Assignment 1, due on 13 Oct 23:00, submit your work through blackboard.

Exercise 1 (Dynamic Pricing in Discrete Time)**.** Assume that we have x_0 items of a ceratin type that we want to sell over a period of N days. At each day, we may sell at most one item. At the k^{th} day, knowing the current number x_k of remaining unsold items, we can set the selling price u_k of a unit item to a nonnegative number of our choice; then, the probability $q_k(u_k)$ of selling an item on the k^{th} day depends on u_k as follows:

$$q_k(u_k) = \alpha \exp(-u_k),$$

where $0 < \alpha < 1$ is a given scalar. The objective is to find the optimal price setting policy so as to maximize the total expected revenue over N days. Let $V_k(x_k)$ be the optimal expected cost from day k to the end if we have x_k unsold units.

1. Assuming that for all k, the value function $V_k(x_k)$ is monotonically nondecreasing as a function of x_k, prove that for $x_k > 0$, the optimal prices have the form
$$\mu_k^*(x_k) = 1 + J_{k+1}(x_k) - V_{k+1}(x_k - 1),$$
and that
$$V_k(x_k) = \alpha \exp(-\mu_k^*(x_k)) + V_{k+1}(x_k).$$

2. Prove simultaneously by induction that, for all k, the value function $V_k(x_k)$ is indeed monotonically nondecreasing as a function of x_k, that the optimal price $\mu_k^*(x_k)$ is monotonically nonincreasing as a function of x_k, and that $V_k(x_k)$ is given in closed form by
$$V_k(x_k) = \begin{cases} (N-k)\alpha\exp(-1), & \text{if } x_k \geq N-k, \\ \sum\limits_{i=k}^{N-x_k} \alpha \exp(-\mu_i^*(x_k)) + x_k\alpha\exp(-1), & \text{if } 0 < x_k < N-k, \\ 0, & \text{if } x_k = 0. \end{cases}$$

Exercise 2 (Label correcting with negative arc lengths)**.** Consider the problem of finding a shortest path from node s to node t, and assume that all cycle lengths are nonnegative (instead of all arc lengths being nonnegative). Suppose that a scalar u_j is known for each node j, which is an underestimate of the shortest distance from j to t (u_j can be taken $-\infty$ if no underestimate is known). Consider a modified version of the typical iteration of the label correcting algorithm discussed above, where Step 2 is replaced by the following:

Modified Step 2: If $d_i + a_{ij} < \min\{d_j, \text{ UPPER } -u_j\}$, set $d_j = d_i + a_{ij}$ and set $i = \text{ParentOf}(j)$. In addition, if $j \neq t$, place j in OPEN if it is not already in OPEN, while if $j = t$, set UPPER to the new value $d_i + a_{it}$ of d_t.

1. Show that the algorithm terminates with a shortest path, assuming there is at least one path from s to t.

2. Why is the Label Correcting Algorithm given in class a special case of the one here?

Exercise 3. We have a set of N objects, denoted $1, 2, \ldots, N$, which we want to group in clusters that consist of consecutive objects. For each cluster $i, i+1, \ldots, j$, there is an associated cost a_{ij}. We want to find a grouping of the objects in clusters such that the total cost is minimum. Formulate the problem as a shortest path problem, and write a DP algorithm for its solution. (Note: An example of this problem arises in typesetting programs, such as TEX/LATEX, that break down a paragraph into lines in a way that optimizes the paragraph's appearance).

Exercise 4. In the framework of the basic problem (i.e., the DP problem described in page 29 in lecture notes), consider the case where the cost is of the form

$$\mathbb{E}_{w_0,\ldots,w_{N-1}} \left\{ \alpha^N g_N(x_N) + \sum_{k=0}^{N-1} \alpha^k g_k(x_k, u_k, w_k) \right\}$$

where α is a discount factor with $0 < \alpha < 1$. Show that an alternate form of the DP algorithm is given by

$$V_N(x_N) = g_N(x_N),$$
$$V_k(x_k) = \min_{u_k \in U_k(x_k)} \mathbb{E}_{w_k} \left\{ g_k(x_k, u_k, w_k) + \alpha V_{k+1}(f_k(x_k, u_k, w_k)) \right\}.$$

Exercise 5. In the framework of the basic problem, consider the case where the system evolution terminates at time i when a given value $\bar{w}$ of the disturbance at time i occurs, or when a termination decision $\bar{u}_i$ is made by the controller. If termination occurs at time i, the resulting cost is

$$T + \sum_{k=0}^{i} g_k(x_k, u_k, w_k),$$

where T is a termination cost. If the process has not terminated up to the final time N, the resulting cost is $g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k)$. Reformulate the problem into the framework of the basic problem.

Hint: Augment the state space with a special termination state.

Exercise 6. Consider a device consisting of N stages connected in series, where each stage consists of a particular component. The components are subject to failure, and to increase the reliability of the device duplicate components are provided. For $j = 1, 2, \ldots, N$, let $(1 + m_j)$ be the number of components for the

j th stage (one mandatory component, and m_j backup ones), let $p_j(m_j)$ be the probability of successful operation when $(1+m_j)$ components are used, and let c_j denote the cost of a single backup component at the j th stage. Formulate in terms of DP the problem of finding the number of components at each stage that maximizes the reliability of the device expressed by the product

$$p_1(m_1) \cdot p_2(m_2) \cdots p_N(m_N)$$

subject to the cost constraint $\sum_{j=1}^{N} c_j m_j \leq A$, where $A > 0$ is given.

Exercise 7 (Monotonicity Property of DP)**.** An evident, yet very important property of the DP algorithm is that if the terminal cost g_N is changed to a uniformly larger cost $\bar{g}_N$ (i.e., $g_N(x_N) \leq \bar{g}_N(x_N), \forall x_N$), then clearly the last stage cost-to-go $J_{N-1}(x_{N-1})$ will be uniformly increased (i.e., $J_{N-1}(x_{N-1}) \leq \bar{J}_{N-1}(x_{N-1})$).

More generally, given two functions J_{k+1} and $\bar{J}_{k+1}$, with $J_{k+1}(x_{k+1}) \leq \bar{J}_{k+1}(x_{k+1})$ for all x_{k+1}, we have, for all x_k and $u_k \in U_k(x_k)$,

$$\mathbb{E}_{w_k}\left[g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k))\right] \leq \mathbb{E}_{w_k}\left[g_k(x_k, u_k, w_k) + \bar{J}_{k+1}(f_k(x_k, u_k, w_k))\right]$$

Suppose now that in the basic problem the system and cost are time invariant; that is, $S_k \triangleq S, C_k \triangleq C, D_k \triangleq D, f_k \triangleq f, U_k \triangleq U, P_k \triangleq P$, and $g_k \triangleq g$. Show that if in the DP algorithm we have $J_{N-1}(x) \leq J_N(x)$ for all $x \in S$, then

$$J_k(x) \leq J_{k+1}(x), \quad \text{for all } x \in S \text{ and } k.$$

Similarly, if we have $J_{N-1}(x) \geq J_N(x)$ for all $x \in S$, then

$$J_k(x) \geq J_{k+1}(x), \quad \text{for all } x \in S \text{ and } k.$$

Exercise 8. In the framework of the basic problem, consider the case where the cost has the following multiplicative form

$$\mathbb{E}_w\left[g_N(x_N) \prod_{k=1}^{N_1} g_k(x_k, u_k, w_k)\right].$$

Develop a DP-like algorithm for this problem assuming $g_k(x_k, u_k, w_k) \geq 0$ for all x_k, u_k and w_k.

Exercise 9. An unemployed worker receives a job offer at each time period, which she may accept or reject. The offered salary takes one of n possible values $w^1, \ldots, w^n$, with given probabilities, independently of preceding offers. If she accepts the offer, she must keep the job for the rest of her life at the same salary level. If she rejects the offer, she receives unemployment compensation c for the current period and is eligible to accept future offers. Assume that income is discounted by a factor $\alpha < 1$.

Hint: Define the states $s^i, i = 1, \ldots, n$, corresponding to the worker being unemployed and being offered a salary w^i, and $\bar{s}^i, i = 1, \ldots, n$, corresponding to the worker being employed at a salary level w^i.

(a) Show that there is a threshold $\bar{w}$ such that it is optimal to accept an offer if and only if its salary is larger than $\bar{w}$, and characterize $\bar{w}$.

(b) Consider the variant of the problem where there is a given probability p_i that the worker will be fired from her job at any one period if her salary is w^i. Show that the result of part (a) holds in the case where p_i is the same for all i. Argue what would happen in the case where p_i depends on i.