ECON5529

Problem Set #3

Due December 4th

Please type up your answers (Latex, Mathematica, Word, etc.) and email them as a pdf file to scharfenakere@umkc.edu.

ANALYTICAL

1. Find the maximum entropy distributions over the following sets with the described constraints.
   a) The whole real line constrained to have a given value of its variance.
   b) The nonnegative real half-line constrained to have a given geometric and arithmetic mean.
   c) The whole real line constrained to have a given mean and absolute mean.
2. The entropy difference between two distributions is measured by the Kullback-Leibler divergence. Given data x in the form of a vector of binned frequencies of n observations, $\bar{f}[x_k] = \left\{\frac{x_1}{n}, ..., \frac{x_K}{n}\right\}$, we can always view this data as a sample of a multinomial model with predicted frequencies $\hat{f}[x_k] = \left\{p_1, \ldots, p_K\right\}$. The likelihood function for a sample $x = \{x_1, \ldots, x_K\}$ (where x_j is the number of observations in bin j) from a multinomial distribution, writing $n = \sum_{k=1}^{K} x_k$ for the sample size, is:

$$P[x|p] = \frac{n!}{x_1! \ldots x_K!} p_1^{x_1} \ldots p_K^{x_K} \tag{0.1}$$

   Using Stirling's approximation, $\log[x!] \approx x\log[x] - x$, prove that

$$P[x|p] = e^{-n\sum_{k=1}^{K}\tilde{f}[x_k]\log\left[\frac{\tilde{f}[x_k]}{\hat{f}[x_k]}\right]}$$
$$= e^{-n\mathrm{KL}[\tilde{f}[x_k],\hat{f}[x_k]]} \tag{0.2}$$

That is, show that the log of the multinomial likelihood is equivalent to the Kullback-Leibler divergence between the empirical and theoretical bin frequencies.

3. Shannon's theorem is a useful and intuitive point of departure for modeling the uncertainty in an economic agents' decision environment. It quantifies the amount of choice variability for a given decision environment and is an increasing function of the number of choices available. Because Shannon entropy is a measure of uncertainty it can provide a meaningful probabilistic description of a decision environment by being incorporated into the behavior of agents as a behavioral constraint. Assuming the typical agent conditions their discrete choices a on perceived signals x from their social environment, we can summarize individual behavior as responding to a payoff function $v[a,x]$. The typical agent's mixed strategy in this scenario is defined by the conditional probability distribution $p[a|x] : A \times X \to (0,1)$ over the actions $a \in A$ that maximizes the expected payoff, $\sum_a p[a|x]\,v[a,x]$. The agent's expected payoff maximization program subject to a constraint on the minimum entropy of the mixed strategy distribution is:

$$\begin{aligned} \underset{\{p[a|x]\geq 0\}}{\mathrm{Max}} & \sum_a p[a|x]\,v[a,x] \\ \text{subject to } & \sum_a p[a|x] = 1 \\ & -\sum_a p[a|x]\mathrm{Log}[p[a|x]] \geq H_{min} \end{aligned} \tag{0.3}$$

   a) Find the maximum entropy distribution over the action set conditional on x, that solve the programming problem for $p[a|x]$.
   b) If the payoff function is equal to the outcome $v[a,x] = x$ and the action set consists of only two actions $a = \{a_1, a_2\}$, what is the MAXENT distribution for $p[a|x]$?
   c) For this MAXENT distribution determine what happens in limit when the shadow price of reducing the entropy constraint goes to zero and infinity.