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RESEARCH ARTICLE

A generalization of Peleg's representation theorem on constant-sum weighted majority games

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Abstract
We propose a variant of the nucleolus associated with distorted satisfaction of each coalition in TU games. This solution is referred to as the α-nucleolus in which α is a profile of distortion rates of satisfaction of all the coalitions. We apply the α-nucleolus to constant-sum weighted majority games. We show that under assumptions of distortions of satisfaction of winning coalitions the α-nucleolus is the unique normalized homogeneous representation of constant-sum weighted majority games which assigns a zero to each null player. As corollary of this result, we derive the well-known Peleg's representation theorem.

Keywords Constant-sum weighted majority games · Homogeneous representation · α-Nucleolus · Distorted satisfaction · Peleg's representation theorem

JEL Classification C71

1 Introduction

Constant-sum weighted majority games have been known as the most classical games applied to voting systems, e.g., von Neumann and Morgenstern (1944). These games are simple games in which a coalition wins iff the sum of the weights of its members is larger than half the sum of weights of all the players. Since these games are derived from a profile of weights of players and a quota, a pair of the weight profile and the quota is called a representation. In particular, if a quota is the minimum of the sum of the weights attainable by a minimal winning coalition, then a profile of weights is called a normalized representation. In addition, if each minimal winning coalition carries the same weight, then it is called a normalized homogeneous representation. If such a homogeneous representation exists, a constant-sum weighted majority game

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is called homogeneous. Since the seminal work of von Neumann and Morgenstern (1944), whether a unique normalized homogeneous representation in constant-sum weighted majority games exists has been an important open question until Peleg (1968) solved this problem. Peleg (1968) showed that in constant-sum weighted majority games the nucleolus (Schmeidler 1969) is always the unique normalized homogeneous representation which assigns a zero to each null player. The nucleolus is an established solution for coalitional games with transferable utility (for short, TU games), and it is a game-theoretic expression of the ‘difference principle of social justice’ à la Rawls (1971). As Maschler (1992) pointed out, Peleg’s representation theorem shows one of the nicest results of solutions for TU games applied to voting systems.

In this note, we generalize Peleg’s representation theorem on constant-sum weighted majority games (Peleg 1968). In constant-sum weighted majority games, the nucleolus is a single-weight that lexicographically maximizes each coalition’s satisfaction over the set of weights. Given a profile of weights, satisfaction of each coalition is the difference between the sum of weights of its members and its coalitional worth. Thus, satisfaction of each coalition plays a role in the nucleolus. We consider another scenario of satisfaction of each coalition using the notion of utopia payoffs. The utopia payoff of each player is her marginal contribution to the grand coalition. Reasonable hope of each coalition may be regarded as the sum of utopia payoffs of its members. The sum of utopia payoffs might be the most that each coalition could reasonably hope for in the sense of Milnor (1952). We employ the notion of aspiration levels in computing satisfaction instead of coalitional worth. The aspiration level of each coalition is a value that lies somewhere between its coalitional worth and its reasonable hope (i.e., the sum of utopia payoffs of its members). For each coalition S, we refer to the difference between its reasonable hope and its coalitional worth as the utopia gap of S. For each coalition S, the number α_S is a weight ratio on its utopia gap. The aspiration level of each coalition S is the sum of its coalitional worth and its weighted utopia gap associated with α_S. Given a profile of weights, distorted satisfaction of each coalition is the difference between the sum of weights of its members and its aspiration level. In this sense, for each coalition S the number α_S is a distortion rate of satisfaction. Using these notions, we introduce a variant of the nucleolus, referred to as the α-nucleolus, where α is a profile of distortions of satisfaction of all the coalitions. We show that under assumptions of distortions of satisfaction of winning coalitions the α-nucleolus is the unique normalized homogeneous representation of constant-sum weighted majority games which assigns a zero to each null player. As a corollary of the representation theorem in the present study, we derive that if no coalition has distortion of satisfaction, then the α-nucleolus is the nucleolus. This is Peleg representation theorem.

In the representation theorem in the present study, we put two assumptions on distortions of satisfaction of winning coalitions. Firstly, in constant-sum weighted majority games that are homogeneous, it seems natural for us to consider that each minimal winning coalition that carries the same weight has the same distortion of satisfaction. The first assumption says that a distortion rate of satisfaction of each minimal winning coalition is homogeneous, and it is minimal among all the winning coalitions. Such a homogeneous distortion rate is assumed to be at most one minus the maximal quota derived from normalized representations. Secondly, it also seems natural for us to consider that a weight ratio on the utopia gap of each winning coalition that does not include a null player is invariant under a situation where the null player joins. This is because a null player does not make any change of the utopia gap of each winning coalition that does not include the null player. The second assumption says that a distortion rate of each winning coalition that does not include a null player is invariant under a situation where the null player joins.

This note is organized as follows. In Sect. 2, we define the α-nucleolus. Using the notion of the α-nucleolus, in Sect. 3, we present the unique representation of constant-sum weighted majority games. In Sect. 4, we remark on the representation theorem in the present study.

2 The α-nucleolus

Let N be a non-empty and finite set of agents. A coalitional game with transferable utility for N (a TU game for N, for short) is a function $v\colon 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. For all $S \in 2^N$, $v(S)$ represents what coalition S can achieve on its own. Let $\bar{\mathcal{V}}$ be the class of all TU games. Let $\mathcal{V}$ be a generic subclass of $\bar{\mathcal{V}}$, that is, $\mathcal{V} \subseteq \bar{\mathcal{V}}$. For $x \in \mathbb{R}^N$ and $S \subseteq N$, let $x(S) \equiv \sum_{i \in S} x_i$. For all $v \in \mathcal{V}$, the following notations are introduced: For all $i \in N$, let $M_i^v \equiv v(N) - v(N \backslash \{i\})$ be the marginal contribution of agent i to the grand coalition N. The number M_i^v is also called agent i's utopia payoff of v. For each $S \subseteq N$, let $M^v(S) \equiv \sum_{i \in S} M_i^v$ be the sum of utopia payoffs of the members of S.

The aspiration level of each coalition $S \subseteq N$ is a value that lies somewhere between $v(S)$ and $M^v(S)$. For each coalition $S \subseteq N$, we refer to the difference between $M^v(S)$ and $v(S)$ as the utopia gap of S. For each $S \subseteq N$, let $g^v(S)$ be the utopia gap of S, that is, $g^v(S) \equiv M^v(S) - v(S)$. For each coalition $S \subseteq N$, the number $\alpha_S \in [0, 1]$ is a weight ratio on the utopia gap of S. Let $\alpha \equiv (\alpha_S)_{S \in 2^N}$, where $\alpha_S \in [0, 1]$. Since this profile of weight ratios makes distorted satisfaction mentioned below, we call it a profile of distortions of satisfaction. Given $\alpha = (\alpha_S)_{S \in 2^N}$, where $\alpha_S \in [0, 1]$, the aspiration level of S, denoted $v^\alpha(S)$, is defined by setting for each $S \in 2^N$

$$v^\alpha(S) \equiv v(S) + \alpha_S g^v(S).$$

The α-aspiration game of v is the mapping that associates with each coalition $S \subseteq N$ its aspiration level $v^\alpha(S)$. By the definition of v and g^v, $v^\alpha(\emptyset) = 0$.

Let $\mathbf{Ef}(v)$ be the set of vectors $x \in \mathbb{R}^N$ such that $x(N) = v(N)$. Let $\mathbf{IP}(v)$ be the set of imputations $x \in \mathbb{R}^N$ such that $x(N) = v(N)$ and for all $i \in N$ $x_i \geq v(\{i\})$. On the domain of $\bar{\mathcal{V}}$, given $\alpha \in [0, 1]^{2^N}$, distorted satisfaction of each coalition $S \in 2^N$ with respect to $x \in \mathbf{Ef}(v)$ is defined by setting for each $S \in 2^N$

$$f(S, x; v) \equiv x(S) - v^\alpha(S).$$

Let $e(x) \equiv (f(S, x; v))_{S \in 2^N} \in \mathbb{R}^{2^N}$, given $\alpha \in [0, 1]^{2^N}$. Let $\geq_{lex}$ be the lexicographic ordering of $\mathbb{R}^{2^N}$.[1]

[1] For all $z \in \mathbb{R}^{2^N}$, $\theta(z) \in \mathbb{R}^{2^N}$ is defined by rearranging the coordinates of z in non-decreasing order. For all $z, z' \in \mathbb{R}^{2^N}$, z is lexicographically larger than z' if $\theta_1(z) > \theta_1(z')$ or $[\theta_1(z) = \theta_1(z')$ and $\theta_2(z) > \theta_2(z')]$ or $[\theta_1(z) = \theta_1(z')$ and $\theta_2(z) = \theta_2(z')$ and $\theta_3(z) > \theta_3(z')]$, and so on. Then, we write $z \geq_{lex} z'$.

Definition 1 On the domain of $\bar{\mathcal{V}}$, given $\alpha \in [0, 1]^{2^N}$, the α-prenucleolus, denoted $\mathbf{PN}^\alpha(v)$, is defined as follows:

$$\mathbf{PN}^\alpha(v) \equiv \left\{ x \in \mathbf{Ef}(v) \middle| e(x) \geq_{lex} e(y) \text{ for all } y \in \mathbf{Ef}(v) \right\}.$$

Proposition 1 *On the domain of $\bar{\mathcal{V}}$, given $\alpha \in [0, 1]^{2^N}$, $\mathbf{PN}^\alpha(v)$ is a single point.*

The proof is identical to the uniqueness argument of the prenucleolus in Theorem 5.1.14 in Peleg and Sudhölter (2003).[2]

Remark 1 $\mathbf{PN}(v)$ coincides with the following single-valued solutions.

(i) On the domain of $\bar{\mathcal{V}}$, if $\alpha = \mathbf{0}$, then $\mathbf{PN}(v)$ coincides with the prenucleolus (Schmeidler 1969).[3]
(ii) On the domain of $\mathcal{V}$ such that $v(N) \leq M^v(N)$, if $\alpha = \mathbf{1}$, then $\mathbf{PN}^\alpha(v)$ coincides with the ENSC value (Hou et al. 2018).[4] Notice that the ENSC value[5] is defined by setting for each $i \in N$,

$$\mathbf{ENSC}_i(v) \equiv M_i^v + \frac{v(N) - M^v(N)}{|N|}.$$

Let $\mathcal{V}$ be such that $\mathbf{IP}(v) \neq \emptyset$ for all $v \in \mathcal{V}$. On the domain of $\mathcal{V}$, given $\alpha \in [0, 1]^{2^N}$, distorted satisfaction of each coalition $S \in 2^N$ with respect to $x \in \mathbf{IP}(v)$ is defined by setting for each $S \in 2^N$ $f(S, x; v) \equiv x(S) - v^\alpha(S)$. Let $e(x) \equiv (f(S, x; v))_{S \in 2^N} \in \mathbb{R}^{2^N}$, given $\alpha \in [0, 1]^{2^N}$.

Definition 2 On the domain of $\mathcal{V}$ such that $\mathbf{IP}(v) \neq \emptyset$ for all $v \in \mathcal{V}$, given $\alpha \in [0, 1]^{2^N}$, the α-nucleolus, denoted $\mathbf{N}^\alpha(v)$, is defined as follows:

$$\mathbf{N}^\alpha(v) \equiv \left\{ x \in \mathbf{IP}(v) \middle| e(x) \geq_{lex} e(y) \text{ for all } y \in \mathbf{IP}(v) \right\}.$$

By the argument appearing in Schmeidler (1969) together with Proposition 1, the α-nucleolus is a single point since $\mathbf{IP}(v)$ is nonempty, compact, and convex.

The following example shows that the α-nucleolus does not necessarily coincide with the nucleolus.

Example 1 Let $N = \{1, 2, 3\}$. Let $v : 2^N \to \mathbb{R}$ such that for all $i \in N$ $v(\{i\}) = 0$, $v(\{1, 2\}) = 30$, $v(\{1, 3\}) = 40$, $v(\{2, 3\}) = 80$, $v(N) = 120$, and $v(\emptyset) = 0$. For all $S \in 2^N$, $\alpha_S = 1/2$. By simple calculation, the nucleolus is given by $\mathbf{N}(v) = (20, 45, 55)$, and the α-nucleolus is given by $\mathbf{N}^\alpha(v) = (10, 50, 60)$. Therefore, $\mathbf{N}(v) \neq \mathbf{N}^\alpha(v)$.

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2 Theorem 5.1.14 is itself a consequence of Theorems 5.1.6 and Corollary 5.1.10 in Peleg and Sudhölter (2003).

3 An 2^N-dimensional vector $\mathbf{0} = (0, 0, \dots, 0)$.

4 An 2^N-dimensional vector $\mathbf{1} = (1, 1, \dots, 1)$.

5 "ENSC" means "Egalitarian Non-Separable Contribution".

In this note, we will not proceed further investigation into game-theoretic properties of the α-nucleolus. Our target is to derive a unique homogeneous representation of constant-sum weighted majority games using the α-nucleolus. We will focus on this topic in the next section.

3 The unique representation of constant-sum weighted majority games

Let $G(N, \mathcal{W})$ be a simple game, where $\mathcal{W}$ is the set of winning coalitions, i.e.,

$$N \in \mathcal{W};\ \emptyset \notin \mathcal{W};$$
$$(S \subseteq T \subseteq N \textit{ and } S \in \mathcal{W}) \Longrightarrow T \in \mathcal{W}.$$

Let v_G be the corresponding TU game of $G(N, \mathcal{W})$, where $v_G(S) = 1$ if $S \in \mathcal{W}$; and $v_G(S) = 0$ otherwise. Notice that v_G is monotonic. Let $\mathcal{W}^m$ be the set of minimal winning coalitions, that is,

$$\mathcal{W}^m \equiv \left\{ S \in \mathcal{W} \middle| T \subsetneq S \Rightarrow T \notin \mathcal{W} \right\}.$$

For each $S \subseteq N$ and a profile of weights $w = (w_i)_{i \in N} \in \mathbb{R}^N_+$, let $w(S) \equiv \sum_{i \in S} w_i$. A simple game $G(N, \mathcal{W})$ is a weighted majority game if there exists a quota $q > 0$ and a profile of weights $w = (w_i)_{i \in N} \in \mathbb{R}^N_+$ such that

$$S \in \mathcal{W} \Longleftrightarrow w(S) \geq q.$$

If a simple game $G(N, \mathcal{W})$ is a weighted majority game, then (q, w) is called a representation of $G(N, \mathcal{W})$. The simple game is strong if

$$S \notin \mathcal{W} \Longleftrightarrow N \backslash S \in \mathcal{W}.$$

A player $i \in N$ is a veto player if $i \in \cap_{S \in \mathcal{W}} S$. A player $i \in N$ is a null player if for all $S \subseteq N \backslash \{i\}$ $v_G(S) = v_G(S \cup \{i\})$. Let D be the set of null players.

For $w \in \mathbb{R}^N_+$, let

$$q(w) \equiv \min_{S \in \mathcal{W}^m} w(S).$$

We call $x \in \mathbf{Ef}(v_G)$ a normalized representation of G if $(q(x), x)$ is a representation of G.

A weighted majority game G is constant-sum if $v_G(S) + v_G(N \backslash S) = 1$ for all $S \subseteq N$. Let G^* be a constant-sum weighted majority game for which there exists $w \in \mathbb{R}^N_+$ such that

$$S \in \mathcal{W} \Longleftrightarrow w(S) > \frac{1}{2} w(N).$$

Since G^* is strong, v_{G^*} is superadditive. Therefore, $\mathbf{IP}(v_{G^*}) \neq \emptyset$, which implies that $\mathbf{N}^\alpha(v_{G^*}) \neq \emptyset$. We call $x \in \mathbf{Ef}(v_{G^*})$ a normalized homogeneous representation of G^* if x is a normalized representation of G^* and $x(S) = q(x)$ for all $S \in \mathcal{W}^m$.

Claim 1 *Let G^* be a constant-sum weighted majority game. If a veto player exists in G^*, then the number of veto players is one.*

Proof Suppose that the number of veto players is more than one. It suffices to consider the case where the number of veto players is two. Let k, k' be veto players, i.e., $k, k' \in \cap_{S \in \mathcal{W}} S$. Take $S \notin \mathcal{W}$ such that $k \in S$ and $k' \notin S$. By the definition of G^*, $N \backslash S \in \mathcal{W}$ such that $k \notin N \backslash S$ and $k' \in N \backslash S$, a contradiction. □

Claim 2 *Let G^* be a constant-sum weighted majority game. If no veto player exists in G^*, then for all $i \in N$ $\{i\} \notin \mathcal{W}$.*

Proof Suppose that there exists $\bar{\imath} \in N$ such that $\{\bar{\imath}\} \in \mathcal{W}$. Since $\{\bar{\imath}\} \in \mathcal{W}^m$, $w_{\bar{\imath}} > \frac{1}{2} w(N)$, which implies that $\bar{\imath}$ must be a veto player, a contradiction. □

Lemma 1 (Peleg 1968, Lemma 3.1) *Let G^* be a constant-sum weighted majority game. An imputation $x \in \mathbf{IP}(v_{G^*})$ is a normalized representation of G^* if and only if $q(x) > \frac{1}{2}$.*

We introduce the following property of distortion, referred to as a minimal homogeneous distortion with respect to G^*. This property states that in a constant-sum weighted majority game a distortion rate of satisfaction of each minimal winning coalition is homogeneous, and it is minimal among all the winning coalitions. Such a homogeneous distortion rate is assumed to be at most one minus the maximal quota derived from normalized representations of G^*. Notice that there exists the maximal quota derived from normalized representations of G^*.[6]

Definition 3 (*Minimal homogeneous distortion*) Let G^* be a constant-sum weighted majority game. Let X be the nonempty set of normalized representations of G^*. A profile of distortions of satisfaction α is a minimal homogeneous distortion with respect to G^* if (i) for all $S, S' \in \mathcal{W}^m$ such that $S \neq S'$, $\alpha_S = \alpha_{S'} \leq 1 - \max_{x \in X} q(x)$, and (ii) for all $T \in \mathcal{W}^m$ and all $T' \in \mathcal{W}$ $\alpha_T \leq \alpha_{T'}$.

Lemma 2 *Let G^* be a constant-sum weighted majority game. Assume that a profile of distortions of satisfaction α is a minimal homogeneous distortion with respect to G^*. Then, if an imputation $x \in \mathbf{IP}(v_{G^*})$ is a normalized representation of G^*, $q(\mathbf{N}^\alpha(v_{G^*})) \geq q(x)$.*

[6] Since the nucleolus is a normalized representation (Peleg 1968, Theorem 3.4), the set of normalized representations of G^*, denoted X, is nonempty. Let $r \equiv q(x)$ for all $x \in X \neq \emptyset$. We consider two cases. Case 1: A unique veto player exists in G^*. By Claim 1, let i^* be a unique veto player. Let us consider the following problem: $\max r$ subject to $x_{i^*} \geq r$, $x_i \geq 0$ for all $i \in N \backslash \{i^*\}$, and $x(N) = 1$. The optimal solution is $r = 1$, which is attainable by $x \in \mathbb{R}^N$ such that $x_{i^*} = 1$ and $x_i = 0$ for all $i \in N \backslash \{i^*\}$. Since x is an imputation and $r > 1/2$, there exists $\max q(x)$ in this case. Case 2: No veto player exists in G^*. By Claim 2, a normalized representation is an imputation. Let us consider the following problem: $\max r$ subject to $x(S) \geq r > 1/2$ for all $S \in \mathcal{W}^m$, $x_i \geq 0$ for all $i \in N$, and $x(N) = 1$. Since the nucleolus is feasible for the problem and the objective function is bounded above, there exists $\max q(x)$ in this case. By the argument of the two cases mentioned above, $\max q(x) \in (1/2, 1]$.

Proof We consider two cases.
Case 1: A veto player exists in G^*.

By Claim 1, there exists a unique veto player i^*. For all $x \in \mathbf{IP}(v_{G^*})$,

$$\begin{aligned}\min_{S\in 2^N} f(S, x; v_{G^*}) &= \min_{S\in 2^N}[x(S) - (1-\alpha_S)v_{G^*}(S) - \alpha_S M^{v_{G^*}}(S)]\\ &= \min\left\{q(x)-1,\ \min_{i\in N\setminus\{i^*\}} x_i\right\}\\ &= q(x)-1,\end{aligned}$$

since $q(x) - 1 \in (-1/2, 0]$ by Lemma 1, and $\min_{i\in N\setminus\{i^*\}} x_i \geq 0$. Then

$$\min_{S\in 2^N} f(S, \mathbf{N}^\alpha(v_{G^*}); v_{G^*}) = q(\mathbf{N}^\alpha(v_{G^*})) - 1 \geq \min_{S\in 2^N} f(S, x; v_{G^*}) = q(x) - 1,$$

which implies $q(\mathbf{N}^\alpha(v_{G^*})) \geq q(x)$.
Case 2: No veto player exists in G^*.

By Claim 2, for all $i \in N$ $\{i\} \notin \mathcal{W}$. For all $i \in N$, since $N\setminus\{i\} \in \mathcal{W}$, $M_i^{v_{G^*}} = 0$. Let $\bar{\alpha} = \alpha_S$ for all $S \in \mathcal{W}^m$. For all $x \in \mathbf{IP}(v_{G^*})$,

$$\begin{aligned}\min_{S\in 2^N} f(S, x; v_{G^*}) &= \min_{S\in 2^N}[x(S) - (1-\alpha_S)v_{G^*}(S) - \alpha_S M^{v_{G^*}}(S)]\\ &= \min\left\{q(x)-1+\bar{\alpha},\ \min_{i\in N} x_i\right\}\\ &= q(x)-1+\bar{\alpha},\end{aligned}$$

since $q(x) - 1 + \bar{\alpha} \leq 0$ by the assumption of the minimal homogeneous distortion, and $\min_{i\in N} x_i \geq 0$. Then

$$\begin{aligned}\min_{S\in 2^N} f(S, \mathbf{N}^\alpha(v_{G^*}); v_{G^*}) &= q(\mathbf{N}^\alpha(v_{G^*})) - 1 + \bar{\alpha}\\ &\geq \min_{S\in 2^N} f(S, x; v_{G^*}) = q(x) - 1 + \bar{\alpha},\end{aligned}$$

which implies $q(\mathbf{N}^\alpha(v_{G^*})) \geq q(x)$. □

Proposition 2 *Let G^* be a constant-sum weighted majority game. Assume that a profile of distortions of satisfaction α is a minimal homogeneous distortion with respect to G^*. Then, the α-nucleolus of G^* is a normalized representation of G^*.*

Proof Let $x \in \mathbf{IP}(v_{G^*})$ be a normalized representation of G^*. By Lemma 1, $q(x) > \frac{1}{2}$, and by Lemma 2, $q(\mathbf{N}^\alpha(v_{G^*})) \geq q(x)$, which implies that $q(\mathbf{N}^\alpha(v_{G^*})) > \frac{1}{2}$. Again, by Lemma 1, $\mathbf{N}^\alpha(v_{G^*})$ is a normalized representation of G^*. □

Next, we introduce the following notations. For all players $i, j \in N$ such that $i \neq j$, let

$$\mathcal{T}_{ij}(N) \equiv \left\{S \subseteq N \middle| i \in S \text{ and } j \notin S\right\}.$$

For all $x \in \mathbf{IP}(v_{G^*})$, let $s_{ij}(x) = \min_{S \in \mathcal{T}_{ij}(N)} f(S, x; v_{G^*})$. The α-kernel of v_{G^*} is defined as follows:

$$\mathbf{K}^{\alpha}(v_{G^*}) \equiv \left\{ x \in \mathbf{IP}(v_{G^*}) \middle| \begin{array}{c} \text{for all } i, j \in N \text{ with } i \neq j \; s_{ij}(x) \geq s_{ji}(x) \\ \text{or for all } k \in N \; x_k = v_{G^*}(\{k\}) \end{array} \right\}.$$

The α-kernel is a variant of the kernel (Davis and Maschler 1965) associated with distorted satisfaction of each coalition.

For all $x \in \mathbf{IP}(v_{G^*})$, let

$$\mathcal{F}(x, v_{G^*}) \equiv \left\{ S \in 2^N \backslash \{N, \emptyset\} \middle| f(S, x; v_{G^*}) \leq f(T, x; v_{G^*}) \; \forall T \in 2^N \backslash \{N, \emptyset\} \right\}.$$

The collection $\mathcal{F}$ is separating if $i, j \in N$ such that $i \neq j$ and $\mathcal{F}(x, v_{G^*}) \cap \mathcal{T}_{ij}(N) \neq \emptyset$, then $\mathcal{F}(x, v_{G^*}) \cap \mathcal{T}_{ji}(N) \neq \emptyset$.

Claim 3 *Let G^* be a constant-sum weighted majority game in which no veto player exists. $\mathcal{F}(\mathbf{N}^{\alpha}(v_{G^*}), v_{G^*})$ is separating.*

Proof Fix an arbitrary $\alpha \in [0, 1]^{2^N}$. On the domain of $\mathcal{V}$ such that for all $v \in \mathcal{V}$ $\mathbf{IP}(v) \neq \emptyset$, it is well known that the nucleolus is included in the kernel. By the same argument as in the proof of this inclusion (e.g., Theorem 5.1.17 in Peleg and Sudhölter 2003), it follows that $\mathbf{N}^{\alpha}(v_{G^*}) \subseteq \mathbf{K}^{\alpha}(v_{G^*})$, which implies that $\mathbf{K}^{\alpha}(v_{G^*}) \neq \emptyset$. Since no veto player exists in G^*, $\mathbf{K}^{\alpha}(v_{G^*})$ is the set of $x \in \mathbf{IP}(v_{G^*})$ such that for all $i, j \in N$ with $i \neq j$ $s_{ij}(x) = s_{ji}(x)$. For all $x \in \mathbf{K}^{\alpha}(v_{G^*})$ $\mathcal{F}(x, v_{G^*})$ is separating. Therefore, $\mathcal{F}(\mathbf{N}^{\alpha}(v_{G^*}), v_{G^*})$ is separating. □

Next, we introduce the following property of distortion, referred to as the null player property of distortion with respect to G^*. This property states that in a constant-sum weighted majority game, given an arbitrary null player k, for each winning coalition S that does not include k, a distortion rate of S is the same as that of $S \cup \{k\}$.

Definition 4 (*Null player property of distortion*) Let G^* be a constant-sum weighted majority game. Let D be the set of null players. Fix an arbitrary null player $k \in D$ in G^*. A profile of distortions of satisfaction α satisfies the null player property of distortion with respect to G^* if for each $S \in \mathcal{W}$ with $k \notin S$, $\alpha_S = \alpha_{S \cup \{k\}}$.

We are in the position to present the main result.

Theorem 1 *Let G^* be a constant-sum weighted majority game. Assume that (i) a profile of distortions of satisfaction α is a minimal homogeneous distortion with respect to G^*, and (ii) it satisfies the null player property of distortion with respect to G^*. Then, the α-nucleolus of G^* is the unique normalized homogeneous representation of G^* which assigns a zero to each null player of G^*.*

Proof Let D be the set of null players of G^*. Let y be a normalized homogeneous representation of G^* which satisfies $y_i = 0$ for all $i \in D$. Since y is homogeneous,

$$y(S) = q(y) \quad \text{for all} \quad S \in \mathcal{W}^m.$$

Let $\mathcal{R}$ be the set of imputations $r \in \mathbf{IP}(G^*)$ such that $r(S) \geq q(y)$ for all $S \in \mathcal{W}^m$ and $r_i = 0$ for all $i \in D$. Let $x = \mathbf{N}^\alpha(v_{G^*})$. We consider three steps.

Step 1: For all $k \in D$, $x_k = 0$.

By Claims 1 and 2, we consider two cases of Step 1.

Case 1 of Step 1: A unique veto player exists in G^*.

It is clear that for all $k \in D$, $x_k = 0$. This is because $x = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is assigned to a unique veto player and 0 is assigned to each player except for the veto player.

Case 2 of Step 1: No veto player exists in G^*.

By Claim 2, it is clear that for all $i \in N$, $x_i \geq v_{G^*}(\{i\}) = 0$. It suffices to show that for all $k \in D$, $x_k \leq 0$. Suppose that there exists $k \in D$ such that $x_k > 0$. Fix such a $\bar{k}$. By the assumption (ii), for $\bar{k} \in D$, and each $S \in \mathcal{W}$ with $\bar{k} \notin S$, $\alpha_S = \alpha_{S \cup \{\bar{k}\}}$. Fix an arbitrarily fixed $S^* \in \mathcal{W}$ such that $\bar{k} \notin S^*$. Since $\alpha_{S^*} = \alpha_{S^* \cup \{\bar{k}\}}$ and $x_{\bar{k}} > 0$,

$$\begin{aligned} f(S^*, x; v_{G^*}) &= x(S^*) - 1 + \alpha_{S^*} \\ &< x(S^*) - 1 + \alpha_{S^* \cup \{\bar{k}\}} + x_{\bar{k}} \\ &= f(S^* \cup \{\bar{k}\}, x; v_{G^*}), \end{aligned}$$

which implies that $f(S^*, x; v_{G^*}) < f(S^* \cup \{\bar{k}\}, x; v_{G^*})$. For an arbitrarily fixed $S^{**} \notin \mathcal{W}$ such that $\bar{k} \notin S^{**}$, $f(S^{**}, x; v_{G^*}) = x(S^{**}) < x(S^{**} \cup \{\bar{k}\}) = f(S^{**} \cup \{\bar{k}\}, x; v_{G^*})$. Therefore, for $\bar{k} \in D$ and each $S \subseteq N$ such that $\bar{k} \notin S$,

$$f(S, x; v_{G^*}) < f(S \cup \{\bar{k}\}, x; v_{G^*}),$$

which implies that $\bar{k} \notin \cup_{S \in \mathcal{F}(x, v_{G^*})} S$, a contradiction to Claim 3.

Step 2: $x \in \mathcal{R}$.

By the assumption (i), Lemma 2 and Proposition 2 hold. By the fact that $q(x) \geq q(y)$ by Lemma 2 together with the fact that for all $S \in \mathcal{W}^m$ $x(S) \geq q(x)$ by Proposition 2, $x(S) \geq q(y)$ for all $S \in \mathcal{W}^m$. By this observation together with Step 1, $x \in \mathcal{R}$.

Step 3: $\mathcal{R} = \{x\}$.

Suppose not. Since $\mathcal{R} \neq \emptyset$ by Step 2, $\mathcal{R}$ has an extreme point z such that $z \neq y$. Since $y(S) = q(y)$ for all $S \in \mathcal{W}^m$, there exists $j \notin D$ such that $z_j = 0$. Fix such a $\bar{j}$. Since $\bar{j} \notin D$, there exists $S \in \mathcal{W}^m$ such that $\bar{j} \in S$. Fix such an $\bar{S}$. Since $\bar{S} \backslash \{\bar{j}\} \notin \mathcal{W}$, $R \equiv (N \backslash \bar{S}) \cup \{\bar{j}\} \in \mathcal{W}$. Since $z \in \mathcal{R}$, $z(\bar{S}) \geq q(y) = y(\bar{S})$. By Lemma 1

$$\frac{1}{2} < z(R) = z(N \backslash \bar{S}) = 1 - z(\bar{S}) \leq 1 - y(\bar{S}) < \frac{1}{2},$$

which is impossible. Therefore, $y = x$. □

As a corollary of Theorem 1, we derive the well-known representation theorem on constant-sum weighted majority games (Peleg 1968).

Corollary 1 *If $\alpha = \mathbf{0}$, then the nucleolus of G^* is the unique normalized homogeneous representation of G^* which assigns a zero to each null player of G^* (see* Peleg 1968*, Theorem* 3.5*).*

4 Concluding remarks

Finally, we remark on Theorem 1. Let G^* be a constant-sum weighted majority game. Let $\{\alpha^*, \alpha^{**}\}$ be any pair of profiles of distortions of satisfaction. Assume that (1) α^* and α^{**} are minimal homogeneous distortions with respect to G^*, and (2) α^* and α^{**} satisfy the null player property of distortion with respect to G^*. According to Theorem 1, by uniqueness, the α^*-nucleolus and the α^{**}-nucleolus must coincide for G^*. By this observation together with Corollary 1, the α-nucleolus and the nucleolus must coincide for G^* if α is a minimal homogeneous distortion and it satisfies the null player property. As a consequence, even if coalitions have distortions of satisfaction satisfying the two assumptions proposed in the present study, the nucleolus is the unique normalized homogeneous representation of constant-sum weighted majority games. In this respect, Theorem 1 is a generalization of Peleg's representation theorem. We close this note with the following example that shows the fact mentioned above.

Example 2 Let $G^*(N, \mathcal{W})$ be a constant-sum weighted majority game, where the set of players is given by $N = \{1, 2, 3, 4, 5, 6\}$, the set of minimal winning coalitions is given by $\mathcal{W}^m = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}, \{2, 3, 5\}, \{1, 4, 5\}\}$, and the set of null players is given by $D = \{6\}$. Consider an arbitrary profile of distortions of satisfaction α satisfying that (i) the minimal homogeneous distortion assumption: for all $S, S' \in \mathcal{W}^m$ such that $S \neq S'$, $\alpha_S = \alpha_{S'} \leq 4/9$, and for all $T \in \mathcal{W}^m$ and all $T' \in \mathcal{W}$ $\alpha_T \leq \alpha_{T'}$, and (ii) the null player property of distortion: for each $S \in \mathcal{W}$ with $6 \notin S$, $\alpha_S = \alpha_{S \cup \{6\}}$. Then, the unique normalized homogeneous representation of G^* which assigns a zero to the null player is the α-nucleolus of G^*, that is, $\mathbf{N}^\alpha(v_{G^*}) = (3/9, 2/9, 2/9, 1/9, 1/9, 0) = \mathbf{N}(v_{G^*})$, where $\mathbf{N}(v_{G^*})$ is the nucleolus of G^*.

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