Random Variables and Their Distributions
Week 10

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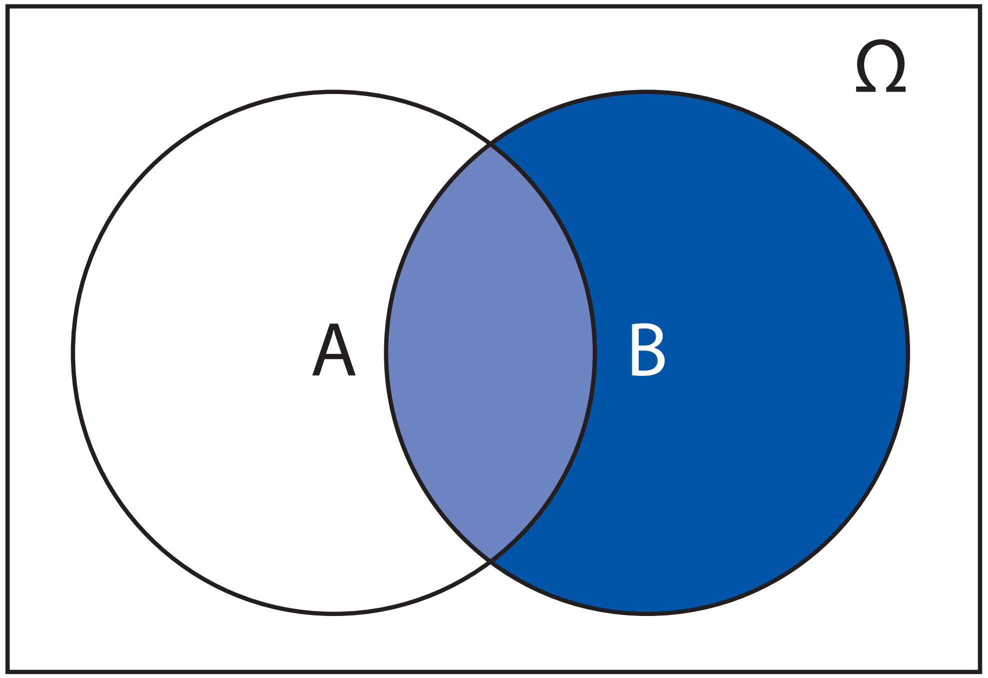
Review of Week 9

- Probability (Chapter 6.1.)
  - Definitions
  - Axioms
  - Permutations
  - Combinations
- Conditional Probability (Chapter 6.2.)
  - Conditional probability
  - Joint probability
  - Independence

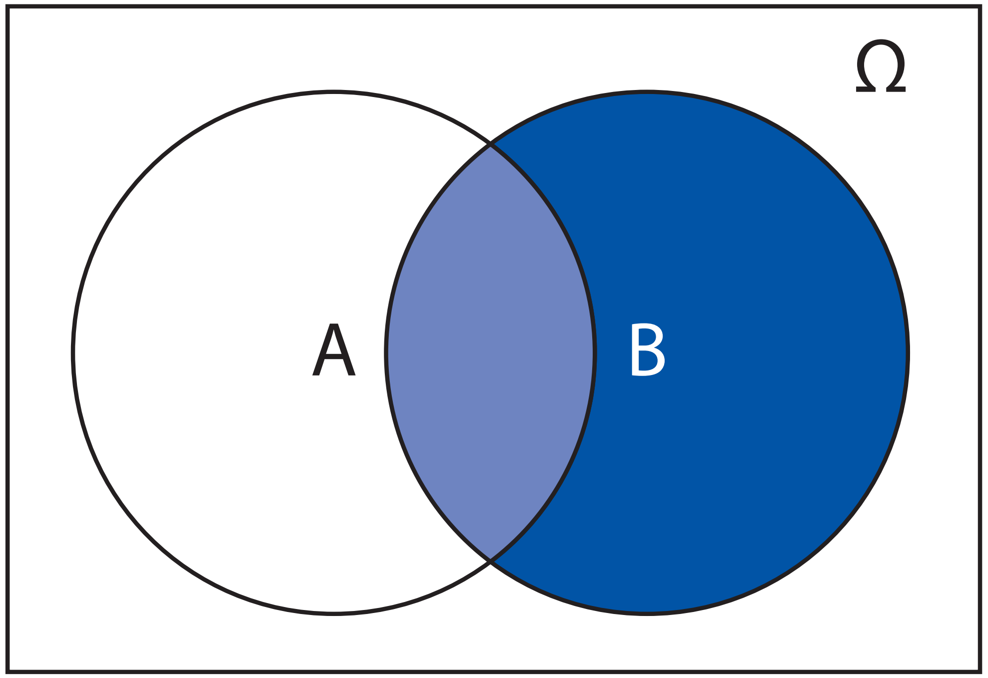
Example 1: Write Down Equivalence/Definition and Plug in Values

- Rolling a dice once
  - A: a multiple of 2; B: a multiple of 3
    - $\Omega =$
    - as a set: $A =$ $B =$
    - as a set: $A^c =$ $B^c =$
    - $P(A) =$ $P(B) =$
    - $P(A^c) =$ $P(B^c) =$
    - $P(A \text{ and } B) =$
    - $P(A|B) =$
    - $P(B|A) =$
    - $P(A) = P(A \text{ and } B) + P(A \text{ and } B^c)$



Example 2: Write Down Equivalence/Definition and Plug in Values

- Flipping coin 3 times
  - A: first flip head; B: second flip tail
    - Ω =
    - as a set: A = B =
    - as a set: A^c = B^c =
    - P(A) = P(B) =
    - P(A^c) = P(B^c) =
    - P(A and B) =
    - P(A|B) =
    - P(B|A) =
    - P(A) = P(A and B) + P(A and B^c)



Example 3: Permutations and Combinations

- You are a producer in a large entertainment company
  - You want to form a 5-member unit from IZ*ONE
    - Assume that you are selecting 5 completely at random
    - What is the chance that you select 1 JP member and 4 KR member?
    - What is the chance that you select at least 2 JP members?
  - You are selecting 7 members from IZ*ONE to cover a BTS song
    - Each will be assigned to a different role (e.g. V, Jin, RM …)
      - How many potential scenarios are there?

Bayes' Rule: Check Textbook 6.2.4

- Bayes' rule (obvious if you know definitions / axioms)

$$\underbrace{\Pr(A \mid B)}_{conditional\ \ probability} = \frac{\overbrace{\Pr(A \text{ and } B)}^{joint\ \ probability}}{\underbrace{\Pr(B)}_{marginal\ \ probability}}$$

$$= \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B \mid A)\Pr(A) + \Pr(B \mid \text{not } A)\Pr(\text{not } A)}$$

- What does Pr(A|B) mean?
  - The chance of A being true given B is observed (i.e. true)
- Why is above the most important result in probability theory?
  - e.g. A is an event happened before B; only B is observed
    - Using Bayes' rule: you can infer about A given B
- Bayesian updating: prior belief Pr(A) -> posterior belief Pr(A|B)

Bayes' Rule: Check Textbook 6.2.4

- Bayes′ rule (obvious if you know definitions / axioms)

$$\underbrace{\Pr(A \mid B)}_{conditional\ \ probability} = \frac{\overbrace{\Pr(A \text{ and } B)}^{joint\ \ probability}}{\underbrace{\Pr(B)}_{marginal\ \ probability}}$$

$$= \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B \mid A)\Pr(A) + \Pr(B \mid \text{not } A)\Pr(\text{not } A)}$$

- Posterior probability Pr(A|B): belief of A after observing evidence B
- Prior probability (belief of A happening w/o evidence): P(A)
- Prior probability (belief of A not happening w/o evidence): P(not A)
- Examples (A: prior event; B: data/evidence)

A (prior event)	I voted for A	having a disease	studied hard
B (data)	A won	positive on medical test	high grade in exam

  - Time order is not necessary but good for examples

Example: Monty Hall Problem

- The most famous probability problem
  - http://www.youtube.com/watch?v=mhlc7peGlGg

  - Even a great mathematician failed!

Example: Monty Hall Problem

- You pick door A. Monty opens door C that has a goat.
  - Should you switch to door B?
- Prior beliefs: $P(A) = P(B) = P(C) = 1/3$
- Data: Monty reveals door C (i.e. MC)
- Posterior belief (inferential goals): $P(A \mid \mathrm{MC})$ and $P(B \mid \mathrm{MC})$
- Question: $P(A \mid \mathrm{MC}) < P(B \mid \mathrm{MC})$
- What do we need? Bayes' rule!
  - Key: Monty's behavior is constrained to open a goat door

- Take home message: Do not believe your intuition, rely on logic.

Contents

- Bayes′ rule (Chapter 6.2.3)
- Random Variables and Probability Distributions (Chapter 6.3)
  - Overview
  - Bernoulli and uniform distributions
  - Binomial distribution
  - Uniform distribution
  - Normal (or Gaussian) distribution
  - Expectation

Random Variables and Probability Distribution

- Random variable assigns a numeric value to each event of the experiment.
  - Coin flip side: head = 1; tail =0
  - #secs took for commuting: any value greater than 0.
- These values represent mutually exclusive and exhaustive events, together forming the entire sample space Ω.
- A discrete RV takes a finite or at most countably infinite #distinct values.
  - coin flip; race; number of years of education
- A continuous RV assumes an uncountably infinite number of values.
  - height, distance from earth, gross domestic product
- Probability distribution: Probability that a random variable takes a certain value or range of values.
  - P(side): P(side=1) = 0.5; P(side=0) = 0.5
  - P(#secs): P(0<#secs<1000) = 0.3; P(1000<#secs<2000) = 0.4 ….

Probability Density / Mass Function

- Probability density function (PDF): f(x) for a continuous random variable
- Probability mass function (PMF): f(x) for a discrete random variable
- Cumulative density function (CDF): $F(x)=P(X\leq x)$
- Cumulative mass function (CMF): $F(x) = P(X \leq x) = \sum_{K \leq x} f(k)$
  - What is the probability that a random variable X takes a value equal to or less than x?
  - Area under the density curve
  - Non-decreasing

Bernoulli Distribution

- $\Omega=\{0,1\}$
- PMF

$$f(x) = \begin{cases} p & \textbf{if } x = 1 \\ 1-p & \textbf{if } x = 0 \\ 0 & \textbf{otherwise.} \end{cases}$$

- CMF

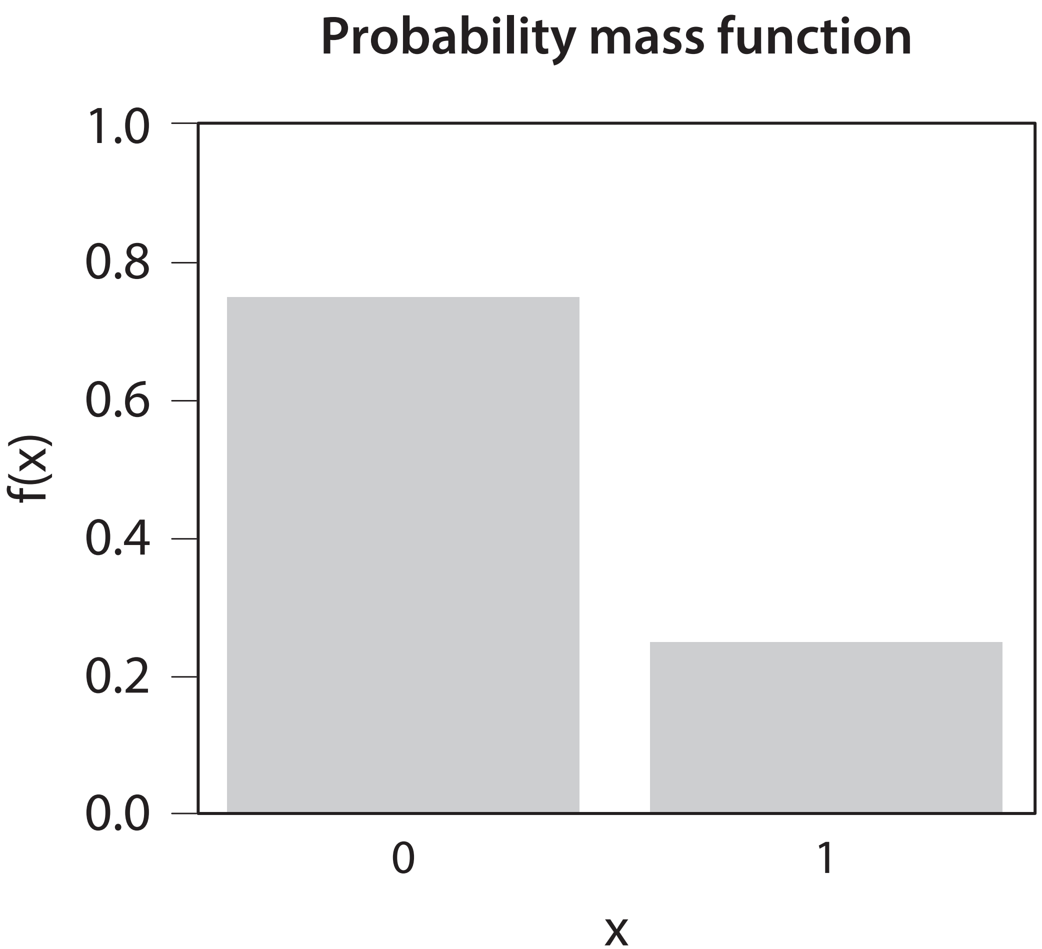
$$F(x) = \begin{cases} p & \textbf{if } x < 0 \\ 1-p & \textbf{if } 0 \le x < 1 \\ 0 & \textbf{if } x \ge 1 \end{cases}$$

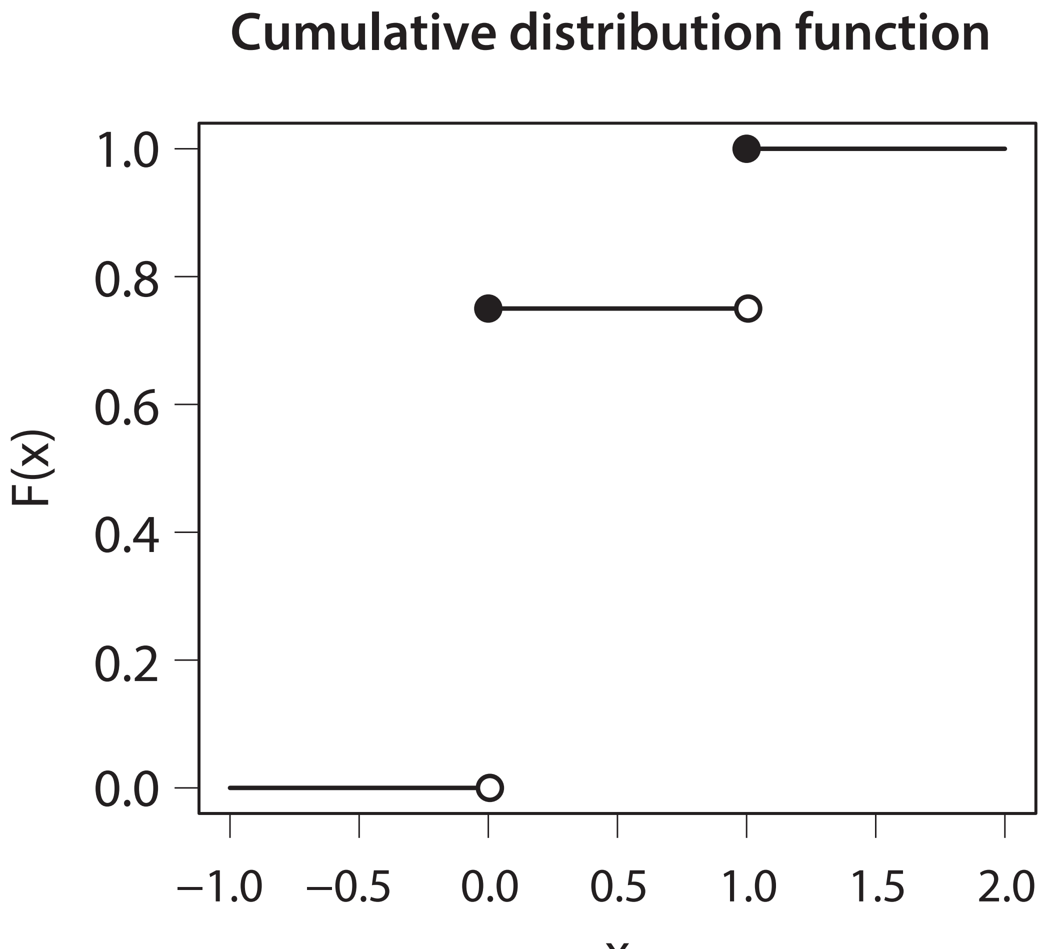
Bernoulli Distribution

- $\Omega=\{0,1\}$
- PMF

$$f(x) = \begin{cases} p & \textbf{if } x = 1 \\ 1-p & \textbf{if } x = 0 \\ 0 & \textbf{otherwise.} \end{cases}$$

- PMF and CDF of Bernoulli distribution for p = 0.25





Binomial Distribution

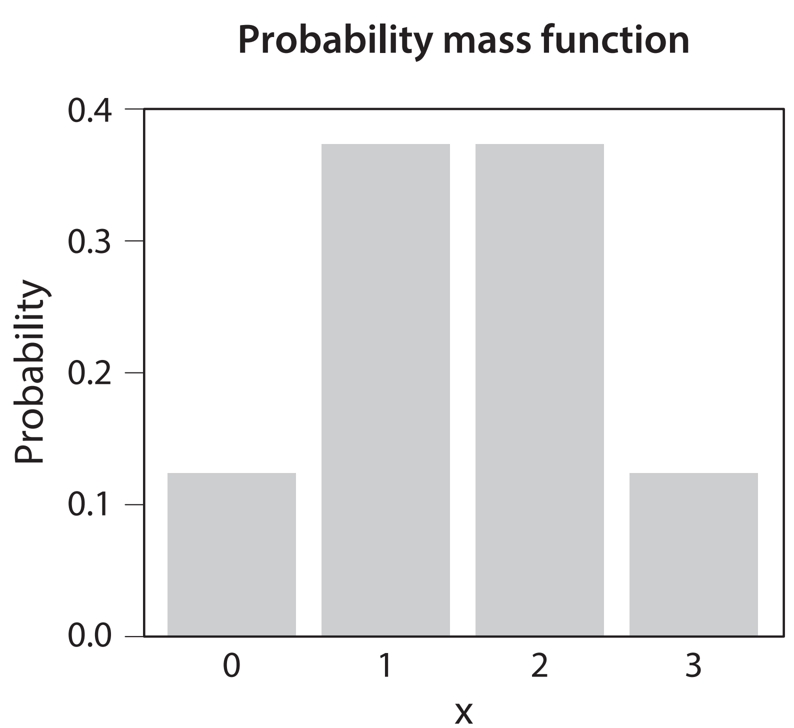
- The number of 1s (one of the binary outcomes) in multiple Bernoulli trials
- Ω={0,1,....,n-1,n}
- PMF

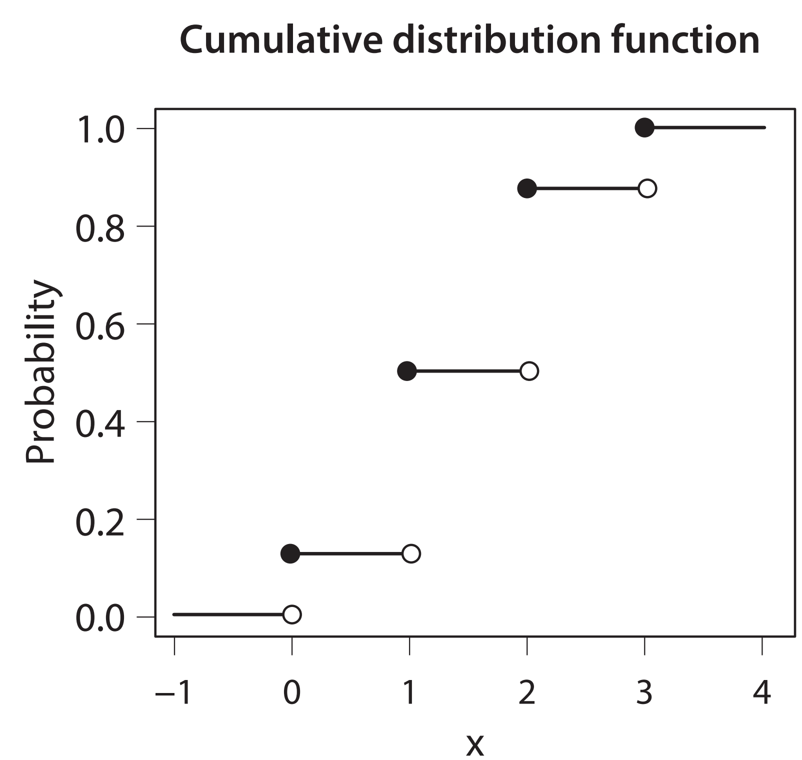
$$f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad \binom{n}{x} =_n C_x$$

- CMF

$$F(x) = P(X \leq x) = \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k}$$

- $p = 0.5$ and $n = 3$





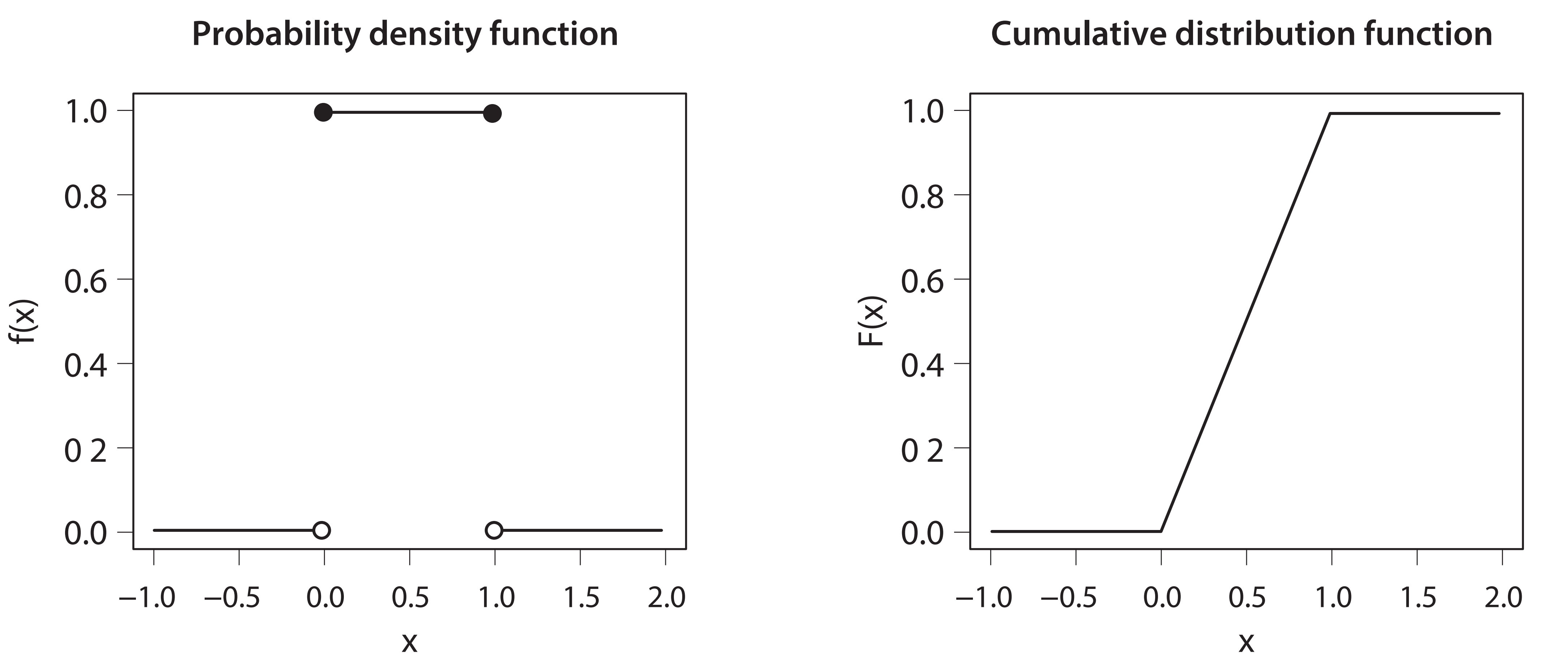
Binomial Distribution

- ex1) In a small department: There are exactly 10 who support candidate A, another 10 people who support candidate B for electing the chair of the department. Suppose that we expect their individual turnout probability is equal to their previous overall turnout rate which was 70%. What is the chance that exactly 7 people vote for candidate A and 7 people vote for candidate B, and the election ends in a tie?

- ex2) In a small department: There are exactly 10 who support candidate A, another 10 people who support candidate B for electing the chair of the department. Suppose that we expect their individual turnout probability is equal to their previous overall turnout rate which was 70%. What is the chance that the election ends in a tie?

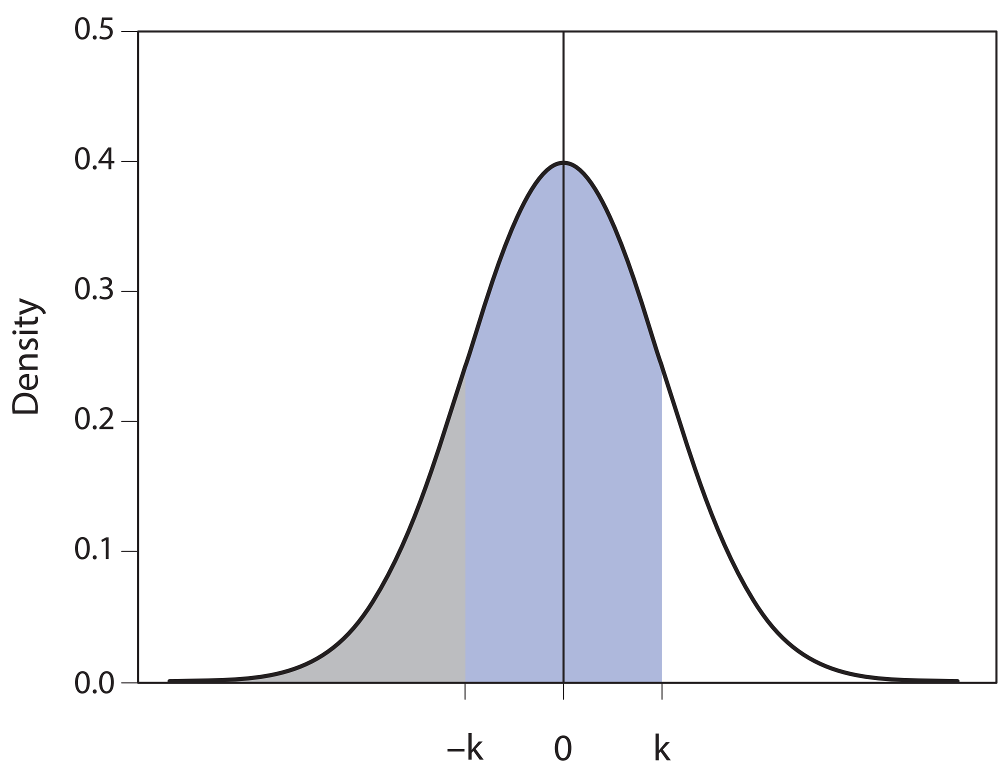
Uniform Distribution

- Every number in an interval has an equal chance of appearance
- $\Omega = \{x \mid a \leq x \leq b\}$
- PDF
$$f(x) = \begin{cases} \frac{1}{b-a} & \textbf{if } a \leq x \leq b \\ 0 & \textbf{otherwise.} \end{cases}$$
- CDF
$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & \textbf{if } a \leq x \leq b \\ 1 & x \geq b \end{cases}$$
- Uniform distribution for the interval [0,1]



Normal Distribution

- Most famous and frequently observed distribution (Why? -> next week)
- Ω=real numbers (continuous number)
- X is normal RV with mean μ and standard deviation σ: $X \sim \mathcal{N}(\mu, \sigma^2)$
- PDF
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$
- $X \sim \mathcal{N}(0,1)$



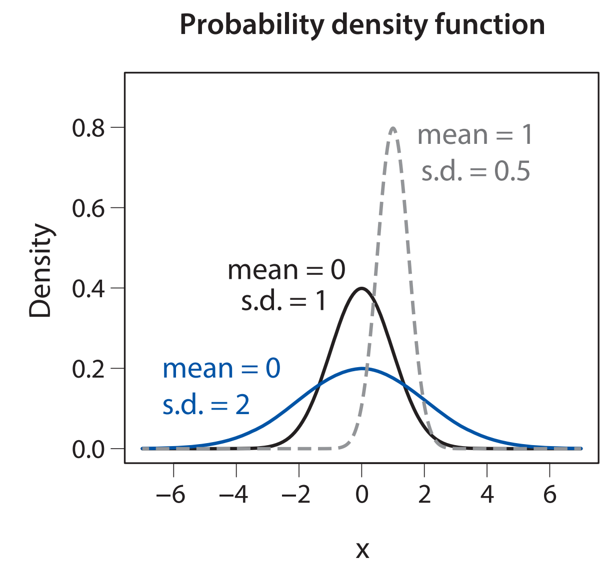
- Singled peaked, symmetric
- about 2/3 are within 1 standard deviation (σ) from the mean
- about 95% are within 2 standard deviations (2σ) from the mean

- CDF $$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(t-\mu)^2\right\} dt$$

Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

- $X \sim \mathcal{N}(\mu, \sigma^2)$
- $S = X + c \rightarrow S \sim \mathcal{N}(\mu + c, \sigma^2)$
- $Y = cX \rightarrow Y \sim \mathcal{N}(c\mu, (c\sigma)^2)$
- **z-score:** $Z = (X - \mu)/\sigma \rightarrow Z \sim \mathcal{N}(0,1)$



- Probability that a normal random variable with mean **μ** and sdv **σ** lies within k standard deviations from the mean for a positive constant $k > 0$

$$\begin{aligned} P(\mu - k\sigma \le X \le \mu + k\sigma) &= P(-k\sigma \le X - \mu \le k\sigma) \\ &= P\left(-k \le \frac{X-\mu}{\sigma} \le k\right) \\ &= P(-k \le Z \le k), \end{aligned}$$

$$P(-k \le Z \le k) = P(Z \le k) - P(Z \le -k) = F(k) - F(-k)$$

Normal Distribution

- Singled peaked, symmetric
- about 2/3 are within 1 standard deviation (σ) from the mean
- about 95% are within 2 standard deviations (2σ) from the mean

```r
## plus minus 1 standard deviation from the mean
pnorm(1) - pnorm(-1)

## [1] 0.6826895

## plus minus 2 standard deviations from the mean
pnorm(2) - pnorm(-2)

## [1] 0.9544997
```

```r
mu <- 5
sigma <- 2
## plus minus 1 standard deviation from the mean
pnorm(mu + sigma, mean = mu, sd = sigma) - pnorm(mu - sigma, mean = mu, sd = sigma)

## [1] 0.6826895

## plus minus 2 standard deviations from the mean
pnorm(mu + 2*sigma, mean = mu, sd = sigma) - pnorm(mu - 2*sigma, mean = mu, sd = sigma)

## [1] 0.9544997
```

Expectation: Definition and General Properties

- Expectation of a random variable X

$$\mathbb{E}(X) = \begin{cases} \sum_x x \times f(x) & \text{if } X \text{ is discrete,} \\ \int x \times f(x)\,dx & \text{if } X \text{ is continuous} \end{cases}$$

- Properties of expectation (a, b: constant–fixed value–)

1. $\mathbb{E}(a) = a$.
2. $\mathbb{E}(aX) = a\mathbb{E}(X)$.
3. $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$.
4. $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$.
5. If X and Y are independent, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$. But generally, $\mathbb{E}(XY) \neq \mathbb{E}(X)\mathbb{E}(Y)$.

Expectation

- Expectation (mean) revisited
  - Expected value of a random variable
    - PMF of any discrete random variable

$$\mathbb{E}(X) = 0 \times f(0) + 1 \times f(1) + \cdots + n \times f(n) \;=\; \sum_{x=0}^{n} x \times f(x)$$

    - PMF e.g. Bernoulli random variable

$$\mathbb{E}(X) = 0 \times P(X=0) + 1 \times P(X=1) = 0 \times f(0) + 1 \times f(1) = 0 \times (1-p) + 1 \times p = p$$

    - PMF e.g. Binomial random variable (Y is a Bernoulli RV with p)

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \mathbb{E}(Y_i) = np$$

    - PDF for a continuous random variable defined in the interval $[a,b]$

$$\mathbb{E}(X) \;=\; \int_a^b x \times f(x)\, dx = \int_a^b \frac{x}{b-a} dx = \left.\frac{x^2}{2(b-a)}\right|_a^b = \frac{a+b}{2}$$

Summary

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- Random Variables and Probability Distributions (Chapter 6.3)
  - Overview
  - Bernoulli and uniform distributions
  - Binomial distribution
  - Uniform distribution
  - Normal (or Gaussian) distribution
  - Expectation

See you next week.