

The formula for the second order Richardson Extrapolation is :

$$A_2(h) = \frac{4f(a+h/2) - 3f(a) - f(a+h)}{h} + O(h^2/12)$$

The error term in $A_2(h)$ is $h^2/12 f'''(a)$

Substituting $A_2(h/2)$ gives an error of $(h^2/48) f'''(a)$

In order to remove this error term we need:

$$4A_2(h/2) - A_2(h)$$

This gives:

$$\begin{aligned} & \frac{16f(a+h/4) - 12f(a) - 4f(a+h/2)}{h/2} - \left[\frac{4f(a+h/2) - 3f(a) - f(a+h)}{h} \right] \\ &= \frac{32f(a+h/4) - 24f(a) - 8f(a+h/2)}{h} - \left[\frac{4f(a+h/2) - 3f(a) - f(a+h)}{h} \right] \\ &= \frac{32f(a+h/4) - 21f(a) - 12f(a+h/2) + f(a+h)}{h} \end{aligned}$$

And we know $A_3(h) = \frac{4A_2(h/2) - A_2(h)}{3} + O(h^2)$

so we have:

$$A_3(h) = \frac{32f(a+h/4) - 21f(a) - 12f(a+h/2) + f(a+h)}{3h}$$

As the equation for the third order accurate Richardson Extrapolation.