Sample variance

$$E(S^2) = E\left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}\right)$$

Let $\sum$ denote $\sum_{i=1}^{n}$

$$= \frac{1}{n-1}E\left(\sum(x_i^2-2x_i\bar{x}+\bar{x}^2)\right)$$

$$= \frac{1}{n-1}E\left(\sum x_i^2-\sum 2x_i\bar{x}+\sum\bar{x}^2\right)$$

$$= \frac{1}{n-1}E\left(\sum x_i^2-2\bar{x}\underbrace{\cancel{\sum x_i}}_{n\bar{x}}+n\bar{x}^2\right)$$

$$= \frac{1}{n-1}\left(E\left(\sum x_i^2\right)-E(n\bar{x}^2)\right)$$

$$= \frac{1}{n-1}\left(\sum E(x_i^2)-nE(\bar{x}^2)\right)$$

$$= \frac{1}{n-1}\left(\sum(\sigma^2+\mu^2)-n\left(\frac{\sigma^2}{n}+\mu^2\right)\right)$$

$$= \frac{1}{n-1}\left(n\sigma^2+n\mu^2-\sigma^2-n\mu^2\right)$$

$$= \frac{1}{n-1}\left((n-1)\sigma^2\right) = \sigma^2$$

population variance

S^2 is unbiased estimator of σ^2

$$Var(x_i) = \sigma^2 = E(x_i^2)-\mu^2$$

$$\Rightarrow E(x_i^2) = \sigma^2+\mu^2$$

$$Var(\bar{x}) = E(\bar{x}^2)-(E(\bar{x}))^2$$

$$\Rightarrow E(\bar{x}^2) = Var(\bar{x})+(E(\bar{x}))^2$$

$$= \frac{\sigma^2}{n}+\mu^2$$

because $Var(\bar{x}) = Var(\sum x_i/n) = \frac{1}{n^2}Var(\sum x_i)$

$$= \frac{1}{n}Var(x_i) = \frac{\sigma^2}{n}$$