Towards efficient algorithmic aspects of algebraic lattices

Thomas Espitau

April 3, 2024, Bordeaux

Seminaire CHARM 2024

Lattices and their reductions

What is this all about?

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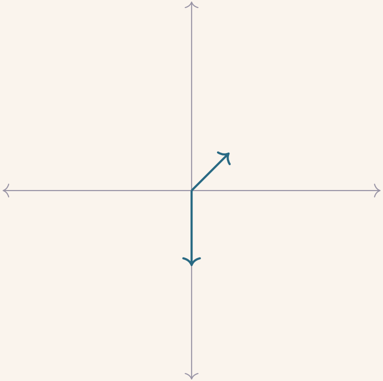

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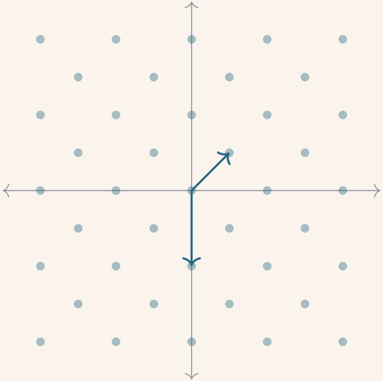

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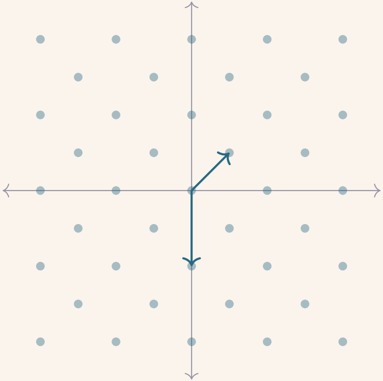

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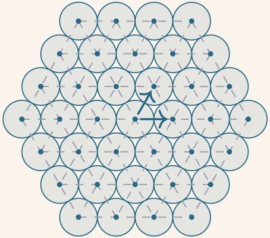

Sphere Packing problem

Hexagonal lattice | Lagrange 1773

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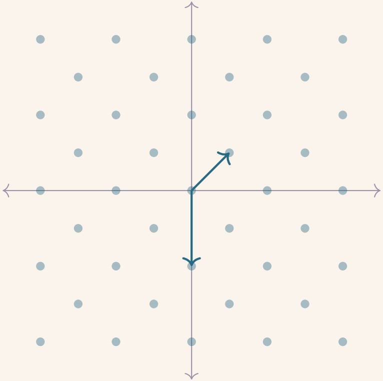

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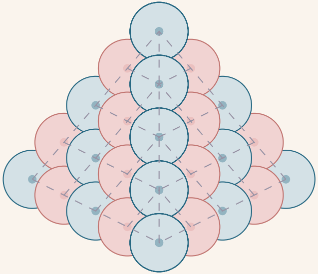

Sphere Packing problem

Kepler's conjecture | Hales 1999

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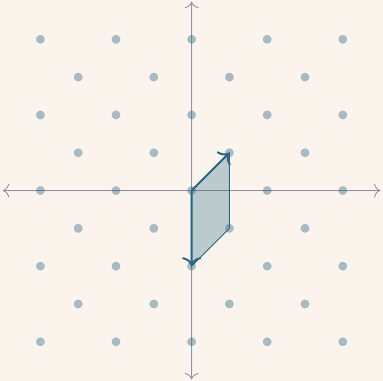

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The (co)volume $\mathrm{covol}(\Lambda)$ of Λ is the quantity

$$\mathrm{covol}(\Lambda) = \sqrt{\det \langle v_i, v_j \rangle}$$

Corresponds to the volume of the fundamental domain $\{\sum x_i v_i \mid 0 \leqslant x_i < 1\}$.

(also the volume (no "co") of the variety $\mathbb{R}^n/\Lambda$)

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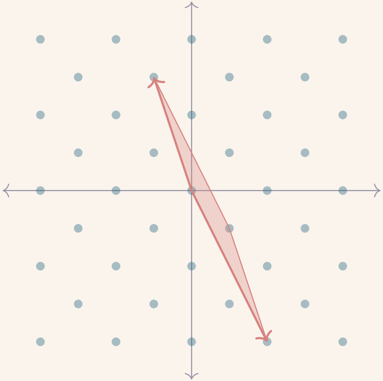

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Independent of the basis

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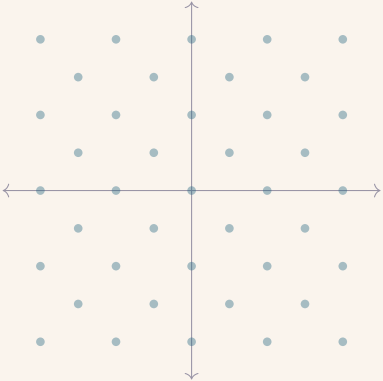

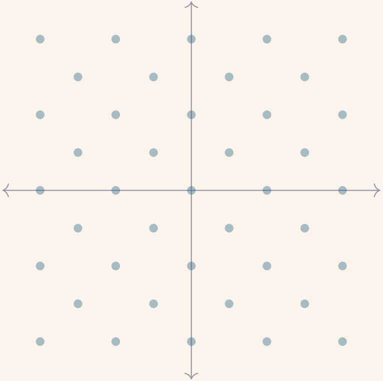

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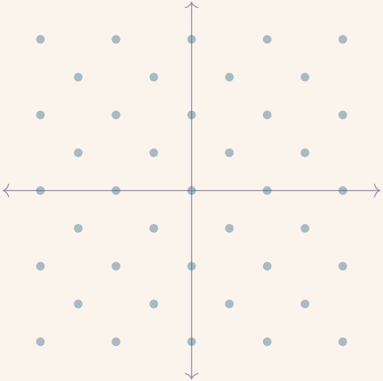

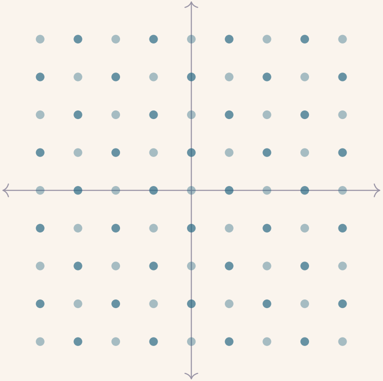

$$\mathrm{covol}(\Lambda) = 2\,\mathrm{covol}(\Lambda')$$

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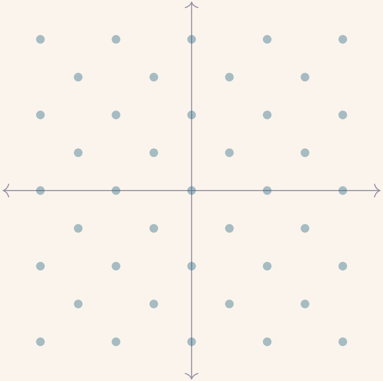

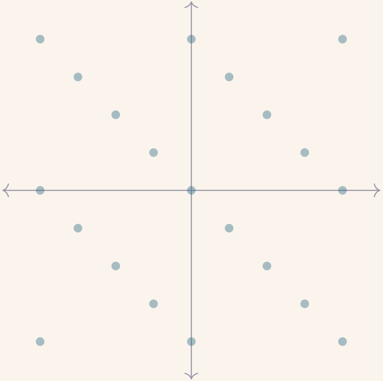

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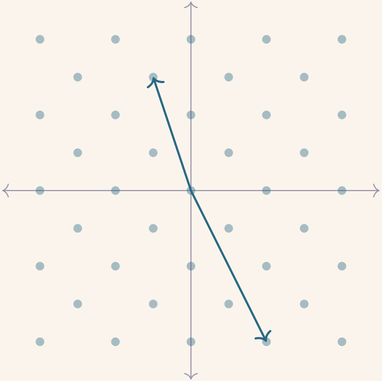

How to get a shorter basis?

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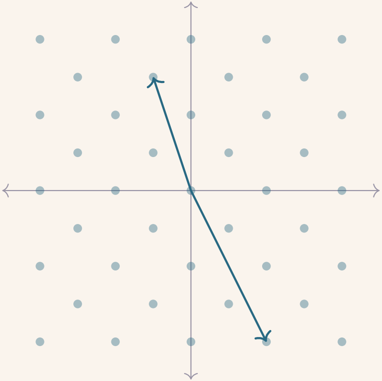

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$\rightarrow$ Use the shortest vector to reduce the longest one.

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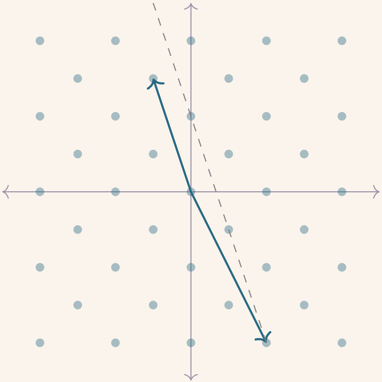

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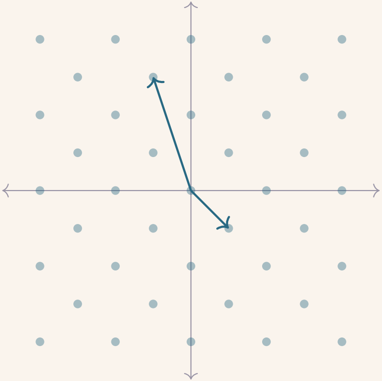

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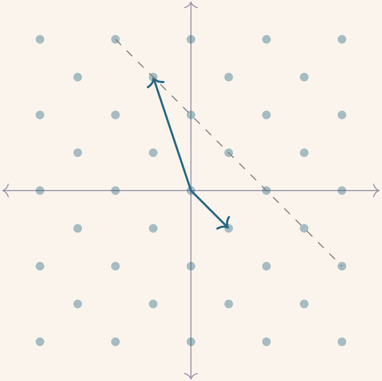

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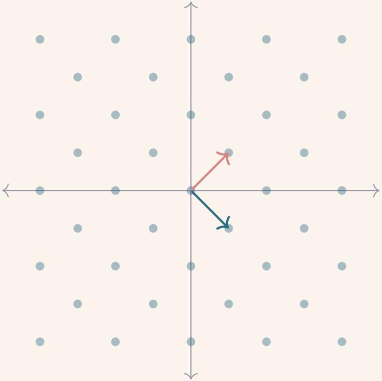

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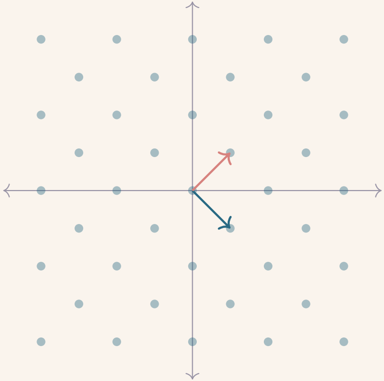

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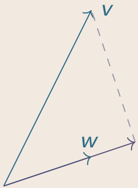
Effective computing of this element:

1. Orthogonal projection $\frac{\langle w,v\rangle}{\langle w,w\rangle} w$



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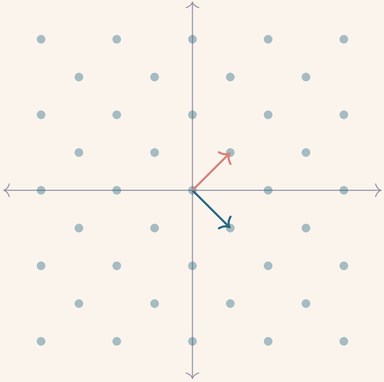

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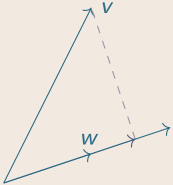

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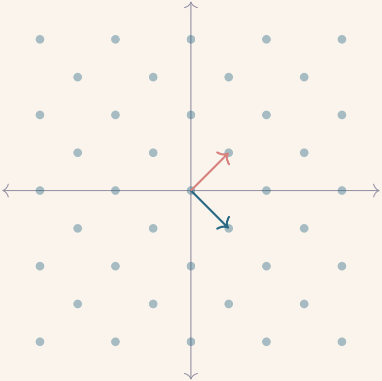

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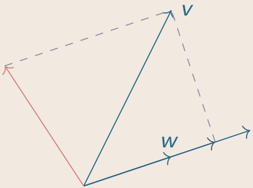

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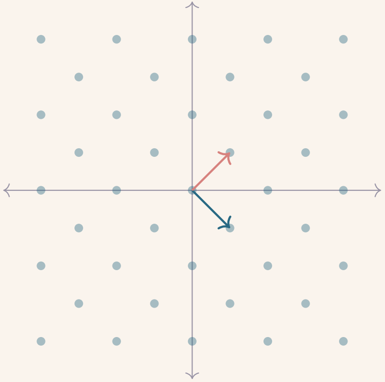

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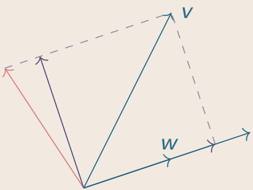

In dim 2... The Gauss(-Lagrange) reduction algorithm

Gauss-Lagrange reduction

if $\|v\| < \|u\|$ **then return** **Gauss**(v, u);
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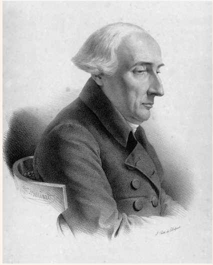

Properties of a Gauss-reduced basis (u, v)

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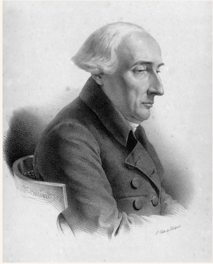

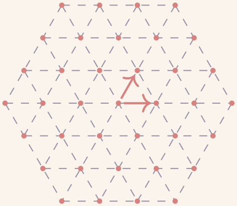

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- $\|u\|^2 \leqslant (4/3)\,\mathrm{covol}(\Lambda)$

And now what?

Minkowski's theorem for first minima: For any lattice Λ of rank d,

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Minkowski-Hermite's theorem for first minima: For any lattice Λ of rank d,

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Finding the shortest/closest vector in a lattice is **hard**

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- Computations in algebraic number theory (ideal computations, HNF, control of size of elements...)

What can we do with a reduction in rank 2 ?

Any basis $(v_1, \ldots, v_d)$ of a lattice Λ yields a filtration given by $(\Lambda_i = v_1\mathbb{Z} \oplus \cdots \oplus v_i\mathbb{Z})$

$$\{0\} = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_{i-1} \subset \Lambda_i \subset \Lambda_{i+1} \subset \cdots \subset \Lambda_d = \Lambda$$

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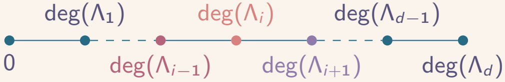

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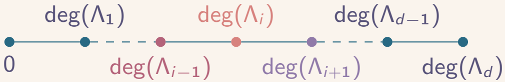

- Where are the natural rank 2 lattices around here?

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(endowed with the *quotient norm*)

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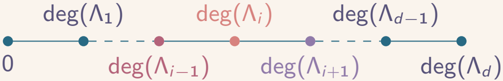

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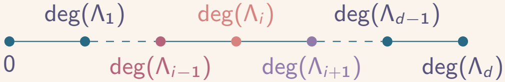

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Action of the reduction

$$2\deg(\Lambda') \leqslant \deg(\Lambda^*) + \log\left(\frac{4}{3}\right)$$

(*by Hermite inequality + log*)

Continuing...

Lifting and replacing in the filtration: find Λ_i' s.t.:

$$\begin{array}{ccccc}
{}^{\Lambda_{i-1}}\!/\!_{\Lambda_{i-1}} & \subset & {}^{\Lambda_i'}\!/\!_{\Lambda_{i-1}} & \subset & {}^{\Lambda_{i+1}}\!/\!_{\Lambda_{i-1}} \\
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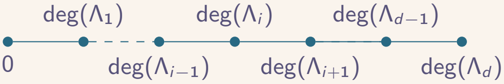
Result on the profile space:



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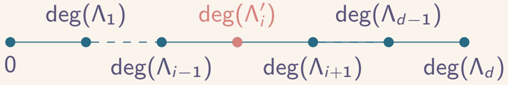

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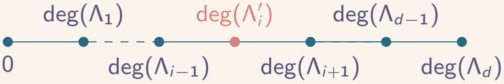

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Gauss's reduction is a *local* tool for densifying the filtration

Effective lifting and size-reduction

Effective lifting

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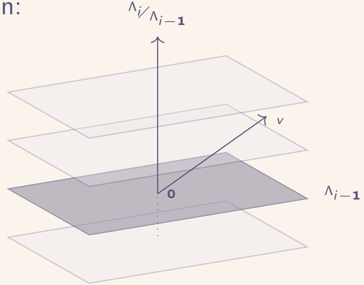

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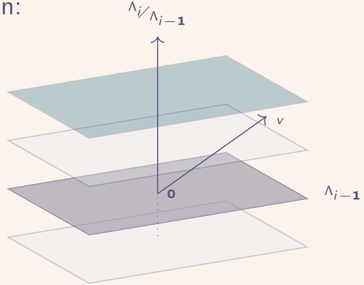

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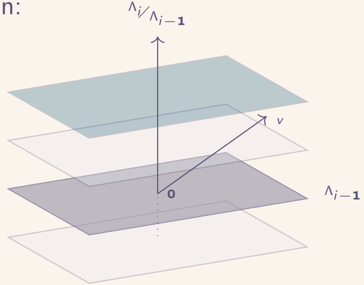

Lifting

1 **for** $j = k-1$ **down to** 1 **do**
2 $\quad v \leftarrow v - \left\lceil \frac{\langle v_k, \pi_j(v_j)\rangle}{\|\pi_j(v_j)\|^2} \right\rfloor \cdot v_j$
3 **end for**

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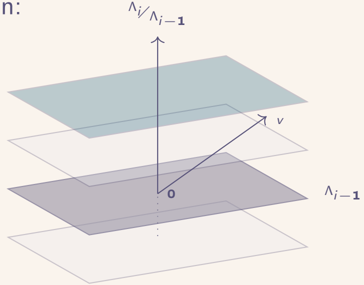

Size-reduction

for $k = 2$ **to** d **do**
  for $j = k-1$ **down to** 1 **do**
    $v_k \leftarrow v_k - \left\lceil \frac{\langle v_k, \pi_j(v_j)\rangle}{\|\pi_j(v_j)\|^2} \right\rfloor \cdot v_j$
  end for
end for
return $(v_1, \ldots, v_d)$

From local to global: an iterative strategy

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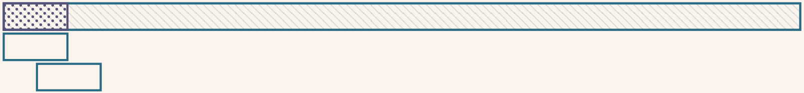

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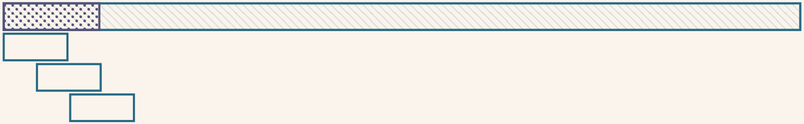

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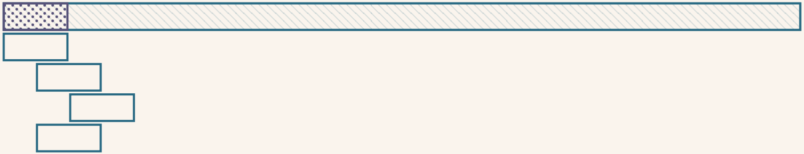

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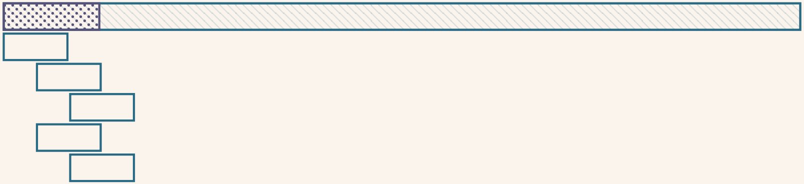

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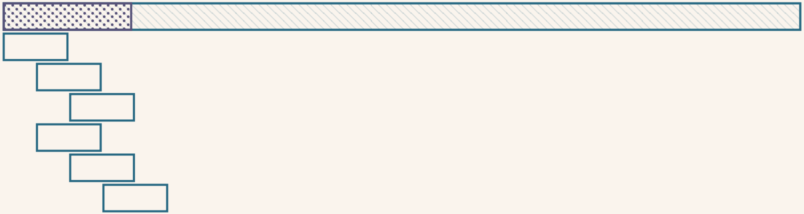

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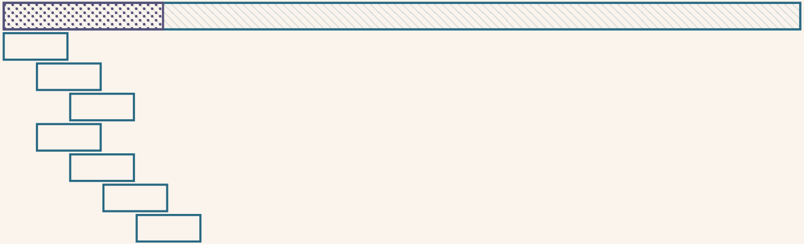

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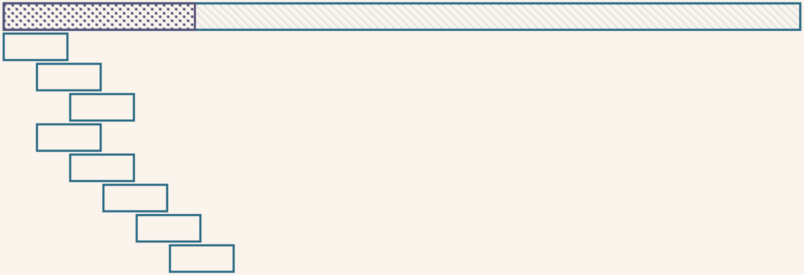

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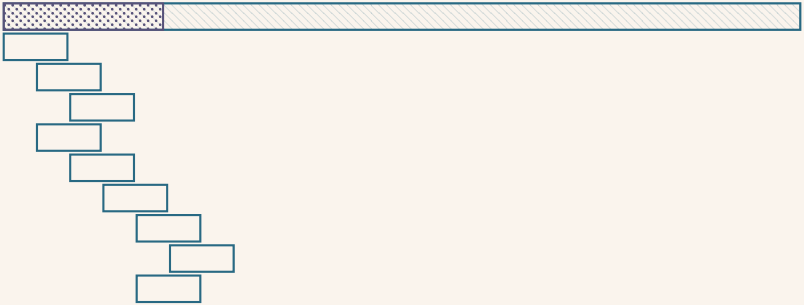

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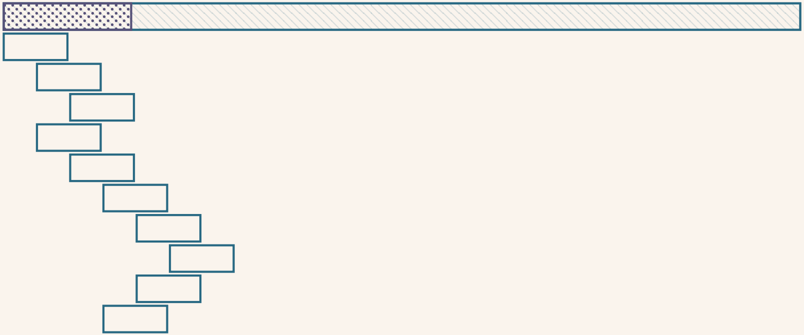

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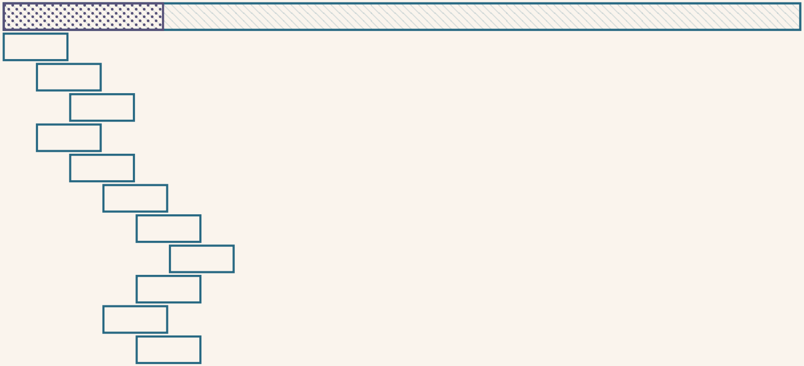

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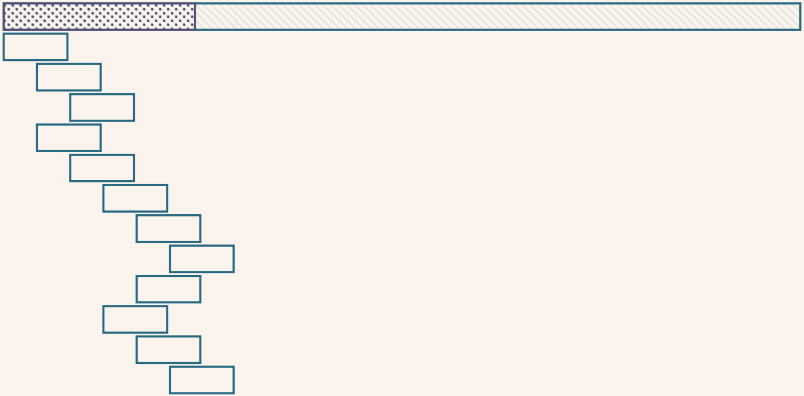

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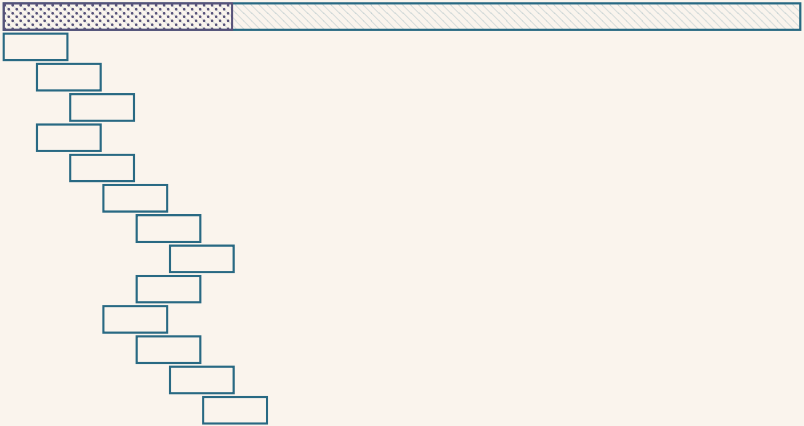

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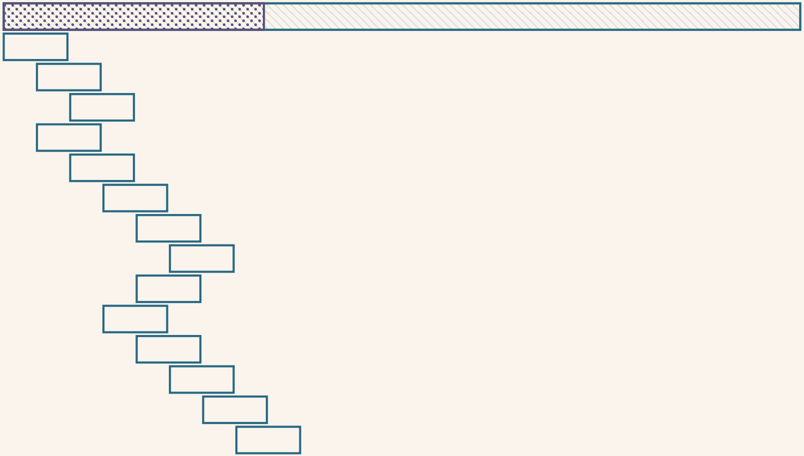

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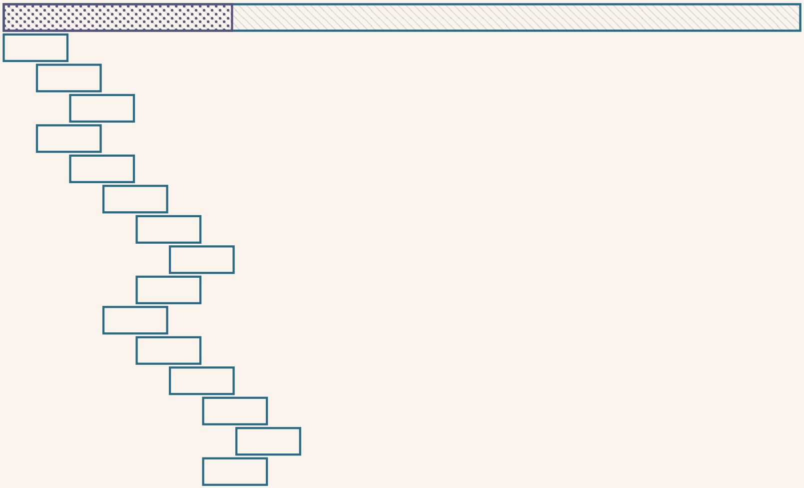

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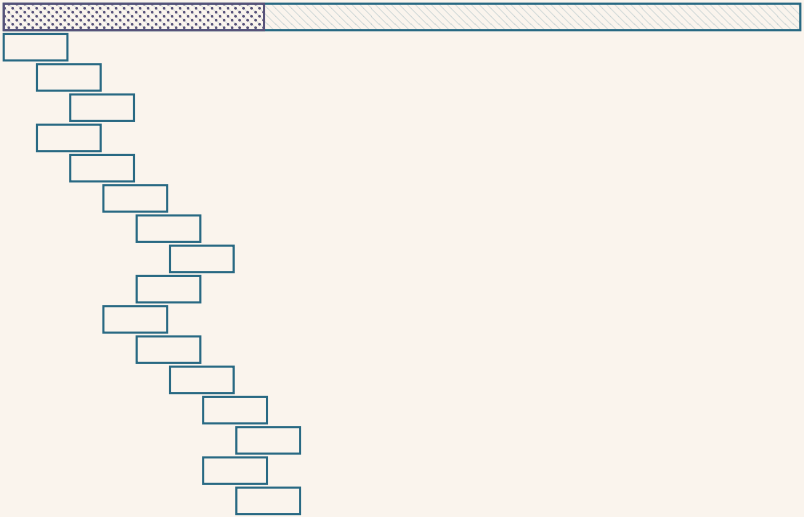

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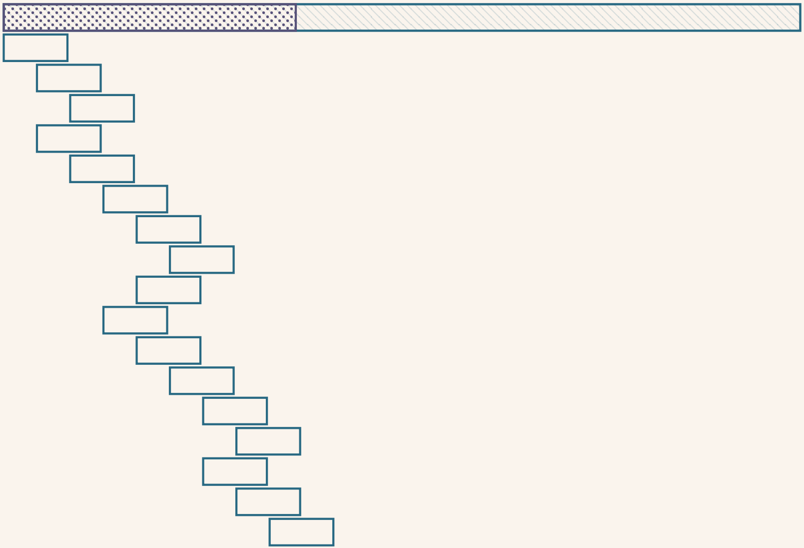

Diagramatically !

Reduction of rank n lattice

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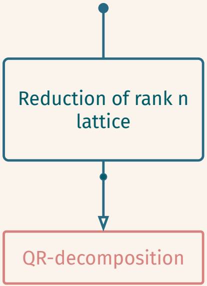

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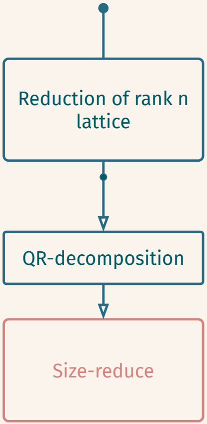

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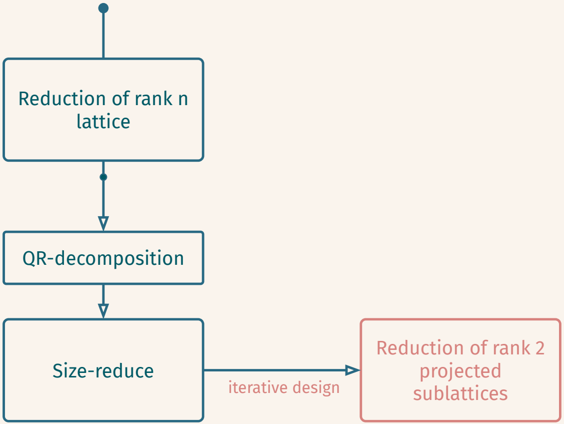

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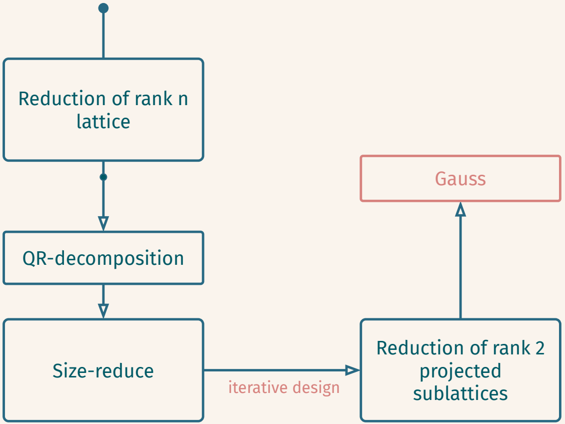

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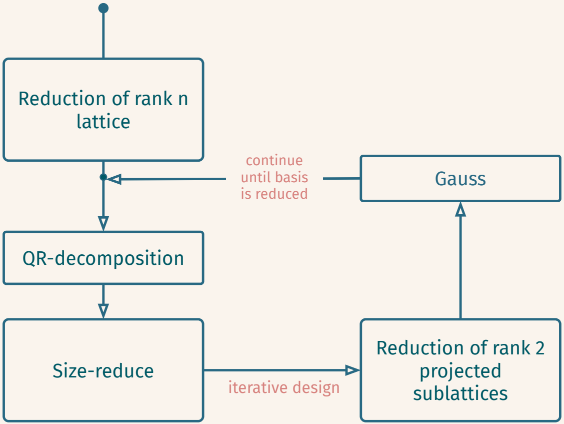

LLL reduced basis

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- Size-Reduction condition
  (*lifts is as good as possible*)

$$\forall i < j, \quad |\langle v_j, \pi_i(v_i) \rangle| \leqslant \frac{1}{2} \|\pi_i(v_i)\|^2$$

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Effective version of Hermite's inequality:

$$\gamma_d \leqslant \gamma_2^{d-1}$$

Guarantees offered by LLL

$$\operatorname{covol}(\Lambda_k) \leqslant \left(\delta - \frac{1}{4}\right)^{-\frac{(d-k)k}{4}} \operatorname{covol}(\Lambda)^{\frac{k}{d}}$$

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Peter van Emde Boas, László Lovász, Hendrik Lenstra and Arjen Lenstra.
(*Bonn* on 27/02/1982)

But... How fast is this reduction?

	Variant	Complexity	
naive arithmetic	Textbook	$\mathrm{O}\left(d^6 \log^3 \\|B\\|_\infty\right)$	naive
		$\mathrm{O}\left(\frac{d^5 \log^2 \\|B\\|_\infty}{d+\log \\|B\\|_\infty} M(d + \log \\|B\\|_\infty)\right)$	refined
	» Bottleneck: *size* of numerators/denominators in *GSO* computations «		
floating point	Nguyen-Stehlé (2009)	$\mathrm{O}\left(d^5(d + \log(\\|B\\|_\infty)) \log(\\|B\\|_\infty)\right)$	lazy size-reduction
	Neumaier-Stehlé (2016)	$\mathrm{O}\left(d^{4+\epsilon} \log(\\|B\\|_\infty)^{1+\epsilon}\right)$	recursive strategy

Generalization: towards algebraic lattices

Number fields and algebraic lattices

Number field

- Finite extension of $\mathbb{Q}$:

$$L \cong {}^{\mathbb{Q}[X]}\!\big/_{(P)}$$

- Ring of integers:

$$\mathcal{O}_L = \{\alpha \,|\, \exists R \in \mathbb{Z}[X] \text{ monic } , R(\alpha) = 0\}$$

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$$\mathcal{O}_{\mathbb{Q}(i\sqrt{5})} = \left\{\frac{a}{2} + \frac{b}{2} i\sqrt{5} \mid a, b \in \mathbb{Z}\right\}$$
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Number field

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(Natural?) Hermitian structure

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Additive notion	*Multiplicative notion*
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Better for the geometry !	*better with the arithmetic !*

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$$\langle x, Hy \rangle = \sum_{\sigma : L \to \mathbb{C}} \langle x, H_\sigma y \rangle_\sigma$$

(Natural?) Hermitian structure

Take your favorite sesquilinear map $g : \Lambda_{\mathbb{R}} \times \Lambda_{\mathbb{R}} \to L_{\mathbb{R}}$ (for instance as vectors $g(x, y) = \sum_i \bar{x}_i y_i$)

- not very *real* ! (distance, etc...)
- How do we go from L to $\mathbb{R}$? Compose with ...

Trace tr	Norm N
Additive notion	*Multiplicative notion*
encode the length of the element when seen as a vector in $\mathbb{C}^{\deg(L)}$.	encode the $\deg(L)$ - "volume" of the lattice spanned by the element)
Better for the geometry !	*better with the arithmetic !*

Lattice

An (algebraic) lattice Λ is a free $\mathcal{O}_L$-module of finite rank, endowed with an inner product on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

Let's look at the trace on $(L \otimes \mathbb{R})^d$

For any $x = (x_1, \ldots, x_d) \in (L \otimes \mathbb{R})^d$ and $y = (y_1, \ldots, y_d) \in (L \otimes \mathbb{R})^d$:

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Corresponds to **Humbert forms** (**"posdef symmetric" matrix over L**)

Algebraic lattice

Algebraic lattice

An algebraic lattice Λ is a projective $\mathcal{O}_L$-module of finite rank, endowed with a Humbert form on the space $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

Algebraic lattice

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Examples I

- $\mathcal{O}_{\mathbb{Q}}^n = \mathbb{Z}^n$ is a rank n lattice for the form Id (!)
- $\mathcal{O}_L$ is a rank 1 lattice for the form $\mathrm{Id}_1 = (1)$
- $\mathcal{O}_L^2$ is a rank 2 lattice for the form Id_2
- $\begin{pmatrix} f \\ g \end{pmatrix} \mathcal{O}_L \oplus \begin{pmatrix} F \\ G \end{pmatrix} \mathcal{O}_L$ is a rank 2 lattice... (if f, g are small it's nothing less than NTRU).

Algebraic lattice

Algebraic lattice

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Examples II

- More complicated, adding projectivity in the mix:

$$\mathfrak{a} \subset (\mathcal{O}_L, \mathrm{Id}_1) \text{ is a sublattice of rank 1 (not free unless } \mathfrak{a} \text{ is principal)}$$

- As $\mathcal{O}_L$ is a Dedekind domain:

$$\mathcal{O}_L \cong v_1\mathfrak{a}_1 \oplus v_2\mathfrak{a}_2 \oplus \cdots \oplus v_n\mathfrak{a}_n$$

is the general form of a projective module of rank n.

Algebraic lattice

An algebraic lattice Λ is a projective $\mathcal{O}_L$-module of finite rank, endowed with a Humbert form on the space $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

Generic form of an algebraic lattice

$$\boxed{\left(v_1\mathfrak{a}_1 \oplus v_2\mathfrak{a}_2 \oplus \cdots \oplus v_n\mathfrak{a}_n \ , \ \left(H_\sigma\right)_{\sigma : L \to \mathbb{C}}\right)}$$

Algebraic lattice

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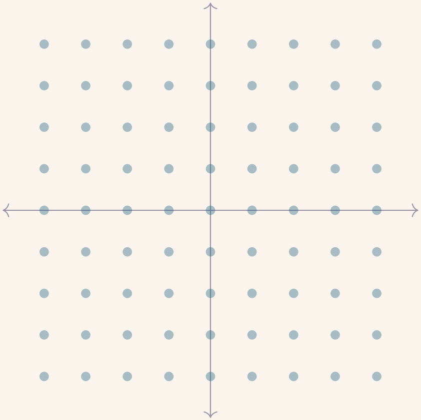
Generic form of an algebraic lattice

$$\Big(\underbrace{v_1\mathfrak{a}_1 \oplus v_2\mathfrak{a}_2 \oplus \cdots \oplus v_n\mathfrak{a}_n}_{\text{algebraic datum}} \,,\ \underbrace{(H_\sigma)_{\sigma:L\to\mathbb{C}}}_{\text{metric datum}} \Big)$$

A bit more examples, this time with drawings



The Gaussian integers

- $\mathcal{O}$, ring of integer of
$$L = \mathbb{Q}(i) = {}^{\mathbb{Z}[T]}\!/_{T^2+1}$$
- $\mathcal{O} = \mathbb{Z}[i] := \{a + ib \mid a, b \in \mathbb{Z}^2\}$
- We take the identity form Id, so that the L-inner product is simply the multiplication: $(x, y) \mapsto \bar{x}y$.

(in dim 2,*already* in $\mathbb{Q}$: unclear *why* we need to norm or trace)

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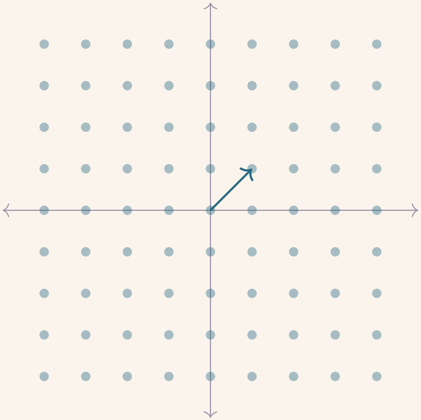

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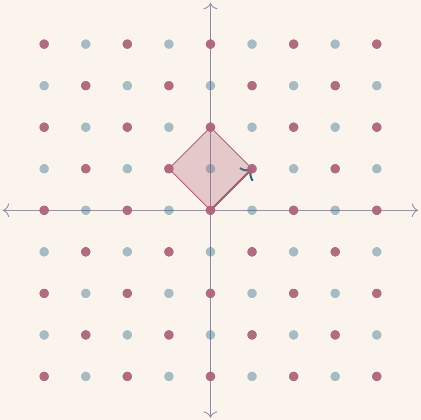

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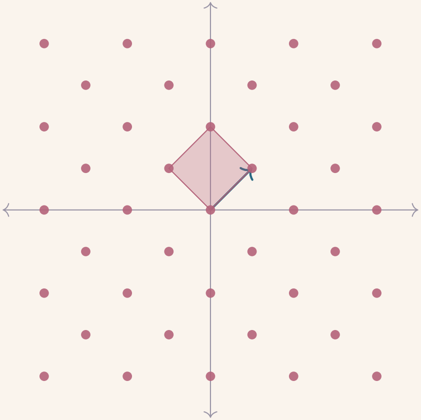

$\mathfrak{a}$ is both a dim 1 lattice (over $\mathcal{O}$) and a 2 dimensional lattice (over $\mathbb{Z}$)

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On the recursive structure of algebraic lattices: baby Galois descent

$$\begin{tikzcd}
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K \arrow[u] & \mathcal{O}_K \arrow[u] \arrow[urr, dash, dashed] & & \\
\mathbb{Q} \arrow[u] & \mathbb{Z} \arrow[u] \arrow[uurr, dash, dashed] & &
\end{tikzcd}$$

- The structure of an algebraic module is not unique. It depends on the base ring.
- For a tower $\mathbb{Q} \subset K \subset L$, an $\mathcal{O}_L$ lattice can be *descended* to:
  - an $\mathcal{O}_K$ lattice (of rank $\times[L : K]$)
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- The form is descended *canonically*
- Over $\mathbb{Z}$, recovers the *trace norm* (here we know how to do the reduction !)

How to do reduction at the top level, for the norm?

A very philosophical question

What is the *right* notion of λ_1 ?

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What is the *right* notion of λ_1 ?

- Is it the shortest vector? (vector taken for trace norm) (in this case use $\mathbb{Z}$-lattice reduction for tr)
- Is it the densest (free? projective?) sublattice of rank 1? (vector/vector+ideal taken by the volume) ... but How?

A very philosophical question

What is the *right* notion of λ_1 ?

"I can't cut the grass until I find the lawnmower and I can't find the lawnmower until I cut the grass"

Reduction of algebraic lattices

How to attack the reduction problem?

$\rightarrow$ **Idea**: Try to keep the core design principles of what we saw.

How to attack the reduction problem?

(Pseudo)-basis $(v_1\mathfrak{a}_1, \ldots, v_d\mathfrak{a}_d)$ gives

$$\Lambda_i = v_1\mathcal{O}_L \oplus \cdots \oplus v_i\mathcal{O}_L$$

$$\{0\} \subset \Lambda_1 \subset \cdots \subset \Lambda_i \subset \cdots \subset \Lambda_d = \Lambda$$

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- Work on $\mathcal{O}_L$-filtrations of the lattice

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→ **Idea**: Try to keep the core design principles of what we saw.

- Work on $\mathcal{O}_L$-filtrations of the lattice
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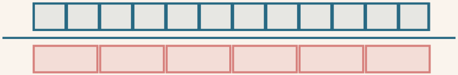

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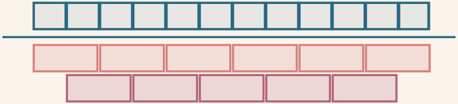

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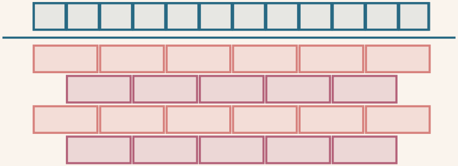

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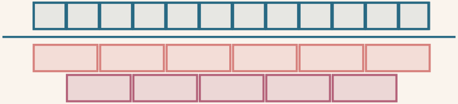
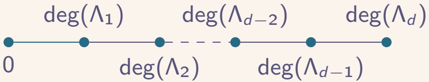


A round of local reductions acts as a *discretized* Laplacian operator on the profile space':



- (discrete) *diffusion property* of the solution of the heat equation

$$\frac{\partial u}{\partial t} = \alpha \Delta u$$

- Characteristic time is quadratic in the diameter of the space $\to O(d^2)$ steps

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- Over $\mathbb{Z}$: requires integral rounding

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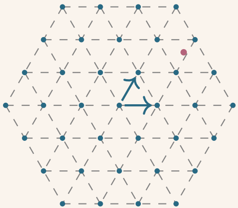
How to attack the reduction problem?

- Over $\mathbb{Z}$: requires integral rounding

- Translated over $\mathcal{O}_L$: find the closest element in the ring: instance of CVP



- Approx-CVP suffices (just do the coefficient-wise rounding!)

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In $L = \mathbb{Q}[\zeta_{16}]$:

$$x = \frac{65210}{3}\zeta^7 + \frac{78658}{3}\zeta^6 - 41412\zeta^5 + \frac{16567}{3}\zeta^4 + 36970\zeta^3 - \frac{100235}{3}\zeta^2 - \frac{145843}{12}\zeta + \frac{86961}{2}.$$

- We have: $N_{L/\mathbb{Q}}(x)^{\frac{1}{8}} \approx 6.6758$
- Embeddings:

$$|\sigma_1(x)| = |\sigma_{15}(x)| \approx 2771.189$$
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Unit rounding for cyclotomics

There is a quasi-linear randomized algorithm that given $x \in (\mathbb{R} \otimes K)^\times$ finds unit $u \in \mathcal{O}_K^\times$ such that for any field embedding $\sigma : K \to \mathbb{C}$:

$$\sigma\left(xu^{-1}\right) = 2^{\mathrm{O}\left(\sqrt{f \log f}\right)} N_{K/\mathbb{Q}}(x)^{\frac{1}{\varphi(f)}}$$

How to attack the reduction problem?

In $L = \mathbb{Q}[\zeta_{16}]$:

$$x = \frac{65210}{3}\zeta^7 + \frac{78658}{3}\zeta^6 - 41412\zeta^5 + \frac{16567}{3}\zeta^4 + 36970\zeta^3 - \frac{100235}{3}\zeta^2 - \frac{145843}{12}\zeta + \frac{86961}{2}.$$

- We have: $N_{L/\mathbb{Q}}(x)^{\frac{1}{8}} \approx 6.6758$
- When using **Unit**:

$$\left|\sigma_1\left(\frac{x}{u}\right)\right| = \left|\sigma_{15}\left(\frac{x}{u}\right)\right| \approx 7.83729$$
$$\left|\sigma_3\left(\frac{x}{u}\right)\right| = \left|\sigma_{13}\left(\frac{x}{u}\right)\right| \approx 7.33868$$
$$\left|\sigma_5\left(\frac{x}{u}\right)\right| = \left|\sigma_{11}\left(\frac{x}{u}\right)\right| \approx 5.93346$$
$$\left|\sigma_7\left(\frac{x}{u}\right)\right| = \left|\sigma_9\left(\frac{x}{u}\right)\right| \approx 5.82028.$$

→ **Idea**: Try to keep the core design principles of what we saw.

- Work on $\mathcal{O}_L$-filtrations of the lattice
- Reduce to a rank 2 oracle on projected quotients
- Size-reduction?

Unit rounding for cyclotomics

There is a quasi-linear randomized algorithm that given $x \in (\mathbb{R} \otimes K)^\times$ finds unit $u \in \mathcal{O}_K^\times$ such that for any field embedding $\sigma : K \to \mathbb{C}$:

$$\sigma\left(xu^{-1}\right) = 2^{\mathrm{O}\left(\sqrt{f \log f}\right)} N_{K/\mathbb{Q}}(x)^{\frac{1}{\varphi(f)}}$$

How to attack the reduction problem?

Size-Reduce

```
U ← Id_{d,d}
for i = 1 to d do
    D ← D_i(Unit(R_{i,i}));
    (U, R) ← (U, R) · D^{-1} ;
    for j = i − 1 down to 1 do
        Σ_{ℓ=0}^{n−1} r_ℓ X^ℓ ← R_{i,j} / R_{j,j}
        μ ← Σ_{ℓ=0}^{n−1} ⌊r_ℓ⌉ X^ℓ
        (U, R) *= T_{i,j}(−μ)
    end for
end for
return U
```

→ **Idea**: Try to keep the core design principles of what we saw.

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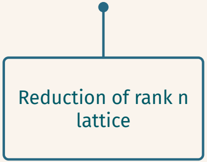
As a flowchart



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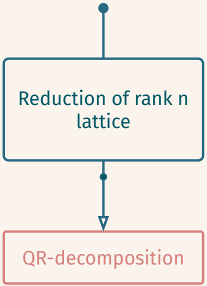

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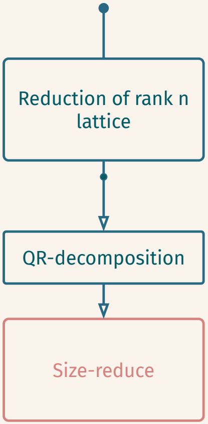

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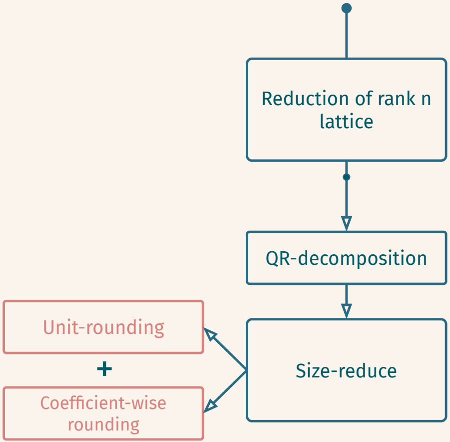

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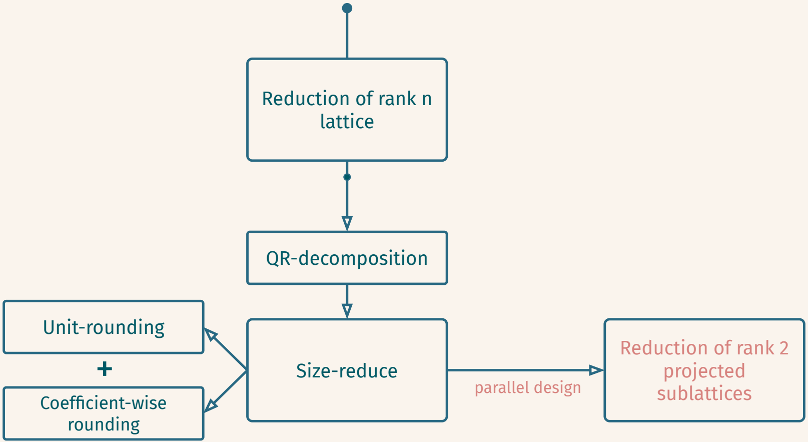

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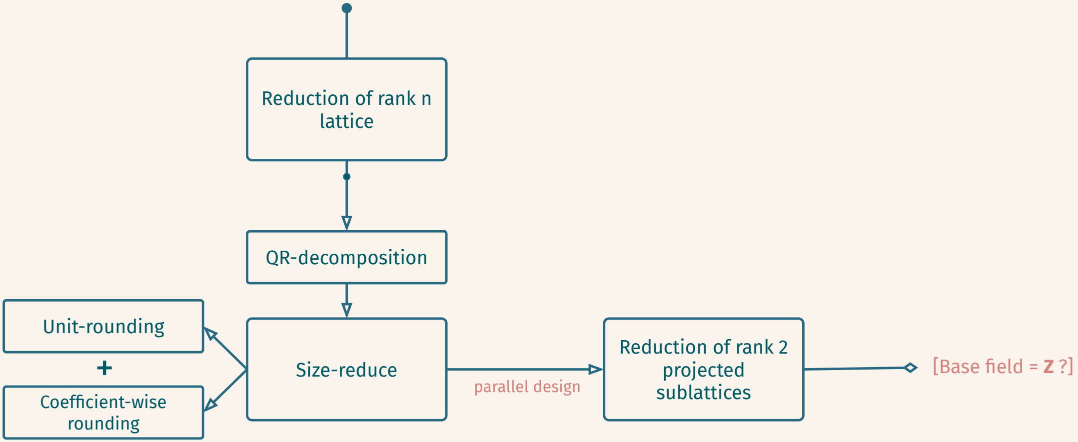

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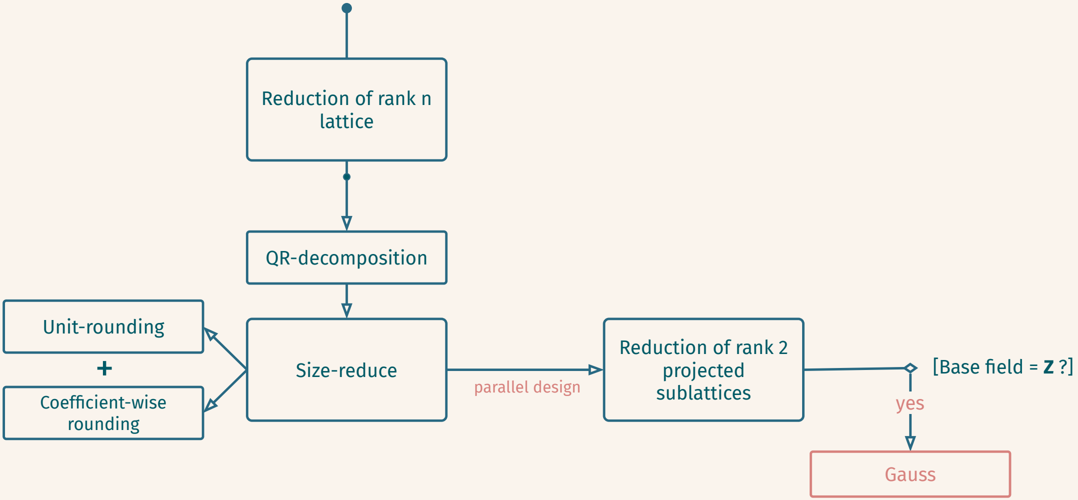

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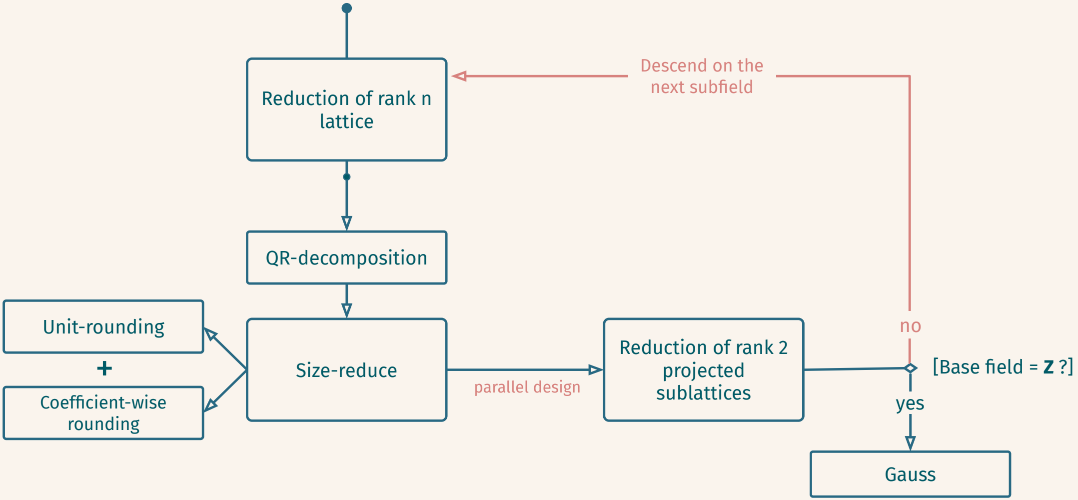

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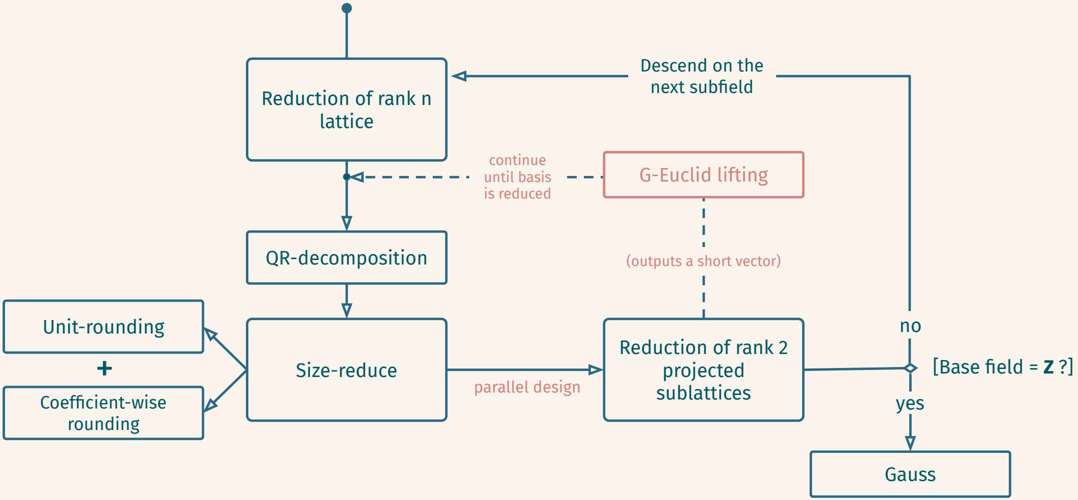

General strategy

$$\begin{tikzcd} \mathcal{O}_{K_h} \\ \mathcal{O}_{K_{h-1}} \arrow[u, hook] \\ \vdots \arrow[u, hook] \\ \mathbb{Z} \arrow[u, hook] \end{tikzcd}$$

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Find $\square$, $\triangle$ s.t. $\left| \begin{pmatrix} a & \square \\ b & \triangle \end{pmatrix} \right| = 1$, i.e

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Solve a *Bezout* equation

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Use **Size-reduce** !

Generalized Euclidean algorithm

G-Euclide, Lift

Function ***G-Euclide***:

- **if** $K_h = \mathbb{Q}$ **then return** **ExGcd**(a, b);
- $\mu, \nu \leftarrow$ **G-Euclide**$\left(K_{h-1}^{\uparrow}, N_{K_h/K_{h-1}}(a), N_{K_h/K_{h-1}}(b)\right)$;
- $\mu', \nu' \leftarrow \mu\, a^{-1} N_{K_h/K_{h-1}}(a), \nu\, b^{-1} N_{K_h/K_{h-1}}(b)$;
- $W \leftarrow \begin{pmatrix} a & \nu' \\ b & \mu' \end{pmatrix}$;
- $V \leftarrow$ **Size-Reduce**(**Orthogonalize**(W)); **return** $W \cdot V[2]$

Function ***Lift***:

- $a, b \leftarrow$ **Ascend**$(K_h, U[1])$; $\mu, \nu \leftarrow$ **G-Euclide**$\left(K_{h-1}^{\uparrow}, a, b\right)$;
- $U \leftarrow \begin{pmatrix} a & \nu \\ b & \mu \end{pmatrix}$;
- **return** U

On the complexity of the reduction

Complexity [E-Kirchner-Fouque 2019]

Let f be a log-smooth integer. The complexity of the algorithm **Reduce** on rank two modules over $K = \mathbb{Q}[x]/\Phi_f(x)$, represented as a matrix M whose number of bits in the input coefficients is uniformly bounded by $B > n$, is *heuristically* a $\tilde{\mathrm{O}}\left(n^2 B\right)$ with $n = \varphi(f)$. The first column of the reduced matrix has its coefficients uniformly bounded by $2^{\tilde{\mathrm{O}}(n)} \operatorname{covol}(M)^{\frac{1}{2n}}$.

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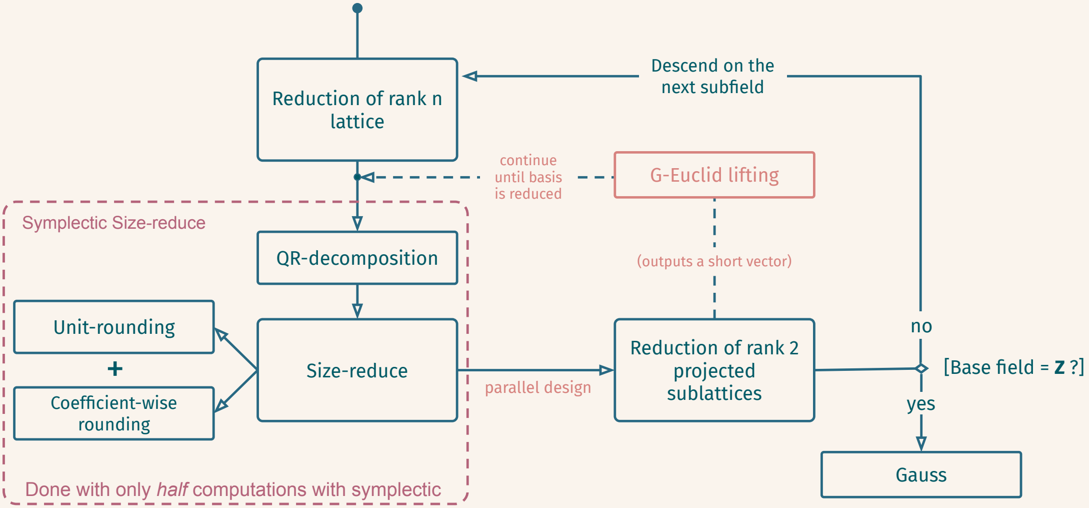
- Sum at each level to conclude on the complexity.

What if i want an actual SVP?

A reduction a la Lovasz

There exists a reduction to γ-HSVP over $\mathcal{O}_L$ from γ^2-SVP over $\mathcal{O}_L$ using at most $2\,\mathrm{rk}$ calls to the Hermite-SVP oracle.

Wait there is more !



Faster with symplectic symmetries

A nano-primer on symplectic geometry

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Euclidean space

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- Symmetric bilinear Form $\langle \cdot, \cdot \rangle$

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$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

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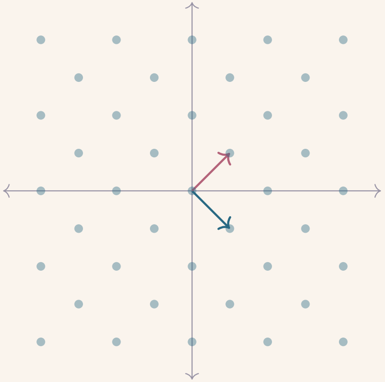
More on Darboux base

sorry, symplectic spaces are of even dimension, so the first non-trivial case is of dim 4... no cute drawing today!



Standard symplectic form

$\omega(u, u) = \det((u, u))$

- Darboux "elements" (u, v) goes by pair of Gram matrix $j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Note $j^2 = -\operatorname{Id}$.
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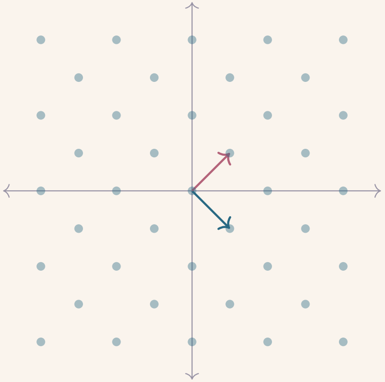

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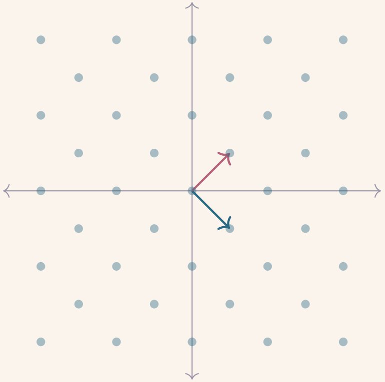

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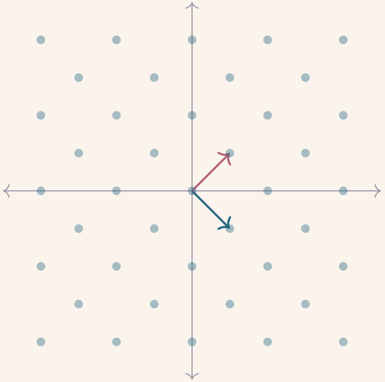

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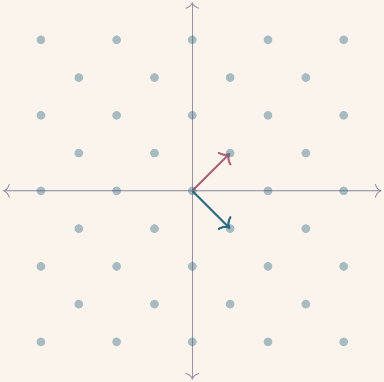

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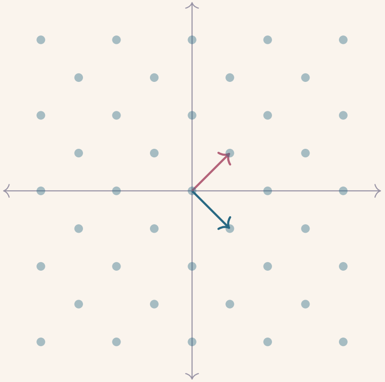

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4. Take a x_i such that $\omega_(x_3, x_4) \neq 0$, scale, etc.

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Symplectic structure is compatible
with the QR-decomposition
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- For any $A \in \mathrm{GL}_2(L)$ with $\det A = 1$

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Local reductions occurring during the reduction, swaps and transvections can preserve the J-symplectism.

Symplectic size-reduction

Symplectic-Size-Reduce

1 Set A, U such that $\begin{pmatrix} A & AU \\ 0 & A^{-s} \end{pmatrix} = R$;

2 $V \leftarrow$ **Size-Reduce**(A);

3 **return** $\begin{pmatrix} V & -V\lfloor U \rceil \\ 0 & V^{-s} \end{pmatrix}$

- R is upper-triangular, only depends on the first half of Q,
- We compute only the part above the antidiagonal of AU. Enough to compute the part above the antidiagonal of $A^{-1}(AU)$, which is persymmetric.

Symplectic bases $\Rightarrow$ Gram-Schmidt (Gram-Darboux) vectors are paired.

Size-reduction is cut in half ! (size-red half, apply blindly)

Speeding up lattice reduction with symplectic symmetries

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LLL can be made 2 times faster !

(thank you for attention, bye!)

Going further with symplectic symmetries in a recursive tower

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$J_h \in \bigwedge^2(K_h^2)$ is the determinant form:

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Symplectic group after descent

A $2d_h \times 2d_h$ matrix M' is symplectic if it preserves the J'_h form, that is if $J'_h \circ M' = J'_h$.

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A $2d_h \times 2d_h$ matrix M' is symplectic if it preserves the J'_h form, that is if $J'_h \circ M' = J'_h$.

Compatibility

Let M be a 2×2 matrix over K_h which is J_h-symplectic, then its descent $M' \in K_{h-1}^{2d_h \times 2d_h}$ is J'_h-symplectic.

Going further with symplectic symmetries in a recursive tower

$J_h \in \bigwedge^2(K_h^2)$ is the determinant form:

$$J_h\left(\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}\right) = x_0 y_1 - x_1 y_0$$

$\rightarrow$ Descend the form J_h in J'_h to K_{h-1} by composition with a non-trivial linear form $\tau : K_h \rightarrow K_{h-1}$

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Compatibility with decompositions

Fix a basis of the symplectic space where the matrix corresponding to J'_h is $\begin{pmatrix} 0 & R_{d_h} \\ -R_{d_h} & 0 \end{pmatrix}$.

Then, for any M a J'_h-symplectic matrix and QR its QR decomposition, both Q and R are J'_h-symplectic.

Improved complexity [E-Kirchner-Fouque 2019]

Select an integer f a power of $q = \mathrm{O}(\log f)$ and let $n = \varphi(f)$. The complexity for reducing matrices M of dimension two over $L = \mathbb{Q}[x]/\Phi_f(x)$ with B the number of bits in the input coefficients is *heuristically*

$$\tilde{\mathrm{O}}\left(n^{2+\varepsilon(q)}B\right) + n^{\mathrm{O}(\log\log n)}, \qquad \varepsilon(q) = \frac{\log(1/2 + 1/2q)}{\log q} < 0$$

and the first column of the reduced matrix has coefficients bounded by

$$2^{\tilde{\mathrm{O}}(n)}\left|N_{K_h/\mathbb{Q}}(\det M)\right|^{\frac{1}{2n}}.$$

From that preliminary work...

Exploitation of the symplectic symmetries:

1. **Decrease** the complexity, but...
2. **Increase** the approximation factor

From that preliminary work...

- Getting further using higher-order symplectic structures

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- 2-elements representation: multiplying $\mathfrak{a} = \alpha_1 \mathcal{O}_L + \alpha_2 \mathcal{O}_L, \mathfrak{b} = \beta_1 \mathcal{O}_L + \beta_2 \mathcal{O}_L$ consists in the reduction of the ideal generated by $(\alpha_i \beta_j)_{1 \leqslant i,j \leqslant 2}$ (module spanned by 4 elements)

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Cross recursive algorithms: reduction and ideal multiplication

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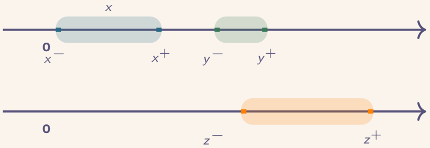
- Certification using *Interval arithmetic*



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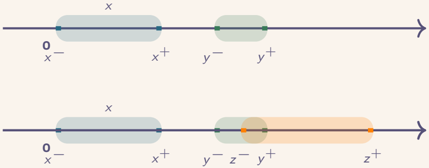

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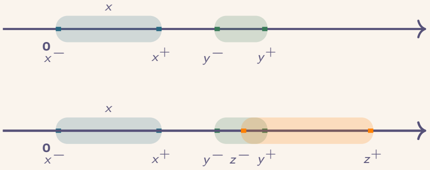

Interval Arithmetic used in such a way allows detecting lack of precision at runtime of a numerical algorithm.

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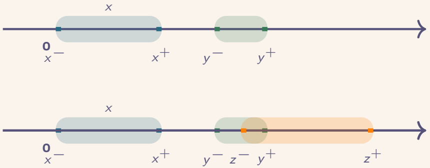

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$\longrightarrow$ Certified reduction + use quasi-optimal precision with adaptive strategy

Proved, *certified* and *efficient* framework
for reducing *general* algebraic lattices

Thank you !