COURSE OUTCOME 1 and 2

Problems to be discussed by the faculty:

Example 1:

A box contains 25 parts of which 10 are defective. Two parts are being drawn simultaneously in a random manner from the box. The probability of both parts being good is

A) 7/20 B) 42/125 C) 25/29 D) 5/9

Solution: Number of ways of drawing 2 parts from 25 parts=$25c_2$

Number of ways of drawing 2 good parts from the 15 good parts=$15c_2$

Probability that both parts are good=$\frac{15_{c_2}}{25_{c_2}} = \frac{7}{20}$

Example 2:

In a housing society, half of the families have a single child per family, while the remain half have two children per family. The probability that a child picked at random, has a sibling

Solution:

The child picked at random will have a sibling if the family has two children

Probability of this event=1/2.

Example 3:

An unbiased coin is tossed an infinite number of times. The probability that the fourth head appears at the 10^{th} toss is

A) 0.067 B) 0.073 C) 0.082 D) 0.091

Solution:

Total number of possibilities for the first ten slips is 2^{10}=1024.

For the fourth head to occur on 10^{th} slip.

We need first 3 heads to occur in the first 9 slips. This is given by

$9c_3$=84.

There is only one way for 4^{th} head occur on 10^{th} slip =84*(1/1024) = 21/256=0.082.

(or) for the 4th head to occur at the 10th toss, you have to first get 3 heads and 6 tails in the 1st 9 toss, and then a head at the 10th toss.

So prob. = (9C3)(.5)^3 (1-.5)^6 (0.5)=.082.

Example: 4

A fair dice is tossed 10 times. What is probability that only the first two tosses will yield heads

A) (1/2)^2 B) 10C2 (1/2)^2 C) (1/2)^10 D) 10C2(1/2)^10

Solution: $(1/2)^2(1/2^8=(1/2)^{10}$

Problems to be discussed by the students:

1) Suppose it is known that the probability that the component survives for more than 6000 hours is 0.42. Suppose also that the probability that the component survives no longer than 4000 hours is 0.04.

   a) What is the probability that the life of the component is less than or equal to 6000 hours

   b) What is the probability that the life of the component is greater than 4000 hours

2) If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems, and a dictionary what is the probability that

   (a) the dictionary is selected

   (b) 2 novels and 1 book of poems are selected

   (c) a novel, a book of poems and the dictionary is selected

   (d) all three books are novels

3) A die is loaded in such a way that an even number is twice as likely to occur as an odd number.

   a) If E is the event that a number less than 4 occurs on a single toss of the die, find P(E).

   (b) Let A be the event that an even number turns up and let B be the event that a number divisible by 3 occurs. Find $P(A \cup B)$ and $P(A \cap B)$.

Problems to be discussed by the faculty:

Example 5:

Suppose that in a senior college class of 500 students it is found that 210 smoke, 258 drink alcoholic beverages, 216 eat between meals, 122 smoke and drink alcoholic beverages, 83 eat between meals and drink alcoholic beverages, 97 smoke and eat between meals, and 52 engage in all three of these bad health practices. If a member of this senior class is selected at random, find the probability that the student

a) Smokes but does not drink alcoholic beverages.
b) eats between meals and drinks alcoholic beverages but does not smoke;
c) Neither smokes nor eats between meals.
d) Probability that the student does not have any habit

Solution: Let A, B, and C be the events that the student selected at random is found to be smoke, drink, alcoholic beverages and eat between meals, respectively.

From the given data

$P(A)=210/500$, $P(B)=258/500$, $P(C0=216/500$, $P(A\cap B)=122/500$, $P(B\cap C)=83/500$, $P(A\cap c)=97/500$ and $P(A\cap B\cap C)=52/500$

a) Probability that the student selected at random smoke but does not drink alcoholic beverages

$=P(A\cap \bar{B})=P(A)-P(A\cap B)=21/500-(122/500)=88/500$

b) Probability that the student selected at random eat between meals and drink alcoholic beverages but does not smoke

$=P(C\cap B\cap \bar{A})=P(\bar{A}\cap B\cap C)=P(B\cap C)-P(A\cap B\cap C)=(83/500)-(52/500)=31/500.$

c) Probability that the student neither smokes nor eats between meals

$=P(\bar{A}\cap \bar{C})=P(\overline{A\cup C})=1-P(A\cup C)=1-[P(A)+P(C)-P(A\cap C)]=1-[(210/500)+(216/500)-(97/500)]=171/500.$

d) Probability that the student does not have any habit

$=P(\bar{A}\cap \bar{B}\cap \bar{C})=P(\overline{A\cup B\cup C})=1-P(A\cup B\cup C)=1-[P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)]=1-(434/500)=66/500.$

Note: i) $P(A \text{ and } B)=P(A\cap B)=P(\text{both } A \text{ and } B)$

ii) $P(\text{either } A \text{ or } B)=P(\text{atleast one})=P(AUB)=P(A)+P(B)-P(A\cap B)$

iii) $P(\text{only } A)=P(A\cap \bar{B})=P(A)-P(A\cap B)$

iv) $P(\text{only } B)=P(B\cap \bar{A})=P(B)-P(A\ A\cap B)$

v) $P(\text{anyone})=(P(A\cap \bar{B})U\ P(B\cap \bar{A}))=P(A\cap \bar{B})+P(B\cap \bar{A})=P(\text{only one})$

Problems to be discussed by the students:

4. The probability that a new airport will get an award for its design is 0.16, the probability that it will get an award for the efficient use of materials is 0.24, and the probability that it will get both awards is 0.11.
    a) what is the probability that it will get at least one of the two awards?
    b) what is the probability that it will get only one of two awards?
    c) what is the probability that it will get neither award
    d) what is the probability that it will get award for its design only?

5. Consider randomly selecting a student at a certain university, and Let A denote the event the selected individual has a Visa Credit card and B be the analogous event for a MasterCard. Suppose that $P(A)=0.5$, $P(B)=0.4$ and $P(A \cap B)=0.25$
    a. Compute the probability that the selected individual has at least one of the two types of cards
    b. Compute the probability that the selected individual has neither type of card

6. A final year student after being interviewed at two companies, he assesses that his probability of getting an offer from company A is 0.8 and the probability that he gets offer from company B is 0.6. If, on the other hand he believes that the probability that he will get offers from both companies is 0.5. Obtain the probability that he will get
    a. at least one offer from these two companies
    b. offer from neither company
    c. offer from company A only
    d. offer from only one company

Problems to be discussed by the faculty:

Example 6

The chance of a student passing an exam is 20%. The chance of passing an exam and getting above 90% marks in it is 5% given that the student passes the examination, the probability that the student gets above 90% marks is

A) 1/18 B) 1/4 C) 2/9 D) 5/18

Solution: Let A and B denote the events of a student passing an exam and a student getting above 90% marks in the exam respectively.

P(A)=20/100=0.2, P(B)=5/100=0.05.

Given that a student passes the examination, the probability that the students gets above 90% marks

=P(B/A)=0.05/0.2=1/4.

Example 7

P(X) = 1/4, P(Y) = 1/3, P(X$\cap Y$) = 1/12 The value of P(Y/X) is

A) 1/4 B) 4/25 C) 1/3 D) 29/50

Solution:

$$P\left(\frac{Y}{X}\right) = \frac{P(Y \cap X)}{P(X)} = \frac{\frac{1}{12}}{\frac{1}{4}} = 1/3.$$

Problems to be discussed by the students:

7. The Probability that a regularly scheduled flight departs on time is P(D)=0.83; the probability that it arrives on time is P(A)=0.82; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78.$ Find the probability that a plane
   (a) arrives on time given that it departed on time.
   (b) departed on time given that it has arrived on time,
   (c) neither departed on time nor arrived on time.

8. A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that
   a) Both the ambulance and the fire engine will be available.
   b) Ambulance or fire engine available.

9. The odds that a book will be reviewed favorably by three independent critics are 5 to 2, 4 to 3, and 3 to 4. Find the probability that of the three reviews, a majority will be favorable.

Problems to be discussed by the faculty:

Example 8

Two cards are drawn at random from an ordinary deck of 52 playing cards. What is the probability of getting two aces if

a) The first card is replaced before the second card is drawn; 4c1/52c1 .4c1/52c1
b) The first card is not replaced before the second card is drawn?(4c1/52c1)

Solution:

a) Since there are four aces among the 52 cards, we get (4/52).(4/52)=1/169.

b) Since there are only three aces among the 51 cards that remain after one ace has been removed from the deck, we get

$$\frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$$

Note that $\frac{4}{52} \cdot \frac{4}{52} \neq \frac{1}{221}$

So independence is violated when the sampling is without replacement.

Example 9

Three vendors are asked to supply a very high precision component.The respective probabilities of their meeting the strick design specifications are 0.8, 0.7 and 0.5 Each vendor supplies one component .The probability out of total three components supplied by the vendors atleast one will meet the design specifications is ---------

Solution: Let A, B, C be the event that the high precision component supplied by the three vendors meets the design specifications. The events A, B and C are independent.

Given that P(A)=0.8, P(B)=0.7 and P(C)=0.5

Probability that at least one meet the design specifications=1-Probability that none of them meet the design specifications

$=1-P(\bar{A} \cap \bar{B} \cap \bar{C}) = 1 - P(\bar{A})P(\bar{B})P(\bar{C})=1-(0.2)(0.3)(0.5)=0.97.$

Example 10

A fair coin is tossed till a head appears in the first time. The probability that the number of required tosses is odd, is

A) 1/3 B) 1/2 C) 2/3 D) ¾

Solution: Add upto probability of the coin coming up head for the first time 1, 3,5, …

$P_0=(1/2)^1+(1/2)^3+(1/2)^5+\ldots=(1/2)/(1-(1/2)^2)=(1/2)/(1-1/4)=2/3.$

Example 11

A fair dice is tossed till two times. The probability that the second toss results in a value that is higher than the first toss is

A) 2/36 B) 2/6 C) 5/12 D) ½

Solution:

(1/6)*(5/6)+(1/6)*(4/6)+(1/6)*(3/6)+(1/6)*(2/6)+(1/6)*(1/6)=15/36=5/12   (or)

Pr(Second >first)+Pr(Second<first)+Pr(Second=first)=1

By symmetry

P(Second>first)=1-P(Second=first)

P(Second>First)=(1-1/6)/2=5/12.

Example 12

A 1-h rain fall of 10cm magnitude at a station has a return period of 50 years. The probability that a 1-h rain fall of 10cm magnitude or more will occur in each of two successive years is

A) 0.004   B) 1.0   C)1.5   D) 2.0

Solution: (1/50)*(1/50)=1/2500=0.004

Problems to be discussed by the students:

10. Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

11. Pollution of the rivers in the United States has been a problem for many years. Consider the following events.

    A= {the rivers is polluted.}
    B={A sample of water tested detects pollution.}
    C= {Fishing permitted.}
    Assume
    P(A)=0.3, P(B/A)=0.75
    $P(B/A^1)=0.20, P(C/A\cap B)=0.20,\ P(C/A^1\cap B)=0.15, P(C/A\cap B^1)=0.80, and$
    $P(C/A^1\cap B^1)=0.90$

    a) Find $P(A\cap B\cap C)$.
    b) Find $P(B^1\cap C)$.
    c) Find P(C)
    d) Find the probability that the river is polluted, given that fishing permitted and the sample tested did not detect pollution.

Problems to be discussed by the faculty:

Example 13

A group contains equal no of men and women of those 20 % of the men, 50 % of women are unemployed. If a person is selected at random from these. The probability of selected person being employed is -----

Solution:

Let E be the event that the probability of selected person being employed is
If man is selected then probability of him to be employed=20/100.
Similarly if woman was selected the probability of her to be employed=50/100
So
probability=P(M).P(E/M)+P(W).P(E/W)=(1/2)*(20/100)+(1/2)*(50/100)=(1/10)+(5/20)=7/20.
Required probability =1-P(E)=1-0.35=0.65.

Example 14

The probability that student knows the correct answer to a multiple choice question is 2/3. If the student does not know the answer, the student guesses the answer. The probability of guessed answer is correct is ¼ . Given that student has answered the question correctly. The conditional probability that student knows the correct answer is

A) 2/3 B) 3/4 C) 5/6 D) 8/9

Solution: Let A be the event that the student knows the correct answer. Then $\bar{A}$ represent the event that the student guesses.

$P(A)=2/3$, $P(\bar{A})=1-P(A)=1/3$

Let E be the event that the answer is correct

Given that $P(E/\bar{A})=1/4$

Since the student answers correctly when he knows the correct answer $P(E/A)=1$

Probability that the student knows the correct answer given that he answers correctly

$$P(A/E) = \frac{P(A \cap E)}{P(E)} = \frac{P(A \cap E)}{P(A \cap E) + P(\bar{A} \cap E)}$$

$$= \frac{P(A)P(\frac{E}{A})}{P(A)P(\frac{E}{A}) + P(\bar{A})P\left(\frac{E}{\bar{A}}\right)} = 8/9$$

Problems to be discussed by the students:

12. In a certain assembly plant, three machines B_1, B_2, B_3, make 30%, 45% and 25% respectively, of the products. It is known that 2%, 3% and 2% of the products made by each machine, respectively, are defective. Now suppose that a finished product is randomly selected.
    a) What is the probability that it is defective?
    b) If the product selected is found to be defective what the probability that it was made by machines B_1, B_2 and B_3.

13. Amy commutes to work by two different routes A and B. If she comes home by route A, then she will be home no later than 6 P. M. with probability 0.8, but if she comes home by route B, then she will be home no later than 6 P. M. with probability 0.7. In

the past, the proportion of times that Amy chose route A is 0.4. If Amy is home after 6 P. M. today, what is the probability that she took route B?

14. Two firms V and W consider bidding on a road building job, which may or may not be awarded depending on the amounts of the bids. Firm V submits a bid and the probability is 3/4 that it will get the job provided firm W does not bid. The probability is 3/4 that W will bid, and if it does, the probability that V will get the job is only 1/3.
    a. What is the probability that V will get the job?
    b. If V gets the job, what is the probability that W did not bid?

Problems to be discussed by the faculty:

Example 15

An important factor in solid missile fuel is the particle size distribution. Significant problems occur if the particle sizes are too large. From the production data in the part, it has been determined that the particle size (in micrometers) distribution is characterized by

$$f(x) = 3x^{-4}, \;\; x > 1$$
$$= 0 \text{ elsewhere}$$

a) Verify that this is a valid density function

b) Evaluate the Cumulative distribution function F(x)

c) What is the probability that a random particle from the manufactured fuel exceeds 4 micrometers?

d) What is the probability that a random particles' size is between 2 and 4 micrometers?

Solution:

a) f(x) is a valid density function if $\int_{-\infty}^{\infty} f(x)dx = 1$ x≤X

$$\int_{0}^{\infty} f(x)dx = \int_{1}^{\infty} 3x^{-4} = 3\left[\frac{x^{-3}}{-3}\right]_{1}^{\infty} = 1$$

Therefore, f(x) is a valid density function.

b) Cumulative distribution function F(x) is given by

$$f(x) = P(X \le x) = \int_{1}^{x} 3x^{-4}dx = 1 - \frac{1}{x^3}$$

$$\therefore \mathrm{F(x)} = \begin{cases} 0, \;\; \mathrm{x} < 1 \\ 1 - \frac{1}{\mathrm{x}^3}, \mathrm{x} \ge 1 \end{cases}$$

c) Probability that a random particle's size exceeds 4 micrometers

$$= P(X > 4) = 1 - F(4) = 1 - (1 - (1/64) = 0.0156.$$

d) Probability that a random particle's size is between 2 and 4 micrometers

$$= P(2 \le X \le 4) = F(4) - F(2) = (1 - 1/64) - (1 - 1/8) = 1/8 - 1/64 \; = 0.109.$$

Example 16

Given that $f(x) = k/2^x$ is a probability distribution for a random variable that can take on the values x=0, 1, 2, 3 and 4. Find K.

a) Find K
b) Find the Cumulative probability distribution F(x)

Solution: a) Since f(x) is the probability mass function

$$\sum_{\mathrm{x}=0}^{4} \mathrm{f(x)} = 1$$

⇨ $f(0) + f(1) + f(2) + f(3) + f(4) = 1$

⇨ $k/1 + k/2 + k/4 + k/8 + k/16 = 1$

⇨ $K[31/16] = 1$

⇨ $K = 16/31.$

$$b)\ F(0) = f(0) = k = 16/31$$
$$F(1) = f(0) + f(1) = K + (K/2) = (3/2)K = 24/31$$
$$F(2) = f(0) + f(1) + f(2) = K + (K/2) + (K/4) = (7/4) * K = 28/31$$
$$F(3) = f(0) + f(1) + f(2) + f(3) = K + (K/2) + (K/4) + (K/8) = 30/31.$$
$$F(4) = f(0) + f(1) + f(2) + f(3) + f(4) = K + (K/2) + (K/4) + (K/8) + (K/16) = 1$$
$$c)\ f(2) = F(2) - F(1) = (28/31) - (24/31) = 4/31.$$

Problems to be discussed by the students:

15. A shipment of 8 similar microcomputers to a retail outlet contains 3 defectives. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.
16. Consider the density function

$$f(x) = \begin{cases} k\sqrt{x}, & 0 < x < 1 \\ 0, & elsewhere \end{cases}$$

a) Evaluate k.

b) Evaluate P(0.3<X<0.6) using the density function

17. For the density function

$$f(x) = \begin{cases} \frac{x^2}{3} & -1 < x < 2 \\ 0 & elsewhere \end{cases}$$

i) To evaluate $P(0 < X \leq 1)$

ii) Find cumulative distribution function

18. If a random variable has the probability density function

$$f(x) = \begin{cases} k(x^2 - 1) & -1 \leq x \leq 2 \\ 0 & elsewhere \end{cases}$$

Find the value of 'k' and $p(\frac{1}{2} \leq x \leq \frac{5}{2})$

Problems to be discussed by the faculty:

∫ uv dx = u ∫ v dx – ∫ (u' ∫ v dx) dx

Example 17

For the continuous probability function $f(x) = kx^2 e^{-x}$ when $x \geq 0$, find

(i) k

(ii) Mean

(iii)Variance

Solution:

(i) We have $\int_{-\infty}^{\infty} f(x)dx = 1$

$\therefore \int_0^{\infty} kx^2 e^{-x} dx = 1 \ (\because x \geq 0)$

$i.e, k[x^2(-e^{-x}) - 2x(e^{-x}) + 2(-e^{-x})]_0^{\infty} = 1$

$i.e, k[(-e^{-x})(x^2 + 2x + 2]_0^{\infty} = 1$

$k(0+1) = 1$ or $k = \frac{1}{2}$

(ii) mean= $\int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} kx^3 e^{-x} dx$

$= k[x^3(-e^{-x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6(e^{-x})]_0^{\infty}$

$= k[(-e^{-x})(x^3 - 3x^2 + 6x - 6)]_0^{\infty} = k0 + 6] = 6k$

$\mu = 6(\frac{1}{2}) = 3 \quad (\because k = \frac{1}{2})$

(iii) Variance= $\int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_0^{\infty} x^2 kx^3 e^{-x} dx - 3^2$

$= k\int_0^{\infty} x^4 e^{-x} dx - 9$

$= k[x^4(-e^{-x}) - 4x^3(e^{-x}) + 12x^2(-e^{-x}) - 24(-e^{-x}) + 24e^{-x}]_0^{\infty} - 9$

$= k[(-e^{-x})(x^4 + 4x^3 + 12x^2 + 24 - 24]_0^{\infty} - 9$

$= \frac{1}{2}[0 + 24] - 9 = 12 - 9 = 3$

Problems to be discussed by the student:

19. A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value and variance of the number of good components in this sample.

20. Let X be random variable with following probability distribution:

X	-3	6	9
f(x)	1/6	½	1/3

Find $\mu g(x)$ where $g(X)=(2X+1)^2$.

Problems to be discussed by the faculty:

Example 18

It has been claimed that in 60% of all solar-heat installations the utility bill is reduced by at least one-third. Accordingly, what are the probabilities that the utility bill will be reduced by at least one-third in

a) Four of five installations

$P(X=4)=5c_4\ (0.60)^4(0.40)^{5-4}$

b) at least four of five installations

c) at the most two installations

d) what are the mean and variance of the number of installations

Solution:

Let X be the number of solar installation where the utility bill is reduced by at least one third. Then the distribution of X is binomial with n=5 and P=0.6

$P(X=x) = 5_{c_x}(0.6)^x(0.4)^{5-x}$, $x=0,1,2,3,4,5$

a) Probability that in four of 5 installations utility bill is reduced by one third is

$=P(X=4)=5_{c_4}(0.6)^4(0.4)^{5-4}=0.259$

b) Probability that in at least 4 installations utility bill is reduced by one third is

$=P(X\geq 4)=P(X=4)+P(X=5)=5_{c_4}(0.6)^4(0.4)^{5-4}+5_{c_5}(0.6)^5(0.4)^{5-5}$

c) Probability that in at most two installations utility bill is reduced by one third is

$=P(X\leq 2)=P(X=0)+P(X=1)+P(X=2)$

$=5_{c_0}(0.6)^0(0.4)^{5-0}+5_{c_1}(0.6)^1(0.4)^{5-1}+5_{c_2}(0.6)^2(0.4)^{5-2}$

d) Mean=np=5(0.6)=3

Variance=np(1-p)=5(0.6)(0.4)=1.2.

Problems to be discussed by the student:

21. A traffic control engineer reports that 75% of the vehicles passing through a checkpoint are from within the state. What is the probability that fewer than 4 of the next 9 vehicles are from out of state?
22. A quality assurance inspector tests 20 circuit boards a day. If 10% of the boards have defects, what is the probability that the inspector will find
    a) Expected number of defective boards on any given day?
    b) Exactly 2 defectives?
    c) Between 1 and 4 defectives?

23. The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease,

    a. Find the mean and variance of the number of patients who recovered from the blood disease
    b. Find the probability that exactly 5 survive from the disease
    c. Find the probability that at least 10 survive from the disease
    d. Find the probability that 3 to 8 survive form the disease
    e. Find the probability that at most 3 survive the disease
    f. Find the probability that none will survive the disease

Problems to be discussed by the faculty:

Example 19

If a bank received on the average 6 bad checks per day, what are the probabilities that it will receive

a) 4 bad checks on any given day?
b) 10 bad checks over any 2 consecutive days
c) No bad check on any given day
d) What are the mean and variance of the number of bad check per day?
   Mean=average=np=6

Solution: Let X be the number of bad checks received per day. Then the distribution of X is Poisson with parameter λ=6.

$$P(X = x) = \frac{e^{-6}6^{x}}{x!}, x = 0,1,2,\ldots$$

$$\text{a) } P(\text{4 bad checks on any given day}) = P(X = 4) = \frac{e^{-6}6^{4}}{4!} = 0.134$$

. b) $P(\text{10 bad checks over any 2 consecutive days}) = P(X = 10) = \frac{e^{-12}12^{x}}{x!},\ x = 0,1,2,\ldots.$

$(\text{Here } \lambda = 12)$

. c) $P(\text{no bad check on any day}) = P(X = 0) = \frac{e^{-6}6^{0}}{0!} = e^{-6}$

d) $\text{Mean and variance of the number of bad checks per day} = \lambda = 6.$

Example 20

In the inspection of tin plate produced by a continuous electrolytic process, 0.2 imperfections is spotted per minute, on average. Find the probabilities of spotting

a) One imperfection in 3 minutes
b) at least two imperfections in 5 minutes
c) at most one imperfection in 15 minutes

Solution: Let X be the number of imperfections spotted per minute. Then the distribution of X is Poisson with λ=0.2

a) $\text{Average number of imperfections in 3 minutes} = \lambda t = 3(0.2) = 0.6$

   Probability of one imperfection in 3 minutes= $\frac{e^{-6}(0.6)^{3}}{3!} = 0.329 = P(X = 1)$

b) Average number of imperfections in 5 minutes= $\lambda t = 5(0.2) = 1.0$

Probability of at least two imperfections per 5 minutes

$$= P(X \geq 2) = 1 - P(X < 2) = 1 - P(X = 0) - P(X = 1)$$
$$= 1 - e - 1 - (e - 1(1.0)/1) = 0.264.$$

c) Average number of imperfections in 15 minutes= $\lambda t = 15(0.2) = 3.0$

Probability of at most one imperfection in 15 minutes

$$= P(X \leq 1) = P(X = 0) + P(X = 1) = e - 3 + ((e - 331/1!) = 0.199.$$

Problems to be discussed by the student:

24. In a manufacturing process where glass products are produced, defects or bubbles occur, occasionally rendering the piece undesirable for marketing. It is known that, on average, 1 in every 1000 of these items produced has one or more bubbles. What is the probability that a random sample of 8000 will yield
    a. fewer than 7 items possessing bubbles
    b. none will possess a bubble
25. During a laboratory experiment the average number of radioactive particles passing through a counter in 1 Millisecond is 4. What is the Probability that
    a) 6 particles enter the counter in a given Millisecond
    b) none will enter the counter in a given millisecond
    c) at least 2 will enter the counter in a given millisecond
    d) at most 2 will enter the counter in a given millisecond
26. Consider a quality assurance department that performs random tests of individual hard disk. Their policy is to shut down the manufacturing process if an inspector finds more than four bad sectors on a disk.
    a) What is the probability of shutting down the process of the mean number of bad sectors is 2?
    b) what is the probability of finding 3 bad sectors on a hard disk?

Problems to be discussed by the faculty:

Example 21

With an eye toward improving performance, industrial engineers studied the ability of scanners to read the bar codes of various food and household products. The maximum reduction in power, just before the scanner cannot read the bar code at a fixed dictionary is called the maximum attenuation. This quantity, measured in decibles, varies from product to product: After collecting the data, the engineers decided to model the variation in maximum attenuation as a normal distribution with mean 10.1 dB and standard deviation 2.7 dB.

a) For the next food product, what is the probability that its maximum attenuation is between 8.5 dB and 13.0 dB?
b) According to the normal model, what proportion of the products has maximum attenuation between 8.5 dB and 13.0 dB?
c) What proportion of the products has maximum attenuation greater than 15.1 dB?

Solution: Let X be the maximum attenuation of the next product, Then X is a normal variable with μ=10.1 and σ=2.7.

$$z = \frac{X - 10.1}{2.7}$$

a) Probability that the maximum attenuation of the next product is between 8.5 dB and 13.0 dB.

$$= P(8.5 \leq X \leq 13.0) = P(\frac{8.5 - 10.1}{2.7} \leq X \leq \frac{13.0 - 10.1}{2.7}) = P(-0.59 \leq Z \leq 1.07) = P(Z \leq 1.07) - P(Z \leq -0.59)$$

$$= 0.8577 - 0.2776 = 0.5801.$$

b) 0.5801 is the proportion of the product having maximum attenuation between 8.5 and 13.0 dB
c) Proportion of the products having maximum attenuation greater than 15.1 dB

$$= P(X > 15.1) = P(Z > (15.1 - 10.1)/(2.7)) = P(Z > 1.85) = 1 - 0.9678 = 0.0322$$

Example 22

The actual amount of instant coffee that a filling machine puts into "4-ounce" jars may be looked upon as a random variable having a normal distribution with σ=0.04 Ounce. If only 2% of the jars are to contain less than 4 ounce, what should be the mean fill of these jars?

Solution: Let X be the amount in ounces of instant coffee that is put into jars. Then the distribution of X is normal with mean μ and standard deviation 0.04 ounce.

We are given that P(X<μ)=0.02

Then P$(\frac{X-\mu}{0.04}<\frac{4-\mu}{0.04}) = 0.02$

P$(Z<\frac{4-\mu}{0.04}) = 0.02 \Rightarrow \frac{4-\mu}{0.04} = -2.05 \Rightarrow \mu$=4.082 ounce.

Problems to be discussed by the students:

27. Given a Standard Normal distribution, find the area under the curve which lies
    a) To the left of z=1.43;
    b) to the right z=-0.89
    c) between z=-2.16 and z=-0.65
    d) to the left of z=-1.39
    e) to the right of z=1.96
    f) between z=-0.48 and z= -1.74
28. Given a standard normal distribution find the value of k such that
    a. $P(z < k) = 0.0427$;
    b. $P(z > k) = 0.2946$;
    c. P(-0.93<Z<k)=0.7235.
29. In an industrial process the diameter of a ball bearing is an important component part. The buyer sets specifications on the diameter to be 3.0$\pm$0.01 cm. The implication is that no part falling outside these specifications will be accepted. It is known that in the process the diameter of a ball bearing has a normal distribution with mean μ=3.0 and standard deviation σ =0.005. On the average, what % of manufactured ball bearings will be scraped?
30. In a test on 2000 electric bulbs it was found that the life of a particular make was normally distributed with an average life of 2040 hours and S.D. of 60 hours. Estimate the number of bulbs likely to burn for
    a. more than 2150 hours
    b. less than 1950 hours
    c. More than 1920 hours and but less than 2160 hours.
    d. Exactly 1960 hours.

Problems to be discussed by the faculty:

Example 23

At a receiving dock on an average 3 trucks arrive per hour to be unloaded at a warehouse. What are the probabilities that the time between the arrivals of successive trucks will be

a) less than 5 minutes b) at least 45 minutes c) is between 5 to 30 minutes

Solution: Assuming that the arrivals follow Poisson process, time interval between successive arrivals has an exponential distribution with mean 1/3 hours.

∴ Parameter of the exponential distribution β=3

The probability density is $f(x)=3e^{-3x}$

a) P(inter arrival time is less than 5 minutes)=P(inter arrival time is less than 1/12 hours)

$$= \int_{0}^{1/12} 3e^{-3x} = 1 - e^{-1/4} = 0.221$$

b) P(inter arrival time is at least 45 minutes)=P(inter arrival time is at least ¾ hours)

$$= \int_{3/4}^{\infty} 3e^{-3x} = e^{-9/4} = 0.105$$

c) P(inter arrival time is between 5 to 30 minutes)

$$= \int_{1/12}^{1/2} 3e^{-3x} = \frac{3}{-3}[e^{-3x}]_{1/12}^{1/2} = -[e^{-\frac{3}{2}} - e^{-\frac{3}{12}}]$$

Problems to be discussed by the students:

31. Suppose that a study of a certain computer system reveals that the response time, in seconds, has an exponential distribution with a mean of 3 seconds.
    a. What is the probability that response time exceeds 5 seconds?
    b. What is the probability that response time is less than 10 seconds?
    c. What are the mean and variance of response time?

32. The life of a certain type of device has an advertised failures rate of 0.01 per hour. The failure rate is constant and the exponential distribution applies.
    a. What is the mean time to failure?
    b. What is the probability that 200 hours will pass before a failure is observed?

33. The amount of time that a surveillance camera will run without having to be reset is a random variable having the exponential distribution with mean 50 days. Find the probabilities that such a camera will

    a. have to be reset in less than 20 days;
    b. Not have to be reset in at least 60 days.

Problems to be discussed by the faculty:

Example 24

Let X, Y and Z be three jointly continuous random variables with joint PDF

$$f_{XYZ}(x,y,z) = \begin{cases} c(x+2y+3z) & 0 \le x,y,z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the constant c.
2. Find the marginal PDF of X.

Solution:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XYZ}(x,y,z)dxdydz \\ &= \int_{0}^{1}\int_{0}^{1}\int_{0}^{1} c(x+2y+3z)\ dxdydz \\ &= \int_{0}^{1}\int_{0}^{1} c\left(\frac{1}{2}+2y+3z\right)\ dydz \\ &= \int_{0}^{1} c\left(\frac{3}{2}+3z\right)\ dz \\ &= 3c. \end{aligned}$$

Thus, $c = \frac{1}{3}$.

2. To find the marginal PDF of X, we note that $R_X = [0,1]$. For $0 \le x \le 1$, we can write

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XYZ}(x,y,z)dydz \\ &= \int_{0}^{1}\int_{0}^{1} \frac{1}{3}(x+2y+3z)\ dydz \\ &= \int_{0}^{1} \frac{1}{3}(x+1+3z)\ dz \\ &= \frac{1}{3}\left(x+\frac{5}{2}\right). \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} \frac{1}{3}\left(x+\frac{5}{2}\right) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 25

Given the joint density function

$$f(x,y) = \begin{cases} \dfrac{x(1+3y^2)}{4} & , \quad 0 < x < 2, 0 < y < 1 \\ 0 \quad \text{elsewhere}, \end{cases}$$

Find g(x), h(y), f(x/y), and evaluate

$$P(\tfrac{1}{4} < X < \tfrac{1}{2}/Y = \tfrac{1}{3})$$

Solution: By definition

$$g(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 \frac{x(1+3y^2)}{4}dy$$

$$= \left(\frac{xy}{4} + \frac{xy^3}{4}\right)\Big|_{y=0}^{y=1} = \frac{x}{2}, \quad 0 < x < 2$$

and

$$h(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^2 \frac{x(1+3y^2)}{4}dx$$

$$= \left(\frac{x^2}{8} + \frac{3x^2y^2}{8}\right)\Big|_{x=0}^{x=2} = \frac{1+3y^2}{2}, \quad 0 < y < 1$$

Therefore,

$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{x(1+3y^2)/4}{(1+3y^2)/2} = \frac{x}{2}, \qquad 0 < x < 2$$

and

$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{x(1+3y^2)/4}{(1+3y^2)/2} = \frac{x}{2}, \qquad 0 < x < 2$$

$$P\left(\frac{1}{4} < X < \frac{1}{2}\middle|Y = \frac{1}{3}\right) = \int_{1/4}^{1/2} \frac{x}{2}dx = \frac{3}{64}.$$

Example 26

Two scanners are needed for an experiment. Of the five available, two have electronic defects, another one has a defect in the memory, and two are in good working order. Two units are selected at random.

a. Find the joint probability distribution of X_1=the number with electronic defects, and X_2 = the number with a defect in memory.
b. Find the probability of 0 or 1 total defects among the two selected.
c. Find the marginal probability distribution of X_1.
d. Find the conditional probability distribution of X_1 given X_2=0.

Solution:

a) Joint distribution of X_1 and X_2

		X_1		
		0	1	2
X_2	0	1/10	4/10	1/10
	1	2/10	2/10	0

or symbolically the joint distribution of X_1 and X_2 is

$$\frac{2_{c_{x_1}} \times 1_{c_{x_2}} \times 2_{c_{x_1-x_2}}}{5_{c_2}} \quad \text{for } x_1 = 0,1,2, x = 0,1 \text{ and } 0 \le x_1 + x_2 \le 2$$

b) Probability of total defects is 0 and 1
$=P(x_1=0 \text{ and } x_2=0)+ P(x_1=0 \text{ and } x_2=1)+P(x_1=1 \text{ and } x_2=0)$
$=1/10+2/10+4/10 =0.7$

c) Marginal probability distribution of X_1 and X_2
Let $f_1(x_1)$ and $f_2(x_2)$ be the marginal distribution of X_1 and X_2 respectively. They are shown in the following table.

			X_1		$f_2(x_2)$
		0	1	2	
X_2	0	1/10	4/10	1/10	
	1	2/10	2/10	0	
$f_1(x_1)$		3/10	8/10	1/10	

d) The conditional distribution of X_1 given X_2=x is defined as

$$f_1\left({}^{x_1}/_{x_2}\right) = \frac{f(x_1, x_2)}{f_2(x_2)}$$

The conditional distribution of X_1 given X_2=0 is given by

$$f_1\left({}^{0}/_{0}\right) = \frac{f(0,0)}{f_2(0)} = \frac{1/10}{6/10} = 1/6$$

$$f_1\left({}^{1}/_{0}\right) = \frac{f(1,0)}{f_2(0)} = \frac{4/10}{6/10} = 4/6$$

$$f_1\left({}^{2}/_{0}\right) = \frac{f(2,0)}{f_2(0)} = \frac{1/10}{6/10} = 1/6$$

Example 27

N people sit around a round table, where N>5. Each person tosses a coin. Anyone whose outcome is different from his/her two neighbors will receive a present. Let X be the number of people who receive presents. Find EX and Var(X).

Solution:

Number the N people from 1 to N. Let X_i be the indicator random variable for the ith person, that is, $X_i = 1$ if the ith person receives a present and zero otherwise. Then

$$X = X_1 + X_2 + \ldots + X_N.$$

First note that $P(X_i = 1) = \frac{1}{4}$. This is the probability that the person to the right has a different outcome times the probability that the person to the left has a different outcome. In other words, if we define H_i and T_i be the events that the ith person's outcome is heads and tails respectively, then we can write

$$\begin{aligned} EX_i &= P(X_i = 1) \\ &= P(H_{i-1}, T_i, H_{i+1}) + P(T_{i-1}, H_i, T_{i+1}) \\ &= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}. \end{aligned}$$

Thus, we find

$$EX = EX_1 + EX_2 + \ldots + EX_N = \frac{N}{4}.$$

Next, we can write

$$\text{Var}(X) = \sum_{i=1}^{N} \text{Var}(X_i) + \sum_{i=1}^{N} \sum_{j \neq i} \text{Cov}(X_i, X_j).$$

Since $X_i \sim Bernoulli(\frac{1}{4})$, we have

$$\text{Var}(X_i) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}.$$

It remains to find $\text{Cov}(X_i, X_j)$. First note that X_i and X_j are independent if there are at least two people between the ith person and the jth person. In other words, if $2 < |i - j| < N - 2$, then X_i and X_j are independent, so

$$\text{Cov}(X_i, X_j) = 0, \text{ for } 2 < |i - j| < N - 2.$$

Also, note that there is a lot of symmetry in the problem:

$$\begin{aligned} \text{Cov}(X_1, X_2) = \text{Cov}(X_2, X_3) = \text{Cov}(X_3, X_4) = \ldots = \text{Cov}(X_{N-1}, X_N) = \text{Cov}(X_N, X_1), \\ \text{Cov}(X_1, X_3) = \text{Cov}(X_2, X_4) = \text{Cov}(X_3, X_5) = \ldots = \text{Cov}(X_{N-1}, X_1) = \text{Cov}(X_N, X_2). \end{aligned}$$

Thus, we can write

$$\begin{aligned} \text{Var}(X) &= N\text{Var}(X_1) + 2N\text{Cov}(X_1, X_2) + 2N\text{Cov}(X_1, X_3) \\ &= \frac{3N}{16} + 2N\text{Cov}(X_1, X_2) + 2N\text{Cov}(X_1, X_3). \end{aligned}$$

So we need to find $\mathrm{Cov}(X_1, X_2)$ and $\mathrm{Cov}(X_1, X_3)$. We have

$$\begin{aligned} E[X_1X_2] &= P(X_1 = 1, X_2 = 1) \\ &= P(H_N, T_1, H_2, T_3) + P(T_N, H_1, T_2, H_3) \\ &= \frac{1}{16} + \frac{1}{16} = \frac{1}{8}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{Cov}(X_1, X_2) &= E[X_1X_2] - E[X_1]E[X_2] \\ &= \frac{1}{8} - \frac{1}{16} = \frac{1}{16}, \end{aligned}$$

$$\begin{aligned} E[X_1X_3] &= P(X_1 = 1, X_3 = 1) \\ &= P(H_N, T_1, H_2, T_3, H_4) + P(T_N, H_1, T_2, H_3, T_4) \\ &= \frac{1}{32} + \frac{1}{32} = \frac{1}{16}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{Cov}(X_1, X_3) &= E[X_1X_3] - E[X_1]E[X_3] \\ &= \frac{1}{16} - \frac{1}{16} = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{Var}(X) &= \frac{3N}{16} + 2N\mathrm{cov}(X_1, X_2) + 2N\mathrm{cov}(X_1, X_3) \\ &= \frac{3N}{16} + \frac{2N}{16} \\ &= \frac{5N}{16}. \end{aligned}$$

Problems to be discussed by the students:

34. A privately owned liquor store operates both a drive-in facility and walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportion of the time that the drive-in and walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$f(x, y) = \begin{cases} \frac{2}{3}(x + 2y), & 0 \le x \le 1, 0 \le y \le 1, \\ 0, & elsewhere. \end{cases}$$

a) Find the marginal density of X

b) Find the marginal density of y

c) Find the probability that the drive-in facility is busy less than one-half of the time.

d) Find the probability that the drive in facility is fully less than one fourth of the time.

Example 28

For a random vector $\mathbf{X}$, show

$$\mathbf{C_X} = \mathbf{R_X} - \mathbf{EXEX^T}.$$

Solution:

We have

$$\begin{aligned}
\mathbf{C_X} &= \mathbf{E}[(\mathbf{X}-\mathbf{EX})(\mathbf{X}-\mathbf{EX})^{\mathbf{T}}] \\
&= \mathbf{E}[(\mathbf{X}-\mathbf{EX})(\mathbf{X^T}-\mathbf{EX^T})] \\
&= \mathbf{E}[\mathbf{XX^T}] - \mathbf{EXEX^T} - \mathbf{EXEX^T} + \mathbf{EXEX^T} \qquad \text{(by linearity of expectation)} \\
&= \mathbf{R_X} - \mathbf{EXEX^T}.
\end{aligned}$$

Example 29

Let $\mathbf{X}$ be an n-dimensional random vector and the random vector $\mathbf{Y}$ be defined as

$$\mathbf{Y} = \mathbf{AX} + \mathbf{b},$$

where $\mathbf{A}$ is a fixed m by n matrix and $\mathbf{b}$ is a fixed m-dimensional vector. Show that

$$\mathbf{C_Y} = \mathbf{AC_XA^T}.$$

Solution:

Note that by linearity of expectation, we have

$$\mathbf{EY} = \mathbf{AEX} + \mathbf{b}.$$

By definition, we have

$$\begin{aligned}
\mathbf{C_Y} &= \mathbf{E}[(\mathbf{Y}-\mathbf{EY})(\mathbf{Y}-\mathbf{EY})^{\mathbf{T}}] \\
&= \mathbf{E}[(\mathbf{AX}+\mathbf{b}-\mathbf{AEX}-\mathbf{b})(\mathbf{AX}+\mathbf{b}-\mathbf{AEX}-\mathbf{b})^{\mathbf{T}}] \\
&= E[\mathbf{A}(\mathbf{X}-\mathbf{EX})(\mathbf{X}-\mathbf{EX})^{\mathbf{T}}\mathbf{A^T}] \\
&= \mathbf{AE}[(\mathbf{X}-\mathbf{EX})(\mathbf{X}-\mathbf{EX})^{\mathbf{T}}]\mathbf{A^T} \qquad \text{(by linearity of expectation)} \\
&= \mathbf{AC_XA^T}.
\end{aligned}$$

Problems to be discussed by the students:

35.

Let X, Y and Z be three jointly continuous random variables with joint PDF

$$f_{XYZ}(x,y,z) = \begin{cases} \frac{1}{3}(x+2y+3z) & 0 \le x,y,z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the joint PDF of X and Y, $f_{XY}(x,y)$.