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Probability Distributions

Week 11

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Contents

Random Variables and Probability Distribution

- Random variable assigns a numeric value to each event of the experiment.
  - Coin flip side: head = 1; tail =0
  - #secs took for commuting: any value greater than 0.
- These values represent mutually exclusive and exhaustive events, together forming the entire sample space Ω.
- A discrete RV takes a finite or at most countably infinite #distinct values.
  - coin flip; race; number of years of education
- A continuous RV assumes an uncountably infinite number of values.
  - height, distance from earth, gross domestic product
- Probability distribution: Probability that a random variable takes a certain value or range of values.
  - P(side): P(side=1) = 0.5; P(side=0) = 0.5
  - P(#secs): P(0<#secs<1000) = 0.3; P(1000<#secs<2000) = 0.4 ....

Probability Density / Mass Function

- Probability mass function (PMF): f(x) for a discrete random variable
- Probability density function (PDF): f(x) for a continuous random variable
- Recall $P(\Omega) = 1$: total sum of f(x) (PMF), or the area of f(x) (PDF), must equal to 1.
- Cumulative mass function (CMF):

$$F(x) = P(X \leq x) = \sum_{K \leq x} f(k)$$

- Cumulative density function (CDF):

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

  - What is the probability that a random variable X takes a value equal to or less than x?
  - Area under the density curve
  - Non-decreasing

Bernoulli Distribution

- $\Omega=\{0,1\}$
- PMF

$$f(x) = \begin{cases} p & \textbf{if } x = 1 \\ 1-p & \textbf{if } x = 0 \\ 0 & \textbf{otherwise.} \end{cases}$$

- CMF

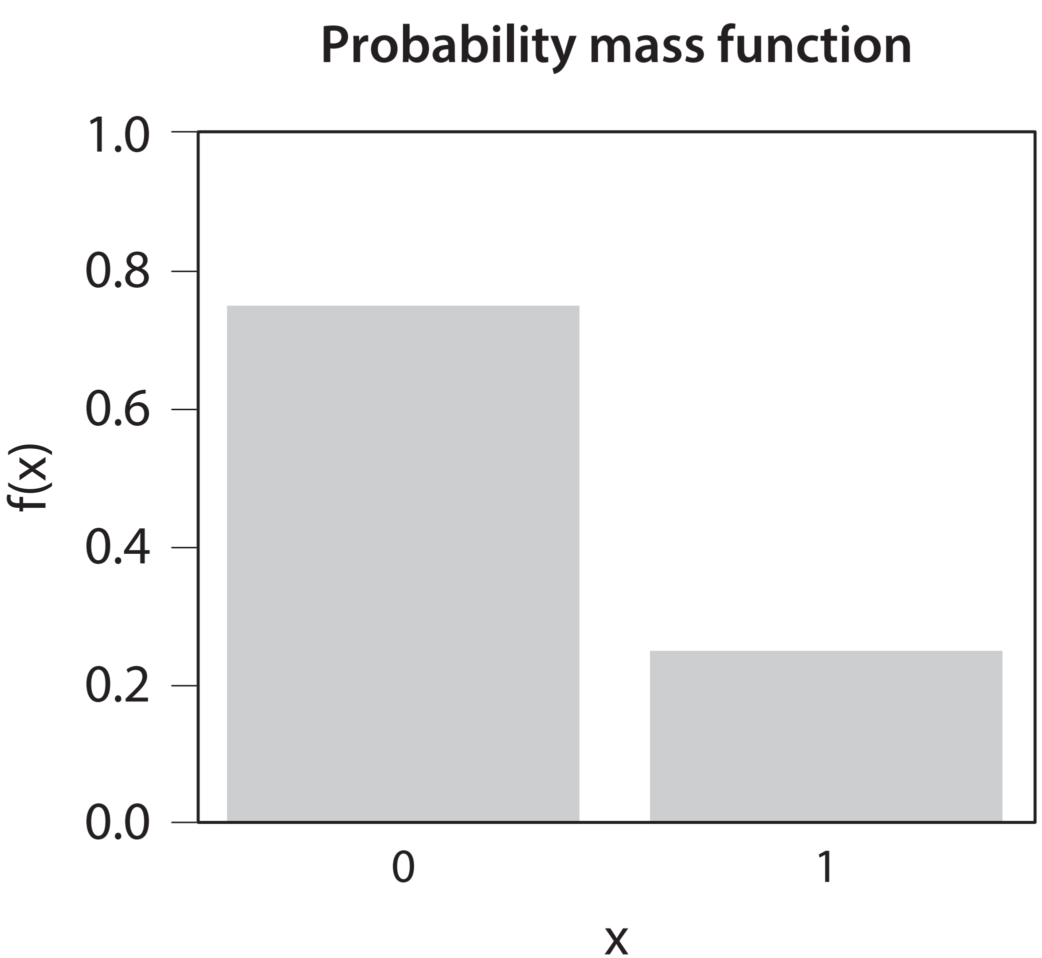
$$F(x) = \begin{cases} 0 & \textbf{if } x < 0 \\ 1-p & \textbf{if } 0 \le x < 1 \\ 1 & \textbf{if } x \ge 1 \end{cases}$$

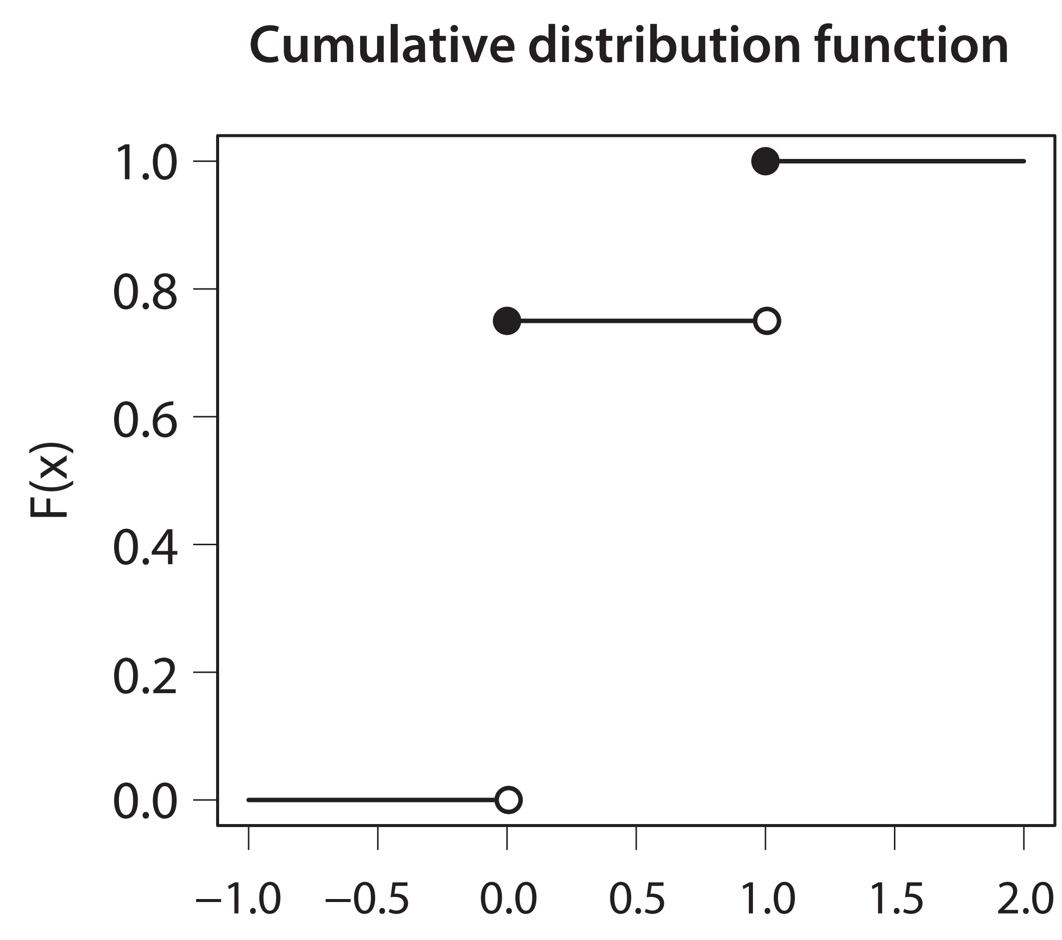
Bernoulli Distribution

- $\Omega=\{0,1\}$
- PMF

$$f(x) = \begin{cases} p & \textbf{if } x = 1 \\ 1-p & \textbf{if } x = 0 \\ 0 & \textbf{otherwise.} \end{cases}$$

- PMF and CDF of Bernoulli distribution for p = 0.25





Binomial Distribution

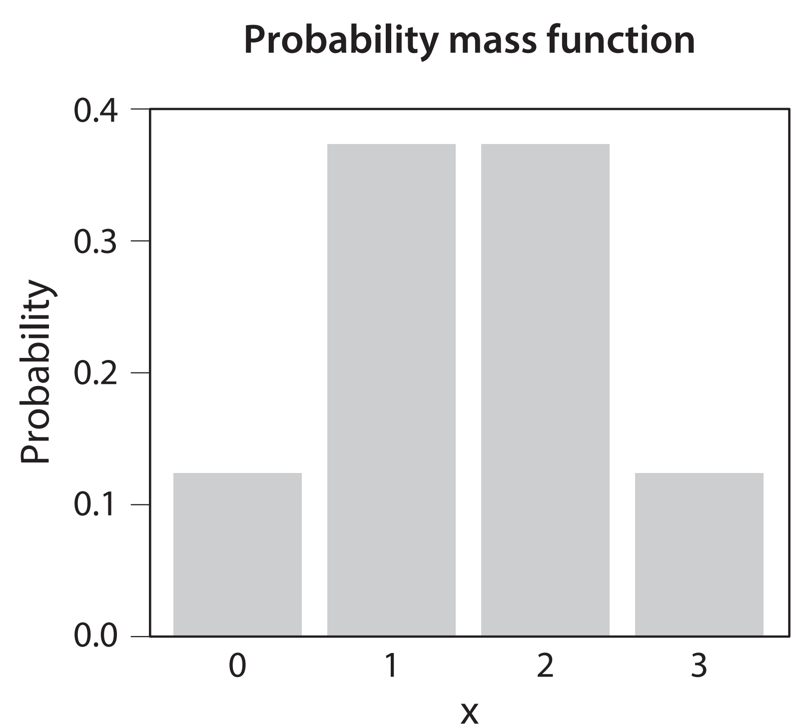
- The number of 1s (one of the binary outcomes) in multiple Bernoulli trials
- Ω={0,1,….,n-1,n}
- PMF

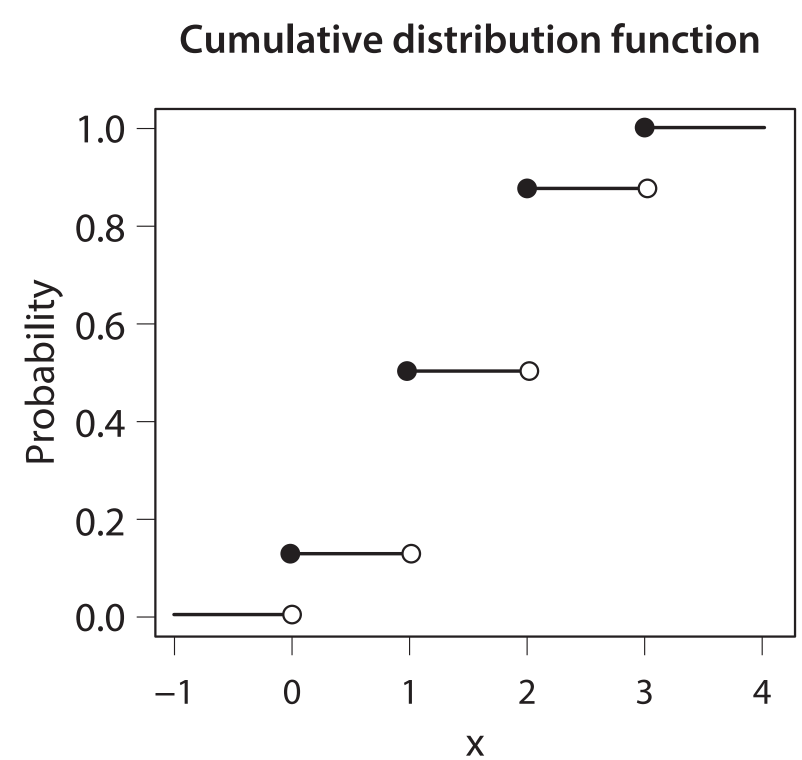
$$f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad \binom{n}{x} = {}_n C_x$$

- CMF

$$F(x) = P(X \leq x) = \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k}$$

- $p = 0.5$ and $n = 3$





Binomial Distribution

- ex1) In a small department: There are exactly 10 who support candidate A, another 10 people who support candidate B for electing the chair of the department. Suppose that we expect their individual turnout probability is equal to their previous overall turnout rate which was 70%. What is the chance that exactly 7 people vote for candidate A and 7 people vote for candidate B, and the election ends in a tie?

- ex2) In a small department: There are exactly 10 who support candidate A, another 10 people who support candidate B for electing the chair of the department. Suppose that we expect their individual turnout probability is equal to their previous overall turnout rate which was 70%. What is the chance that the election ends in a tie?

Uniform Distribution

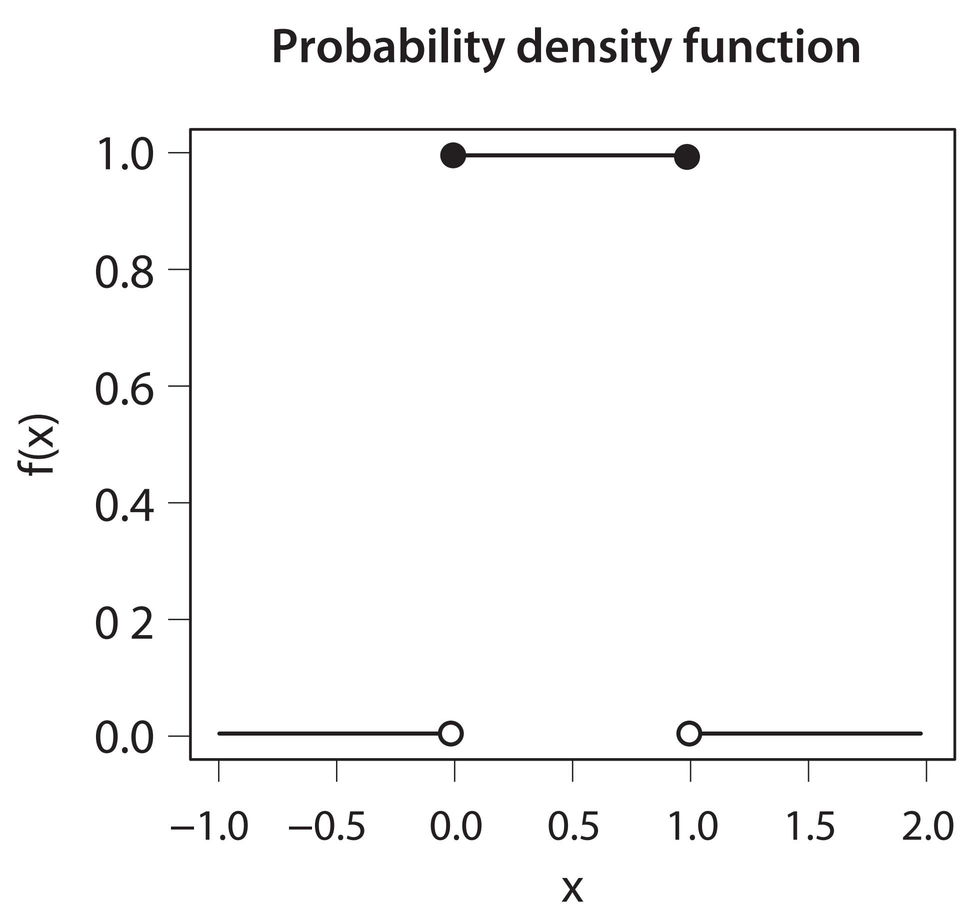
- Every number in an interval has an equal chance of appearance
- $\Omega =$ **set of real numbers in a range** $[a, b]$
- PDF

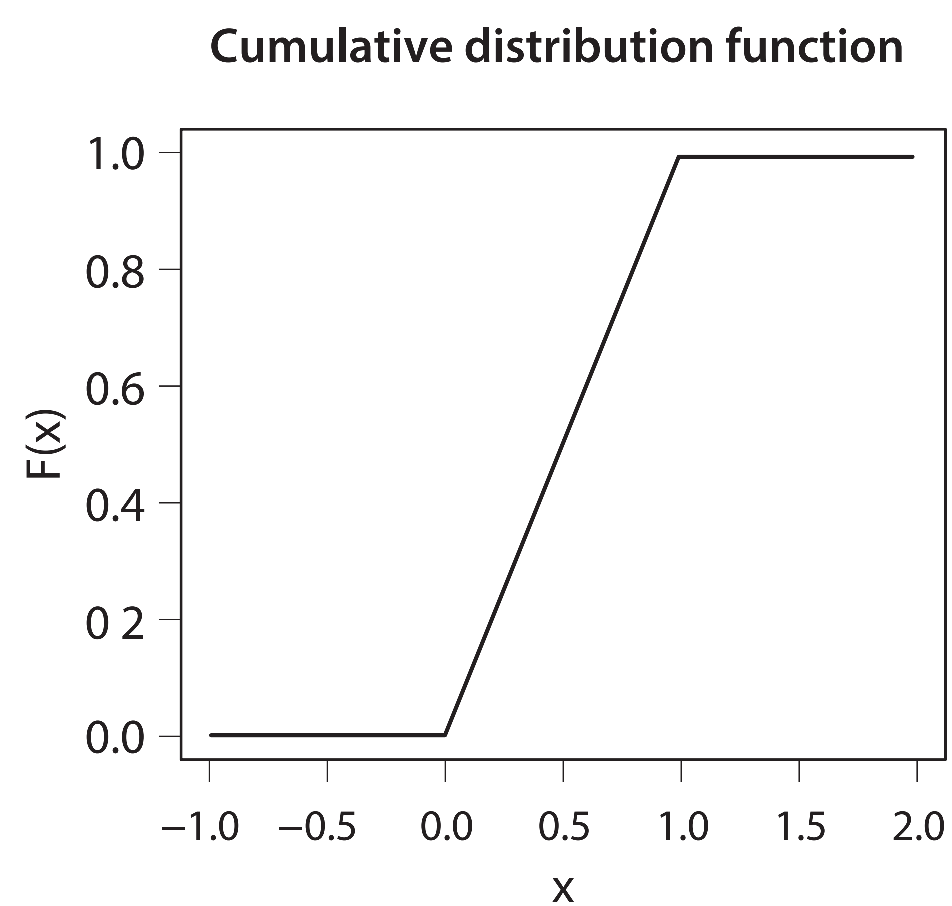
$$f(x) = \begin{cases} \frac{1}{b-a} & \textbf{if } a \le x \le b \\ 0 & \textbf{otherwise.} \end{cases}$$

- CDF

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & \textbf{if } a \le x \le b \\ 1 & x \ge b \end{cases}$$

- Uniform distribution for the interval [0,1]

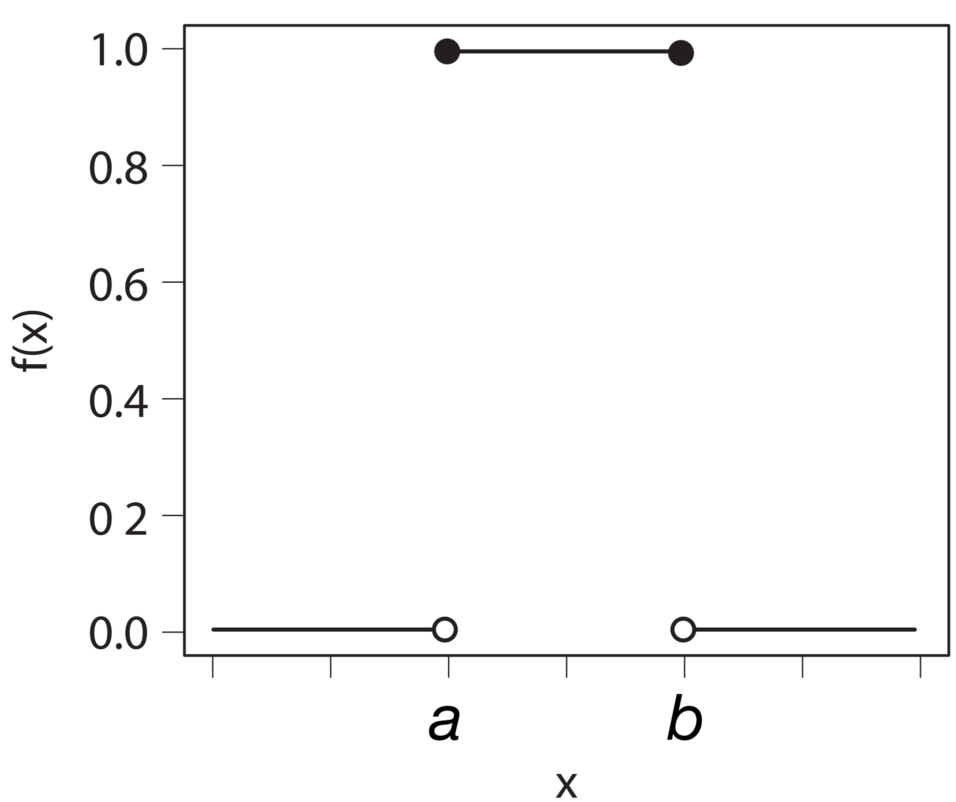




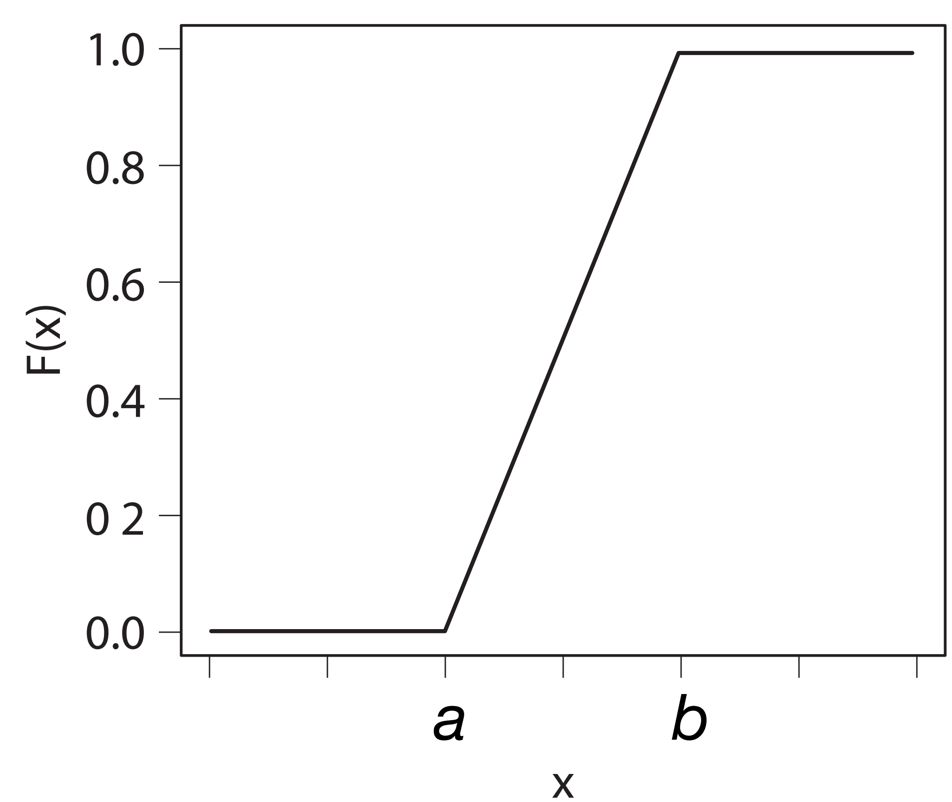
Uniform Distribution

- Every number in an interval has an equal chance of appearance
- $\Omega =$ **set of real numbers**
- PDF
$$f(x) = \begin{cases} \frac{1}{b-a} & \textbf{if } a \le x \le b \\ 0 & \textbf{otherwise.} \end{cases}$$
- CDF
$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & \textbf{if } a \le x \le b \\ 1 & x \ge b \end{cases}$$
- Uniform distribution for the interval $[a,b]$

Probability density function

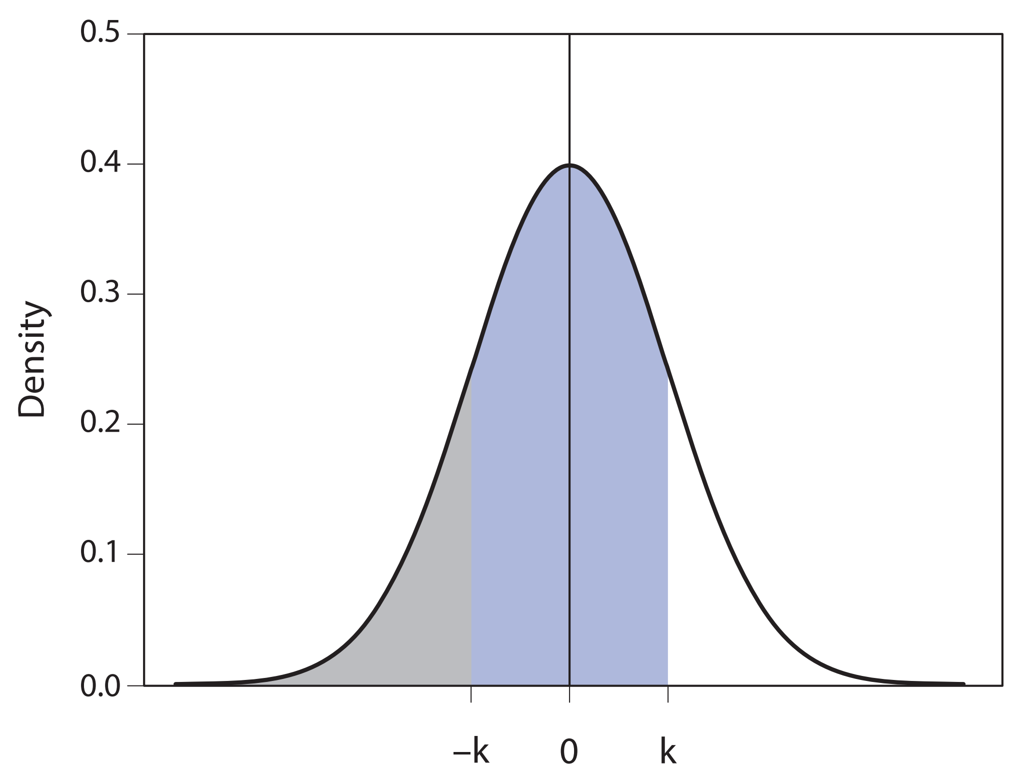


Cumulative distribution function



Normal Distribution

- Most famous and frequently observed distribution (Why?)
- Ω=real numbers (continuous number)
- X is normal RV with mean μ and standard deviation σ: $X \sim \mathcal{N}(\mu, \sigma^2)$
- PDF
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$
- e.g. $X \sim \mathcal{N}(0,1)$



- Singled peaked, symmetric
- about 2/3 are within 1 standard deviation (σ) from the mean
- about 95% are within 2 standard deviations (2σ) from the mean

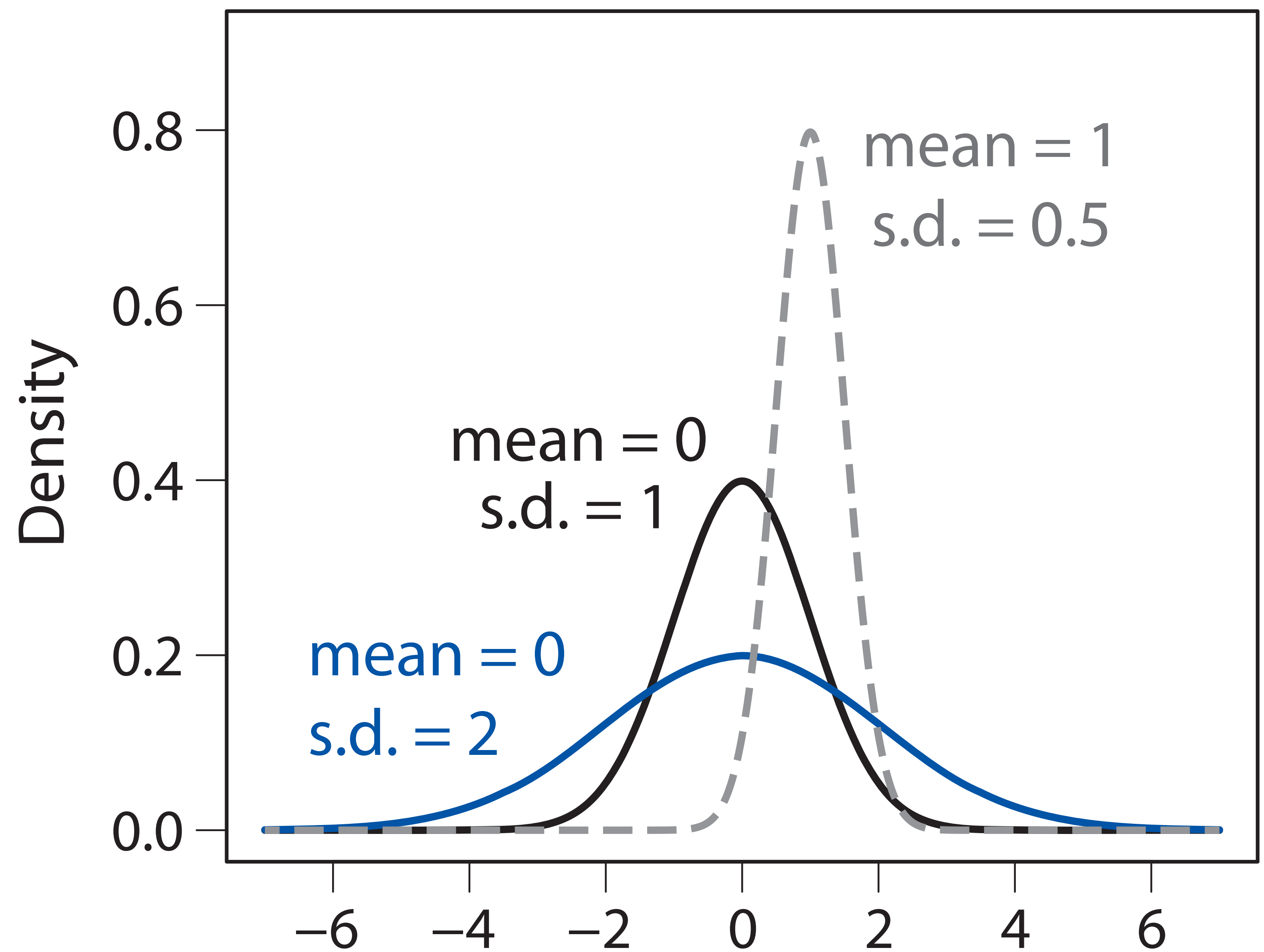
- CDF $$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(t-\mu)^2\right\} dt$$

Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

- $X \sim \mathcal{N}(\mu, \sigma^2)$
- $S = X + c \rightarrow S \sim \mathcal{N}(\mu + c, \sigma^2)$
- $Y = cX \rightarrow Y \sim \mathcal{N}(c\mu, (c\sigma)^2)$
- $T = aX + b \rightarrow T \sim \mathcal{N}(a\mu + b, (a\sigma)^2)$

Probability density function



Normal Distribution

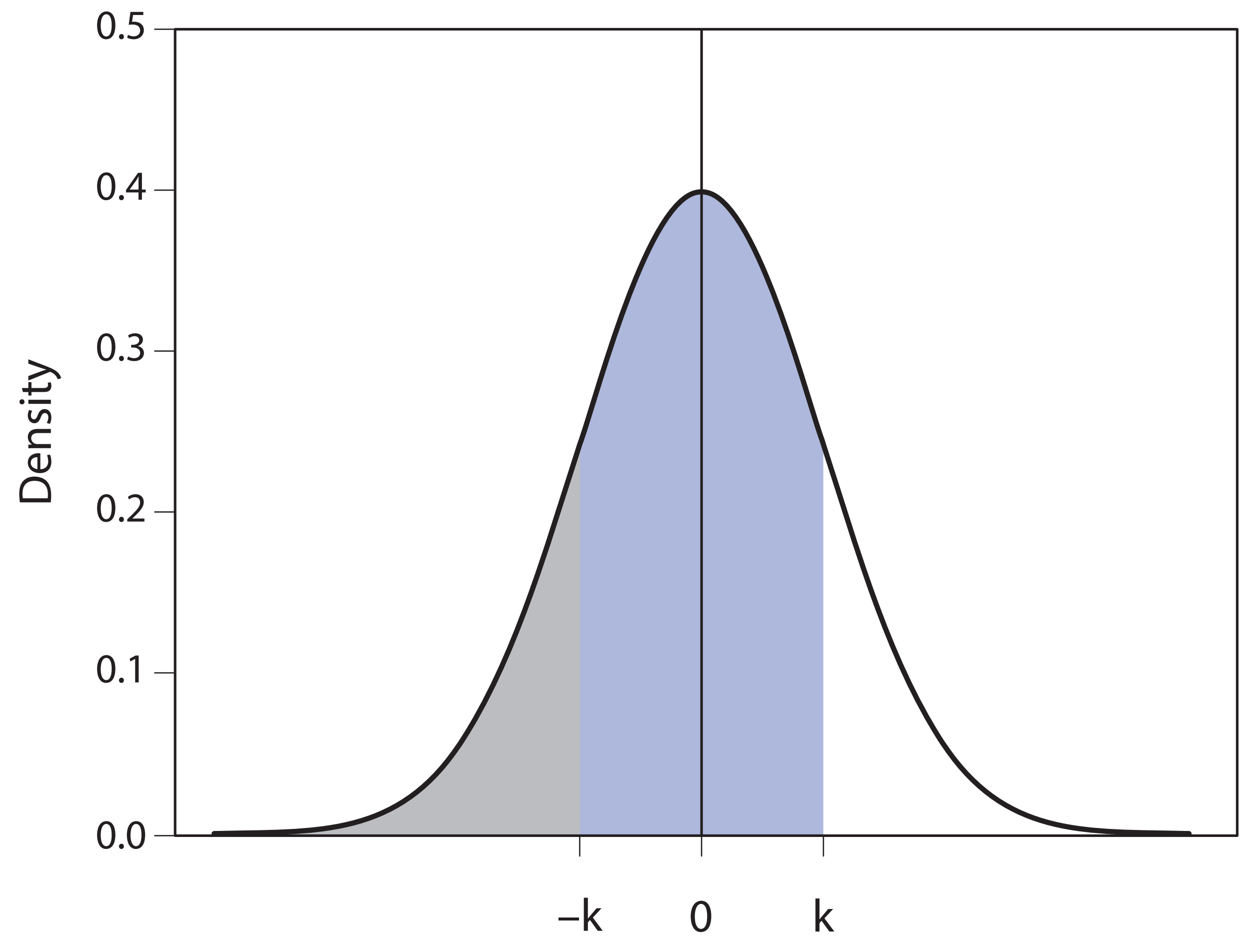
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

- $X \sim \mathcal{N}(\mu, \sigma^2)$
- $S = X + c \rightarrow S \sim \mathcal{N}(\mu + c, \sigma^2)$
- $Y = cX \rightarrow Y \sim \mathcal{N}(c\mu, (c\sigma)^2)$
- $T = aX + b \rightarrow T \sim \mathcal{N}(a\mu + b, (a\sigma)^2)$
- **z-score:** $Z = (X - \mu)/\sigma \rightarrow Z \sim \mathcal{N}(0,1)$

https://www.youtube.com/watch?v=nqNzKeVCYBU

Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$



- Probability that a normal random variable with mean **μ** and sdv **σ** lies within k standard deviations from the mean for a positive constant $k > 0$

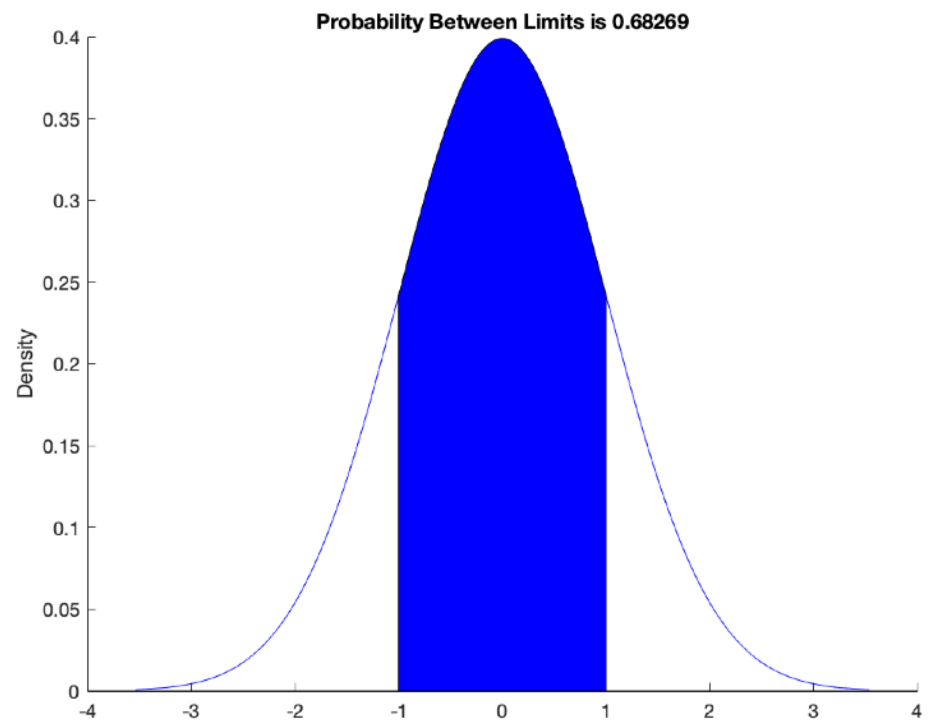
$$\begin{aligned} P(\mu - k\sigma \leq X \leq \mu + k\sigma) &= P(-k\sigma \leq X - \mu \leq k\sigma) \\ &= P\left(-k \leq \frac{X-\mu}{\sigma} \leq k\right) \\ &= P(-k \leq Z \leq k), \end{aligned}$$

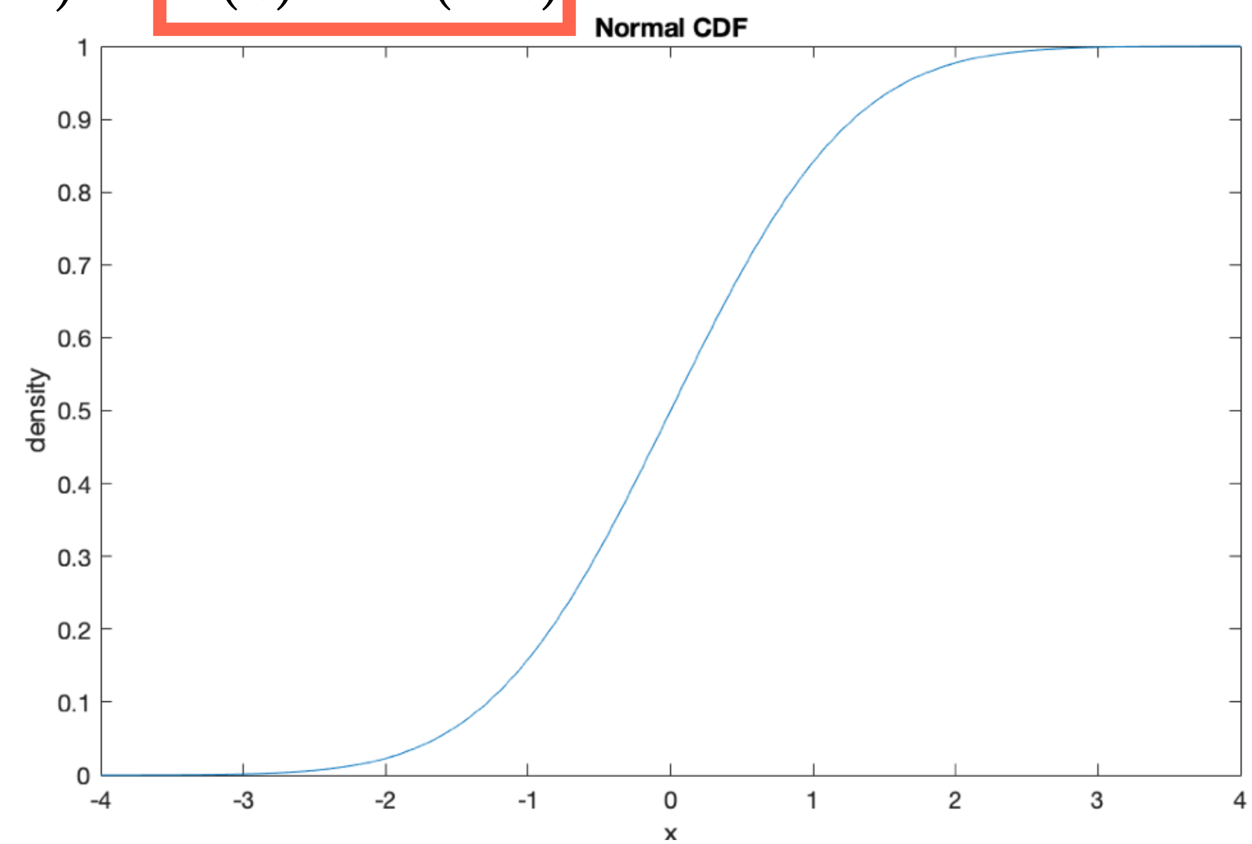
$$P(-k \leq Z \leq k) = P(Z \leq k) - P(Z \leq -k) = \boxed{F(k) - F(-k)}$$

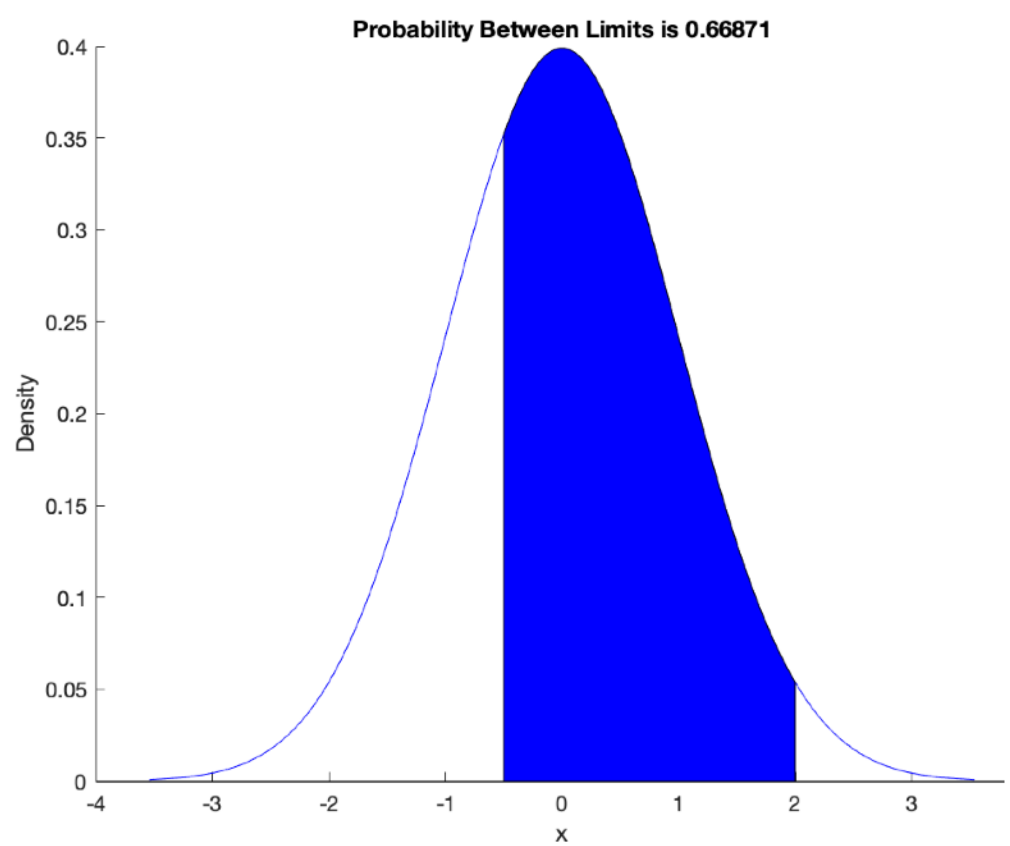
Normal Distribution

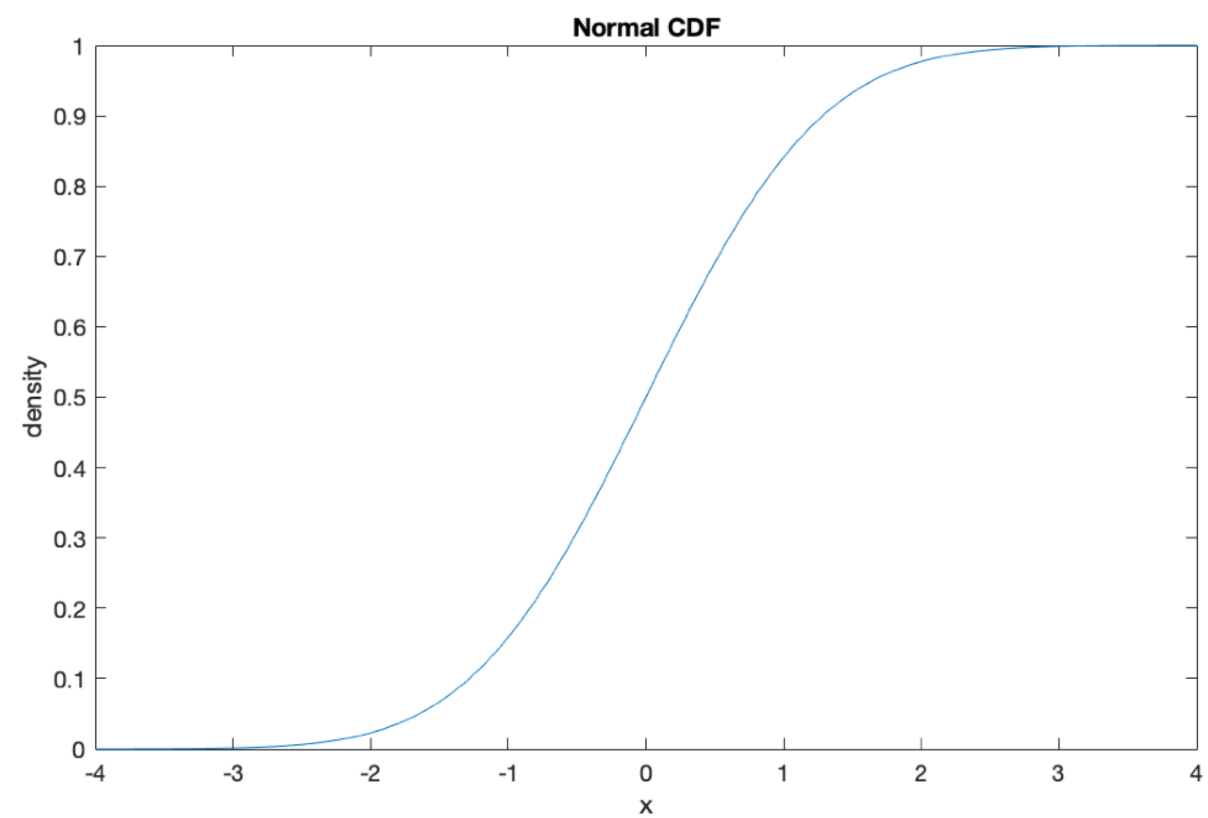
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

$$P(-k \le Z \le k) = P(Z \le k) - P(Z \le -k) = \boxed{F(k) - F(-k)}$$









Normal Distribution

- Singled peaked, symmetric
- about 2/3 are within 1 standard deviation (σ) from the mean
- about 95% are within 2 standard deviations (2σ) from the mean

```
## plus minus 1 standard deviation from the mean
pnorm(1) - pnorm(-1)

## [1] 0.6826895

## plus minus 2 standard deviations from the mean
pnorm(2) - pnorm(-2)

## [1] 0.9544997
```

```
mu <- 5
sigma <- 2
## plus minus 1 standard deviation from the mean
pnorm(mu + sigma, mean = mu, sd = sigma) - pnorm(mu - sigma, mean = mu, sd = sigma)

## [1] 0.6826895

## plus minus 2 standard deviations from the mean
pnorm(mu + 2*sigma, mean = mu, sd = sigma) - pnorm(mu - 2*sigma, mean = mu, sd = sigma)

## [1] 0.9544997
```

Expectation: Definition and General Properties

- Expectation (population mean) of a random variable X
  - Fixed value given a probability distribution (different from sample means)

$$\mathbb{E}(X) = \begin{cases} \sum_x x \times f(x) & \text{if } X \text{ is discrete,} \\ \int x \times f(x)\,dx & \text{if } X \text{ is continuous} \end{cases}$$

- Properties of expectation (a, b: constant values; X,Y: independent RVs)

1. $\mathbb{E}(a) = a$.
2. $\mathbb{E}(aX) = a\mathbb{E}(X)$.
3. $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$.
4. $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$.
5. If X and Y are independent, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$. But generally, $\mathbb{E}(XY) \neq \mathbb{E}(X)\mathbb{E}(Y)$.

Expectation: Examples

$$\mathbb{E}(X) = \begin{cases} \sum_x x \times f(x) & \text{if } X \text{ is discrete,} \\ \int x \times f(x)\,dx & \text{if } X \text{ is continuous} \end{cases}$$

- Expectation (population mean)
  - Expected value of a random variable
    - e.g. PMF: Bernoulli random variable

$$\mathbb{E}(X) = 0 \times P(X=0) + 1 \times P(X=1) = 0 \times f(0) + 1 \times f(1) = 0 \times (1-p) + 1 \times p = p$$

    - e.g. PMF: Binomial random variable

$$\mathbb{E}(X) = 0 \times f(0) + 1 \times f(1) + \cdots + n \times f(n) = \sum_{x=0}^{n} x \times f(x)$$

    - e.g. PMF: Binomial random variable (Y_i is a Bernoulli RV with p)

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \mathbb{E}(Y_i) = np$$

    - e.g. PDF: uniform random variable defined in the interval $[a,b]$

$$\mathbb{E}(X) = \int_a^b x \times f(x)\,dx = \int_a^b \frac{x}{b-a}dx = \left.\frac{x^2}{2(b-a)}\right|_a^b = \frac{a+b}{2}$$

Summary

- Random Variables and Probability Distributions (Chapter 6.3)
  - Overview
  - Bernoulli / Binomial distribution
  - Uniform / Normal distribution
  - Expectation

See you next week.