Assignment 2. Due on 17 Nov 11:59 PM.

Exercise 1. Consider a problem of expanding over N time periods the capacity of a production facility. Let us denote by x_k the production capacity at the beginning of period k, and by $u_k \geq 0$ the addition to capacity during the k th period. Thus, capacity evolves according to

$$x_{k+1} = x_k + u_k, \quad k = 0, 1, \ldots, N-1$$

The demand at the k th period is denoted w_k and has a known probability distribution that does not depend on either x_k or u_k. Also, successive demands are assumed to be independent and bounded. We denote:

$C_k(u_k)$: Expansion cost associated with adding capacity u_k.

$P_k(x_k + u_k - w_k)$: Penalty associated with capacity $x_k + u_k$ and demand w_k.

$S(x_N)$: Salvage value of final capacity x_N.

Thus, the cost function has the form

$$\mathbb{E}_{w_0, \ldots, w_{N-1}}\left[-S(x_N) + \sum_{k=0}^{N-1}\left(C_k(u_k) + P_k(x_k + u_k - w_k)\right)\right]$$

(a) Derive the DP algorithm for this problem.

(b) Assume that S is a concave function with $\lim_{x \to \infty} \mathrm{d}S(x)/\mathrm{d}x = 0, P_k$ are convex functions, and the expansion cost C_k is of the form

$$C_k(u) = \begin{cases} K + c_k u & \text{if } u > 0 \\ 0 & \text{if } u = 0 \end{cases}$$

where $K \geq 0, c_k > 0$ for all k. Show that the optimal policy is of the (s, S) type assuming

$$c_k y + \mathbb{E}\left[P_k(y - w_k)\right] \to \infty \text{ as } |y| \to \infty$$

Exercise 2 (Single-leg Revenue Management problem)**.** For a single leg RM problem assume that:

- There are $n = 10$ classes.
- Demand D_j is calculated through discretizing a truncated normal with mean $\mu = 10$ and standard deviation $\sigma = 2$, on support $[0, 20]$. Specifically, take:

$$\mathbb{P}(D_j = k) = \frac{\Phi((k + 0.5 - 10)/2) - \Phi((k - 0.5 - 10)/2)}{\Phi((20.5 - 10)/2) - \Phi((-0.5 - 10)/2)}, \quad k = 0, \ldots, 20$$

Note that this discretization and re-scaling verifies: $\sum_{k=0}^{20} \mathbb{P}(D_j = k) = 1$.

- Total capacity available is $C = 100$.
- Prices are $p_1 = 500, p_2 = 480, p_3 = 465, p_4 = 420, p_5 = 400, p_6 = 350, p_7 = 320, p_8 = 270, p_9 = 250$, and $p_{10} = 200$.

Write a code to compute optimal protection levels $y_1^*, \ldots, y_9^*$; and find the total expected revenue $V_{10}(100)$. Note that you can take advantage of the structure of the optimal policy to simplify its computation.

Exercise 3 (Heuristic for the single-leg RM problem)**.** In the airline industry, the singleleg RM problem is typically solved using a heuristic; the so-called EMSR-b (expected marginal seat revenue - version b). There is no much reason for this other than the tradition of its usage, and the fact that it provides consistently good results. Here is a description:

Consider stage $j+1$ in which we want to determine protection level y_j. Define the aggregated future demand for classes $j, j-1, \ldots, 1$, by $S_j = \sum_{k=1}^{j} D_k$, and let the weighted-average revenue from classes $1, \ldots, j$, denoted $\bar{p}_j$, be defined by

$$\bar{p}_j = \frac{\sum_{k=1}^{j} p_k \mathbb{E}\left[D_k\right]}{\sum_{k=1}^{j} \mathbb{E}\left[D_k\right]}$$

Then the EMSR-b protection level for class j and higher, y_j, is chosen by

$$\mathbb{P}\left(S_j > y_j\right) = \frac{p_{j+1}}{\bar{p}_j}$$

It is common when using EMSR-b to assume demand for each class j is independent and normally distributed with mean μ_j and variance σ_j^2, in which case

$$y_j = \mu + z_\alpha \sigma$$

where $\mu = \sum_{k=1}^{j} \mu_k$ is the mean and $\sigma^2 = \sum_{k=1}^{j} \sigma_k^2$ is the variance of the aggregated demand to come at stage $j+1$, and

$$z_\alpha = \Phi^{-1}\left(1 - p_{j+1}/\bar{p}_j\right)$$

Apply this heuristic to compute protection levels $y_1, \ldots, y_9$ using the data of the previous exercise and assuming that demand is normal (no truncation, no discretization), and compare the outcome with the optimal protection levels computed before.

Exercise 4. A driver is looking for parking on the way to his destination. Each parking place is free with probability p independently of whether other parking places are free or not. The driver cannot observe whether a parking place is free until he reaches it. If he parks k places from his destination, he incurs a cost k. If he reaches the destination without having parked, the cost is C.

(a) Let F_k be the minimal expected cost if he is k parking places from his destination, where $F_0 = C$. Show that

$$F_k = p \min\{k, F_{k-1}\} + qF_{k-1}, \quad k = 1, 2, \ldots$$

where $q = 1 - p$.

(b) Show that an optimal policy is of the form: "Never park if $k \geq k^*$, but take the first free place if $k < k^*$, where k is the number of parking places from the destination, and

$$k^* = \min\left\{i : i \text{ integer}, \, q^{i-1} < (pC + q)^{-1}\right\}$$