3 The Burgers equation

3.1 Physical considerations

The Burgers equation is a simple equation to understand the main properties of the Navier-Stokes equations. In this one-dimensional equation the pressure is neglected but the effects of the nonlinear and viscous terms remain, hence as in the Navier-Stokes equations a Reynolds number can be defined. This number expresses the ratio between the advective and the viscous contribution in a flow. The present book deals with flows at high Reynolds numbers where the nonlinear terms play a fundamental role, and the physics is more complicated than that when the viscous term dominates. The simulation of the flow evolution then necessitates the use of accurate and robust numerical methods. In 3D turbulent flows, where the number of degrees of freedom is greater than in high Re laminar flows, to get solutions it is necessary to introduce some sort of closure to account for the impossibility to resolve the small scales. Before applying any new idea about numerical methods to 3D flows, the good sense suggests to find the simplest equation to test these ideas. This consideration explains why the Burgers equation was often used to check new numerical methods or closure for turbulent flows. The equation is

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial uu}{\partial x}=+\nu\frac{\partial^2 u}{\partial x^2} \qquad |x|\leq 1 \qquad 3.1$$

Since a numerical method should account for the physics linked to the equation it is important to understand the role each term in Eq.(3.1) is playing. This effort is facilitated if the equations are written not only in the physical but also in the wave number space. To get the equation on the wave number space periodicity in the x direction is assumed and the initial velocity profile should be periodic. The simplest initial condition is to assign a single wave number

$$u(x,0)=-sin(\pi x) \qquad 3.2$$

The solution can be expressed by a truncated Fourier series

$$u(x,t)=\sum_{k=-N/2-1}^{N/2} u_k(t)e^{ik^*x} \qquad 3.3$$

with $k^*=k2\pi/L_x$. By substituting Eq.(3.3) into Eq.(3.1) the equation becomes

$$\frac{du_k}{dt}=-\frac{ik^*}{2}\sum_{p=-N}^{N} u_p u_{k-p}-\nu k^2 u_k \qquad 3.4$$

In physical space Eq.(3.1) shows that the effect of the viscous terms is a diffusion of u, that is the variable u has to migrate into the external space. This physical process occurs if the domain is large enough. On the other hand, for instance, if periodicity is limiting the space the physical process is affected by this geometrical limitation. From physical reasoning it turns out that if the equation is solved in a region completely filled by the initial condition, there is an unphysical diffusion of $u(x) > 0$ in the region of $u(x) < 0$ resulting in a greater decay of the peak velocity and hence of $\frac{\partial u}{\partial x}|_{x=0}$. On the contrary this does not occur when an external region encircles the space where the initial conditions are assigned. This reasoning can be checked by numerical simulations, and it suggests that 2D and 3D turbulence simulations have physical validity until the size of the energy containing eddies are smaller than the computational box.

Eq.(3.4) shows that viscosity reduces the amplitude of each wave number u_k in time. In fact, by neglecting the nonlinear terms, Eq.(3.4) becomes

$$\frac{du_k}{dt} = -\nu k^2 u_k \qquad 3.5$$

with solution

$$u_k(t) = u_k(0)e^{-2\nu k^2 t} \qquad 3.6$$

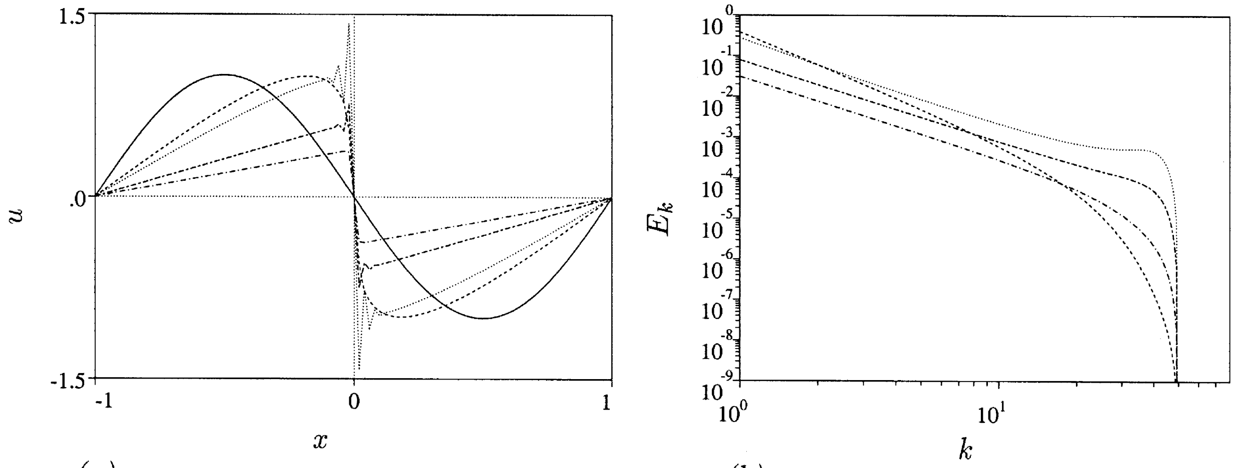
Eq.(3.6) shows that the high wave numbers components are dumped at a rate that increases with the wave number, and with ν . This physical consideration is important in numerical simulations; in fact when the resolution is not satisfactory to reproduce high gradients or equivalently the smallest scales, the truncation errors produce unphysical components with amplitudes at high wave numbers. Eq.(3.6) shows that these errors at high k decay fast. If the simulations are performed at a value of the viscosity at which the resolution is not satisfactory, during a transient, energy is transferred at small scales and it piles up at high wave numbers. This energy is dissipated very fast and at a certain time the form of the spectrum is close to that of a fully resolved simulation. Only from that instant the simulation has a physical meaning. This procedure is often used in the DNS of isotropic turbulence since a transient is necessary to generate the phases of the velocities that were random in the initial field . In other words in this transient the structures of turbulence are produced, and then the simulation cannot be fully resolved. In chapter 8 this procedure is described.

Eq.(3.4) shows that a convolution product represents the nonlinear terms in wave number space and thus it turns out that the amplitude at each wave number is affected by the amplitude of all the other wave numbers. The evolution equation for the energy $E_k = u_{-k}u_k$ is obtained by multiplying the equation for u_k by u_{-k} and that for u_{-k} by u_k, it follows

$$\frac{dE_k}{dt} = T(k) - \nu k^2 E_k \qquad 3.7$$

$T(k)$ expresses that the transfer of energy occurs by interactions among wavenumbers k, p, q such that $p + q + k = 0$. In one-dimensional and in three-dimensional space the energy goes from low to high wavenumbers. On the contrary in two-dimensional space energy is transferred from high to low wave numbers. For the Burgers equation, the transfer of energy to small wave numbers, in physical space is equivalent to the formation of very sharp gradients that are the steeper the smaller is ν.

To understand the physics of a non-resolved simulation that was described for the 3D case, it is useful to see what occurs for the Burgers equation. It has to be expected that in the region of high gradients, if the grid is not sufficiently small, oscillations grow which successively are dumped by viscosity. In the wave number representation this process is



(a) *(b)*

Fig. 3.1 a) Velocity profiles , b) Energy spectrum ; —— $t = 0$, ---- $t = 1.$, ······· $t = 1.5$, ---— $t = 4.$, —·— $t = 7.$

shown by an energy pile-up at high wave numbers. Fig.3.1 indeed shows the two point of view by a simulation with 101 grid points equally distributed at $\nu = 10^2\pi$. At $t = 1.5$, when the transfer of energy to small scales is not completed, the insufficient resolution produces wiggles with high amplitude, localized around $x = 0$; these wiggles correspond to the bump in the energy spectrum. At $t = 7$ these unphysical energy components are not any longer observable, and Fig.3.1 shows that the spectrum and the velocity profile are smooth. Since the method conserves, in the inviscid limit, global energy, this unphysical transient does not bring the calculation to diverge, but the real physics is disrupted.

The evolution of the velocity gradient at $x = 0$ characterizes the physics of the Burgers equation and, as it was shown by Orlandi & Briscolini (1983), if it is scaled by $\pi\nu$, it has a self-similar behavior. Two simulations were performed at $\pi\nu = 10^{-2}$ and 10^{-3} with a fully resolved grid; the results in Fig.3.2 show that at the same value of πt the same peak value of the normalized velocity gradient is reached, even if one started from different values. A self-similar decay in time is also predicted. In the same figure a simulation, non-fully resolved, shows that the unphysical dissipation, during the transient, affects the whole simulation.

Local and total momentum conservation are the fundamental properties of the Navier-Stokes equations in the inviscid limit and these should be preserved for the set of the discretized equations. This property holds in the physical as well as in the wave number space. The preservation of these properties can be easily shown for the Burgers equation to facilitate the comprehension of the fulfillment in 3D. By recalling that $u_k = u_{-k}$ immediately it turns out, in Eq.(3.4), that $\iota \sum_{k=-N}^{N} k \sum_{p=-N}^{N} u_p u_{k-p} = 0$. From the physical point of view it means that the nonlinear terms redistribute the wave number amplitude among different wave numbers with a total null contribution. The same conclusion is obtained by integrating Eq.(3.1) in the whole space, for $\nu = 0$ and with periodic or with Dirichlet conditions.

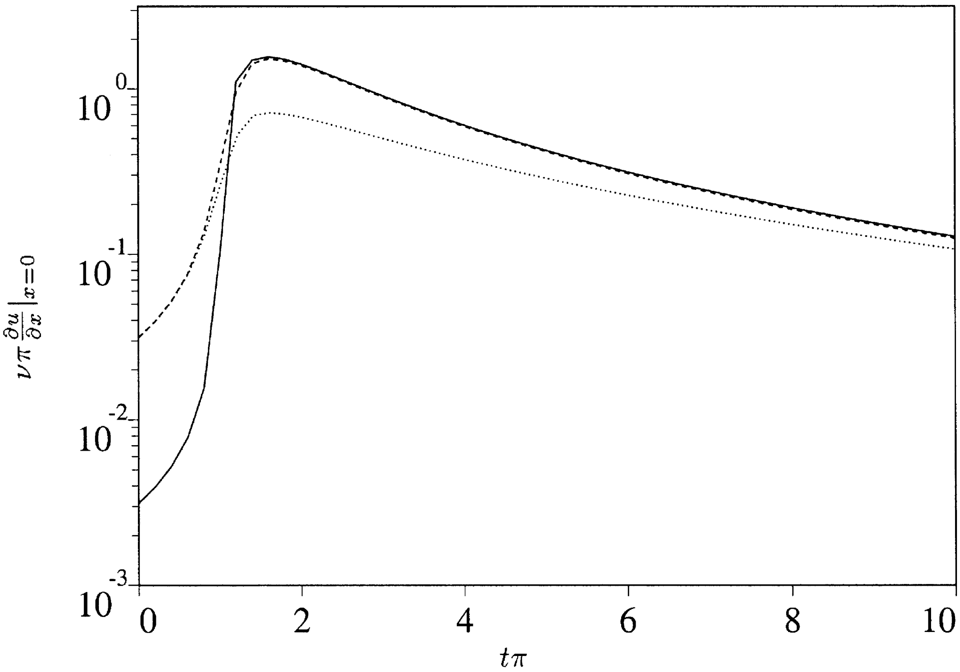


Fig. 3.2 time evolution of the velocity gradient at $x = 0$, ——— $\pi\nu = 10^{-3}$, $n = 1600, \alpha =$ 3., - - - - $\pi\nu = 10^{-2}$, $n = 800, \alpha = 3.$, ········ $\pi\nu = 10^{-2}$, $n = 100, \alpha = 0$.

The conservation of energy is a further property of the Burgers and Navier-Stokes equations. The expression of $T(k)$ shows that $\sum_{-N}^{N} T(k) = 0$, that is, that the energy is transferred by triadic interactions among different wave numbers, and hence for $\nu = 0$ the total energy is conserved. The energy equation in physical space, by defining $q = u^2/2$, is

$$\frac{\partial q}{\partial t} + \frac{2}{3}\frac{\partial uq}{\partial x} = +\nu\frac{\partial^2 q}{\partial x^2} - \nu\left(\frac{\partial u}{\partial x}\right)^2 \qquad 3.8$$

By integrating the second term in Eq.(3.8) it turns out that the total contribution of the energy advection is zero. On the right hand side appears the viscous energy diffusion, and the energy dissipation. For the Burgers as well as for 3D flows the integral of the turbulent diffusion is null for periodical conditions and hence the total energy decays in time because of the negative right hand side that is called the rate of energy dissipation.

From these physical considerations two rules hold: the first suggests that for incompressible viscous flows, without discontinuities for the presence of ν, it is convenient to write the transport equations of each variable, in the continuous as well as in the discrete form, such that global conservation of the total quantity and of the high moments occurs. The second rule forces to have a sufficient number of computational points across the sharp gradients to fully resolve them. Since the occurrence of these thin layers depends on ν, it turns out that the simulation of equations with a small ν requires more care.

3.2 Spatial discretization

In accordance with the previous considerations the Burgers equation evolves from the