ECON5529
Bayesian Theory

Lecture 10

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Economic Time Series

- Much important and interesting economic data, particularly macroeconomic data, is in the form of observations over time, or *time series*.
- Recall, if we believe that there is some information in the order of time series data $\{y_1, \ldots, y_T\}$, then time series data is not exchangeable, since scrambling the data will destroy the temporal correlations.
- If we believe, however, that the only correlations that are important are between one period and the next, then the "stacked" observations, $\{\{y_1, y_2\}, \{y_2, y_3\}, \ldots, \{y_{T-1}, y_T\}\}$ are exchangeable, since we would view any sequence of two successive observations of the data as carrying the same information.
- This “stacking” of observations is the underlying hypothesis about autoregressive processes. Therefore, an AR(1) model assumes the current value y_t is linearly dependent only on its preceding value y_{t-1} and is of the form:

$$y_t = \alpha + \rho\, y_{t-1} + \epsilon_t \qquad t = 1, 2, \ldots T, \;\; \epsilon_t \sim \text{White Noise}$$

- White noise implies $E[\epsilon_t] = 0$, $E\left[\epsilon_t^2\right] = \sigma_\epsilon^2$, and $E\left[\epsilon_t\, \epsilon_{t-j}\right] = 0$. If the autoregressive parameter $|\rho| < 1$ the process is said to be stationary. If $\rho = 1$ the process is nonstationary and called a random walk or unit root process.
- We can solve a stationary AR(1) process by noting that this is a first order difference equation which can be solved by recursive substitution

$$y_t = \alpha + \rho\, y_{t-1} + \epsilon_t$$
$$y_{t-1} = \alpha + \rho\, y_{t-2} + \epsilon_{t-1}$$

- which implies

$$y_t = \alpha + \rho\,(\alpha + \rho\, y_{t-2} + \epsilon_{t-1}) + \epsilon_t$$
$$y_t = \alpha + \rho\,\alpha + \rho^2\, y_{t-2} + \rho\,\epsilon_{t-1} + \epsilon_t$$

- Continuing in this way ...

$$y_t = (\alpha + \epsilon_t) + \rho\,(\alpha + \epsilon_{t-1}) + \rho^2(\alpha + \epsilon_{t-2}) + \ldots$$
$$y_t = \frac{\alpha}{1-\rho} + \epsilon_t + \rho\,\epsilon_{t-1} + \rho^2\,\epsilon_{t-2} + \ldots$$

- Thus, the expected value of the series is

$$E[y_t] = E\left[\frac{\alpha}{1-\rho} + \epsilon_t + \rho\,\epsilon_{t-1} + \rho^2\,\epsilon_{t-2} + \ldots\right] = \frac{\alpha}{1-\rho}$$

- When $|\rho| < 1$ the model will have a constant expected value $E[y_t] = E[y_{t-s}] = \frac{\alpha}{1-\rho}$, $\forall\, s$.

Stationarity

- Most economic time series is non-stationary and dominated by a trend due to e.g. population growth, inflation, and technological advances.
- Non-stationary time series make it difficult to study regularities other than the trend, for example, cyclical fluctuations, trend reversion after a shock, or correlation with other time series.
- For example, a researcher who discovers a strong positive correlation between the number of people who drowned by falling into a pool and the number of films Nicolas Cage appeared in, may insist we abolish Nicolas Cage productions in a futile attempt to reduce pool drownings.
- The orthodox statistical method for dealing with non-stationary time series is to apply *ad hoc* filtering devices such as *detrending* and *seasonal adjustment*.
- The philosophy behind filtering time series is to produce new and different data which represents an estimate of what the series would look like with the contaminating effects removed.

Making the prior explicit

- Frequently time series are “decomposed” in order to isolate and study the *trend* and *cycle* components.
- In some cases reframing statistical procedures in Bayesian terms to make the prior explicit can provide valuable insights into what the procedure is actually accomplishing.
- As an example, consider the “Hodrick-Prescott filter” often used in modern econometrics to decompose time series into a trend and a cyclical component.
- The data for the Hodrick-Prescott filter is a time series $\{y_1, \ldots, y_T\}$. The idea is to use this data to find a time trend $\{\tau_1, \ldots, \tau_T\}$ in order to identify the “cyclical” or “residual” deviation of the series from the trend, $\{c_1, \ldots, c_T\}$, where $c_t = y_t - \tau_t$.
- The algorithm implemented in many statistical software packages returns the result of the minimization problem:

$$\underset{\{\tau_t\}}{\text{Min}}\left(\sum_{t=1}^{T} (y_t - \tau_t)^2 - \lambda \sum_{t=2}^{T-1} ((\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}))^2\right)$$

- The first term penalizes the cyclical component, the second term (the sum of the squares of the trend component’s second differences) penalizes variations in the growth rate of the trend component.
- The variable λ is treated as a parameter to be *chosen* by the researcher that determines the strength of the penalty. Hodrick and Prescott recommend $\lambda = 1600$ for quarterly time series data and 6.25 for annual data.
- The H-P filter is frequently used in macroeconomics where deviations from trend real GDP are interpreted as a measure of how far above or below “potential GDP” the economy is operating.
- We can see from this definition that what the H-P filter does is to trade off fit (the first component in the minimization) against the smoothness of the trend, which is measured by the cumulative variation of the second difference of the trend series.

- If $\tau_{t+1} - \tau_t = b$ where b is a constant, the second component of the minimization would be zero, and the trend would be a straight line with slope b.
- We saw this already with the employment and income data where we estimated the trend using OLS and plotted the residuals which appeared as cycles.
- In general , if the data are logarithms of some time series then the constant slope b is the average rate of growth of the time series over the observation period, and the $\{c_t\}$ would be deviations from a constant trend rate of growth.
- The H-P filter allows the trend to vary over the observation period, but penalizes changes in trend by the parameter λ. If $\lambda = 0$, the best fit would be achieved by taking $\tau_t = y_t$, but then the filter would not succeed in identifying a trend different from the data at all.

Bayesian interpretation of the H-P filter

- From a Bayesian point of view the minimization in the H-P filter looks like the maximization of a posterior probability where the first term $\sum_{t=1}^{T} (y_t - \tau_t)^2$ looks like the likelihood function of the trend as a model based on the sum of squared errors.
- The term $\lambda \sum_{t=2}^{T-1} ((\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}))^2$ looks like a prior that puts a higher probability on smooth trend models and a lower probability on jagged trend models that allowed frequent changes in the underlying trend growth rate.
- Thus we could re-phrase the H-P filter in Bayesian terms as modeling the data $\{y_t\}$ by a trend $\{\tau_t\}$ with a likelihood

  $$p[D:\{y_t\} \mid H:\{\tau_t\}] = \prod_{t=1}^{T} e^{-(y_t - \tau_t)^2}$$

- or log likelihood

  $$\mathrm{Log}[p[D:\{y_t\} \mid H:\{\tau_t\}]] = -\sum_{t=1}^{T} (y_t - \tau_t)^2$$

- and a log prior

  $$p[H:\{\tau_t\}] = -\lambda \sum_{t=2}^{T-1} ((\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}))^2$$

- This leads to the log posterior:

  $$\mathrm{Log}[p[H:\{\tau_t\} \mid D:\{y_t\}]] = -\sum_{t=1}^{T} (y_t - \tau_t)^2 - \lambda \sum_{t=2}^{T-1} ((\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}))^2$$

- Maximizing the log posterior is equivalent to the minimization problem in the algorithm defining the H-P filter. Therefore, in this case the non-Bayesian procedure can be rationalized in terms of a well-defined Bayesian prior.
- The Bayesian approach to the H-P filter emphasizes several important points.
  - First, there is no “right” choice for the parameter λ, which is a free parameter in the Bayesian prior.
  - Second, there is no "objective" way to decompose time series into a trend and a cyclical residual.

Detrending Time Series

- There are three frequently used methods for detrending economic time series data: first differencing, log-linear detrending, and the HP-Filter.
- Neoclassic growth theory most frequently suggests the log-linear method while in econometrics and financial analysis the first difference method is preferred.

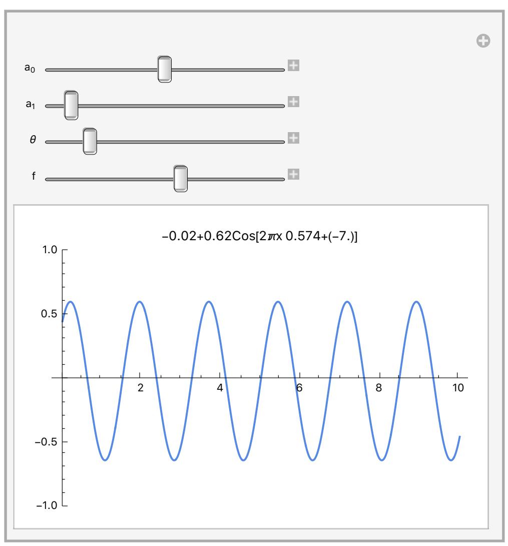
- From a signal processing perspective, the first difference is the ***smallest*** time-window and includes a ***high-pass filter*** that gives the resulting time series the closest appearance to white noise.
- On the other hand, log-linear detrending has the ***largest*** time-window since it covers the entire trended series and thus has a ***low-pass filter***.
- The HP-Filter has a nonlinear user-specified window and thus has a ***band-pass filter*** that can be used to extract harmonics in the range of business cycle frequencies.

Frequency Domain

- Dynamic interactions in complex systems often produce different types of cyclical motions in different frequency regions. For example seasonal fluctuations with a one-year period, business cycle fluctuations of a 5-8 year period, Goodwin cycles in the labor market, and "long wave" cycles of a period of around 50 years. For this reason studying time series in the frequency domain can offer an illuminating perspective on data.
- The transformation of time series data into the *frequency domain* is done through the ***Fourier transform*** which takes a periodic signal as a function of time $x[t]$ and transforms it into a complex valued function of angular frequency $x[\omega]$.
- In this form the time series data are broken down sinusoids of different frequencies. That is, the Fourier transformed data appear as a collection of cyclical oscillations at a spectrum of various frequencies.
- The core building block of the frequency domain is the cosine function, which represents regular repeating fluctuations.

$$y_t = a_0 + a_1 \mathrm{Cos}[2\,\pi\, f\, t + \theta]$$

- Where a_0 is the ***Direct Current*** (DC) component of the signal, a_1 is the ***amplitude*** of the harmonic, f is the ***frequency*** or ***harmonic number***, and θ is the ***phase angle*** of the harmonics.



Frequencies, periods, and phases

- The arguments to trigonometric functions are angles expressed in *radians*, the angle covered by the radius of the circle laid out along the circle. It takes $2\,\pi$ radians to go around the circle. Thus, in the display the frequency f is measured in *cycles per unit time* because of the factor $2\,\pi$ multiplying it. This measure of cyclical frequency is called the *angular* or *radial frequency:*

$$\omega = \frac{2\pi}{T} = 2\pi f$$

- Where T is the time period measured in seconds and f is the ordinary frequency measured in hertz (cycles per second).
- We also often think of cycles in terms of their *period*, that is, the length of time it takes to complete one cycle, or the inverse of the frequency, $T = \frac{1}{f}$. Thus a frequency of 1/4 cycles per quarter corresponds to a period of 4 quarters, or one year.
- It is always possible to change the unit of cycles using the conversion:
- Angular frequency : $\omega = \frac{2\pi}{T} = 2\pi f$

  Ordinary frequency : $f = \frac{1}{T}$

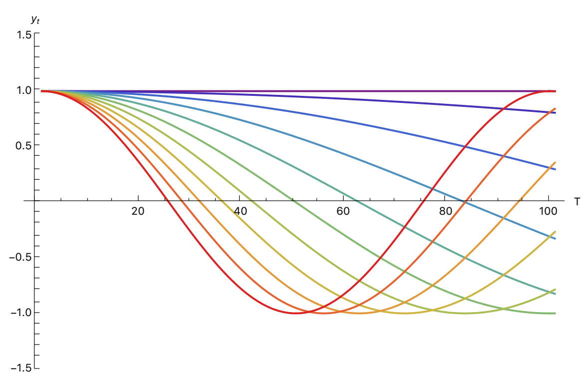
  Period : $T = \frac{1}{f}$

Fourier and Harmonic Analysis

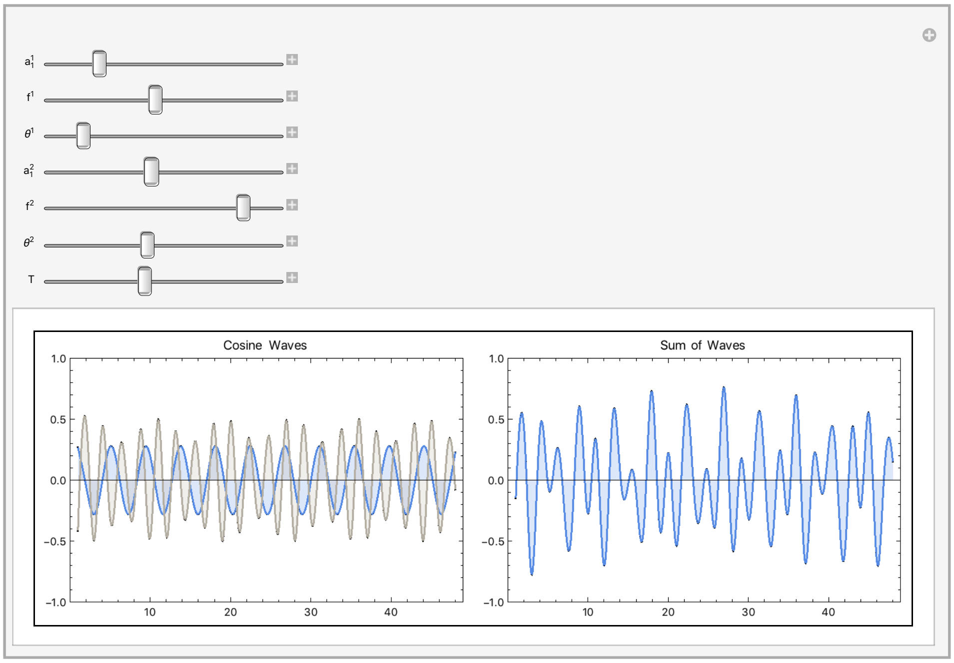
- Suppose we have data in the form of a time series $\{y_0, \ldots, y_{T-1}\}$ with T finite observations. Joseph Fourier (1768-1830) noticed that it is possible to fit arbitrary time series of this type with cosine functions of varying frequencies. His idea was to model the time series data as a sum of k cosine functions of various frequencies.

$$y_t = a_0 + \sum_{k=0}^{\frac{T-1}{2}} a_k \operatorname{Cos}\left[\frac{2\pi k t}{T-1} + \theta_k\right]$$

- The coefficient a_k is the *amplitude* of the frequency k cycle, and the parameter θ_k is the *phase* of the frequency k cycle in this modeling.
- The lowest frequency corresponding to $k = 0$ is 0 cycles per time unit. The next frequency is $\frac{1}{T}$ cycles per time unit, then $\frac{2}{T}$ cycles per time unit, and so on up to the *highest* frequency $\frac{T-1}{2(T-1)} = 1/2$ cycles per time unit, which is called the *Nyquist frequency*.



- The period corresponding to the Nyquist frequency is 2 time units, which is the shortest cyclical period we could hope to get information, since we do not observe the system frequently enough.
- We can visualize a time series of $k = 2$ Cosine waves.



- When $f_k = 0$, the corresponding term is a constant a_0 called the *DC* (Direct Current) component. Since there are T data points it is generally possible to fit the data exactly with $T/2$ cosine functions with frequencies $f_k = 0, 1, \ldots, T/2$ by choosing a_k, θ_k appropriately. This implies that, given the a_k, θ_k it is possible to recover the original data.
- Usually the time series model is written using the sine and cosine sum identities:

$$\mathrm{Sin}[\alpha+\beta] = \mathrm{Sin}[\alpha]\,\mathrm{Cos}[\beta] + \mathrm{Cos}[\alpha]\,\mathrm{Sin}[\beta]$$

$$\mathrm{Cos}[\alpha+\beta] = \mathrm{Cos}[\alpha]\,\mathrm{Cos}[\beta] - \mathrm{Sin}[\alpha]\,\mathrm{Sin}[\beta]$$

- So that we can express y_t equivalently as:

$$y_t = \sum_{k=0}^{\frac{T-1}{2}} \left(a_k \,\mathrm{Cos}\left[\frac{2\,\pi\,k\,t}{T-1}\right] + b_k \,\mathrm{Sin}\left[\frac{2\,\pi\,k\,t}{T}\right]\right)$$

$$= \sum_{k=0}^{\frac{T-1}{2}} \sqrt{a_k^2 + b_k^2} \left(\frac{a_j}{\sqrt{a_k^2+b_k^2}} \,\mathrm{Cos}\left[\frac{2\,\pi\,k\,t}{T-1}\right] + \frac{b_j}{\sqrt{a_k^2+b_k^2}} \,\mathrm{Sin}\left[\frac{2\,\pi\,k\,t}{T}\right]\right)$$

$$= \sum_{k=0}^{\frac{T-1}{2}} \sqrt{a_k^2 + b_k^2} \left(\mathrm{Cos}[\theta_k]\,\mathrm{Cos}\left[\frac{2\,\pi\,k\,t}{T-1}\right] + \mathrm{Sin}[\theta_k]\,\mathrm{Sin}\left[\frac{2\,\pi\,k\,t}{T}\right]\right)$$

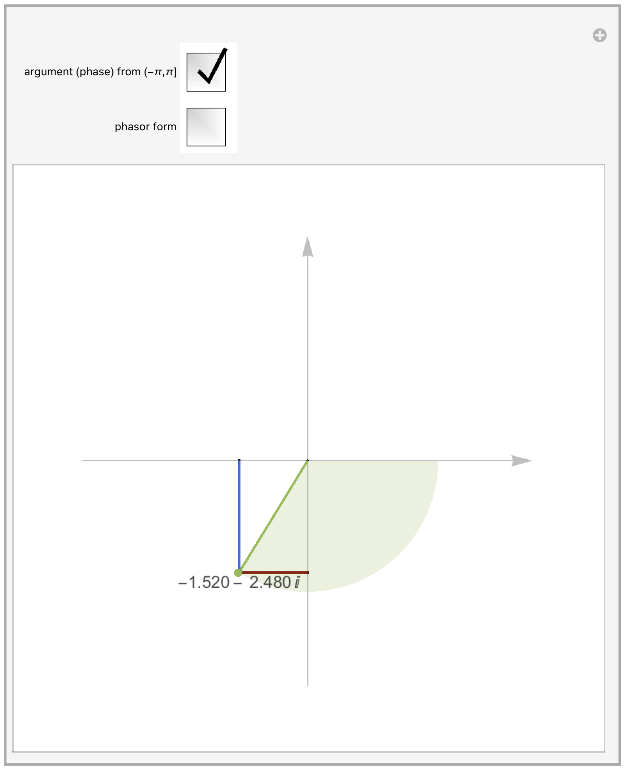
$$= \sum_{k=0}^{\frac{T-1}{2}} a_k \,\mathrm{Cos}\left[\frac{2\,\pi\,k\,t}{T-1} + \theta_k\right]$$

- Where a_0, a_k, and b_k are called the Fourier coefficients. Since this harmonic Fourier series is linear, the Fourier coefficients can be estimated easily.

From Harmonics to Complex Numbers

- Leonhard Euler (1707-1783) discovered remarkable properties of *complex numbers* that greatly simplify calculations with trigonometric functions.

- The complex numbers are pairs of real numbers, (a, b) called the *real* and *imaginary* parts of the numbers, and written as $a + b\, i$, where $i^2 = -1$, so that $i = \sqrt{-1}$.
- Euler showed that if we define $e^{i\phi} = \mathrm{Cos}[\phi] + i\,\mathrm{Sin}[\phi]$, all of the properties of the exponential function carry over, due to the properties of the trigonometric functions. For example $e^{i\phi_1} e^{i\phi_2} = e^{i(\phi_1+\phi_2)}$ just as is the case with real numbers.
- The complex number $a + b\, i = r\, e^{i\phi}$ where $r = \sqrt{a^2 + b^2}$ is the absolute value (also called the *modulo* or *magnitude*) of the complex number and $\phi = \mathrm{ArcTan}[\frac{b}{a}]$ is its *phase*.



- The coefficients of the Fourier representation of a time series are particularly simple to write and compute in terms of complex exponentials. The Fourier coefficients can be represented as:

  $$X_k = \sum_{t=0}^{T-1} y_t\, e^{-\frac{i\,2\,\pi\,k\,t}{T-1}}$$

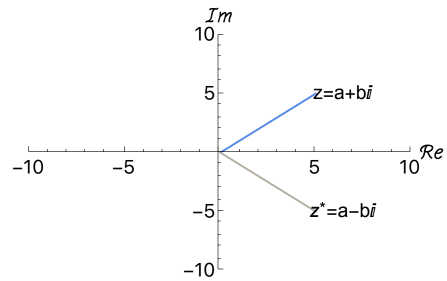
- This is the ***discrete Fourier transform*** which transforms a time series into a function of frequency and amounts to multiplying the original *time domain* data $\{y_0, \ldots, y_{T-1}\}$ by a matrix of complex numbers. This can be computed extremely rapidly by the algorithm known as the *Fast Fourier Transform*.
- It is always possible to completely recover the time domain data using the inverse Fourier transform:

  $$y_t = \sum_{t=0}^{T-1} X_k\, e^{\frac{i\,2\,\pi\,k\,t}{T-1}}$$

The Discrete Fourier Transform and The Periodogram

- When we apply the Discrete Fourier Transform to the time domain data $\{y_0, \ldots, y_{T-1}\}$, the result is a list or vector of complex numbers $\{X_0, \ldots, X_{T-1}\}$, which is the *frequency domain* representation of the data.

- Since each of the complex numbers in the frequency domain has two components, while each of the real numbers in the time domain is a single real number, there is some redundancy in the frequency domain.
- When the time domain data is a list of real numbers, the frequency domain representation has the property that $X_{T-k} = X_k^*,\ k = 1, \ldots, T-1$ where $(a + b\,i)^* = a - b\,i$ is the *complex conjugate* operation. So half of the transform coefficients just repeat the information in the first half.



- Arthur Schuster introduced the *periodogram* in 1905, as a means for detecting periodicity in time series data and estimating its unknown frequency.
- Using principles of Bayesian inference, E.T. Jaynes poses the problem as analyzing fixed time series data consisting of samples of a continuous function contaminated with additive independent Gaussian noise with an unknown variance of σ^2.
- Presuming the possible periodic signal is sinusoidal with unknown amplitude, frequency, and phase, the periodogram exhausts all the information in the data relevant to assessing the possibility that a signal is present, and to estimating the frequency and amplitude of such a signal.
- The *periodogram* is calculated as the squared magnitude of the *Discrete Fourier Transform:*

$$C[X_k] = \frac{1}{T-1} \left| \sum_{t=0}^{T-1} y_t\, e^{-\frac{i\,2\,\pi\,k\,t}{T-1}} \right|^2 = \frac{1}{T-1} \left| \mathrm{FFT} \right|^2$$

Bayesian Spectral Analysis

- As E.T. Jaynes and G. Larry Bretthorst point out, the basic time series model we are always considering is fixed discrete data $D = \{y_1, y_2, \ldots y_n\}$ sampled at discrete uniform interval times $\{t_1, t_2, \ldots, t_n\}$, with a model equation:

  $D = y_t = f[t] + \epsilon$
- Where $f[t]$ is the signal and ϵ is the noise. Different models will correspond to different choices of the signal $f[t]$ which will then determine the residual noise.
- The simplest class of a harmonic models is the estimation of a single stationary harmonic signal plus noise from a univariate time series.
- In terms of Bayes' theorem we are interested in $p[H \mid D]$. The Hypothesis in this case can be stated generally as:

  $H = y_t = \sum_{i=0}^{n-1} \alpha_i\, \Phi_i[t, \{k\}]$
- where $f[t]$ is an analytic representation of the time series, $\Phi_i[t, \{\omega\}]$ is a particular model function such as a sinusoid (Cos or Sin function) which is explicitly written as a function of time and a set of harmonic frequencies, α_i is the power or amplitude with which Φ_i enters the model, and $\{k\}$ is the set of frequency parameters we are interested in estimating.
- In terms of Bayes' theorem we are interested in the posterior probability $p[\{k\} \mid D]$.

- The model hypothesis we will continue to use is:

$$H = y_t = f[t] = \sum_{k=0}^{\frac{T-1}{2}} a_k \operatorname{Cos}\left[\frac{2\pi k t}{T-1} + \theta_k\right]$$

- Assuming the errors are normally distributed the likelihood is

$$p[D \mid H] = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\epsilon_i^2}{2\sigma^2}}$$

- Since the error is defined as $\epsilon = y_t - f[t]$ we can plug our hypothesis in directly

$$p[D \mid H] = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(y_i - \sum_{k=0}^{\frac{T-1}{2}} a_k \operatorname{Cos}\left[\frac{2\pi k t}{T-1} + \theta_k\right]\right)^2}{2\sigma^2}}$$

- Bretthorst shows that using this expression of $p[D \mid H]$ and treating the amplitudes a_k as nuisance parameters, the probability for the frequency of a *single* periodic sinusoidal signal is approximated as

$$p[X_k \mid y_t] \propto e^{\frac{C[X_k]}{\sigma^2}}$$

- This equation assumes that the noise variance σ^2 is a known quantity. By treating the σ^2 as a nuisance parameter with a specified prior distribution we can integrate it out.
- Bretthorst (1988) shows that multiplying the expression for $p[D \mid H]$ by Jeffreys' prior $\left(\frac{1}{\sigma}\right)$ and integrating over positive values of σ^2 gives the resulting posterior distribution which can be expressed as a ***Student-t distribution***. Thus, the estimated frequency of a single sinusoidal signal is given approximately by:

$$p[X_k \mid y_t] \propto \left(1 - \frac{2\,C[X_k]}{T\,\overline{d^2}}\right)^{\frac{2-T}{2}}$$

- Where $\overline{d^2} = \frac{1}{T}\sum_i y_i^2$ is the mean square average of the data. This equation is a more conservative estimate of the spectrum as it treats anything that cannot be fitted by the model as noise. This model is appropriate when we have a diffuse prior over σ^2.
- In practice it is frequently necessary to *zero pad* the FFT to obtain a sufficient density of points in the periodogram. Zero padding adds a buffer zone of zeros at the far end of the data stream.

White Noise Macroeconomics

- The Ragnar Frisch school of business cycle theory claims that persistent cycles can be maintained by external shocks to an otherwise stable system in equilibrium.
- Such business cycle models are called *noise-driven damped oscillators* or *harmonic oscillators under Brownian motion*.
- The idea is that under general equilibrium conditions the system is a stable linear system which, aside from continuous stochastic shocks, are otherwise in equilibrium.
- In general, dynamical systems are characterized by their *attractors* and/or *repellers* which are called *sinks* and *sources*. Such dynamics are either motions towards or away from fixed points.
- There are also attractors and repellers for fixed rates of change that are characterized by well defined periodic motions called *orbits*.
- If we generate a time series from independent normal random shocks, the spectrum is flat.

- In fact, physicists since 1930 have proved that such oscillators cannot sustain harmonics on the scale we tend to see in macroeconomic data.
- Ping Chen argues that the “color chaos” of the macroeconomic spectrum is the signature of an inherently unstable, chaotic system generating endogenous chaotic motions with characteristic frequencies determined by structural interactions, for example, between consumption, investment, and government spending.
- Therefore, Chen argues that business cycles can only be explained by nonlinear economic dynamics. Nonlinear deterministic systems are a more general case of linear systems and can be used to model complex economic dynamics. Such deterministic systems are capable of generating endogenous orbits that closely mimic linear systems disturbed by exogenous stochastic processes.
- Theoretically, the idea that complex oscillations, such as those that characterize the business cycle, can be generated endogenously by the system, suggests that economic variables can be in multiple stable, albeit disequilibrium, states.
- However, there is another class of dynamics called *chaotic attractors* which are aperiodic and can exhibit complex yet deterministic patterns. Such systems are defined by their nonlinear dynamical structures.

Example: Henon Map

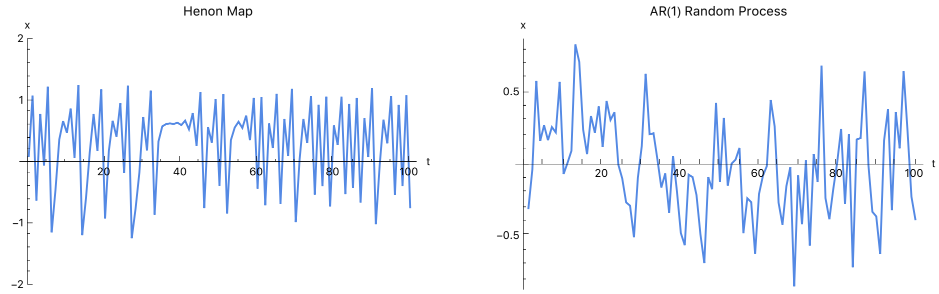
- Some low dimensional nonlinear systems are easily capable of generating complex time series with endogenous self sustaining oscillations. For example we can compare an AR[1] random process time series with a time series generated from the chaotic Hénon map.
- The Hénon map is a discrete time dynamical system that exhibits chaotic behavior. It is defined by the two equations:

$$x_t = 1 + \alpha\, x_{t-1} + y_t$$

$$y_t = \beta\, x_{t-1}$$

```
Clear[x, y, a, b, Henon];
x[0] = .1;
y[0] = .1;
a = 1.4;
b = 0.3;
x[t_] := x[t] = 1 + y[t - 1] - a * x[t - 1] ^ 2
y[t_] := y[t] = b * x[t - 1]
x1[t_] := x[t + 1]
x2[t_] := x[t + 2]
Henon = Table[{t, x[t]}, {t, 0, 100, 1}];
Rndm = RandomFunction[ARProcess[{.1, .2}, .1], {1, 100}];
Rndm = Table[{i, Rndm["Path"][[All, 2]][[i]]}, {i, 1, 100}];
Rndmlag = Table[{Rndm[[All, 2]][[i]], Rndm[[All, 2]][[i + 1]], Rndm[[All, 2]][[i + 2]]},
   {i, 1, Length[Rndm[[All, 2]]] - 2}];
```

```
GraphicsGrid[{{ListLinePlot[Henon, PlotLabel → "Henon Map",
   AxesLabel → {"t", "x"}, PlotRange → {Automatic, {-2, 2}}], ListLinePlot[Rndm,
   PlotLabel → "AR(1) Random Process", AxesLabel → {"t", "x"}]}}, ImageSize → 800]
```




```
GraphicsGrid[{{ListPointPlot3D[
   Table[{x[t], x1[t], x2[t]}, {t, 0, 1000}], AxesLabel → {"xt", "xt+1", "xt+2"}],
  ListPointPlot3D[Table[{Rndm[[All, 2]][[i]], Rndm[[All, 2]][[i + 1]],
     Rndm[[All, 2]][[i + 2]]}, {i, 1, Length[Rndm[[All, 2]]] - 2}],
   AxesLabel → {"xt", "xt+1", "xt+2"}]}}, ImageSize → 800]
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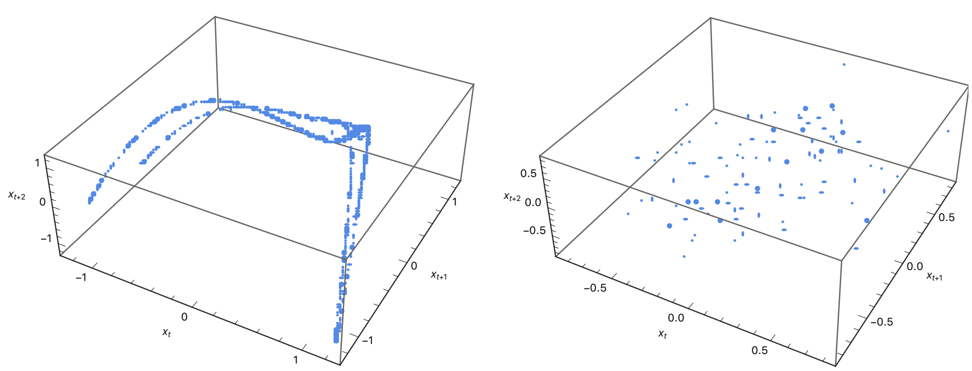

- The existence of chaotic systems raises the question of whether the real phenomena economists tend to model as stochastic processes may not in fact arise from relatively low-dimensional deterministic dynamical systems.
- If economic data are generated by a deterministic system of low dimensionality there are methods for discovering what this system is by reconstructing its state space.
- If, however, the data are generated by a deterministic system of high dimensionality, then there is no hope of identifying the system and it might as well be considered a stochastic process.

Phase Shift, Coherence, and Gain

- The relation between an input signal $x[t]$ and an output signal $y[t]$ is defined by the *transfer function* $H[t]$

  $x[t] \to H[t] \to y[t]$

- The transfer function is a "black box" function that defines the relationship between the input "impulse" signal and the output "response" signal.

$$H[t] = \frac{y[t]}{x[t]} \text{ or } y[t] = H[t]\, x[t]$$

- Spectral analysis can be easily extended to analyze the *cross spectrum* between two signals.

$$P_{x\,y}[k] = \sum_{0}^{T-1} R_{x\,y}[t]\, e^{-\frac{i\,2\,\pi\,k\,t}{T-1}}$$

- where $R_{x\,y}[t]$ is called the cross-correlation sequence and is defined as

$$R_{x\,y}[t] = E[(x_{t+\tau} - \mu_x)\,(y_t - \mu_t)] = E[(x_t - \mu_x)\,(y_{t-\tau} - \mu_t)]$$

- The cross spectrum is calculated as:

$$P_{x\,y}[k] = \mathcal{F}[x_t]^{*}\,\mathcal{F}[y_t]$$

- Where $\mathcal{F}^{*}$ is the complex conjugate of the Fourier transform of one time series and $\mathcal{F}$ is the Fourier transform of the other series.
- We can use the cross spectrum to calculate

The *gain* : $\dfrac{P_{x\,y}}{P_{x\,x}}$

The *coherence* between two signals is : $\dfrac{\left|P_{x\,y}\right|^{2}}{P_{x\,x}\,P_{y\,y}}$

The *phase shift* : $\text{Arctan}\left[\dfrac{\mathcal{Re}[P_{x\,y}]}{\mathcal{Im}[P_{x\,y}]}\right]$