04-L

栈与队列

直方图内最大矩形

就这么着，我有了一所严丝密缝、涂抹灰泥的木板房子，七英尺宽，十五英尺长，立柱有八英尺高...

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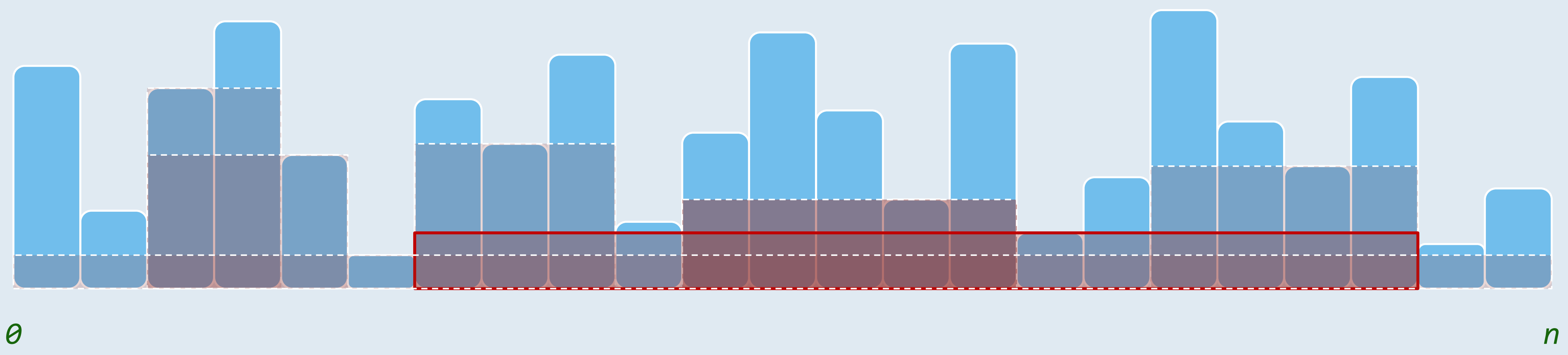
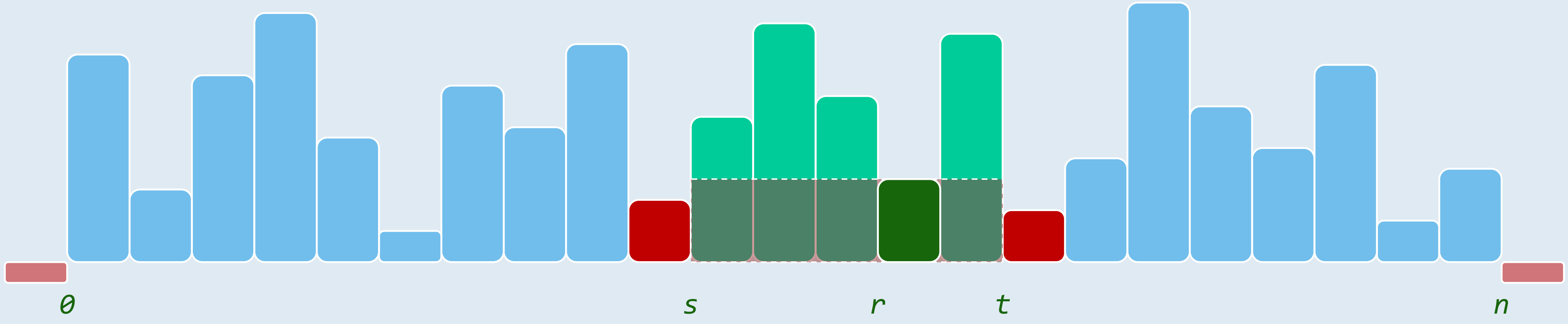
Maximum Rectangle



- Let H[0,n) be a histogram of non-negative integers
- How to find the largest orthogonal rectangle in H[]?
- To eliminate possible ambiguity

  we can, for example, choose the LEFTMOST one

Maximal Rectangles



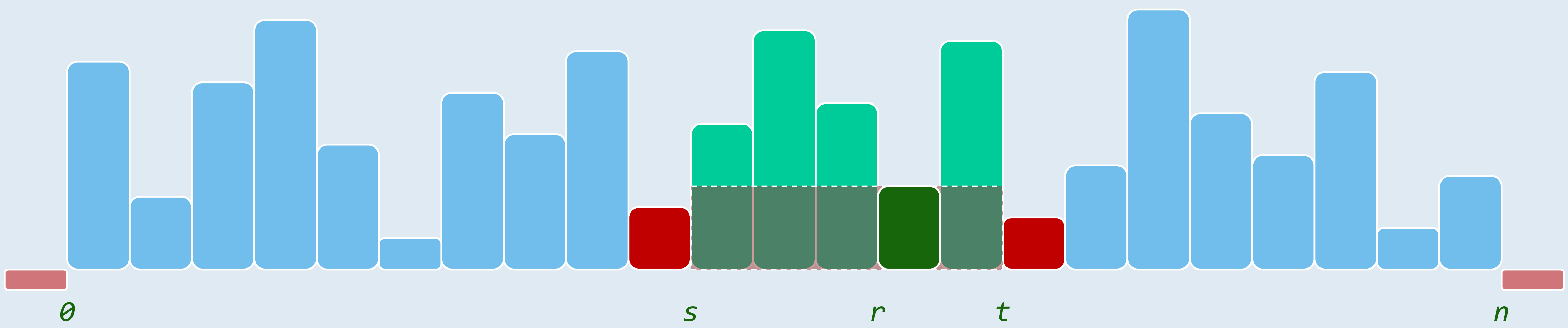
❖ **Maximal rectangle supported by H[r]:** $maxRect(r) = H[r] \cdot (t(r) - s(r))$

where

$$s(r) = \max\{\ k \mid 0 \le k \le r \text{ and } H[k-1] < H[r]\ \}$$

$$t(r) = \min\{\ k \mid r < k \le n \text{ and } H[r] > H[k]\ \}$$

Brute-force



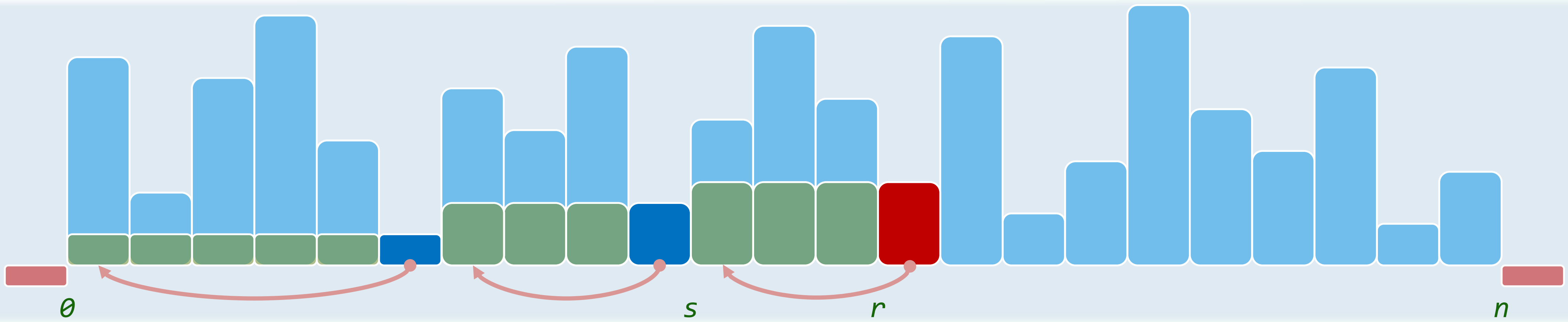
- Determining s(r) and t(r) for all r's requires $\mathcal{O}(n^2)$ time

$$s(r) = \max\{\ k \mid 0 \le k \le r \text{ and } H[k-1] < H[r]\ \}$$

$$t(r) = \min\{\ k \mid r < k \le n \text{ and } H[r] > H[k]\ \}$$

- Actually, we can do this more efficiently ...

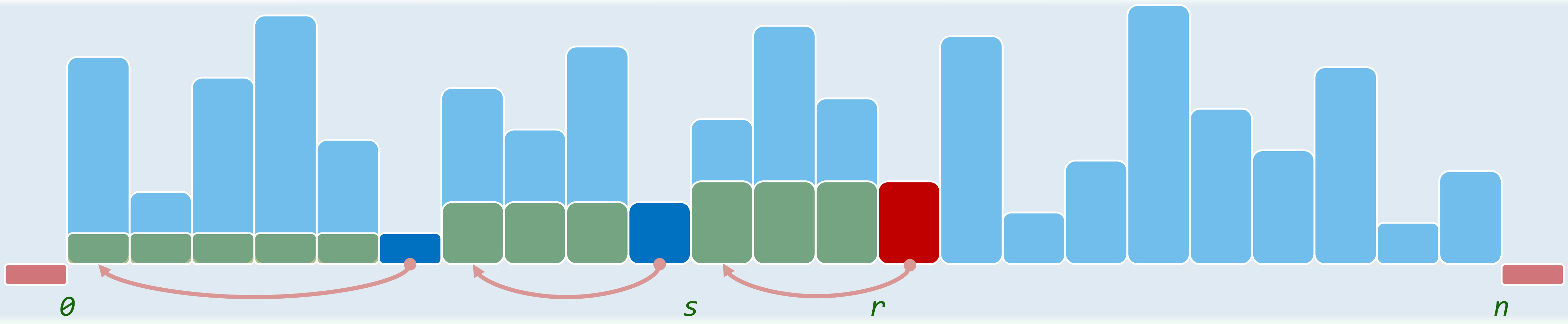
Using Stack: Algorithm



- All s(r)'s can be determined by a **LINEAR** scan of the histogram

```
int* s = new int[n]; Stack<Rank> S;
for ( int r = 0; r < n; r++ ) //try using SENTINEL for simplicity by yourself
   while ( !S.empty() && H[S.top()] >= H[r] ) S.pop(); //until H[top] < H[r]
   s[r] = S.empty() ? 0 : 1 + S.top();    S.push(r); //S is always ASCENDING
while( !S.empty() ) S.pop();
```

Using Stack: Loop Invariant & Correctness



❖ **After each iteration of the outer loop, we always have**

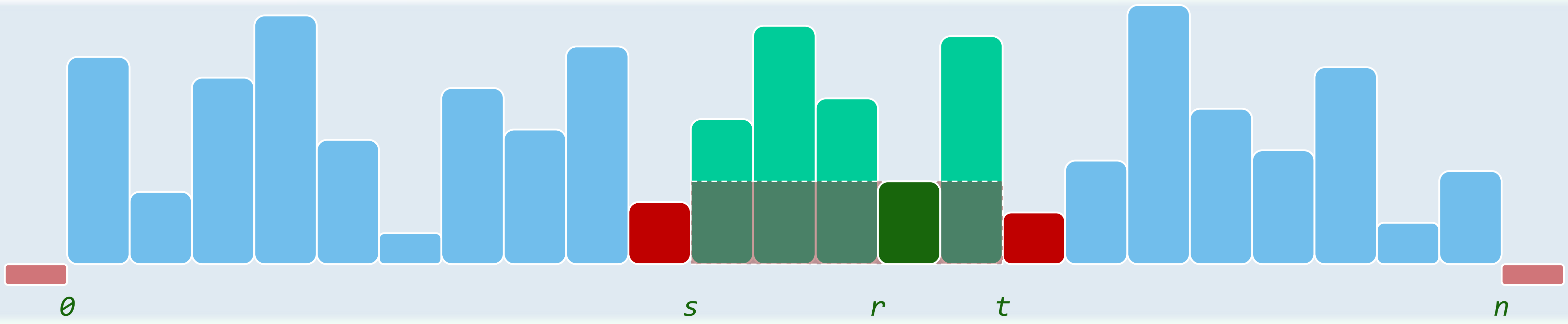
$$S[S.size()-1] = S.top() = r \qquad \textbf{and}$$

$$rank(S[i-1]) + 1 = s[S[i]] = \max\{\ k \mid 0 \le k < S[i] \text{ and } H[k] < H[S[i]]\ \} \quad (\text{for } 0 \le i < S.size())$$

❖ **Each r should be pushed into the stack and right before that, we have**

$$s[r] = 1 + S.top()$$

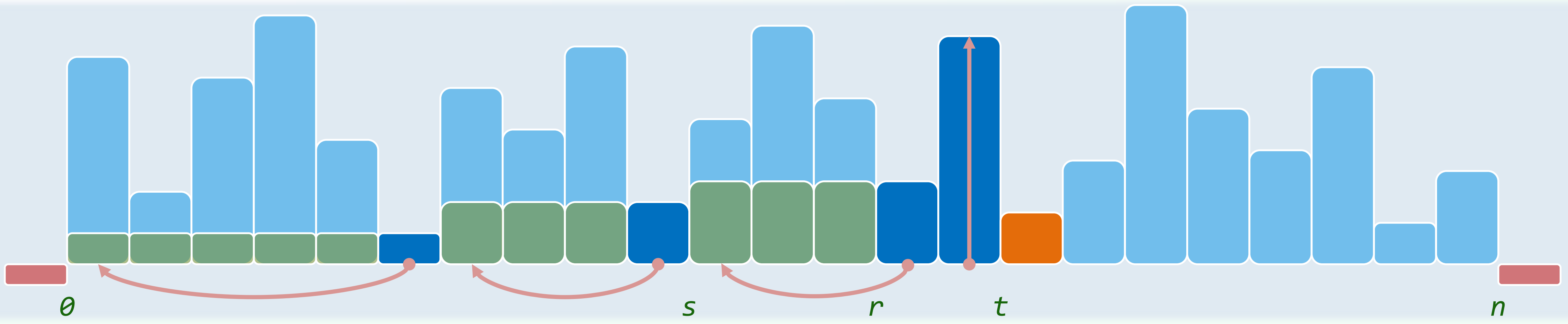
Using Stack: Complexity



- And t(r)'s can be determined by another scan in the REVERSED direction
- Hence all maximal rectangles can be computed in $\mathcal{O}(n)$ time (by AMORTIZATION)
- However, what if the histogram is given in an ON-LINE manner?

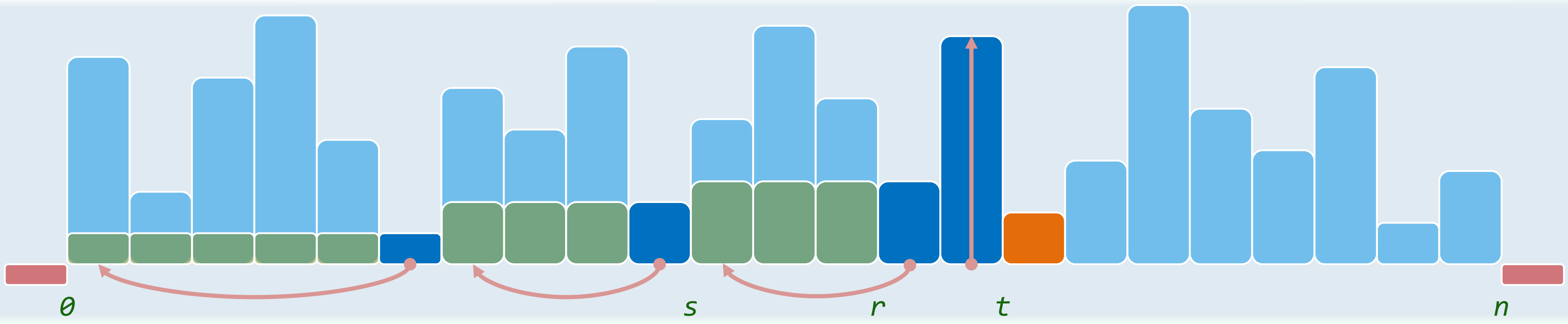
  In this case, the t(r)'s CAN'T be determined until the ENTIRE input is ready
- Is it possible to compute BOTH s(r)'s and t(r)'s by a SINGLE scan? //on-fly

One-Pass Scan: Algorithm




```
❖ Stack<int> SR;  __int64 maxRect = 0; //SR.2ndTop() == s(r)-1 & SR.top() == r
❖ for ( int t = 0; t <= n; t++ ) //amortized-O(n)
     while ( !SR.empty() && ( t == n || H[SR.top()] > H[t] ) )
        int r = SR.pop(); int s = SR.empty() ? 0 : SR.top() + 1;
        maxRect = max( maxRect, H[r] * ( t - s ) );
     if ( t < n ) SR.push( t );
  return maxRect;
```

One-Pass Scan: Loop Invariant & Correctness



- Again, after each iteration of the outer loop, we always have

  $SR[SR.size()-1] = SR.top() = t$ **and**

  $rank(SR[i-1]) + 1 = s[SR[i]] \quad (\text{for } 0 \leq i < SR.size())$

- At the end of each iteration of the inner loop, we have

  $t[SR.top()] = t$ **and** $s[SR.top()] = 1 + SR[SR.size()-2]$