Probability Distributions
Week 11

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Contents

- Random Variables and Probability Distributions (Chapter 6.3)
  - Overview
  - Bernoulli / Binomial distribution
  - Uniform / Normal distribution
  - Expectation and Variance
- Election prediction (6.3.6)
- The Law of large numbers (6.4.1)
- Central limit theorem (6.4.2)

Random Variables and Probability Distribution

- Random variable assigns a numeric value to each event of the experiment.
  - Coin flip side: head = 1; tail =0
  - #secs took for commuting: any value greater than 0.
- These values represent mutually exclusive and exhaustive events, together forming the entire sample space Ω.
- A discrete RV takes a finite or at most countably infinite #distinct values.
  - coin flip; race; number of years of education
- A continuous RV assumes an uncountably infinite number of values.
  - height, distance from earth, gross domestic product
- Probability distribution: Probability that a random variable takes a certain value or range of values.
  - P(side): P(side=1) = 0.5; P(side=0) = 0.5
  - P(#secs): P(0<#secs<1000) = 0.3; P(1000<#secs<2000) = 0.4 ....

Probability Density / Mass Function

- Probability mass function (PMF): f(x) for a discrete random variable
- Probability density function (PDF): f(x) for a continuous random variable
- Recall $P(\Omega) = 1$: total sum of f(x) (PMF), or the area of f(x) (PDF), must equal to 1.
- Cumulative mass function (CMF):

$$F(x) = P(X \leq x) = \sum_{K \leq x} f(k)$$

- Cumulative density function (CDF):

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

  - What is the probability that a random variable X takes a value equal to or less than x?
  - Area under the density curve
  - Non-decreasing

Bernoulli Distribution

- $\Omega=\{0,1\}$
- PMF

$$f(x) = \begin{cases} p & \textbf{if } x = 1 \\ 1-p & \textbf{if } x = 0 \\ 0 & \textbf{otherwise.} \end{cases}$$

- CMF

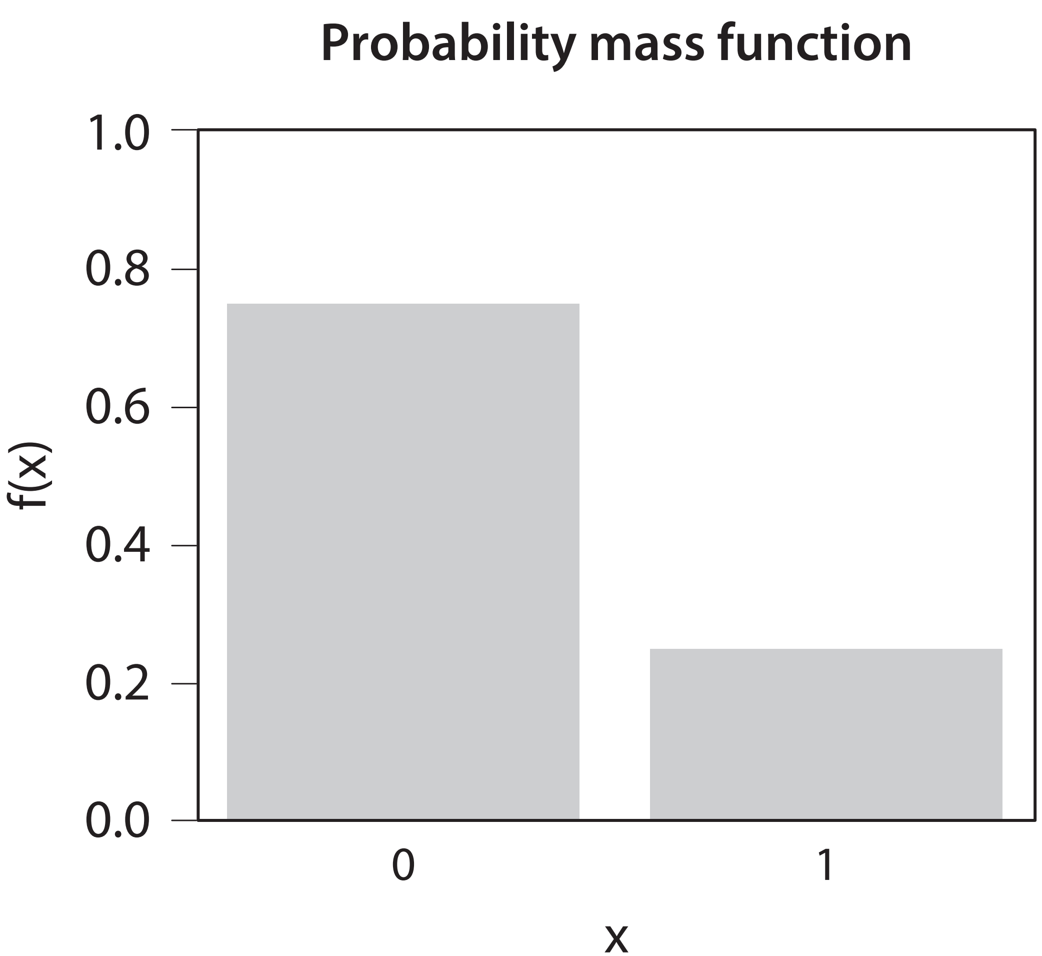
$$F(x) = \begin{cases} 0 & \textbf{if } x < 0 \\ 1-p & \textbf{if } 0 \le x < 1 \\ 1 & \textbf{if } x \ge 1 \end{cases}$$

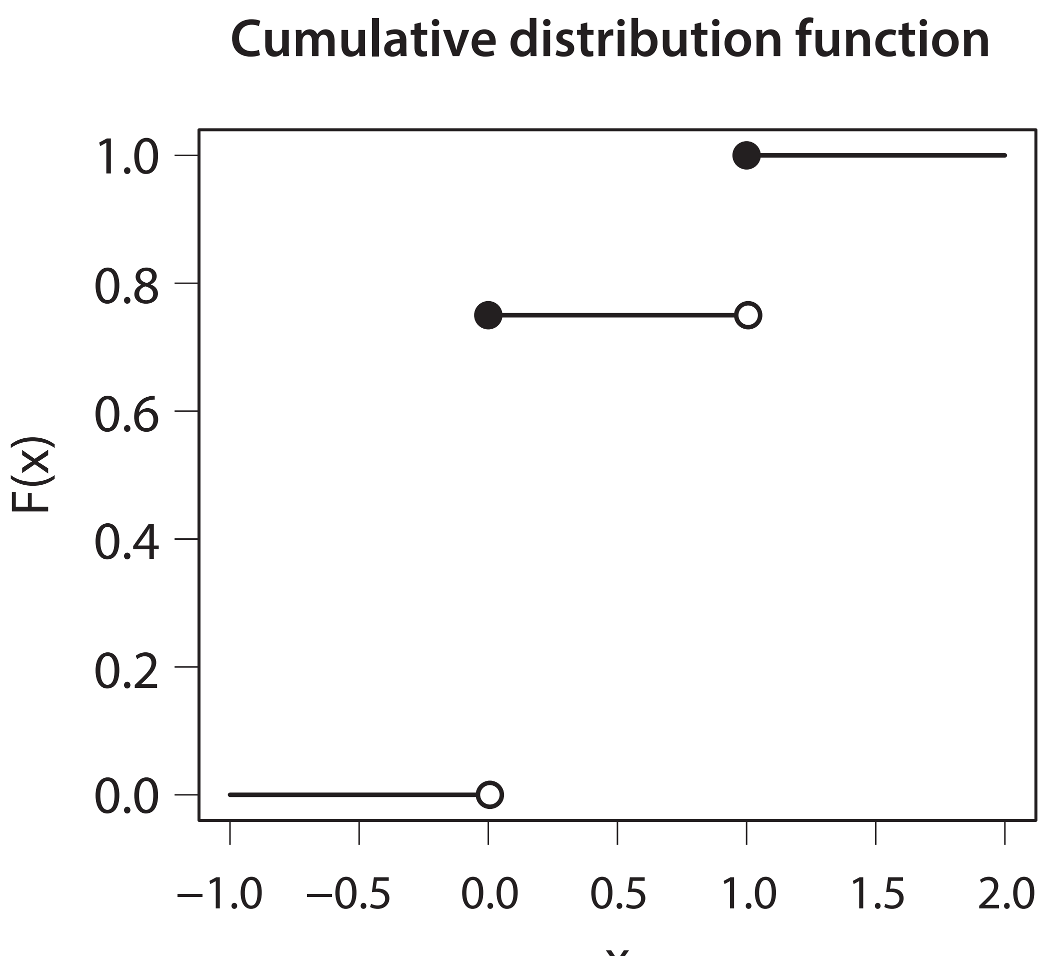
Bernoulli Distribution

- $\Omega=\{0,1\}$
- PMF

$$f(x) = \begin{cases} p & \textbf{if } x = 1 \\ 1-p & \textbf{if } x = 0 \\ 0 & \textbf{otherwise.} \end{cases}$$

- PMF and CDF of Bernoulli distribution for p = 0.25





Binomial Distribution

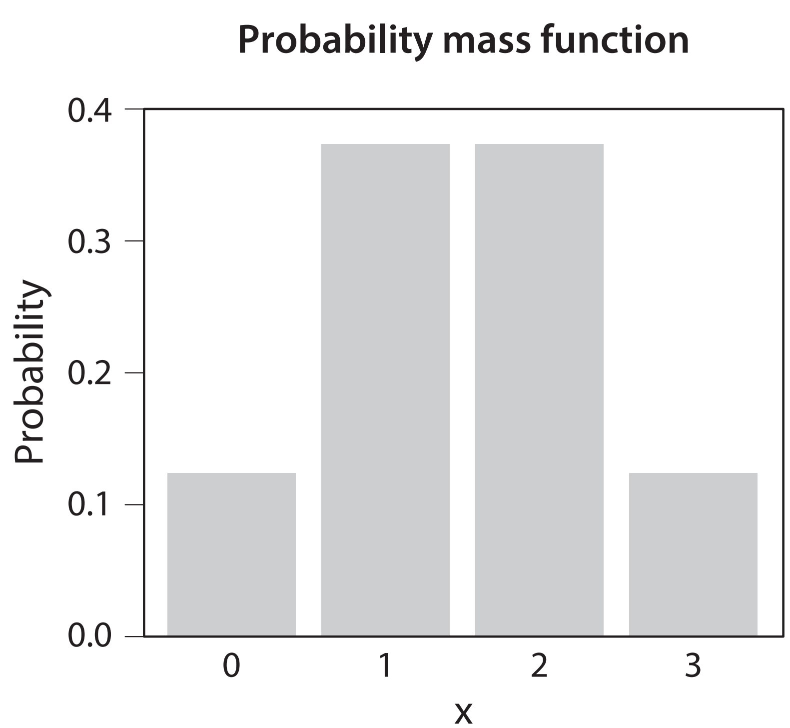
- The number of 1s (one of the binary outcomes) in multiple Bernoulli trials
- $\Omega=\{0,1,\ldots,n-1,n\}$
- PMF

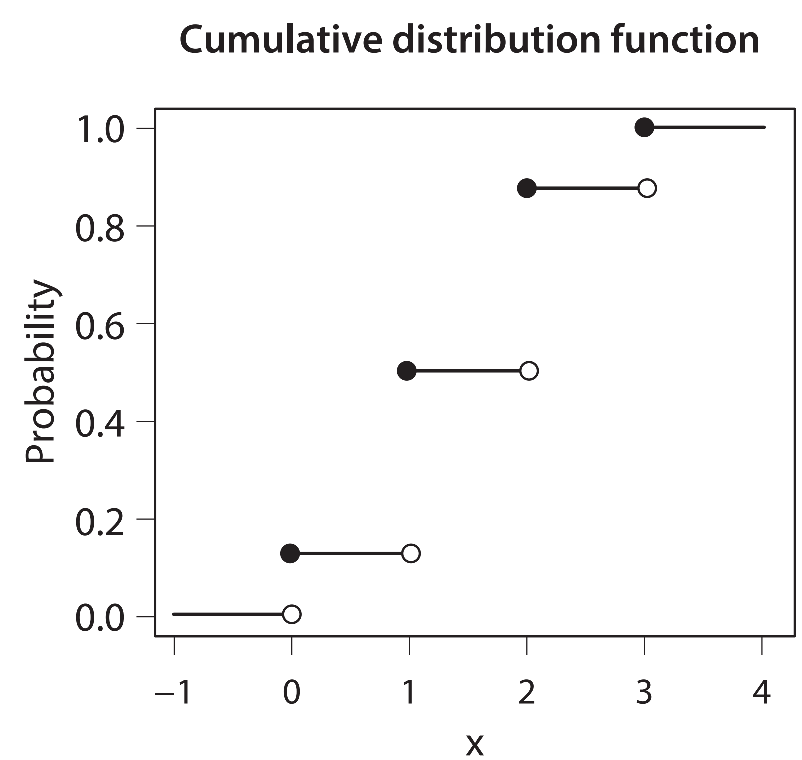
$$f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad \binom{n}{x} =_n C_x$$

- CMF

$$F(x) = P(X \le x) = \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k}$$

- $p = 0.5$ and $n = 3$





Binomial Distribution

- ex1) In a small department: There are exactly 10 who support candidate A, another 10 people who support candidate B for electing the chair of the department. Suppose that we expect their individual turnout probability is equal to their previous overall turnout rate which was 70%. What is the chance that exactly 7 people vote for candidate A and 7 people vote for candidate B, and the election ends in a tie?

- ex2) In a small department: There are exactly 10 who support candidate A, another 10 people who support candidate B for electing the chair of the department. Suppose that we expect their individual turnout probability is equal to their previous overall turnout rate which was 70%. What is the chance that the election ends in a tie?

Uniform Distribution

- Every number in an interval has an equal chance of appearance
- $\Omega =$ **set of real numbers**
- PDF

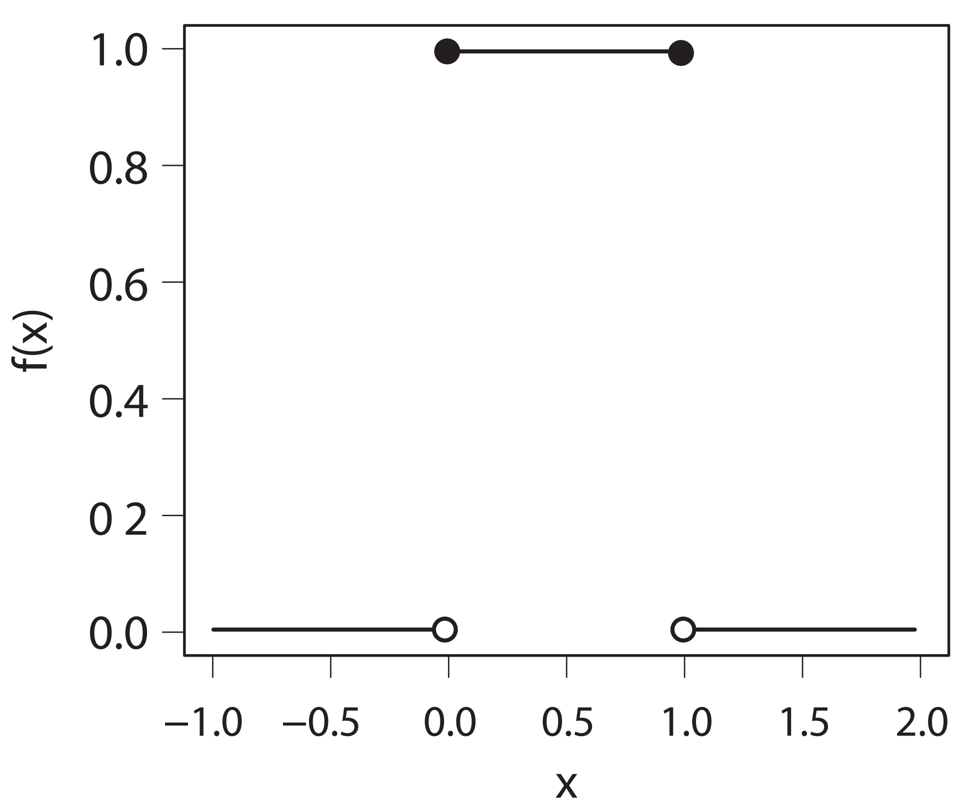
$$f(x) = \begin{cases} \frac{1}{b-a} & \textbf{if } a \le x \le b \\ 0 & \textbf{otherwise.} \end{cases}$$

- CDF

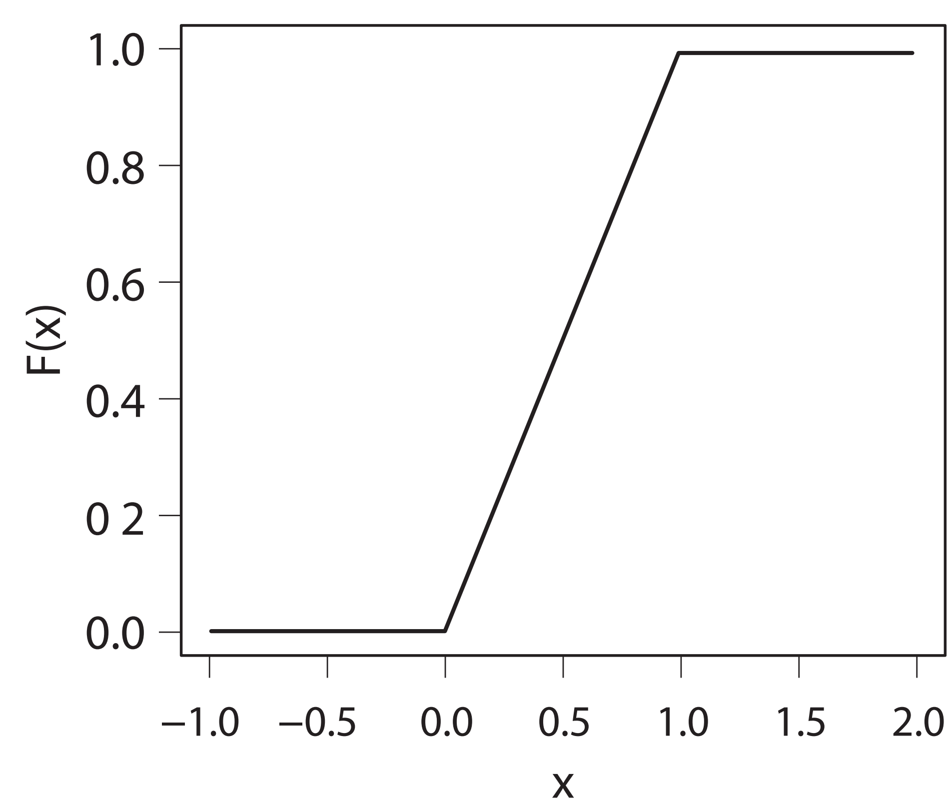
$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & \textbf{if } a \le x \le b \\ 1 & x \ge b \end{cases}$$

- Uniform distribution for the interval [0,1]

Probability density function

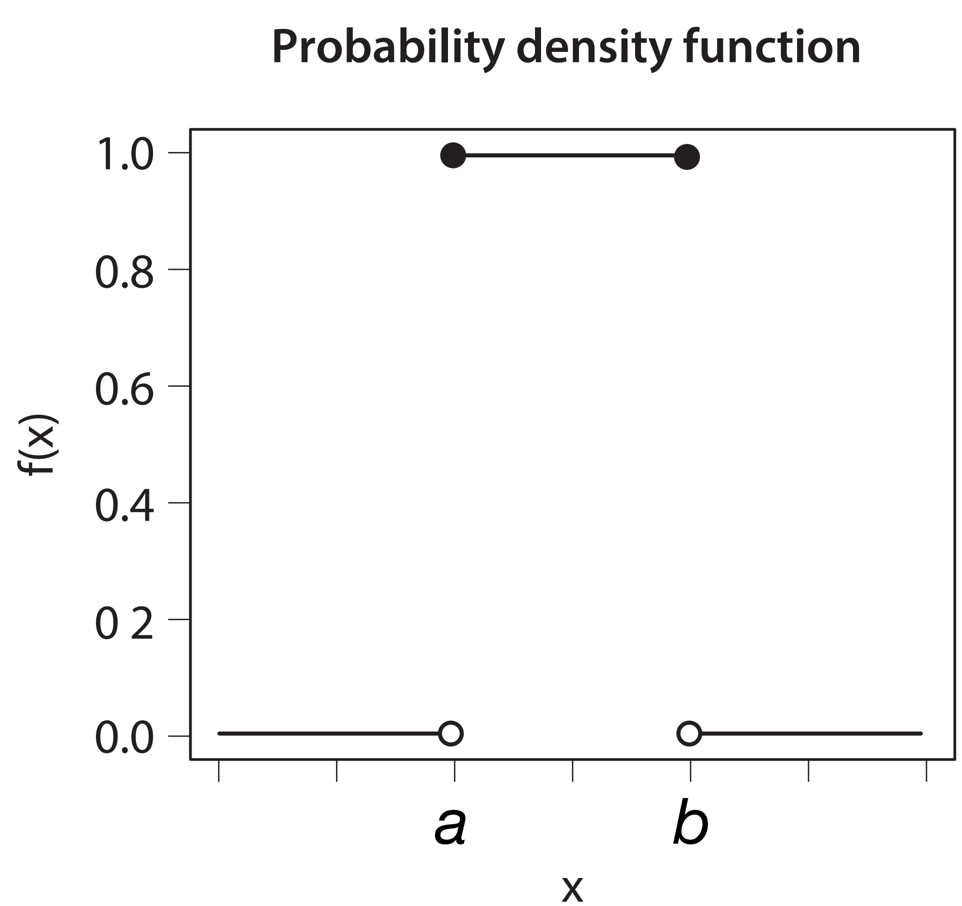


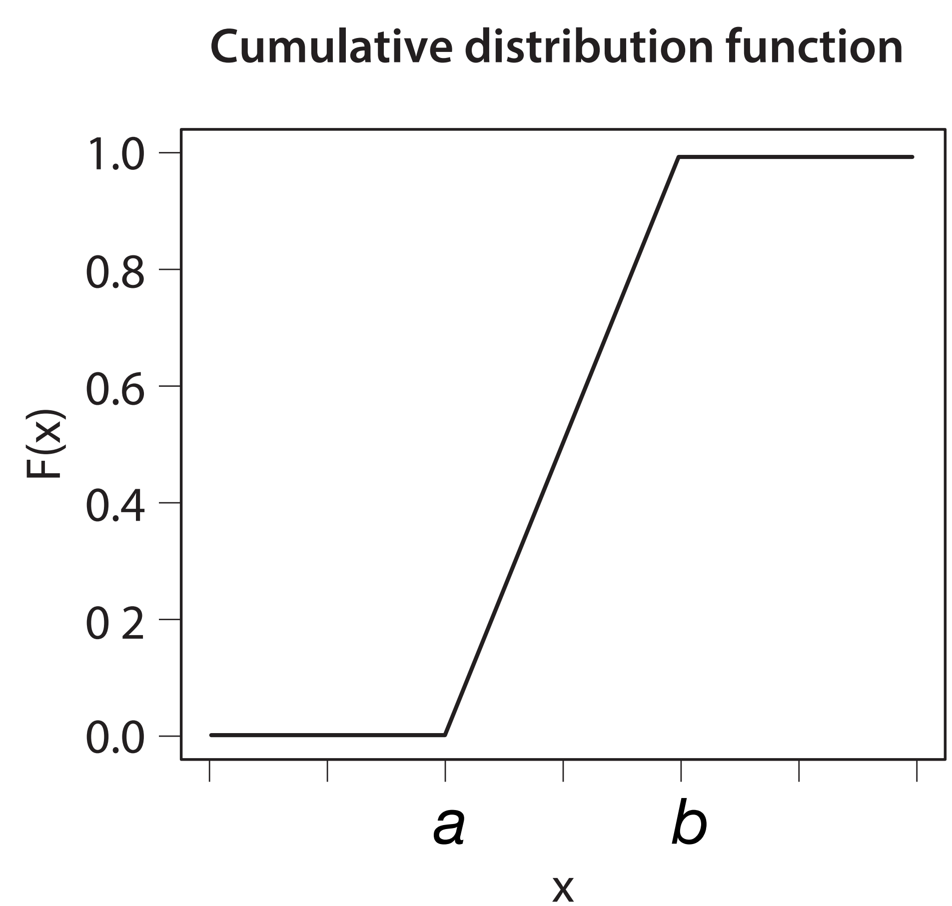
Cumulative distribution function



Uniform Distribution

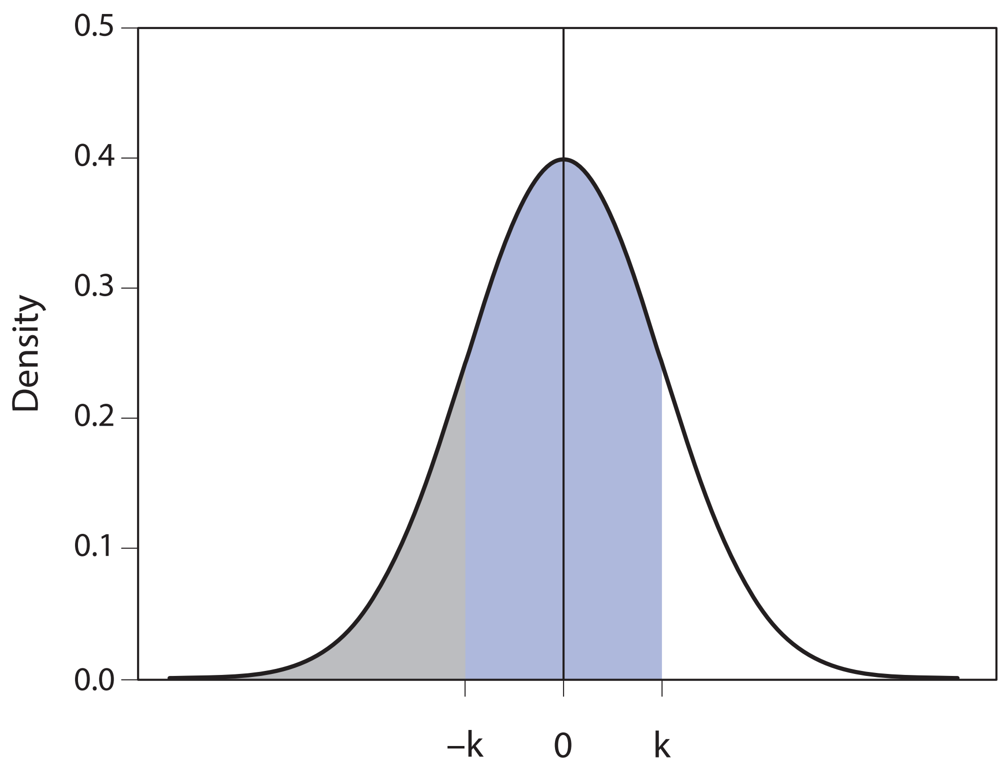
- Every number in an interval has an equal chance of appearance
- $\Omega =$ **set of real numbers**
- PDF
$$f(x) = \begin{cases} \frac{1}{b-a} & \textbf{if } a \le x \le b \\ 0 & \textbf{otherwise.} \end{cases}$$
- CDF
$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & \textbf{if } a \le x \le b \\ 1 & x \ge b \end{cases}$$
- Uniform distribution for the interval $[a,b]$





Normal Distribution

- Most famous and frequently observed distribution (Why?)
- Ω=real numbers (continuous number)
- X is normal RV with mean μ and standard deviation σ: $X \sim \mathcal{N}(\mu, \sigma^2)$
- PDF $$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$
- e.g. $X \sim \mathcal{N}(0,1)$



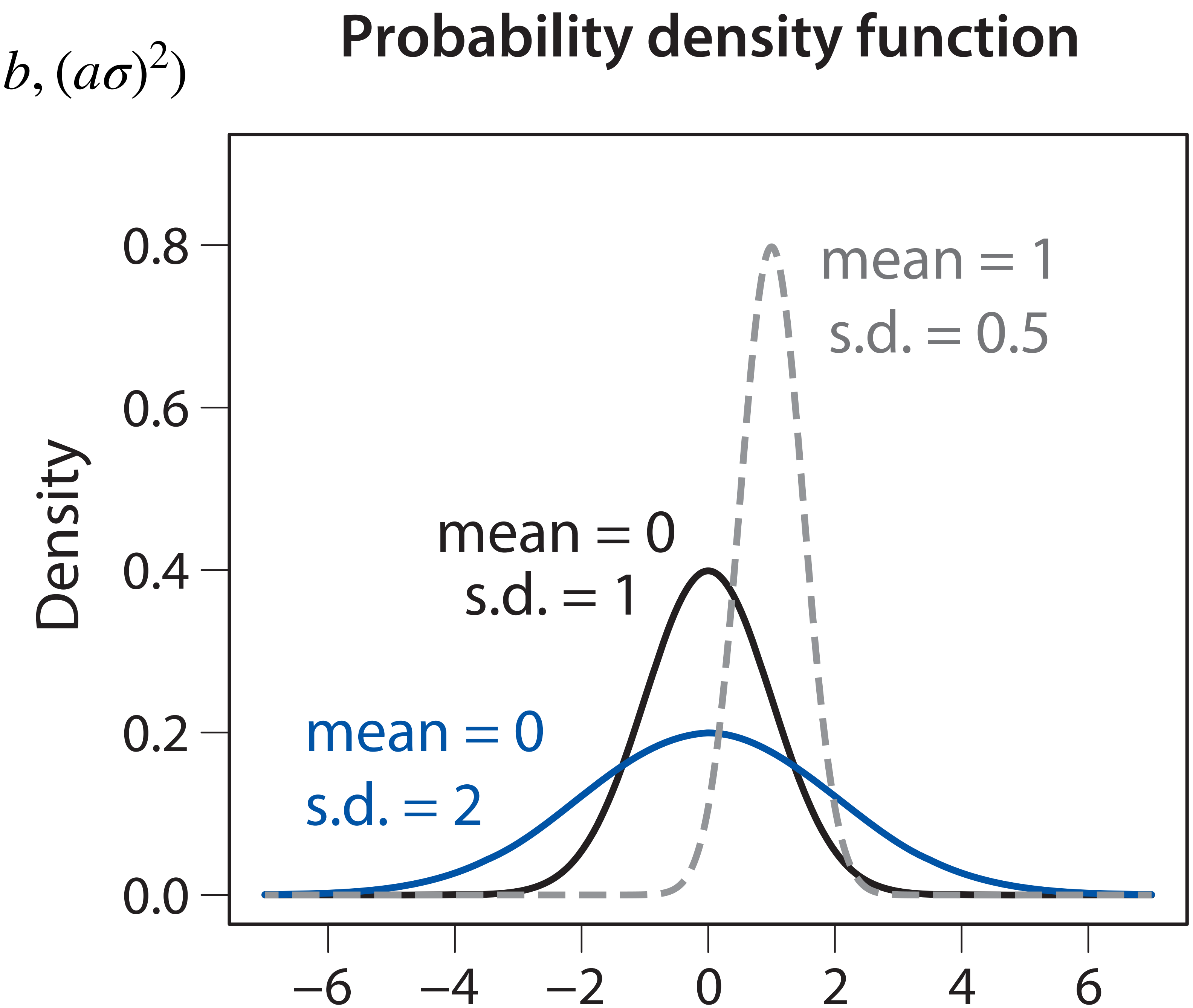
- Singled peaked, symmetric
- about 2/3 are within 1 standard deviation (σ) from the mean
- about 95% are within 2 standard deviations (2σ) from the mean

- CDF $$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(t-\mu)^2\right\} dt$$

Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

- $X \sim \mathcal{N}(\mu, \sigma^2)$
- $S = X + c \rightarrow S \sim \mathcal{N}(\mu + c, \sigma^2)$
- $Y = cX \rightarrow Y \sim \mathcal{N}(c\mu, (c\sigma)^2)$
- $T = aX + b \rightarrow T \sim \mathcal{N}(a\mu + b, (a\sigma)^2)$



Normal Distribution

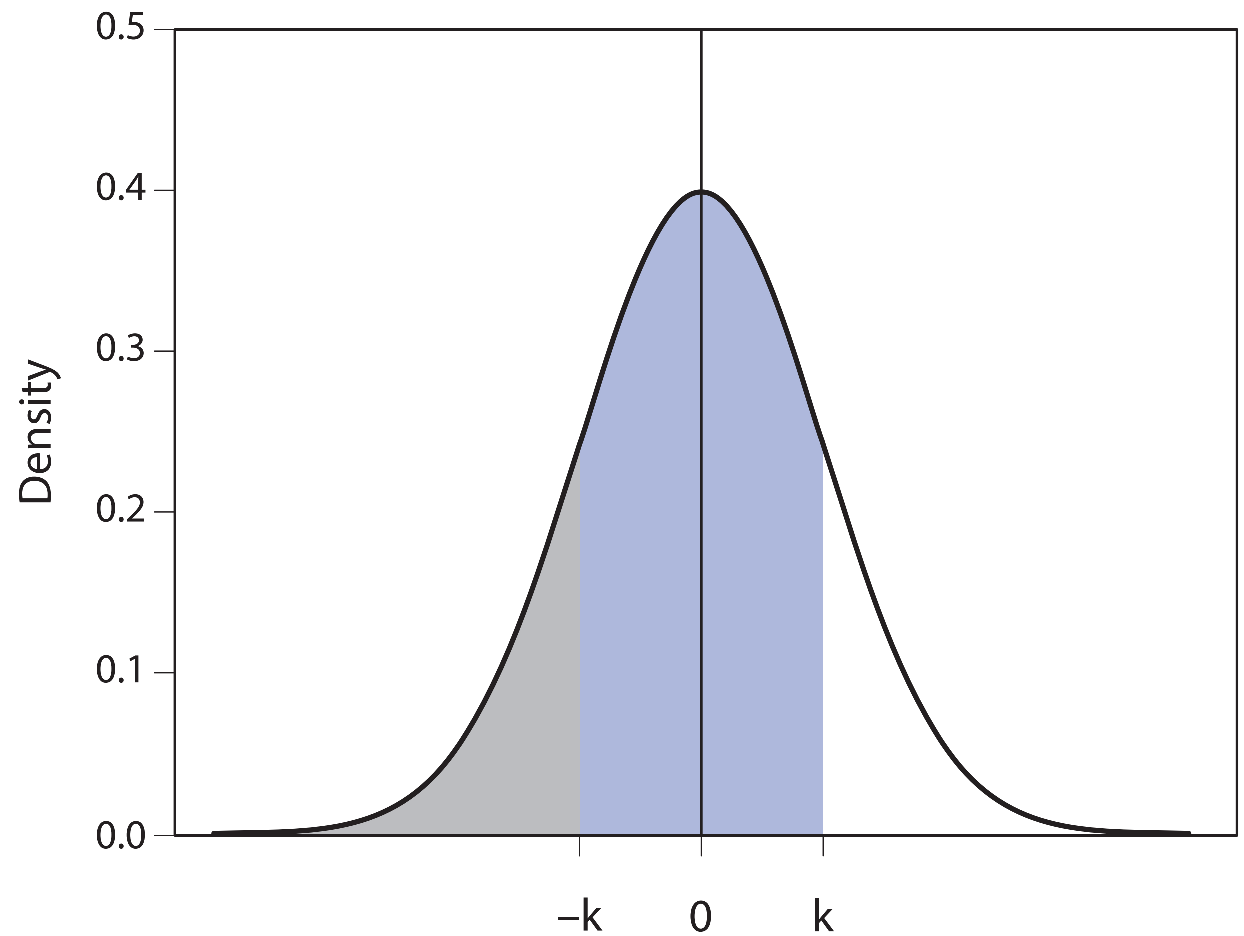
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

- $X \sim \mathcal{N}(\mu, \sigma^2)$
- $S = X + c \rightarrow S \sim \mathcal{N}(\mu + c, \sigma^2)$
- $Y = cX \rightarrow Y \sim \mathcal{N}(c\mu, (c\sigma)^2)$
- $T = aX + b \rightarrow T \sim \mathcal{N}(a\mu + b, (a\sigma)^2)$
- **z-score:** $Z = (X - \mu)/\sigma \rightarrow Z \sim \mathcal{N}(0,1)$

https://www.youtube.com/watch?v=nqNzKeVCYBU

Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$



- Probability that a normal random variable with mean **μ** and sdv **σ** lies within k standard deviations from the mean for a positive constant $k > 0$

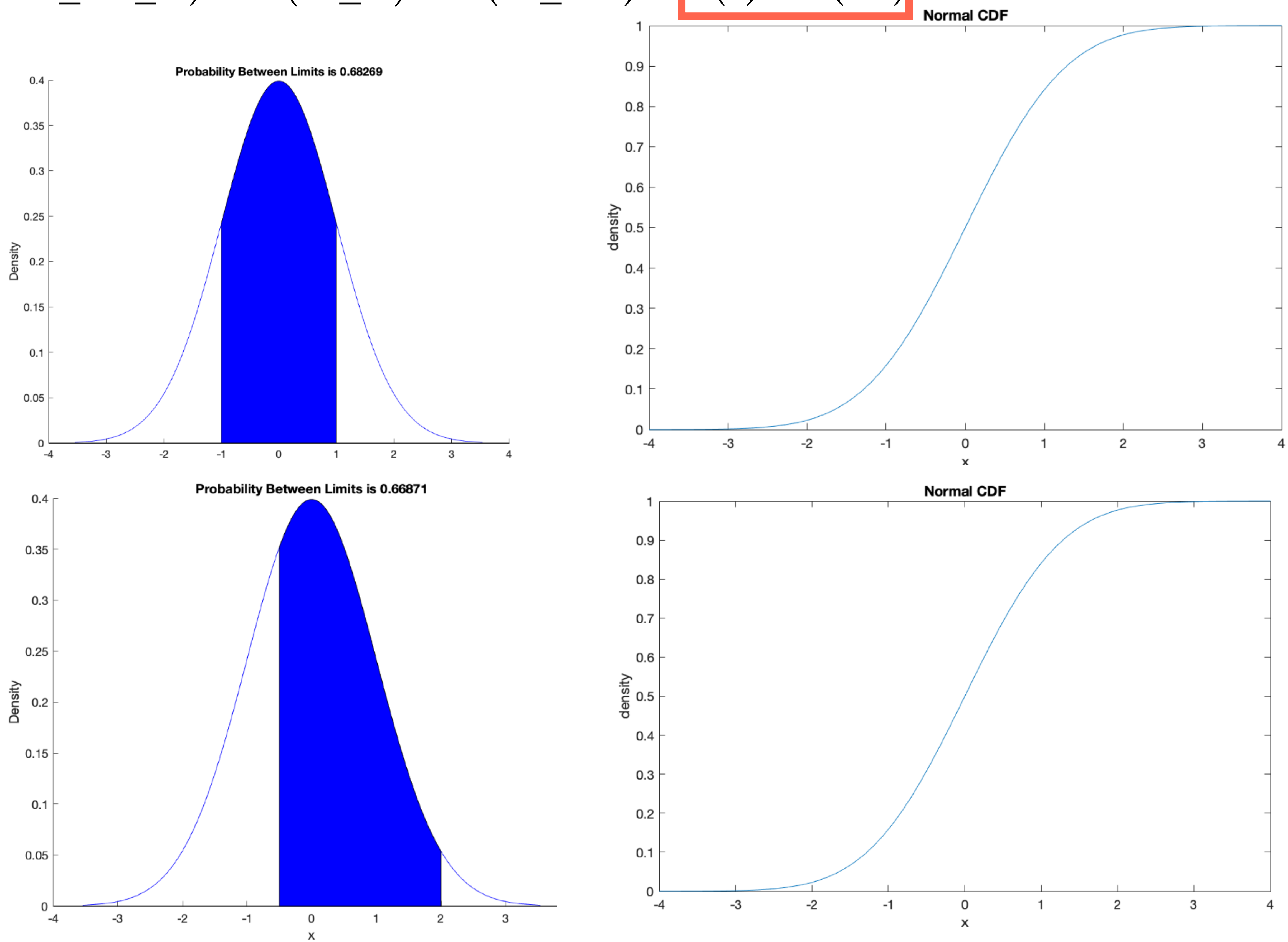
$$\begin{aligned} P(\mu - k\sigma \le X \le \mu + k\sigma) &= P(-k\sigma \le X - \mu \le k\sigma) \\ &= P\left(-k \le \frac{X-\mu}{\sigma} \le k\right) \\ &= P(-k \le Z \le k), \end{aligned}$$

$$P(-k \le Z \le k) = P(Z \le k) - P(Z \le -k) = \boxed{F(k) - F(-k)}$$

Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

$$P(-k \leq Z \leq k) = P(Z \leq k) - P(Z \leq -k) = \boxed{F(k) - F(-k)}$$



Normal Distribution

- Singled peaked, symmetric
- about 2/3 are within 1 standard deviation (σ) from the mean
- about 95% are within 2 standard deviations (2σ) from the mean

```
## plus minus 1 standard deviation from the mean
pnorm(1) - pnorm(-1)

## [1] 0.6826895

## plus minus 2 standard deviations from the mean
pnorm(2) - pnorm(-2)

## [1] 0.9544997
```

```
mu <- 5
sigma <- 2
## plus minus 1 standard deviation from the mean
pnorm(mu + sigma, mean = mu, sd = sigma) - pnorm(mu - sigma, mean = mu, sd = sigma)

## [1] 0.6826895

## plus minus 2 standard deviations from the mean
pnorm(mu + 2*sigma, mean = mu, sd = sigma) - pnorm(mu - 2*sigma, mean = mu, sd = sigma)

## [1] 0.9544997
```

Expectation: Definition and General Properties

- Expectation (population mean) of a random variable X
  - Fixed value given a probability distribution (different from sample means)

$$\mathbb{E}(X) = \begin{cases} \sum_x x \times f(x) & \text{if } X \text{ is discrete,} \\ \int x \times f(x)\, dx & \text{if } X \text{ is continuous} \end{cases}$$

- Properties of expectation (a, b: constant values; X,Y: independent RVs)

1. $\mathbb{E}(a) = a$.
2. $\mathbb{E}(aX) = a\mathbb{E}(X)$.
3. $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$.
4. $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$.
5. If X and Y are independent, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$. But generally, $\mathbb{E}(XY) \neq \mathbb{E}(X)\mathbb{E}(Y)$.

Expectation: Examples

- Expectation (mean) revisited
  - Expected value of a random variable
    - e.g. PMF: Bernoulli random variable

$$\mathbb{E}(X) = 0 \times P(X = 0) + 1 \times P(X = 1) = 0 \times f(0) + 1 \times f(1) = 0 \times (1 - p) + 1 \times p = p$$

    - e.g. PMF: Binomial random variable

$$\mathbb{E}(X) = 0 \times f(0) + 1 \times f(1) + \cdots + n \times f(n) = \sum_{x=0}^{n} x \times f(x)$$

    - e.g. PMF: Binomial random variable (Y is a Bernoulli RV with p)

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \mathbb{E}(Y_i) = np$$

    - e.g. PDF: uniform random variable defined in the interval $[a,b]$

$$\mathbb{E}(X) = \int_a^b x \times f(x)\, dx = \int_a^b \frac{x}{b-a} dx = \left.\frac{x^2}{2(b-a)}\right|_a^b = \frac{a+b}{2}$$

Variance: Definition and General Properties

- Population variance (different from sample variance)

$$\begin{aligned}\mathbb{V}(X) &= \mathbb{E}[\{X - \mathbb{E}(X)\}^2] \\ &= \mathbb{E}[X^2 - 2X\mathbb{E}(X) + \{\mathbb{E}(X)\}^2] \\ &= \mathbb{E}(X^2) - 2\mathbb{E}(X)\mathbb{E}(X) + \{\mathbb{E}(X)\}^2 \\ &= \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2.\end{aligned}$$

- e.g. PMF: Bernoulli random variable

$$\mathbb{V}(X) = \mathbb{E}(X) - \{\mathbb{E}(X)\}^2 = p(1-p)$$

- e.g. PDF: Uniform random variable

$$\begin{aligned}\mathbb{V}(X) &= \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2 = \int_a^b \frac{x^2}{b-a}\,dx - \left(\frac{a+b}{2}\right)^2 \\ &= \left.\frac{x^3}{3(b-a)}\right|_a^b - \left(\frac{a+b}{2}\right)^2 = \frac{1}{12}(b-a)^2.\end{aligned}$$

- Square root of population variance is population standard deviation

https://towardsdatascience.com/why-sample-variance-is-divided-by-n-1-89821b83ef6d

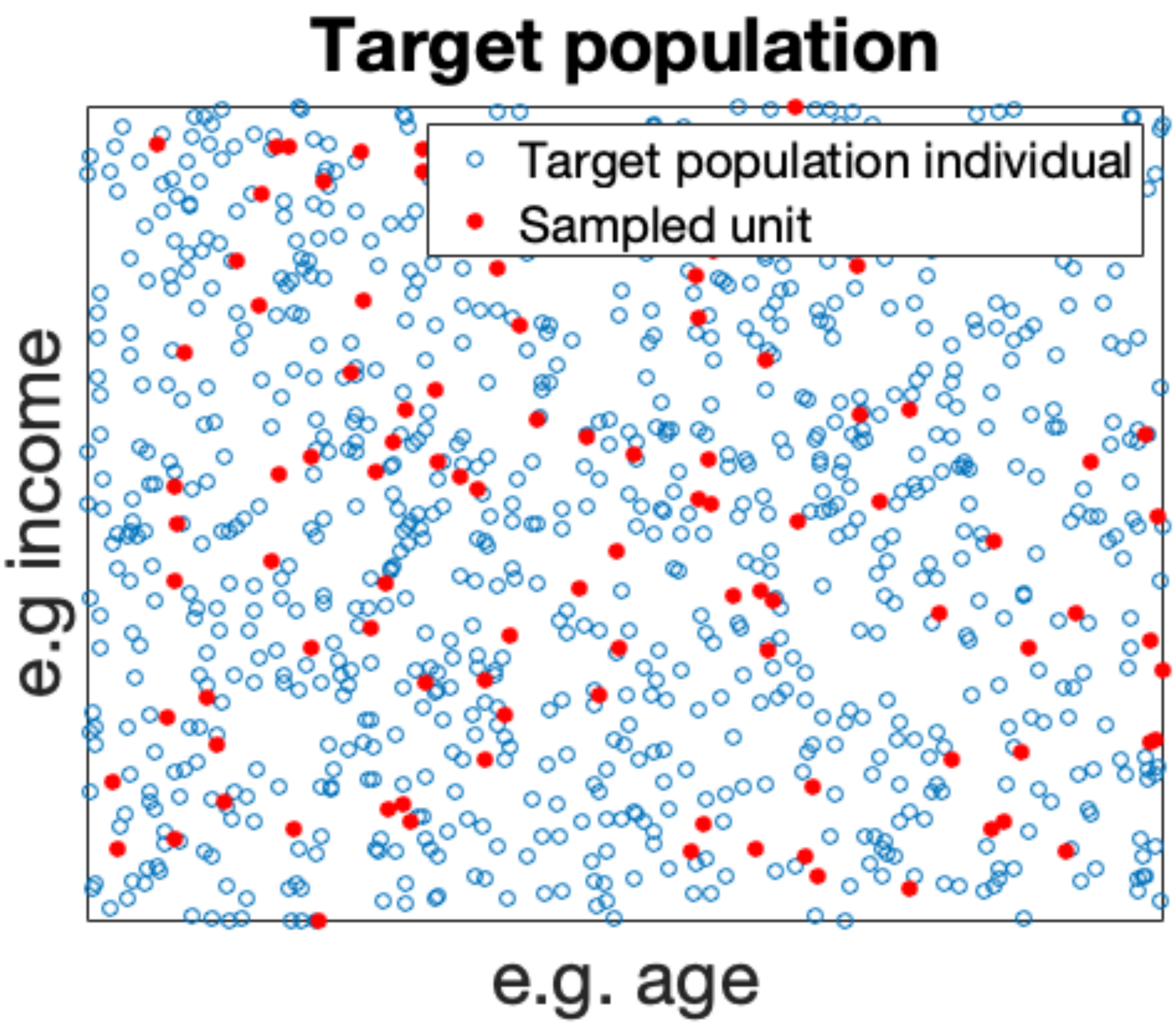
Variance: Definition and General Properties

1. $\mathbb{V}(a) = 0$.
2. $\mathbb{V}(aX) = a^2\mathbb{V}(X)$.
3. $\mathbb{V}(X + b) = \mathbb{V}(X)$.
4. $\mathbb{V}(aX + b) = a^2\mathbb{V}(X)$.
5. If X and Y are independent, $\mathbb{V}(X + Y) = \mathbb{V}(X) + \mathbb{V}(Y)$.

Contents

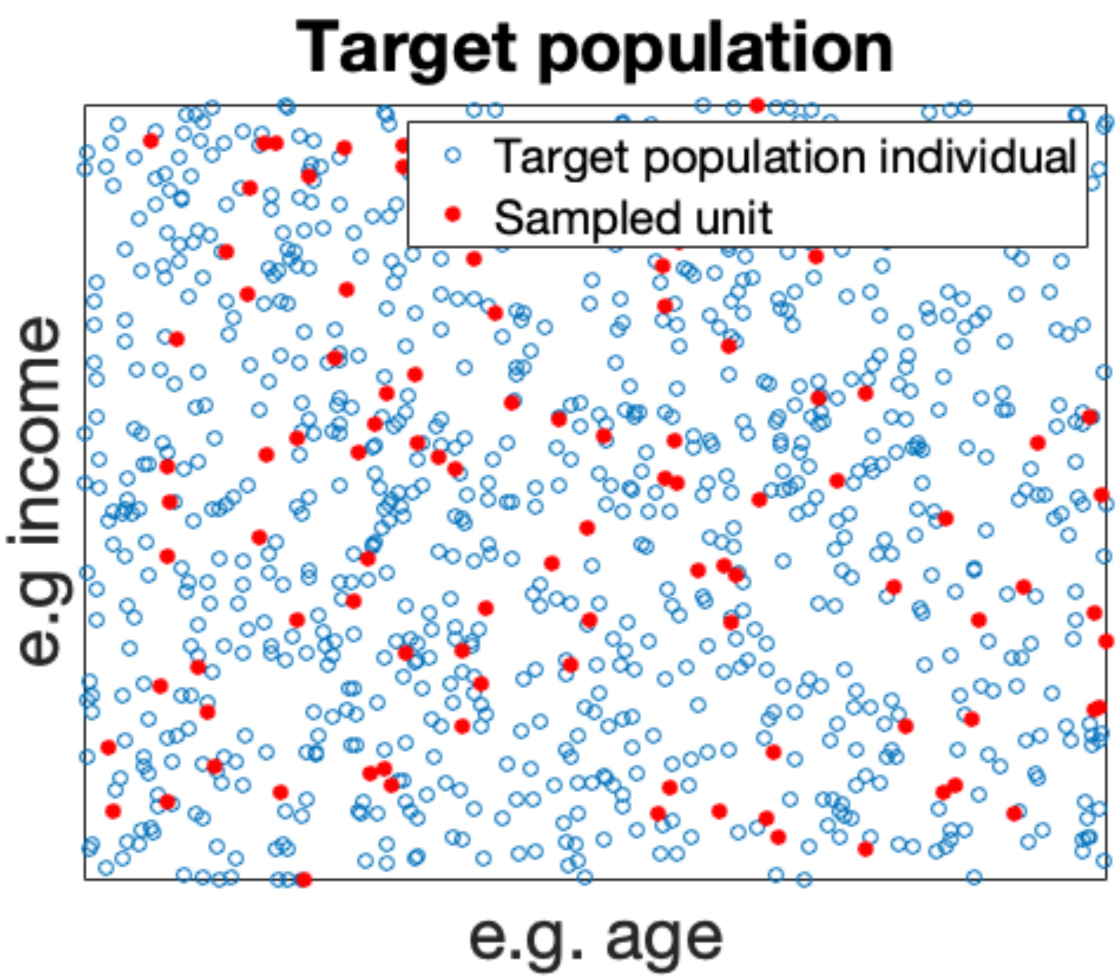
Election Prediction (6.3.6)

- Binomial outcomes in 2 candidate elections
  - Vote share = sum of individual binary choices



Election Prediction (6.3.6)

- e.g. even for the case of random sampling: **sampling variability**



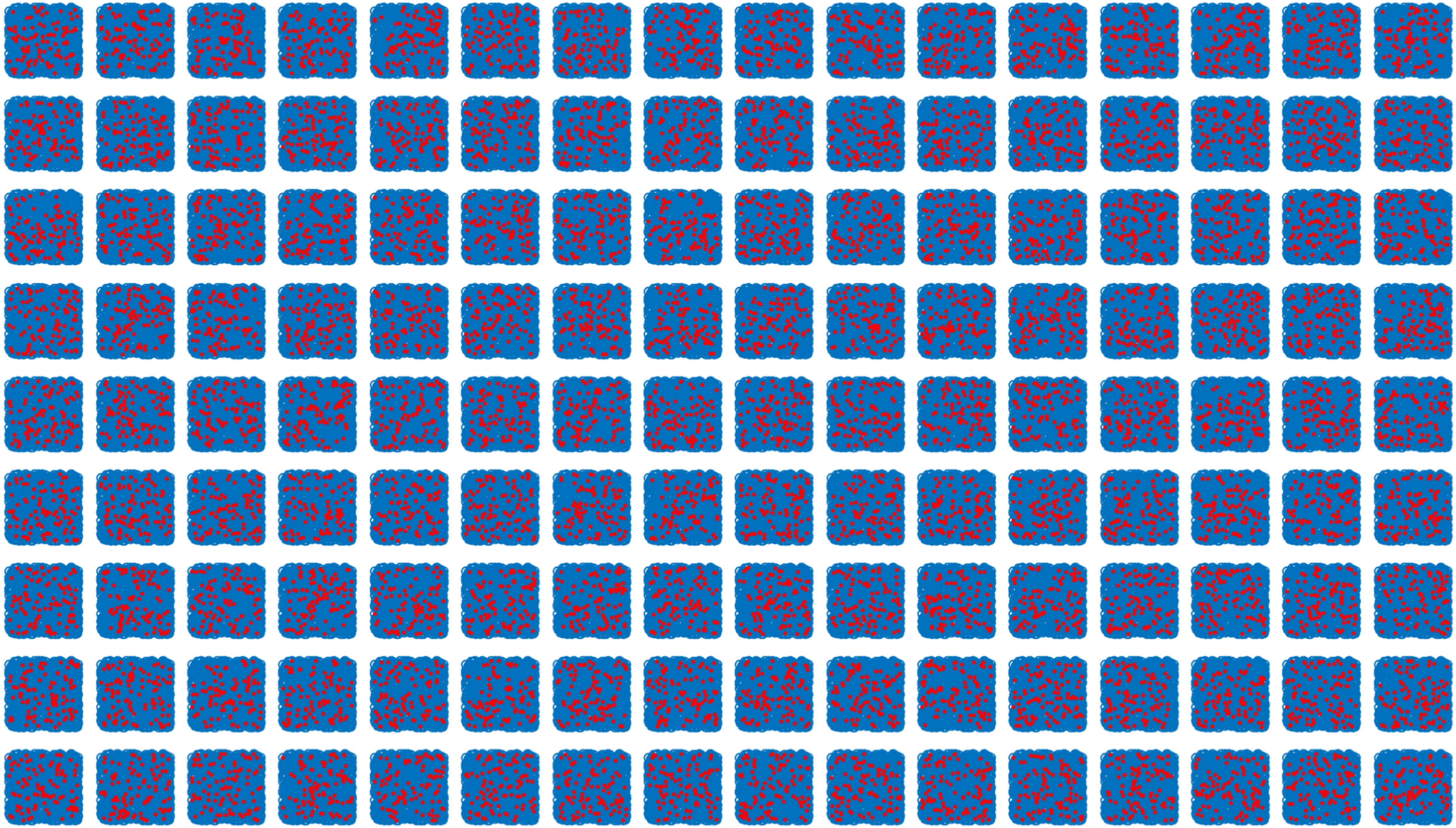
VS

- We cannot ignore such possibility!

- From candidate preference of individuals in your sample:
  - Infer **target population preference** -> **Uncertainty** must be involved!
  - Time to use random variables (binomial distribution)

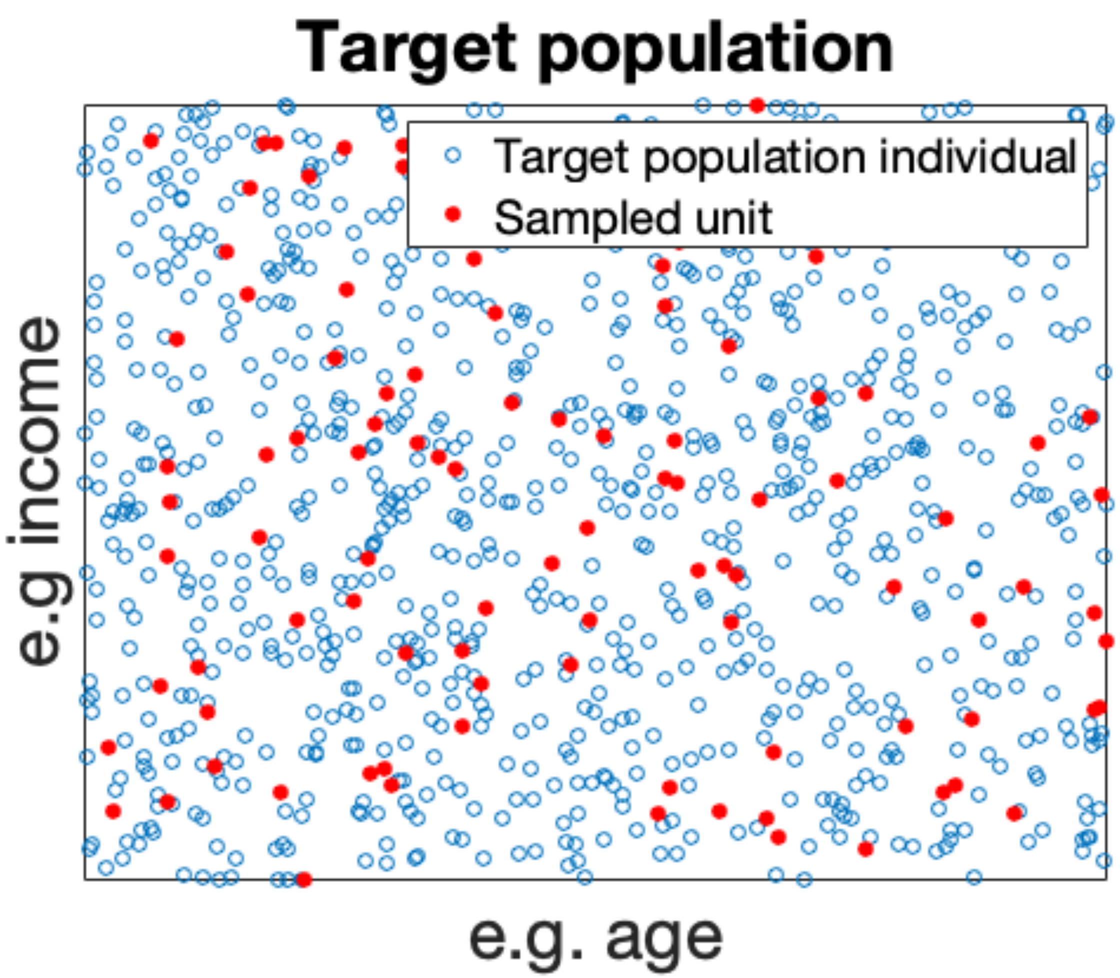
Election Prediction (6.3.6)

- e.g. even for the case of random sampling: **sampling variability**
  - e.g. selecting 100 individuals from 1,000 (each exhibits different share)

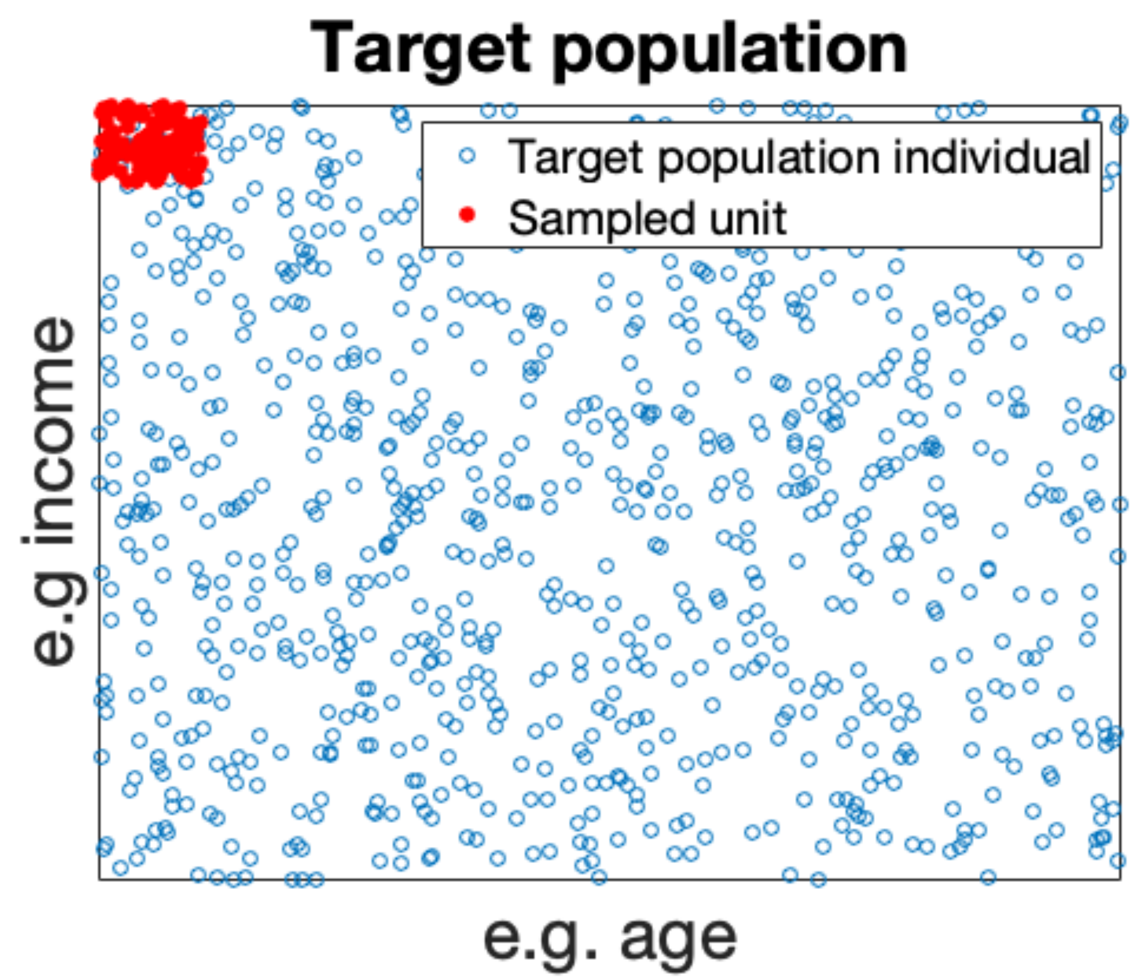


Election Prediction (6.3.6)

- e.g. even for the case of random sampling: **sampling variability**



VS

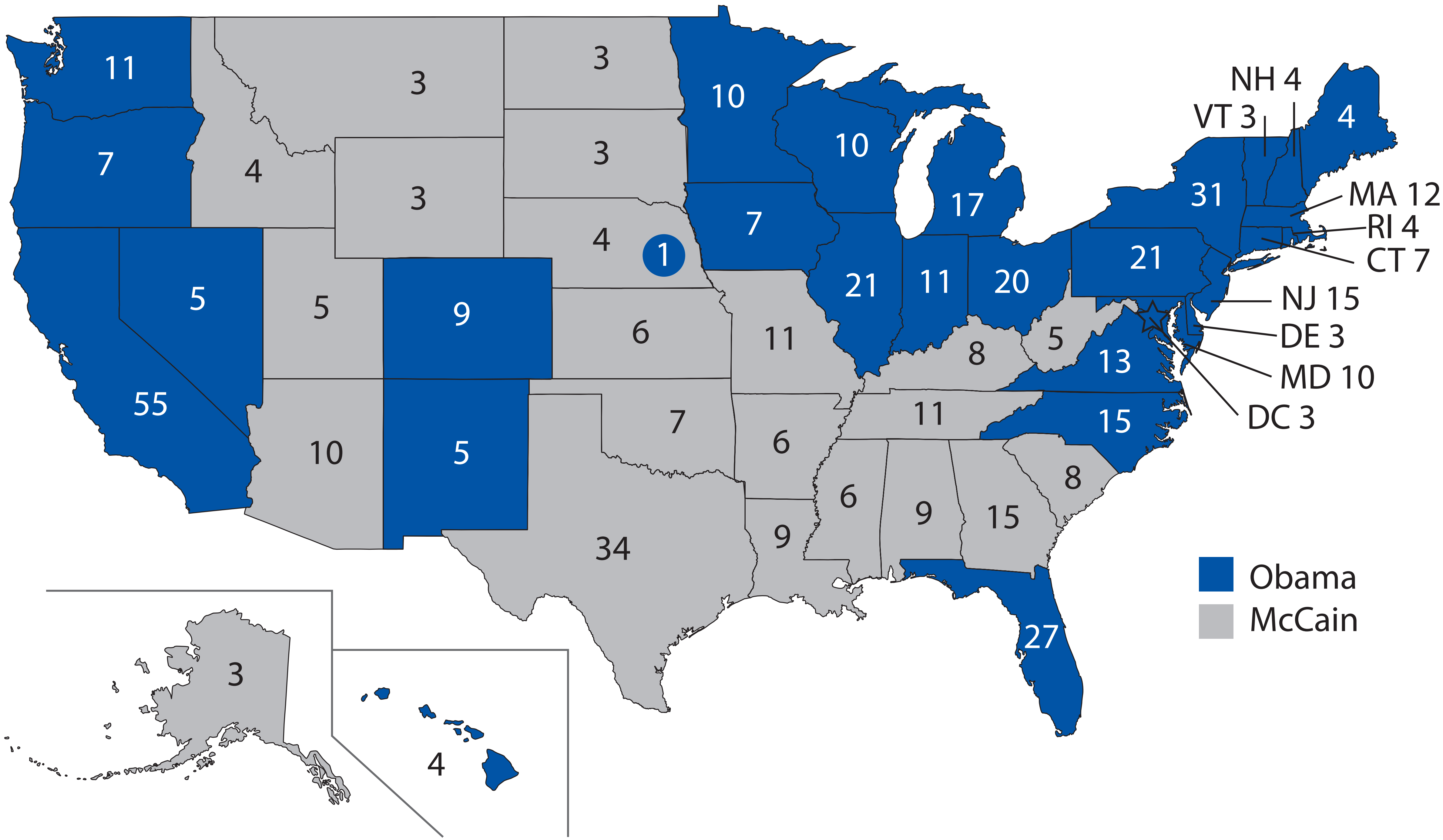


- We cannot ignore such possibility!

- From candidate preference of individuals in your sample:
  - Infer **target population preference** -> **Uncertainty** must be involved!
  - Time to use random variables (binomial distribution)

Election Prediction (4.1.3)

- Electoral college system: winner-take-all for Electoral College votes /state



Election Prediction (6.3.6)

- Given the population-level voting propensity (ignore 3rd candidate)
  - Possible outcomes of 1,000 individual / state samples:

```
pres08 <- read.csv("pres08.csv")
## two-party vote share
pres08$p <- pres08$Obama / (pres08$Obama + pres08$McCain)
```

$$\mathbb{E}(\text{Obama's votes}) = \sum_{j=1}^{51} v_j \times P(\text{Obama wins state } j) = \sum_{j=1}^{51} v_j \times P(S_j > 500)$$

$$\mathbb{V}(\text{Obama's predicted votes}) = \sum_{j=1}^{51} \mathbb{V}(v_j \mathbf{1}\{S_j > 500\})$$

$$= \sum_{j=1}^{51} v_j^2 P(S_j > 500)\big(1 - P(S_j > 500)\big)$$

Election Prediction (6.3.6)

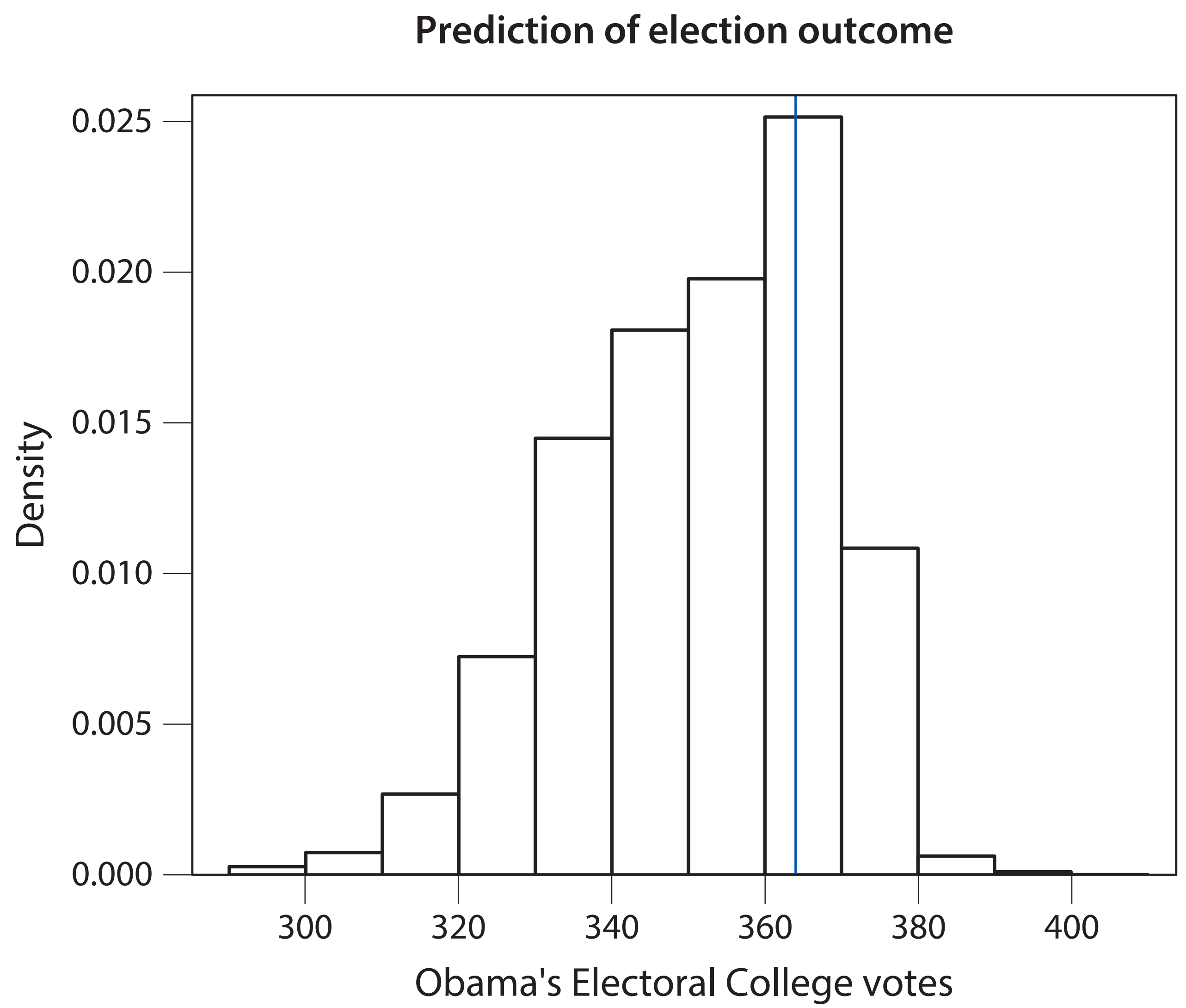
- Given the population-level voting propensity (ignore 3rd candidate)
  - Possible outcomes of 1,000 individual / state samples:

```
pres08 <- read.csv("pres08.csv")
## two-party vote share
pres08$p <- pres08$Obama / (pres08$Obama + pres08$McCain)
```

```
n.states <- nrow(pres08) # number of states
n <- 1000 # number of respondents
sims <- 10000 # number of simulations
## Obama's electoral votes
Obama.ev <- rep(NA, sims)
for (i in 1:sims) {
    ## samples number of votes for Obama in each state
    draws <- rbinom(n.states, size = n, prob = pres08$p)
    ## sums state's Electoral College votes if Obama wins the majority
    Obama.ev[i] <- sum(pres08$EV[draws > n / 2])
}
hist(Obama.ev, freq = FALSE, main = "Prediction of election outcome",
     xlab = "Obama's Electoral College votes")
abline(v = 364, col = "blue") # actual result
```

Election Prediction (6.3.6)

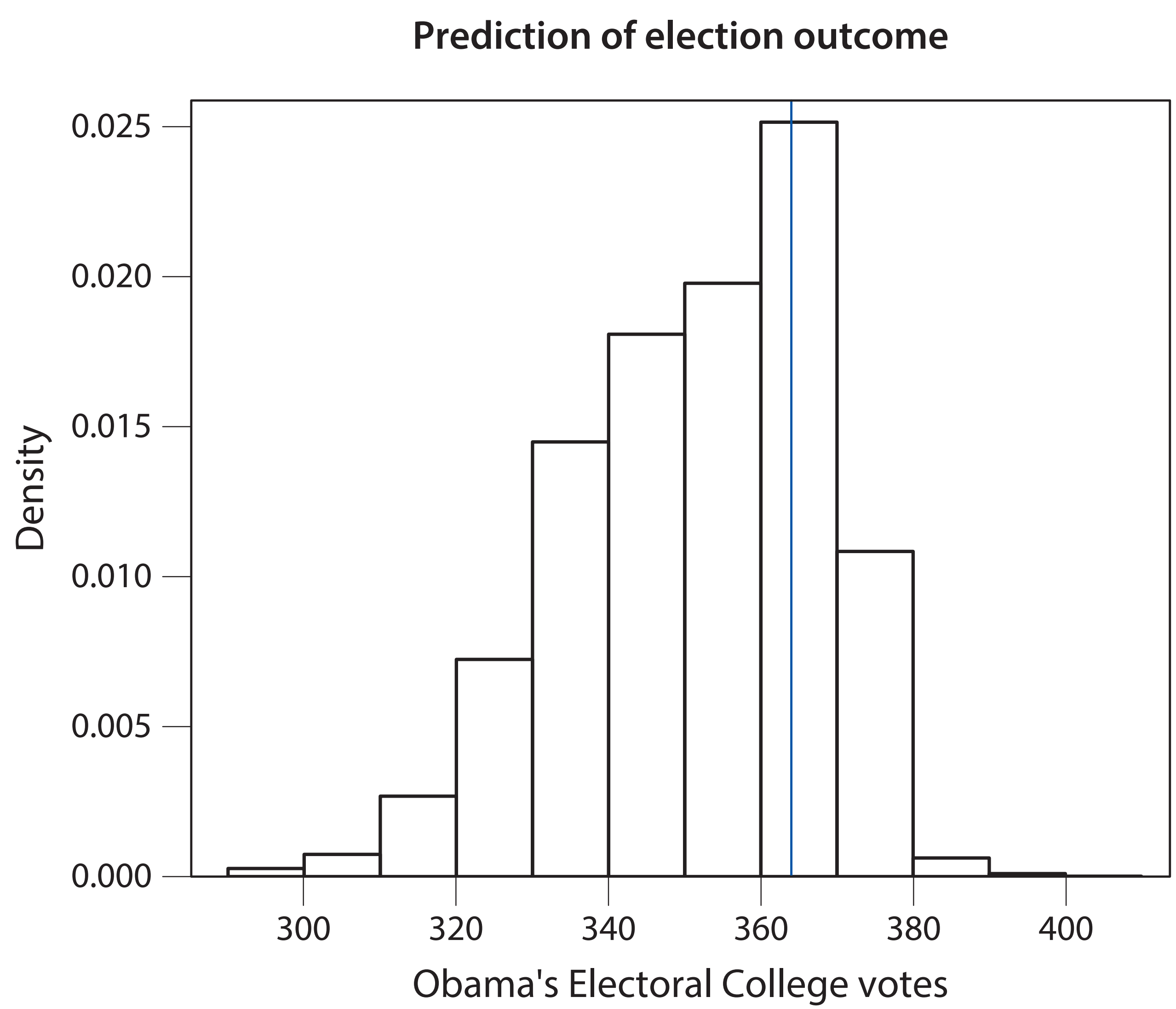
- Given the population-level voting propensity (ignore 3rd candidate)
  - Possible outcomes of 1,000 individual / state samples:




```
summary(Obama.ev)

##      Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##     291.0   340.0   353.0   352.2   364.0   401.0
```

Election Prediction (4.1.3)



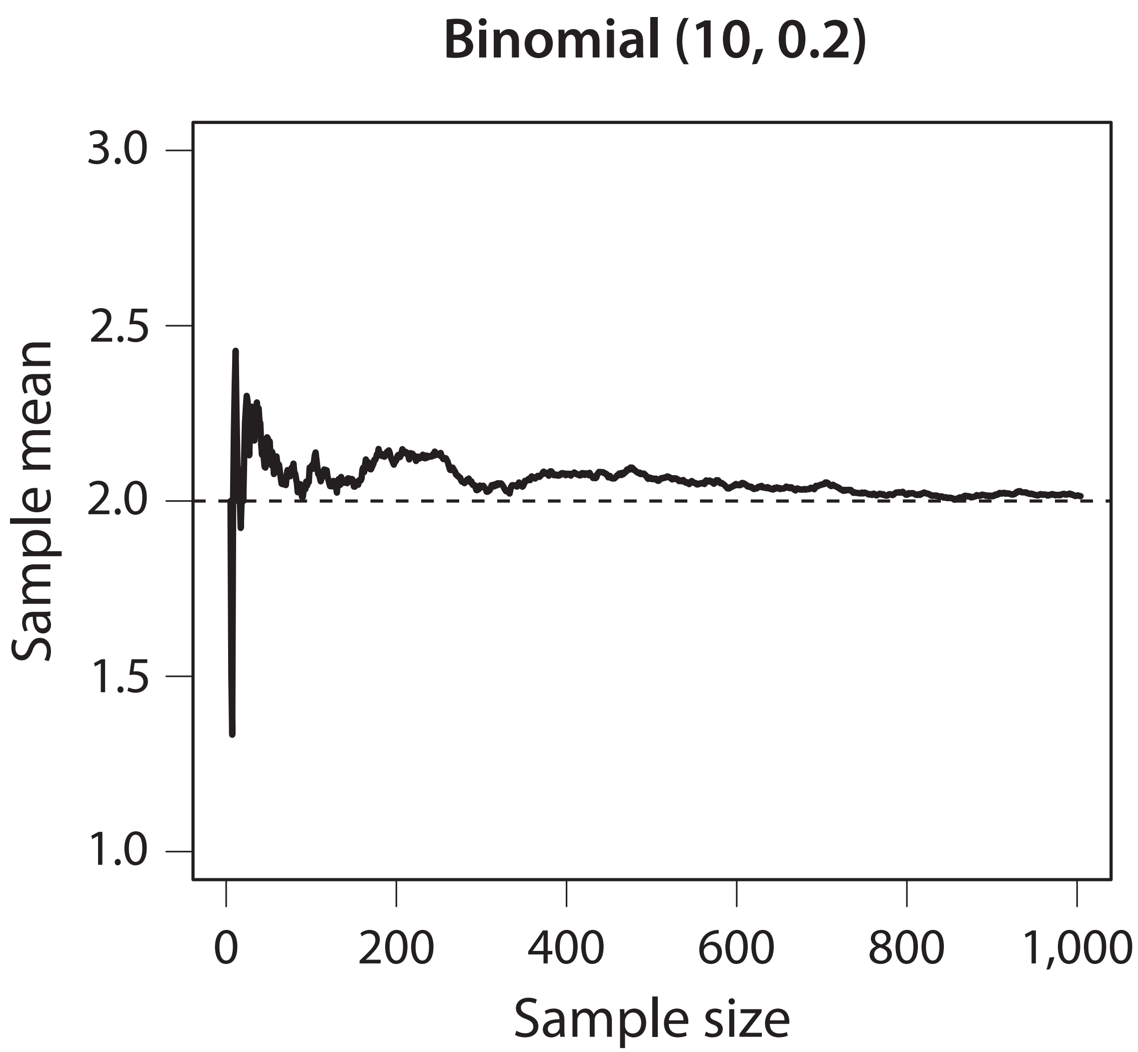
- Prediction error (ε) = actual outcome - predicted outcome
- Average predictor error is called bias
- Root-mean-squared error: $\sqrt{\sum_{i=1}^{n} \epsilon_i^2 / n}$

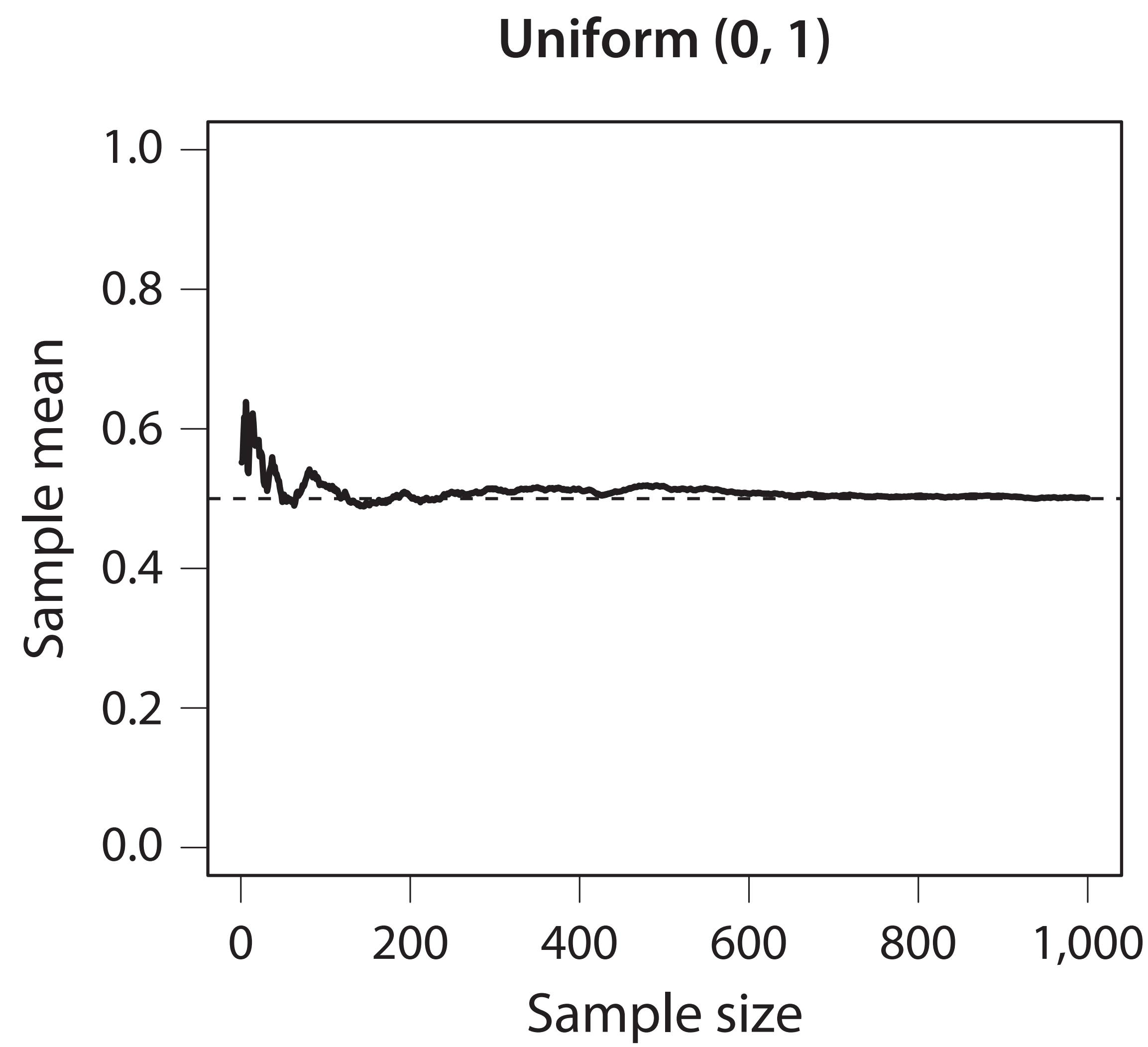
Contents

The Law of Large Numbers (6.4.1)

- As sample size grows, sample mean approaches the population mean

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathbb{E}(X)$$





The Law of Large Numbers (6.4.1)

- e.g. Randomized control trials (Randomized experiments)
  - Difference in the means estimator (sample average treatment effect)
    - Sample average difference $\longrightarrow$??
      - Recall
$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathbb{E}(X)$$
      - Sample ATE converges to Population average treatment effect

$$\overline{Y(1)} - \overline{Y(0)} = \frac{1}{n_1}\sum_{i=1}^{n_1} Y(1)_i - \frac{1}{n_0}\sum_{i=1}^{n_0} Y(0)_i \to \mathbb{E}(Y(1)) - \mathbb{E}(Y(0))$$

See you next week.