MAE 502 Course Project: Racing the Circuit of America

1 Introduction

In this project, you will work in groups and be asked to control a rear-wheel-drive front-wheel-steer bicycle model based on real Formula One car specifications. The goal is to finish a lap on the race track within the shortest possible time. Through the project you shall gain better understanding about vehicle modeling and control.

2 Track

The race track we are using is the Circuit of America in Austin, which hosts the Formula One United States Grand Prix since 2012. The overall length of the track is 5.5 [km]. The start and finish line located both at the origin, and the vehicle is driving in counter clockwise direction along the track.

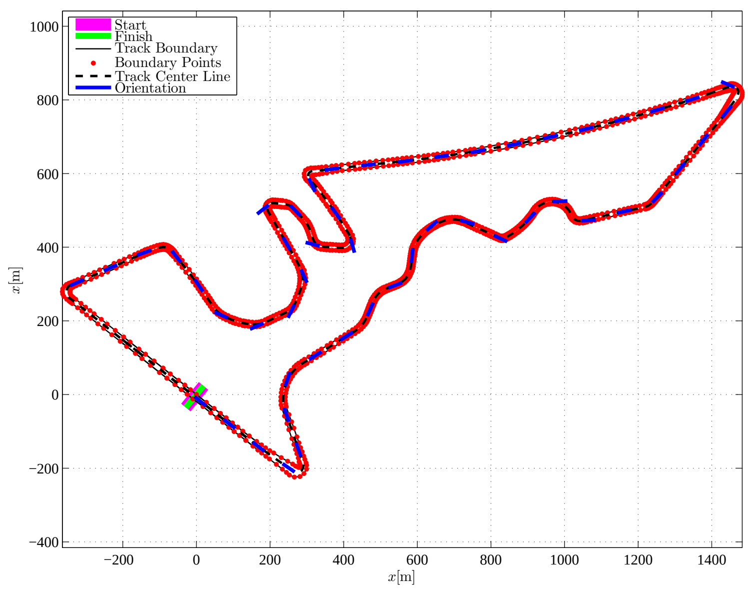

Figure 1: Track: Circuit of America

The GPS coordinates of the track are collected from Google Earth and converted to Cartesian coordinates using the Haversine formula; see Figure 1 for the results. Altogether, 597 pairs of left and right boundary points are collected, and they are marked as red dots in Figure 1. The points are not evenly distributed, as more points are used to capture sharp turns. The centerline is calculated by averaging the corresponding left and right boundary points. The orientation of the centerline is also estimated at each data point as

$$\mathbf{t}(s) = \begin{bmatrix} \cos\left(\theta(s)\right) \\ \sin\left(\theta(s)\right) \end{bmatrix}. \tag{1}$$

Some of these $\mathbf{t}$ vectors are marked as blue line segments in Figure 1. For simplicity, the elevation is omitted.

3 Models

3.1 Dynamics

We model the Formula One car as a rear-wheel-drive bicycle model with skates; see Figure 3.

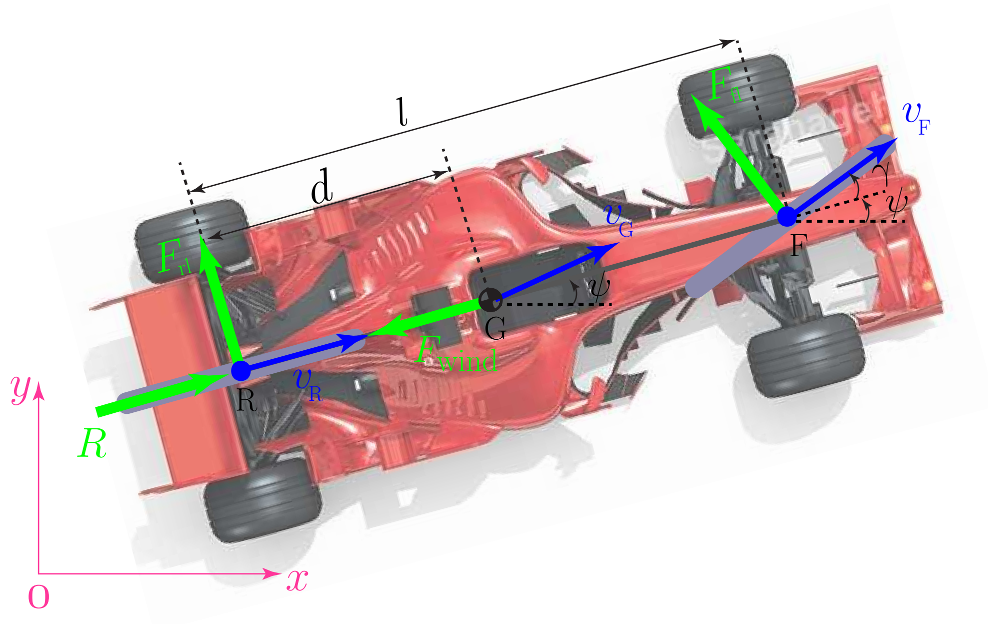

Figure 2: Rear wheel drive bicycle model

We choose the generalized coordinates the location of the center of mass (COM) (x, y) and the yaw angle ψ, and define a pseudo velocity

$$\sigma = \dot{x}\cos\psi + \dot{y}\sin\psi, \tag{2}$$

which is the longitudinal velocity of the COM. The active forces are the driving force R applied at the rear wheel, and the air resistance F_{wind} applied at the COM. The driving force R act at the center of the rear axle while the air drag force

$$F_{\text{wind}} = k\sigma^2, \tag{3}$$

acts at the COM, and k is given in Table 1.

The model has the following constraints, resulted from the no-side-slip condition at both the front skate and the rear skate:

$$\begin{aligned} &\dot{x}\sin\psi - \dot{y}\cos\psi + \dot{\psi}d = 0, \\ &\dot{x}\sin(\psi+\gamma) - \dot{y}\cos(\psi+\gamma) - \dot{\psi}(l-d)\cos\gamma = 0. \end{aligned} \tag{4}$$

Using (2,4) and the Appell equation, we obtain the dynamics

$$
\begin{aligned}
\dot{x} &= \left(\cos\psi - \frac{d}{l}\sin\psi\tan\gamma\right)\sigma \\
\dot{y} &= \left(\sin\psi + \frac{d}{l}\cos\psi\tan\gamma\right)\sigma \\
\dot{\psi} &= \frac{\tan\gamma}{l}\sigma \\
\dot{\sigma} &= \frac{R - F_{\text{wind}} - m_0\dfrac{\tan\gamma}{\cos^2\gamma}\dot{\gamma}\sigma}{m + m_0\tan^2\gamma}.
\end{aligned}
\tag{5}
$$

Using the Lagrange equation of second kind with multipliers, we get the lateral forces at the front and the rear wheel as

$$
\begin{aligned}
F_{\text{rl}} &= \frac{m}{l}\tan\gamma\left(1 - \frac{d}{l}\right)\sigma^2 - \frac{m_0 - m\dfrac{d}{l}}{m + m_0\tan^2\gamma}\left((R - F_{\text{wind}})\tan\gamma + \frac{m\dot{\gamma}\sigma}{\cos^2\gamma}\right) \\
F_{\text{fl}} &= \frac{1}{\cos\gamma}\left[m\frac{d}{l^2}\sigma^2\tan\gamma + \frac{m_0}{m + m_0\tan^2\gamma}\left((R - F_{\text{wind}})\tan\gamma + \frac{m\dot{\gamma}\sigma}{\cos^2\gamma}\right)\right]
\end{aligned}
\tag{6}
$$

The longitudinal control is given by $R = u_{\text{long}}$. Due to the presence of $\dot{\gamma}$, we can define γ as one additional state and define $\dot{\gamma} = u_{\text{steer}}$ as the actual control. Thus, the extended control system can be given as

$$
\begin{aligned}
\dot{x} &= \left(\cos\psi - \frac{d}{l}\sin\psi\tan\gamma\right)\sigma \\
\dot{y} &= \left(\sin\psi + \frac{d}{l}\cos\psi\tan\gamma\right)\sigma \\
\dot{\psi} &= \frac{\tan\gamma}{l}\sigma \\
\dot{\sigma} &= \frac{-k\sigma^2}{m + m_0\tan^2\gamma} + \frac{1}{m + m_0\tan^2\gamma}u_{\text{long}} + \frac{m_0\sigma\tan\gamma}{m\cos^2\gamma + m_0\sin^2\gamma}u_{\text{steer}} \\
\dot{\gamma} &= u_{\text{steer}}.
\end{aligned}
\tag{7}
$$

3.2 Input Limitations and Constraints

In model (3), we limit the driving force as $R \in [-10000, 5500]$[N], which allows a top speed of about 285 [km/h] and a maximum deceleration of 1.5 g. We also limit the steering angle as $\gamma \in [-0.5, 0.5]$[rad] and $\dot{\gamma} \in [-1, 1]$ [rad/s]. The lateral forces are limited by the coefficient of friction as 5000 [N] at the front and 5500 [N] at the rear. Note that the down force of the wings significantly increases these compared to what would result from the vehicle's weight. All the parameters of the model are listed in Table 1.

Vehicle Parameter	**Value**	**Field Name**
Gravitational constant (g)	9.81 [g/s^2]	`'g'`
Mass (m)	660 [kg]	`'m'`
Air drag coefficient (C_{D})	1	`'Cd'`
Air density at 25 $°C$ (ρ)	1.184 [kg/m^3]	`'rho'`
Frontal area (A)	1.5 [m^2]	`'A'`
$k = \frac{1}{2} C_{\mathrm{d}} \rho A$	0.88 [kg/m]	`'k'`
Wheelbase l	3.4 [m]	'w'
Distance of the mass center from the real axle (d)	1.6 [m]	`'a'`
Distance of the mass center from the front axle	1.8[m]$= l - d$	`'b'`
Moment of inertia about z axis (J_{G})	450 [kg $\cdot$ m^2]	`'JG'`
$m_0 = \frac{J_{\mathrm{G}} + md^2}{l^2}$	185.09 [kg]	`'m0'`
Driving force range ($[R_{\min}, R_{\max}]$)	[-10000,5500] [N]	`'R_min'`,`'R_max'`
Front wheel Lateral force limits ($F_{\mathrm{fl,max}}$)	5000 [N]	`'Ffl_max'`
Rear wheel Lateral force limits ($F_{\mathrm{rl,max}}$)	5500 [N]	`'Frl_max'`
Steering wheel range ($[\gamma_{\min}, \gamma_{\max}]$)	$[-0.5, 0.5]$ [rad]	`'gamma_max'`,`'gamma_min'`

Table 1: Data of the Formula One car [1]

4 Competition Rules

4.1 Key Dates

The initial submission deadline is **Friday, 11:59 EDT, April 25th, 2025**. The instructor will test all the submissions. Should there be any issues during this pre-challenge test on the program, the instructors will let the team know via email.
The second and final submission deadline is **Tuesday, 11:59 EDT, April 29th, 2025**. Each team will have a second chance to submit their final controller via **Via UB Learn**.
The challenge will be live on **Wednesday, April 30th, 2025** during lecture time, and the final ranking will be announced.

4.2 Controller Submission Format

You will be asked to provide your control inputs R, $\dot{\gamma}$ as functions of time. You could either use the raw data or use it directly as a function handle. Note that if the input is defined as an explicit function (e.g., a MATLAB function file that defines the analytical expression of the input), this is also acceptable.
We remark that since we are using explicitly ODE numerical solver with Runge Kutta method with no uncertainties, if you generate your design with a feedback controller, you could still convert your closed loop controller action into open loop time profile and it should still work exactly the same way. The request for having input submission as open-loop control profiles is to facilitate competition presentation.
Specifically, each input can be given as a function of time through interpolation as

```
R_time=[0,t1,t2,......];
R_sample=[R1,R2,R3,......];
R=@(t) interp1(R_time,R_sample,t);
```

Note that if you only provide γ information, then $\dot{\gamma}$ will be generated using numerical approximation. Note that to obtain the derivative $\dot{\gamma}(0)$ numerically, γ also needs to be defined at a negative time moment, that is,

```
gamma_time=[-0.1,0,t1,t2,......];
gamma_sample=[R0,R0,R1,R2,R3,......];
gamma=@(t) interp1(R_time,R_sample,t);
```

4.3 Setup

The control input will be given to the simulation code and run in real time. All teams' vehicles will start from the center of the start line (cf. magenta line in Figure 1) with the peak speed (i.e., $v_{\max} = \sqrt{R_{\max}/k}$), unless the team specify their initial speed differently. The order of the presentations will be determined randomly and announced at the beginning of the challenge.

4.4 Evaluation

You will be scored according to your lap time. All teams that complete a lap without getting off the track or violating any force constraints will be given a basic score of 60. If the lap is not finished, the score will be based on the distance traveled. The bonus of 40 points will be given based on the lap time achieved, and the fastest team will get 100 points.

5 Competition Package

The provided Development package contains both reference and simulation code.

Ref

The `'Ref'` folder contains relevant literature on race car modeling and control. The teams are encouraged to consult them to get initial ideas for the control design but should search the literature for other available techniques. The references are by no means complete.

Code

The simulation codes can be found in `'Code'` folder are developed and will be run in MATLAB. The simulation code package provided on Ublearn contains the following 9 `m` files for simulation and 2 `mat` files for data that are necessary for the competition.

`LapTimeMain.m` The main file of simulation. Please assign your input R, γ, and $\dot{\gamma}$ as functions of time in this file.

`CarSimRealTime.m` Simulation/motion main file. To test your lap time, please make sure you use the original file.

`car_RWD.m` The function that gives the right-hand side of the rear-wheel-drive bicycle model; cf. (5). It will be used in `CarSimRealTime.m`.

`Cartisan2Track.m` The function that localizes the vehicle on the track. That is, it determines how far the vehicle traveled in terms of the arc-length along the centerline, denoted as s, and how much the vehicle center of mass is away from the center line, denoted as n, which is positive if the vehicle is to the left of the center line. It is used by `car_RWD.m`, and s and n are available for control.

`Force_rwd.m` The function that gives lateral forces given the current states. It is used in `CarSimRealTime.m` to check whether the lateral forces exceed the limits.

`InputChecker.m` The function that checks and caps (if needed) the inputs. It is used in `LapTimeMain.m`.

`plot_results.m` The script that generate post simulation plots.

`figureset.m` The helper function that set size of figures, used by `plot_results.m`.

`F1CarData.mat` Vehicle data; cf. Table 1.

`CircuitOfAmerica.mat` Track data; cf. Figure 1. The description is given in Table 2.

Examples

Two examples are provided to facilitate the development. It is expected that your team's performance should be at least as good as the baseline. You are encouraged to search the literature and implement different solutions in this competition.

Table 2: Track structure data description

Field name	Description
‘bl’	left boundary data
‘br’	right boundary data
‘cline’	center line data
‘bfl’	finish line on the left boundary
‘bfr’	finish line on the right boundary
‘bstl’	start line on the left boundary
‘bstr’	start line on the right boundary
‘arc_s’	centerline arclength
‘theta’	centerline orientation angle
‘fun_bl’	left boundary position as function of centerline arclength
‘fun_br’	right boundary position as function of centerline arclength
‘fun_width’	width as function of centerline arclength
‘center’	centerline position as function of centerline arclength
‘ftheta’	centerline orientation angle as function of centerline arclength
‘t’	centerline orientation vector as function of centerline arclength

Open loop example

`LapTimeMain_OpenLoopExample.m` will load a submission from a previous competition. The input is provided in the same format as the final submission.

Controlled example

`LapTimeMain_ControlledExample.m` will load a baseline close loop design. It uses the path generated by `Path_generation.m` that includes the base path and speed profile to follow. The feedback controllers are a PI controller longitudinal speed tracking and a steering controller based on the Stanley controller [2]. They are implemented with `car_RWD_with_control.m` by extending the car model to include controllers. It uses helper functions in `Utilities`.

References

[1] D.J.N. Limebeer and G. Perantoni. Optimal control of a formula one car on a three-dimensional track part 2: Optimal control. *Journal of Dynamic Systems, Measurement, and Control*, 137(5):051019, 2015.

[2] Sebastian Thrun, Mike Montemerlo, Hendrik Dahlkamp, David Stavens, Andrei Aron, James Diebel, Philip Fong, John Gale, Morgan Halpenny, Gabriel Hoffmann, et al. Stanley: The robot that won the darpa grand challenge. *Journal of field Robotics*, 23(9):661–692, 2006.

[3] Wubing B Qin, Yiming Zhang, Dénes Takács, Gábor Stépán, and Gábor Orosz. Nonholonomic dynamics and control of road vehicles: moving toward automation. *Nonlinear Dynamics*, 110(3):1959–2004, 2022.

6 Model Details

We are using a rear wheel drive 2D skate model for a Formula 1 car, where both front and rear wheels are modelled as skate that does not allow side slip. The model is derived using Appell's equation of motion link. Details about the background on Appell's equation of motion may be found in [3].

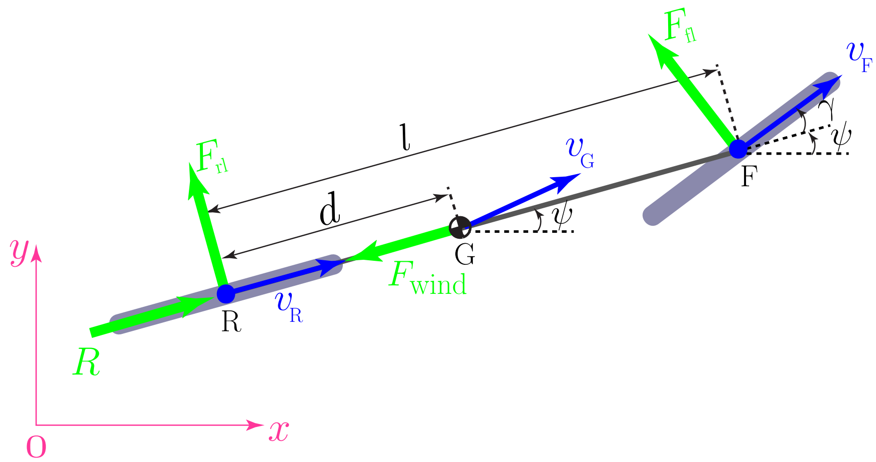

Figure 3: Rear wheel drive bicycle model

Choose the generalized coordinate as $q_1 = x$, $q_2 = y$ and $q_3 = \psi$. Constraints

$$\begin{aligned} &\dot{x}\sin\psi - \dot{y}\cos\psi + \dot{\psi}d = 0 \\ &\dot{x}\sin(\psi+\gamma) - \dot{y}\cos(\psi+\gamma) - \dot{\psi}(l+d)\cos\gamma = 0 \end{aligned} \tag{8}$$

Generalized force is given by

$$Q_1 = (R - F_{\text{wind}})\cos\psi, \quad Q_2 = (R - F_{\text{wind}})\sin\psi. \tag{9}$$

The kinetic energy is given by

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}J_{\text{G}}\dot{\psi}^2 \tag{10}$$

Define pseudo velocities as

$$\sigma = \dot{x}\cos\psi + \dot{y}\sin\psi. \tag{11}$$

and

$$\dot{\sigma} = \ddot{x}\cos\psi + \ddot{y}\sin\psi - \dot{x}\dot{\psi}\sin\psi + \dot{y}\dot{\psi}\cos\psi \tag{12}$$

Then we can have the constraints expressed as

$$\begin{aligned} \dot{x} &= \left(\cos\psi - \frac{d}{l}\sin\psi\tan\gamma\right)\sigma \\ \dot{y} &= \left(\sin\psi + \frac{d}{l}\cos\psi\tan\gamma\right)\sigma \\ \dot{\psi} &= \frac{\tan\gamma}{l}\sigma \end{aligned} \tag{13}$$

Acceleration energy is given by

$$
\begin{aligned}
S &= \frac{1}{2}m\underline{a}_{\mathrm{G}}^2 + \frac{1}{2}J_{\mathrm{G}}\underline{\alpha}^2 + \underbrace{(\underline{\alpha}\cdot\underline{\omega}\cdot\underline{H}_{\mathrm{G}})}_{=0} \\
&= \frac{1}{2}m(\ddot{x}+\ddot{y})^2 + \frac{1}{2}J_{\mathrm{G}}\ddot{\psi}^2 + \underbrace{\cdots}_{\text{Terms does not contain accelerations}} \\
(\text{Plug in} \quad \ddot{x},\ddot{y},\ddot{\psi}) &= \frac{1}{2}\left(m + \underbrace{\frac{md^2+J_{\mathrm{G}}}{l^2}}_{:=m_0}\tan^2\gamma\right)\dot{\sigma}^2 + \underbrace{\frac{md^2+J_{\mathrm{G}}}{l^2}}_{:=m_0}\frac{\tan\gamma}{\cos^2\gamma}\dot{\gamma}\sigma\dot{\sigma} + \underbrace{\cdots}_{\text{Terms does not contain}\,\sigma}
\end{aligned}
\tag{14}
$$

Pseudo force

$$
\begin{aligned}
&\delta P = \underline{R} - \underline{F}_{\mathrm{wind}}\delta\underline{\dot{r}}_{\mathrm{G}} = \begin{bmatrix}(R-F_{\mathrm{wind}})\cos\psi & (R-F_{\mathrm{wind}})\sin\psi\end{bmatrix}\delta\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = R\delta(\dot{x}\cos\psi + \dot{y}\sin\psi) = R\delta\sigma \\
&\Rightarrow \quad \pi = R - F_{\mathrm{wind}}.
\end{aligned}
\tag{15}
$$

Thus, Appell equation is given by

$$
\frac{\partial S}{\partial\dot{\sigma}} = \pi \quad \Leftrightarrow \quad \left(m + m_0\tan^2\gamma\right)\dot{\sigma} + m_0\frac{\tan\gamma}{\cos^2\gamma}\dot{\gamma}\sigma = R - F_{\mathrm{wind}}
\tag{16}
$$

The dynamics is given by

$$
\begin{aligned}
\dot{x} &= \left(\cos\psi - \frac{d}{l}\sin\psi\tan\gamma\right)\sigma \\
\dot{y} &= \left(\sin\psi + \frac{d}{l}\cos\psi\tan\gamma\right)\sigma \\
\dot{\psi} &= \frac{\tan\gamma}{l}\sigma \\
\dot{\sigma} &= \frac{R - F_{\mathrm{wind}} - m_0\dfrac{\tan\gamma}{\cos^2\gamma}\dot{\gamma}\sigma}{m + m_0\tan^2\gamma}.
\end{aligned}
\tag{17}
$$

Then use Lagrange equation of second kind with multipliers one has

$$
\begin{aligned}
m\ddot{x} &= (R-F_{\mathrm{wind}})\cos\psi + \nu_1\sin\psi + \nu_2\sin(\psi+\gamma) \\
m\ddot{y} &= (R-F_{\mathrm{wind}})\sin\psi - \nu_1\cos\psi - \nu_2\cos(\psi+\gamma) \\
J_{\mathrm{G}}\ddot{\psi} &= \nu_1 d - \nu_2(l-d)\cos\gamma \\
\Leftrightarrow \quad & \begin{bmatrix} m\ddot{x} - (R-F_{\mathrm{wind}})\cos\psi \\ m\ddot{y} - (R-F_{\mathrm{wind}})\sin\psi \end{bmatrix} = \underbrace{\begin{bmatrix}\sin\psi & \sin(\psi+\gamma) \\ -\cos\psi & -\cos(\psi+\gamma)\end{bmatrix}}_{\det(\cdot)=\sin\gamma}\begin{bmatrix}\nu_1\\ \nu_2\end{bmatrix}
\end{aligned}
\tag{18}
$$

Based on Free body diagram, we have

$$
\begin{aligned}
\nu_1 &= -F_{\mathrm{R}} = \frac{(R-F_{\mathrm{wind}})\cos\gamma}{\sin\gamma} - \frac{m}{\sin\gamma}\left(\ddot{x}\cos(\psi+\gamma) + \ddot{y}\sin(\psi+\gamma)\right) \\
\nu_2 &= -F_{\mathrm{F}} = -\frac{(R-F_{\mathrm{wind}})}{\sin\gamma} + \frac{m}{\sin\gamma}\left(\ddot{x}\cos\psi + \ddot{y}\sin\psi\right)
\end{aligned}
\tag{19}
$$

From the constraints we have

$$
\begin{aligned}
\ddot{x} &= \left(\cos\psi - \frac{d}{l}\sin\psi\tan\gamma\right)\dot{\sigma} + \left(-\dot{\psi}\sin\psi - \frac{d}{l}\dot{\psi}\cos\psi\tan\gamma - \frac{d}{l}\frac{\sin\psi}{\cos^2\gamma}\dot{\gamma}\right)\sigma \\
\ddot{y} &= \left(\sin\psi + \frac{d}{l}\cos\psi\tan\gamma\right)\dot{\sigma} + \left(\dot{\psi}\cos\psi - \frac{d}{l}\dot{\psi}\sin\psi\tan\gamma + \frac{d}{l}\frac{\cos\psi}{\cos^2\gamma}\dot{\gamma}\right)\sigma
\end{aligned} \tag{20}
$$

Therefore we have

$$
\begin{aligned}
\ddot{x}\cos(\psi+\gamma) + \ddot{y}\sin(\psi+\gamma) &= \dot{\sigma}\left(\cos\gamma + \frac{d}{l}\tan\gamma\sin\gamma\right) + \sigma\left(\dot{\psi}\sin\gamma - \frac{d}{l}\dot{\psi}\sin\gamma + \frac{d}{l}\frac{\sin\gamma}{\cos^2\gamma}\dot{\gamma}\right) \\
&= \dot{\sigma}\cos\gamma\left(1 + \frac{d}{l}\tan^2\gamma\right) + \frac{1}{l}\left(1 - \frac{d}{l}\right)\sin\gamma\tan\gamma\sigma^2 + \frac{d}{l}\frac{\sin\gamma}{\cos^2\gamma}\dot{\gamma}\sigma \\
\ddot{x}\cos\psi + \ddot{y}\sin\psi &= \dot{\sigma} + \dot{x}\dot{\psi}\sin\psi - \dot{y}\dot{\psi}\cos\psi \\
&= \dot{\sigma} - \frac{d}{l^2}\tan^2\gamma\sigma^2
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\nu_1 &= \frac{(R - F_{\text{wind}})\cos\gamma}{\sin\gamma} - \frac{m}{\sin\gamma}\left(\dot{\sigma}\cos\gamma\left(1 + \frac{d}{l}\tan^2\gamma\right) + \frac{1}{l}\left(1 - \frac{d}{l}\right)\sin\gamma\tan\gamma\sigma^2 + \frac{d}{l}\frac{\sin\gamma}{\cos^2\gamma}\dot{\gamma}\sigma\right) \\
&= \underbrace{\frac{(R - F_{\text{wind}})}{\tan\gamma} - \frac{m}{\tan\gamma}\dot{\sigma}\left(1 + \frac{d}{l}\tan^2\gamma\right)}_{\text{Singularity possible}} - \frac{m}{l}\left(1 - \frac{d}{l}\right)\tan\gamma\sigma^2 - m\frac{d}{l}\frac{1}{\cos^2\gamma}\dot{\gamma}\sigma \\
\nu_2 &= -\frac{(R - F_{\text{wind}})}{\sin\gamma} + \frac{m}{\sin\gamma}\left(\dot{\sigma} - \frac{d}{l^2}\tan^2\gamma\sigma^2\right) \\
&= \underbrace{-\frac{(R - F_{\text{wind}})}{\sin\gamma} + \frac{m}{\sin\gamma}\dot{\sigma}}_{\text{Singularity possible}} - m\frac{d}{l^2}\frac{\tan\gamma}{\cos\gamma}\sigma^2
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
&\frac{(R-F_{\text{wind}})}{\tan\gamma}-\frac{m}{\tan\gamma}\dot{\sigma}\left(1+\frac{d}{l}\tan^2\gamma\right)\\
&\qquad=\frac{(R-F_{\text{wind}})(m+m_0\tan^2\gamma)-m\left((R-F_{\text{wind}})-m_0\dfrac{\tan\gamma}{\cos^2\gamma}\dot{\gamma}\sigma\right)\left(1+\dfrac{d}{l}\tan^2\gamma\right)}{m\tan\gamma+m_0\tan^3\gamma}\\
&\qquad=\frac{(R-F_{\text{wind}})\left(m_0-m\dfrac{d}{l}\right)\tan^2\gamma+mm_0\dfrac{\tan\gamma}{\cos^2\gamma}\dot{\gamma}\sigma\left(1+\dfrac{d}{l}\tan^2\gamma\right)}{m\tan\gamma+m_0\tan^3\gamma}\\
&\qquad=\frac{(R-F_{\text{wind}})\left(m_0l-md\right)\sin\gamma\cos\gamma+mm_0\dot{\gamma}\sigma\left(l+d\tan^2\gamma\right)}{ml\cos^2\gamma+m_0l\sin^2\gamma}\\
&-\frac{(R-F_{\text{wind}})}{\sin\gamma}+\frac{m}{\sin\gamma}\dot{\sigma}\\
&\qquad=\frac{m(R-F_{\text{wind}})-mm_0\dfrac{\tan\gamma}{\cos^2\gamma}\dot{\gamma}\sigma-(R-F_{\text{wind}})(m+m_0\tan^2\gamma)}{m\sin\gamma+m_0\tan^2\gamma\sin\gamma}\\
&\qquad=\frac{-mm_0\dot{\gamma}\sigma-(R-F_{\text{wind}})m_0\sin\gamma\cos\gamma}{m\cos^3\gamma+m_0\sin^2\gamma\cos\gamma}\\
&\qquad=-\frac{mm_0\dot{\gamma}\sigma}{m\cos^3\gamma+m_0\sin^2\gamma\cos\gamma}-\frac{(R-F_{\text{wind}})m_0\sin\gamma}{m\cos^2\gamma+m_0\sin^2\gamma}
\end{aligned}
\tag{23}
$$

Finally we have

$$
\begin{aligned}
\nu_1&=-\frac{m}{l}\left(1-\frac{d}{l}\right)\tan\gamma\sigma^2+\frac{(R-F_{\text{wind}})\left(m_0l-md\right)\sin\gamma\cos\gamma}{ml\cos^2\gamma+m_0l\sin^2\gamma}+\frac{m(m_0l-md)\dot{\gamma}\sigma}{ml\cos^2\gamma+m_0l\sin^2\gamma}\\
&=-\frac{m}{l}\tan\gamma\left(1-\frac{d}{l}\right)\sigma^2+\frac{m_0-m\dfrac{d}{l}}{m+m_0\tan^2\gamma}\left((R-F_{\text{wind}})\tan\gamma+\frac{m\dot{\gamma}\sigma}{\cos^2\gamma}\right)\\
\nu_2&=-m\frac{d}{l^2}\frac{\tan\gamma}{\cos\gamma}\sigma^2-\frac{mm_0\dot{\gamma}\sigma}{m\cos^3\gamma+m_0\sin^2\gamma\cos\gamma}-\frac{(R-F_{\text{wind}})m_0\sin\gamma}{m\cos^2\gamma+m_0\sin^2\gamma}\\
&=-\frac{1}{\cos\gamma}\left[m\frac{d}{l^2}\tan\gamma\sigma^2+\frac{m_0}{m+m_0\tan^2\gamma}\left((R-F_{\text{wind}})\tan\gamma+\frac{m\dot{\gamma}\sigma}{\cos^2\gamma}\right)\right]
\end{aligned}
\tag{24}
$$

Verified through MAPLE derivation and MATLAB simulation.