocks.

```
vector[3] mu = ...;
real x[3] = normal_rng(mu, 3);
```

Argument types

In the case of PRNG functions, arguments marked `ints` may be integers or integer arrays, whereas arguments marked `reals` may be integers or reals, integer or real arrays, vectors, or row vectors.

pseudotype	allowable PRNG arguments
`ints`	`int`, `int[]`
`reals`	`int`, `int[]`, `real`, `real[]`, `vector`, `row_vector`

Dimension matching

In general, if there are multiple non-scalar arguments, they must all have the same dimensions, but need not have the same type. For example, the `normal_rng` function may be called with one vector argument and one real array argument as long as they have the same number of elements.

```
vector[3] mu = ...;
real sigma[3] = ...;
real x[3] = normal_rng(mu, sigma);
```

Return type

The result of a vectorized PRNG function depends on the size of the arguments and the distribution's support. If all arguments are scalars, then the return type is a scalar. For a continuous distribution, if there are any non-scalar arguments, the return type is a real array (`real[]`) matching the size of any of the non-scalar arguments, as all non-scalar arguments must have matching size. Discrete distributions return `ints` and continuous distributions return `reals`, each of appropriate size. The symbol `R` denotes such a return type.

12. Binary Distributions

Binary probability distributions have support on $\{0, 1\}$, where 1 represents the value true and 0 the value false.

12.1. Bernoulli Distribution

Probability Mass Function

If $\theta \in [0, 1]$, then for $y \in \{0, 1\}$,

$$\text{Bernoulli}(y \mid \theta) = \begin{cases} \theta & \text{if } y = 1, \text{ and} \\ 1 - \theta & \text{if } y = 0. \end{cases}$$

Sampling Statement

`y ~ bernoulli(theta)`

Increment target log probability density with `bernoulli_lpmf(y | theta)` dropping constant additive terms.

Stan Functions

`real bernoulli_lpmf(ints y | reals theta)`
The log Bernoulli probability mass of y given chance of success theta

`real bernoulli_cdf(ints y, reals theta)`
The Bernoulli cumulative distribution function of y given chance of success theta

`real bernoulli_lcdf(ints y | reals theta)`
The log of the Bernoulli cumulative distribution function of y given chance of success theta

`real bernoulli_lccdf(ints y | reals theta)`
The log of the Bernoulli complementary cumulative distribution function of y given chance of success theta

`R bernoulli_rng(reals theta)`
Generate a Bernoulli variate with chance of success theta; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

12.2. Bernoulli Distribution, Logit Parameterization

Stan also supplies a direct parameterization in terms of a logit-transformed chance-of-success parameter. This parameterization is more numerically stable if the chance-

of-success parameter is on the logit scale, as with the linear predictor in a logistic regression.

Probability Mass Function
If $\alpha \in \mathbb{R}$, then for $y \in \{0, 1\}$,

$$\text{BernoulliLogit}(y \mid \alpha) = \text{Bernoulli}(y|\text{logit}^{-1}(\alpha)) = \begin{cases} \text{logit}^{-1}(\alpha) & \text{if } y = 1, \text{ and} \\ 1 - \text{logit}^{-1}(\alpha) & \text{if } y = 0. \end{cases}$$

Sampling Statement
`y ~ bernoulli_logit(alpha)`

Increment target log probability density with `bernoulli_logit_lpmf(y | alpha)` dropping constant additive terms.

Stan Functions
`real bernoulli_logit_lpmf(ints y | reals alpha)`
The log Bernoulli probability mass of y given chance of success `inv_logit(alpha)`

`R bernoulli_logit_rng(reals alpha)`
Generate a Bernoulli variate with chance of success $\text{logit}^{-1}(\alpha)$; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

12.3. Bernoulli-Logit Generalized Linear Model (Logistic Regression)

Stan also supplies a single function for a generalized linear model with Bernoulli likelihood and logit link function, i.e. a function for a logistic regression. This provides a more efficient implementation of logistic regression than a manually written regression in terms of a Bernoulli likelihood and matrix multiplication.

Probability Mass Function
If $x \in \mathbb{R}^{n \cdot m}, \alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^m$, then for $y \in \{0, 1\}^n$,

$$\text{BernoulliLogitGLM}(y \mid x, \alpha, \beta) = \prod_{1 \le i \le n} \text{Bernoulli}(y_i \mid \text{logit}^{-1}(\alpha_i + x_i \cdot \beta))$$

$$= \prod_{1 \le i \le n} \begin{cases} \text{logit}^{-1}(\alpha_i + \sum_{1 \le j \le m} x_{ij} \cdot \beta_j) & \text{if } y_i = 1, \text{ and} \\ 1 - \text{logit}^{-1}(\alpha_i + \sum_{1 \le j \le m} x_{ij} \cdot \beta_j) & \text{if } y_i = 0. \end{cases}$$

Sampling Statement
`y ~ bernoulli_logit_glm(x, alpha, beta)`

Increment target log probability density with `bernoulli_logit_glm_lpmf(y | x, alpha, beta)` dropping constant additive terms.

Stan Functions

`real` **`bernoulli_logit_glm_lpmf`**`(int y | matrix x, real alpha, vector beta)`

The log Bernoulli probability mass of `y` given chance of success `inv_logit(alpha + x * beta)`, where the same intercept `alpha` and dependant variable value `y` are used for all observations. The number of columns of `x` needs to match the size of the coefficient vector `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

`real` **`bernoulli_logit_glm_lpmf`**`(int y | matrix x, vector alpha, vector beta)`

The log Bernoulli probability mass of `y` given chance of success `inv_logit(alpha + x * beta)`, where an intercept `alpha` is used that is allowed to vary by observation. The dependant variable value `y` is used for all observations. The number of rows of `x` must match the size of `alpha` and the number of columns of `x` needs to match the size of the coefficient vector `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

`real` **`bernoulli_logit_glm_lpmf`**`(int[] y | row_vector x, real alpha, vector beta)`

The log Bernoulli probability mass of `y` given chance of success `inv_logit(alpha + x * beta)`, where the same intercept `alpha` and same independent variables values `x` are used for all observations. The number of columns of `x` needs to match the size of the coefficient vector `beta`.

`real` **`bernoulli_logit_glm_lpmf`**`(int[] y | row_vector x, vector alpha, vector beta)`

The log Bernoulli probability mass of `y` given chance of success `inv_logit(alpha + x * beta)`, where an intercept `alpha` is used that is allowed to vary by observation. The same independent variables values `x` are used for all observations. The size of `y` must match the size of `alpha` and the number of columns of `x` needs to match the size of the coefficient vector `beta`.

`real` **`bernoulli_logit_glm_lpmf`**`(int[] y | matrix x, real alpha, vector beta)`

The log Bernoulli probability mass of `y` given chance of success `inv_logit(alpha + x * beta)`, where the same intercept `alpha` is used for all observations. The number of rows of the independent variable matrix `x` needs to match the size of the dependent variable vector `y` and the number of columns of `x` needs to match the size of the coefficient vector `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

`real` **`bernoulli_logit_glm_lpmf`**`(int[] y | matrix x, vector alpha, vector beta)`
The log Bernoulli probability mass of y given chance of success `inv_logit(alpha + x * beta)`, where an intercept `alpha` is used that is allowed to vary by observation. The number of rows of the independent variable matrix `x` needs to match the size of the dependent variable vector `y` and `alpha` and the number of columns of `x` needs to match the size of the coefficient vector `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

13. Bounded Discrete Distributions

Bounded discrete probability functions have support on $\{0, \ldots, N\}$ for some upper bound N.

13.1. Binomial Distribution

Probability Mass Function

Suppose $N \in \mathbb{N}$ and $\theta \in [0, 1]$, and $n \in \{0, \ldots, N\}$.

$$\text{Binomial}(n \mid N, \theta) = \binom{N}{n} \theta^n (1 - \theta)^{N-n}.$$

Log Probability Mass Function

$$\begin{aligned} \log \text{Binomial}(n \mid N, \theta) &= \log \Gamma(N + 1) - \log \Gamma(n + 1) - \log \Gamma(N - n + 1) \\ &\quad + n \log \theta + (N - n) \log(1 - \theta), \end{aligned}$$

Gradient of Log Probability Mass Function

$$\frac{\partial}{\partial \theta} \log \text{Binomial}(n \mid N, \theta) = \frac{n}{\theta} - \frac{N - n}{1 - \theta}$$

Sampling Statement

`n ~ binomial(N, theta)`

Increment target log probability density with `binomial_lpmf(n | N, theta)` dropping constant additive terms.

Stan Functions

`real binomial_lpmf(ints n | ints N, reals theta)`
The log binomial probability mass of n successes in N trials given chance of success theta

`real binomial_cdf(ints n, ints N, reals theta)`
The binomial cumulative distribution function of n successes in N trials given chance of success theta

`real` **`binomial_lcdf`**`(ints n | ints N, reals theta)`
The log of the binomial cumulative distribution function of n successes in N trials given chance of success theta

`real` **`binomial_lccdf`**`(ints n | ints N, reals theta)`
The log of the binomial complementary cumulative distribution function of n successes in N trials given chance of success theta

`R` **`binomial_rng`**`(ints N, reals theta)`
Generate a binomial variate with N trials and chance of success theta; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

13.2. Binomial Distribution, Logit Parameterization

Stan also provides a version of the binomial probability mass function distribution with the chance of success parameterized on the unconstrained logistic scale.

Probability Mass Function

Suppose $N \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and $n \in \{0, \ldots, N\}$. Then

$$\begin{aligned} \text{BinomialLogit}(n \mid N, \alpha) &= \text{Binomial}(n \mid N, \text{logit}^{-1}(\alpha)) \\ &= \binom{N}{n} \left(\text{logit}^{-1}(\alpha)\right)^n \left(1 - \text{logit}^{-1}(\alpha)\right)^{N-n}. \end{aligned}$$

Log Probability Mass Function

$$\begin{aligned} \log \text{BinomialLogit}(n \mid N, \alpha) &= \log \Gamma(N+1) - \log \Gamma(n+1) - \log \Gamma(N-n+1) \\ &\quad + n \log \text{logit}^{-1}(\alpha) + (N-n) \log \left(1 - \text{logit}^{-1}(\alpha)\right), \end{aligned}$$

Gradient of Log Probability Mass Function

$$\frac{\partial}{\partial \alpha} \log \text{BinomialLogit}(n \mid N, \alpha) = \frac{n}{\text{logit}^{-1}(-\alpha)} - \frac{N-n}{\text{logit}^{-1}(\alpha)}$$

Sampling Statement

`n ~` **`binomial_logit`**`(N, alpha)`

Increment target log probability density with `binomial_logit_lpmf(n | N, alpha)` dropping constant additive terms.

Stan Functions

`real` **`binomial_logit_lpmf`**`(ints n | ints N, reals alpha)`

The log binomial probability mass of n successes in N trials given logit-scaled chance of success alpha

13.3. Beta-Binomial Distribution

Probability Mass Function

If $N \in \mathbb{N}$, $\alpha \in \mathbb{R}^+$, and $\beta \in \mathbb{R}^+$, then for $n \in 0, \ldots, N$,

$$\text{BetaBinomial}(n \mid N, \alpha, \beta) = \binom{N}{n} \frac{\text{B}(n + \alpha, N - n + \beta)}{\text{B}(\alpha, \beta)},$$

where the beta function $\text{B}(u, v)$ is defined for $u \in \mathbb{R}^+$ and $v \in \mathbb{R}^+$ by

$$\text{B}(u, v) = \frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u + v)}.$$

Sampling Statement

`n ~` **`beta_binomial`**`(N, alpha, beta)`

Increment target log probability density with `beta_binomial_lpmf(n | N, alpha, beta)` dropping constant additive terms.

Stan Functions

`real` **`beta_binomial_lpmf`**`(ints n | ints N, reals alpha, reals beta)`

The log beta-binomial probability mass of n successes in N trials given prior success count (plus one) of alpha and prior failure count (plus one) of beta

`real` **`beta_binomial_cdf`**`(ints n, ints N, reals alpha, reals beta)`

The beta-binomial cumulative distribution function of n successes in N trials given prior success count (plus one) of alpha and prior failure count (plus one) of beta

`real` **`beta_binomial_lcdf`**`(ints n | ints N, reals alpha, reals beta)`

The log of the beta-binomial cumulative distribution function of n successes in N trials given prior success count (plus one) of alpha and prior failure count (plus one) of beta

`real` **`beta_binomial_lccdf`**`(ints n | ints N, reals alpha, reals beta)`

The log of the beta-binomial complementary cumulative distribution function of n successes in N trials given prior success count (plus one) of alpha and prior failure count (plus one) of beta

`R` **`beta_binomial_rng`**`(ints N, reals alpha, reals beta)`

Generate a beta-binomial variate with N trials, prior success count (plus one) of alpha, and prior failure count (plus one) of beta; may only be used in transformed data and

generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

13.4. Hypergeometric Distribution

Probability Mass Function

If $a \in \mathbb{N}$, $b \in \mathbb{N}$, and $N \in \{0, \ldots, a+b\}$, then for $n \in \{\max(0, N-b), \ldots, \min(a, N)\}$,

$$\text{Hypergeometric}(n \mid N, a, b) = \frac{\binom{a}{n}\binom{b}{N-n}}{\binom{a+b}{N}}.$$

Sampling Statement

`n ~ hypergeometric(N, a, b)`

Increment target log probability density with `hypergeometric_lpmf(n | N, a, b)` dropping constant additive terms.

Stan Functions

`real hypergeometric_lpmf(int n ~|~ int N, int a, int b)`

The log hypergeometric probability mass of n successes in N trials given total success count of a and total failure count of b

`int hypergeometric_rng(int N, int a, int2 b)`

Generate a hypergeometric variate with N trials, total success count of a, and total failure count of b; may only be used in transformed data and generated quantities blocks

13.5. Categorical Distribution

Probability Mass Functions

If $N \in \mathbb{N}$, $N > 0$, and if $\theta \in \mathbb{R}^N$ forms an N-simplex (i.e., has nonnegative entries summing to one), then for $y \in \{1, \ldots, N\}$,

$$\text{Categorical}(y \mid \theta) = \theta_y.$$

In addition, Stan provides a log-odds scaled categorical distribution,

$$\text{CategoricalLogit}(y \mid \beta) = \text{Categorical}(y \mid \text{softmax}(\beta)).$$

See the definition of softmax for the definition of the softmax function.

Sampling Statement

`y ~ categorical(theta)`

Increment target log probability density with `categorical_lpmf(y | theta)` dropping constant additive terms.

Sampling Statement
`y ~ categorical_logit(beta)`

Increment target log probability density with `categorical_logit_lpmf(y | beta)` dropping constant additive terms.

Stan Functions
All of the categorical distributions are vectorized so that the outcome y can be a single integer (type `int`) or an array of integers (type `int[]`).

`real categorical_lpmf(ints y | vector theta)`
The log categorical probability mass function with outcome(s) y in $1:N$ given N-vector of outcome probabilities theta. The parameter theta must have non-negative entries that sum to one, but it need not be a variable declared as a simplex.

`real categorical_logit_lpmf(ints y | vector beta)`
The log categorical probability mass function with outcome(s) y in $1:N$ given log-odds of outcomes beta.

`int categorical_rng(vector theta)`
Generate a categorical variate with N-simplex distribution parameter theta; may only be used in transformed data and generated quantities blocks

`int categorical_logit_rng(vector beta)`
Generate a categorical variate with outcome in range $1:N$ from log-odds vector beta; may only be used in transformed data and generated quantities blocks

13.6. Categorical Logit Generalized Linear Model (Softmax Regression)

Stan also supplies a single function for a generalized linear model with categorical likelihood and logit link function, i.e. a function for a softmax regression. This provides a more efficient implementation of softmax regression than a manually written regression in terms of a Categorical likelihood and matrix multiplication.

Note that the implementation does not put any restrictions on the coefficient matrix β. It is up to the user to use a reference category, a suitable prior or some other means of identifiability. See Multi-logit in the Stan User's Guide.

Probability Mass Functions
If $N, M, K \in \mathbb{N}$, $N, M, K > 0$, and if $x \in \mathbb{R}^{M \cdot K}, \alpha \in \mathbb{R}^N, \beta \in \mathbb{R}^{K \cdot N}$, then for $y \in \{1, \ldots, N\}^M$,

$$\text{CategoricalLogitGLM}(y \mid x, \alpha, \beta) = \prod_{1 \leq i \leq M} \text{CategoricalLogit}(y_i \mid \alpha + x_i \cdot \beta) = \prod_{1 \leq i \leq M} \text{Categorical}(y_i \mid s$$

See the definition of softmax for the definition of the softmax function.

Sampling Statement
`y ~ categorical_logit_glm(x, alpha, beta)`

Increment target log probability density with `categorical_logit_glm(y | x, alpha, beta)` dropping constant additive terms.

Stan Functions
`real categorical_logit_glm_lpmf(int y | row_vector x, vector alpha, matrix beta)`
The log categorical probability mass function with outcome `y` in $1:N$ given N-vector of log-odds of outcomes `alpha + x * beta`. The size of the independent variable row vector `x` needs to match the number of rows of the coefficient matrix `beta`. The size of the intercept vector `alpha` must match the number of columns of the coefficient matrix `beta`.

`real categorical_logit_glm_lpmf(int y | matrix x, vector alpha, matrix beta)`
The log categorical probability mass function with outcomes `y` in $1:N$ given N-vector of log-odds of outcomes `alpha + x * beta`. The same vector of intercepts `alpha` and the same dependent variable value `y` are used for all instances. The number of columns of the independent variable `x` needs to match the number of rows of the coefficient matrix `beta`. The size of the intercept vector `alpha` must match the number of columns of the coefficient matrix `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

`real categorical_logit_glm_lpmf(int[] y | row_vector x, vector alpha, matrix beta)`
The log categorical probability mass function with outcomes `y` in $1:N$ given N-vector of log-odds of outcomes `alpha + x * beta`. The same vector of intercepts `alpha` and same row vector of the independent variables `x` are used for all instances. The size of the independent variable matrix `x` needs to match the number of rows of the coefficient vector `beta`. The size of the intercept vector `alpha` must match the number of columns of the coefficient vector `beta`.

`real categorical_logit_glm_lpmf(int[] y | matrix x, vector alpha, matrix beta)`
The log categorical probability mass function with outcomes `y` in $1:N$ given N-vector of log-odds of outcomes `alpha + x * beta`. The same vector of intercepts `alpha` is used for all instances. The number of rows of the independent variable matrix `x` needs to match the size of the dependent variable vector `y`. The number of columns of independnt variable `x` needs to match the number of rows of the coefficient matrix

beta. The size of the intercept vector alpha must match the number of columns of the coefficient matrix beta. If x and y are data (not parameters) this function can be executed on a GPU.

13.7. Ordered Logistic Distribution

Probability Mass Function

If $K \in \mathbb{N}$ with $K > 2$, $c \in \mathbb{R}^{K-1}$ such that $c_k < c_{k+1}$ for $k \in \{1, \ldots, K-2\}$, and $\eta \in \mathbb{R}$, then for $k \in \{1, \ldots, K\}$,

$$\text{OrderedLogistic}(k \mid \eta, c) = \begin{cases} 1 - \text{logit}^{-1}(\eta - c_1) & \text{if } k = 1, \\ \text{logit}^{-1}(\eta - c_{k-1}) - \text{logit}^{-1}(\eta - c_k) & \text{if } 1 < k < K, \text{and} \\ \text{logit}^{-1}(\eta - c_{K-1}) - 0 & \text{if } k = K. \end{cases}$$

The $k = K$ case is written with the redundant subtraction of zero to illustrate the parallelism of the cases; the $k = 1$ and $k = K$ edge cases can be subsumed into the general definition by setting $c_0 = -\infty$ and $c_K = +\infty$ with $\text{logit}^{-1}(-\infty) = 0$ and $\text{logit}^{-1}(\infty) = 1$.

Sampling Statement

k ~ **ordered_logistic**(eta, c)

Increment target log probability density with ordered_logistic_lpmf(k | eta, c) dropping constant additive terms.

Stan Functions

real **ordered_logistic_lpmf**(ints k | vector eta, vectors c)

The log ordered logistic probability mass of k given linear predictors eta, and cutpoints c.

int **ordered_logistic_rng**(real eta, vector c)

Generate an ordered logistic variate with linear predictor eta and cutpoints c; may only be used in transformed data and generated quantities blocks

13.8. Ordered Logistic Generalized Linear Model (Ordinal Regression)

Probability Mass Function

If $N, M, K \in \mathbb{N}$ with $N, M > 0$, $K > 2$, $c \in \mathbb{R}^{K-1}$ such that $c_k < c_{k+1}$ for $k \in \{1, \ldots, K-2\}$, and $x \in \mathbb{R}^{N \cdot M}$, $\beta \in \mathbb{R}^M$, then for $y \in \{1, \ldots, K\}^N$,

$$\text{OrderedLogisticGLM}(y \mid x, \beta, c) = \prod_{1 \le i \le N} \text{OrderedLogistic}(y_i \mid x_i \cdot \beta, c) = \prod_{1 \le i \le N} \begin{cases} 1 - \text{logit}^{-1}(x_i \cdot \beta \\ \text{logit}^{-1}(x_i \cdot \beta - c \\ \text{logit}^{-1}(x_i \cdot \beta - c \end{cases}$$

The $k = K$ case is written with the redundant subtraction of zero to illustrate the parallelism of the cases; the $y = 1$ and $y = K$ edge cases can be subsumed into the general definition by setting $c_0 = -\infty$ and $c_K = +\infty$ with $\text{logit}^{-1}(-\infty) = 0$ and $\text{logit}^{-1}(\infty) = 1$.

Sampling Statement

`y ~ ordered_logistic_glm(x, beta, c)`

Increment target log probability density with `ordered_logistic_lpmf(y | x, beta, c)` dropping constant additive terms.

Stan Functions

`real ordered_logistic_glm_lpmf(int y | row_vector x, vector beta, vector c)`

The log ordered logistic probability mass of y, given linear predictors `x * beta`, and cutpoints c. The size of the independent variable row vector `x` needs to match the size of the coefficient vector `beta`. The cutpoints `c` must be ordered.

`real ordered_logistic_glm_lpmf(int y | matrix x, vector beta, vector c)`

The log ordered logistic probability mass of y, given linear predictors `x * beta`, and cutpoints c. The same value of the independent variable `y` is used for all instances. The number of columns of the independent variable row vector `x` needs to match the size of the coefficient vector `beta`. The cutpoints `c` must be ordered. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

`real ordered_logistic_glm_lpmf(int[] y | row_vector x, vector beta, vector c)`

The log ordered logistic probability mass of y, given linear predictors `x * beta`, and cutpoints c. The same row vector of the independent variables `x` is used for all instances. The size of the independent variable row vector `x` needs to match the size of the coefficient vector `beta`. The cutpoints `c` must be ordered.

`real ordered_logistic_glm_lpmf(int[] y | matrix x, vector beta, vector c)`

The log ordered logistic probability mass of y, given linear predictors `x * beta`, and cutpoints c. The number of rows of the independent variable matrix `x` needs to match the size of the dependent variable vector `y`. The number of columns of the independent variable row vector `x` needs to match the size of the coefficient vector `beta`. The cutpoints `c` must be ordered. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

13.9. Ordered Probit Distribution

Probability Mass Function

If $K \in \mathbb{N}$ with $K > 2$, $c \in \mathbb{R}^{K-1}$ such that $c_k < c_{k+1}$ for $k \in \{1, \ldots, K-2\}$, and $\eta \in \mathbb{R}$, then for $k \in \{1, \ldots, K\}$,

$$\text{OrderedProbit}(k \mid \eta, c) = \begin{cases} 1 - \Phi(\eta - c_1) & \text{if } k = 1, \\ \Phi(\eta - c_{k-1}) - \Phi(\eta - c_k) & \text{if } 1 < k < K, \text{and} \\ \Phi(\eta - c_{K-1}) - 0 & \text{if } k = K. \end{cases}$$

The $k = K$ case is written with the redundant subtraction of zero to illustrate the parallelism of the cases; the $k = 1$ and $k = K$ edge cases can be subsumed into the general definition by setting $c_0 = -\infty$ and $c_K = +\infty$ with $\Phi(-\infty) = 0$ and $\Phi(\infty) = 1$.

Sampling Statement

`k ~ ordered_probit(eta, c)`

Increment target log probability density with `ordered_probit_lpmf(k | eta, c)` dropping constant additive terms.

Stan Functions

`real ordered_probit_lpmf(ints k | vector eta, vectors c)`
The log ordered probit probability mass of k given linear predictors eta, and cutpoints c.

`int ordered_probit_rng(real eta, vector c)`
Generate an ordered probit variate with linear predictor eta and cutpoints c; may only be used in transformed data and generated quantities blocks

14. Unbounded Discrete Distributions

The unbounded discrete distributions have support over the natural numbers (i.e., the non-negative integers).

14.1. Negative Binomial Distribution

For the negative binomial distribution Stan uses the parameterization described in Gelman et al. (2013). For alternative parameterizations, see section negative binomial glm.

Probability Mass Function

If $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$, then for $n \in \mathbb{N}$,

$$\text{NegBinomial}(n \mid \alpha, \beta) = \binom{n + \alpha - 1}{\alpha - 1} \left(\frac{\beta}{\beta + 1}\right)^{\alpha} \left(\frac{1}{\beta + 1}\right)^{n}.$$

The mean and variance of a random variable $n \sim \text{NegBinomial}(\alpha, \beta)$ are given by

$$\mathbb{E}[n] = \frac{\alpha}{\beta} \quad \text{and} \quad \text{Var}[n] = \frac{\alpha}{\beta^2}(\beta + 1).$$

Sampling Statement

`n ~ neg_binomial(alpha, beta)`

Increment target log probability density with `neg_binomial_lpmf(n | alpha, beta)` dropping constant additive terms.

Stan Functions

`real neg_binomial_lpmf(ints n | reals alpha, reals beta)`
The log negative binomial probability mass of `n` given shape `alpha` and inverse scale `beta`

`real neg_binomial_cdf(ints n, reals alpha, reals beta)`
The negative binomial cumulative distribution function of `n` given shape `alpha` and inverse scale `beta`

`real neg_binomial_lcdf(ints n | reals alpha, reals beta)`
The log of the negative binomial cumulative distribution function of `n` given shape `alpha` and inverse scale `beta`

`real` **`neg_binomial_lccdf`**`(ints n | reals alpha, reals beta)`
The log of the negative binomial complementary cumulative distribution function of `n` given shape `alpha` and inverse scale `beta`

`R` **`neg_binomial_rng`**`(reals alpha, reals beta)`
Generate a negative binomial variate with shape `alpha` and inverse scale `beta`; may only be used in transformed data and generated quantities blocks. `alpha` / `beta` must be less than 2^{29}. For a description of argument and return types, see section vectorized function signatures.

14.2. Negative Binomial Distribution (alternative parameterization)

Stan also provides an alternative parameterization of the negative binomial distribution directly using a mean (i.e., location) parameter and a parameter that controls overdispersion relative to the square of the mean. Section combinatorial functions, below, provides a second alternative parameterization directly in terms of the log mean.

Probability Mass Function
The first parameterization is for $\mu \in \mathbb{R}^+$ and $\phi \in \mathbb{R}^+$, which for $n \in \mathbb{N}$ is defined as

$$\text{NegBinomial2}(n \mid \mu, \phi) = \binom{n + \phi - 1}{n} \left(\frac{\mu}{\mu + \phi}\right)^n \left(\frac{\phi}{\mu + \phi}\right)^{\phi}.$$

The mean and variance of a random variable $n \sim \text{NegBinomial2}(n \mid \mu, \phi)$ are

$$\mathbb{E}[n] = \mu \quad \text{and} \quad \text{Var}[n] = \mu + \frac{\mu^2}{\phi}.$$

Recall that $\text{Poisson}(\mu)$ has variance μ, so $\mu^2/\phi > 0$ is the additional variance of the negative binomial above that of the Poisson with mean μ. So the inverse of parameter ϕ controls the overdispersion, scaled by the square of the mean, μ^2.

Sampling Statement
`n ~` **`neg_binomial_2`**`(mu, phi)`

Increment target log probability density with `neg_binomial_2_lpmf(n | mu, phi)` dropping constant additive terms.

Stan Functions
`real` **`neg_binomial_2_lpmf`**`(ints n | reals mu, reals phi)`
The negative binomial probability mass of `n` given location `mu` and precision `phi`.

`real` **`neg_binomial_2_cdf`**`(ints n, reals mu, reals phi)`
The negative binomial cumulative distribution function of `n` given location `mu` and precision `phi`.

`real` **`neg_binomial_2_lcdf`**`(ints n | reals mu, reals phi)`
The log of the negative binomial cumulative distribution function of `n` given location `mu` and precision `phi`.

`real` **`neg_binomial_2_lccdf`**`(ints n | reals mu, reals phi)`
The log of the negative binomial complementary cumulative distribution function of `n` given location `mu` and precision `phi`.

`R` **`neg_binomial_2_rng`**`(reals mu, reals phi)`
Generate a negative binomial variate with location `mu` and precision `phi`; may only be used in transformed data and generated quantities blocks. `mu` must be less than 2^{29}. For a description of argument and return types, see section vectorized function signatures.

14.3. Negative Binomial Distribution (log alternative parameterization)

Related to the parameterization in section negative binomial, alternative parameterization, the following parameterization uses a log mean parameter $\eta = \log(\mu)$, defined for $\eta \in \mathbb{R}$, $\phi \in \mathbb{R}^+$, so that for $n \in \mathbb{N}$,

$$\text{NegBinomial2Log}(n \mid \eta, \phi) = \text{NegBinomial2}(n \mid \exp(\eta), \phi).$$

This alternative may be used for sampling, as a function, and for random number generation, but as of yet, there are no CDFs implemented for it.

Sampling Statement
`n ~` **`neg_binomial_2_log`**`(eta, phi)`

Increment target log probability density with `neg_binomial_2_log_lpmf(n | eta, phi)` dropping constant additive terms.

Stan Functions
`real` **`neg_binomial_2_log_lpmf`**`(ints n | reals eta, reals phi)`
The log negative binomial probability mass of `n` given log-location `eta` and inverse overdispersion parameter `phi`. This is especially useful for log-linear negative binomial regressions.

`R` **`neg_binomial_2_log_rng`**`(reals eta, reals phi)`
Generate a negative binomial variate with log-location `eta` and inverse overdispersion control `phi`; may only be used in transformed data and generated quantities blocks. `eta` must be less than $29 \log 2$. For a description of argument and return types, see section vectorized function signatures.

14.4. Negative-Binomial-2-Log Generalized Linear Model (Negative Binomial Regression)

Stan also supplies a single function for a generalized linear model with negative binomial likelihood and log link function, i.e. a function for a negative binomial regression. This provides a more efficient implementation of negative binomial regression than a manually written regression in terms of a negative binomial likelihood and matrix multiplication.

Probability Mass Function

If $x \in \mathbb{R}^{n \cdot m}$, $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}^m$, $\phi \in \mathbb{R}^+$, then for $y \in \mathbb{N}^n$,

$$\text{NegBinomial2LogGLM}(y \mid x, \alpha, \beta, \phi) = \prod_{1 \leq i \leq n} \text{NegBinomial2}(y_i \mid \exp(\alpha_i + x_i \cdot \beta), \phi).$$

Sampling Statement

`y ~ neg_binomial_2_log_glm(x, alpha, beta, phi)`

Increment target log probability density with `neg_binomial_2_log_glm_lpmf(y | x, alpha, beta, phi)` dropping constant additive terms.

Stan Functions

`real neg_binomial_2_log_glm_lpmf(int y | matrix x, real alpha, vector beta, real phi)`

The log negative binomial probability mass of `y` given log-location `alpha + x * beta` and inverse overdispersion parameter `phi`, where the same intercept `alpha`, inverse overdispersion parameter `phi` and dependant variable `y` are used for all observations. The number of columns of `x` needs to match the size of the coefficient vector `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

`real neg_binomial_2_log_glm_lpmf(int y | matrix x, vector alpha, vector beta, real phi)`

The log negative binomial probability mass of `y` given log-location `alpha + x * beta` and inverse overdispersion parameter `phi`, where the same inverse overdispersion parameter `phi` and independent variable value `y` are used for all observations. Intercept `alpha` is used that is allowed to vary by observation. The number of rows of `x` must match the size of `alpha` and the number of columns of `x` needs to match the size of the coefficient vector `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

`real neg_binomial_2_log_glm_lpmf(int[] y | row_vector x, real alpha, vector beta, real phi)`

The log negative binomial probability mass of `y` given log-location `alpha + x * beta` and inverse overdispersion parameter `phi`, where the same intercept `alpha`, inverse

overdispersion parameter `phi` and independent variable values `x` are used for all observations. The number of columns of `x` needs to match the size of the coefficient vector `beta`.

`real` **`neg_binomial_2_log_glm_lpmf`**`(int[] y | row_vector x, vector alpha, vector beta, real phi)`
The log negative binomial probability mass of `y` given log-location `alpha + x * beta` and inverse overdispersion parameter `phi`, where the same inverse overdispersion parameter `phi` and independent variable values `x` are used for all observations. Intercept `alpha` is used that is allowed to vary by observation. The size of `y` must match the size of `alpha` and the number of columns of `x` needs to match the size of the coefficient vector `beta`.

`real` **`neg_binomial_2_log_glm_lpmf`**`(int[] y | matrix x, real alpha, vector beta, real phi)`
The log negative binomial probability mass of `y` given log-location `alpha + x * beta` and inverse overdispersion parameter `phi`, where the same intercept `alpha` and inverse overdispersion parameter `phi` is used for all observations. The number of rows of the independent variable matrix `x` needs to match the size of the dependent variable vector `y` and the number of columns of `x` needs to match the size of the coefficient vector `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

`real` **`neg_binomial_2_log_glm_lpmf`**`(int[] y | matrix x, vector alpha, vector beta, real phi)`
The log negative binomial probability mass of `y` given log-location `alpha + x * beta` and inverse overdispersion parameter `phi`, where the same inverse overdispersion parameter `phi` is used for all observations and an intercept `alpha` is used that is allowed to vary by observation. The number of rows of the independent variable matrix `x` needs to match the size of the dependent variable vector `y` and `alpha` and the number of columns of `x` needs to match the size of the coefficient vector `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

14.5. Poisson Distribution

Probability Mass Function
If $\lambda \in \mathbb{R}^+$, then for $n \in \mathbb{N}$,

$$\text{Poisson}(n|\lambda) = \frac{1}{n!}\,\lambda^n \exp(-\lambda).$$

Sampling Statement
`n ~` **`poisson`**`(lambda)`

Increment target log probability density with `poisson_lpmf(n | lambda)` dropping constant additive terms.

Stan Functions

`real` **`poisson_lpmf`**`(ints n | reals lambda)`
The log Poisson probability mass of n given rate lambda

`real` **`poisson_cdf`**`(ints n, reals lambda)`
The Poisson cumulative distribution function of n given rate lambda

`real` **`poisson_lcdf`**`(ints n | reals lambda)`
The log of the Poisson cumulative distribution function of n given rate lambda

`real` **`poisson_lccdf`**`(ints n | reals lambda)`
The log of the Poisson complementary cumulative distribution function of n given rate lambda

`R` **`poisson_rng`**`(reals lambda)`
Generate a Poisson variate with rate lambda; may only be used in transformed data and generated quantities blocks. lambda must be less than 2^{30}. For a description of argument and return types, see section vectorized function signatures.

14.6. Poisson Distribution, Log Parameterization

Stan also provides a parameterization of the Poisson using the log rate $\alpha = \log \lambda$ as a parameter. This is useful for log-linear Poisson regressions so that the predictor does not need to be exponentiated and passed into the standard Poisson probability function.

Probability Mass Function

If $\alpha \in \mathbb{R}$, then for $n \in \mathbb{N}$,

$$\text{PoissonLog}(n|\alpha) = \frac{1}{n!} \exp\left(n\alpha - \exp(\alpha)\right).$$

Sampling Statement

`n ~` **`poisson_log`**`(alpha)`

Increment target log probability density with `poisson_log_lpmf(n | alpha)` dropping constant additive terms.

Stan Functions

`real` **`poisson_log_lpmf`**`(ints n | reals alpha)`
The log Poisson probability mass of n given log rate alpha

`R` **`poisson_log_rng`**`(reals alpha)`
Generate a Poisson variate with log rate alpha; may only be used in transformed data

and generated quantities blocks. alpha must be less than $30 \log 2$. For a description of argument and return types, see section vectorized function signatures.

14.7. Poisson-Log Generalized Linear Model (Poisson Regression)

Stan also supplies a single function for a generalized linear model with Poisson likelihood and log link function, i.e. a function for a Poisson regression. This provides a more efficient implementation of Poisson regression than a manually written regression in terms of a Poisson likelihood and matrix multiplication.

Probability Mass Function

If $x \in \mathbb{R}^{n \cdot m}$, $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}^m$, then for $y \in \mathbb{N}^n$,

$$\text{PoisonLogGLM}(y|x, \alpha, \beta) = \prod_{1 \leq i \leq n} \text{Poisson}(y_i | \exp(\alpha_i + x_i \cdot \beta)).$$

Sampling Statement

`y ~ poisson_log_glm(x, alpha, beta)`

Increment target log probability density with `poisson_log_glm_lpmf(y | x, alpha, beta)` dropping constant additive terms.

Stan Functions

`real poisson_log_glm_lpmf(int y | matrix x, real alpha, vector beta)`

The log Poisson probability mass of `y` given the log-rate `alpha + x * beta`, where the same intercept `alpha` and dependent variable value `y` are used for all observations. The number of columns of `x` needs to match the size of the coefficient vector `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

`real poisson_log_glm_lpmf(int y | matrix x, vector alpha, vector beta)`

The log Poisson probability mass of `y` given the log-rate `alpha + x * beta`, where an intercept `alpha` is used that is allowed to vary with the different observations. Dependent variable value `y` is used for all observations. The number of rows of `x` must match the size of `alpha` and the number of columns of `x` needs to match the size of the coefficient vector `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

`real poisson_log_glm_lpmf(int[] y | row_vector x, real alpha, vector beta)`

The log Poisson probability mass of `y` given the log-rate `alpha + x * beta`, where the same intercept `alpha` is used for all observations. The number of columns of `x` needs to match the size of the coefficient vector `beta`.

real **poisson_log_glm_lpmf**(int[] y | row_vector x, vector alpha, vector beta)
The log Poisson probability mass of y given the log-rate alpha + x * beta, where an intercept alpha is used that is allowed to vary with the different observations. The number of rows of x must match the size of alpha and the number of columns of x needs to match the size of the coefficient vector beta.

real **poisson_log_glm_lpmf**(int[] y | matrix x, real alpha, vector beta)
The log Poisson probability mass of y given the log-rate alpha + x * beta, where the same intercept alpha is used for all observations. The number of rows of the independent variable matrix x needs to match the size of the dependent variable vector y and the number of columns of x needs to match the size of the coefficient vector beta. If x and y are data (not parameters) this function can be executed on a GPU.

real **poisson_log_glm_lpmf**(int[] y | matrix x, vector alpha, vector beta)
The log Poisson probability mass of y given the log-rate alpha + x * beta, where an intercept alpha is used that is allowed to vary with the different observations. The number of rows of the independent variable matrix x needs to match the size of the dependent variable vector y and alpha and the number of columns of x needs to match the size of the coefficient vector beta. If x and y are data (not parameters) this function can be executed on a GPU.

15. Multivariate Discrete Distributions

The multivariate discrete distributions are over multiple integer values, which are expressed in Stan as arrays.

15.1. Multinomial Distribution

Probability Mass Function

If $K \in \mathbb{N}$, $N \in \mathbb{N}$, and $\theta \in K$-simplex, then for $y \in \mathbb{N}^K$ such that $\sum_{k=1}^{K} y_k = N$,

$$\text{Multinomial}(y|\theta) = \binom{N}{y_1, \ldots, y_K} \prod_{k=1}^{K} \theta_k^{y_k},$$

where the multinomial coefficient is defined by

$$\binom{N}{y_1, \ldots, y_k} = \frac{N!}{\prod_{k=1}^{K} y_k!}.$$

Sampling Statement

`y ~ multinomial(theta)`

Increment target log probability density with `multinomial_lpmf(y | theta)` dropping constant additive terms.

Stan Functions

`real multinomial_lpmf(int[] y | vector theta)`

The log multinomial probability mass function with outcome array `y` of size K given the K-simplex distribution parameter `theta` and (implicit) total count `N = sum(y)`

`int[] multinomial_rng(vector theta, int N)`

Generate a multinomial variate with simplex distribution parameter `theta` and total count N; may only be used in transformed data and generated quantities blocks

15.2. Multinomial Distribution, Logit Parameterization

Stan also provides a version of the multinomial probability mass function distribution with the K-simplex for the event count probabilities per category given on the unconstrained logistic scale.

Probability Mass Function

If $K \in \mathbb{N}$, $N \in \mathbb{N}$, and $\text{softmax}^{-1}(\theta) \in K$-simplex, then for $y \in \mathbb{N}^K$ such that $\sum_{k=1}^{K} y_k = N$,

$$\text{MultinomialLogit}(y|\theta) = \text{Multinomial}(y|\text{softmax}^{-1}(\theta)) = \binom{N}{y_1, \ldots, y_K} \prod_{k=1}^{K} [\text{softmax}^{-1}(\theta)_k]^{y_k},$$

where the multinomial coefficient is defined by

$$\binom{N}{y_1, \ldots, y_k} = \frac{N!}{\prod_{k=1}^{K} y_k!}.$$

Sampling Statement

`y ~ multinomial_logit(theta)`

Increment target log probability density with `multinomial_logit_lpmf(y | theta)` dropping constant additive terms.

Stan Functions

`real multinomial_logit_lpmf(int[] y | vector theta)`

The log multinomial probability mass function with outcome array `y` of size K given the K-simplex distribution parameter $\text{softmax}^{-1}(\theta)$ and (implicit) total count `N = sum(y)`

`int[] multinomial_logit_rng(vector theta, int N)`

Generate a multinomial variate with simplex distribution parameter $\text{softmax}^{-1}(\theta)$ and total count N; may only be used in transformed data and generated quantities blocks

Continuous Distributions

16. Unbounded Continuous Distributions

The unbounded univariate continuous probability distributions have support on all real numbers.

16.1. Normal Distribution

Probability Density Function

If $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{Normal}(y|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right).$$

Sampling Statement

`y ~ normal(mu, sigma)`

Increment target log probability density with `normal_lpdf(y | mu, sigma)` dropping constant additive terms.

Stan Functions

`real normal_lpdf(reals y | reals mu, reals sigma)`
The log of the normal density of y given location mu and scale sigma

`real normal_cdf(reals y, reals mu, reals sigma)`
The cumulative normal distribution of y given location mu and scale sigma; normal_cdf will underflow to 0 for $\frac{y-\mu}{\sigma}$ below -37.5 and overflow to 1 for $\frac{y-\mu}{\sigma}$ above 8.25; the function `Phi_approx` is more robust in the tails, but must be scaled and translated for anything other than a standard normal.

`real normal_lcdf(reals y | reals mu, reals sigma)`
The log of the cumulative normal distribution of y given location mu and scale sigma; normal_lcdf will underflow to $-\infty$ for $\frac{y-\mu}{\sigma}$ below -37.5 and overflow to 0 for $\frac{y-\mu}{\sigma}$ above 8.25; `log(Phi_approx(...))` is more robust in the tails, but must be scaled and translated for anything other than a standard normal.

`real normal_lccdf(reals y | reals mu, reals sigma)`
The log of the complementary cumulative normal distribution of y given location mu and scale sigma; normal_lccdf will overflow to 0 for $\frac{y-\mu}{\sigma}$ below -37.5 and underflow to $-\infty$ for $\frac{y-\mu}{\sigma}$ above 8.25; `log1m(Phi_approx(...))` is more robust in the tails, but must be scaled and translated for anything other than a standard normal.

R **normal_rng**(reals mu, reals sigma)
Generate a normal variate with location mu and scale sigma; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

Standard Normal Distribution
The standard normal distribution is so-called because its parameters are the units for their respective operations—the location (mean) is zero and the scale (standard deviation) one. The standard normal is parameter-free, and the unit parameters allow considerable simplification of the expression for the density.

$$\text{StdNormal}(y) \;=\; \text{Normal}(y \mid 0, 1) \;=\; \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y^2}{2}\right).$$

Up to a proportion on the log scale, where Stan computes,

$$\log \text{Normal}(y \mid 0, 1) \;=\; \frac{-y^2}{2} + \text{const.}$$

With no logarithm, no subtraction, and no division by a parameter, the standard normal log density is much more efficient to compute than the normal log density with constant location 0 and scale 1.

Sampling Statement
y ~ **std_normal**()

Increment target log probability density with std_normal_lpdf(y) dropping constant additive terms.

Stan Functions
real **std_normal_lpdf**(reals y)
The standard normal (location zero, scale one) log probability density of y.

real **std_normal_cdf**(reals y)
The cumulative standard normal distribution of y; std_normal_cdf will underflow to 0 for y below -37.5 and overflow to 1 for y above 8.25; the function Phi_approx is more robust in the tails.

real **std_normal_lcdf**(reals y)
The log of the cumulative standard normal distribution of y; std_normal_lcdf will underflow to $-\infty$ for y below -37.5 and overflow to 0 for y above 8.25; log(Phi_approx(...)) is more robust in the tails.

real **std_normal_lccdf**(reals y)
The log of the complementary cumulative standard normal distribution of y;

std_normal_lccdf will overflow to 0 for y below -37.5 and underflow to $-\infty$ for y above 8.25; `log1m(Phi_approx(...))` is more robust in the tails.

`real` **`std_normal_rng`**`()`
Generate a normal variate with location zero and scale one; may only be used in transformed data and generated quantities blocks.

16.2. Normal-Id Generalized Linear Model (Linear Regression)

Stan also supplies a single function for a generalized linear lodel with normal likelihood and identity link function, i.e. a function for a linear regression. This provides a more efficient implementation of linear regression than a manually written regression in terms of a normal likelihood and matrix multiplication.

Probability Distribution Function
If $x \in \mathbb{R}^{n \cdot m}, \alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^m, \sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}^n$,

$$\text{NormalIdGLM}(y|x, \alpha, \beta, \sigma) = \prod_{1 \leq i \leq n} \text{Normal}(y_i|\alpha_i + x_i \cdot \beta, \sigma).$$

Sampling Statement
`y ~` **`normal_id_glm`**`(x, alpha, beta, sigma)`

Increment target log probability density with `normal_id_glm_lpdf(y | x, alpha, beta, sigma)` dropping constant additive terms.

Stan Functions
`real` **`normal_id_glm_lpdf`**`(real y | matrix x, real alpha, vector beta, real sigma)`
The log normal probability density of `y` given location `alpha + x * beta` and scale `sigma`, where the same intercept `alpha`, scale `sigma` and dependent variable value `y` are used for all observations. The number of columns of `x` needs to match the size of the coefficient vector `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

`real` **`normal_id_glm_lpdf`**`(real y | matrix x, vector alpha, vector beta, real sigma)`
The log normal probability density of `y` given location `alpha + x * beta` and scale `sigma`, where the same scale `sigma` and dependent variable valuey are used for all observations and an intercept `alpha` is used that is allowed to vary by observation. The number of rows of the independent variable matrix `x` needs to match the size of the intercept vector `alpha` and the number of columns of `x` needs to match the size of the coefficient vector `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

`real` **`normal_id_glm_lpdf`**`(vector y | row_vector x, real alpha, vector beta, real sigma)`
The log normal probability density of `y` given location `alpha + x * beta` and scale `sigma`, where the same intercept `alpha`, scale `sigma` and independent variable values `x` are used for all observations. The number of columns of `x` needs to match the size of the coefficient vector `beta`.

`real` **`normal_id_glm_lpdf`**`(vector y | row_vector x, vector alpha, vector beta, real sigma)`
The log normal probability density of `y` given location `alpha + x * beta` and scale `sigma`, where the same scale `sigma` and independent variable values `x` are used for all observations and an intercept `alpha` is used that is allowed to vary by observation. The size of the dependent variable vector `y` needs to match the size of the intercept vector `alpha` and the number of columns of `x` needs to match the size of the coefficient vector `beta`.

`real` **`normal_id_glm_lpdf`**`(vector y | matrix x, real alpha, vector beta, real sigma)`
The log normal probability density of `y` given location `alpha + x * beta` and scale `sigma`, where the same intercept `alpha` and scale `sigma` is used for all observations. The number of rows of the independent variable matrix `x` needs to match the size of the dependent variable vector `y` and the number of columns of `x` needs to match the size of the coefficient vector `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

`real` **`normal_id_glm_lpdf`**`(vector y | matrix x, vector alpha, vector beta, real sigma)`
The log normal probability density of `y` given location `alpha + x * beta` and scale `sigma`, where the same scale `sigma` is used for all observations and an intercept `alpha` is used that is allowed to vary by observation. The number of rows of the independent variable matrix `x` needs to match the size of the dependent variable vector `y` and the size of the intercept vector alpha. The number of columns of `x` needs to match the size of the coefficient vector `beta`. If `x` and `y` are data (not parameters) this function can be executed on a GPU.

16.3. Exponentially Modified Normal Distribution

Probability Density Function

If $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, and $\lambda \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{ExpModNormal}(y|\mu,\sigma,\lambda) = \frac{\lambda}{2}\ \exp\left(\frac{\lambda}{2}\left(2\mu + \lambda\sigma^2 - 2y\right)\right) \operatorname{erfc}\left(\frac{\mu + \lambda\sigma^2 - y}{\sqrt{2}\sigma}\right).$$

Sampling Statement
`y ~ exp_mod_normal(mu, sigma, lambda)`

Increment target log probability density with `exp_mod_normal_lpdf(y | mu, sigma, lambda)` dropping constant additive terms.

Stan Functions
`real exp_mod_normal_lpdf(reals y | reals mu, reals sigma, reals lambda)`
The log of the exponentially modified normal density of y given location mu, scale sigma, and shape lambda

`real exp_mod_normal_cdf(reals y, reals mu, reals sigma, reals lambda)`
The exponentially modified normal cumulative distribution function of y given location mu, scale sigma, and shape lambda

`real exp_mod_normal_lcdf(reals y | reals mu, reals sigma, reals lambda)`
The log of the exponentially modified normal cumulative distribution function of y given location mu, scale sigma, and shape lambda

`real exp_mod_normal_lccdf(reals y | reals mu, reals sigma, reals lambda)`
The log of the exponentially modified normal complementary cumulative distribution function of y given location mu, scale sigma, and shape lambda

`R exp_mod_normal_rng(reals mu, reals sigma, reals lambda)`
Generate a exponentially modified normal variate with location mu, scale sigma, and shape lambda; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

16.4. Skew Normal Distribution

Probability Density Function
If $\xi \in \mathbb{R}$, $\omega \in \mathbb{R}^+$, and $\alpha \in \mathbb{R}$, then for $y \in \mathbb{R}$,

$$\text{SkewNormal}(y \mid \xi, \omega, \alpha) = \frac{1}{\omega\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-\xi}{\omega}\right)^2\right) \left(1 + \text{erf}\left(\alpha\left(\frac{y-\xi}{\omega\sqrt{2}}\right)\right)\right).$$

Sampling Statement
`y ~ skew_normal(xi, omega, alpha)`

Increment target log probability density with `skew_normal_lpdf(y | xi, omega, alpha)` dropping constant additive terms.

Stan Functions

`real` **`skew_normal_lpdf`**`(reals y | reals xi, reals omega, reals alpha)`
The log of the skew normal density of y given location xi, scale omega, and shape alpha

`real` **`skew_normal_cdf`**`(reals y, reals xi, reals omega, reals alpha)`
The skew normal distribution function of y given location xi, scale omega, and shape alpha

`real` **`skew_normal_lcdf`**`(reals y | reals xi, reals omega, reals alpha)`
The log of the skew normal cumulative distribution function of y given location xi, scale omega, and shape alpha

`real` **`skew_normal_lccdf`**`(reals y | reals xi, reals omega, reals alpha)`
The log of the skew normal complementary cumulative distribution function of y given location xi, scale omega, and shape alpha

`R` **`skew_normal_rng`**`(reals xi, reals omega, real alpha)`
Generate a skew normal variate with location xi, scale omega, and shape alpha; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

16.5. Student-T Distribution

Probability Density Function

If $\nu \in \mathbb{R}^+$, $\mu \in \mathbb{R}$, and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{StudentT}(y|\nu,\mu,\sigma) = \frac{\Gamma\left((\nu+1)/2\right)}{\Gamma(\nu/2)} \, \frac{1}{\sqrt{\nu\pi}\,\sigma} \left(1 + \frac{1}{\nu}\left(\frac{y-\mu}{\sigma}\right)^2\right)^{-(\nu+1)/2}.$$

Sampling Statement

`y ~` **`student_t`**`(nu, mu, sigma)`

Increment target log probability density with `student_t_lpdf(y | nu, mu, sigma)` dropping constant additive terms.

Stan Functions

`real` **`student_t_lpdf`**`(reals y | reals nu, reals mu, reals sigma)`
The log of the Student-t density of y given degrees of freedom nu, location mu, and scale sigma

`real` **`student_t_cdf`**`(reals y, reals nu, reals mu, reals sigma)`
The Student-t cumulative distribution function of y given degrees of freedom nu, location mu, and scale sigma

`real` **`student_t_lcdf`**`(reals y | reals nu, reals mu, reals sigma)`
The log of the Student-t cumulative distribution function of y given degrees of freedom nu, location mu, and scale sigma

`real` **`student_t_lccdf`**`(reals y | reals nu, reals mu, reals sigma)`
The log of the Student-t complementary cumulative distribution function of y given degrees of freedom nu, location mu, and scale sigma

`R` **`student_t_rng`**`(reals nu, reals mu, reals sigma)`
Generate a Student-t variate with degrees of freedom nu, location mu, and scale sigma; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

16.6. Cauchy Distribution

Probability Density Function
If $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{Cauchy}(y|\mu,\sigma) = \frac{1}{\pi\sigma}\,\frac{1}{1+((y-\mu)/\sigma)^2}.$$

Sampling Statement
`y ~` **`cauchy`**`(mu, sigma)`

Increment target log probability density with `cauchy_lpdf(y | mu, sigma)` dropping constant additive terms.

Stan Functions
`real` **`cauchy_lpdf`**`(reals y | reals mu, reals sigma)`
The log of the Cauchy density of y given location mu and scale sigma

`real` **`cauchy_cdf`**`(reals y, reals mu, reals sigma)`
The Cauchy cumulative distribution function of y given location mu and scale sigma

`real` **`cauchy_lcdf`**`(reals y | reals mu, reals sigma)`
The log of the Cauchy cumulative distribution function of y given location mu and scale sigma

`real` **`cauchy_lccdf`**`(reals y | reals mu, reals sigma)`
The log of the Cauchy complementary cumulative distribution function of y given location mu and scale sigma

`R` **`cauchy_rng`**`(reals mu, reals sigma)`
Generate a Cauchy variate with location mu and scale sigma; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

16.7. Double Exponential (Laplace) Distribution

Probability Density Function

If $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{DoubleExponential}(y|\mu,\sigma) = \frac{1}{2\sigma} \exp\left(-\frac{|y-\mu|}{\sigma}\right).$$

Note that the double exponential distribution is parameterized in terms of the scale, in contrast to the exponential distribution (see section exponential distribution), which is parameterized in terms of inverse scale.

The double-exponential distribution can be defined as a compound exponential-normal distribution (Ding and Blitzstein 2018). Using the inverse scale parameterization for the exponential distribution, and the standard deviation parameterization for the normal distribution, one can write

$$\alpha \sim \text{Exponential}\left(\frac{1}{2\sigma^2}\right)$$

and

$$\beta \mid \alpha \sim \text{Normal}(\mu, \sqrt{\alpha}),$$

then

$$\beta \sim \text{DoubleExponential}(\mu,\sigma).$$

This may be used to code a non-centered parameterization by taking

$$\beta^{\text{raw}} \sim \text{Normal}(0,1)$$

and defining

$$\beta = \mu + \alpha\,\beta^{\text{raw}}.$$

Sampling Statement

`y ~ double_exponential(mu, sigma)`

Increment target log probability density with `double_exponential_lpdf(y | mu, sigma)` dropping constant additive terms.

Stan Functions

`real double_exponential_lpdf(reals y | reals mu, reals sigma)`
The log of the double exponential density of y given location mu and scale sigma

`real double_exponential_cdf(reals y, reals mu, reals sigma)`
The double exponential cumulative distribution function of y given location mu and scale sigma

`real` **`double_exponential_lcdf`**`(reals y | reals mu, reals sigma)`
The log of the double exponential cumulative distribution function of y given location mu and scale sigma

`real` **`double_exponential_lccdf`**`(reals y | reals mu, reals sigma)`
The log of the double exponential complementary cumulative distribution function of y given location mu and scale sigma

`R` **`double_exponential_rng`**`(reals mu, reals sigma)`
Generate a double exponential variate with location mu and scale sigma; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

16.8. Logistic Distribution

Probability Density Function
If $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{Logistic}(y|\mu,\sigma) = \frac{1}{\sigma} \exp\left(-\frac{y-\mu}{\sigma}\right) \left(1 + \exp\left(-\frac{y-\mu}{\sigma}\right)\right)^{-2}.$$

Sampling Statement
`y ~` **`logistic`**`(mu, sigma)`

Increment target log probability density with `logistic_lpdf(y | mu, sigma)` dropping constant additive terms.

Stan Functions
`real` **`logistic_lpdf`**`(reals y | reals mu, reals sigma)`
The log of the logistic density of y given location mu and scale sigma

`real` **`logistic_cdf`**`(reals y, reals mu, reals sigma)`
The logistic cumulative distribution function of y given location mu and scale sigma

`real` **`logistic_lcdf`**`(reals y | reals mu, reals sigma)`
The log of the logistic cumulative distribution function of y given location mu and scale sigma

`real` **`logistic_lccdf`**`(reals y | reals mu, reals sigma)`
The log of the logistic complementary cumulative distribution function of y given location mu and scale sigma

`R` **`logistic_rng`**`(reals mu, reals sigma)`
Generate a logistic variate with location mu and scale sigma; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

16.9. Gumbel Distribution

Probability Density Function

If $\mu \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{Gumbel}(y|\mu,\beta) = \frac{1}{\beta} \exp\left(-\frac{y-\mu}{\beta} - \exp\left(-\frac{y-\mu}{\beta}\right)\right).$$

Sampling Statement

`y ~ gumbel(mu, beta)`

Increment target log probability density with `gumbel_lpdf(y | mu, beta)` dropping constant additive terms.

Stan Functions

`real` **`gumbel_lpdf`**`(reals y | reals mu, reals beta)`
The log of the gumbel density of y given location mu and scale beta

`real` **`gumbel_cdf`**`(reals y, reals mu, reals beta)`
The gumbel cumulative distribution function of y given location mu and scale beta

`real` **`gumbel_lcdf`**`(reals y | reals mu, reals beta)`
The log of the gumbel cumulative distribution function of y given location mu and scale beta

`real` **`gumbel_lccdf`**`(reals y | reals mu, reals beta)`
The log of the gumbel complementary cumulative distribution function of y given location mu and scale beta

`R` **`gumbel_rng`**`(reals mu, reals beta)`
Generate a gumbel variate with location mu and scale beta; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

17. Positive Continuous Distributions

The positive continuous probability functions have support on the positive real numbers.

17.1. Lognormal Distribution

Probability Density Function

If $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{LogNormal}(y|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma}\,\frac{1}{y}\,\exp\left(-\frac{1}{2}\left(\frac{\log y - \mu}{\sigma}\right)^2\right).$$

Sampling Statement

`y ~ lognormal(mu, sigma)`

Increment target log probability density with `lognormal_lpdf(y | mu, sigma)` dropping constant additive terms.

Stan Functions

`real lognormal_lpdf(reals y | reals mu, reals sigma)`
The log of the lognormal density of y given location mu and scale sigma

`real lognormal_cdf(reals y, reals mu, reals sigma)`
The cumulative lognormal distribution function of y given location mu and scale sigma

`real lognormal_lcdf(reals y | reals mu, reals sigma)`
The log of the lognormal cumulative distribution function of y given location mu and scale sigma

`real lognormal_lccdf(reals y | reals mu, reals sigma)`
The log of the lognormal complementary cumulative distribution function of y given location mu and scale sigma

`R lognormal_rng(reals mu, reals sigma)`
Generate a lognormal variate with location mu and scale sigma; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

17.2. Chi-Square Distribution

Probability Density Function

If $\nu \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{ChiSquare}(y|\nu) = \frac{2^{-\nu/2}}{\Gamma(\nu/2)} y^{\nu/2-1} \exp\left(-\frac{1}{2} y\right).$$

Sampling Statement

`y ~ chi_square(nu)`

Increment target log probability density with `chi_square_lpdf(y | nu)` dropping constant additive terms.

Stan Functions

`real chi_square_lpdf(reals y | reals nu)`
The log of the Chi-square density of y given degrees of freedom nu

`real chi_square_cdf(reals y, reals nu)`
The Chi-square cumulative distribution function of y given degrees of freedom nu

`real chi_square_lcdf(reals y | reals nu)`
The log of the Chi-square cumulative distribution function of y given degrees of freedom nu

`real chi_square_lccdf(reals y | reals nu)`
The log of the complementary Chi-square cumulative distribution function of y given degrees of freedom nu

`R chi_square_rng(reals nu)`
Generate a Chi-square variate with degrees of freedom nu; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

17.3. Inverse Chi-Square Distribution

Probability Density Function

If $\nu \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{InvChiSquare}(y \mid \nu) = \frac{2^{-\nu/2}}{\Gamma(\nu/2)} y^{-\nu/2-1} \exp\left(-\frac{1}{2}\frac{1}{y}\right).$$

Sampling Statement

`y ~ inv_chi_square(nu)`

Increment target log probability density with `inv_chi_square_lpdf(y | nu)` dropping constant additive terms.

Stan Functions

`real` **`inv_chi_square_lpdf`**`(reals y | reals nu)`
The log of the inverse Chi-square density of y given degrees of freedom nu

`real` **`inv_chi_square_cdf`**`(reals y, reals nu)`
The inverse Chi-squared cumulative distribution function of y given degrees of freedom nu

`real` **`inv_chi_square_lcdf`**`(reals y | reals nu)`
The log of the inverse Chi-squared cumulative distribution function of y given degrees of freedom nu

`real` **`inv_chi_square_lccdf`**`(reals y | reals nu)`
The log of the inverse Chi-squared complementary cumulative distribution function of y given degrees of freedom nu

`R` **`inv_chi_square_rng`**`(reals nu)`
Generate an inverse Chi-squared variate with degrees of freedom nu; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

17.4. Scaled Inverse Chi-Square Distribution

Probability Density Function

If $\nu \in \mathbb{R}^+$ and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{ScaledInvChiSquare}(y|\nu,\sigma) = \frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)}\,\sigma^{\nu}\,y^{-(\nu/2+1)}\,\exp\left(-\frac{1}{2}\,\nu\,\sigma^2\,\frac{1}{y}\right).$$

Sampling Statement

`y ~` **`scaled_inv_chi_square`**`(nu, sigma)`

Increment target log probability density with `scaled_inv_chi_square_lpdf(y | nu, sigma)` dropping constant additive terms.

Stan Functions

`real` **`scaled_inv_chi_square_lpdf`**`(reals y | reals nu, reals sigma)`
The log of the scaled inverse Chi-square density of y given degrees of freedom nu and scale sigma

`real` **`scaled_inv_chi_square_cdf`**`(reals y, reals nu, reals sigma)`
The scaled inverse Chi-square cumulative distribution function of y given degrees of freedom nu and scale sigma

`real` **`scaled_inv_chi_square_lcdf`**`(reals y | reals nu, reals sigma)`

The log of the scaled inverse Chi-square cumulative distribution function of y given degrees of freedom nu and scale sigma

`real` **`scaled_inv_chi_square_lccdf`**`(reals y | reals nu, reals sigma)`
The log of the scaled inverse Chi-square complementary cumulative distribution function of y given degrees of freedom nu and scale sigma

`R` **`scaled_inv_chi_square_rng`**`(reals nu, reals sigma)`
Generate a scaled inverse Chi-squared variate with degrees of freedom nu and scale sigma; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

17.5. Exponential Distribution

Probability Density Function
If $\beta \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{Exponential}(y|\beta) = \beta \exp(-\beta y).$$

Sampling Statement
`y ~` **`exponential`**`(beta)`

Increment target log probability density with `exponential_lpdf(y | beta)` dropping constant additive terms.

Stan Functions
`real` **`exponential_lpdf`**`(reals y | reals beta)`
The log of the exponential density of y given inverse scale beta

`real` **`exponential_cdf`**`(reals y, reals beta)`
The exponential cumulative distribution function of y given inverse scale beta

`real` **`exponential_lcdf`**`(reals y | reals beta)`
The log of the exponential cumulative distribution function of y given inverse scale beta

`real` **`exponential_lccdf`**`(reals y | reals beta)`
The log of the exponential complementary cumulative distribution function of y given inverse scale beta

`R` **`exponential_rng`**`(reals beta)`
Generate an exponential variate with inverse scale beta; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

17.6. Gamma Distribution

Probability Density Function

If $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{Gamma}(y|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, y^{\alpha-1} \exp(-\beta\, y).$$

Sampling Statement

`y ~ gamma(alpha, beta)`

Increment target log probability density with `gamma_lpdf(y | alpha, beta)` dropping constant additive terms.

Stan Functions

`real gamma_lpdf(reals y | reals alpha, reals beta)`
The log of the gamma density of y given shape alpha and inverse scale beta

`real gamma_cdf(reals y, reals alpha, reals beta)`
The cumulative gamma distribution function of y given shape alpha and inverse scale beta

`real gamma_lcdf(reals y | reals alpha, reals beta)`
The log of the cumulative gamma distribution function of y given shape alpha and inverse scale beta

`real gamma_lccdf(reals y | reals alpha, reals beta)`
The log of the complementary cumulative gamma distribution function of y given shape alpha and inverse scale beta

`R gamma_rng(reals alpha, reals beta)`
Generate a gamma variate with shape alpha and inverse scale beta; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

17.7. Inverse Gamma Distribution

Probability Density Function

If $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{InvGamma}(y|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, y^{-(\alpha+1)} \exp\left(-\beta\, \frac{1}{y}\right).$$

Sampling Statement

`y ~ inv_gamma(alpha, beta)`

Increment target log probability density with `inv_gamma_lpdf(y | alpha, beta)` dropping constant additive terms.

Stan Functions

`real` **`inv_gamma_lpdf`**`(reals y | reals alpha, reals beta)`
The log of the inverse gamma density of y given shape alpha and scale beta

`real` **`inv_gamma_cdf`**`(reals y, reals alpha, reals beta)`
The inverse gamma cumulative distribution function of y given shape alpha and scale beta

`real` **`inv_gamma_lcdf`**`(reals y | reals alpha, reals beta)`
The log of the inverse gamma cumulative distribution function of y given shape alpha and scale beta

`real` **`inv_gamma_lccdf`**`(reals y | reals alpha, reals beta)`
The log of the inverse gamma complementary cumulative distribution function of y given shape alpha and scale beta

`R` **`inv_gamma_rng`**`(reals alpha, reals beta)`
Generate an inverse gamma variate with shape alpha and scale beta; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

17.8. Weibull Distribution

Probability Density Function

If $\alpha \in \mathbb{R}^+$ and $\sigma \in \mathbb{R}^+$, then for $y \in [0, \infty)$,

$$\text{Weibull}(y|\alpha,\sigma) = \frac{\alpha}{\sigma}\left(\frac{y}{\sigma}\right)^{\alpha-1}\exp\left(-\left(\frac{y}{\sigma}\right)^{\alpha}\right).$$

Note that if $Y \propto \text{Weibull}(\alpha,\sigma)$, then $Y^{-1} \propto \text{Frechet}(\alpha,\sigma^{-1})$.

Sampling Statement

`y ~` **`weibull`**`(alpha, sigma)`

Increment target log probability density with `weibull_lpdf(y | alpha, sigma)` dropping constant additive terms.

Stan Functions

`real` **`weibull_lpdf`**`(reals y | reals alpha, reals sigma)`
The log of the Weibull density of y given shape alpha and scale sigma

`real` **`weibull_cdf`**`(reals y, reals alpha, reals sigma)`
The Weibull cumulative distribution function of y given shape alpha and scale sigma

`real` **`weibull_lcdf`**`(reals y | reals alpha, reals sigma)`
The log of the Weibull cumulative distribution function of y given shape alpha and scale sigma

`real` **`weibull_lccdf`**`(reals y | reals alpha, reals sigma)`
The log of the Weibull complementary cumulative distribution function of y given shape alpha and scale sigma

`R` **`weibull_rng`**`(reals alpha, reals sigma)`
Generate a weibull variate with shape alpha and scale sigma; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

17.9. Frechet Distribution

Probability Density Function
If $\alpha \in \mathbb{R}^+$ and $\sigma \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$,

$$\text{Frechet}(y|\alpha,\sigma) = \frac{\alpha}{\sigma}\left(\frac{y}{\sigma}\right)^{-\alpha-1}\exp\left(-\left(\frac{y}{\sigma}\right)^{-\alpha}\right).$$

Note that if $Y \propto \text{Frechet}(\alpha,\sigma)$, then $Y^{-1} \propto \text{Weibull}(\alpha,\sigma^{-1})$.

Sampling Statement
`y ~` **`frechet`**`(alpha, sigma)`

Increment target log probability density with `frechet_lpdf(y | alpha, sigma)` dropping constant additive terms.

Stan Functions
`real` **`frechet_lpdf`**`(reals y | reals alpha, reals sigma)`
The log of the Frechet density of y given shape alpha and scale sigma

`real` **`frechet_cdf`**`(reals y, reals alpha, reals sigma)`
The Frechet cumulative distribution function of y given shape alpha and scale sigma

`real` **`frechet_lcdf`**`(reals y | reals alpha, reals sigma)`
The log of the Frechet cumulative distribution function of y given shape alpha and scale sigma

`real` **`frechet_lccdf`**`(reals y | reals alpha, reals sigma)`
The log of the Frechet complementary cumulative distribution function of y given shape alpha and scale sigma

`R` **`frechet_rng`**`(reals alpha, reals sigma)`
Generate a Frechet variate with shape alpha and scale sigma; may only be used in

transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

17.10. Rayleigh Distribution

Probability Density Function

If $\sigma \in \mathbb{R}^+$, then for $y \in [0, \infty)$,

$$\text{Rayleigh}(y|\sigma) = \frac{y}{\sigma^2} \exp(-y^2/2\sigma^2).$$

Sampling Statement

`y ~ rayleigh(sigma)`

Increment target log probability density with `rayleigh_lpdf(y | sigma)` dropping constant additive terms.

Stan Functions

`real rayleigh_lpdf(reals y | reals sigma)`
The log of the Rayleigh density of y given scale sigma

`real rayleigh_cdf(real y, real sigma)`
The Rayleigh cumulative distribution of y given scale sigma

`real rayleigh_lcdf(real y | real sigma)`
The log of the Rayleigh cumulative distribution of y given scale sigma

`real rayleigh_lccdf(real y | real sigma)`
The log of the Rayleigh complementary cumulative distribution of y given scale sigma

`R rayleigh_rng(reals sigma)`
Generate a Rayleigh variate with scale sigma; may only be used in generated quantities block. For a description of argument and return types, see section vectorized PRNG functions.

18. Positive Lower-Bounded Distributions

The positive lower-bounded probabilities have support on real values above some positive minimum value.

18.1. Pareto Distribution

Probability Density Function

If $y_{\min} \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}^+$, then for $y \in \mathbb{R}^+$ with $y \geq y_{\min}$,

$$\text{Pareto}(y|y_{\min}, \alpha) = \frac{\alpha \, y_{\min}^{\alpha}}{y^{\alpha+1}}.$$

Sampling Statement

`y ~ pareto(y_min, alpha)`

Increment target log probability density with `pareto_lpdf(y | y_min, alpha)` dropping constant additive terms.

Stan Functions

`real pareto_lpdf(reals y | reals y_min, reals alpha)`
The log of the Pareto density of y given positive minimum value y_min and shape alpha

`real pareto_cdf(reals y, reals y_min, reals alpha)`
The Pareto cumulative distribution function of y given positive minimum value y_min and shape alpha

`real pareto_lcdf(reals y | reals y_min, reals alpha)`
The log of the Pareto cumulative distribution function of y given positive minimum value y_min and shape alpha

`real pareto_lccdf(reals y | reals y_min, reals alpha)`
The log of the Pareto complementary cumulative distribution function of y given positive minimum value y_min and shape alpha

`R pareto_rng(reals y_min, reals alpha)`
Generate a Pareto variate with positive minimum value y_min and shape alpha; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

18.2. Pareto Type 2 Distribution

Probability Density Function

If $\mu \in \mathbb{R}$, $\lambda \in \mathbb{R}^+$, and $\alpha \in \mathbb{R}^+$, then for $y \geq \mu$,

$$\text{Pareto\_Type\_2}(y|\mu,\lambda,\alpha) = \frac{\alpha}{\lambda}\left(1 + \frac{y-\mu}{\lambda}\right)^{-(\alpha+1)}.$$

Note that the Lomax distribution is a Pareto Type 2 distribution with $\mu = 0$.

Sampling Statement

`y ~ pareto_type_2(mu, lambda, alpha)`

Increment target log probability density with `pareto_type_2_lpdf(y | mu, lambda, alpha)` dropping constant additive terms.

Stan Functions

`real pareto_type_2_lpdf(reals y | reals mu, reals lambda, reals alpha)`
The log of the Pareto Type 2 density of y given location mu, scale lambda, and shape alpha

`real pareto_type_2_cdf(reals y, reals mu, reals lambda, reals alpha)`
The Pareto Type 2 cumulative distribution function of y given location mu, scale lambda, and shape alpha

`real pareto_type_2_lcdf(reals y | reals mu, reals lambda, reals alpha)`
The log of the Pareto Type 2 cumulative distribution function of y given location mu, scale lambda, and shape alpha

`real pareto_type_2_lccdf(reals y | reals mu, reals lambda, reals alpha)`
The log of the Pareto Type 2 complementary cumulative distribution function of y given location mu, scale lambda, and shape alpha

`R pareto_type_2_rng(reals mu, reals lambda, reals alpha)`
Generate a Pareto Type 2 variate with location mu, scale lambda, and shape alpha; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

18.3. Wiener First Passage Time Distribution

Probability Density Function

If $\alpha \in \mathbb{R}^+$, $\tau \in \mathbb{R}^+$, $\beta \in [0,1]$ and $\delta \in \mathbb{R}$, then for $y > \tau$,

$$\text{Wiener}(y|\alpha,\tau,\beta,\delta) = \frac{\alpha^3}{(y-\tau)^{3/2}} \exp\left(-\delta\alpha\beta - \frac{\delta^2(y-\tau)}{2}\right) \sum_{k=-\infty}^{\infty} (2k+\beta)\phi\left(\frac{2k\alpha+\beta}{\sqrt{y-\tau}}\right)$$

where $\phi(x)$ denotes the standard normal density function; see (Feller 1968), (Navarro and Fuss 2009).

Sampling Statement

`y ~ wiener(alpha, tau, beta, delta)`

Increment target log probability density with `wiener_lpdf(y | alpha, tau, beta, delta)` dropping constant additive terms.

Stan Functions

`real wiener_lpdf(reals y | reals alpha, reals tau, reals beta, reals delta)`

The log of the Wiener first passage time density of y given boundary separation alpha, non-decision time tau, a-priori bias beta and drift rate delta

Boundaries

Stan returns the first passage time of the accumulation process over the upper boundary only. To get the result for the lower boundary, use

$$\text{wiener}(y|\alpha,\tau,1-\beta,-\delta)$$

For more details, see the appendix of Vandekerckhove and Wabersich (2014).

19. Continuous Distributions on [0, 1]

The continuous distributions with outcomes in the interval $[0, 1]$ are used to characterized bounded quantities, including probabilities.

19.1. Beta Distribution

Probability Density Function
If $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$, then for $\theta \in (0, 1)$,

$$\text{Beta}(\theta|\alpha, \beta) = \frac{1}{\text{B}(\alpha, \beta)} \theta^{\alpha-1} (1 - \theta)^{\beta-1},$$

where the beta function $\text{B}()$ is as defined in section combinatorial functions.

Warning: If $\theta = 0$ or $\theta = 1$, then the probability is 0 and the log probability is $-\infty$. Similarly, the distribution requires strictly positive parameters, $\alpha, \beta > 0$.

Sampling Statement
`theta ~ beta(alpha, beta)`

Increment target log probability density with `beta_lpdf(theta | alpha, beta)` dropping constant additive terms.

Stan Functions
`real beta_lpdf(reals theta | reals alpha, reals beta)`
The log of the beta density of `theta` in $[0, 1]$ given positive prior successes (plus one) alpha and prior failures (plus one) beta

`real beta_cdf(reals theta, reals alpha, reals beta)`
The beta cumulative distribution function of `theta` in $[0, 1]$ given positive prior successes (plus one) alpha and prior failures (plus one) beta

`real beta_lcdf(reals theta | reals alpha, reals beta)`
The log of the beta cumulative distribution function of `theta` in $[0, 1]$ given positive prior successes (plus one) alpha and prior failures (plus one) beta

`real beta_lccdf(reals theta | reals alpha, reals beta)`
The log of the beta complementary cumulative distribution function of `theta` in $[0, 1]$ given positive prior successes (plus one) alpha and prior failures (plus one) beta

`R beta_rng(reals alpha, reals beta)`
Generate a beta variate with positive prior successes (plus one) alpha and prior failures

(plus one) beta; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

19.2. Beta Proportion Distribution

Probability Density Function

If $\mu \in (0,1)$ and $\kappa \in \mathbb{R}^+$, then for $\theta \in (0,1)$,

$$\text{Beta\_Proportion}(\theta|\mu,\kappa) = \frac{1}{\text{B}(\mu\kappa,(1-\mu)\kappa)}\,\theta^{\mu\kappa-1}\,(1-\theta)^{(1-\mu)\kappa-1},$$

where the beta function $\text{B}()$ is as defined in section combinatorial functions.

Warning: If $\theta = 0$ or $\theta = 1$, then the probability is 0 and the log probability is $-\infty$. Similarly, the distribution requires $\mu \in (0,1)$ and strictly positive parameter, $\kappa > 0$.

Sampling Statement

`theta ~` **`beta_proportion`**`(mu, kappa)`

Increment target log probability density with `beta_proportion_lpdf(theta | mu, kappa)` dropping constant additive terms.

Stan Functions

`real` **`beta_proportion_lpdf`**`(reals theta | reals mu, reals kappa)`
The log of the beta_proportion density of `theta` in $(0,1)$ given mean mu and precision kappa

`real` **`beta_proportion_lcdf`**`(reals theta | reals mu, reals kappa)`
The log of the beta_proportion cumulative distribution function of `theta` in $(0,1)$ given mean mu and precision kappa

`real` **`beta_proportion_lccdf`**`(reals theta | reals mu, reals kappa)`
The log of the beta_proportion complementary cumulative distribution function of `theta` in $(0,1)$ given mean mu and precision kappa

`R` **`beta_proportion_rng`**`(reals mu, reals kappa)`
Generate a beta_proportion variate with mean mu and precision kappa; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

20. Circular Distributions

Circular distributions are defined for finite values y in any interval of length 2π.

20.1. Von Mises Distribution

Probability Density Function

If $\mu \in \mathbb{R}$ and $\kappa \in \mathbb{R}^+$, then for $y \in \mathbb{R}$,

$$\text{VonMises}(y|\mu,\kappa) = \frac{\exp(\kappa\cos(y-\mu))}{2\pi I_0(\kappa)}.$$

In order for this density to properly normalize, y must be restricted to some interval $(c, c+2\pi)$ of length 2π, because

$$\int_c^{c+2\pi} \text{VonMises}(y|\mu,\kappa)\,dy = 1.$$

Similarly, if μ is a parameter, it will typically be restricted to the same range as y.

If $\kappa > 0$, a von Mises distribution with its 2π interval of support centered around its location μ will have a single mode at μ; for example, restricting y to $(-\pi, \pi)$ and taking $\mu = 0$ leads to a single local optimum at the mode μ. If the location μ is not in the center of the support, the density is circularly translated and there will be a second local maximum at the boundary furthest from the mode. Ideally, the parameterization and support will be set up so that the bulk of the probability mass is in a continuous interval around the mean μ.

For $\kappa = 0$, the Von Mises distribution corresponds to the circular uniform distribution with density $1/(2\pi)$ (independently of the values of y or μ).

Sampling Statement

`y ~ von_mises(mu, kappa)`

Increment target log probability density with `von_mises_lpdf(y | mu, kappa)` dropping constant additive terms.

Stan Functions

`R von_mises_lpdf(reals y | reals mu, reals kappa)`
The log of the von mises density of y given location mu and scale kappa. For a description of argument and return types, see section vectorized function signatures.

`R von_mises_rng(reals mu, reals kappa)`
Generate a Von Mises variate with location mu and scale kappa (i.e. returns values in

the interval $[(\mu \mod 2\pi) - \pi, (\mu \mod 2\pi) + \pi]$); may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

Numerical Stability

Evaluating the Von Mises distribution for $\kappa > 100$ is numerically unstable in the current implementation. Nathanael I. Lichti suggested the following workaround on the Stan users group, based on the fact that as $\kappa \to \infty$,

$$\text{VonMises}(y|\mu, \kappa) \to \text{Normal}(\mu, \sqrt{1/\kappa}).$$

The workaround is to replace `y ~ von_mises(mu,kappa)` with

```
if (kappa < 100)
  y ~ von_mises(mu, kappa);
else
  y ~ normal(mu, sqrt(1 / kappa));
```

21. Bounded Continuous Distributions

The bounded continuous probabilities have support on a finite interval of real numbers.

21.1. Uniform Distribution

Probability Density Function
If $\alpha \in \mathbb{R}$ and $\beta \in (\alpha, \infty)$, then for $y \in [\alpha, \beta]$,

$$\text{Uniform}(y|\alpha, \beta) = \frac{1}{\beta - \alpha}.$$

Sampling Statement
`y ~ uniform(alpha, beta)`

Increment target log probability density with `uniform_lpdf(y | alpha, beta)` dropping constant additive terms.

Stan Functions
`real uniform_lpdf(reals y | reals alpha, reals beta)`
The log of the uniform density of y given lower bound alpha and upper bound beta

`real uniform_cdf(reals y, reals alpha, reals beta)`
The uniform cumulative distribution function of y given lower bound alpha and upper bound beta

`real uniform_lcdf(reals y | reals alpha, reals beta)`
The log of the uniform cumulative distribution function of y given lower bound alpha and upper bound beta

`real uniform_lccdf(reals y | reals alpha, reals beta)`
The log of the uniform complementary cumulative distribution function of y given lower bound alpha and upper bound beta

`R uniform_rng(reals alpha, reals beta)`
Generate a uniform variate with lower bound alpha and upper bound beta; may only be used in transformed data and generated quantities blocks. For a description of argument and return types, see section vectorized PRNG functions.

22. Distributions over Unbounded Vectors

The unbounded vector probability distributions have support on all of $\mathbb{R}^K$ for some fixed K.

22.1. Multivariate Normal Distribution

Probability Density Function

If $K \in \mathbb{N}$, $\mu \in \mathbb{R}^K$, and $\Sigma \in \mathbb{R}^{K \times K}$ is symmetric and positive definite, then for $y \in \mathbb{R}^K$,

$$\text{MultiNormal}(y|\mu,\Sigma) = \frac{1}{(2\pi)^{K/2}} \frac{1}{\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(y-\mu)^\top \Sigma^{-1} (y-\mu)\right),$$

where $|\Sigma|$ is the absolute determinant of Σ.

Sampling Statement

`y ~ multi_normal(mu, Sigma)`

Increment target log probability density with `multi_normal_lpdf(y | mu, Sigma)` dropping constant additive terms.

Stan Functions

The multivariate normal probability function is overloaded to allow the variate vector y and location vector μ to be vectors or row vectors (or to mix the two types). The density function is also vectorized, so it allows arrays of row vectors or vectors as arguments; see section vectorized function signatures for a description of vectorization.

`real multi_normal_lpdf(vectors y | vectors mu, matrix Sigma)`
The log of the multivariate normal density of vector(s) y given location vector(s) mu and covariance matrix Sigma

`real multi_normal_lpdf(vectors y | row_vectors mu, matrix Sigma)`
The log of the multivariate normal density of vector(s) y given location row vector(s) mu and covariance matrix Sigma

`real multi_normal_lpdf(row_vectors y | vectors mu, matrix Sigma)`
The log of the multivariate normal density of row vector(s) y given location vector(s) mu and covariance matrix Sigma

`real multi_normal_lpdf(row_vectors y | row_vectors mu, matrix Sigma)`
The log of the multivariate normal density of row vector(s) y given location row vector(s) mu and covariance matrix Sigma

Although there is a direct multi-normal RNG function, if more than one result is required, it's much more efficient to Cholesky factor the covariance matrix and call `multi_normal_cholesky_rng`; see section multi-variate normal, cholesky parameterization.

`vector` **`multi_normal_rng`**`(vector mu, matrix Sigma)`
Generate a multivariate normal variate with location mu and covariance matrix Sigma; may only be used in transformed data and generated quantities blocks

`vector` **`multi_normal_rng`**`(row_vector mu, matrix Sigma)`
Generate a multivariate normal variate with location mu and covariance matrix Sigma; may only be used in transformed data and generated quantities blocks

`vectors` **`multi_normal_rng`**`(vectors mu, matrix Sigma)`
Generate an array of multivariate normal variates with locations mu and covariance matrix Sigma; may only be used in transformed data and generated quantities blocks

`vectors` **`multi_normal_rng`**`(row_vectors mu, matrix Sigma)`
Generate an array of multivariate normal variates with locations mu and covariance matrix Sigma; may only be used in transformed data and generated quantities blocks

22.2. Multivariate Normal Distribution, Precision Parameterization

Probability Density Function

If $K \in \mathbb{N}$, $\mu \in \mathbb{R}^K$, and $\Omega \in \mathbb{R}^{K \times K}$ is symmetric and positive definite, then for $y \in \mathbb{R}^K$,

$$\text{MultiNormalPrecision}(y|\mu,\Omega) = \text{MultiNormal}(y|\mu,\Omega^{-1})$$

Sampling Statement

`y ~` **`multi_normal_prec`**`(mu, Omega)`

Increment target log probability density with `multi_normal_prec_lpdf(y | mu, Omega)` dropping constant additive terms.

Stan Functions

`real` **`multi_normal_prec_lpdf`**`(vectors y | vectors mu, matrix Omega)`
The log of the multivariate normal density of vector(s) y given location vector(s) mu and positive definite precision matrix Omega

`real` **`multi_normal_prec_lpdf`**`(vectors y | row_vectors mu, matrix Omega)`
The log of the multivariate normal density of vector(s) y given location row vector(s) mu and positive definite precision matrix Omega

`real` **`multi_normal_prec_lpdf`**`(row_vectors y | vectors mu, matrix Omega)`
The log of the multivariate normal density of row vector(s) y given location vector(s) mu and positive definite precision matrix Omega

real **multi_normal_prec_lpdf**(row_vectors y | row_vectors mu, matrix Omega)
The log of the multivariate normal density of row vector(s) y given location row vector(s) mu and positive definite precision matrix Omega

22.3. Multivariate Normal Distribution, Cholesky Parameterization

Probability Density Function

If $K \in \mathbb{N}$, $\mu \in \mathbb{R}^K$, and $L \in \mathbb{R}^{K \times K}$ is lower triangular and such that $LL^\top$ is positive definite, then for $y \in \mathbb{R}^K$,

$$\text{MultiNormalCholesky}(y|\mu, L) = \text{MultiNormal}(y|\mu, LL^\top).$$

If L is lower triangular and LL^{top} is a $K \times K$ positive definite matrix, then $L_{k,k}$ must be strictly positive for $k \in 1{:}K$. If an L is provided that is not the Cholesky factor of a positive-definite matrix, the probability functions will raise errors.

Sampling Statement

y ~ **multi_normal_cholesky**(mu, L)

Increment target log probability density with multi_normal_cholesky_lpdf(y | mu, L) dropping constant additive terms.

Stan Functions

real **multi_normal_cholesky_lpdf**(vectors y | vectors mu, matrix L)
The log of the multivariate normal density of vector(s) y given location vector(s) mu and lower-triangular Cholesky factor of the covariance matrix L

real **multi_normal_cholesky_lpdf**(vectors y | row_vectors mu, matrix L)
The log of the multivariate normal density of vector(s) y given location row vector(s) mu and lower-triangular Cholesky factor of the covariance matrix L

real **multi_normal_cholesky_lpdf**(row_vectors y | vectors mu, matrix L)
The log of the multivariate normal density of row vector(s) y given location vector(s) mu and lower-triangular Cholesky factor of the covariance matrix L

real **multi_normal_cholesky_lpdf**(row_vectors y | row_vectors mu, matrix L)
The log of the multivariate normal density of row vector(s) y given location row vector(s) mu and lower-triangular Cholesky factor of the covariance matrix L

vector **multi_normal_cholesky_rng**(vector mu, matrix L)
Generate a multivariate normal variate with location mu and lower-triangular Cholesky factor of the covariance matrix L; may only be used in transformed data and generated quantities blocks

`vector` **`multi_normal_cholesky_rng`**`(row_vector mu, matrix L)`
Generate a multivariate normal variate with location mu and lower-triangular Cholesky factor of the covariance matrix L; may only be used in transformed data and generated quantities blocks

`vectors` **`multi_normal_cholesky_rng`**`(vectors mu, matrix L)`
Generate an array of multivariate normal variates with locations mu and lower-triangular Cholesky factor of the covariance matrix L; may only be used in transformed data and generated quantities blocks

`vectors` **`multi_normal_cholesky_rng`**`(row_vectors mu, matrix L)`
Generate an array of multivariate normal variates with locations mu and lower-triangular Cholesky factor of the covariance matrix L; may only be used in transformed data and generated quantities blocks

22.4. Multivariate Gaussian Process Distribution

Probability Density Function
If $K, N \in \mathbb{N}$, $\Sigma \in \mathbb{R}^{N \times N}$ is symmetric, positive definite kernel matrix and $w \in \mathbb{R}^K$ is a vector of positive inverse scales, then for $y \in \mathbb{R}^{K \times N}$,

$$\text{MultiGP}(y|\Sigma, w) = \prod_{i=1}^{K} \text{MultiNormal}(y_i|0, w_i^{-1}\Sigma),$$

where y_i is the ith row of y. This is used to efficiently handle Gaussian Processes with multi-variate outputs where only the output dimensions share a kernel function but vary based on their scale. Note that this function does not take into account the mean prediction.

Sampling Statement
`y ~` **`multi_gp`**`(Sigma, w)`

Increment target log probability density with `multi_gp_lpdf(y | Sigma, w)` dropping constant additive terms.

Stan Functions
`real` **`multi_gp_lpdf`**`(matrix y | matrix Sigma, vector w)`
The log of the multivariate GP density of matrix y given kernel matrix Sigma and inverses scales w

22.5. Multivariate Gaussian Process Distribution, Cholesky parameterization

Probability Density Function

If $K, N \in \mathbb{N}$, $L \in \mathbb{R}^{N \times N}$ is lower triangular and such that $LL^\top$ is positive definite kernel matrix (implying $L_{n,n} > 0$ for $n \in 1{:}N$), and $w \in \mathbb{R}^K$ is a vector of positive inverse scales, then for $y \in \mathbb{R}^{K \times N}$,

$$\text{MultiGPCholesky}(y \mid L, w) = \prod_{i=1}^{K} \text{MultiNormal}(y_i \mid 0, w_i^{-1} LL^\top),$$

where y_i is the ith row of y. This is used to efficiently handle Gaussian Processes with multi-variate outputs where only the output dimensions share a kernel function but vary based on their scale. If the model allows parameterization in terms of Cholesky factor of the kernel matrix, this distribution is also more efficient than MultiGP(). Note that this function does not take into account the mean prediction.

Sampling Statement

`y ~ multi_gp_cholesky(L, w)`

Increment target log probability density with `multi_gp_cholesky_lpdf(y | L, w)` dropping constant additive terms.

Stan Functions

`real multi_gp_cholesky_lpdf(matrix y | matrix L, vector w)`

The log of the multivariate GP density of matrix y given lower-triangular Cholesky factor of the kernel matrix L and inverses scales w

22.6. Multivariate Student-T Distribution

Probability Density Function

If $K \in \mathbb{N}$, $\nu \in \mathbb{R}^+$, $\mu \in \mathbb{R}^K$, and $\Sigma \in \mathbb{R}^{K \times K}$ is symmetric and positive definite, then for $y \in \mathbb{R}^K$,

$$\begin{aligned} &\text{MultiStudentT}(y \mid \nu, \mu, \Sigma) \\ &= \tfrac{1}{\pi^{K/2}} \, \tfrac{1}{\nu^{K/2}} \, \tfrac{\Gamma((\nu+K)/2)}{\Gamma(\nu/2)} \, \tfrac{1}{\sqrt{|\Sigma|}} \left(1 + \tfrac{1}{\nu} \, (y-\mu)^\top \, \Sigma^{-1} \, (y-\mu)\right)^{-(\nu+K)/2}. \end{aligned}$$

Sampling Statement

`y ~ multi_student_t(nu, mu, Sigma)`

Increment target log probability density with `multi_student_t_lpdf(y | nu, mu, Sigma)` dropping constant additive terms.

Stan Functions

`real` **`multi_student_t_lpdf`**`(vectors y | real nu, vectors mu, matrix Sigma)`
The log of the multivariate Student-*t* density of vector(s) y given degrees of freedom nu, location vector(s) mu, and scale matrix Sigma

`real` **`multi_student_t_lpdf`**`(vectors y | real nu, row_vectors mu, matrix Sigma)`
The log of the multivariate Student-*t* density of vector(s) y given degrees of freedom nu, location row vector(s) mu, and scale matrix Sigma

`real` **`multi_student_t_lpdf`**`(row_vectors y | real nu, vectors mu, matrix Sigma)`
The log of the multivariate Student-*t* density of row vector(s) y given degrees of freedom nu, location vector(s) mu, and scale matrix Sigma

`real` **`multi_student_t_lpdf`**`(row_vectors y | real nu, row_vectors mu, matrix Sigma)`
The log of the multivariate Student-*t* density of row vector(s) y given degrees of freedom nu, location row vector(s) mu, and scale matrix Sigma

`vector` **`multi_student_t_rng`**`(real nu, vector mu, matrix Sigma)`
Generate a multivariate Student-*t* variate with degrees of freedom nu, location mu, and scale matrix Sigma; may only be used in transformed data and generated quantities blocks

`vector` **`multi_student_t_rng`**`(real nu, row_vector mu, matrix Sigma)`
Generate a multivariate Student-*t* variate with degrees of freedom nu, location mu, and scale matrix Sigma; may only be used in transfomed data and generated quantities blocks

`vectors` **`multi_student_t_rng`**`(real nu, vectors mu, matrix Sigma)`
Generate an array of multivariate Student-*t* variates with degrees of freedom nu, locations mu, and scale matrix Sigma; may only be used in transformed data and generated quantities blocks

`vectors` **`multi_student_t_rng`**`(real nu, row_vectors mu, matrix Sigma)`
Generate an array of multivariate Student-*t* variates with degrees of freedom nu, locations mu, and scale matrix Sigma; may only be used in transformed data andgenerated quantities blocks

22.7. Gaussian Dynamic Linear Models

A Gaussian Dynamic Linear model is defined as follows, For $t \in 1, \ldots, T$,

$$
\begin{aligned}
y_t &\sim N(F'\theta_t, V) \\
\theta_t &\sim N(G\theta_{t-1}, W) \\
\theta_0 &\sim N(m_0, C_0)
\end{aligned}
$$

where y is $n \times T$ matrix where rows are variables and columns are observations. These functions calculate the log-likelihood of the observations marginalizing over the latent states ($p(y|F, G, V, W, m_0, C_0)$). This log-likelihood is a system that is calculated using the Kalman Filter. If V is diagonal, then a more efficient algorithm which sequentially processes observations and avoids a matrix inversions can be used (Durbin and Koopman 2001, sec. 6.4).

Sampling Statement

`y ~ gaussian_dlm_obs(F, G, V, W, m0, C0)`

Increment target log probability density with `gaussian_dlm_obs_lpdf(y | F, G, V, W, m0, C0)` dropping constant additive terms.

Stan Functions

The following two functions differ in the type of their V, the first taking a full observation covariance matrix V and the second a vector V representing the diagonal of the observation covariance matrix. The sampling statement defined in the previous section works with either type of observation V.

`real gaussian_dlm_obs_lpdf(matrix y | matrix F, matrix G, matrix V, matrix W, vector m0, matrix C0)`

The log of the density of the Gaussian Dynamic Linear model with observation matrix y in which rows are variables and columns are observations, design matrix F, transition matrix G, observation covariance matrix V, system covariance matrix W, and the initial state is distributed normal with mean m0 and covariance C0.

`real gaussian_dlm_obs_lpdf(matrix y | matrix F, matrix G, vector V, matrix W, vector m0, matrix C0)`

The log of the density of the Gaussian Dynamic Linear model with observation matrix y in which rows are variables and columns are observations, design matrix F, transition matrix G, observation covariance matrix with diagonal V, system covariance matrix W, and the initial state is distributed normal with mean m0 and covariance C0.

23. Simplex Distributions

The simplex probabilities have support on the unit K-simplex for a specified K. A K-dimensional vector θ is a unit K-simplex if $\theta_k \geq 0$ for $k \in \{1, \ldots, K\}$ and $\sum_{k=1}^{K} \theta_k = 1$.

23.1. Dirichlet Distribution

Probability Density Function

If $K \in \mathbb{N}$ and $\alpha \in (\mathbb{R}^+)^K$, then for $\theta \in K$-simplex,

$$\text{Dirichlet}(\theta|\alpha) = \frac{\Gamma\left(\sum_{k=1}^{K} \alpha_k\right)}{\prod_{k=1}^{K} \Gamma(\alpha_k)} \prod_{k=1}^{K} \theta_k^{\alpha_k - 1}.$$

Warning: If any of the components of θ satisfies $\theta_i = 0$ or $\theta_i = 1$, then the probability is 0 and the log probability is $-\infty$. Similarly, the distribution requires strictly positive parameters, with $\alpha_i > 0$ for each i.

Meaning of Dirichlet Parameters

A symmetric Dirichlet prior is $[\alpha, \ldots, \alpha]^\top$. To code this in Stan,

```
data {
  int<lower = 1> K;
  real<lower = 0> alpha;
}
generated quantities {
  vector[K] theta = dirichlet_rng(rep_vector(alpha, K));
}
```

Taking $K = 10$, here are the first five draws for $\alpha = 0.001$. For $\alpha = 1$, the distribution is uniform over simplexes.

```
1) 0.17 0.05 0.07 0.17 0.03 0.13 0.03 0.03 0.27 0.05
2) 0.08 0.02 0.12 0.07 0.52 0.01 0.07 0.04 0.01 0.06
3) 0.02 0.03 0.22 0.29 0.17 0.10 0.09 0.00 0.05 0.03
4) 0.04 0.03 0.21 0.13 0.04 0.01 0.10 0.04 0.22 0.18
5) 0.11 0.22 0.02 0.01 0.06 0.18 0.33 0.04 0.01 0.01
```

That does not mean it's uniform over the marginal probabilities of each element. As the size of the simplex grows, the marginal draws become more and more concentrated below (not around) $1/K$. When one component of the simplex is large, the others

must all be relatively small to compensate. For example, in a uniform distribution on 10-simplexes, the probability that a component is greater than the mean of $1/10$ is only 39%. Most of the posterior marginal probability mass for each component is in the interval $(0, 0.1)$.

When the α value is small, the draws gravitate to the corners of the simplex. Here are the first five draws for $\alpha = 0.001$.

```
1) 3e-203 0e+00 2e-298 9e-106 1e+000 0e+00 0e+000 1e-047 0e+00 4e-279
2) 1e+000 0e+00 5e-279 2e-014 1e-275 0e+00 3e-285 9e-147 0e+00 0e+000
3) 1e-308 0e+00 1e-213 0e+000 0e+000 8e-75 0e+000 1e+000 4e-58 7e-112
4) 6e-166 5e-65 3e-068 3e-147 0e+000 1e+00 3e-249 0e+000 0e+00 0e+000
5) 2e-091 0e+00 0e+000 0e+000 1e-060 0e+00 4e-312 1e+000 0e+00 0e+000
```

Each row denotes a draw. Each draw has a single value that rounds to one and other values that are very close to zero or rounded down to zero.

As α increases, the draws become increasingly uniform. For $\alpha = 1000$,

```
1) 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10
2) 0.10 0.10 0.09 0.10 0.10 0.10 0.11 0.10 0.10 0.10
3) 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10
4) 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10
5) 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10 0.10
```

Sampling Statement
`theta ~ dirichlet(alpha)`

Increment target log probability density with `dirichlet_lpdf(theta | alpha)` dropping constant additive terms.

Stan Functions
`real dirichlet_lpdf(vector theta | vector alpha)`
The log of the Dirichlet density for simplex theta given prior counts (plus one) alpha

`vector dirichlet_rng(vector alpha)`
Generate a Dirichlet variate with prior counts (plus one) alpha; may only be used in transformed data and generated quantities blocks

24. Correlation Matrix Distributions

The correlation matrix distributions have support on the (Cholesky factors of) correlation matrices. A Cholesky factor L for a $K \times K$ correlation matrix Σ of dimension K has rows of unit length so that the diagonal of $LL^\top$ is the unit K-vector. Even though models are usually conceptualized in terms of correlation matrices, it is better to operationalize them in terms of their Cholesky factors. If you are interested in the posterior distribution of the correlations, you can recover them in the generated quantities block via

```
generated quantities {
  corr_matrix[K] Sigma;
  Sigma = multiply_lower_tri_self_transpose(L);
}
```

24.1. LKJ Correlation Distribution

Probability Density Function

For $\eta > 0$, if Σ a positive-definite, symmetric matrix with unit diagonal (i.e., a correlation matrix), then

$$\text{LkjCorr}(\Sigma|\eta) \propto \det(\Sigma)^{(\eta-1)}.$$

The expectation is the identity matrix for any positive value of the shape parameter η, which can be interpreted like the shape parameter of a symmetric beta distribution:

- if $\eta = 1$, then the density is uniform over correlation matrices of order K;
- if $\eta > 1$, the identity matrix is the modal correlation matrix, with a sharper peak in the density at the identity matrix for larger η; and
- for $0 < \eta < 1$, the density has a trough at the identity matrix.
- if η were an unknown parameter, the Jeffreys prior is proportional to $\sqrt{2\sum_{k=1}^{K-1}\left(\psi_1\left(\eta + \frac{K-k-1}{2}\right) - 2\psi_1\left(2\eta + K - k - 1\right)\right)}$, where $\psi_1()$ is the trigamma function

See (Lewandowski, Kurowicka, and Joe 2009) for definitions. However, it is much better computationally to work directly with the Cholesky factor of Σ, so this distribution should never be explicitly used in practice.

Sampling Statement
`y ~ lkj_corr(eta)`

Increment target log probability density with `lkj_corr_lpdf(y | eta)` dropping constant additive terms.

Stan Functions
`real lkj_corr_lpdf(matrix y | real eta)`
The log of the LKJ density for the correlation matrix y given nonnegative shape eta. The only reason to use this density function is if you want the code to run slower and consume more memory with more risk of numerical errors. Use its Cholesky factor as described in the next section.

`matrix lkj_corr_rng(int K, real eta)`
Generate a LKJ random correlation matrix of order K with shape eta; may only be used in transformed data and generated quantities blocks

24.2. Cholesky LKJ Correlation Distribution

Stan provides an implicit parameterization of the LKJ correlation matrix density in terms of its Cholesky factor, which you should use rather than the explicit parameterization in the previous section. For example, if `L` is a Cholesky factor of a correlation matrix, then

```
L ~ lkj_corr_cholesky(2.0); # implies L * L' ~ lkj_corr(2.0);
```

Because Stan requires models to have support on all valid constrained parameters, `L` will almost always[1] be a parameter declared with the type of a Cholesky factor for a correlation matrix; for example,

```
parameters {   cholesky_factor_corr[K] L;   # rather than corr_matrix[K] S
```

Probability Density Function
For $\eta > 0$, if L is a $K \times K$ lower-triangular Cholesky factor of a symmetric positive-definite matrix with unit diagonal (i.e., a correlation matrix), then

$$\text{LkjCholesky}(L|\eta) \propto |J| \det(LL^\top)^{(\eta-1)} = \prod_{k=2}^{K} L_{kk}^{K-k+2\eta-2}.$$

See the previous section for details on interpreting the shape parameter η. Note that even if $\eta = 1$, it is still essential to evaluate the density function because the density of L is not constant, regardless of the value of η, even though the density of $LL^\top$ is constant iff $\eta = 1$.

[1] It is possible to build up a valid `L` within Stan, but that would then require Jacobian adjustments to imply the intended posterior.

A lower triangular L is a Cholesky factor for a correlation matrix if and only if $L_{k,k} > 0$ for $k \in 1{:}K$ and each row L_k has unit Euclidean length.

Sampling Statement
`L ~ lkj_corr_cholesky(eta)`

Increment target log probability density with `lkj_corr_cholesky_lpdf(L | eta)` dropping constant additive terms.

Stan Functions
`real lkj_corr_cholesky_lpdf(matrix L | real eta)`
The log of the LKJ density for the lower-triangular Cholesky factor L of a correlation matrix given shape eta.

`matrix lkj_corr_cholesky_rng(int K, real eta)`
Generate a random Cholesky factor of a correlation matrix of order K that is distributed LKJ with shape eta; may only be used in transformed data and generated quantities blocks

25. Covariance Matrix Distributions

The covariance matrix distributions have support on symmetric, positive-definite $K \times K$ matrices.

25.1. Wishart Distribution

Probability Density Function

If $K \in \mathbb{N}$, $\nu \in (K-1, \infty)$, and $S \in \mathbb{R}^{K \times K}$ is symmetric and positive definite, then for symmetric and positive-definite $W \in \mathbb{R}^{K \times K}$,

$$\text{Wishart}(W|\nu, S) = \frac{1}{2^{\nu K/2}} \frac{1}{\Gamma_K\left(\frac{\nu}{2}\right)} |S|^{-\nu/2} |W|^{(\nu-K-1)/2} \exp\left(-\frac{1}{2} \text{tr}\left(S^{-1}W\right)\right),$$

where $\text{tr}()$ is the matrix trace function, and $\Gamma_K()$ is the multivariate Gamma function,

$$\Gamma_K(x) = \frac{1}{\pi^{K(K-1)/4}} \prod_{k=1}^{K} \Gamma\left(x + \frac{1-k}{2}\right).$$

Sampling Statement

`W ~ wishart(nu, Sigma)`

Increment target log probability density with `wishart_lpdf(W | nu, Sigma)` dropping constant additive terms.

Stan Functions

`real wishart_lpdf(matrix W | real nu, matrix Sigma)`
The log of the Wishart density for symmetric and positive-definite matrix W given degrees of freedom nu and symmetric and positive-definite scale matrix Sigma

`matrix wishart_rng(real nu, matrix Sigma)`
Generate a Wishart variate with degrees of freedom nu and symmetric and positive-definite scale matrix Sigma; may only be used in transformed data and generated quantities blocks

25.2. Inverse Wishart Distribution

Probability Density Function

If $K \in \mathbb{N}$, $\nu \in (K-1, \infty)$, and $S \in \mathbb{R}^{K \times K}$ is symmetric and positive definite, then for symmetric and positive-definite $W \in \mathbb{R}^{K \times K}$,

$$\text{InvWishart}(W|\nu, S) = \frac{1}{2^{\nu K/2}} \frac{1}{\Gamma_K\left(\frac{\nu}{2}\right)} |S|^{\nu/2} |W|^{-(\nu+K+1)/2} \exp\left(-\frac{1}{2} \text{tr}(SW^{-1})\right).$$

Sampling Statement

`W ~ inv_wishart(nu, Sigma)`

Increment target log probability density with `inv_wishart_lpdf(W | nu, Sigma)` dropping constant additive terms.

Stan Functions

`real inv_wishart_lpdf(matrix W | real nu, matrix Sigma)`
The log of the inverse Wishart density for symmetric and positive-definite matrix W given degrees of freedom nu and symmetric and positive-definite scale matrix Sigma

`matrix inv_wishart_rng(real nu, matrix Sigma)`
Generate an inverse Wishart variate with degrees of freedom nu and symmetric and positive-definite scale matrix Sigma; may only be used in transformed data and generated quantities blocks

Additional Distributions

26. Hidden Markov Models

An elementary first-order Hidden Markov model is a probabilistic model over N observations, y_n, and N hidden states, x_n, which can be fully defined by the conditional distributions $p(y_n \mid x_n, \phi)$ and $p(x_n \mid x_{n-1}, \phi)$. Here we make the dependency on additional model parameters, ϕ, explicit. When x is continuous, the user can explicitly encode these distributions in Stan and use Markov chain Monte Carlo to integrate x out.

When each state x takes a value over a discrete and finite set, say $\{1, 2, ..., K\}$, we can take advantage of the dependency structure to marginalize x and compute $p(y \mid \phi)$. We start by defining the conditional observational distribution, stored in a $K \times N$ matrix ω with

$$\omega_{kn} = p(y_n \mid x_n = k, \phi).$$

Next, we introduce the $K \times K$ transition matrix, Γ, with

$$\Gamma_{ij} = p(x_n = j \mid x_{n-1} = i, \phi).$$

Each row defines a probability distribution and must therefore be a simplex (i.e. its components must add to 1). Currently, Stan only supports stationary transitions where a single transition matrix is used for all transitions. Finally we define the initial state K-vector ρ, with

$$\rho_k = p(x_0 = k \mid \phi).$$

The Stan functions that support this type of model are special in that the user does not explicitly pass y and ϕ as arguments. Instead, the user passes $\log \omega$, Γ, and ρ, which in turn depend on y and ϕ.

26.1. Stan functions

`real` **`hmm_marginal`**`(matrix log_omega, matrix Gamma, vector rho)`
Returns the log probability density of y, with x_n integrated out at each iteration. The arguments represent (1) the log density of each output, (2) the transition matrix, and (3) the initial state vector.

- *`log_omega`*: $\log \omega_{kn} = \log p(y_n \mid x_n = k, \phi)$, log density of each output,
- *`Gamma`*: $\Gamma_{ij} = p(x_n = j | x_{n-1} = i, \phi)$, the transition matrix,
- *`rho`*: $\rho_k = p(x_0 = k \mid \phi)$, the initial state probability.

`int[] hmm_latent_rng(matrix log_omega, matrix Gamma, vector rho)`
Returns a length N array of integers over $\{1, ..., K\}$, sampled from the joint posterior distribution of the hidden states, $p(x \mid \phi, y)$. May be only used in transformed data and generated quantities.

`matrix hmm_hidden_state_prob(matrix log_omega, matrix Gamma, vector rho)`
Returns the matrix of marginal posterior probabilities of each hidden state value. This will be a $K \times N$ matrix. The i^{th} column is a simplex of probabilities for the n^{th} variable. Moreover, let A be the output. Then $A_{ij} = p(x_j = i \mid \phi, y)$. This function may only be used in transformed data and generated quantities.

Appendix

27. Mathematical Functions

This appendix provides the definition of several mathematical functions used throughout the manual.

27.1. Beta

The beta function, $\mathrm{B}(a,b)$, computes the normalizing constant for the beta distribution, and is defined for $a > 0$ and $b > 0$ by

$$\mathrm{B}(a,b) \;=\; \int_0^1 u^{a-1}(1-u)^{b-1}\,du \;=\; \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,,$$

where $\Gamma(x)$ is the Gamma function.

27.2. Incomplete Beta

The incomplete beta function, $\mathrm{B}(x;a,b)$, is defined for $x \in [0,1]$ and $a,b \geq 0$ such that $a+b \neq 0$ by

$$\mathrm{B}(x;\,a,b) \;=\; \int_0^x u^{a-1}\,(1-u)^{b-1}\,du,$$

where $\mathrm{B}(a,b)$ is the beta function defined in appendix. If $x = 1$, the incomplete beta function reduces to the beta function, $\mathrm{B}(1;a,b) = \mathrm{B}(a,b)$.

The regularized incomplete beta function divides the incomplete beta function by the beta function,

$$I_x(a,b) \;=\; \frac{\mathrm{B}(x;\,a,b)}{B(a,b)}\,.$$

27.3. Gamma

The gamma function, $\Gamma(x)$, is the generalization of the factorial function to continuous variables, defined so that for positive integers n,

$$\Gamma(n+1) = n!$$

Generalizing to all positive numbers and non-integer negative numbers,

$$\Gamma(x) = \int_0^\infty u^{x-1}\exp(-u)\,du.$$

27.4. Digamma

The digamma function Ψ is the derivative of the $\log\Gamma$ function,

$$\Psi(u) \;=\; \frac{d}{du}\log\Gamma(u) \;=\; \frac{1}{\Gamma(u)}\;\frac{d}{du}\Gamma(u).$$

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