Analysis of Algorithms - Assignment 1

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Problem 1: Array Selection Algorithm

1.1 Optimized Selection Algorithm

The algorithm described below is the **Median-of-Medians** algorithm, which finds the k-th smallest element in an unsorted array of n elements in a recursive style.

Given an array A of n distinct elements and an integer k (where $1 \leq k \leq n$), the following algorithm returns the k-th smallest element in A:

1. **Divide into Groups:**
   Divide the n elements of array A into $\lceil \frac{n}{5} \rceil$[1]groups. Each group will contain 5 elements, except possibly the last group, which may contain between 1 and 5 elements.

2. **Find Group Medians:**
   For each of the $\lceil \frac{n}{5} \rceil$ groups, sort the elements within the group and find the median. This can be done with a constant number of comparisons, since each group has at most 5 elements.

3. **Find Median-of-Medians (Pivot Selection):**
   Create a new array M consisting of all the medians found in Step 2. The size of M is $\lceil \frac{n}{5} \rceil$. Recursively apply the algorithm to M to find its **median**[2]. Let this element be x, which will serve as the pivot.

4. **Partition:**
   Partition the original array A into three subarrays based on the pivot x:

   - $L = \{a \in A \mid a < x\}$ (elements strictly less than x)
   - $E = \{a \in A \mid a = x\}$ (elements equal to x)
   - $G = \{a \in A \mid a > x\}$ (elements strictly greater than x)

   Let $|L|$, $|E|$, and $|G|$ denote the number of elements in these subarrays, respectively.

5. **Recursive Search:**
   Determine which subarray contains the k-th smallest element and recursively apply the algorithm to that subarray:

   - If $k \leq |L|$, the k-th smallest element is in L. Recursively apply the algorithm to find the k-th smallest element in L.
   - If $|L| < k \leq |L| + |E|$, the k-th smallest element is x. Return x.
   - Otherwise, $k > |L| + |E|$, the k-th smallest element is in G. Recursively apply the algorithm to find the $(k - |L| - |E|)$-th smallest element in G.

[1]Some might be curious why is 5 here; this is because 5 is the smallest group size that satisfies linear worst-case complexity. A more detailed discussion can be found in the justification section.

[2]It's worth noting that the **median** here is not necessarily the median of array A.

1.2 Recurrence Relation Analysis

Let $T(n)$ be the worst-case number of comparisons required by the algorithm described above for an input of size n.

1. **Divide into Groups:**
   This step involves no comparisons, just array manipulation. $O(1)$.

2. **Find Group Medians:**
   - There are $\lceil \frac{n}{5} \rceil$ groups.
   - Each group has at most 5 elements. Find median of at most 5 elements requires a constant number of comparisons[3].
   - Thus, finding all group medians requires $O(\lceil \frac{n}{5} \rceil \cdot 1) = O(n)$ comparisons.

3. **Find Median-of-Medians (Pivot Selection):**
   We recursively apply the algorithm to the array M of size $\lceil \frac{n}{5} \rceil$ to find its median (i.e., the $\lceil \frac{\lceil \frac{n}{5} \rceil}{2} \rceil$-th smallest element), which contributes $T(\lceil \frac{n}{5} \rceil)$ to the recurrence.

4. **Partition:**
   Partitioning the n elements around based on pivot x takes $O(n)$ comparisons.

5. **Recursive Search:**
   - Consider the pivot x. It is the median of the $\lceil \frac{n}{5} \rceil$ group medians.
   - At least half of these group medians are less than or equal to x.
     That is, at least $\lceil \frac{\lceil \frac{n}{5} \rceil}{2} \rceil$ elements are $\leq x$.
   - Each such group median m_i comes from a group of 5 elements, and at least 3 elements in that group are $\leq m_i$.
   - Therefore, at least $3 \cdot \lceil \frac{\lceil \frac{n}{5} \rceil}{2} \rceil$ elements in the original array A are $\leq x$.
     - Since $\lceil \frac{n}{x} \rceil \geq \frac{n}{x}$, we have $\lceil \frac{\lceil \frac{n}{5} \rceil}{2} \rceil \geq \frac{n}{10}$
     - Thus, as least $\frac{3n}{10}$ elements are $\leq x$.
     - Which implies that $|L| + |E| \geq \frac{3n}{10}$.
   - Similarly, at least $\frac{3n}{10}$ elements are $\geq x$. Which implies that $|G| + |E| \geq \frac{3n}{10}$.
   - For these bounds, we can deduce the maximum size of the recursive call:
     - $|L| = |A| - |G| - |E| \leq n - \frac{3n}{10} = \frac{7n}{10}$.
     - $|G| = |A| - |L| - |E| \leq n - \frac{3n}{10} = \frac{7n}{10}$.
     - Therefore, the maximum size of the recursive call is on a subarray of size at most $\frac{7n}{10}$. Which contributes $T(\frac{7n}{10})$ to the recurrence.

   Combining all above, we have the recurrence relation for the worst-case comparison number:

$$T(n) \leq T(\lceil \frac{n}{5} \rceil) + T(\frac{7n}{10}) + O(n)$$

   For simplicity, ignoring ceilings, we have:

$$T(n) \leq T(\frac{n}{5}) + T(\frac{7n}{10}) + O(n)$$

[3] It can be proven that using tournament sort, the median of 5 elements can be found in the worst case of 6 comparisons.

1.3 Complexity Analysis

The complexity of the above **Median-of-Medians** algorithm can be analyzed using substitution method. We can first re-write the recurrence relation as:

$$T(n) \leq T(\frac{n}{5}) + T(\frac{7n}{10}) + cn$$

for some constant $c > 0$.

Assume that $T(n) \leq dn$ for some constant $d > 0$ and for all $n \geq 1$, we need to find a d such that this assumption holds. Substitude dn into the recurrence:

$$\begin{aligned} T(n) &\leq d\frac{n}{5} + d\frac{7n}{10} + cn \\ &= \frac{9dn}{10} + cn \leq dn \end{aligned}$$

Which gives $d \geq 10c$. Thus, we can choose $d = 10c$ to make the assumption holds. For base case ($n \leq 5$), since it takes constant time, we can always choose d to be large enough to ensure the solution holds.

To this end, we have proved that the worst-case time complexity of the **Median-of-Medians** algorithm is $O(n)$.

Justification

In this additional section, we discuss why the group size is 5.

In the previous section, we proved that for a group size of 5, at most, a subproblem of size $\frac{7n}{10}$ needs to be solved. Now, let's assume the group size is k. This can be rephrased as: among $\lceil \frac{n}{k} \rceil$ medians, $\lceil \frac{\lceil \frac{n}{k} \rceil}{2} \rceil$ are smaller than the pivot. where, in each group, $\lceil \frac{k}{2} \rceil$ elements are smaller or equal than the pivot. This totals $\lceil \frac{\lceil \frac{n}{k} \rceil}{2} \rceil \lceil \frac{k}{2} \rceil \geq \frac{n}{4}$[4], meaning the maximum size of the subproblem is $n - \frac{n}{4} = \frac{3n}{4}$. In the recurrence relation, we need to handle subproblems $T(\frac{n}{k})$ and $T(\frac{3n}{4})$. For this recurrence relation to resolve to $O(n)$, it needs to be ensured that $\frac{n}{k} + \frac{3n}{4} < n$, that is, $k > 4$.

[4]This argument is still imprecise, but we can test $k = 2 \sim 4$ to get the same result.

Problem 2: Search in Bitonic Sequence

The running time of the **Search in Bitonic** algorithm can be broken down as follows:

1. **Divide:**
   - Dividing the array into two halves involves calculating a middle index, which takes $O(1)$ time.
   - To determine if a subarray is bitonic or monotonic, we can compare a few key elements (a_1, a_2, a_{n-1}, a_n), which takes $O(1)$ time. So analyzing the monotonicity of both halves takes also $O(1)$ time.
2. **Conquer:**
   - **Scenario 1**: One half is bitonic (size $N/2$), the other is monotonic (size $N/2$). The recursive search on the bitonic half contributes $T(N/2)$. The binary search on the monotonic half takes $O(\log(N/2)) = O(\log N)$ time. Total time for this scenario: $T(N/2) + O(\log N)$.
   - **Scenario 2**: Both halves are monotonic (each size $N/2$).
     Total time for this scenario: $O(\log N)$.
3. **Combine:**
   Combining the answers from the left and right halves takes $O(1)$ time.

The worst-case scenario occurs when the algorithm always has to recurse on a bitonic half. Therefore, the worst-case recurrence relation is:

$$T(N) = T(\frac{N}{2}) + O(\log N)$$

For simplicity, we can re-write it as:

$$T(N) = T(\frac{N}{2}) + c\log N$$

For some constant $c > 0$. This recurrence can be solved by unrolling it:

$$\begin{aligned}
T(N) &= T(\frac{N}{2}) + c\log N \\
&= T(\frac{N}{4}) + c\log N + c\log\frac{N}{2} \\
&= T(\frac{N}{8}) + c\log N + c\log\frac{N}{2} + c\log\frac{N}{4} \\
&\dots
\end{aligned}$$

This pattern continues until the base case, $T(1)$ or $T(2)$, which takes $O(1)$ time. The recursion depth is $k = \log N$. Thus, we have:

$$\begin{aligned} T(N) &= T(\frac{N}{2^k}) + c\sum_{i=0}^{k-1} \log(\frac{N}{2^i}) \\ &= T(1) + c\sum_{i=0}^{\log N - 1} (\log N - i) \\ &= O(1) + c\frac{\log N(\log N + 1)}{2} \\ &= O(1) + O(\log^2 N) = O(\log^2 N) \end{aligned}$$

Therefore, the worst-case time complexity of the **Search in Bitonic** algorithm is $O(\log^2 N)$.

Problem 3: Master Theorem

1. $T(n) = 3T(n/2) + n^2$
   - $a = 3, b = 2, f(n) = n^2$
   - $f(n) = \Theta(n^2) \to d = 2$
   - $a = 3 < 4 = 2^2 = b^d$
   - $T(n) = \Theta(n^d) = \Theta(n^2)$ (case 1)

2. $T(n) = 5T(n/4) + n(\log n)^2$

   Let $T_L(n) = 5T_L(n/4) + n$
   - $a = 5, b = 4, f(n) = n$
   - $f(n) = \Theta(n) \to d = 1$
   - $a = 5 > 4^1 = b^d$
   - $T_L(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_4 5})$ (case 3)

   Let $T_U(n) = 5T_U(n/4) + n^{\log_4 5}$
   - $a = 5, b = 4, f(n) = n^{\log_4 5}$
   - $f(n) = \Theta(n^{\log_4 5}) \to d = \log_4 5$
   - $a = 5 = 4^{\log_4 5} = b^d$
   - $T_U(n) = \Theta(n^d \log n) = \Theta(n^{\log_4 5} \log n)$ (case 2)

   Since we have $T_L(n) < T(n) < T_U(n)$, we can give the following asymptotic bound:

   $$T(n) = \Omega(n^{\log_4 5}) \text{ and } T(n) = O(n^{\log_4 5} \log n)$$

   Well, actually we can also derive an asymptotically tight bound for $T(n)$ by substitution and unrolling the recurrence:

   $$\begin{aligned}
   T(n) &= 5T(n/4) + n(\log n)^2 \\
   n &\mapsto 4^k \\
   T(4^k) &= 5T(4^{k-1}) + 4^k(\log 4^k)^2 \\
   &= 5T(4^{k-1}) + 4^k(2k)^2 \\
   \frac{T(4^k)}{4^k} &= \frac{5}{4}\frac{T(4^{k-1})}{4^{k-1}} + (2k)^2 \\
   S(k) &:= \frac{T(4^k)}{4^k} \\
   S(k) &= \frac{5}{4}S(k-1) + (2k)^2 \\
   &= (\frac{5}{4})^k S(0) + \sum_{i=1}^{k}(\frac{5}{4})^{k-i}(2i)^2 \\
   &= (\frac{5}{4})^k T(1) + 4(\frac{5}{4})^k \sum_{i=1}^{k}(\frac{4}{5})^i i^2
   \end{aligned}$$

$$0 < x < 1 \to \sum_{i=1}^{\infty} x^i i^2 = \frac{x(1+x)}{(1-x)^3} \to 0 < \sum_{i=1}^{k} (\frac{4}{5})^i i^2 < \frac{\frac{4}{5}(1+\frac{4}{5})}{(1-\frac{4}{5})^3} = 180$$

$$\to S(k) = (\frac{5}{4})^k [T(1) + 4\sum_{i=1}^{k} (\frac{4}{5})^i i^2] = \Theta((\frac{5}{4})^k)$$

$$T(n) = nS(\frac{\log n}{2}) = \Theta(n(\frac{5}{4})^{\frac{\log n}{2}}) = \Theta(n(\frac{\sqrt{5}}{2})^{\log n})$$

3. $T(n) = 3T(\frac{n}{3} - 3) + \frac{n}{3}$

   Let $T_L(n) = 3T_L(\frac{n}{4}) + \frac{n}{3}$[1]

   - $a = 3, b = 4, f(n) = \frac{n}{3}$
   - $f(n) = \Theta(n) \to d = 1$
   - $a = 3 < 4^1 = b^d$
   - $T_L(n) = \Theta(n^d) = \Theta(n)$ (case 1)

   Let $T_U(n) = 3T_U(\frac{n}{3}) + \frac{n}{3}$

   - $a = 3, b = 3, f(n) = \frac{n}{3}$
   - $f(n) = \Theta(n) \to d = 1$
   - $a = 3 = 3^1 = b^d$
   - $T_U(n) = \Theta(n^d \log n) = \Theta(n \log n)$ (case 2)

   Since we have $T_L(n) < T(n) < T_U(n)$, we can give the following asymptotic bound:

$$T(n) = \Omega(n) \text{ and } T(n) = O(n \log n)$$

4. $T(n) = 2T(\sqrt{n}) + \log n$

   By making the substitution $n \mapsto 2^k$ and $S(k) = T(2^k)$, the recurrence relation can be re-write as:

$$T(2^k) = 2T(2^{k-1}) + \log 2^k$$
$$S(k) = 2S(k-1) + k$$

   By unrolling the recurrence, we have:

$$\begin{aligned} S(k) &= 2S(k-1) + k \\ &= 2^k S(0) + \sum_{i=1}^{k} 2^{k-i} i \\ &= 2^k T(1) + 2^k \sum_{i=1}^{k} \frac{i}{2^i} \end{aligned}$$

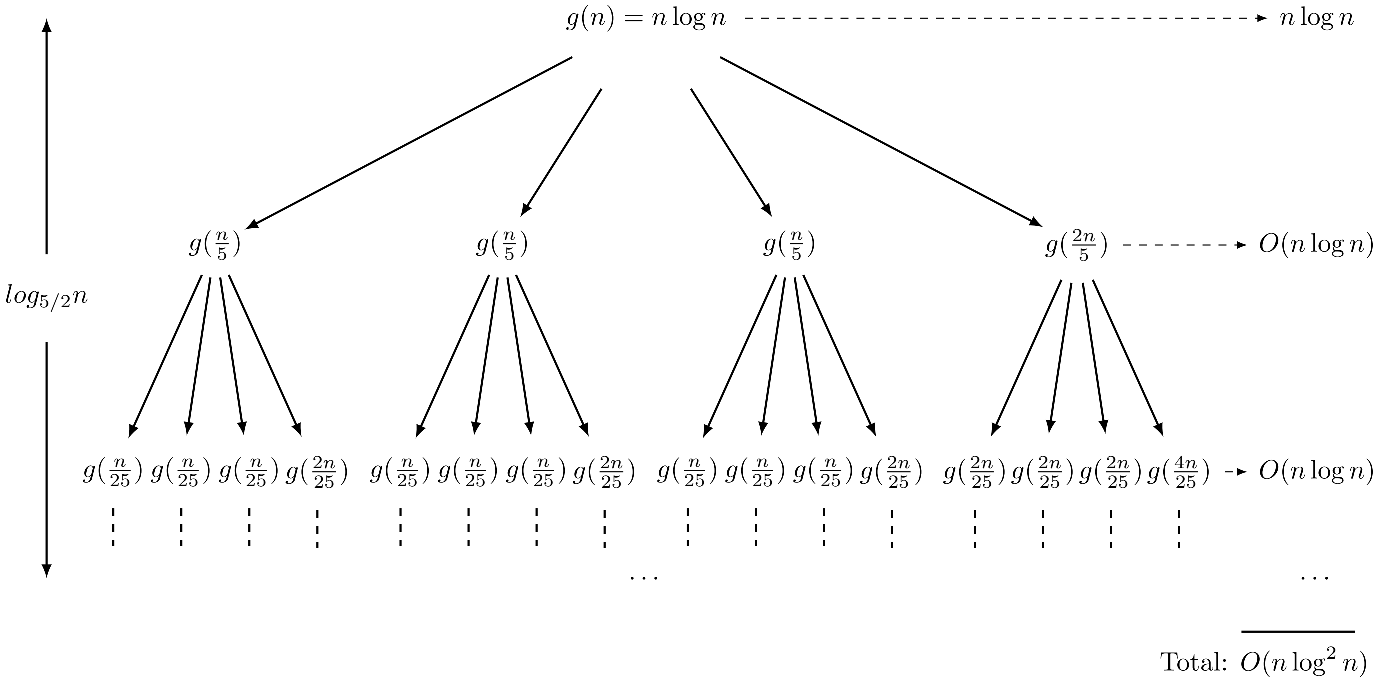
$$0 < x < 1 \to \sum_{i=1}^{\infty} x^i i = \frac{x}{(1-x)^2} \to 0 < \sum_{i=1}^{k} \frac{i}{2^i} < \frac{0.5}{0.5^2} = 2$$

$$\to S(k) = \Theta(2^k)$$

$$T(n) = S(\frac{\log n}{2}) = \Theta(2^{\frac{\log n}{2}}) = \Theta(\sqrt{n})$$

[1]Since for sufficiently large n ($n \geq 36$), $\frac{n}{4} \leq \frac{n}{3} - 3$ is guaranteed, it can therefore be considered not to affect the asymptotic properties.

Problem 4: Recursion Tree



The derivation of the cost for each layer is as follows: Let the set $M = \{m_1, m_2, ...\}$ denote the set of subproblem sizes for each layer. It can be proven that $\sum_{i=1}^{|M|} m_i \leq n$ (this is because for any problem decomposition, the subproblem sizes are $3n/5 + 2n/5 = n$).

Since we have $g(n) + g(m) \leq g(n+m)$ for $g(n) = n \log n$ and any $n, m \geq 2$, one can prove that $\sum_{i=1}^{|M|}(m_i \log m_i) \leq (\sum_{i=1}^{|M|} m_i) \log(\sum_{i=1}^{|M|} m_i) \leq n \log n$ holds if all $m_i \geq 2$.

Intuitively, the solution to the recurrence relation is at most the number of layers multiplied by the cost of each layer, i.e., $O(\log_{5/2} n \times n \log n) = O(n \log^2 n)$. Here we use the substitution method to prove that this is indeed an asymptotic upper bound for the recurrence solution:

Assume that $T(n) \leq dn \log^2 n$ holds for some constant $d > 0$ and for all $n \geq 1$, we need to find a d such that this assumption holds. Substitude it into the recurrence relation:

$$
\begin{aligned}
T(n) &= 3T(\frac{n}{5}) + T(\frac{2n}{5}) + n \log n \\
&\leq 3d\frac{n}{5} \log^2 \frac{n}{5} + d\frac{2n}{5} \log^2 \frac{2n}{5} + n \log n \\
&= 3d\frac{n}{5}(\log^2 n - 2 \log n \log 5 + \log^2 5) \\
&\quad + d\frac{2n}{5}(\log^2 n - 2 \log n \log \frac{5}{2} + \log^2 \frac{5}{2}) + n \log n \\
&= dn \log^2 n + n \log n[1 - \frac{2d}{5}(5 \log 5 - 2 \log 2)] + \frac{dn}{5}(3 \log^2 5 + 2 \log^2 \frac{5}{2})
\end{aligned}
$$

To ensure the RHS $\leq dn\log^2 n$, we need to have:

$$\begin{aligned} n\log n[1-\frac{2d}{5}(5\log 5-2\log 2)]+\frac{dn}{5}(3\log^2 5+2\log^2\frac{5}{2}) &\leq 0 \\ \log n[1-\frac{2d}{5}(5\log 5-2\log 2)]+\frac{d}{5}(3\log^2 5+2\log^2\frac{5}{2}) &\leq 0 \text{ (Divide by } n) \end{aligned}$$

In order for this inequality to hold for sufficiently large n, we must make the coefficient of $\log n$ negative or zero:

$$\begin{aligned} 1-\frac{2d}{5}(5\log 5-2\log 2) &\leq 0 \\ \frac{2d}{5}(5\log 5-2\log 2) &\geq 1 \\ d &\geq \frac{5}{2(5\log 5-2\log 2)} \end{aligned}$$

Therefore, we can choose $d=\frac{5}{2(5\log 5-2\log 2)}$ to ensure the solution holds. For base case $(n\leq 1)$[1], since it takes constant time, we can always choose d to be large enough to ensure the solution $T(n)\leq dn\log^2 n$ holds.

To this end, we have proved that the asymptotic complexity of the recurrence relation is $O(n\log^2 n)$.

[1] Here, we do not need to guarantee $\frac{n}{5}\geq 2$, because the inequality $g(n)+g(m)\leq g(n+m)$ was not used when solving the recurrence relation using the substitution method.

Problem 5: Password Generation Counting

5.1 Recurrence Relation Analysis

Let $T(n)$ be the **number of operations** required to count the **number of valid passwords** $S(n)$ of digital sum n. The following algorithm gives a recursive solution to this problem:

- If $n = 1$, or n has been visited before, return $S(n)$. It's trivial that $S(1) = 1$.
- Otherwise, recursively call the algorithm to count the number of valid passwords of digital sum $n - 1$, and return the sum $S(n) = \sum_{i=1}^{\min(9,n-1)} S(n-i)$.

The recurrence relation can be written as:

$$T(n) = T(n-1) + O(1)$$

This is because in each step n, we only need to call the algorithm once on $n - 1$, which contributes $T(n-1)$ to the recurrence, The remaining operation is just a single summation of at most ten elements $\sum_{i=1}^{\min(9,n-1)} S(i)$ which takes constant time, because $S(1)$ through $S(n-1)$ have all been pre-calculated after calling the algorithm for $n - 1$.

The base case is $T(1) = c$, since there is only one valid password of digital sum 1, which is '1'.

The total number of operations $T(n) = O(n)$ can be immediately derived by unrolling the recurrence relation.

5.2 Pseudocode

Algorithm 1 Count Passwords with Sum S

```
Input: Target sum n
Output: Total number of different passwords
Global/Shared: 'S' array of size (n + 1), initialized with 0, S[1] ← 1
function CountPasswordsRecursive(n)
    if n < 1 then
        return 0                                          ▷ Boundary Case
    end if
    if S[n] > 0 then
        return S[n]                                       ▷ Pre-calculated
    end if
    ans ← 0
    for i from 1 to 9 do
        ans ← ans + CountPasswordsRecursive(n − i)
    end for
    S[n] ← ans                                            ▷ Store result
    return ans
end function
```

Problem 6: Ancient Pyramid Treasure Hunt

6.1 State Definition

First, it can be proven that any valid path from the bottom-left corner to the top-right corner is also a valid path from the top-right corner to the bottom-left corner; therefore, we can transform the problem into finding the maximum number of non-duplicate treasures that can be collected by two people starting simultaneously from the bottom-left corner and arriving at the top-right corner.

A direct solution would be to record the current positions of both people and the corresponding number of (non-duplicate) treasures collected, which would have an $O(n^4)$ complexity; A straightforward observation is that when the position of one person is determined, the number of possible positions for the other person is only $O(n)$ (instead of $O(n^2)$). This is because when both people start at the same time, they would have taken the same number of steps, leading to $x_1 + y_1 = x_2 + y_2$.

Therefore, we can use a three-dimensional state array $dp[x_1][y_1][x_2]$ to record number of unique treasures obtained, and calculate y_2 state-wisely during transitions. Since each state will be computed at most once, the time complexity of this algorithm is $O(n^3)$.

6.2 State Transition & Recurrence Relation

For any valid state (x_1, y_1, x_2, y_2), since each person can only move from the left or from below, there are four possible preceding states:

- $(x_1 - 1, y_1, x_2 - 1, y_2)$
- $(x_1, y_1 - 1, x_2 - 1, y_2)$
- $(x_1 - 1, y_1, x_2, y_2 - 1)$
- $(x_1, y_1 - 1, x_2, y_2 - 1)$

Each person moves to the position corresponding to the current state (they might overlap) and collects unique treasures. Therefore, the state transition equation can be written as:

$$
\begin{aligned}
dp[x_1][y_1][x_2] = \max(& \\
& dp[x_1 - 1][y_1][x_2 - 1], \\
& dp[x_1][y_1 - 1][x_2 - 1], \\
& dp[x_1 - 1][y_1][x_2], \\
& dp[x_1][y_1 - 1][x_2] \\
&) + \text{unique treasures collected}
\end{aligned}
$$

Where the last dimension y_2 was compressed. Next, one can traverse the states in a temporal order to ensure that when any state is reached, its subproblems have already been visited.

alternatively, one can use a recursive solution, starting from the top-right corner (n, n, n), and whenever a state (x_1, y_1, x_2) is accessed, either return $dp[x_1][y_1][x_2]$ (if this state has already been visited) or recursively visit its predecessor states, combine the answers, and record the result to $dp[x_1][y_1][x_2]$.

Finally, invalid states should be handled. For iterative solutions, states where (any one person) is at a wall can be skipped during the loop; for recursive solutions, recursion can return immediately when an invalid state is accessed.

6.3 Boundary Case & Base Case

Boundary conditions and base cases can be simply handled by array initialization:

- Initialize the dp array to $-\infty$.
- Set $dp[0][0][0] \leftarrow 0$.

These impossible (but not wall) cases will always be negative during calculation and will not contribute to the final answer. One might think that some positions reachable from the bottom-left corner might not be reachable from the top-right corner, but it can be proven that these positions will also not contribute to the final answer (because only complete valid paths from bottom-left to top-right will).