

10-A4

## 高级搜索树

### 伸展树：分摊分析

所谓物价，其实就是我称之为生命的那部分，必须在交换时支付：要么立即支付，要么以后支付

转圆石于千仞之山者，势也

邓俊辉

deng@tsinghua.edu.cn

# S的势能

❖ (任何时刻的) 任何一棵伸展树  $S$ , 都可以**假想地**被认为具有势能:

$$\Phi(S) = \log \left( \prod_{v \in S} \text{size}(v) \right) = \sum_{v \in S} \log(\text{size}(v)) = \sum_{v \in S} \text{rank}(v)$$

❖ 直觉: 越**平衡/倾侧**的树, 势能越**小/大**

- 单链:  $\Phi(S) = \log n! = \mathcal{O}(n \log n)$

- 满树: 
$$\begin{aligned} \Phi(S) &= \log \prod_{d=0}^h (2^{h-d+1} - 1)^{2^d} \leq \log \prod_{d=0}^h (2^{h-d+1})^{2^d} \\ &= \log \prod_{d=0}^h 2^{(h-d+1) \cdot 2^d} = \sum_{d=0}^h (h-d+1) \cdot 2^d = (h+1) \cdot \sum_{d=0}^h 2^d - \sum_{d=0}^h d \cdot 2^d \\ &= (h+1) \cdot (2^{h+1} - 1) - [(h-1) \cdot 2^{h+1} + 2] = 2^{h+2} - h - 3 = \mathcal{O}(n) \end{aligned}$$

# T的上界

❖ 考查对伸展树  $S$  的  $m \gg n$  次连续访问（不妨仅考查  $\text{search}()$ ）

❖ 若记： $A^{(k)} = T^{(k)} + \Delta\Phi^{(k)}$ ,  $k = 0, 1, 2, \dots, m$

则有： $A - \mathcal{O}(n \log n) \leq \underline{T} = A - \Delta\Phi \leq A + \mathcal{O}(n \log n)$

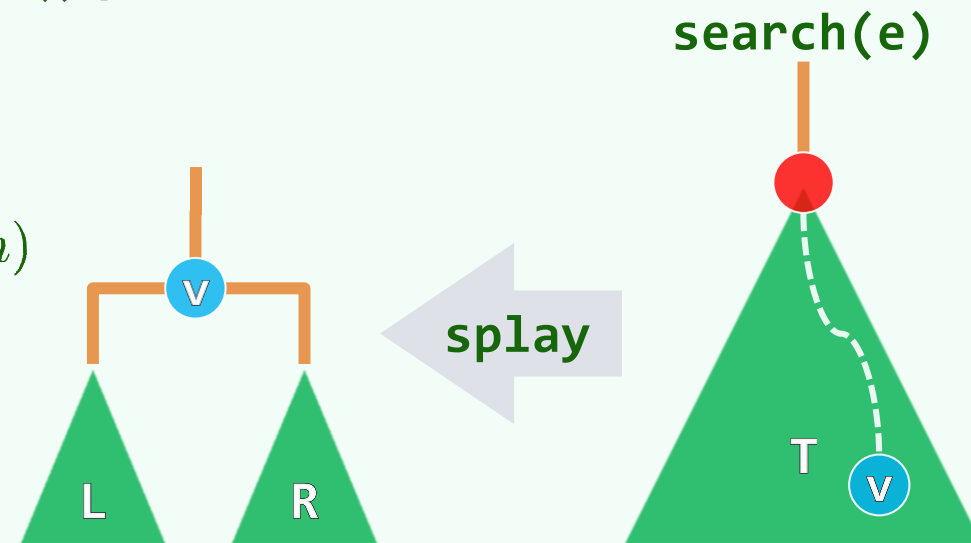
❖ 故若能证明： $A = \mathcal{O}(m \log n)$

则必有： $T = \mathcal{O}(m \log n)$

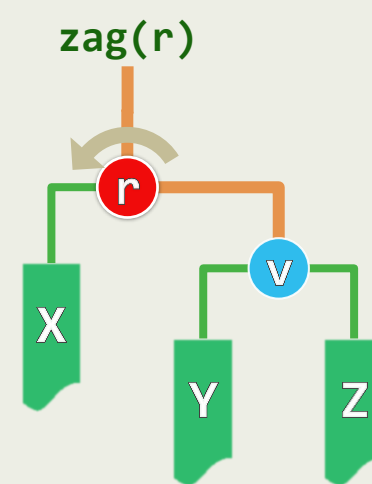
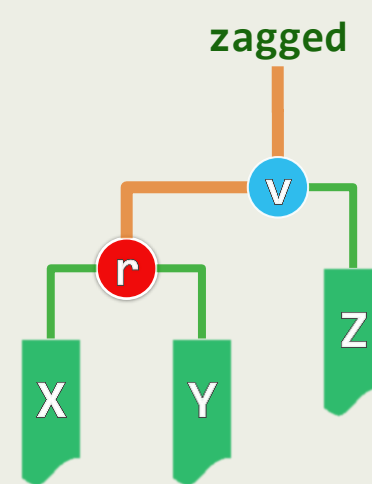
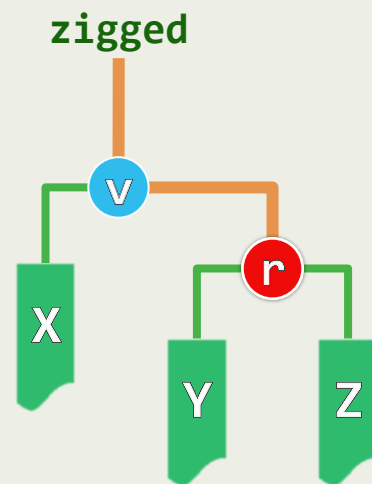
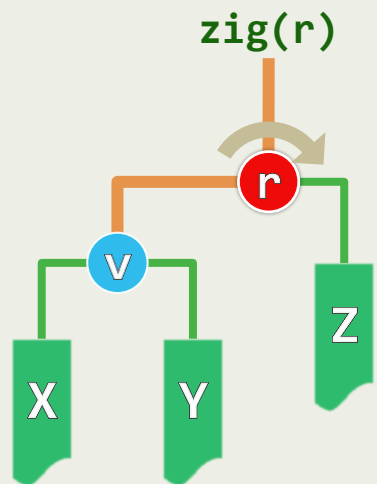
❖ 好消息是，尽管  $T^{(k)}$  的变化幅度可能很大，我们却能证明：

$A^{(k)}$  都不致超过节点  $v$  的势能变化量，即： $\mathcal{O}(\text{rank}^{(k)}(v) - \text{rank}^{(k-1)}(v)) = \mathcal{O}(\log n)$

❖ 事实上， $A^{(k)}$  不过是  $v$  的若干次连续伸展操作（时间成本）的累积，这些操作无非三种情况...

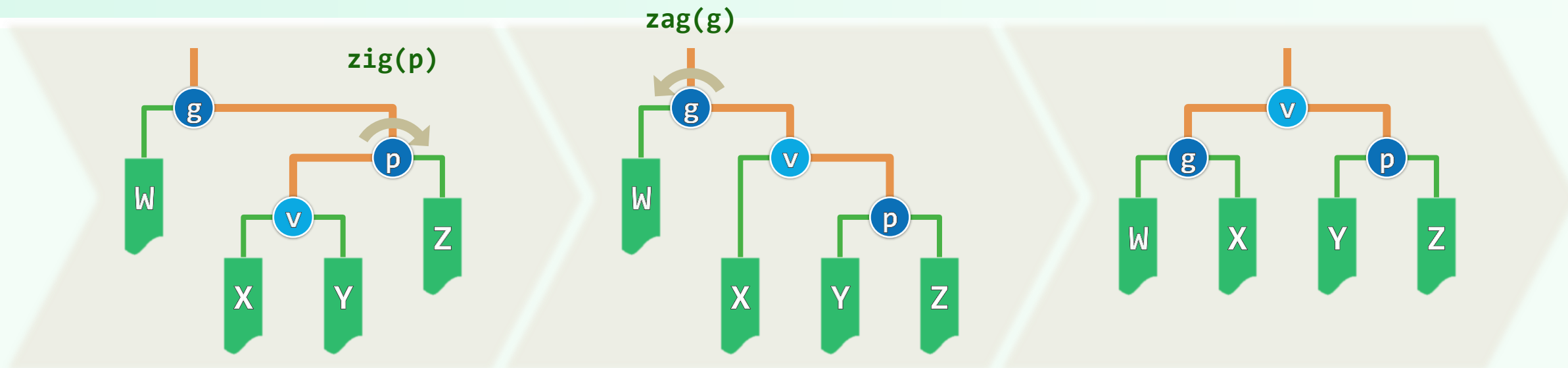


# Zig / Zag



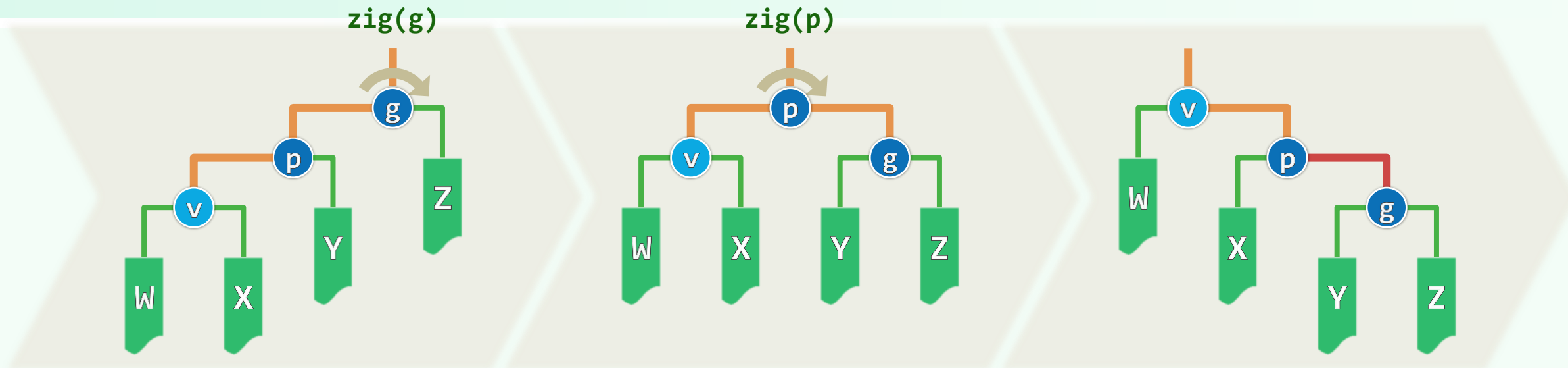
$$\begin{aligned}
 A_i^{(k)} &= T_i^{(k)} + \Delta\Phi(S_i^{(k)}) = 1 + \Delta\text{rank}_i(v) + \Delta\text{rank}_i(r) \\
 &= 1 + [\text{rank}_i(v) - \text{rank}_{i-1}(v)] + \underline{[\text{rank}_i(r) - \text{rank}_{i-1}(r)]} \\
 &< 1 + [\text{rank}_i(v) - \text{rank}_{i-1}(v)]
 \end{aligned}$$

## zig-zag / zag-zig



$$\begin{aligned}
 A_i^{(k)} &= T_i^{(k)} + \Delta\Phi(S_i^{(k)}) = 2 + \Delta\text{rank}_i(g) + \Delta\text{rank}_i(p) + \Delta\text{rank}_i(v) \\
 &= 2 + [\text{rank}_i(g) - \cancel{\text{rank}_{i-1}(g)}] + [\text{rank}_i(p) - \underline{\text{rank}_{i-1}(p)}] + [\cancel{\text{rank}_i(v)} - \underline{\text{rank}_{i-1}(v)}] \\
 &< 2 + \underline{\text{rank}_i(g) + \text{rank}_i(p)} - 2 \cdot \text{rank}_{i-1}(v) \quad (\because \text{rank}_{i-1}(p) > \text{rank}_{i-1}(v)) \\
 &< 2 + \underline{2 \cdot \text{rank}_i(v) - 2} - 2 \cdot \text{rank}_{i-1}(v) \quad (\because \frac{\log G_i + \log P_i}{2} \leq \log \frac{G_i + P_i}{2} < \log \frac{V_i}{2}) \\
 &= 2 \cdot (\text{rank}_i(v) - \text{rank}_{i-1}(v))
 \end{aligned}$$

## zig-zig / zag-zag



$$\begin{aligned}
 A_i^{(k)} &= T_i^{(k)} + \Delta\Phi(S_i^{(k)}) = 2 + \Delta\text{rank}_i(g) + \Delta\text{rank}_i(p) + \Delta\text{rank}_i(v) \\
 &= 2 + [\text{rank}_i(g) - \cancel{\text{rank}_{i-1}(g)}] + [\text{rank}_i(p) - \underline{\text{rank}_{i-1}(p)}] + [\cancel{\text{rank}_i(v)} - \underline{\text{rank}_{i-1}(v)}] \\
 &< 2 + \text{rank}_i(g) + \underline{\text{rank}_i(p)} - 2 \cdot \text{rank}_{i-1}(v) \quad (\because \text{rank}_{i-1}(p) > \text{rank}_{i-1}(v)) \\
 &< 2 + \text{rank}_i(g) + \underline{\text{rank}_i(v)} - 2 \cdot \text{rank}_{i-1}(v) \quad (\because \text{rank}_i(p) < \text{rank}_i(v)) \\
 &< 3 \cdot (\text{rank}_i(v) - \text{rank}_{i-1}(v)) \quad \left( \because \frac{\log G_i + \log V_{i-1}}{2} \leq \log \frac{G_i + V_{i-1}}{2} < \log \frac{V_i}{2} \right)
 \end{aligned}$$