

Independently drawn observations:

$X_1, X_2, \dots, X_n$  from population (probability distribution) with mean  $E(X)$  and variance  $V(X)$

We are interested in the distribution of the sample mean  $\bar{X}_n$  (i.e. the mean of the observations  $X_1, X_2, \dots, X_n$ )

Expectation of  $\bar{X}_n$ :

$$E(\bar{X}_n) = E\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n} \sum_{i=1}^n E(X_i) = E(X)$$

because

$$E(a(X_1 + X_2)) = a E(X_1 + X_2) = a E(X_1) + a E(X_2)$$

And for all  $i$ ,  $X_i$  is drawn from an identical distribution (population) with mean:  $E(X)$

Variance of  $\bar{X}_n$

$$V(\bar{X}_n) = V\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n^2} \sum_{i=1}^n V(X_i) = \frac{1}{n} V(X)$$

because

$$V(a(X_1 + X_2)) = a^2 V(X_1 + X_2) = a^2 V(X_1) + a^2 V(X_2)$$

And for all  $i$ ,  $X_i$  is drawn from an identical distribution (population) with variance  $V(X)$

For large enough sample ( $n$  is large),  
the distribution of sample mean  $\bar{X}_n$   
will converge to normal, and

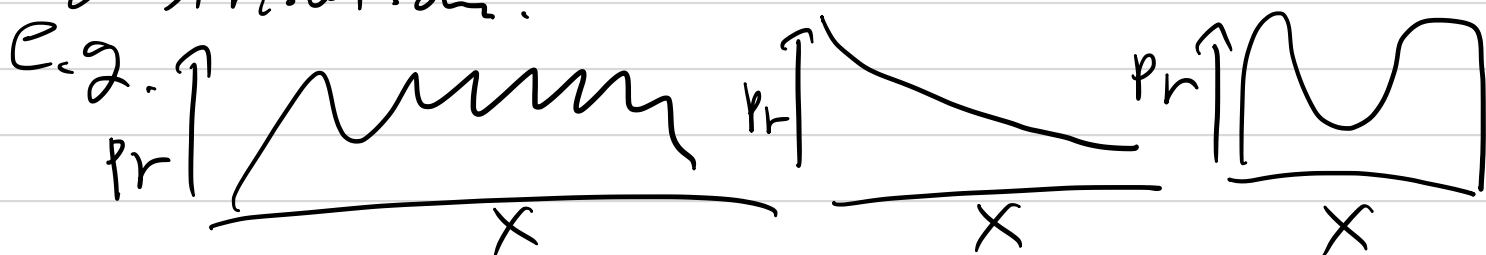
$$\frac{\bar{X}_n - E(X)}{\sqrt{V(X)/n}} \rightsquigarrow N(0,1)$$

converge in distribution  
as  $n$  grows

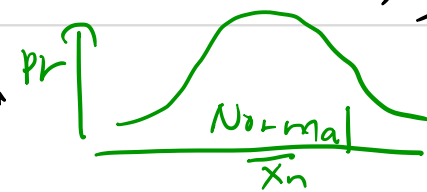
↑ This is same as

$$\frac{\bar{X}_n - E(\bar{X}_n)}{\sqrt{\text{Var}(\bar{X}_n)}}$$

Recall that population can follow ANY  
distribution:



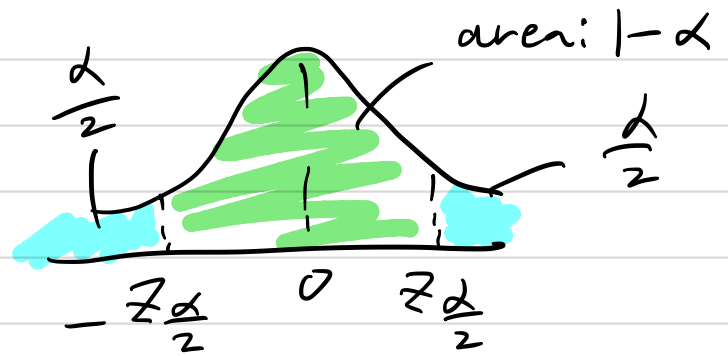
HOWEVER once you obtain the samples  
and draw the distribution of their  
MEANS, due to Central Limit Theorem:  
it will always follow  $N(E(X), V(X)/n)$   
when  $n$  is large.



# Hypothesis testing

$$H_0: \mu = \mu_0 \longrightarrow H_1: \mu \neq \mu_0$$

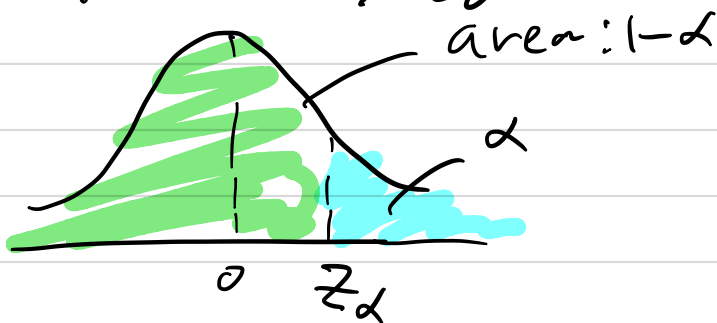
population mean



$$H_1: \mu < \mu_0$$



$$H_1: \mu > \mu_0$$



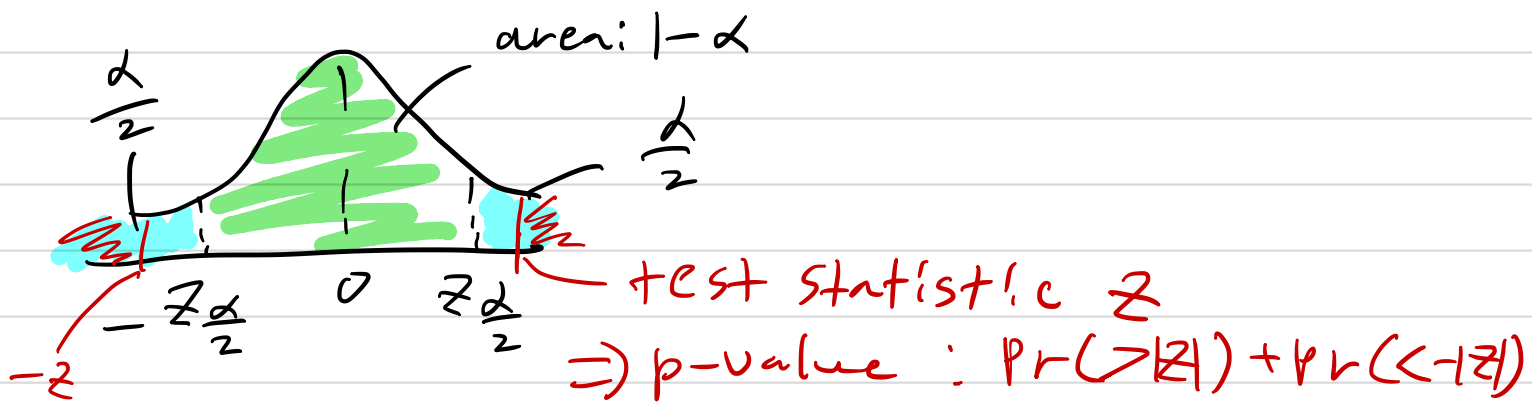
If your test statistic belongs to

{  fail to reject the null!

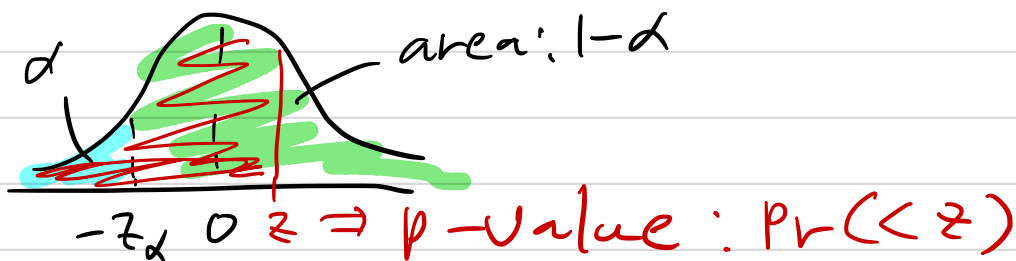
{  reject the null

Recall: Hypothesis testing is to check how likely we observe THE sample mean given  $H_0$  is TRUE:  $\mu = \mu_0$

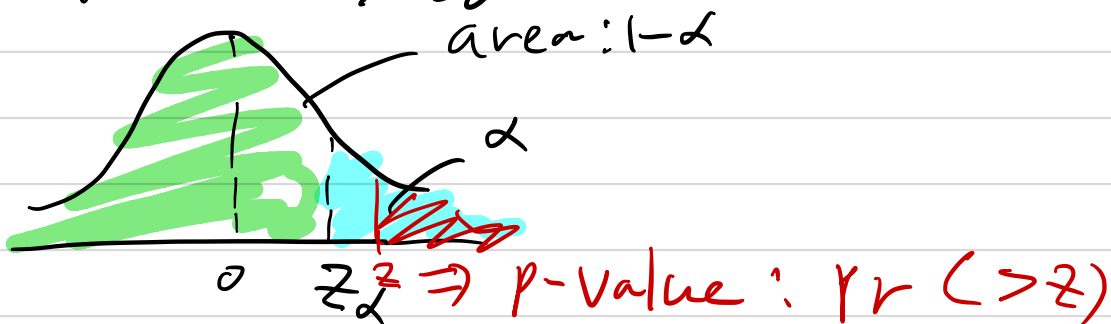
$$H_1: \mu \neq \mu_0$$



$$H_1: \mu < \mu_0$$



$$H_1: \mu > \mu_0$$



If  $p\text{-value} \leq \alpha \Rightarrow$  reject the null!

$p\text{-value} > \alpha \Rightarrow$  fail to reject the null!

"=" case is negligible since it is almost impossible to get a p-value exactly same as  $\alpha$ .

Tea tasting experiment

$H_0$ :  $\theta$  chance of being correct  $= 0.5$

$H_1$ :  $\theta > 0.5$

p-value for getting everything correct  
if  $H_0$  is true:  $\theta = 0.5$

The setting: Complete Randomization w/ 12  
M: 4 and T: 4

Let the subject know M: 4 and T: 4

# Entire possibilities:  $8C_4 = 70$

# Correct choices

|         | 0           | 2           | 4           | 6           | 8           |
|---------|-------------|-------------|-------------|-------------|-------------|
| # Cases | $4C_0 4C_4$ | $4C_1 4C_3$ | $4C_2 4C_2$ | $4C_3 4C_1$ | $4C_4 4C_0$ |
|         | 1           | 16          | 36          | 16          | 1           |

Probability  $\frac{1}{70}$   $\frac{16}{70}$   $\frac{36}{70}$   $\frac{16}{70}$   $\frac{1}{70}$

Because

If  $H_0$  is true:  $\theta = 0.5$ , then every case among the 70 cases will have equal probability to occur!

p-value (# correct choices = 8) =  $\frac{1}{70}$

p-value ( "  $\geq 6$  ) =  $\frac{16}{70} + \frac{1}{70} = \frac{17}{70}$

Because we are conducting right-sided test!  
(i.e.,  $H_1: \theta > \mu_0$ )