Cramer's Rule

Cramers rule is a mathematical theorem used to solve systems of linear equations with as many equations as unknowns, using determinants.

* It applies to square matrices where the determinant of the coefficient matrix is non zero.

Conditions

① The system must be a square system (same number of equations and unknowns)

② The determinant of the coefficient matrix must be non-zero.

③ Only applicable to linear equations.

Linear equations because ⇒

- Variables raised only to the power of 1.
- No multiplication between variables (xy, x^2 etc.)
- Graphically, each equation (in 3D) represents a plane, the solution is the point where all three planes intersect.

Application :-

- Used in solving small systems of linear equation.
- Used in theoritical mathematics and engineering.
- Basis for understanding matrix-based solutions in linear algebra.

EX-9.3

use cramer's rule to solve,

$$0.3x_1 + 0.52x_2 + x_3 = -0.01$$
$$0.5x + x_2 + 1.9x_3 = 0.67$$
$$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44.$$

The determinant D can be written as,

$$D = \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix}$$

$$= 0.3(0.5 - 0.57) - 0.52(0.25 - 0.19) - 1(0.15 - 0.1)$$

$$= -2.2 \times 10^{-3}$$

$$Dx_1 = \begin{vmatrix} -0.01 & 0.52 & 1 \\ 0.67 & 1 & 1.9 \\ -0.44 & 0.3 & 0.5 \end{vmatrix}$$

$$= 0.03278.$$

$$Dx_2 = \begin{vmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & -0.44 & 0.5 \end{vmatrix}$$

$$= 0.0649$$

$$Dx_3 = \begin{vmatrix} 0.3 & 0.52 & -0.01 \\ 0.5 & 1 & 0.67 \\ 0.1 & 0.3 & -0.44 \end{vmatrix} = -0.04356.$$

$$x_1 = \frac{Dx_1}{D} = \frac{0.03278}{-0.0022} = -14.9$$

$$x_2 = \frac{Dx_2}{D} = \frac{0.0649}{-0.0022} = -29.5$$

$$x_3 = \frac{Dx_3}{D} = \frac{-0.04356}{-0.0022} = 19.8$$

When Cramer's rule becomes impractical?

⇒ For more than three equations Cramer's rule becomes impractical because, as the number of equations increases, the determinants are time consuming to evaluate by hand (or by computer)

Iterative Process / Direct Substitution Method

Considering an equation $f(x)=0$ which can take in the form $x=\phi(x)$. If $\phi'(x)<1$, $\phi(x)$ is convergent.

$$x_1 = \phi(x_0)$$
$$x_2 = \phi(x_1)$$
$$x_3 = \phi(x_2) \ldots$$

Find root of $x = 0.5x^2 + 0.25$ using iterative method.

$f(x) = x - 0.5x^2 - 0.25$ as $x - 0.5x^2 - 0.25 = 0$

$f(0) = -0.25 < 1$

$f(1) = 0.75 > 1$

The root is between 0 and 1.

So $x_0 = \frac{0+1}{2} = 0.5$

Now, $x = \varphi(x)$

$\varphi(x) = 0.5(0.5)^2 + 0.25$

$= 0.375.$

$x_2 = 0.5(0.375)^2 + 0.25 = 0.32031$

$x_3 = 0.30130$

$x_4 = 0.29539$

$x_5 = 0.29362$

$x_6 = 0.29310$

$x_7 = 0.29295.$

The converges to 0.293

Solve $x^3 + x - 1 = 0$.

here, $x = 1 - x^3$

$f(x) = x^3 + x - 1$

$f(0) = -1$

$f(1) = 1$

root বের করার Polynomial equation solve করার জন্য x এর root value জানার $f(x) = 0$ লিখে এর derivative করে $x = \boxed{}$ এর মান যদি 1 এর কম হয় [illegible] convergence [illegible] divergence হবে

So the root is between 0, 1

So $x_0 = \frac{0+1}{2} = 0.5$

$\phi(x) = 1 - x^3$ (x = যা আসছে তা $\phi(x)$)

$\phi'(x) = 0 - 3x^2$

$\phi'(x)$ at $x_0 = 0.5 = \underline{\underline{-0.75 < 1}}$

So the iterative process can be applied.

$x_{i+1} = \phi(x_i)$

$x_1 = 1 - (0.5)^3 = 0.875$

$x_2 = 1 - (0.875)^3 = 0.33008$

$x_3 = 1 - (0.33008)^3 = 0.96404$

$\cancel{x_4 = 1 - (0.96404)^3 = 0.10405}$

$\cancel{x_5 = 1 - (0.10405)^3 =}$

Divergence and convergence for $2x^2 - 4x + 1 = 0$

$2x^2 + 1 = 4x$

$\Rightarrow x = \frac{x^2}{2} + \frac{1}{4}$

Convergence = 1st step − 2nd step > 2nd step − 3rd step.

n	x_n
1	1.0
2	0.75
3	0.53125
⋮	⋮
6	0.30330

Divergence 3rd step - 2nd step > 2nd step - 1st step.

n	x_n
1	2.0
2	2.25
3	2.78125
4	4.11768
5	8.72763
6	38.33574

root : বলতে x এর এমন একটা মান বের করতে হবে যেন function এর value 0 এর কাছাকাছি মান আসে।

Bracketing Method

The bracketing method is a numerical technique for finding the root of an equation $f(x)=0$ by identifying two initial values a and b such that:

$f(a).f(b)<0$. So thus the function must be continious in the interval.

Types of Bracketing methods ⇒

☐ Bisection Method (split the interval and chooses half with sign change)

☐ False Position (Regula Falsi)

uses secant line between $f(a)$ and $f(b)$ to estimate root.

④ Brent's Method:
Combines bisection, secant and interpolation for faster convergence

Bisection Method:-

If $f(x)$ is real and continuous in the interval from x_l to x_u and $f(x_l)$ and $f(x_u)$ have opposite sign. that is $f(x_l)\,f(x_u)<0$ then there is at least one real root between x_l and x_u.

Bisection Method Algorithm:-

① Choose lower x_l and upper x_u guesses for the root such that the function changes sign over the interval. This can be checked by ensuring that $f(x_l).f(x_u)<0$

② An estimate of the root x_r is determined by

$$x_r = \frac{x_l + x_u}{2}$$

③ Make the following evaluations to determine in which subinterval the root lies:

ⓐ If $f(x_l).f(x_r)<0$ the roots lies in the lower subinterval Therefore set $x_u = x_r$ and return to step 2.

ⓑ If $f(x_l).f(x_r)>0$ the root lies in the upper subinterval. Therefore set $x_l = x_r$ and return to step 2.

ⓒ If $f(x_l).f(x_r)=0$, the roots equals x; terminate the computation

Example-5.3 Use Bisection method to solve the following problem up to approximate percent relative error $\varepsilon_a = 0.42\%$. $f(c) = \frac{gm}{c}\left(1 - e^{-\frac{10c}{68.1}}\right) - 40$

$$f(c) = \frac{667.38}{c}\left(1 - e^{-0.146843c}\right) - 40.$$

Solution:-

Let $x_l = 12$, $x_u = 16$

Algo step1:- $f(12) = 6.0669$

$f(16) = -2.2687$.

$f(12).f(16) = 6.0669 \times -2.2687 = -13.76$.

$\therefore f(x_l).f(x_u) < 0$; $x_u = x_r$

Algo step2:- Now, $x_r = \frac{12+16}{2} = 14$.

$f(14) = 1.5687$

$F(12).f(14) = 6.0669 \times 1.5687 = 9.52 > 0$; $x_l = x_r$.

$\therefore x_l = 14$.

$x_r = \frac{14+16}{2} = 15$.

$f(14)\,f(15) = 1.5687 \times -0.4248 = -0.666 < 0$; $x_u = x_r$

$\therefore x_u = 15$.

Now root $x_r = \frac{14+15}{2} = 14.5$

I am going to continue this process until the appropriate error $\varepsilon_a < = 0.422$

$$\varepsilon_a = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| \times 100\%.$$

$f(x_l) \cdot f(x_r)$ positive → lower change; negative → upper change

Iteration	x_l	x_u	x_r	ε_a (%)
1	12	16	14	
2	14	16	15	6.677
3	14	15	14.5	3.448
4	14.5	15	14.75	1.695
5	14.75	15	14.875	0.840
6	14.75	14.875	14.8125	0.422.

Thus after 6 iterations, ε_a falls below 0.422%.

Difference between Iteration method and Bisection method ⇒

Bisection Method	Iteration Method
① Always converges	① May not converges
② Slower (linear convergence)	② Faster (if converges)
③ Needs interval with sign change	③ Needs good initial guess.
④ safe and slow	④ Fast but sensetive

$F(x) = x - \cos x$ we have to solve in both iteration method and Bisection method ⇒

Iteration method

Let us assume,

$F(0) = -1.00 < 0$

$F(1) = 0.00015 > 0$

so the root lies between the 0 and 1.

$x_0 = \frac{0+1}{2} = 0.5$

$x - \cos x = 0.$

$\therefore x = \cos x$

$\phi(x) = \cos x.$

$x_1 = \cos(0.5 \times 57.3) = 0.8775$

$x_2 = \cos(0.8775 \times 57.3) = 0.63897$

$x_3 = 0.80267$

$x_4 = 0.69474$

$x_5 = 0.76818$

$x_6 = 0.71913$

$x_7 = 0.75234$

$x_8 = 0.73005$

$x_9 = 0.7451$

0.6795

It is going to a fix point so the root of $x = \cos x$ is approximates 0.7391

Iteration	x_L	x_u	x_r	$F(x_L)$	$F(x_u)$	$F(x_r)$	ϵ
1	0	1	0.5	-1	0.459	[illegible]	
2	0	0.5	0.25	-1	0.1224	0.0311	100
3	0	0.25	0.125	-1	0.0311	-0.8672	50
	0.125	0.25	0.1875	-0.8672	0.0311	-0.795	

Iteration	x_L	x_u	x_r	$F(x_L)$	$F(x_u)$	$F(x_r)$	ϵ_a%
1	0	1	0.5	-1	0.459	-0.3776	
2	0.5	1	0.75	-0.3776	0.459	0.0183	33.33%
3	0.5	0.75	0.625	-0.3776	0.0183	-0.1859	20%
4	0.625	0.75	~~[illegible]~~ 0.6875	-0.1859	0.0183	~~[illegible]~~ -0.0853	9.09%
5	0.6875	0.75	0.7188	-0.0853	0.0183	-0.338	4.35%
6	0.7188	0.75	0.7344	-0.338	0.0183	-0.0078	2.12%
7	0.7344	0.75	0.7422	-0.0078	0.0183	0.0053	1.05%
8	0.7344	0.7422	0.7383	-0.0078	0.0053	-0.0013	0.53%
9	0.7383	0.7422	0.7403	-0.0013	0.0053	0.0020	0.27%
10	0.7383	0.7403	0.7393	-0.0013	0.0020	0.0004	0.135
11	0.7383	0.7393	0.7388	-0.0013	0.0004	-0.0004	0.067

As the error becomes very small so the root we can say $x = 0.7388$.

False-Position Method (Linear interpolation method, regula Falsi)

False Position method joins $f(x_l)$ and $f(x_u)$ by a straight line. The intersection of this line and x-axis represents an improved estimates of the root.

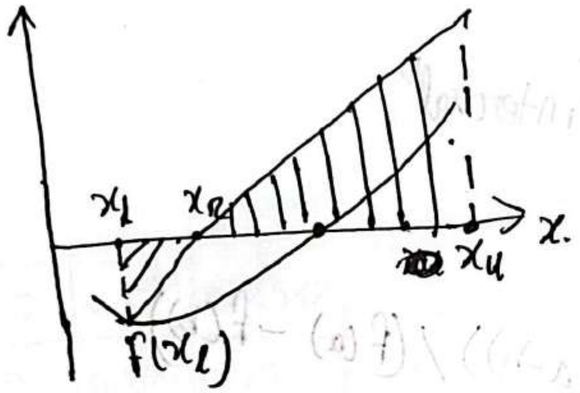


Fig: Graphical depiction of the method of false Position.

The intersection of the straight line with the x-axis can be estimated as,

$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u}$$

$$\Rightarrow f(x_l)\cdot(x_r - x_u) = f(x_u)\cdot(x_r - x_l)$$

x_r ডান side এ নিলে

$$\Rightarrow x_r\left[f(x_l) - f(x_u)\right] = x_u f(x_l) - x_l f(x_u)$$

$$\Rightarrow x_r = \frac{x_u f(x_l)}{f(x_l) - f(x_u)} - \frac{x_l f(x_u)}{f(x_l) - f(x_u)}$$

x_u + যোগ
x_u − বিয়োগ

$$\Rightarrow x_r = x_u + \frac{x_u \cdot f(x_l)}{f(x_l) - f(x_u)} - x_u - \frac{x_l f(x_u)}{f(x_l) - f(x_u)}$$

$$\Rightarrow x_r = x_u + \frac{x_u f(x_l)}{f(x_l) - f(x_u)} - \frac{x_l f(x_u)}{f(x_l) - f(x_u)}$$

$$\Rightarrow x_r = x_u - \frac{f(x_u)\cdot(x_l - x_u)}{f(x_l) - f(x_u)}$$

False Position Method Algorithm:-

Input: $f(x)$, a, b, ε, N

If $f(a) * f(b) \geq 0$:

  Print "Invalid interval"

  Exit

For $i=1$ to N:

  $x_r = b - (f(b) * (a-b)) / (f(a) - f(b))$

If $|f(x_r)| < \varepsilon$:

  Return x_r

If $f(a) * f(x_r) < 0$:

  $b = x_r$

Else:

  $a = x_r$

Return x_r.

Example:- $f(x) = x^3 - 4x + 1$

Let us assume,

$x_u = 0$, $x_l = 1$

$x_r =$

$f(x_u) = 1$, $f(x_l) = -2$

As $f(x_l) \cdot f(x_u) < 0$ so the root lies between them

$$x_r = 1 - \frac{(-2)(-1)}{1+2} \qquad x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

$$= 1 - \frac{2}{3}$$

$$= \frac{1}{3}$$

Iteration	x_l	x_u	x_r	$f(x_l)$	$f(x_u)$	$f(x_r)$	ε
1	0	1	1/3	1	-2	-0.2963	
2	0	0.333	0.2571	1	-0.2963	-0.0115	29.63%
3	0	0.2571	0.2542	1	-0.0115	-0.004	1.14%
4	0	0.2542	0.2532	1	-0.004	0.0035	0.3949%
5	0.2532	0.2542	0.2537	0.0035	-0.004	0.0017	0.1971%
6	0.2537	0.2542	0.2538	0.0017	-0.004	0.0010	0.0394%
7	0.2538	0.2542	0.2539	0.0010	-0.004	0.0008	0.0394%

so thus the root is 0.2539

A case where Bisection is preferable than false positioning:-

Although false position method would seem to be the bracketing method of preference, there are certain cases where it performs poorly.

Example-5.6 :- $f(x) = x^{10} - 1$

Let us assume the lower limit $x_L = 0$ and upper limit $x_u = 1.3$.

Using Bisection the result can be summarized as,

Iteration	x_l	x_u	x_r	$F(x_l)$	$F(x_u)$	$F(x_r)$	ε
1	0	1.3	0.65	-1	12.7858	-0.9865	
2	0.65	1.3	0.9750	-0.9865	12.7858	-0.2237	33.3%
3	0.9750	1.3	1.1375	-0.2237	12.758	2.6267	14.2857%
4	0.975	1.1375	1.0563	-0.2237	2.6267	0.7285	7.7%
5	0.975	1.0563	1.0157	-0.2237	0.7285	0.1686	3.99%

Using False Position

Iteration	x_l	x_u	x_r	ε
1	0	1.3	0.0943	
2	0.09430	1.3	0.18176	48.1
3	0.18176	1.3	0.26287	30.9
4	0.26287	1.3	0.33811	22.3
5	0.33811	1.3	0.40788	17.1

After 5 iteration, appropriate error from bisection is 40 and from false position is 17.1.

This proves that, there are cases where bisection is preferable

Before integrating. $f_1(x) = f(a) + \frac{f(b)-f(a)}{(b-a)}(x-a)$ can be expressed,

$$f_1(x) = \frac{f(b)-f(a)}{b-a}x. + f(a) - \frac{f(b)-f(a)}{b-a}.a.$$

Grouping the last two terms gives

$$f_1(x) = \frac{f(b)-f(a)}{b-a}.x + \frac{bf(a)-af(a)-af(b)+af(a)}{b-a.}$$

$$= \frac{f(b)-f(a)}{b-a}.x + \frac{bf(a)-af(a)}{b-a.}$$

which can be integrated between $x=a$ and $x=b$.

$$I = \frac{f(b)-f(a)}{b-a}\frac{x^2}{2} + \frac{bf(a)-af(a)}{b-a}.x\Big|_a^b$$

$$= \frac{f(b)-f(a)}{b-a.}\frac{b^2-a^2}{2} + \frac{bf(a)-af(a)}{b-a}(b-a)$$

since, $b^2-a^2=(b-a)(b+a)$

$$I = \frac{f(b)-f(a)}{2}(b+a) + bf(a)-af(b)$$

$$= b-a\left[\frac{f(a)+f(b)}{2}\right]$$

This is the derivation of trapezoid rule.

Newton-Raphson Method:-

It is a powerful iterative technique to find roots of nonlinear equations of the form $f(x)=0$.

Formula of root $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

x_n = current approximation

x_{n+1} = next approximation

$f(x_n)$ = Function

$f'(x_n)$ = derivative of the function.

Algorithm Steps:-

① Choose an initial guess x_0.

② Compute next value using $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

③ Repeat the formula until the solution converges:

$|x_{n+1} - x_n| < \varepsilon$ (small tolerance)

Solve $x^2 - 2 = 0$.

$f(x) = x^2 - 2$

$f'(x) = 2x$

x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x)}{f'(x_n)}$
1	-1	2	1.5
1.5	0.25	3	1.4167
1.4167	0.0069	2.8334	1.4143
1.4143	0.0001	2.8286	1.4143

repeated the root.

Converges to $\sqrt{2} \approx 1.4142$

Adv :-

- Fast convergence (quadratic)
- Simple to implement
- Great for well-behaved functions.

Disadv:-

- Needs derivative $f'(x)$
- Can diverge if initial guess is poor.
- Fails at points where $f'(x)=0$.

Method	Convergence rate	speed
Bisection	Linear	Slow
Secant	Super linear	Moderate
Newton-Raphson	Quadratic	Fast

Real Application:-

- Used in calculators to compute square roots and inverse trigonometric function.
- Efficient for embedded systems with limited resources (e.g. old Nokia phones).

Method can diverge

1. Poor initial guess.
2. $f'(x)=0$ or very small
3. Function not continuous or badly behave.
4. Root is multiple → The slope becomes flat → convergence become slow or fail.

Gauss Elimination Method:- REF ⇒ ~~Reduce~~ Row ~~ele~~ echelon form.

The Gauss Elimination Method is a systematic technique for solving systems of linear equations. It reduces a system to row-echelon form using row operations, then solve it using back-substitution.

Method:-

Forward Elimination :- convert the coefficient matrix into upper triangular form by eliminating variable below (diagonal) the pivot.

Back substitution:- Solve the last equation first and substitute back to find other unknowns.

Solve this example. $2x + 3y + z = 1$
$4x + 7y + 5z = 2$
$6x + 18y + 6z = 5$

Matrix $\left[\begin{array}{ccc:c} 2 & 3 & 1 & 1 \\ 4 & 7 & 5 & 2 \\ 6 & 18 & 6 & 5 \end{array}\right]$

Now, $R_2 - 2 \times R_1$,

$$\left[\begin{array}{ccc:c} 2 & 3 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 6 & 18 & 6 & 5 \end{array}\right]$$

again, $R_3 - 3R_1 = R_3'$

$$\left[\begin{array}{ccc:c} 2 & 3 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 9 & 3 & 2 \end{array}\right]$$

[পরেরবার R_1 দিয়ে লব করলে যেটা 0 করছি উঠে Problem হতে পারে]

Now making the matrix upper triangular by

$R_3' = R_3 - 9R_2$

$$\left[\begin{array}{ccc:c} 2 & 3 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -24 & 2 \end{array}\right]$$

$-24z = 2$

$\therefore z = \frac{-1}{12}$

$y + 3z = 0$

$\Rightarrow y = -3z = -\cancel{3} \cdot \frac{-1}{\cancel{12}\,4}$

$= \frac{1}{4}$

$2x + 3y + z = 1$

$\Rightarrow 2x + 3 \cdot 1/4 - 1/12 = 1$

$\Rightarrow x = \frac{1}{6}$

$\therefore$ So the answer is $x = \frac{1}{6}$, $y = \frac{1}{4}$, $z = \frac{-1}{12}$

Gauss-Jordan Elimination

It is a step by step method used to solve a system of linear equations by transforming the argumented matrix into Reduced Row Echelon Form (RREF) using row operation.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 150 \end{bmatrix}$$

1) R_1 stays same
2) $R_2 = R_2 - 2 \times R_1 = [0, -1, 1 \; | \; 0]$
3) $R_3 = R_3 - R_1 = [0, -1, 1 \; | \; 50]$
4) $R_3 = R_3 - R_2 = [0, 0, 0 \; | \; 50]$ contradiction

This is inconsistent method and there is no solution as one row is totally zero so it is inconsistent.

A math for both Gauss-Jordan Elimination and Gauss Elimination.

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Solution: First we have to express the coefficients and the right-hand side as argumented matrix:

$$\begin{bmatrix} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

$$r_1' = \frac{R_1}{3}$$

$$\begin{bmatrix} 1 & -0.03333 & -0.06667 & 2.61667 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

Then, $r_2' = r_2 - r_1 \times 0.1$

$r_3' = r_3 - r_1 \times 0.3$

$$\begin{bmatrix} 1 & -0.03333 & -0.066667 & 2.61667 \\ 0 & 7.00333 & -0.29333 & -19.5617 \\ 0 & -0.190000 & 10.0200 & 70.6150 \end{bmatrix}$$

Next, $r_2' = \frac{r_2}{7.00333}$

$$\begin{bmatrix} 1 & -0.03333 & -0.06667 & 2.61667 \\ 0 & 1 & -0.04188 & -2.79320 \\ 0 & -0.19000 & 10.0200 & 70.6150 \end{bmatrix}$$

$r_1' = r_1 + 0.03333 r_2$

$r_3' = r_3 + 0.1900 r_2$

$$\begin{bmatrix} 1 & 0 & -0.0680629 & 2.52356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 10.01200 & 70.0843 \end{bmatrix}$$

$R_3' = R_3 / 10.01200$

$$\begin{bmatrix} 1 & 0 & -0.0680629 & 2.52356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 1 & 7.0000 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 8 & 2 \\ 0 & -18 & -7 \\ 0 & 0 & -7/2 \end{bmatrix}$$

$R_1' = R_1 + 0.0680 R_3$

$R_2' = R_2 + 0.048 R_3$

R₁ কে 0 করে R₂ এর সাপেক্ষে যে কিছু টা করা.

R₁ এর 2 কে 0 করে R₃ এর কোন Point o...

Compare

$R_1 \Rightarrow R_1$

All time

$\frac{2}{-7/2}$ R_3

R₃ এর কোন Point আসে

$$\begin{bmatrix} 1 & 0 & 0 & 3.000 \\ 0 & 1 & 0 & -2.5000 \\ 0 & 0 & 1 & 7.000 \end{bmatrix}$$

$\therefore x = 3,\ y = -2.5,\ z = 7.$

Difference between Gauss elimination and Gauss Jordan

1) When an unknown is eliminated in the Gauss-Jordan method it is eliminated from all other equations, but in Gauss Elimination, just the subsequents ones are eliminated.

2) In Gauss-Jordan method, the elimination steps result in an identity matrix whereas, in Gauss Elimination, the elimination steps results in a triangular matrix.

3) It is not necessary to employ back substitution to obtain the solution in Gauss-Jordan method, but in

Gauss elimination, the back substitution is necessary for the solution.

Interpolation:-

The value of 'x' for which the value of 'y' is to be estimated is an intermediate value in the given set of values of 'x' then the method of determining y is called interpolation.

$x \rightarrow$ argument
$y \rightarrow$ entry

interpolation is a numerical method use to estimate the value of a function at a point within the range of known data points

interpolation methods $\Rightarrow$

1) Newton's forward interpolation
2) Newton's backward interpolation

if the interval is equal.

3) Binomial method $\rightarrow$ missing terms.
4) Lagrange's interpolation $\rightarrow$ unequal interval.

Newton's Forward and Backward Interpolation:-

Let $x_0, x_1, x_2, \ldots x_{n-1}, x_n$ be a set of equidistant values of variable x.

$\therefore x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \ldots = x_n - x_{n-1} = h$

Let $u = \frac{x - x_0}{h}$

Newton's forward difference formula,

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots + \frac{u(u-1)(u-2)\dots(u-n+1)}{n!}\Delta^n y_0.$$

Horizontal difference (First-order difference) :

There are difference along the rows of a table.

$$\Delta y_i = y_{i+1} - y_i$$

x	y
1	2
2	4
3	7
4	11

$\Delta y_0 = 4 - 2 = 2$

$\Delta y_1 = 7 - 4 = 3$

$\Delta y_2 = 11 - 7 = 4.$

diagonal difference (Higher-order difference) :

There are taken diagonally between the horizontal differences.

second order difference,

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = 3 - 2 = 1$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = 4 - 3 = 1$$

third order difference,

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = 1-1=0$$

So the full difference table looks like:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	2	2	1	0
2	4	3	1	
3	7	4		
4	11			

$f(x) = x^2+1$

x	$f(x)$
0	1
1	2
2	5
3	10
4	17

First difference,

$\Delta f_0 = f(1) - f(0) = 2-1 = 1$

$\Delta f_1 = 3$, $\Delta f_2 = 5$, $\Delta f_3 = 7$

second difference,

$\Delta^2 f_0 = 2$, $\Delta^2 f_1 = 2$, $\Delta^2 f_2 = 2$

The second difference are constant which confirms degree 2 polynomial.

third difference,

$\Delta^3 f_0 = 2-2=0$

And all higher difference are zero

So for a polynomial of degree n, the nth order finite difference is constant and all higher-order differences are zero.

For degree 1, first difference constant.

" deg 2, second " "

Newton's forward interpolation to find $f(2.5)$ value for x

$f(x)=x^3$

x	$y=f(x)$
1	1
2	8
3	27
4	64

$u=\frac{x-x_0}{h}$ and

Condition checking,

$h=x_1-x_0=2-1=1=x_2-x_1=3-2=1$

the interval of x is equal and as 2.5 is closer to the begining of the table we are going to use,

Newton's Forward interpolation.

The forward difference table,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	1	7	12	6
2	8	19	18	
3	27	37		
4	64			

Newton's forward interpolation formula,

$$f(x)=y_0+u\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$x_0=1$

$h=1$

$$u=\frac{x-x_0}{h}=\frac{2.5-1}{1}=1.5$$

$$f(2.5) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= 1 + (1.5)(7) + \frac{1.5 \times 0.5}{2}(12) + \frac{1.5(0.5)(-0.5)}{6}6$$

$$= 15.625$$

since the actual function,

$$f(x) = x^3 = (2.5)^3 = 15.625.$$

Newton Backward Formula:-

$$f(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

Given table,

x	$f(x) = x^3 = y$
1	1
2	8
3	27
4	64

$h = x_1 - x_0 = 2 - 1 = 1 = x_2 - x_1 = 3 - 2 = 1$

the difference is equal. And as 3.5 is near the end so we will use Backward formula.

$x_n = 4$

$$u = \frac{x - x_n}{h} = \frac{3.5 - 4}{1} = -0.5$$

Building the difference table,

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
1	1			
2	8	7		
3	27	19	12	
4	64	37	18	6

Plugging the values in formula,

$$f(3.5) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$= 64 + (-0.5)(37) + \frac{-0.5(0.5)}{2}(18) + \frac{-0.5(0.5)(1.5)}{6}(6)$$

$$= 42.875$$

As the function,

$$f(x) = x^3$$

$$f(3.5) = (3.5)^3 = \underline{42.875}$$

Comparison table for Forward and Backward interpolation

Forward interpolation	Backward interpolation
when x is near the beginning of the data.	when x is near the end of the data
First value x_0	Last value x_n
$u = \frac{x-x_0}{h}$	$u = \frac{x-x_n}{h}$
Forward difference (Δy)	Backward difference (∇y)
$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots$	$f(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2}\nabla^2 y_n$
values of x close to x_0	values of x close to x_n
From top to bottom (forward steps)	From bottom to top (backward steps)

Interpolation is finding the unknown value of a function between known data point.

Adv of Lagrange interpolation ⇒

1. No difference table need.
2. Works for unequal interval.
3. simple to apply
4. Accurate for small datasets.
5. Good for computer.

Newton-Cote integration Formula:

The newton cotes method helps us estimate the area under a curve by using simple shapes like straight lines or curves; that connect some points on the function.

General Newton-Cote formula

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i)$$

① $x_0, x_1, \ldots, x_n$ are equally spaced points in $[a,b]$

② w_i are weights depending on the interpolation.

③ The degree n of the formula is the number of points used minus 1. or the subinterval.

Types of Newton Cote formula (closed form):

① Trapezoidal Rule $(n=1)$:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a)+f(b)\right]$$

uses a straight line to connect two points.

② Simpson's 1/3 Rule $(n=2)$:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

use Parabola through 3 points.

③ Simpson's 3/8 Rule ($n=3$):

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right]$$

use cubic polynomial through 4 points.

④ Boole's Rule ($n=4$):

$$\int_a^b f(x)\,dx \approx \frac{2h}{45}\left[7f(x_0)+32f(x_1)+12f(x_2)+32f(x_3)+7f(x_4)\right]$$

uses 5 points and fits a higher degree polynomial.

$h=\frac{b-a}{n}$; this is the width of each small subinterval.

Trapezoidal Rule:- Derivation of Trapezoid rule

The trapezoidal rule is the first of the Newton-Cotes closed integration formulas. It corresponds to the case where the polynomial in the eqn is first order.

$$I=\int_a^b f(x)\,dx=\int_a^b f_1(x)\,dx.$$

A straight line can be represented as

$$f_1(x)=f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$$

The area under this straight line is an estimate of the integral of $f(x)$ between limits a and b.

$$I=\int_a^b\left[f(a)+\frac{f(b)-f(a)}{b-a}(x-a)\right]dx$$

$$= \left[f(a)x + \frac{f(b)-f(a)}{b-a} \cdot \frac{(x-a)^2}{2} \right]_a^b$$

$$= f(a).b - f(a).a + \frac{f(b)-f(a)}{(b-a)} \cdot \frac{(b-a)^2}{2}$$

$$= f(a)(b-a) + \frac{[f(b)-f(a)] \cdot (b-a)}{2}$$

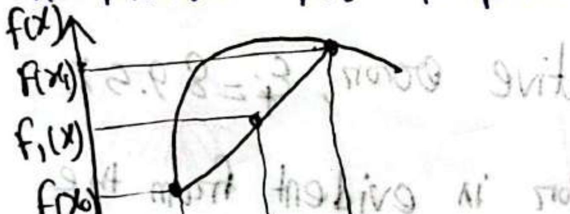
$$= \frac{2f(a)(b-a) + f(b)(b-a) - f(a)(b-a)}{2}$$

$$= \frac{f(a)(b-a) + f(b)(b-a)}{2}$$

$$\therefore I = (b-a) \cdot \frac{f(a)+f(b)}{2}$$

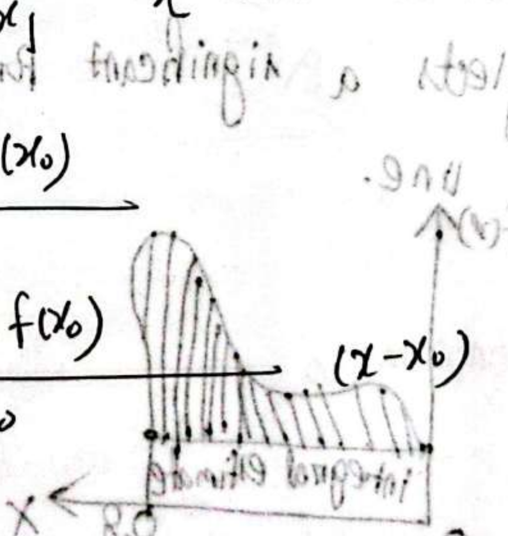
$I \approx$ width $\times$ average height.

This is a formula for trapezoidal rule.



$$\frac{f_1(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\Rightarrow f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0)$$



Ex-21.1

use trapezoidal rule to numerically integrate $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$.

from $a = 0$ to $b = 0.8$ exact value $= 1.640533$

$$\varepsilon_t = \left| \frac{V_A - V_E}{V_E} \right| \times 100\%$$

V_A = actual value observed

V_E = Expected value.

Solution:

The function values.

$f(0) = 0.2$

$f(0.8) = 0.232$

using trapezoidal rule we get,

$$I \simeq 0.8 \frac{0.2 + 0.232}{2} = 0.1728.$$

which represents an error of

$$E_t = 1.640533 - 0.1728 = 1.467733$$

which corresponds to percent relative error, $\varepsilon_t = 89.5\%$.

The reason for this large error is evident from the figure as we can see that the area under the straight line neglects a significant portion of the integral lying above the line.

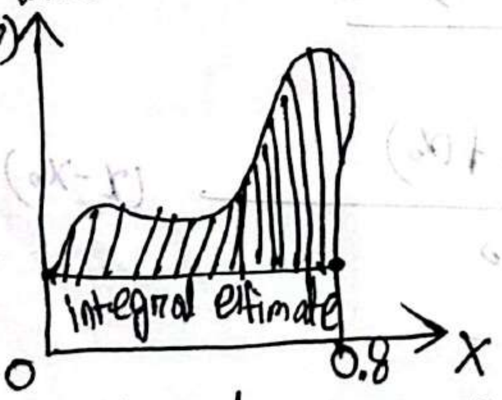


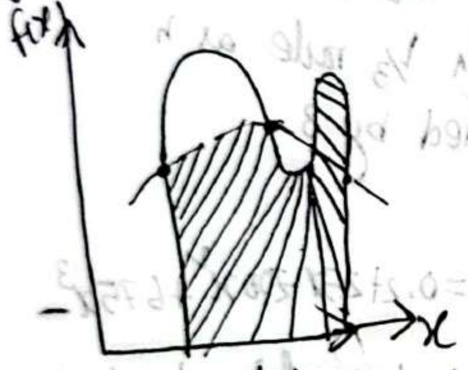
Fig: single Application of Trapezoidal rule

How to remove Trapezoidal rule?

⇒ By using Simphon's $\frac{1}{3}$ Rule

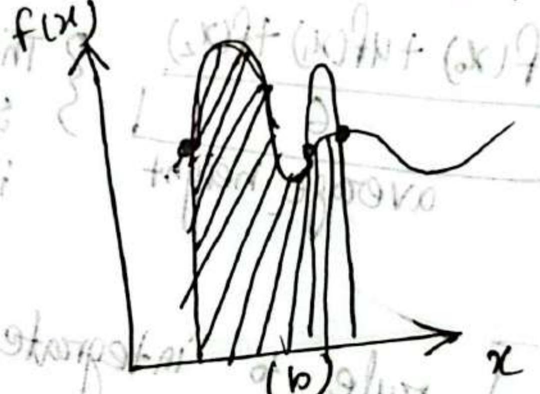
Simpson's Rule:-

Another way to obtain a more accurate estimate of an integral is to use higher-order polynomials to connect the points. For example if there is an extra point midway between f(a) and f(b), the three points can be connected with a parabola. If there are two points equally spaced between f(a) and f(b), the four points can be connected with a third-order polynomial. The formula that result from taking the integrals under these polynomials are called Simphon's rule



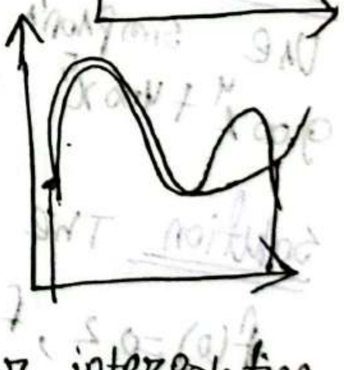
(a)

[$\frac{1}{3}$ rule]



(b)

[$\frac{3}{8}$ rule]



Simphon $\frac{1}{3}$ Rule:-

Simphon's $\frac{1}{3}$ rule results when a second order interpolating polynomial is substituted into

$$I = \int_a^b f(x)\,dx = \int_a^b f_2(x)\,dx$$

If a and b are designated as x_0 and x_2 and $f_2(x)$ is represented by a second order larger polynomial, the integral becomes,

$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_1-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

After integration and algebraic manipulation the following formula results

$$I \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2) \right] \; ; h = \frac{b-a}{2}$$

$$I \cong \underbrace{(b-a)}_{\text{width}} \; \underbrace{\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}}_{\text{average height}}$$

} This is called simpson's 1/3 rule as h is divided by 3.

Example-21.4

Use simpson's 1/3 rule to integrate $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a = 0$ to $b = 0.8$ exact integral is 1.640533

Solution The function values,

$f(0) = 0.2$, $f(0.4) = 2.456$ $f(0.8) = 0.232$

Using simpson's 1/3 rule,

$$I \cong 0.8 \, \frac{0.2 + 4(2.456) + 0.232}{6}$$

$$= 1.367467$$

which represents an exact error of

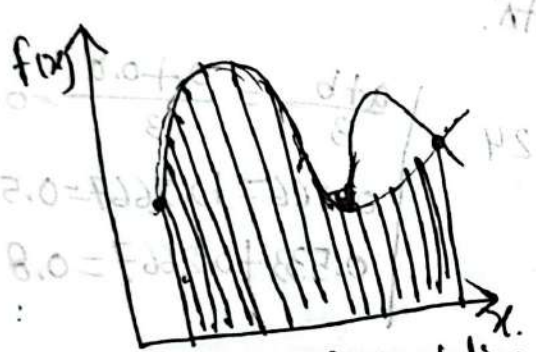
$$E_t = 1.640533 - 1.367467$$
$$= 0.2730667.$$

$$\varepsilon_t = \frac{E}{\text{exact value}} \times 100\% = 16.6\%.$$

which is 5 times more accurate than for a single application of the trapezoidal rule.

Simpson's 3/8 Rule:-

If there are two points equally spaced between $f(a)$ and $f(b)$ the four points can be connected with a third order polynomial.



Graphical representation of simpson 3/8 rule

a third order Lagrange polynomial can be fit to four points and integrated: $I \cong \int_a^b f_3(x)\,dx$

$$I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right] \text{ where } h = \frac{b-a}{3}$$

This equation is called simpsons 3/8 rule because h is manipulated by 3/8. It is the third Newton-Cotes closed integration formula.

The 3/8 rule can also be expressed as,

$$I \cong \underbrace{(b-a)}_{\text{width}} \underbrace{\frac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8}}_{\text{average height}}$$

Example 21.6

Use Simpson's 3/8 rule to integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a=0$ to $b=0.8$

Solution: A single application of Simpson's 3/8 rule requires four equally spaced points:

$f(0) = 0.2$; $f(0.2667) = 1.432724$

$f(0.5333) = 3.487177$

$f(0.8) = 0.232$

$\frac{a+b}{3} = \frac{0+0.8}{3} = 0.2667$

$0.2667 + 0.2667 = 0.533$

$0.533 + 0.2667 = 0.8$

Using Simpson's 3/8 rule,

$$I \cong 0.8 \, \frac{0.2 + 3(1.432724 + 3.487177) + 0.232}{8}$$

$$= 1.519170$$

error, $E_t = 1.640533 - 1.519170 = 0.1213630$

$\therefore \varepsilon_t = 7.4\%$

Use it in Conjunction with simpson's 3/8 rule to integrate the same function for five segments.

Solution:-

The data needed for a five segment application $0.8/5 = 0.16$, $h = 0.16$.

$f(0) = 0.2$	$f(0.48) = 3.186015$
$f(0.16) = 1.296919$	$f(0.64) = 3.181929$
$f(0.32) = 1.743393$	$F(0.80) = 0.232$

The integral for the first two segment is obtained using simpson's 1/3 rule:

$$I = 0.32\,\frac{0.2 + 4(1.296919) + 1.743393}{6}$$

$$= 0.3803237$$

$$= (b-a)\,\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}$$

for first two segment,

$x_0 = 0.2$, $x_1 = 0.16$, $x_2 = 0.32$

$a = 0$, $b = 0.32$

For last three segments the 3/8 rule can be used to obtain,

$$I = 0.48\,\frac{1.743393 + 3(3.186015 + 3.181929) + 0.232}{8}$$

$$= 1.264754$$

$$= (b-a)\,\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}$$

$x_0 = 0.32$, $x_1 = 0.48$, $x_2 = 0.64$, $x_3 = 0.8$

$a = 0.32$, $b = 0.8$

$b - a = 0.8 - 0.32 = 0.48$

The total integral is computed by summing the two results:

$$I = 0.3803237 + 1.264753 = 1.645077$$

$E_t = 1.640533 - 1.645077$

$\varepsilon_t = -0.28\%$

Runge-Kutta Methods

$$\frac{dy}{dx} = f(x, y)$$

New value = old value + slope × step size

$$y_{i+1} = y_i + \phi h$$

ordinary differential equation:-

An ordinary differential equation is an equation that relates a function with its derivative. It is ordinary because it involves derivative with respect to a single variable.

Euler Method

Euler method is a numerical technique to approximate the solution of ordinary differential equation,

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Euler method estimates the value of y at the next point using the slope at the current point.

Eulers formula: $y_{n+1} = y_n + h.f(x_n, y_n) \mid y(x_0) = y_0$

① x_n, y_n are the current point
② h is the step size
③ $f(x,y)$ is the derivative function
④ y_{n+1} is the estimated value at $x_{n+1} = x_n + h$

Ex 25.1 Use Eulers Method to numerically integrate the given equation:

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

from $x=0$ to $x=4$ with a step size of 0.5. The initial condition at $x=0$ is $y=1$. Recall the exact eqn is given by $y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$

Solution $y_{i+1} = y_i + f(x_i, y_i)h$ can be use to implement Eulers method,

$$y(0.5) = y(0) + f(0,1)\,0.5$$

where $y(0)=1$ and the slope estimate at $x=0$ is

$$f(0,1) = -2(0)^3 + 12(0)^2 - 20(0) + 8.5 = 8.5$$

Therefore,

$$y(0.5) = 1 + 8.5(0.5) = 5.25$$

The true solution at $x=0.5$ is

$$y = -0.5(0.5)^4 + 4(0.5)^3 - 10(0.5)^2 + 8.5(0.5) + 1$$

$$= 3.21875$$

Thus the error $\varepsilon_t = \frac{\text{true} - \text{approximate}}{\text{true}} \times 100\%$

$$= \frac{3.21875 - 5.25}{3.21875} \times 100\%$$

$$= -63.1\%$$

For second step,

$$y(1) = y(0.5) + f(0.5, 5.25)0.5$$

$$= 5.25 + [-2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5]0.5$$

$$= 5.875$$

The true solution at $x = 1.0$ is,

$$y = -0.5(1.0)^4 + 4(1)^3 - 10(1)^2 + 8.5 \times 1 + 1$$

$$= 3.0.$$

Therefore, $\varepsilon_t = \frac{3 - 5.875}{3} \times 100\%$

$$= -95.8\%$$

The computation is repeated and the results are shown in the table and figure below.

x	y_{true}	y_{Euler}	Percent relative error
0	1	1	-63.1%
0.5	3.21875	5.25	-95.8%
1	3	5.875	-131%
1.5	2.21875	5.125	-125%
2	2	4.5	-74.7%
2.5	2.71875	4.75	-46.9%
3	4	5.875	-51.0%
3.5	4.71875	7.125	-133.3%
4	3	7	

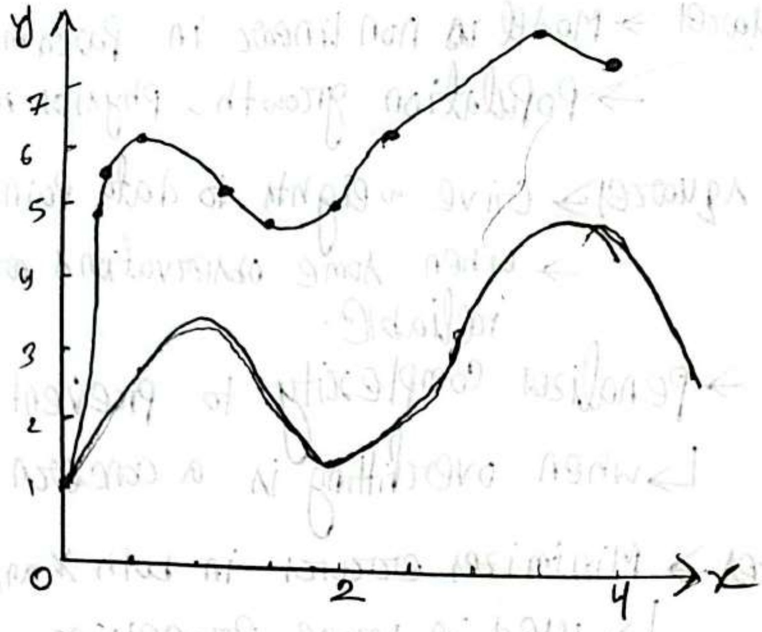


Figure ⇒ Comparison of true solution with a numerical solution using Eulers method for the given ~~its~~ integral.

Least Squares Regression :

Least square regression is a method used to find the best-fitting line (or curve) through a set of data points by minimizing the sum of the ~~##~~ squared errors (difference between actual and predicated values) We use square so that we can find the positive and negative difference both

types of Least-square Regression :

1. Linear Least squares ⇒ straight line $Y = B_0 + B_1 X$ → relationship is linear.
2. Multiple Linear regression → fits multiple input : $Y = B_0 + B_1 X_1 + B_2 X_2 + \dots$
   → Predict house Price :
3. Polynomial regression ⇒ fits nonlinear curves ⇒ $Y = a + bx + cx^2 + \dots$
   → U shaped or curved trends.

④ Nonlinear least squares → Model is non linear in parameters
→ Population growth, physics models.

⑤ Weighted Least squares → Give weights to data points.
→ when some observations are more reliable.

⑥ Ridge regression → penalizes complexity to prevent overfitting
↳ when overfitting is a concern

⑦ Total Least Squares → Minimizes errors in both X and Y directions
↳ used in image processing or when X values have noise.

Linear Regression:-

Linear Regression is a supervised machine learning algorithm that predicts a continious output based on the linear relationship between input and output variables.

$$Y = B_0 + B_1 X + \epsilon$$

Y = output (dependent variable)
X = input (independent variable)

B_0 and B_1 are intercept (value of Y when X=0) and slope (change in Y per unit change in X) respectively

ϵ = error or residual between the model and the observation.

Criteria for a 'best' fit:

For fitting a best line through the data would be to minimize sum of the residual errors (for all available data)

$$\sum_{i=1}^{n} |e_i| = \sum_{i=1}^{n} |y_i - a_0 - a_1 x_i| \quad \text{--- (1)} \quad \text{where } n = \text{total no of Points.}$$

Let, $$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2 \quad \text{--- (2)}$$

$$\frac{\delta S_r}{\delta a_0} = -2 \sum (y_i - a_0 - a_1 x_i) \quad \text{--- (3)}$$

$$\frac{\delta S_r}{\delta a_1} = -2 \sum (y_i - a_0 - a_1 x_i) x_i \quad \text{--- (4)}$$

$$(3) \Rightarrow 0 = \sum y_i - \sum a_0 - \sum a_1 x_i \quad \text{--- (5)}$$

$$(4) \Rightarrow 0 = \sum y_i x_i - \sum a_0 x_i - \sum a_1 x_i^2 \quad \text{--- (6)}$$

$$(5) \Rightarrow n a_0 + (\sum x_i) a_1 = \sum y_i \quad \text{--- (7)}$$

$$(6) \Rightarrow (\sum x_i) a_0 + (\sum x_i^2) a_1 = \sum x_i y_i$$

From (6) and (7) we get,

This can be asked as "Find out the value of a_0 and a_1 in the case of least square regression."

$$a_1 = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

$$(7) \Rightarrow n a_0 + (\sum x_i) a_1 = \sum y_i$$

$$\Rightarrow n a_0 = \sum y_i - (\sum x_i) a_1$$

$$\Rightarrow a_0 = \frac{\sum y_i}{n} - \frac{(\sum x_i) a_1}{n} \quad \therefore a_0 = \bar{y} - a_1 \bar{x}$$

CamScanner

Ex-17.1

Fit a straight line to the x and y values in the first two columns of the table
- computation for an error analysis of the linear fit:

x_i	y_i	$(y_i - \bar{y})^2$	$(y_i - a_0 - a_1 x_i)^2$
1	0.5	8.5765	0.1687
2	2.5	0.8622	0.5625
3	2.0	2.0408	0.3473
4	4.0	0.3265	0.3265
5	3.5	0.0051	0.5896
6	6.0	6.6122	0.7972
7	5.5	4.2908	0.1993
Σ 28	24.0	22.7143	2.9911

Solution:- Here the quantities are,

$n = 7$, $\Sigma x_i y_i = 119.5$ $\quad \Sigma(x_i^2) = 140$

$\Sigma x_i = 28$ $\quad \bar{x} = \frac{28}{7} = 4$

$\Sigma y_i = 24$ $\quad \bar{y} = \frac{24}{7} = 3.428571$

$$\text{Now, } a_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum(x_i^2) - (\sum x_i)^2}$$

$$= \frac{7(119.5) - 28(24)}{7(140) - (28)^2}$$

$$= 0.8392857$$

$$a_0 = \bar{y} - a_1\bar{x}$$

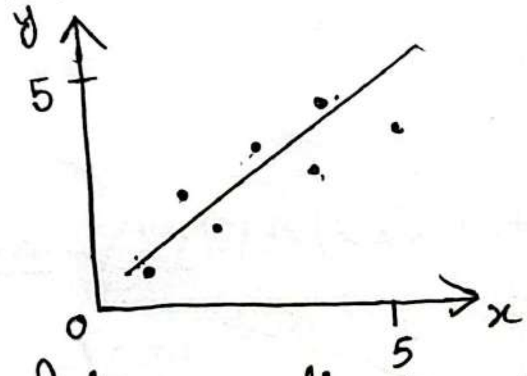
$$= 3.428571 - 0.8392857(4)$$

$$= 0.07142857$$

There the least square fit is,

$$y = a_0 + a_1 x$$

$$\Rightarrow y = 0.07142857 + 0.8392857x$$



More satisfactory result using the least square fit.

Training a linear regression model using Gradient Descent:-

x	y
1	2
2	4
3	6

First let us predict,

$m=0,\ b=0,\ \alpha=0.01$

$\hat{y}_1 = 0\times1+0=0$

$\hat{y}_2 = 0\times2+0=0$

$\hat{y}_3 = 0\times3+0=0.$

Error, $y_1-\hat{y}_1 = 2-0=2=e_1$

$e_2=4-0=4$

$e_3=6-0=6.$

Gradients,

$$\frac{\partial Loss}{\partial m} = \frac{-2}{n}\sum(\overset{x.e}{x_i)(y_i-\hat{y}})$$

$$= \frac{-2}{3}\left[(1\times2)+(2\times4)+(3\times6)\right]$$

$$= -18.67$$

$$\frac{\partial loss}{\partial b} = \frac{-2}{n}\sum(\overset{e}{y_i-\hat{y}})$$

$$= \frac{-2}{3}(2+4+6) = -8$$

Updated,

$$m := m-\alpha\cdot\frac{\partial Loss}{\partial m} = 0-0.01(-18.67)$$

$$= 0.1867$$

$= \cancel{0-0.01}$

$b := b - \alpha \frac{\partial Loss}{\partial b}$

$= 0 - 0.01(-8)$

$= 0.08$

iterations2,

$\hat{y}_1 = 0.1867 \times 1 + 0.08 = 0.2667$

$\hat{y}_2 = 0.1867 \times 2 + 0.08 = 0.4534$

$\hat{y}_3 = 0.1867 \times 3 + 0.08 = 0.6401$

Errors,

$e_1 = 2 - 0.2667 = 1.7333$

$e_2 = 4 - 0.4534 = 3.5466$

$e_3 = 6 - 0.6401 = 5.3599$

Gradients,

$\frac{\partial loss}{\partial m} = \frac{-2}{3}(1 \times 1.733 + 2 \times 3.5466 + 3 \times 5.3599) -$

$= -16.60$

$\frac{\partial loss}{\partial b} = \frac{-2}{3}(1.7333 + 3.5466 + 5.3599) = -7.09$

update,

$m := 0.1867 + 0.01 \times 16.60 = 0.1867 + 0.166 = 0.3527$

$b := 0.08 + 0.01 \times 7.09 = 0.1509$

Eventually the model will learn,

$y \approx 2x.$

Multiple Linear Rigression:

Multiple linear regression Predicts a continuous output Y based on two or more input variables.

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \dots + \beta_n X_n + \varepsilon$$

Y = target variable

$X_1, X_2, \dots X_n$ = input features.

β_0 = intercept (Constant term)

$\beta_1, \dots, \beta_n$ = Coefficient (weights) for each features

ε = Error term (difference between actual and predicted)

What is R^2 (R-squared)?

R^2 is the coefficient of determination a metric that tells you how well your regression model fits the data.

$$R^2 = 1 - \frac{SS_{res}}{SS_{tot}}$$

SS_{res} = Residual sum of squares $= \sum (y_i - \hat{y}_i)^2$

SS_{tot} = Total sum of squares $= \sum (y_i - \bar{y})^2$

R^2 value	Meaning
1.0	→ Perfect Prediction - all Points lie on the regression line
0.8	→ 80% of variation explained by model.
0.0	→ model explain nothing (Just guessing mean)
<0	→ worse than just using mean.

Polynomial Regression:-

Polynomial regression is an extensions of Linear regression, where the relationship between the independent variable and dependent variable is modeled as nth degree polynomial.

Linear regression may not capture non-linear patterns in data. Polynomial regression allows more flexibility with curved trends.

The least squares procedure can be readily extended to fit the data to a higher-order polynomial. For example, lets suppose that we fit a second-order polynomial or quadratic:

$$y = a_0 + a_1 x + a_2 x^2 + \varepsilon$$

For this case the sum of the squares of the residual is:

$$S_r = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2 \text{ ——— } (1)$$

Taking the derivative of eqn (1) with respect to each of the unknown coefficient of the polynomial, as in.

$$\frac{\delta S_r}{\delta a_0} = -2 \sum (y_i - a_0 - a_1 x_i - a_2 x_i^2)$$

$$\frac{\delta S_r}{\delta a_1} = -2 \sum x_i (y_i - a_0 - a_1 x_i - a_2 x_i^2)$$

$$\frac{\delta S_r}{\delta a_2} = -2 \sum x_i^2 (y_i - a_0 - a_1 x_i - a_2 x_i^2)$$

These equations can be set equal to zero and rearranged to develop the following set of normal equations:

$$(n)a_0 + (\sum x_i)a_1 + (\sum x_i^2)a_2 = \sum y_i$$

$$(\sum x_i)a_0 + (\sum x_i^2)a_1 + (\sum x^3)a_2 = \sum x_i y_i$$

$$(\sum x_i^2)a_0 + (\sum x_i^3)a_1 + (\sum x_i^4)a_3 = \sum x_i^2 y_i$$

standard error is formulated as:

$$S_{y/x} = \sqrt{\frac{S_r}{n-(m+1)}}$$

Example 17.5

Fit a second order polynomial to the data in the first two columns of the table.

x_i	y_i	$(y_i-\bar{y})^2$	$(y_i-a_0-ax_i-a_2x_i^2)^2$
0	2.1	544.44	0.14332
1	7.7	314.47	1.00286
2	13.6	140.03	1.08158
3	27.2	3.12	0.80491
4	40.9	239.22	0.61951
5	61.1	1272.11	0.09439
Σ	152.6	2513.39	3.74657

$\cancel{(3\times1)\,(3\times3)}$

From the given data,

order, $m=2$ $\quad \Sigma x_i = 15$, $\quad \Sigma x_i^4 = 979$.

$n = 6$ $\quad \Sigma y_i = 152.6$ $\quad \Sigma x_i y_i = 585.6$

$\bar{x} = 2.5$ $\quad \Sigma x_i^2 = 55$ $\quad \Sigma x_i^2 y_i = 2488.8$

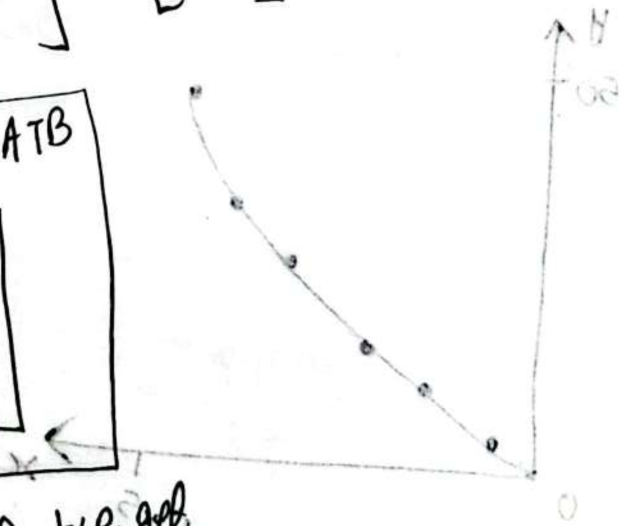
$\bar{y} = 25.433$ $\quad \Sigma x_i^3 = 225$

Simulation linear equation,

$$\begin{bmatrix} n & \Sigma x_i & \Sigma x_i^2 \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 \\ \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \\ \Sigma x_i^2 y_i \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix}}_{MATA} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 152.6 \\ 585.6 \\ 2488.8 \end{bmatrix}}_{MATB}$$

$$\boxed{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = (MATA)^{-1} \times MATB = \begin{bmatrix} 2.47857 \\ 2.35929 \\ 1.86071 \end{bmatrix}}$$



using Gauss Elimination we get,

$a_0 = 2.47857$, $a_1 = 2.35929$, $a_2 = 1.86071$

The least square ~~quadrict~~ · quadratic equation for two case,

$$y = a_0 + a_1 x + a_2 x^2$$
$$= 2.47857 + 2.35929x + 1.86071x^2$$

The standard error of estimate based on the regression Polynomial is

$$S_{y/x} = \sqrt{\frac{3.74657}{6-3}} = 1.12$$

$(y-\hat{y}_i)^2$ ↳ Actual function square.

The coefficient of determination,

$$r^2 = 1 - \frac{3.74657}{2513.39}$$
$$= 0.99851$$

The correlation coefficient is $r = \sqrt{0.99851}$
$$= 0.9993$$

there result indicate that 99.851 Percent of the original uncertainty has been explained by the model.

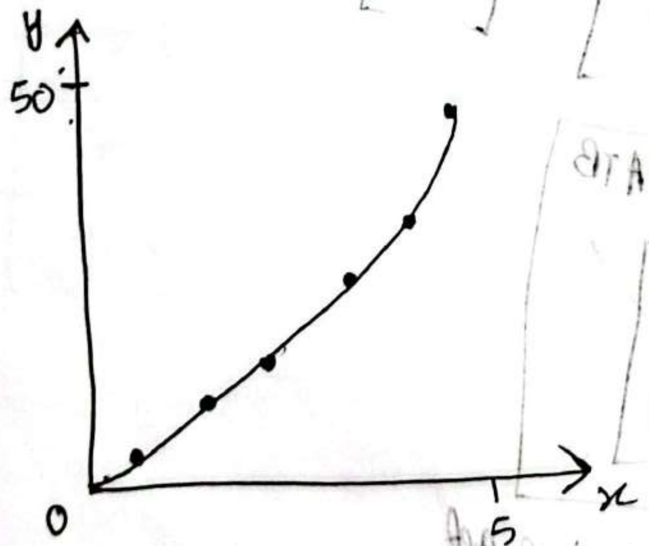


Fig:- Fit of a second order Polynomial.

Logistic Regression:

Logistic regression is a type of machine learning algorithm used for classification Problem and predicts Probability between 0 and 1. and output is binary as yes/no, Pass/fail etc.

It use sigmoid function,

$$P(y=1|x) = \frac{1}{1+e^{-(\beta_0+\beta_1 x_1+\beta_2 x_2+\dots\beta_n x_n)}}$$

Always have to predict,

$\beta_0 = -4.5$
$\beta_1 = 0.03$
$\beta_2 = 0.05$
$\beta_3 = 0.1$

where,
$x_1, x_2 \dots$ input features
β_0 intercept
β_i model coefficients.

if $P \geq 0.5 \Rightarrow 1$ (Positive) else 0 (negative) here using threshold.

Aspect	Linear regression	Logistic regression
output	Continious value	Probability (0-1)
used for	Regression Problem	classification Problem.
Algorithm	Least square	Maximum likelihood
Function	Linear	Sigmoid / logistic.
Example	predict house Price	Predict if house will sell first (yes/no)

Age	Income	Time on website	Buy	
22	25	8	0	
35	45	22	1	$\Rightarrow 0.73$
50	60	10	1	$\Rightarrow 0.73$
29	30	5	0	$\Rightarrow 0.16383$

Here I have considered,
Age $= x_1$, Income $= x_2$, Time on website $= x_3$, Buy $=$ Prediction

$$P(\text{Buy}) = \frac{1}{1+e^{-(\beta_0+\beta_1x_1+\beta_2x_2+\beta_3x_3)}}$$

consider,
$\beta_0 = -4.5$
$\beta_1 = 0.03$
$\beta_2 = 0.05$
$\beta_3 = 0.1$

For, $x_1 = 22$, $x_2 = 25$, $x_3 = 8$

$$\beta_0+\beta_1x_1+\beta_2x_2+\beta_3x_3 = -4.5+(0.03\times22)+(0.05\times25)+(0.1\times8)$$
$$= -1.79$$

$$P = \frac{1}{1+e^{-(-1.79)}} = 0.143 < 0.5$$ so it will

show negative result.

Hours studied	Attendance	Pass	
2	70	0	$\Rightarrow 0.28$
4	80	0	$\Rightarrow 0.4$
6	78	0	$\Rightarrow 0.39$
8	85	0	$\Rightarrow 0.49$
10	90	1	

$\beta_0 = -4.5$
$\beta_1 = 0.03$
$\beta_2 = 0.05$
$\beta_2 = 0.1$ ধরলে

$B_2 = 0.1, B_1 = 0.03$

Hours	Attendance	Pass	
2	70	1	$\Rightarrow 0.92$
4	80		
6	78	1	$\Rightarrow 0.97$
8	85		
10	90		

Linear vs logistic regression $\Rightarrow$

linear	logistic
① Predicts continious value	① Predict categorical outcomes
② $y = a_0 + a_1 x_1$	② $y = \frac{1}{1+e^{-(a_0+a_1x)}}$
③ output can be any real number between $-\infty$ to $+\infty$	③ output can only between 0 and 1.
④ straight line graph	④ s shaped curve graph.
⑤ Predicting house price in area	⑤ Predict if the house will sell.
⑥ It is regression means continious Prediction	⑥ classification $\Rightarrow$ binary prediction it we

Stochastic Gradient Descent

stochastic Gradient descent updates the model for each training example, not the whole dataset making it faster and suitable for large datasets. It updates parameter to reduce error.

Formula;

$$w := \hat{w} - \eta \frac{\partial L(w)}{\partial w}$$

$w:$ = updated weight
w = weight parameter (Predicted)
η = learning rate (small positive number)
$L(w)$ = loss function error.

loss function $L(w) = (y - wx)^2$

. derivative of $L(w) = 2(y - wx) \cdot (-x) = -2x(y - wx)$

learning rate (0.1)

x	y
1	2
2	4
3	6.

$y = wx$

First iteration,

Let us predict,

$x = 1$
$y = 2$
learning rate = 0.1
initial weigh = 0

$\hat{w}$ = Predicted weight $= wx$
$= 0.1 = 0$

gradient,

$-2x(y - wx)$
$= -2 \cdot 1(2 - 0 \cdot 1) = -4$

update weight $:= 0 - 0.1(-4)$
$= 0.4$

iteration 2, after iteration 1, initial weight = 0.4

updated predicted weight $w = 0.4 \times 1 = 0.4$

Gradient $= -2x(y - wx) = -2 \cdot 1 (2 - 0.4 \times 1) = -3.2$

updated weight $= 0.4 - 0.1(-3.2)$

$= 0.72$

After iteration 2, ~~predicted~~ updated initial weigh $= 0.72$

predicted weigh $= 0.72 \times 1 = 0.72$

Gradient $= -2x(y - wx)$

$= -2 \cdot 1(2 - 0.72 \cdot 1)$

$= -2(2 - 0.72) = -2.56$

updated weight $= \hat{w} - \eta \cdot \frac{dL(w)}{dw}$

$= 0.72 - 0.1(-2.56)$

$= 0.97600$

iteration	weight (w)	prediction ($\hat{w}$)	Gradient	update w
1	0	0	-4	0.4
2	0.4	0.4	-3.2	0.72
3	0.72	0.72	-2.56	0.976

The weight slowly converges towards the correct value using the actual gradient.

CS CamScanner

Stochastic Gradient Descent VS Batch Gradient Descent.

Feature	Batch Gradient Descent	Stochastic Gradient Descent
Update rule	→ use entire dataset to compute gradient	→ use 1 random point per update
Speed	→ Slower per iteration (more data to process)	→ faster per iteration
Convergence	→ Smooth but might get stuck in local minima	→ Noisy path, may escape local minima..
Memory usage	→ High (loads all data)	→ Low (use one example at a time)
Stability	→ stable, steady convergence	→ Noisy, but can converge faster overall.
Use case	→ small / medium dataset	→ large scale / online learning

Milen's Method

Milen's predictor formula, $y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y_{n-1} + 2y'_n\right]$

Milen's corrector formula, $y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$

Given $\frac{dy}{dx} = \frac{1}{2}(x+y)$, $y(0)=2$, $y(0.5)=2.636$, $y(1.0)=3.595$, $y(1.5)=4.968$ Find $y(2)$ by Milen's Method.

putting $n=3$ into equation,

$$y_{4p} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 - 2y'_3\right] \quad \text{—— (1)}$$

Given that, $x_0 = 0$, $y_0 = 2$

$x_1 = 0.5$, $y_1 = 2.636$

$x_2 = 1.0$, $y_2 = 3.595$

$x_3 = 1.5$, $y_3 = 4.968$

$y_{4,p}$ cause we have three values here $n=3$.

The differential equation, $y' = \frac{1}{2}(x+y)$

$y'_1 = \frac{1}{2}(x_1+y_1) = \frac{1}{2}(0.5+2.636) = 1.568$

$y'_2 = \frac{1}{2}(x_2+y_2) = \frac{1}{2}(1.0+3.595) = 2.2975$

$y'_3 = \frac{1}{2}(x_3+y_3) = \frac{1}{2}(1.5+4.968) = 3.234$

utting the values in equation (1)

$$y_{4p} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 - 2y'_3\right]$$

$$= 2 + \frac{4\times 0.5}{3}\left[2\times 1.568 - 2.2975 + 2\times 3.234\right]$$

$$= 6.871$$

o our predictor value is 6.871.

$y_4' = \frac{1}{2}(x_4+y_4)$

$= \frac{1}{2}(2+6.871) = 4.4355$

Now we will correct it to get the actual value by Milen's corrector formula,

$y_{n+1,C} = y_{n-1} + \frac{h}{3}[y_2' + 4y_3' + y_4']$

$= y_2 + \frac{0.5}{3}[2.2975 + 4\times 3.324 + 4.4355]$

$= 6.8731$

If four values of y are not given you have to use Taylor's method to find remaining values of y.

Picard Method

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx$$

Approximation minimum 3rd approximation বের করে মান বের করতে হবে।

$$y_1 = y_0 + \int_{x_0}^{x} (x, y_0)\,dx \quad [\text{First approximation}]$$

$$y_2 = y_0 + \int_{x_0}^{x} (x, y_1)\,dx \quad [\text{second approximation}]$$

$$y_n = y_0 + \int_{x_0}^{x} (x, y_n)\,dx \quad [\text{nth approximation}]$$

$\frac{dy}{dx} = 1+xy$ and $y(0)=1$, calculate $y(0.1)$, $y(0.2)$ using picard.

$x_0 = 0,\ y_0 = 1,\quad f(x,y) = 1+xy$

1st approximation,

$$y_1 = 1 + \int_0^x (1+xy_0)\,dx$$

$$= 1 + \int_0^x 1 + x.1\,dx$$

$$= 1 + \int_0^x 1+x\,dx = 1 + \left[x + \frac{x^2}{2}\right]_0^x$$

$$= 1 + x + \frac{x^2}{2}$$

2nd approximation,

$$y_2 = 1 + \int_0^x (1+xy_1)\,dx$$

$$= 1 + \int_0^x 1 + x\left(1+x+x^2/2\right)dx$$

$$= 1 + \int_0^x 1 + x + x^2 + x^3/2\,dx.$$

$= 1 + \left[x + x^2/2 + x^3/3 + x^4/8 \right]_0^x dx$

$= 1 + x + x^2/2 + x^3/3 + x^4/8$

3rd approximation,

$y_3 = 1 + \int_0^x (1 + x y_2)\, dx$

$= 1 + \int_0^x (1 + x(1 + x + x^2/2 + x^3/3 + x^4/8))\, dx$

$= 1 + \int_0^x 1 + x + x^2 + x^3/2 + x^4/3 + x^5/8 \; dx$

$= 1 + \left[x + x^2/2 + x^3/3 + x^4/8 + x^5/15 + x^6/48 \right]_0^x$

$= 1 + x + x^2/2 + x^3/3 + x^4/8 + x^5/15 + x^6/48$

$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{15} + \frac{(0.1)^6}{48}$

$= 1.10534$

$y(0.2) = 1.22286.$

Lagrange Method Interpolation

The Lagrange interpolation formula is a way to find a polynomial called Lagrange polynomial that takes certain values at arbitrary point.

General form, $$P(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3}y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}y_3$$

১টা বাদ দিয়ে সব x এর value x থেকে বাদ করব যা বাদ দিয়েছিলাম তা থেকে সব x এর value বাদ দিবো নিচে সাথে যত নাম্বার x বাদ দিব তত নম্বর y গুণ।

Using Lagranges interpolation formula find $y(10)$ from the following table:

x	5	6	9	11
y	12	13	14	16

$x_0 = 5,\ x_1 = 6,\ x_2 = 9,\ x_3 = 11$

$y_0 = 12,\ y_1 = 13,\ y_2 = 14,\ y_3 = 16.$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Putting, $x = 10$ and all values,

$$f(10) = \frac{(10-6)(10-9)(10-11)}{18} \times 12 + \frac{(10-5)(10-9)(10-11)}{15} \times 13$$
$$+ \frac{(10-5)(10-6)(10-11)}{-24} \times 14 + \frac{(x-5)(x-6)(x-9)}{60} \times 16$$

$$= -\frac{12}{9} - \frac{13}{3} + \frac{70}{12} + \frac{130}{30}$$

$$= 14.6663.$$

5.10 Using Lagranges interpolation find f'(0.12)

x	$f(x)$
0.05	0.05
0.10	0.0999
0.20	0.1987
0.26	0.2571

$$f(0.12) = \frac{(0.12-0.10)(0.12-0.2)(0.12-0.26)}{(0.05-0.1)(0.05-0.2)(0.05-0.26)} \times 0.05$$

$$+ \frac{(0.12-0.05)(0.12-0.2)(0.12-0.26)}{(0.1-0.05)(0.1-0.2)(0.1-0.26)} \times 0.0999 + \frac{(0.12-0.05)(0.12-0.1)(0.12-0.2...}{(0.2-0.05)(0.2-0.1)(0.2-0...}$$

$$\times 0.1987 + \frac{(0.12-0.05)(0.12-0.1)(0.12-0.2)}{(0.26-0.05)(0.26-0.1)(0.26-0.2)} \times 0.2571$$

$$= -0.0071 + 0.0979 + 0.0433 - 0.0143$$

$$= 0.1198$$

Advantages: ① The formula is used to find the value of the function even when the arguments are not equally spaced.

② The formula is used to find the value of independent variable x corresponding to a given value of a function.

Rungee-Kutta Method

Apply RK method to find an approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x + y$ and $y = 1$.

if $x_0 = 0$, $h = x_1$

so $h = 0.2$

$y_1 = y_0 + k$

$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$

$k_3 = hf(x_0 + h/2, y_0 + k_2/2)$

$k_4 = hf(x_0 + h, y_0 + k_3)$

So now, $k_1 = 0.2(x_0 + y_0)$

$= 0.2(0 + 1)$

$= 0.2$

$k_2 = 0.2(x_0 + h/2 + y_0 + k_1/2)$

$= 0.2(0 + 0.1 + 1 + 0.1)$

$= 0.24$

$k_3 = 0.2(0.1 + 1 + 0.12)$

$= 0.244$

$k_4 = 0.2(0.2 + 1 + 0.244)$

$= 0.28880$

$k = \frac{1}{6}(0.2 + 2 \times 0.24 + 0.244 \times 2 + 0.28880)$

$= 0.24280$

$y_1 = 1 + 0.24280$

$= 1.24280.$

RBF => row ~~et~~ echelon form => upper triangular matrix.

RRBF => reduced row/ ~~elch~~ echelon form.

RBF => row echelon matrix => upper triangular matrix => all elements below pivot is zero

Reduced row echelon matrix => identity matrix => All elements below and above pivot is zero.

* An ordinary differential equation is an equation that involves one independent variable, one or more of it's derivative. It shows how one quantity is change in respect to other.

A car is moving on a road. Its acceleration is not constant $\frac{dv}{dt} = a(t) = 2t$, the $v(0) = 0$, step size $h = 0.1$ what is the velocity of the car at $t = 0.4$ seconds.

Step	t_n	v_n	$f(t_n) = 2t$	$v_{n+1} = v_n + h \cdot f$
				0
0	0.0	0	0	0.02
1	0.1	0	0.2	0.06
2	0.2	0.02	0.4	0.12
3	0.3	0.06	0.6	0.2
4	0.4	0.12	0.8	

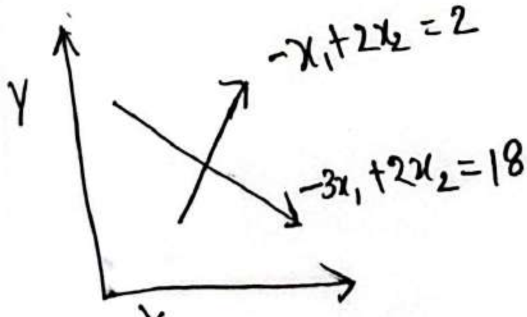
Approximate ~~val~~ velocity at $t = 0.4$ second is $0.200\, m\, s^{-1}$

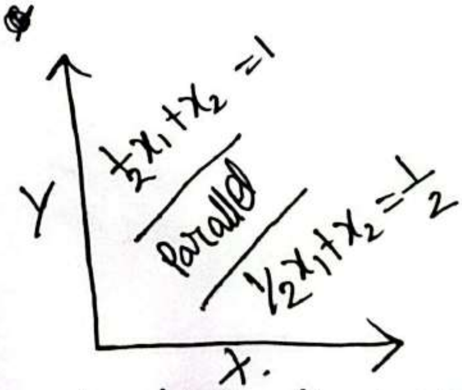
Range-Kutta Method

$$\frac{dy}{dx} = f(x, y)$$

New value = old value + slope x step size

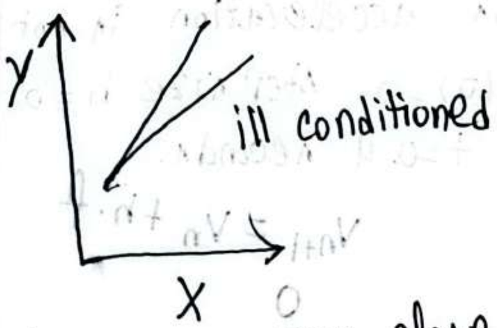
$$y_{i+1} = y_i + \phi h$$

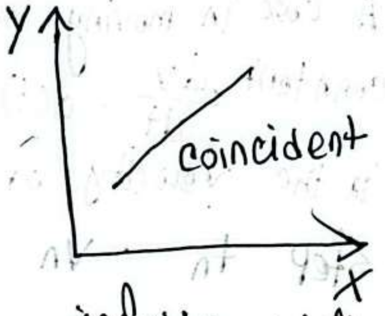




- There is only one solution or root

As the lines are parallel, there is no solution.





slope is very close, point of intersection difficult to detect.

infinite solution

(II) and (IV) are singular