Rank Analysis of Incomplete Block Designs: I. The Method of Paired Comparisons

Author(s): Ralph Allan Bradley and Milton E. Terry

Source: *Biometrika*, Dec., 1952, Vol. 39, No. 3/4 (Dec., 1952), pp. 324-345
Published by: Oxford University Press on behalf of Biometrika Trust
Stable URL: http://www.jstor.com/stable/2334029

RANK ANALYSIS OF INCOMPLETE BLOCK DESIGNS

I. THE METHOD OF PAIRED COMPARISONS

BY RALPH ALLAN BRADLEY AND MILTON E. TERRY*

Virginia Agricultural Experiment Station of the Virginia Polytechnic Institute

1. INTRODUCTION

The analysis of experiments involving paired comparisons has received considerable attention in statistical methodology. Thurstone (1927) has considered the problem on the assumptions that a linear variate is involved and that perceptible differences exist among the items presented for comparison. More recently, Mosteller (1951 a, b) has elaborated upon Thurstone's method and, having postulated a sensation continuum over which sensations are jointly normally distributed, has developed a χ^2 test following transformation of the observed variates.

Kendall & Babington Smith (1940) proposed a method of analysis for paired comparisons which does not depend on assumptions of a linear variate or of normality, and the procedure may be described as a combinatorial type test. They form a coefficient of agreement which essentially measures discrepancies from perfect agreement, although the model used in the test is not explicitly formulated. In subjective tests the consistency of a judge is measured in terms of circular triads. We note that tests of consistency and tests of agreement, when differences are known to exist, may also be considered to be tests of null hypotheses upon the postulation of absence of differences.

Guttman (1946) has developed a method of quantifying paired comparisons. His problem was to determine a numerical value for each of a number of items which will best represent the comparisons in some sense. The problem may be considered to be one of estimation as distinct from the problem of testing hypotheses.

When only two items are to be compared in a ranking experiment, a test of the hypothesis of no-difference between them on some characteristic may be based on the binomial distribution. The estimation of the probabilities that the items are superior in a given comparison may be accomplished, and these estimates afford a method of rating the items or a method of quantification. In the present paper a generalization of the binomial model and distribution is obtained.

A mathematical model is formulated and maximum-likelihood estimates of treatment ratings provide a simple solution to Guttman's problem of quantification. Likelihood ratio statistics are used for tests of a specified class of hypotheses. Although these tests basically agree with those of Kendall & Babington Smith, they subdivide the possible results from an experiment of a given size into more distinct subclasses, thus perhaps indicating better sensitivity.

The test procedure is flexible. In subjective testing the experimenter may assume *a priori* that the standards of judging are uniform or that they vary by judges, by time, or by both. That is, true treatment ratings may be considered to be constant throughout an

* This project was supported by funds from the Research and Marketing Act of 1946, under Contract No. A-1s-32683, with the Bureau of Agricultural Economics.

experiment or to be functions of judges and of time. In experiments concerned with the detection of treatment differences the latter alternatives are important. Certain test characteristics may be rather intangible and difficult to describe, with the added difficulty that personal preferences may influence judges' decisions. Even in simpler cases the research worker may desire to forgo the tedious procedure of training and co-ordinating judges. In these situations tests of treatment differences may be performed and a measure of the agreement among judges obtained, although estimates of over-all treatment ratings are not usually meaningful.

Tables for the test procedures for small treatment and sample sizes are provided and asymptotic distributions are considered. The use of the tables and the method of analysis are illustrated with an example of a taste-testing experiment on pork roasts from animals fed on one of three corn rations with peanut supplemements.

2. MATHEMATICAL MODEL

Let us consider t treatments in an experiment involving paired comparisons. We shall first consider that these treatments have true ratings (or preferences), $\pi_1, \ldots, \pi_t$, on a particular subjective continuum throughout an experiment. The continuum is specialized by the requirements that every $\pi_i \geqslant 0$ and that $\Sigma\pi_i = 1$, the latter condition being added for convenience. Further definition follows with the assumption that, when treatment i appears with treatment j in a block, the probability that treatment i obtains top rating (or a rank of 1) is $\pi_i/(\pi_i+\pi_j)$. Later generalization will require the addition of a second subscript on the parameters indicative of judges or time.

Now r_{ijk} will designate the rank of the ith treatment in the kth repetition of the block in which treatment i appears with treatment j. Clearly $r_{jik} = 3-r_{ijk}$. Estimates of $\pi_1, \ldots, \pi_t$ will be denoted by $p_1, \ldots, p_t$ respectively, and n will be reserved to denote the number of repetitions of the design when a repetition is defined to be a set of all pairs of treatments.

In certain cases, as noted above, repetitions of the design may be performed by different judges or at different times. We shall discuss, in a subsequent section, the analysis when true ratings $\pi_{1u}, \ldots, \pi_{tu}$ exist in the uth of g groups not necessarily identical from group to group.

3. THE LIKELIHOOD FUNCTION

We may now obtain the likelihood function, assuming probability independence between blocks or pairs of treatments. Consider the probability of the observed rankings in the kth repetition for the block in which treatments i and j are compared. The probability of the observed result is

$$\left(\frac{\pi_i}{\pi_i+\pi_j}\right)^{2-r_{ijk}} \left(\frac{\pi_j}{\pi_i+\pi_j}\right)^{2-r_{jik}} = \frac{\pi_i^{2-r_{ijk}}\pi_j^{2-r_{jik}}}{(\pi_i+\pi_j)}.$$

For if the ith treatment obtains top ranking, $r_{ijk} = 1$ and $r_{jik} = 2$, and the expression above becomes $\pi_i/(\pi_i+\pi_j)$; alternatively, $r_{ijk} = 2$, $r_{jik} = 1$, and the probability is $\pi_j/(\pi_i+\pi_j)$. Multiplying the appropriate expressions for all comparisons within a repetition and for all n repetitions, we reach the likelihood function in the general form

$$L = \prod_i \pi_i^{2n(t-1)-\sum\limits_{j\neq i}\sum\limits_k r_{ijk}} \prod_{i<j} (\pi_i+\pi_j)^{-n}. \tag{1}$$

When repetitions of the design are performed by groups with distinct parameters, the likelihood function will be a product over the groups of functions of the form of (1).

4. Likelihood ratio tests and estimation

A general class of tests of the null hypothesis,

$$H_0 : \pi_i = 1/t \quad (i = 1, \ldots, t), \tag{2}$$

against alternative hypotheses

$$H_a : \pi_i = \pi(h) \quad (h = 1, \ldots, m);$$
$$i = s_{h-1}+1, \ldots, s_h, \quad \text{where} \quad s_0 = 0,\ s_m = t \text{ and } \sum_h (s_h - s_{h-1})\,\pi(h) = 1, \tag{3}$$

are possible using likelihood ratio tests. That is, tests of the null hypothesis of identical treatment ratings may be performed when the alternative hypothesis specifies that the treatments have identical ratings within each of m groups of treatments while the groups themselves may differ. Alternative hypotheses involving only a subset of parameters do not lead to parameter-free tests. Two special cases of this general class of tests will be considered in the next section.

If $p(h)$ is the maximum-likelihood estimate of $\pi(h)$, these estimates are obtained from the equations

$$\left[\left\{2n(t-1)(s_h - s_{h-1}) - \sum_{i=s_{h-1}+1}^{s_h} \sum_{j\neq i}\sum_k r_{ijk} - \tfrac{1}{2}n(s_h - s_{h-1})(s_h - s_{h-1} - 1)\right\} \Big/ p(h)\right]$$
$$- n(s_h - s_{h-1}) \sum_{f \neq h} (s_f - s_{f-1})/\{p(h) + p(f)\} = 0 \quad (h = 1, \ldots, m), \tag{4}$$

and

$$\sum_h (s_h - s_{h-1})\, p(h) = 1. \tag{5}$$

Equations (4) are obtained from the reduction of the first-order maximizing conditions on the logarithm of the likelihood function when a Lagrange multiplier is used for the restraint on the parameters (3).

The general test statistic,* a monotone function of the likelihood ratio, is

$$B = n \sum_{h<f} (s_h - s_{h-1})(s_f - s_{f-1}) \log\{p(h) + p(f)\}$$
$$- \sum_h \left\{2n(t-1)(s_h - s_{h-1}) - \sum_{i=s_{h-1}+1}^{s_h} \sum_{j\neq i}\sum_k r_{ijk} - \tfrac{1}{2}n(s_h - s_{h-1})(s_h - s_{h-1} - 1)\right\} \log p(h). \tag{6}$$

B is implicitly a function of the treatment sums of ranks.

Solution of equations (4) and (5) provides estimates of the true treatment ratings. Pairwise comparison of these estimates provides a quantitative measure of the ratings of a pair of items relative to the test attribute.

The estimates p_i of π_i may be used for pairwise comparisons of treatments in the sense that the ratio p_i/p_j measures the relative frequency of occurrence of rank 1 for treatment i as compared with treatment j for this particular paired comparison. If the estimates are converted to logarithms, the values $\log p_i$ occur on a linear scale and permit over-all comparisons of the experimental treatments. Any consideration of differences among treatments should be based on the values of the $\log p_i$'s.

5. Special tests

Two special alternative hypotheses are of particular interest.

Case (i). H_1 : no π_i is assumed equal to any π_j ($i \neq j$). That is, in the general hypothesis (3), $m = t$.

* We use logarithms to base 10 unless otherwise specified.

In this case equations (4) and (5) are simplified and become

$$\frac{a_i}{p_i} - n \sum_{j \neq i} (p_i + p_j)^{-1} = 0 \quad (i, j = 1, \ldots, t), \tag{7}$$

and

$$\sum_i p_i = 1. \tag{8}$$

We define

$$a_i = 2n(t-1) - \sum_{j \neq i} \sum_{k=1}^{n} r_{ijk}. \tag{9}$$

Both equations (4) and (7) contain one degree of dependence and imply respectively equations (5) and (8).

The test statistic becomes

$$B_1 = n \sum_{i<j} \log (p_i + p_j) - \sum_i \left\{ 2n(t-1) - \sum_{j \neq i} \sum_{k=1}^{n} r_{ijk} \right\} \log p_i. \tag{10}$$

The preparation of tables for the exact distribution of B_1 is discussed in the following section.

Case (ii). $H_2: \pi_i = \pi \ (i = 1, \ldots, s)$; $\pi_i = \dfrac{1 - s\pi}{(t-s)} \ (i = s+1, \ldots, t)$. This is a reduction of the general hypothesis to the case in which $m = 2$.

This special test is similar to certain single degree of freedom comparisons in the analysis of variance. It is possible to compare two groups of items so long as all experimental items are included in one or another of the groups; however, it should be noted that one may always disregard all pairs in the experiment involving one or more extraneous items and proceed with tests based on comparisons within any subgroup of items.

For the special case (ii), the maximum-likelihood equations (4) and (5) may be solved simply and the test statistic written as an explicit function of treatment sums of ranks. When p is the estimate of π, we have

$$p = \frac{ns(4t - s - 3) - 2 \sum_{i=1}^{s} \sum_{j \neq i} \sum_k r_{ijk}}{ns(5st - 2t^2 - 6s + 3t) - 2(2s - t) \sum_{i=1}^{s} \sum_{j \neq i} \sum_k r_{ijk}}, \tag{11}$$

and the statistic, by substitution in (6), is

$$\begin{aligned} B_2 = {} & \left\{ \sum_{i=1}^{s} \sum_{j \neq i} \sum_k r_{ijk} + \tfrac{1}{2} ns(s-1) - 2n(t-1)s \right\} \log \left\{ \frac{(t-s)p}{(t-2s)p + 1} \right\} \\ & + \left\{ \sum_{i=s+1}^{t} \sum_{j \neq i} \sum_k r_{ijk} + \tfrac{1}{2} n(t-s)(t-s-1) - 2n(t-1)(t-s) \right\} \log \left\{ \frac{1 - sp}{(t-2s)p + 1} \right\}. \end{aligned} \tag{12}$$

A discussion of the distribution of B_2 is included with that of the distribution of B_1.

6. TABLES FOR B_1 AND B_2

It is possible to generate all combinations of treatment sums of ranks for any given number of treatments and repetitions of the paired comparison design. The probability of each such combination may be obtained under the null hypothesis of equality of true treatment ratings.

If three items are compared in a single repetition, the possible sets of rank sums are 2, 3, 4 and 3, 3, 3. Each of the six permutations of the elements of the first set has a probability 1/8, while the probability of the second set is 2/8. The treatment sums of ranks for two repetitions and three treatments are obtained by adding 2, 3, 4 and 3, 3, 3 in turn to corresponding elements in the sets of sums of ranks consisting of all permutations of 2, 3, 4 and to 3, 3, 3. In the sets of sums of ranks so produced, all permutations of a given set of treatment sums of ranks are taken to be equivalent. The probability of a given permutation is obtained by multiplying the basic probabilities of the combination and the permutation used to produce the given permutation. The probability of a given new combination of rank sums is obtained by adding the probabilities obtained for each permutation of the elements of the combination.

The procedure may be arranged systematically as shown in Table 1.

Table 1. *The generation of treatment sums of ranks and probabilities for three treatments and two repetitions*

Probabilities		1/8	1/8	1/8	1/8	1/8	1/8	2/8
	Rank sums	2, 3, 4	2, 4, 3	3, 2, 4	3, 4, 2	4, 2, 3	4, 3, 2	3, 3, 3
6/8	2, 3, 4	4, 6, 8	4, 7, 7	5, 5, 8	5, 7, 6	6, 5, 7	6, 6, 6	5, 6, 7
2/8	3, 3, 3	5, 6, 7	5, 7, 6	6, 5, 7	6, 7, 5	7, 5, 6	7, 6, 5	6, 6, 6

The combination 5, 6, 7, say, appears in its various permutations in nine places in this table. In row 1, column 4, for example, 5, 7, 6 appears and its probability is 6/64 obtained by multiplying marginal probabilities of row and column. The probability of the combination 5, 6, 7 is then the sum of the nine individual probabilities and has the value 36/64. When three repetitions with three treatments are considered, the generating rows at the top of the table are unchanged, but the columns at the left above are replaced by the possible combinations of sums of ranks obtained for two repetitions with their corresponding probabilities. This procedure is continued for larger numbers of treatments and repetitions.

When the sets of possible combinations of treatment sums of ranks are obtained with their probabilities of occurrence, for each such set we may substitute in equations (7), (8) and (9) and obtain estimates $p_1, \ldots, p_t$. The solution of these equations is tedious; in some cases elementary methods are applicable, in others it is necessary to use repeated approximations in an iterative procedure. In the later work, many of the procedures have been programmed on I.B.M. equipment. When we substitute $p_1, \ldots, p_t$ in (10), the statistic B_1 is evaluated.

Tables for the distribution of B_1 for three and four items and up to ten repetitions of the design are given in Appendix A. The possible sets of treatment sums of ranks are given in the left-hand columns. Corresponding estimates, $p_1, \ldots, p_t$, are then given with the value of B_1. The final column shows significance levels, P, in the form of cumulative probabilities. These probabilities are obtained from the individual probabilities of the possible sets, accumulated beginning with small values of B_1 which are most discordant under the null hypothesis.

The distribution of the statistic B_2 may be recovered from tables for B_1. When t and n are specified, it is easy to compute values of p and B_2 using (11) and (12). Probabilities may be obtained by elementary considerations.

B_2 is no longer symmetric over the treatments, and certain permutations of treatment sums of ranks must be considered for each entry in the tables of Appendix A. However, each such permutation is equally likely, and its probability may be obtained from the cumulative probabilities for B_1. (Sets grouped together with equal values of B_1 always have equal probabilities.)

For any hypothesis for which B_2 is the appropriate statistic it is possible to evaluate p, B_2 and the corresponding probability of each value of B_2 as obtained from the distribution of B_1. Tables for the distribution of B_2 will be prepared at a later date and are not presently available.

7. THE COMBINATION OF EXPERIMENTS

As noted in the introduction, it may happen that an experiment is performed in groups* of repetitions of sizes, n_u $(u = 1, \ldots, g)$, with $\sum\limits_1^g n_u = n$. Two possible methods of performing an over-all test of significance are available and depend on the specification of the alternative hypotheses. We shall illustrate these methods with reference to the important special test (i), noting that similar procedures may be developed for all tests of the general form specified in § 4.

(i) *Pooled analysis*

If an experimenter is willing to assume that true treatment ratings, $\pi_1, \ldots, \pi_t$, exist as the alternative hypothesis for all groups of repetitions, no new analysis is required. Total treatment sums of ranks are obtained by addition of corresponding group treatment sums of ranks over the g groups. The experiment is treated as though one group of n repetitions of the design had been employed and the tables of Appendix A may be used.

(ii) *Combined analysis*

In many cases the alternative hypothesis that the same true ratings exist for all groups is not realistic. If the detection of treatment differences is the main concern of the experimenter, a pooled analysis may be inappropriate and even give a non-significant result, while each group alone exhibits significant treatment differences. This is particularly likely to happen where judge preferences may prohibit the setting up of uniform ranking criteria.

Let us specify an alternative hypothesis as follows:

(*a*) Within the uth of g groups, true ratings $\pi_{1u}, \ldots, \pi_{tu}, \sum\limits_{i=1}^{t} \pi_{iu} = 1$ exist, and these ratings may change from group to group.

(*b*) Group experiments are independent in probability. Then, in addition, we define B_1^u to be the likelihood ratio statistic corresponding to B_1 (6) for the uth group $(u = 1, \ldots, g)$. The statistics B_1^u are self-weighting. That is, the groups may be combined and an over-all test of significance performed depending on a statistic

$$B_1^c = \sum_{u=1}^{g} B_1^u. \tag{13}$$

This statistic is again a monotone function of the likelihood ratio and does not depend on values of n_u other than in the evaluation of B_1^u.

Tables for the distribution of B_1^c are discussed in the following section.

One note should be added. The decision to pool or combine group results should be made from *a priori* knowledge of group behaviour. When group data may be pooled, it is possible

* These groups may represent judges in sensory difference experimentation, different localities, days, or non-treatment experimental techniques.

that the pooled value of B_1 exhibits higher significance than the corresponding value of B_1^c. However, it is easy to show that

$$B_1 \geqslant B_1^c \tag{14}$$

in every situation.

Estimates of the parameters, $p_1, \ldots, p_t$, should usually be obtained by groups when groups are combined, but over-all estimates are available when groups are pooled.

8. Tables for B_1^c

The probability of a specified value of B_1^c may be simply obtained by elementary probability. Suppose values of n_u are equal in sets of sizes $g_1, \ldots, g_w$, $\sum_1^w g_i = g$, and that values of B_1^u within these sets are equal in subsets of sizes $g_{i1}, \ldots, g_{iw_i}$, $\sum_{j=1}^{w_i} g_{ij} = g_i$ $(i = 1, \ldots, w)$. The probability of a specified value of B_1^c is

$$P(B_1^c) = \prod_{i=1}^{w} g_i! \left\{ \prod_{j=1}^{w_i} g_{ij}! \right\}^{-1} \prod_{u=1}^{g} P(B_1^u). \tag{15}$$

Values of B_1^u and $P(B_1^u)$ may be obtained from the table of Appendix A. B_1^c is calculated by addition as in (13), and its probability is evaluated by use of (15).

Using the results above, we have computed tables for B_1^c for certain experiments wherein there are equal numbers of repetitions in each group. Only values at approximately the 0·10 level of significance or higher have been recorded, and these are selected for easy interpolation. These tables are shown in Appendix B.

9. A coefficient of agreement

A measure of consistency of ranking from group to group is naturally provided by the difference between the pooled value of B_1 and B_1^c. Small values of $B_1 - B_1^c$ (note that $B_1 - B_1^c \geqslant 0$) will exhibit good agreement in ranking from group to group, while large values indicate discordant rankings.

If we set up the hypotheses

$$\left.\begin{aligned} &H_0 : \pi_{iu} = \pi_i \quad (u = 1, \ldots, g;\ i = 1, \ldots, t) \\ \text{and}\quad &H_1 : \pi_{iu} \quad\quad (u = 1, \ldots, g;\ i = 1, \ldots, t) \text{ unrestricted by groups,} \end{aligned}\right\} \tag{16}$$

then

$$-2 \log_e \lambda = 2(B_1 - B_1^c) \log_e 10, \tag{17}$$

where λ is the likelihood ratio statistic for comparison of H_0 and H_1. $B_1 - B_1^c$ is then a monotone function of the likelihood ratio statistic.

The distribution of $B_1 - B_1^c$ for small samples will depend on parameters $\pi_1, \ldots, \pi_t$ under H_0 and is therefore not a parameter-free test. A conditional test, which is exact, may be formed and has some value. Suppose B_1^c is fixed at the observed value. Corresponding to B_1^c we have group sums of ranks. If there is no agreement from group to group, all permutations of group sums of ranks are equally likely, and for each permutation a pooled value of B_1, and consequently the difference $B_1 - B_1^c$, may be obtained. Thus for fixed B_1^c the distribution of $B_1 - B_1^c$ can be derived under an assumption of no agreement from group to group. This conditional test reverses the hypotheses (16), and small values of $B_1 - B_1^c$ show significant agreement from group to group.

A large sample test of the hypotheses of (16) is discussed in the following section.

10. LARGE SAMPLE DISTRIBUTIONS

If λ is the likelihood ratio, it is known (Wilks, 1946, pp. 150–2, § 7·2) that $-2\log_e\lambda$ is distributed as χ^2 under very general conditions. This result can be employed in the special cases considered above.

Case (i). In the first special test (§ 5)

$$-2\log_e\lambda_1 = nt(t-1)\log_e 2 - 2B_1\log_e 10 \tag{18}$$

is distributed in the limit as χ^2_{t-1}, i.e. as χ^2 with $t-1$ degrees of freedom. (It has been noted that B_1 as tabled is a linear function of logarithms to base 10.)

The authors have been unsuccessful in an attempt to evaluate the moments of $-2\log_e\lambda_1$ by theoretical methods. However, numerical values of the mean and variance of this statistic have been computed for small numbers of items and repetitions. These are given in Table 2.

Table 2. *Mean and variance of* $-2\log_e\lambda_1$

n	$t=3$		$t=4$	
	Mean	Variance	Mean	Variance
1	3·12	3·24	4·55	9·96
2	3·39	7·27	3·59	9·51
3	2·80	7·50	3·33	7·80
4	2·54	6·51	3·22	7·10
5	2·40	5·83	3·13	6·66
6	2·32	5·38	—	—
7	2·27	5·15	—	—
8	2·23	4·95	—	—
9	2·20	4·82	—	—
10	2·18	4·71	—	—
∞	2·00	4·00	3·00	6·00

It may be observed that even for these numbers of items and repetitions there is definite evidence of rapid convergence to the limiting values for the means and somewhat slower convergence for the variances. For small samples, on the average $-2\log_e\lambda_1$ will be a little too large, and use of the large sample approximation will tend to lead to the announcement of too many significant results. The approximation appears to be fairly good for practical purposes if the number of repetitions is not too small (say $n \geqslant 15$).

We may note that the computations are fairly difficult if the approximate test must be used. To compute B_1 it is necessary to solve equations (7) and (8) and substitute in the formula (10) for B_1. The equations are most easily solved by obtaining a first approximation by comparison with available tables (multiples of $\Sigma r_1, \ldots, \Sigma r_t$ yield identical estimates $p_1, \ldots, p_t$) and using an iterative procedure. The iterations consist of obtaining second approximations such as

$$p_i^{(1)} = a_i\left\{\frac{n}{p_1^{(1)}+p_i^{(0)}} + \ldots + \frac{n}{p_{i-1}^{(1)}+p_i^{(0)}} + \frac{n}{p_i^{(0)}+p_{i+1}^{(0)}} + \ldots + \frac{n}{p_i^{(0)}+p_t^{(0)}}\right\}^{-1},$$

where the superscript in parentheses indicates the order of iteration.

For the combined test, from the additive property of χ^2, the limiting distribution of

$$\begin{aligned} -2\log_e \lambda_1^c &= -2\sum_{u=1}^{g} \log_e \lambda_1^u \\ &= nt(t-1)\log_e 2 - 2B_1^c \log_e 10 \end{aligned} \tag{19}$$

is that of χ^2 with $g(t-1)$ degrees of freedom. The notation in (19) corresponds to that of § 8 (*b*).

If we consider the test specification (16), a parameter-free test of agreement may be formed for the large sample distribution. It follows that $2(B_1 - B_1^c)\log_e 10$ has the χ^2-distribution with $(g-1)(t-1)$ degrees of freedom in the limit. Large values of this statistic show discordant ranking from group to group.

The large sample test may be summarized as in Table 3.

Table 3. *Large sample analysis*

(Note that $\log_e 2 = 0{\cdot}69315$ and $2\log_e 10 = 4{\cdot}60518$)

Statistic	Hypotheses	Limiting distribution
$nt(t-1)\log_e 2 - 2B_1 \log_e 10$	$\begin{cases} H_0: \pi_i = 1/t \\ H_1: \pi_i \end{cases}$	χ^2_{t-1}
$2(B_1 - B_1^c)\log_e 10$	$\begin{cases} H_0: \pi_{iu} = \pi_i \\ H_1: \pi_{iu} \end{cases}$	$\chi^2_{(g-1)(t-1)}$
$nt(t-1)\log_e 2 - 2B_1^c \log_e 10$	$\begin{cases} H_0: \pi_{iu} = 1/t \\ H_1: \pi_{iu} \end{cases}$	$\chi^2_{g(t-1)}$

Case (*ii*). In the second special test, the statistic

$$-2\log_e \lambda_2 = 2ns(t-s)\log_e 2 - 2B_2 \log_e 10, \tag{20}$$

has in the limit the distribution of χ^2 with one degree of freedom.

11. The experimental procedure and analysis illustrated*

In a recent taste-testing experiment, pork roasts were compared by ranking in pairs on their flavour characteristics. The roasts were obtained from three groups of hogs which had been fattened on three different rations: corn (maize), corn plus a peanut supplement, and corn plus a large peanut supplement. The object was to determine whether the addition of peanuts to the diet was recognizable in the fresh-pork roasts or not. One would like to ask expert judges to rank pairs on the basis of flavour attributable to the peanut diet; however, this characteristic proved too intangible to define, and each judge was asked to rank pairs on the basis of his own preferences. This leads *a priori* to a combined analysis (§ 8) for the experimental data.

When a new procedure is proposed, it is useful for applied work to show a systematic listing of the steps involved. We now indicate these steps with reference to the results of

* The illustrative example is taken from preliminary experimental results of L. L. Davis, C. M. Kincaid and H. R. Thomas at the Virginia Agricultural Experiment Station.

two of the several judges used in the experiment described above. Each judge performed five repetitions of the paired design ($t = 3$, $n = 5$).

Procedure

Step 1 (experimental). A competent panel of judges was selected and so instructed that they all had experience with the experimental material.

Step 2. For each judge and for each repetition six small containers were coded. Two samples from roasts from each of the three treatment groups of animals were placed in the containers and the three requisite pairs formed. Code numbers were recorded and the pairs presented to the judges in a random order together with score cards.

Step 3. For each pair a judge tasted each sample and recorded the value 1 for the sample preferred and 2 for the other sample.

Step 4 (analysis). The experimenter collected and decoded the data for each judge and recorded the results as follows. C denotes the corn ration, Cp the corn plus peanut supplement ration, and CP the corn plus large peanut supplement ration. The treatment sums of ranks, Σr_i, for C, Cp, CP are respectively 19, 13, 13 and 13, 15, 17 for the two judges.

Table 4. *Rankings for two judges in the pork experiment*

Repetition ...	1			2			3			4			5		
	C	Cp	CP	C	Cp	CP	C	Cp	CP	C	Cp	CP	C	Cp	CP
Pair	Judge 1														
C, Cp	2	1	—	2	1	—	2	1	—	2	1	—	2	1	—
C, CP	2	—	1	1	—	2	2	—	1	2	—	1	2	—	1
Cp, CP	—	2	1	—	1	2	—	2	1	—	1	2	—	2	1
	Judge 2														
C, Cp	2	1	—	2	1	—	1	2	—	1	2	—	1	2	—
C, CP	1	—	2	1	—	2	1	—	2	1	—	2	2	—	1
Cp, CP	—	1	2	—	1	2	—	2	1	—	2	1	—	1	2

Step 5. Since it was agreed that the results of the two judges should be combined, we enter the table of Appendix A at $n = 5$. For judge 1, $p_C = 0{\cdot}05$, $p_{Cp} = 0{\cdot}47$, $p_{CP} = 0{\cdot}47$, $B_1 = 2{\cdot}917$, the significance level is $0{\cdot}057$; for judge 2, $p_C = 0{\cdot}53$, $p_{Cp} = 0{\cdot}30$, $p_{CP} = 0{\cdot}17$, $B_1 = 4{\cdot}034$ and the significance level is $0{\cdot}404$.

Step 6. The combined statistic B_1^c of equation (13) was obtained and has the value $2{\cdot}917 + 4{\cdot}034 = 6{\cdot}951$. From the table of Appendix B under the two equal groups, $n = 10$, the significance level for the combined test was found to be $0{\cdot}069$. It was concluded that it had not been demonstrated that ration differences detectable by these judges were present at any usual significance level.

Step 6*a*. If a decision to pool the data had been made, treatment sums of ranks added over the judges would have been 32, 28, 30, and the table of Appendix A would have been used for $n = 10$. We would have found $p_C = 0{\cdot}24$, $p_{Cp} = 0{\cdot}43$, $p_{CP} = 0{\cdot}32$, $B_1 = 8{\cdot}797$ and the significance level would have been 0·630. Since it seems extremely unlikely on the basis of this method that treatment differences are present, it is not here meaningful to compare treatments by use of their estimated ratings.

$B_1 - B_1^c = 8{\cdot}7973 - 6{\cdot}9516 = 1{\cdot}8459$ and is indicative of poor agreement of the preferences of the two judges. In fact use of the large sample approximation (Table 3) gives $\chi^2 = 8{\cdot}50$ with 2 degrees of freedom, a result significant at the 2 % level.

12. Discussion

The authors have not attempted to obtain the power of this rank-order test procedure. The method is clear, but any consideration of exact power would require tables for each specified set of parameters of the alternative hypotheses of substantially greater complexity than those for the null distributions. In addition, the simplifications due to symmetry over treatments in certain null cases would disappear. The merits of the test procedure are then dependent on the properties of the maximum-likelihood methods used.

Experiments using the above methods at the Virginia Polytechnic Institute and elsewhere have been satisfactorily conducted. The simplicity and appropriateness of the experimental design, together with the simplicity of the analysis, wherein one has only to add small integers and consult prepared tables, appear to be important factors in the appeal of the methods. The comprehensive tables already prepared are easy to read, and the extension of these tables is proceeding. New computing equipment is expected to speed the tabling work.

One of the questions asked in connexion with this work pertains to the possibility of extending the analysis to incomplete block designs with larger block sizes. We are proceeding with a consideration of such extensions. The method of paired comparisons becomes inefficient where it is possible to rank more than two treatments at a time and where more than a few items or treatments are considered.

Apart from the application of the theoretical considerations for the methods of this paper, it is to be observed that the probabilities of the tables of Appendix A may be useful elsewhere. Whenever ranking methods are used in incomplete blocks of size 2, tests of null hypotheses of treatment equality will depend on the probabilities tabled. The probabilities of individual sets of treatment rank sums may be recovered from the cumulative probabilities given, since all sets bracketed together have equal individual probabilities (that is, sets of rank sums with identical values of B_1 also have identical probabilities). Publication of the totality of possible sets of sums of ranks in Appendix A is necessary for use with Appendix B, and further desirable in that they may form a basis for future tables for methods yet to be devised.

13. Summary

A method of analysis of paired comparisons is provided which permits tests of hypotheses of a general class and the estimation of treatment ratings or preferences. The mathematical model developed is simple and easy to interpret and apply. Ranks are used in incomplete blocks of size 2, and such ranking will permit later generalization to larger block sizes. The method of maximum likelihood is employed and tests depend on the likelihood ratio statistics. Two special tests are featured and test the null hypothesis that true treatment ratings are equal. The alternative hypothesis (i) makes no assumptions of equality of

treatment ratings and (ii) makes the assumption that there are only two groups of treatments wherein within group treatments do not differ in ratings but the two groups themselves may have different ratings.

The procedures shown are applicable in most problems where qualitative measurements alone are reliable and are particularly useful in problems involving subjective ranking by a small panel of judges for the detection of differences. Methods of pooling and of combining the results of several judges are given. The method of combining permits an over-all test of significance without the usual assumption that members of a panel agree upon the nature of the differences to be detected. The decision to pool or to combine is made on the basis of *a priori* knowledge of judge behaviour. If results are combined, estimates of treatment ratings are usually obtained for judges individually, although average estimates for the group of judges may be obtained by reverting to the pooled analysis for this special purpose. An example from taste testing is given.

The large sample distributions of the statistics are discussed, and tables for the exact test procedures are shown in the two appendices following.

In conclusion, the authors would like to express their appreciation to Prof. Lyle L. Davis, food technologist, for advice on experimental techniques and for trial experimentation. We would also acknowledge the computational and clerical assistance of Mr A. F. Teske, Mrs T. S. Russell, Mrs F. A. Spracher, Mrs M. H. Kirkpatrick and Mrs A. L. Ruiz.

REFERENCES

COCHRAN, W. G. & COX, GERTRUDE (1950). *Experimental Designs*. New York: John Wiley and Sons, Inc.

FISHER, R. A. & YATES, F. (1948). *Statistical Tables for Biological, Agricultural and Medical Research*, 3rd ed. Edinburgh: Oliver and Boyd.

GUTTMAN, LOUIS (1946). An approach for quantifying paired comparisons and rank order. *Ann. Math. Statist.* **17**, 144.

KENDALL, M. G. & BABINGTON SMITH, B. (1940). On the method of paired comparisons. *Biometrika*, **31**, 324.

MOSTELLER, FREDERICK (1951*a*). Remarks on the method of paired comparisons. I. The least squares solution assuming equal standard deviations and equal correlations. *Psychometrika*, **16**, 3.

MOSTELLER, FREDERICK (1951*b*). Remarks on the method of paired comparisons. II. The effect of an aberrant standard deviation when equal standard deviations and equal correlations are assumed. *Psychometrika*, **16**, 203.

THURSTONE, L. L. (1927). Psychophysical analysis. *Amer. J. Psychol.* **38**, 368.

THURSTONE, L. L. (1945). The prediction of choice. *Psychometrika*, **10**, 237.

WILKS, S. S. (1946). *Mathematical Statistics*. Princeton University Press.

TABLES FOR THE RANK ANALYSIS OF INCOMPLETE BLOCK DESIGNS

APPENDIX A. *The distribution of the likelihood ratio for general alternatives*

The following table gives the values of the likelihood ratio statistic, B_1, and the likelihood estimates of the true treatment ratings, $p_1, \ldots, p_t$, together with probabilities, P, that B_1 will not be exceeded if the null hypothesis is true. Since low values of B_1 indicate discordant results, P gives the significance level.

n is the number of repetitions of the design and t the number of treatments. The design symbols, t or v, λ, b, r, k are standard and as used, for example, by Fisher & Yates (1948, p. 17, Introduction to Table XVIII) and Cochran & Cox (1950, pp. 270 and 304). λ in this design description should not be confused with the same symbol generally used to indicate a likelihood ratio. Parentheses contain combinations with equal values of B_1. Σr_i is the sum of ranks for treatment i.

In setting up the table, several conventions have been adopted to simplify the printing: (i) Where p_1 is unity and the remaining probabilities are therefore all zero the result is given as 1 —— or 1 ———. (ii) The lowest value of B_1 possible for each n is zero and is printed as 0. (iii) Where there are no entries in the final column above a single entry of ·0000, the corresponding values of P are less than a half unit in the fourth decimal place.

3 *treatments.* (*Design:* $t = 3$, $\lambda = 1$, $b = 3$, $r = 2$, $k = 2$)

Σr_1	Σr_2	Σr_3	p_1	p_2	p_3	B_1	P
			$n = 1$				
2	3	4	1	—	—	0	·7500
3	3	3	·33	·33	·33	0·903	1·0000
			$n = 2$				
4	6	8	1	—	—	0	·0938
4	7	7	1	—	—	0·602	·2812
5	5	8	·50	·50	—		
5	6	7	·59	·28	·13	1·498	·8438
6	6	6	·33	·33	·33	1·806	1·0000
			$n = 3$				
6	9	12	1	—	—	0	·0117
6	10	11	1	—	—	0·829	·0820
7	8	12	·67	·33	—		
7	9	11	·70	·22	·07	1·840	·2226
7	10	10	·71	·14	·14	2·077	·4336
8	8	11	·45	·45	·09		
8	9	10	·50	·31	·19	2·511	·8906
9	9	9	·33	·33	·33	2·709	1·0000

Σr_1	Σr_2	Σr_3	p_1	p_2	p_3	B_1	P
			$n = 4$				
8	12	16	1	—	—	0	·0015
8	13	15	1	—	—	0·977	·0132
9	11	16	·75	·25	—		
8	14	14	1	—	—	1·204	·0220
10	10	16	·50	·50	—		
9	12	15	·77	·19	·04	2·084	·0513
9	13	14	·78	·13	·09	2·468	·1684
10	11	15	·56	·37	·06		
10	12	14	·59	·28	·13	2·997	·3237
10	13	13	·60	·20	·20	3·158	·5347
11	11	14	·43	·43	·14		
11	12	13	·45	·32	·23	3·466	·9155
12	12	12	·33	·33	·33	3·614	1·0000
			$n = 5$				
10	15	20	1	—	—	0	·0002
10	16	19	1	—	—	1·087	·0020
11	14	20	·80	·20	—		
10	17	18	1	—	—	1·461	·0057
12	13	20	·60	·40	—		

Σr_1	Σr_2	Σr_3	p_1	p_2	p_3	B_1	P
				$n = 5$ (*cont.*)			
11	15	19	·81	·16	·03	2·274	·0112
11	16	18	·82	·12	·06	2·767	·0386
12	14	19	·64	·32	·05		
11	17	17	·82	·09	·09	2·917	·0569
13	13	19	·47	·47	·05		
12	15	18	·66	·25	·09	3·372	·1000
12	16	17	·66	·19	·14	3·645	·2464
13	14	18	·51	·38	·10		
13	15	17	·53	·30	·17	4·034	·4039
13	16	16	·54	·23	·23	4·158	·6053
14	14	17	·41	·41	·18		
14	15	16	·43	·32	·25	4·399	·9313
15	15	15	·33	·33	·33	4·515	1·0000
				$n = 6$			
12	18	24	1	—	—	0	·0000
12	19	23	1	—	—	1·174	·0003
13	17	24	·83	·17	—		
12	20	22	1	—	—	1·659	·0010
14	16	24	·67	·33	—		
12	21	21	1	—	—	1·806	·0014
15	15	24	·50	·50	—		
13	18	23	·84	·14	·02	2·431	·0024
13	19	22	·84	·11	·04	3·009	·0082
14	17	23	·69	·27	·04		
13	20	21	·85	·09	·06	3·270	·0178
15	16	23	·55	·41	·04		
14	18	22	·70	·22	·07	3·680	·0282
14	19	21	·71	·18	·11	4·040	·0708
15	17	22	·58	·34	·09		
14	20	20	·71	·14	·14	4·154	·0976
16	16	22	·45	·45	·09		
15	18	21	·59	·28	·13	4·496	·1504
15	19	20	·60	·22	·18	4·710	·3139
16	17	21	·48	·38	·14		
16	18	20	·50	·31	·19	5·022	·4680
16	19	19	·50	·25	·25	5·123	·6575
17	17	20	·40	·40	·20		
17	18	19	·41	·33	·26	5·321	·9421
18	18	18	·33	·33	·33	5·418	1·0000
				$n = 7$			
14	21	28	1	—	—	0	
14	22	27	1	—	—	1·247	·0000
15	20	28	·86	·14	—		
14	23	26	1	—	—	1·819	·0002
16	19	28	·71	·29	—		
				$n = 7$ (*cont.*)			
14	24	25	1	—	—	2·076	·0004
17	18	28	·57	·43	—		
15	21	27	·86	·12	·02	2·563	·0005
15	22	26	·86	·10	·03	3·213	·0016
16	20	27	·73	·24	·03		
15	23	25	·87	·08	·05	3·561	·0039
17	19	27	·60	·36	·03		
15	24	24	·87	·07	·07	3·672	·0053
18	18	27	·48	·48	·04		
16	21	26	·74	·20	·06	3·942	·0076
16	22	25	·75	·17	·08	4·372	·0185
17	20	26	·62	·31	·07		
16	23	24	·75	·14	·11	4·576	·0351
18	19	26	·51	·41	·08		
17	21	25	·64	·26	·10	4·884	·0504
17	22	24	·65	·22	·14	5·173	·1060
18	20	25	·54	·35	·12		
17	23	23	·65	·18	·18	5·266	·1396
19	19	25	·44	·44	·12		
18	21	24	·55	·30	·16	5·544	·1992
18	22	23	·55	·25	·20	5·723	·3716
19	20	24	·46	·37	·17		
19	21	23	·47	·32	·21	5·984	·5201
19	22	22	·47	·26	·26	6·070	·6976
20	20	23	·39	·39	·22		
20	21	22	·40	·33	·27	6·238	·9500
21	21	21	·33	·33	·33	6·322	1·0000
				$n = 8$			
16	24	32	1	—	—	0	
16	25	31	1	—	—	1·309	
17	23	32	·87	·13	—		
16	26	30	1	—	—	1·954	·0000
18	22	32	·75	·25	—		
16	27	29	1	—	—	2·298	·0001
19	21	32	·62	·38	—		
16	28	28	1	—	—	2·408	·0001
20	20	32	·50	·50	—		
17	24	31	·88	·11	·01	2·678	·0001
17	25	30	·88	·09	·03	3·390	·0003
18	23	31	·76	·21	·02		
17	26	29	·88	·08	·04	3·810	·0008
19	22	31	·65	·32	·03		
17	27	28	·88	·07	·05	4·008	·0015
20	21	31	·54	·43	·03		

Σr_1	Σr_2	Σr_3	p_1	p_2	p_3	B_1	P
			$n=8$ (*cont.*)				
18	24	30	·77	·19	·04	4·168	·0020
18	25	29	·77	·16	·07	4·660	·0046
19	23	30	·66	·28	·06		
18	26	28	·78	·13	·09	4·937	·0091
20	22	30	·56	·37	·06		
18	27	27	·78	·11	·11	5·026	·0118
21	21	30	·47	·47	·07		
19	24	29	·67	·24	·08	5·222	·0158
19	25	28	·68	·21	·11	5·572	·0324
20	23	29	·58	·32	·10		
19	26	27	·68	·17	·14	5·741	·0560
21	22	29	·49	·41	·10		
20	24	28	·59	·28	·13	5·994	·0758
20	25	27	·60	·24	·16	6·236	·1419
21	23	28	·51	·35	·14		
20	26	26	·60	·20	·20	6·316	·1808
22	22	28	·43	·43	·14		
21	24	27	·52	·30	·18	6·552	·2447
21	25	26	·52	·26	·22	6·705	·4211
22	23	27	·44	·37	·18		
22	24	26	·45	·32	·23	6·931	·5631
22	25	25	·45	·27	·27	7·005	·7293
23	23	26	·38	·38	·23		
23	24	25	·39	·33	·28	7·152	·9559
24	24	24	·33	·33	·33	7·225	1·0000
			$n=9$				
18	27	36	1	—	—	0	
18	28	35	1	—	—	1·363	
19	26	36	·89	·11	—		
18	29	34	1	—	—	2·070	
20	25	36	·78	·22	—		
18	30	33	1	—	—	2·488	
21	24	36	·67	·33	—		
18	31	32	1	—	—	2·685	
22	23	36	·56	·44	—		
19	27	35	·89	·10	·01	2·780	·0000
19	28	34	·89	·09	·02	3·545	·0001
20	26	35	·79	·19	·02		
19	29	33	·89	·07	·03	4·027	·0002
21	25	35	·68	·29	·02		
19	30	32	·89	·06	·04	4·299	·0003
22	24	35	·58	·39	·03		
20	27	34	·79	·17	·04	4·369	·0004
19	31	31	·89	·05	·05	4·386	·0005
23	23	35	·48	·48	·03		
20	28	33	·80	·15	·05	4·914	·0011
21	26	34	·70	·26	·05		
			$n=9$ (*cont.*)				
20	29	32	·80	·13	·07	5·252	·0022
22	25	34	·60	·34	·06		
20	30	31	·80	·11	·09	5·416	·0038
23	24	34	·51	·43	·06		
21	27	33	·70	·22	·07	5·524	·0048
21	28	32	·71	·20	·09	5·923	·0093
22	26	33	·62	·30	·08		
21	29	31	·71	·17	·12	6·156	·0166
23	25	33	·53	·38	·09		
21	30	30	·71	·14	·14	6·231	·0208
24	24	33	·45	·45	·09		
22	27	32	·63	·26	·11	6·390	·0268
22	28	31	·63	·23	·14	6·687	·0490
23	26	32	·55	·33	·12		
22	29	30	·64	·20	·17	6·832	·0791
24	25	32	·47	·40	·12		
23	27	31	·56	·29	·15	7·049	·1029
23	28	30	·56	·25	·18	7·259	·1772
24	26	31	·49	·35	·16		
23	29	29	·56	·23	·22	7·328	·2202
25	25	31	·42	·42	·16		
24	27	30	·50	·31	·19	7·534	·2867
24	28	29	·50	·27	·23	7·668	·4638
25	26	30	·43	·37	·20		
25	27	29	·44	·32	·24	7·868	·5992
25	28	28	·44	·28	·28	7·933	·7550
26	26	29	·38	·38	·24		
26	27	28	·38	·33	·29	8·063	·9606
27	27	27	·33	·33	·33	8·128	1·0000
			$n=10$				
20	30	40	1	—	—	0	
20	31	39	1	—	—	1·412	
21	29	40	·90	·10	—		
20	32	38	1	—	—	2·173	
22	28	40	·80	·20	—		
20	33	37	1	—	—	2·653	
23	27	40	·70	·30	—		
21	30	39	·90	·09	·01	2·861	
20	34	36	1	—	—	2·923	
24	26	40	·60	·40	—		
20	35	35	1	—	—	3·010	
25	25	40	·50	·50	—		
21	31	38	·90	·08	·02	3·684	
22	29	39	·81	·18	·02		
21	32	37	·90	·07	·03	4·220	
23	28	39	·71	·27	·02		

Σr_1	Σr_2	Σr_3	p_1	p_2	p_3	B_1	P
$n = 10$ (*cont.*)							
22	30	38	·81	·16	·03	4·549	·0000
21	33	36	·90	·06	·03	4·554	·0001
24	27	39	·62	·35	·02		
21	34	35	·90	·05	·04	4·715	·0001
25	26	39	·53	·44	·03		
22	31	37	·81	·14	·05	5·141	·0002
23	29	38	·72	·24	·04		
22	32	36	·82	·12	·06	5·534	·0005
24	28	38	·64	·32	·05		
22	33	35	·82	·11	·08	5·760	·0009
25	27	38	·55	·39	·05		
23	30	37	·73	·21	·06	5·788	·0012
22	34	34	·82	·09	·09	5·834	·0014
26	26	38	·47	·47	·05		
23	31	36	·73	·19	·08	6·238	·0025
24	29	37	·65	·28	·07		
23	32	35	·74	·16	·10	6·525	·0046
25	28	37	·57	·35	·08		
23	33	34	·74	·14	·12	6·665	·0074
26	27	37	·50	·42	·08		
24	30	36	·66	·25	·09	6·745	·0090
24	31	35	·66	·22	12	7·090	·0157
25	29	36	·58	·31	·10		
24	32	34	·67	·19	·14	7·291	·0261
26	28	36	·51	·38	·11		
24	33	33	·67	·17	·17	7·357	·0320
27	27	36	·44	·44	·11		
25	30	35	·59	·28	·13	7·492	·0399
25	31	34	·60	·24	·16	7·752	·0674
26	29	35	·52	·34	·14		
25	32	33	·60	·21	·18	7·879	·1035
27	28	35	·46	·40	·14		
26	30	34	·53	·30	·17	8·069	·1306
26	31	33	·54	·26	·20	8·255	·2112
27	29	34	·47	·35	·17		
26	32	32	·54	·23	·23	8·316	·2571
28	28	34	·41	·41	·18		
27	30	33	·48	·32	·21	8·499	·3250
27	31	32	·48	·28	·24	8·619	·5009
28	29	33	·42	·37	·21		
28	30	32	·43	·32	·24	8·797	·6299
28	31	31	·43	·28	·28	8·856	·7762
29	29	32	·38	·38	·25		
29	30	31	·38	·33	·29	8·973	·9644
30	30	30	·33	·33	·33	9·031	1·0000

4 *treatments*. (*Design*: $t = 4, \lambda = 1, b = 6, r = 3, k = 2$)

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
				$n = 1$					
3	4	5	6	1	—	—	—	0	·3750
3	5	5	5	1	—	—	—	0·903	·6250
4	4	4	6	·33	·33	·33	—		
4	4	5	5	·38	·38	·12	·12	1·579	1·0000
				$n = 2$					
6	8	10	12	1	—	—	—	0	·0058
6	8	11	11	1	—	—	—		
6	9	9	12	1	—	—	—	0·602	·0232
7	7	10	12	·50	·50	—	—		
7	7	11	11	·50	·50	—	—	1·204	·0290
6	9	10	11	1	—	—	—	1·498	·0994
7	8	9	12	·59	·28	·13	—		
6	10	10	10	1	—	—	—	1·806	·1190
8	8	8	12	·33	·33	·33	—		
7	8	10	11	·60	·29	·08	·04	2·359	·2245
7	9	9	11	·62	·17	·17	·05	2·631	·3065
7	9	10	10	·62	·18	·10	·10	2·898	·5643
8	8	9	11	·37	·37	·21	·06		
8	8	10	10	·38	·38	·12	·12	3·158	·6639
8	9	9	10	·40	·23	·23	·14	3·389	·9627
9	9	9	9	·25	·25	·25	·25	3·612	1·0000
				$n = 3$					
9	12	15	18	1	—	—	—	0	·0001
9	12	16	17	1	—	—	—		
9	13	14	18	1	—	—	—	0·829	·0010
10	11	15	18	·67	·33	—	—		
10	11	16	17	·67	·33	—	—	1·659	·0018
10	12	14	18	·70	·22	·07	—	1·840	·0040
9	13	15	17	1	—	—	—		
9	14	14	17	1	—	—	—		
10	13	13	18	·71	·14	·14	—	2·077	·0072
9	13	16	16	1	—	—	—		
11	11	14	18	·45	·45	·09	—		
9	14	15	16	1	—	—	—	2·511	·0144
11	12	13	18	·50	·31	·19	—		
9	15	15	15	1	—	—	—	2·709	·0160
12	12	12	18	·33	·33	·33	—		
10	12	15	17	·71	·23	·05	·02	2·836	·0204
10	12	16	16	·71	·23	·03	·03	3·069	·0270
11	11	15	17	·46	·46	·06	·02		
10	13	14	17	·72	·16	·10	·02	3·248	·0361
11	11	16	16	·46	·46	·04	·04	3·301	·0386
10	13	15	16	·72	·16	·07	·05	3·659	·0766
11	12	14	17	·50	·32	·14	·03		

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
				$n = 3$ (*cont.*)					
10	14	14	16	·72	·11	·11	·05	3·837	·1030
11	13	13	17	·52	·22	·22	·03		
10	14	15	15	·73	·12	·08	·08	4·016	·1398
12	12	13	17	·36	·36	·24	·04		
11	12	15	16	·51	·33	·10	·06	4·060	·1788
11	13	14	16	·53	·24	·16	·07	4·397	·2521
12	12	14	16	·37	·37	·18	·08	4·568	·3531
11	13	15	15	·53	·24	·11	·11		
11	14	14	15	·54	·17	·17	·12	4·727	·4899
12	13	13	16	·38	·27	·27	·09		
12	12	15	15	·38	·38	·12	·12	4·737	·5246
12	13	14	15	·39	·28	·19	·14	5·046	·7759
12	14	14	14	·40	·20	·20	·20	5·197	·8879
13	13	13	15	·29	·29	·29	·14		
13	13	14	14	·29	·29	·21	·21	5·346	1·0000
				$n = 4$					
12	16	20	24	1	—	—	—	0	
12	16	21	23	1	—	—	—		
12	17	19	24	1	—	—	—	0·977	
13	15	20	24	·75	·25	—	—		
12	16	22	22	1	—	—	—		
12	18	18	24	1	—	—	—	1·204	
14	14	20	24	·50	·50	—	—		
13	15	21	23	·75	·25	—	—	1·954	
12	17	20	23	1	—	—	—	2·084	
13	16	19	24	·77	·19	·04	—		
13	15	22	22	·75	·25	—	—	2·181	
14	14	21	23	·50	·50	—	—		
14	14	22	22	·50	·50	—	—	2·408	·0000
12	17	21	22	1	—	—	—		
13	17	18	24	·78	·13	·09	—	2·468	·0004
14	15	19	24	·56	·37	·06	—		
12	18	19	23	1	—	—	—		
12	18	20	22	1	—	—	—	2·997	·0008
14	16	18	24	·59	·28	·13	—		
12	18	21	21	1	—	—	—		
12	19	19	22	1	—	—	—	3·158	·0012
14	17	17	24	·60	·20	·20	—		
15	15	18	24	·43	·43	·14	—		
13	16	20	23	·78	·18	·04	·01	3·187	·0013
15	16	17	24	·45	·32	·23	—	3·466	·0021
12	19	20	21	1	—	—	—		
14	15	20	23	·56	·38	·05	·01	3·568	·0027
13	16	21	22	·78	·18	·03	·02		

$n = 4$ (*cont.*)

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
12	20	20	20	1	—	—	—	3·612	·0029
16	16	16	24	·33	·33	·33	—		
13	17	19	23	·78	·14	·07	·01	3·698	·0033
13	18	18	23	·78	·10	·10	·01	3·854	·0035
14	15	21	22	·56	·38	·04	·02	3·944	·0041
13	17	20	22	·78	·14	·05	·02	4·211	·0059
14	16	19	23	·60	·28	·10	·02		
15	15	19	23	·43	·43	·11	·02	4·368	·0071
13	17	21	21	·78	·14	·04	·04		
13	18	19	22	·78	·11	·08	·03	4·502	·0105
14	17	18	23	·61	·22	·16	·02		
14	16	20	22	·60	·29	·08	·04	4·718	·0129
13	18	20	21	·78	·11	·06	·04	4·794	·0187
15	16	18	23	·46	·33	·18	·03		
14	16	21	21	·60	·29	·06	·06	4·872	·0219
15	15	20	22	·44	·44	·09	·04		
13	19	19	21	·78	·08	·08	·05	4·928	·0257
15	17	17	23	·47	·25	·25	·03		
15	15	21	21	·44	·44	·06	·06	5·026	·0268
13	19	20	20	·79	·09	·06	·06	5·063	·0318
16	16	17	23	·35	·35	·26	·03		
14	17	19	22	·61	·22	·12	·04	5·131	·0372
14	18	18	22	·62	·17	·17	·05	5·262	·0407
14	17	20	21	·61	·23	·09	·07	5·414	·0595
15	16	19	22	·47	·34	·14	·05		
15	17	18	22	·48	·26	·20	·06	5·668	·0905
14	18	19	21	·62	·18	·13	·07		
15	16	20	21	·47	·34	·11	·08	5·694	·1067
14	18	20	20	·62	·18	·10	·10	5·797	·1267
16	16	18	22	·37	·37	·21	·06		
16	17	17	22	·37	·28	·28	·06	5·920	·1523
14	19	19	20	·62	·14	·14	·10		
15	17	19	21	·49	·27	·16	·09	6·064	·1860
15	18	18	21	·49	·21	·21	·09	6·183	·2074
16	16	19	21	·37	·37	·16	·09	6·190	·2506
15	17	20	20	·49	·27	·12	·12		
16	16	20	20	·37	·38	·12	·12	6·316	·2645
16	17	18	21	·39	·29	·22	·10	6·426	·4033
15	18	19	20	·50	·21	·16	·12		
15	19	19	19	·50	·17	·17	·17	6·543	·4325
17	17	17	21	·30	·30	·30	·10		
16	17	19	20	·39	·30	·18	·13	6·665	·5444
16	18	18	20	·40	·23	·23	·14	6·778	·6147
16	18	19	19	·40	·24	·18	·18	6·892	·7915
17	17	18	20	·31	·31	·24	·14		
17	17	19	19	·31	·31	·19	·19	7·005	·8470
17	18	18	19	·32	·25	·25	·19	7·115	·9858
18	18	18	18	·25	·25	·25	·25	7·225	1·0000

$n = 5$

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
15	20	25	30	1	—	—	—	0	
15	20	26	29	1	—	—	—	1·087	
16	19	25	30	·80	·20	—	—		
15	21	24	30	1	—	—	—		
15	20	27	28	1	—	—	—	1·461	
17	18	25	30	·60	·40	—	—		
15	22	23	30	1	—	—	—		
16	19	26	29	·80	·20	—	—	1·474	
15	21	25	29	1	—	—	—	2·274	
16	20	24	30	·81	·16	·03	—		
16	19	27	28	·80	·20	—	—	2·548	
17	18	26	29	·60	·40	—	—		
15	21	26	28	1	—	—	—	2·767	
15	22	24	29	1	—	—	—		
16	21	23	30	·82	·12	·06	—		
17	19	24	30	·64	·31	·05	—		
15	21	27	27	1	—	—	—	2·917	
15	23	23	29	1	—	—	—		
16	22	22	30	·82	·09	·09	—		
18	18	24	30	·47	·47	·05	—		
17	18	27	28	·60	·40	—	—	2·923	
15	22	25	28	1	—	—	—	3·373	
17	20	23	30	·66	·25	·09	—		
16	20	25	29	·81	·16	·02	·00	3·458	
15	22	26	27	1	—	—	—	3·645	
15	23	24	28	1	—	—	—		
17	21	22	30	·67	·19	·14	—		
18	19	23	30	·52	·38	·11	—		
16	20	26	28	·81	·16	·02	·01	3·948	·0000
17	19	25	29	·64	·32	·04	·01		
15	23	25	27	1	—	—	—	4·035	·0001
18	20	22	30	·53	·30	·17	—		
16	21	24	29	·81	·13	·05	·01	4·053	·0001
16	20	27	27	·81	·16	·01	·01	4·098	·0002
18	18	25	29	·47	·47	·04	·01		
15	23	26	26	1	—	—	—	4·158	·0002
15	24	24	27	1	—	—	—		
18	21	21	30	·54	·23	·23	—		
19	19	22	30	·41	·41	·18	—		
16	22	23	29	·82	·10	·07	·01	4·321	·0002
15	24	25	26	1	—	—	—	4·399	·0003
19	20	21	30	·43	·32	·25	—		
17	19	26	28	·64	·32	·03	·01	4·437	·0003
15	25	25	25	1	—	—	—	4·516	·0003
20	20	20	30	·33	·33	·33	—		
17	19	27	27	·64	·31	·02	·02	4·586	·0004
18	18	26	28	·47	·47	·03	·02		
16	21	25	28	·82	·13	·04	·02	4·649	·0005
17	20	24	29	·66	·25	·08	·01		
18	18	27	27	·47	·47	·02	·02	4·735	·0005

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
$n=5$ (*cont.*)									
16	21	26	27	·82	·13	·03	·02	4·917	·0006
18	19	24	29	·52	·38	·09	·01		
16	22	24	28	·82	·10	·06	·02	5·024	·0008
17	21	23	29	·67	·20	·12	·01		
16	23	23	28	·82	·08	·08	·02	5·144	·0009
17	22	22	29	·67	·16	·16	·02		
17	20	25	28	·66	·26	·06	·02	5·239	·0010
16	22	25	27	·82	·11	·05	·03	5·401	·0013
18	20	23	29	·54	·31	·14	·02		
17	20	26	27	·66	·26	·05	·04	5·505	·0017
18	19	25	28	·52	·38	·07	·03		
16	22	26	26	·82	·11	·04	·04	5·521	·0019
19	19	23	29	·42	·42	·14	·02		
16	23	24	27	·82	·08	·06	·03	5·629	·0024
18	21	22	29	·55	·24	·19	·02		
17	21	24	28	·67	·21	·09	·03	5·714	·0027
18	19	26	27	·52	·38	·06	·04	5·770	·0031
16	23	25	26	·82	·09	·05	·04	5·858	·0039
19	20	22	29	·44	·34	·20	·02		
17	22	23	28	·68	·17	·13	·03	5·938	·0044
16	24	24	26	·82	·07	·07	·04	5·967	·0049
19	21	21	29	·44	·27	·27	·02		
16	24	25	25	·82	·07	·05	·05	6·075	·0055
20	20	21	29	·35	·35	·27	·02		
17	21	25	27	·67	·21	·08	·04	6·083	·0068
18	20	24	28	·54	·31	·11	·04		
17	21	26	26	·67	·21	·06	·06	6·201	·0075
19	19	24	28	·42	·42	·12	·04		
17	22	24	27	·68	·17	·10	·05	6·408	·0099
18	21	23	28	·55	·25	·16	·04		
18	20	25	27	·54	·32	·09	·05	6·479	·0112
17	23	23	27	·68	·14	·14	·05	6·512	·0127
18	22	22	28	·56	·20	·20	·04		
18	20	26	26	·54	·32	·07	·07	6·566	·0143
19	19	25	27	·42	·42	·10	·06		
17	22	25	26	·68	·17	·08	·07	6·629	·0180
19	20	23	28	·44	·34	·17	·04		
19	19	26	26	·42	·42	·08	·08	6·682	·0185
17	23	24	26	·68	·14	·11	·07	6·835	·0242
19	21	22	28	·45	·28	·22	·05		
18	21	24	27	·55	·26	·13	·06	6·869	·0271
17	23	25	25	·68	·14	·09	·09	6·939	·0306
20	20	22	28	·36	·36	·23	·05		
17	24	24	25	·68	·11	·11	·09	7·040	·0349
20	21	21	28	·37	·29	·29	·05		
18	22	23	27	·56	·21	·17	·06	7·071	·0394
18	21	25	26	·56	·26	·10	·08	7·087	·0487
19	20	24	27	·45	·35	·14	·06		
$n=5$ (*cont.*)									
19	20	25	26	·45	·35	·11	·09	7·304	·0559
18	22	24	26	·56	·21	·14	·09	7·385	·0729
19	21	23	27	·46	·29	·18	·07		
18	23	23	26	·57	·17	·17	·09	7·482	·0833
19	22	22	27	·46	·23	·23	·07		
18	22	25	25	·57	·21	·11	·11	7·486	·0937
20	20	23	27	·37	·37	·19	·07		
18	23	24	25	·57	·18	·14	·11	7·679	·1249
20	21	22	27	·38	·30	·24	·08		
19	21	24	26	·46	·29	·15	·09	7·694	·1409
18	24	24	24	·57	·14	·14	·14	7·775	·1472
21	21	21	27	·31	·31	·31	·08		
19	21	25	25	·47	·29	·12	·12	7·795	·1673
20	20	24	26	·37	·37	·16	·10		
20	20	25	25	·37	·37	·12	·12	7·895	·1728
19	22	23	26	·47	·24	·19	·10	7·884	·1964
19	22	24	25	·47	·24	·16	·13	8·077	·2666
20	21	23	26	·39	·31	·20	·10		
19	23	23	25	·48	·20	·20	·13	8·169	·3090
20	22	22	26	·39	·25	·25	·11		
19	23	24	24	·48	·20	·16	·16	8·263	·3605
21	21	22	26	·32	·32	·26	·11		
20	21	24	25	·39	·31	·17	·13	8·269	·4125
20	22	23	25	·39	·26	·21	·14	8·449	·4883
20	22	24	24	·40	·26	·17	·17	8·543	·5800
21	21	23	25	·32	·32	·21	·14		
20	23	23	24	·40	·21	·21	·17	8·632	·6904
21	22	22	25	·33	·27	·27	·14		
21	21	24	24	·32	·32	·17	·17	8·634	·7181
21	22	23	24	·33	·27	·22	·18	8·812	·8782
22	22	22	24	·27	·27	·27	·18	8·900	·9423
21	23	23	23	·33	·22	·22	·22		
22	22	23	23	·27	·27	·22	·22	8·988	1·0000
$n=6$									
18	24	30	36	1	—	—	—	0	
18	24	31	35	1	—	—	—	1·174	
18	25	29	36	1	—	—	—		
19	23	30	36	·83	·17	—	—		
18	24	32	34	1	—	—	—	1·659	
18	26	28	36	1	—	—	—		
20	22	30	36	·67	·33	—	—		
18	24	33	33	1	—	—	—	1·806	
18	27	27	36	1	—	—	—		
21	21	30	36	·50	·50	—	—		

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
$n=6$ (*cont.*)									
19	23	31	35	·83	·17	—	—	2·348	
18	25	30	35	1	—	—	—	2·431	
19	24	29	36	·84	·14	·02	—		
19	23	32	34	·83	·17	—	—	2·833	
20	22	31	35	·67	·33	—	—		
19	23	33	33	·83	·17	—	—	2·980	
21	21	31	35	·50	·50	—	—		
18	25	31	34	1	—	—	—	3·009	
18	26	29	35	1	—	—	—		
19	25	28	36	·84	·11	·04	—		
20	23	29	36	·69	·27	·04	—		
18	25	32	33	1	—	—	—	3·270	
18	27	28	35	1	—	—	—		
19	26	27	36	·85	·09	·06	—		
21	22	29	36	·55	·41	·04	—		
20	22	32	34	·67	·33	—	—	3·317	
20	22	33	33	·67	·33	—	—	·3465	
21	21	32	34	·50	·50	—	—		
21	21	33	33	·50	·50	—	—	3·612	
18	26	30	34	1	—	—	—	3·680	
20	24	28	36	·70	·22	·07	—		
19	24	30	35	·84	·14	·02	—	3·685	
18	26	31	33	1	—	—	—	4·040	
18	27	29	34	1	—	—	—		
20	25	27	36	·71	·18	·10	—		
21	23	28	36	·57	·34	·09	—		
18	26	32	32	1	—	—	—	4·154	
18	28	28	34	1	—	—	—		
20	26	26	36	·71	·14	·14	—		
22	22	28	36	·45	·45	·09	—		
19	24	31	34	·84	·14	·01	·01	4·262	
20	23	30	35	·69	·27	·03	·01		
19	25	29	35	·84	·11	·04	·01	4·348	
18	27	30	33	1	—	—	—	4·496	
21	24	27	36	·59	·28	·13	—		
19	24	32	33	·84	·14	·01	·01	4·521	
21	22	30	35	·55	·41	·04	·01		
19	26	28	35	·85	·09	·05	·01	4·703	
18	27	31	32	1	—	—	—	4·710	
18	28	29	33	1	—	—	—		
21	25	26	36	·60	·22	·18	—		
22	23	27	36	·48	·38	·14	—		
19	27	27	35	·85	·07	·07	·01	4·713	
20	23	31	34	·69	·27	·02	·01	4·837	
19	25	30	34	·84	·12	·03	·01	5·012	
20	24	29	35	·70	·23	·06	·01		
18	28	30	32	1	—	—	—	5·023	
22	24	26	36	·50	·31	·19	—		
20	23	32	33	·69	·27	·02	·01	5·096	
21	22	31	34	·55	·41	·03	·01		

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
$n=6$ (*cont.*)									
18	28	31	31	1	—	—	—	5·123	
18	29	29	32	1	—	—	—		
22	25	25	36	·50	·25	·25	—		
23	23	26	36	·40	·40	·20	—		
18	29	30	31	1	—	—	—	5·321	
23	24	25	36	·41	·33	·26	—		
21	22	32	33	·55	·41	·02	·02	5·355	
19	25	31	33	·84	·12	·02	·01	5·368	
21	23	29	35	·58	·34	·07	·01		
18	30	30	30	1	—	—	—	5·419	·0000
24	24	24	36	·33	·33	·33	—		
19	26	29	34	·85	·10	·05	·01	5·456	·0001
20	25	28	35	·71	·19	·09	·01		
19	25	32	32	·84	·12	·02	·02	5·481	·0001
22	22	29	35	·46	·46	·08	·01		
19	27	28	34	·85	·08	·06	·01	5·667	·0001
20	26	27	35	·72	·15	·12	·01		
20	24	30	34	·70	·23	·05	·02	5·672	·0001
19	26	30	33	·85	·10	·04	·02	5·903	·0001
21	24	28	35	·59	·28	·11	·01		
20	24	31	33	·71	·23	·04	·02	6·026	·0001
21	23	30	34	·58	·34	·06	·02		
19	26	31	32	·85	·10	·03	·02	6·114	·0001
22	23	28	35	·48	·38	·12	·01		
20	24	32	32	·71	·23	·03	·03	6·138	·0001
22	22	30	34	·46	·46	·06	·02		
20	25	29	34	·71	·19	·08	·02	6·200	·0002
19	27	29	33	·85	·08	·05	·02	6·204	·0002
21	25	27	35	·60	·23	·15	·01		
19	28	28	33	·85	·07	·07	·02	6·301	·0002
21	26	26	35	·61	·19	·19	·01		
21	23	31	33	·58	·34	·05	·03	6·378	·0002
21	23	32	32	·58	·34	·04	·04	6·490	·0002
22	22	31	33	·46	·46	·05	·03		
20	26	28	34	·72	·16	·10	·02	6·497	·0003
19	27	30	32	·85	·08	·04	·03	6·505	·0003
22	24	27	35	·50	·32	·16	·02		
20	27	27	34	·72	·13	·13	·02	6·593	·0003
22	22	32	32	·46	·46	·04	·04	6·602	·0003
19	27	31	31	·85	·08	·03	·03	6·603	·0004
23	23	27	35	·41	·41	·17	·02		
20	25	30	33	·71	·19	·06	·03	6·640	·0004
21	24	29	34	·60	·29	·09	·02		
19	28	29	32	·85	·07	·05	·03	6·694	·0005
22	25	26	35	·51	·26	·21	·02		
20	25	31	32	·71	·19	·05	·04	6·848	·0006
22	23	29	34	·49	·39	·10	·03		
19	28	30	31	·85	·07	·05	·04	6·883	·0007
23	24	26	35	·42	·34	·22	·02		

$n=6$ (*cont.*)

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
19	29	29	31	·85	·06	·06	·04	6·974	·0008
23	25	25	35	·42	·28	·28	·02		
20	26	29	33	·72	·16	·08	·03	7·022	·0009
21	25	28	34	·61	·24	·13	·03		
19	29	30	30	·85	·06	·05	·05	7·066	·0010
24	24	25	35	·35	·35	·28	·02		
21	24	30	33	·60	·29	·08	·04	7·077	·0011
20	27	28	33	·72	·13	·11	·04	7·206	·0013
21	26	27	34	·61	·20	·16	·03		
21	24	31	32	·60	·29	·06	·05	7·283	·0015
22	23	30	33	·49	·39	·08	·04		
20	26	30	32	·72	·16	·07	·05	7·318	·0018
22	24	28	34	·50	·32	·14	·03		
20	26	31	31	·72	·16	·06	·06	7·413	·0020
23	23	28	34	·41	·41	·14	·03		
22	23	31	32	·49	·39	·07	·05	7·489	·0021
21	25	29	33	·61	·24	·11	·04	7·540	·0023
20	27	29	32	·72	·14	·09	·05	7·587	·0028
22	25	27	34	·51	·27	·18	·03		
20	28	28	32	·72	·11	·11	·05	7·675	·0031
22	26	26	34	·52	·22	·22	·03		
20	27	30	31	·72	·14	·08	·06	7·770	·0037
23	24	27	34	·43	·35	·19	·04		
21	26	28	33	·61	·20	·14	·05	7·806	·0041
21	25	30	32	·61	·25	·09	·06	7·831	·0048
22	24	29	33	·51	·33	·12	·05		
21	27	27	33	·62	·17	·17	·05	7·892	·0050
21	25	31	31	·61	·25	·07	·07	7·926	·0055
23	23	29	33	·41	·41	·12	·05		
20	28	29	31	·73	·12	·10	·06	7·944	·0064
23	25	26	34	·43	·29	·24	·04		
20	28	30	30	·73	·12	·08	·08	8·032	·0070
24	24	26	34	·36	·36	·24	·04		
20	29	29	30	·73	·10	·10	·08	8·117	·0076
24	25	25	34	·36	·30	·30	·04		
22	24	30	32	·51	·33	·10	·06	8·121	·0083
21	26	29	32	·62	·21	·12	·06	8·179	·0098
22	25	28	33	·52	·28	·15	·05		
22	24	31	31	·51	·33	·08	·08	8·215	·0106
23	23	30	32	·42	·42	·10	·07		
23	23	31	31	·42	·42	·08	·08	8·308	·0108
21	27	28	32	·62	·17	·14	·06	8·348	·0130
22	26	27	33	·52	·23	·19	·05		
21	26	30	31	·62	·21	·10	·08	8·359	·0152
23	24	28	33	·43	·35	·16	·05		
22	25	29	32	·52	·28	·13	·07	8·548	·0168
21	27	29	31	·62	·18	·12	·08	8·610	·0204
23	25	27	33	·44	·30	·20	·06		
21	28	28	31	·62	·15	·15	·08	8·692	·0226
23	26	26	33	·44	·25	·25	·06		
21	27	30	30	·62	·18	·10	·10	8·695	·0248
24	24	27	33	·37	·37	·21	·06		

$n=6$ (*cont.*)

Σr_1	Σr_2	Σr_3	Σr_4	p_1	p_2	p_3	p_4	B_1	P
22	25	30	31	·52	·28	·11	·09	8·726	·0294
23	24	29	32	·43	·36	·14	·07		
22	26	28	32	·53	·24	·16	·07	8·795	·0321
21	28	29	30	·62	·15	·12	·10	8·859	·0383
24	25	26	33	·37	·31	·26	·06		
22	27	27	32	·53	·20	·20	·07	8·875	·0399
23	24	30	31	·44	·36	·11	·09	8·903	·0432
21	29	29	29	·62	·12	·12	·12	8·941	·0444
25	25	25	33	·31	·31	·31	·06		
22	26	29	31	·53	·24	·14	·09	9·052	·0536
23	25	28	32	·45	·30	·17	·08		
22	26	30	30	·53	·24	·11	·11	9·136	·0590
24	24	28	32	·37	·37	·18	·08		
22	27	28	31	·53	·20	·17	·10	9·212	·0718
23	26	27	32	·45	·26	·21	·08		
23	25	29	31	·45	·30	·15	·10	9·307	·0796
22	27	29	30	·54	·20	·14	·12	9·375	·0976
24	25	27	32	·38	·32	·22	·08		
23	25	30	30	·45	·31	·12	·12	9·390	·1069
24	24	29	31	·37	·37	·15	·10		
22	28	28	30	·54	·17	·17	·12	9·454	·1175
24	26	26	32	·38	·26	·26	·08		
24	24	30	30	·38	·38	·12	·12	9·474	·1202
22	28	29	29	·54	·17	·14	·14	9·533	·1328
25	25	26	32	·32	·32	·27	·09		
23	26	28	31	·46	·26	·18	·10	9·543	·1455
23	27	27	31	·46	·22	·22	·10	9·620	·1530
23	26	29	30	·46	·26	·15	·13	9·704	·1888
24	25	28	31	·38	·32	·19	·11		
23	27	28	30	·46	·22	·19	·13	9·858	·2382
24	26	27	31	·39	·27	·23	·11		
24	25	29	30	·39	·32	·16	·13	9·864	·2632
23	27	29	29	·46	·22	·16	·16	9·936	·2923
25	25	27	31	·33	·33	·23	·11		
23	28	28	29	·46	·19	·19	·16	10·012	·3264
25	26	26	31	·33	·28	·28	·11		
24	26	28	30	·39	·28	·19	·14	10·093	·3669
24	27	27	30	·40	·23	·23	·14	10·168	·3906
24	26	29	29	·39	·28	·16	·16	10·170	·4382
25	25	28	30	·33	·33	·20	·14		
25	25	29	29	·33	·33	·17	·17	10·247	·4521
24	27	28	29	·40	·24	·20	·17	10·320	·5826
25	26	27	30	·34	·28	·24	·14		
24	28	28	28	·40	·20	·20	·20	10·394	·6081
26	26	26	30	·29	·29	·28	·14		
25	26	28	29	·34	·28	·20	·17	10·471	·6978
25	27	27	29	·34	·24	·24	·17	10·544	·7502
25	27	28	28	·34	·24	·21	·21	10·618	·8726
26	26	27	29	·29	·29	·24	·17		
26	26	28	28	·29	·29	·21	·21	10·692	·9084
26	27	27	28	·29	·25	·25	·21	10·764	·9919
27	27	27	27	·25	·25	·25	·25	10·837	1·0000

Appendix B. *The distribution of the combined likelihood ratio for general alternatives*

This table gives selected values of the combined likelihood ratio statistics, B_1^c, together with cumulative probabilities (levels of significance) associated with each tabular value. Only groups of equal size are considered. Thus if n is the total number of repetitions of the design and three groups are considered, each group contains $n/3$ repetitions.

B_1^c is computed by adding values of B_1 for each group as obtained from Appendix A. Only values significant at approximately the 0·10 level of significance or higher are recorded and these are selected for easy interpolation.

3 *treatments.* (*Design*: $t = 3, \lambda = 1, b = 3, r = 2, k = 2$)

2 equal groups

$n = 4$		$n = 6$		$n = 8$		$n = 10$	
B_1^c	P	B_1^c	P	B_1^c	P	B_1^c	P
0	·009	0·8293	·002	2·4683	·001	4·3987	·001
0·6021	·044	1·6586	·007	3·4452	·006	5·2447	·005
1·2042	·079	1·8402	·010	3·9738	·013	5·6469	·010
1·4984	·185	2·5112	·026	4·1686	·019	6·2895	·020
		2·7093	·048	4·3620	·025	6·4123	·030
		2·9064	·078	4·4424	·034	6·6731	·041
		3·3405	·142	4·5526	·041	6·8013	·052
				4·6696	·050	6·9250	·063
				4·9366	·065	6·9514	·069
				5·0812	·074	7·0179	·082
				5·2422	·086	7·0751	·089
				5·4652	·123	7·1656	·107

4 equal groups

$n = 8$		$n = 12$		$n = 16$		$n = 20$	
B_1^c	P	B_1^c	P	B_1^c	P	B_1^c	P
1·2042	·003	4·3281	·001	7·9978	·001	11·7538	·001
1·8063	·007	5·1807	·005	8·9950	·005	12·7753	·005
2·1005	·019	5·5759	·010	9·4390	·010	13·2106	·010
2·4084	·023	6·0498	·020	9·9076	·020	13·6708	·020
2·7026	·045	6·3897	·030	10·0686	·026	13·8394	·025
2·9968	·062	6·6266	·039	10·1624	·030	13·9669	·030
3·0104	·068	6·6810	·048	10·3965	·040	14·1835	·040
3·3047	·092	6·8791	·061	10·5843	·050	14·3529	·050
3·5989	·159	7·0209	·069	10·7074	·060	14·4811	·060
		7·0995	·080	10·8684	·070	14·6057	·069
		7·2578	·105	10·9854	·080	14·7232	·079
				11·0658	·092	14·8393	·089
				11·1320	·101	14·9368	·100

3 equal groups

$n = 6$		$n = 9$		$n = 12$		$n = 15$	
B_1^c	P	B_1^c	P	B_1^c	P	B_1^c	P
0	·001	2·5112	·001	5·3727	·001	8·0688	·001
1·2042	·016	2·7093	·002	6·2191	·005	9·0642	·005
1·4984	·030	3·3405	·005	6·6031	·010	9·4794	·010
1·8063	·041	3·7357	·011	7·0209	·021	9·8945	·020
2·1005	·101	4·1698	·022	7·1379	·026	10·0456	·025
		4·5097	·032	7·3265	·030	10·1738	·031
		4·5883	·040	7·5199	·039	10·4352	·039
		4·7466	·052	7·6341	·049	10·5704	·052
		4·9835	·063	7·8146	·059	10·6882	·060
		5·0224	·071	7·9079	·068	10·8049	·066
		5·2205	·101	8·0181	·083	10·9031	·080
				8·0781	·088	10·9594	·090
				8·1351	·100	10·9858	·097

5 equal groups

$n = 10$		$n = 15$		$n = 20$		$n = 25$	
B_1^c	P	B_1^c	P	B_1^c	P	B_1^c	P
1·8063	·001	6·0498	·001	10·7582	·001	15·4283	·001
2·4084	·004	7·0762	·005	11·8177	·005	16·4902	·005
2·7026	·009	7·5103	·010	12·2791	·010	16·9568	·010
3·3047	·022	8·0715	·019	12·8047	·020	17·4901	·020
3·5989	·038	8·2687	·026	12·9851	·025	17·6727	·025
3·6125	·040	8·3240	·030	13·0873	·030	17·8042	·030
3·9068	·052	8·5221	·040	13·3387	·040	18·0249	·040
4·2010	·083	8·7425	·049	13·5254	·049	18·2179	·050
4·2146	·085	8·9174	·060	13·6443	·060	18·3626	·060
4·4952	·101	9·0592	·068	13·7889	·070	18·4900	·069
		9·1543	·080	13·9161	·080	18·6137	·080
		9·2961	·089	14·0263	·089	18·7191	·090
		9·3349	·101	14·1000	·100	18·7831	·100