《强化学习与控制》

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Direct RL with Policy Gradient

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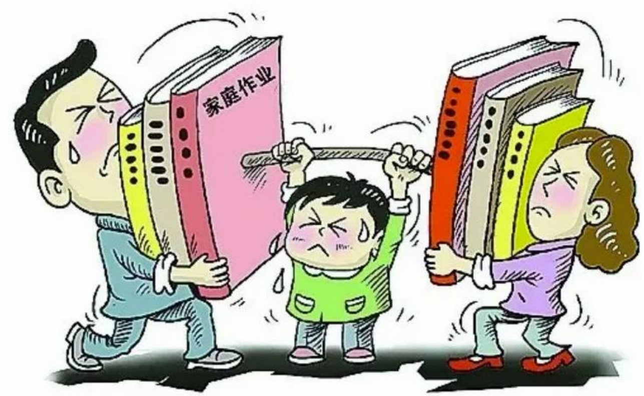
Never lose a holy curiosity

You cannot teach a man anything;
you can only help him discover it in himself.

-- Galileo Galilei (1564 - 1642)



Outline

1. Indirect RL vs Direct RL
2. Likelihood Ratio Gradient
3. AC from Direct RL
4. Optimization Viewpoint

General Optimal Control Problem

Basis of RL problems

- To find an optimal policy to maximize / minimize a weighted sum of expected return
- Subject to (1) data samples from environment interaction (i.e., model-free) or (2) analytical environment model (i.e., model-based)

$$\max_{\pi}/\min_{\pi} \ \mathbb{E}_{s\sim d(s)}\{v^{\pi}(s)\}$$

Subj. to

$$p(s'|s,a)=\mathcal{P}_{ss'}^{a},$$

or

$$\{s_0, a_0, r_0, s_1, a_1, r_1, s_2, a_2, r_2, s_3, a_3, r_3, \cdots\}$$

?

$\pi^*(a|s)$

Indirect RL

VS

Direct RL

Indirect RL vs Direct RL

(1) Indirect RL

- Sufficient & necessary condition of optimality
  - Hamilton-Jacobi-Bellman equation (continuous-time)
  - Bellman equation (discrete-time)

$$\boldsymbol{\pi}^*(\boldsymbol{a}|\boldsymbol{s}) = \text{Solution of HJB/Bellman equation}$$

- Convergence: Bellman operator is γ-contractive

(2) Direct RL

- Search for a parameterized policy that maximizes the overall objective function

$$\theta^* = \arg\max_{\theta} J(\pi(a|s;\theta))$$

- Search θ^* by using numerical optimization technique
- Convergence: Same as optimization algorithms

Classification of Direct RL

Mainstream direct RL methods

Zero-order optimization

- Evolutionary algorithm (e.g., finite difference)
- Bayesian optimization

First-order optimization

- Likelihood ratio gradient
- Natural policy gradient
- Deterministic policy gradient

Second-order optimization

- Newton method
- Quasi-Newton method

General Optimal Control Problem

Overall RL objective function

$$\max_{\theta} J(\theta) = \mathbb{E}_{s_t \sim d(s_t)}\{v^{\pi_\theta}(s_t)\}$$

$$= \int d(s_t) v^{\pi_\theta}(s_t) \mathrm{d}s_t$$

$$= \mathbb{E}_{s_t, a_t, s_{t+1}, \dots \sim \rho_{\pi_\theta}} \left\{ \sum_{\tau=t}^{\infty} \gamma^{\tau-t} r_\tau \right\}$$

ρ_{π_θ} is joint probability of states and actions in the trajectory

- Revisit value function in terms of trajectory concept

$$v^\pi(s) = \mathbb{E}_{a_t, s_{t+1}, a_{t+1}, s_{t+2}, a_{t+2}, \dots \sim \rho_{\pi_\theta}} \left\{ \sum_{\tau=t}^{\infty} \gamma^{\tau-t} r_\tau \,|\, s_t = s \right\}$$

$$q^\pi(s, a) = \mathbb{E}_{s_{t+1}, a_{t+1}, s_{t+2}, a_{t+2}, \dots \sim \rho_{\pi_\theta}} \left\{ \sum_{\tau=t}^{\infty} \gamma^{\tau-t} r_\tau \,|\, s_t = s, a_t = a \right\}$$

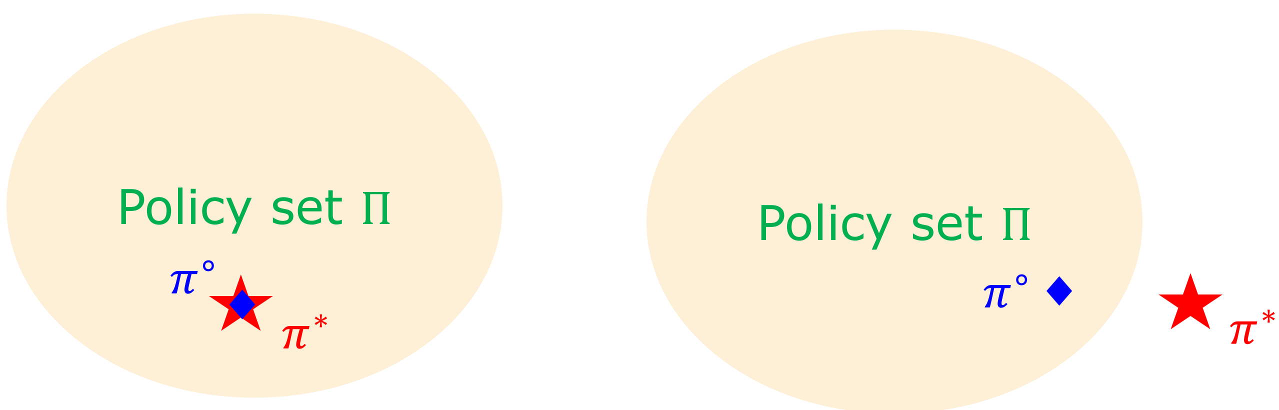
Influence of Initial State Distribution

Define two kinds of "optimal" policies

$$\pi^*(s) = \arg\max_{a \in \mathcal{A}} \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^{a} \left(r + \gamma v^*(s')\right), \forall s \in \mathcal{S}$$

$$\pi^{\circ} = \arg\max_{\pi \in \Pi} J\left(\pi(s)\right)$$

- Π is the allowable policy set from designers
- π^* is optimal policy coming from each state element
- π° is optimal policy from overall RL criterion maximization



Case (1): $\pi^* \in \Pi$ Case (2): $\pi^* \notin \Pi$

Influence of Initial State Distribution

- **Case (1): $\pi^*(s)$ is inside allowable policy set Π**
  - 1st step

$$J(\pi^*) \le J(\pi^\circ) = \max_\pi J(\pi)$$

  - 2nd step

$$\begin{aligned} J(\pi^\circ) &= \max_\pi \mathbb{E}_{s\sim d(s)}\{v^\pi(s)\} \\ &\le \max_\pi \mathbb{E}_{s\sim d(s)}\left\{\max_\pi v^\pi(s)\right\} \\ &= \mathbb{E}_{s\sim d(s)}\left\{\max_\pi v^\pi(s)\right\} \\ &= \mathbb{E}_{s\sim d(s)}\{v^*(s)\} \\ &= J(\pi^*) \end{aligned}$$

  - Conclusion

$$J(\pi^*) = J(\pi^\circ)$$

$$\mathbb{E}_{s\sim d(s)}\left\{\max_\pi v^\pi(s)\right\} = \max_\pi \mathbb{E}_{s\sim d(s)}\{v^\pi(s)\}$$

    - Initial state distribution does not affect optimal policy

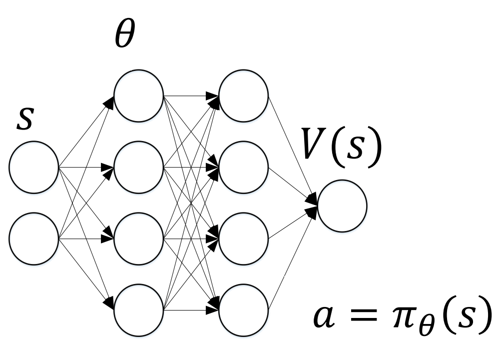
Influence of Initial State Distribution

- **Case (2): $\pi^*(s)$ is NOT inside allowable policy set Π**
  - Only the inequality holds

$$\max_{\pi} \mathbb{E}_{s\sim d(s)}\{v^{\pi}(s)\} \le \mathbb{E}_{s\sim d(s)}\left\{\max_{\pi} v^{\pi}(s)\right\}$$

$$J(\pi^{\circ}) \le J(\pi^*)$$

  - Conclusion
    - Policy $\pi^{\circ}(s) \in \Pi$ gives a less optimal policy than π^*
    - π° becomes dependent of initial state distribution $d(s)$
  - Hint: policy set Π should be as large as possible
    - Neural network is a good choice!



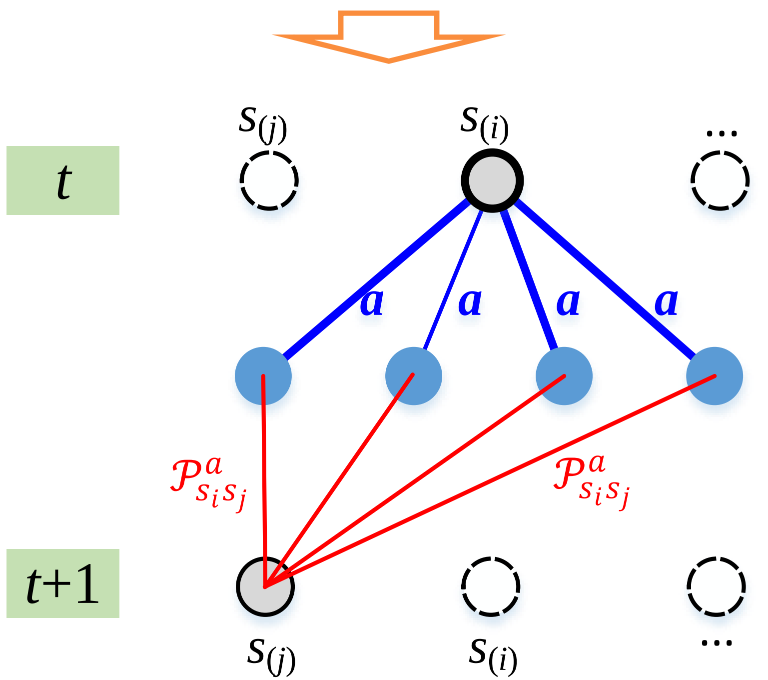
Stationary State Distribution

One-step transition probability

- Given policy $\pi(a|s)$ and environment model $p(s'|s,a)$

$$\mathcal{S} = \{s_{(1)}, s_{(2)}, \cdots, s_{(n)}\}$$

$$\zeta_{i,j} = \sum_{a \in \mathcal{A}} \pi(a|s = s_{(i)}) p(s' = s_{(j)}|s = s_{(i)}, a)$$



Stationary State Distribution

State distribution at time t

- Occurrence frequency of a certain state at time t

$$d_t(s_{(i)}) = \Pr\{s_t = s_{(i)}\}$$

- "Stationary" refers to "stationary in time"

$$\boldsymbol{d}_{t+1} = H_{n\times n}\boldsymbol{d}_t$$

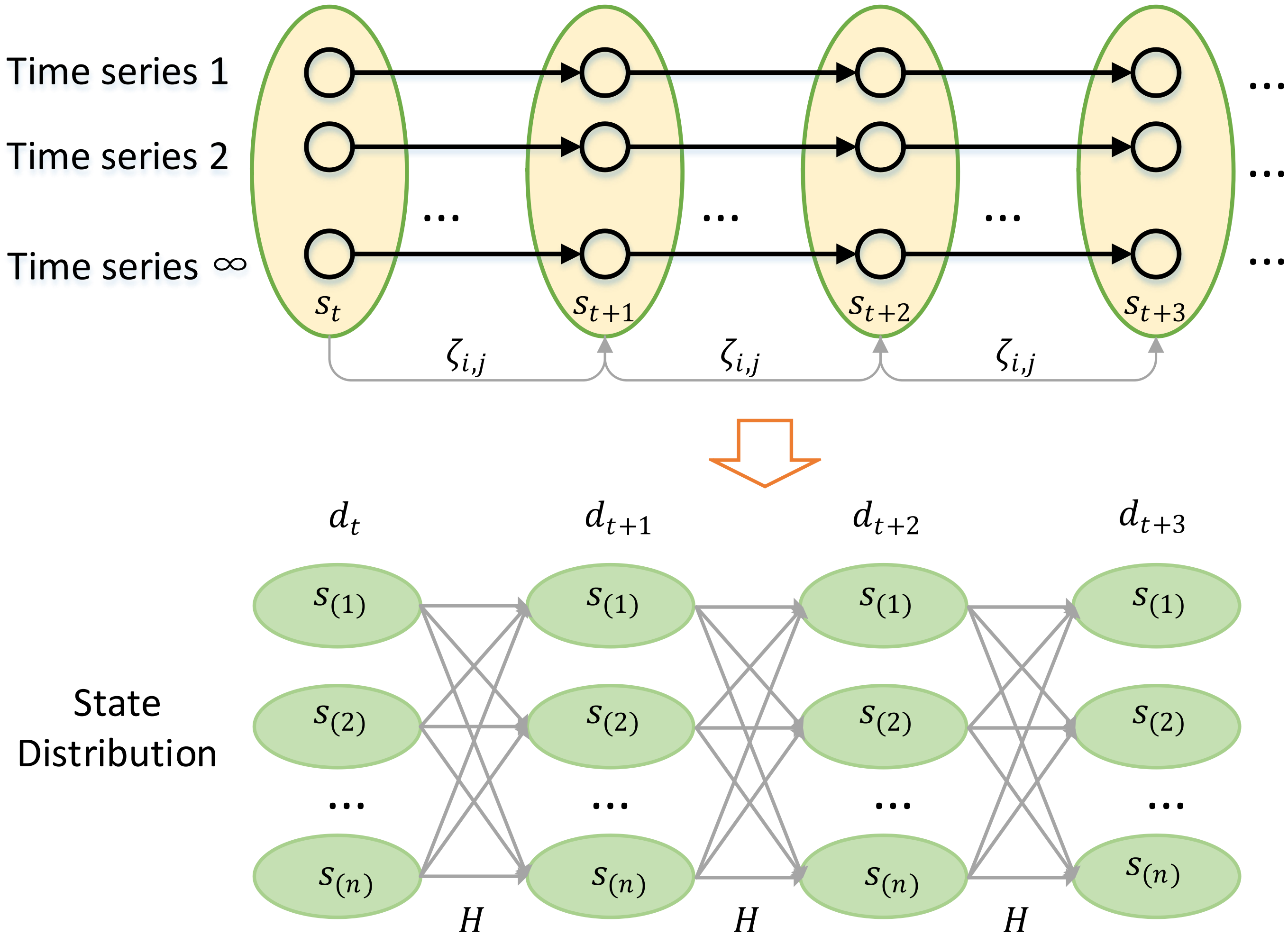
$$H_{n\times n} = \begin{bmatrix} \zeta_{1,1} & \cdots & \zeta_{n,1} \\ \vdots & \ddots & \vdots \\ \zeta_{1,n} & \cdots & \zeta_{n,n} \end{bmatrix} \quad \boldsymbol{d}_t = \begin{bmatrix} d_t(s_{(1)}) & d_t(s_{(2)}) & \cdots & d_t(s_{(n)}) \end{bmatrix}^{\mathrm{T}}$$

$$\boldsymbol{d}(\boldsymbol{s}) = \mathbf{H}\boldsymbol{d}(\boldsymbol{s})$$

Stationary state distribution (**SSD**)

Stationary State Distribution

Random variable vs State distribution



Stationary State Distribution

Some properties of SSD

- (1) Any finite, irreducible, and ergodic Markov chain has a unique SSD

- (2) For any $i, j \in \mathcal{S}$, the following limit exists, independent of initial state s_0

$$\lim_{t\to\infty} \Pr\{s_t = s_{(j)} | s_0 = s_{(i)}\}$$

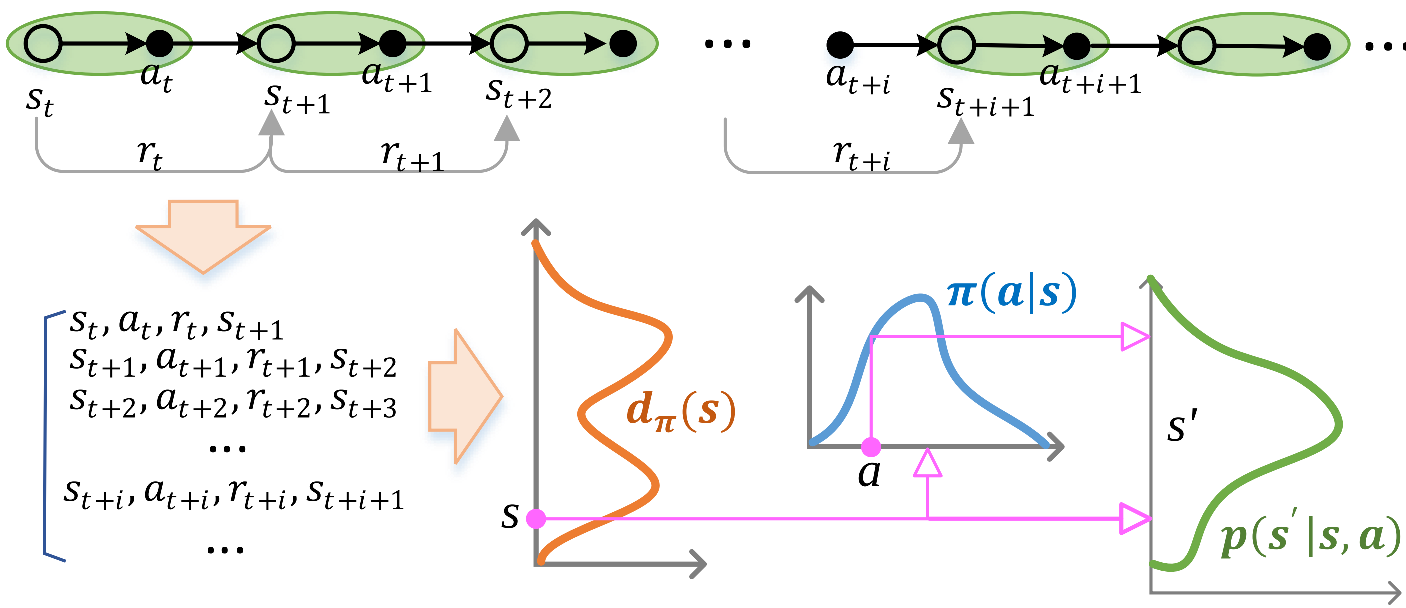
- (3) In an MDP, the SSD under policy π is

$$d_\pi(s_{(j)}) = \lim_{t\to\infty} \Pr\{s_t = s_{(j)} | s_0 = s_{(i)}\}$$

Stationary State Distribution

Graphic understanding

- Limiting distribution that can starts from any initial state distribution
- Temporal order of samples becomes meaningless since each sample could occur randomly with infinite times



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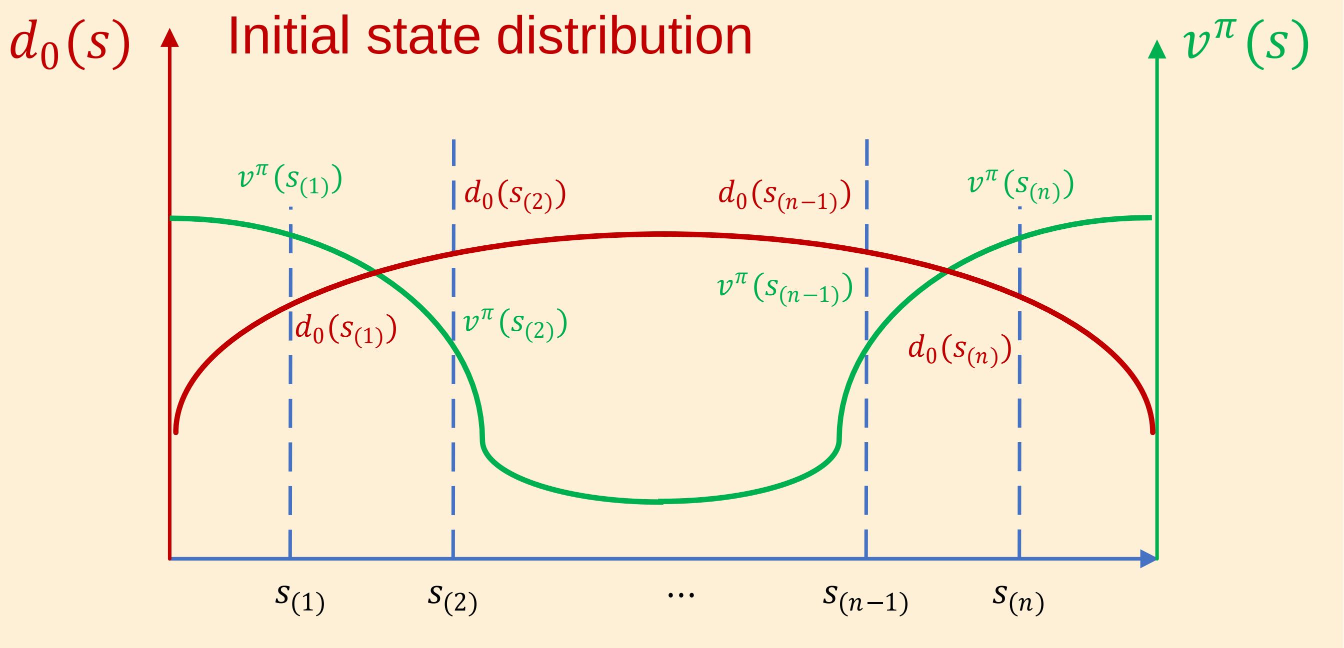
Objective function for Direct RL

Overall RL objective function

- Assume that current time $t = 0$
- Finite state space $\mathcal{S} = \{s_{(1)}, s_{(2)}, \cdots, s_{(n)}\}$

$$J(\theta) = \mathbb{E}_{s_0 \sim d_0(s_0)}\{v^\pi(s_0)\} = \sum_{s_0 \in \mathcal{S}} d_0(s_0) v^\pi(s_0)$$



Likelihood Ratio Gradient

$$\nabla_\theta J(\theta) = \nabla_\theta \sum_{s_0} d_0(s_0) v^\pi(s_0)$$

$$v^\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) q^\pi(s,a)$$

$$= \nabla_\theta \sum_{s_0} d_0(s_0) \sum_{a_0} \pi_\theta(a_0|s_0) q^{\pi_\theta}(s_0, a_0)$$

$d_0(s_0)$ is independent of θ

$$= \sum_{s_0} d_0(s_0) \nabla_\theta \sum_{a_0} \pi_\theta(a_0|s_0) q^{\pi_\theta}(s_0, a_0)$$

Derivation rule

$$= \sum_{s_0} d_0(s_0) \sum_{a_0} [\nabla_\theta \pi_\theta(a_0|s_0) q^{\pi_\theta}(s_0, a_0) + \pi_\theta(a_0|s_0) \nabla_\theta q^{\pi_\theta}(s_0, a_0)]$$

Relation of q-function and v-function

$$= \sum_{s_0} d_0(s_0) \sum_{a_0} \left[\nabla_\theta \pi_\theta(a_0|s_0) q^{\pi_\theta}(s_0, a_0) + \pi_\theta(a_0|s_0) \nabla_\theta \left[r_0 + \gamma \sum_{s_1} p(s_1|s_0, a_0) v^{\pi_\theta}(s_1) \right] \right]$$

Likelihood Ratio Gradient

$$= \sum_{s_0} d_0(s_0) \sum_{a_0} \left[\nabla_\theta \pi_\theta(a_0|s_0) q^{\pi_\theta}(s_0, a_0) \ + \pi_\theta(a_0|s_0) \nabla_\theta \left[r_0 + \gamma \sum_{s_1} p(s_1|s_0, a_0) v^{\pi_\theta}(s_1) \right] \right]$$

$$= \sum_{s_0} d_0(s_0) \sum_{a_0} \left[\nabla_\theta \pi_\theta(a_0|s_0) q^{\pi_\theta}(s_0, a_0) + \pi_\theta(a_0|s_0) \left[\gamma \sum_{s_1} p(s_1|s_0, a_0) \nabla_\theta v^{\pi_\theta}(s_1) \right] \right]$$

$$= \sum_{s_0} d_0(s_0) \sum_{a_0} \left[\nabla_\theta \pi_\theta(a_0|s_0) q^{\pi_\theta}(s_0, a_0) + \pi_\theta(a_0|s_0) \left[\gamma \sum_{s_1} p(s_1|s_0, a_0) \nabla_\theta \sum_{a_1} \pi_\theta(a_1|s_1) q^{\pi_\theta}(s_1, a_1) \right] \right]$$

$$= \sum_{s_0} d_0(s_0) \sum_{a_0} \left[\nabla_\theta \pi_\theta(a_0|s_0) q^{\pi_\theta}(s_0, a_0) \right.$$

Derivation rule

Likelihood Ratio Gradient

Triangular analysis

$$\nabla_\theta J(\theta) = \gamma \sum_{s_0} d_0(s_0) \sum_{a_0} \pi_\theta(a_0|s_0) \sum_{s_1} p(s_1|s_0,a_0) \sum_{a_1} \pi_\theta(a_1|s_1) \nabla_\theta q^{\pi_\theta}(s_1,a_1)$$

$$+\gamma \sum_{s_0} d_0(s_0) \sum_{a_0} \pi_\theta(a_0|s_0) \sum_{s_1} p(s_1|s_0,a_0) \sum_{a_1} \nabla_\theta \pi_\theta(a_1|s_1) q^{\pi_\theta}(s_1,a_1)$$

$$+\sum_{s_0} d_0(s_0) \sum_{a_0} \nabla_\theta \pi_\theta(a_0|s_0) q^{\pi_\theta}(s_0,a_0)$$

Addition is associative

Likelihood Ratio Gradient

Triangular analysis

$$\nabla_\theta J(\theta) = \gamma \sum_{s_0} d_0(s_0) \sum_{a_0} \pi_\theta(a_0|s_0) \sum_{s_1} p(s_1|s_0,a_0) \sum_{a_1} \pi_\theta(a_1|s_1) \nabla_\theta q^{\pi_\theta}(s_1,a_1)$$

$$+ \sum_{s_1} \gamma \sum_{s_0} d_0(s_0) \sum_{a_0} \underline{\pi_\theta(a_0|s_0) p(s_1|s_0,a_0)} \sum_{a_1} \nabla_\theta \pi_\theta(a_1|s_1) q^{\pi_\theta}(s_1,a_1)$$

$$+ \sum_{s_0} d_0(s_0) \sum_{a_0} \nabla_\theta \pi_\theta(a_0|s_0) q^{\pi_\theta}(s_0,a_0)$$

Transition probability for $s_0 \rightarrow s_1$

Roll forward till infinity

$$\nabla_\theta J(\theta) = \sum_{s} \sum_{t=0}^{\infty} \gamma^t p(s_t = s|\pi_\theta) \sum_{a} \nabla_\theta \pi_\theta(a|s) q^{\pi_\theta}(s,a)$$

Likelihood Ratio Gradient

Vanilla Policy Gradient (Sutton et al., 2000)

$$\nabla_\theta J(\theta) = \frac{1}{1-\gamma} \sum_s d_{\pi_\theta}^{\gamma}(s) \sum_a \nabla_\theta \pi_\theta(a|s) q^{\pi_\theta}(s,a)$$

$$d_{\pi_\theta}^{\gamma}(s) \stackrel{\text{def}}{=} (1-\gamma) \sum_{t=0}^{\infty} \gamma^t p(s_t = s | \pi_\theta)$$

Discounted state distribution

When does it become SSD?

$d_0(s)$ is stationary state distribution

$d_0(s)$ is nonstationary but $\gamma \to 1$

Vanilla Policy Gradient

- **Case (1): If** $d_0(s) = d_{\pi_\theta}(s)$**, then** $s_t \sim d_{\pi_\theta}$ **for all** t

$$d_{\pi_\theta}^{\gamma}(s) = (1-\gamma)\sum_{t=0}^{\infty}\gamma^t p(s_\tau = s|\pi_\theta) = (1-\gamma)\sum_{t=0}^{\infty}\gamma^t d_{\pi_\theta}(s) = d_{\pi_\theta}(s)$$

Independent of time

$$\nabla_\theta J(\theta) = \frac{1}{1-\gamma}\sum_{s} d_{\pi_\theta}^{\gamma}(s)\sum_{a}\nabla_\theta \pi_\theta(a|s) q^{\pi_\theta}(s,a)$$

$$\nabla_\theta J(\theta) \propto \mathbb{E}_{\pi_\theta}\{\nabla_\theta \log \pi_\theta(a|s)\, q^{\pi_\theta}(s,a)\}$$

True action-value function

Vanilla Policy Gradient

- **Case (2): $d_0(s)$ is NOT stationary, but $\gamma \to 1$**
  - Property of normalization (Independent of γ)

$$(1-\gamma)\sum_{s} d_{\pi_\theta}^{\gamma}(s) = (1-\gamma)\sum_{t=0}^{\infty}\gamma^t \sum_{s} p(s_t = s|\pi_\theta) = (1-\gamma)\sum_{t=0}^{\infty}\gamma^t = 1$$

  - Limit of approximation

$$\lim_{\gamma\to 1} d_{\pi_\theta}^{\gamma}(s) = \lim_{\gamma\to 1}\frac{d_{\pi_\theta}^{\gamma}(s)}{\sum_s d_{\pi_\theta}^{\gamma}(s)}$$

$$= \lim_{\gamma\to 1}\lim_{N\to\infty}\frac{\sum_{t=0}^{N}\gamma^t p(s_t = s|\pi_\theta)}{\sum_s \sum_{t=0}^{N}\gamma^t p(s_t = s|\pi_\theta)}$$

$$= \lim_{N\to\infty}\frac{\sum_{t=0}^{N} p(s_t = s|\pi_\theta)}{\sum_{t=0}^{N}\sum_s p(s_t = s|\pi_\theta)}$$

$$= \lim_{N\to\infty}\frac{\sum_{t=0}^{N} p(s_t = s|\pi_\theta)}{N+1}$$

$$= d_{\pi_\theta}(s)$$

$$\sum_{s} p(s_t = s|\pi_\theta) = 1$$

$$d_{\pi_\theta}(s) = \lim_{N\to\infty}\frac{\sum_{t=0}^{N} p(s_t = s|\pi_\theta)}{N+1}$$

Vanilla Policy Gradient

- **Case (2):** $d_0(s)$ **is NOT stationary, but** $\gamma \rightarrow 1$

Vanilla Policy Gradient (Sutton et al., 2000)

$$\lim_{\gamma \rightarrow 1} d^{\gamma}_{\pi_\theta}(s) = d_{\pi_\theta}(s)$$

$$\nabla_\theta J(\theta) = \frac{1}{1-\gamma} \sum_s d^{\gamma}_{\pi_\theta}(s) \sum_a \nabla_\theta \pi_\theta(a|s) q^{\pi_\theta}(s,a)$$

$$\nabla_\theta J(\theta) \propto \mathbb{E}_{\pi_\theta}\{\nabla_\theta \log \pi_\theta(a|s)\, q^{\pi_\theta}(s,a)\}$$

True action-value function

Special Case: MC Policy Gradient

Policy Gradient with Monte Carlo Estimation

- Monte Carlo estimation of action-value function

$$q^{\pi}(s,a) \approx \mathrm{Avg}\{G_t | s_t = s, a_t = a\}$$

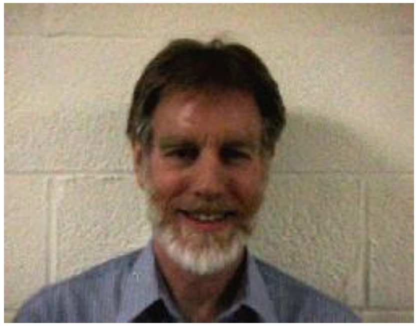
Average returns followed after a particular state-action pair (s, a)

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\{\nabla_\theta \log \pi_\theta(a|s)\, q^{\pi_\theta}(s,a)\}$$

$$q^{\pi_\theta}(s,a) \approx \mathrm{Avg}\{G_t | s_t = s, a_t = a\}$$

$$\theta \leftarrow \theta + \beta \cdot \nabla_\theta \log \pi_\theta(a|s)\, \mathrm{Avg}\{G_t | s_t, a_t\}$$

*REINFORCE (Williams, 1992)



Variance Reduction with Baseline

Baseline technique

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\{q^{\pi_\theta}(s,a)\nabla_\theta \log \pi_\theta(a|s)\}$$

Baseline

$$\nabla_\theta J(\theta) \propto \mathbb{E}_{\pi_\theta}\{(q^{\pi_\theta}(s,a) - \zeta(s))\nabla_\theta \log \pi_\theta(a|s)\}$$

- Unbiased estimation only if the baseline is independent of action
  - Proof

$$\begin{aligned}
\mathbb{E}_{\pi_\theta}\{\zeta(s)\nabla_\theta \log \pi_\theta(a|s)\} &= \sum_s d_{\pi_\theta}(s) \sum_a \pi_\theta(a|s) \cdot \zeta(s) \frac{\nabla_\theta \pi_\theta(a|s)}{\pi_\theta(a|s)} \\
&= \sum_s d_{\pi_\theta}(s)\, \zeta(s) \nabla_\theta \sum_a \pi_\theta(a|s) \\
&= \sum_s d_{\pi_\theta}(s)\, \zeta(s) \nabla_\theta 1 \\
&= \sum_s d_{\pi_\theta}(s)\, \zeta(s) \times 0 \\
&= 0
\end{aligned}$$

Variance Reduction with Baseline

What is the optimal baseline?

$$
\begin{aligned}
\Delta\mathbb{D} &= \mathbb{D}\{\nabla_\theta J_{\mathrm{BL}}\} - \mathbb{D}\{\nabla_\theta J\} \\
&= \mathbb{D}_{\pi_\theta}\left\{\left(q^{\pi_\theta}(s,a) - \zeta(s)\right)\nabla_\theta \log \pi_\theta\right\} - \mathbb{D}_{\pi_\theta}\{q^{\pi_\theta}(s,a)\nabla_\theta \log \pi_\theta\} \\
&= -\mathbb{E}_{\pi_\theta}(\nabla_\theta \log \pi_\theta)^2 \mathbb{E}_{\pi_\theta}\{(2v^{\pi_\theta}(s) - \zeta(s))\zeta(s)\}
\end{aligned}
$$

minimize

$\zeta(s) = v^{\pi_\theta}(s)$

Optimal baseline is state-value function

$$
\Delta\mathbb{D}_{\min} = -\mathbb{E}_{\pi_\theta}(\nabla_\theta \log \pi_\theta)^2 \mathbb{E}_{\pi_\theta}\left\{\left(v^{\pi_\theta}(s)\right)^2\right\} \le 0
$$

Variance Reduction with Baseline

What is the optimal baseline?

- The best choice of baseline is state-value function

$$\zeta(s) = v^{\pi_\theta}(s)$$

$$\nabla_\theta J(\theta) \propto \mathbb{E}_{\pi_\theta}\left\{\underbrace{\left(q^{\pi_\theta}(s,a) - v^{\pi_\theta}(s)\right)}_{A(s,a)\text{: advantage function}} \nabla_\theta \log \pi_\theta(a|s)\right\}$$

Two viewpoints

Baseline

Bootstrapping

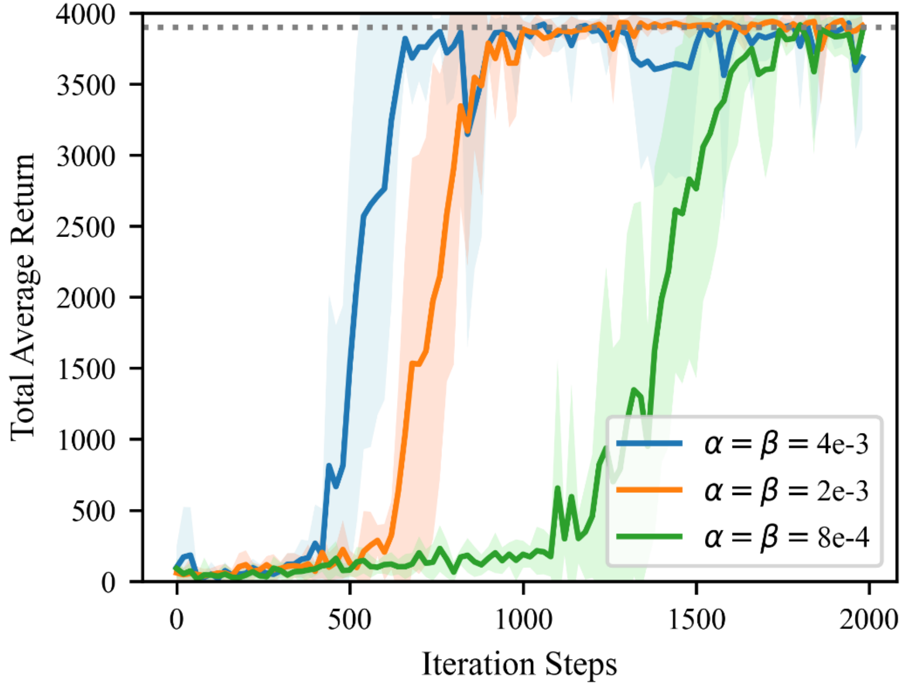
- Replace action-value with state-value

$$\nabla_\theta J(\theta) \propto \mathbb{E}_{\pi_\theta}\left\{\underbrace{\left(r + \gamma v^{\pi_\theta}(s') - v^{\pi_\theta}(s)\right)}_{\text{One-step TD error}} \nabla \log \pi_\theta(a|s)\right\}$$

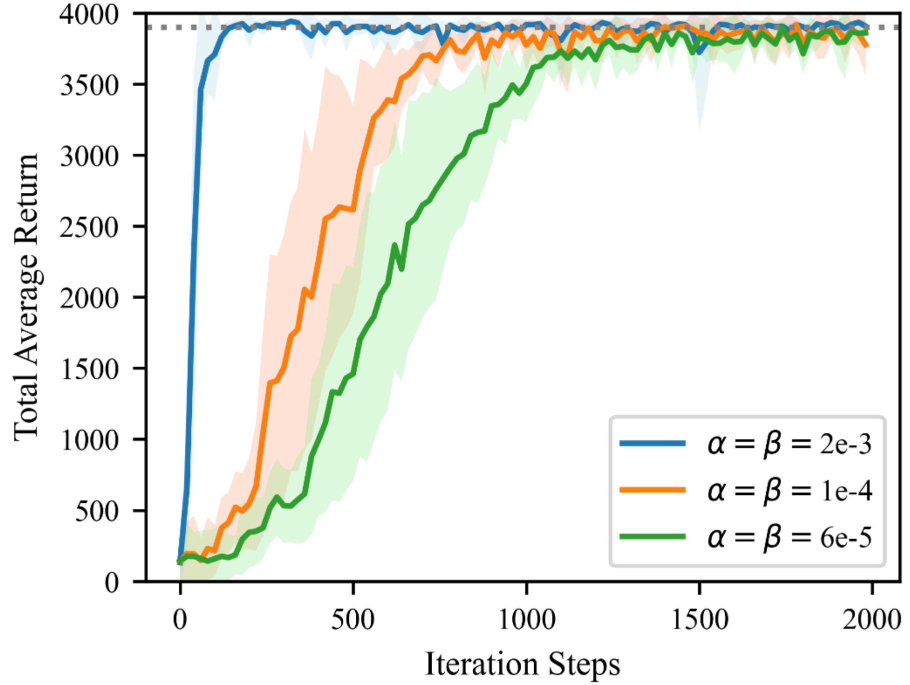
Vanilla policy gradient with state-value function

How Baseline Works?

- **AC (w/o baseline) vs A2C (w/ baseline)**



On-policy AC (w/o baseline)



On-policy A2C (w/ baseline)

On-policy Gradient vs Off-policy Gradient

Off-policy quasi-gradient

- Learn from data generated by old policy and other forms of suboptimal data, including data from expert demonstration

π

$$\max_{\theta} \mathbb{E}_{\mathcal{D}_\pi \sim \pi}\{v^{\pi_\theta}(s)\}$$

Sample-based estimation

$$\nabla J \propto \frac{1}{|\mathcal{D}_\pi|} \sum_{\mathcal{D}_\pi} Q(s,a) \nabla \log \pi_\theta(a|s)$$

On-policy PG

π

On-policy gradient

Off-policy gradient

b

Store IS ratio in buffer

$$\max_{\theta} \mathbb{E}_{\mathcal{D}_b \sim b}\{v^{\pi_\theta}(s)\}$$

Sample-based estimation

$$\nabla J \propto \frac{1}{|\mathcal{D}_b|} \sum_{\mathcal{D}_b} \frac{\pi_\theta(a|s)}{b(a|s)} Q(s,a) \nabla \log \pi_\theta(a|s)$$

Off-policy PG

π b

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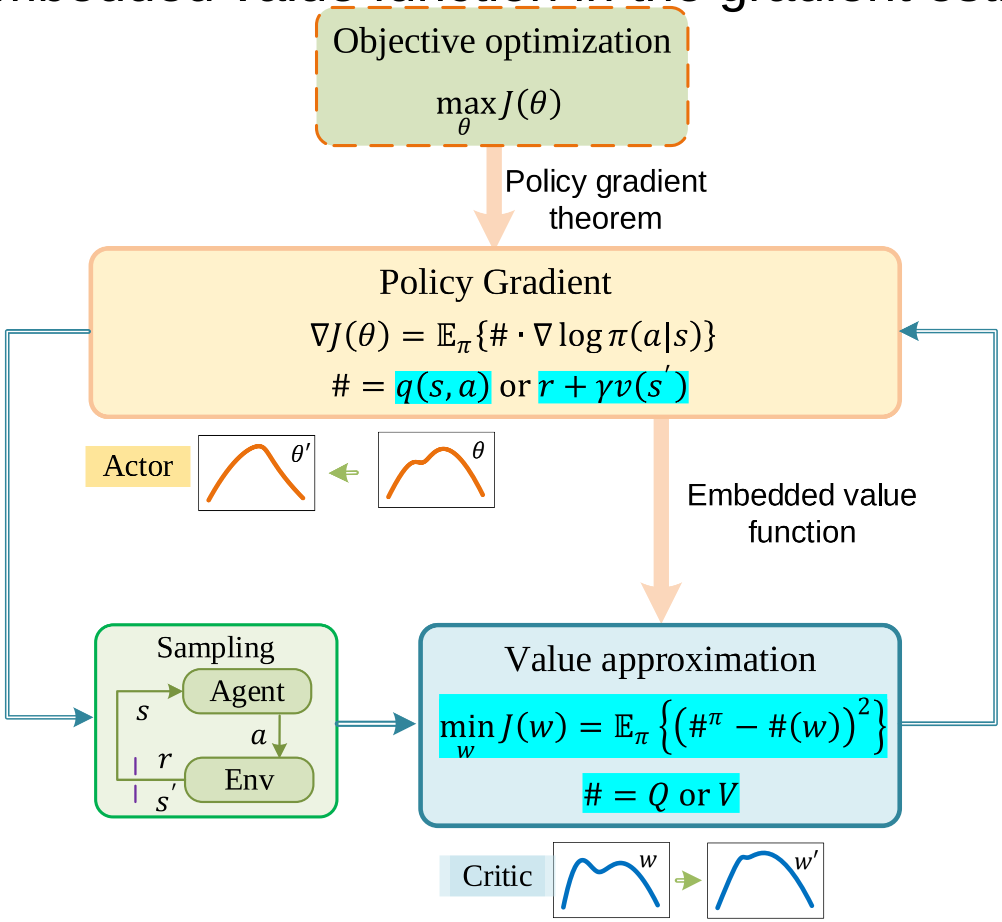
Understand actor-critic with direct RL

- Actor: gradient-based policy updates
- Critic: embedded value function in the gradient estimation



Actor-Critic RL

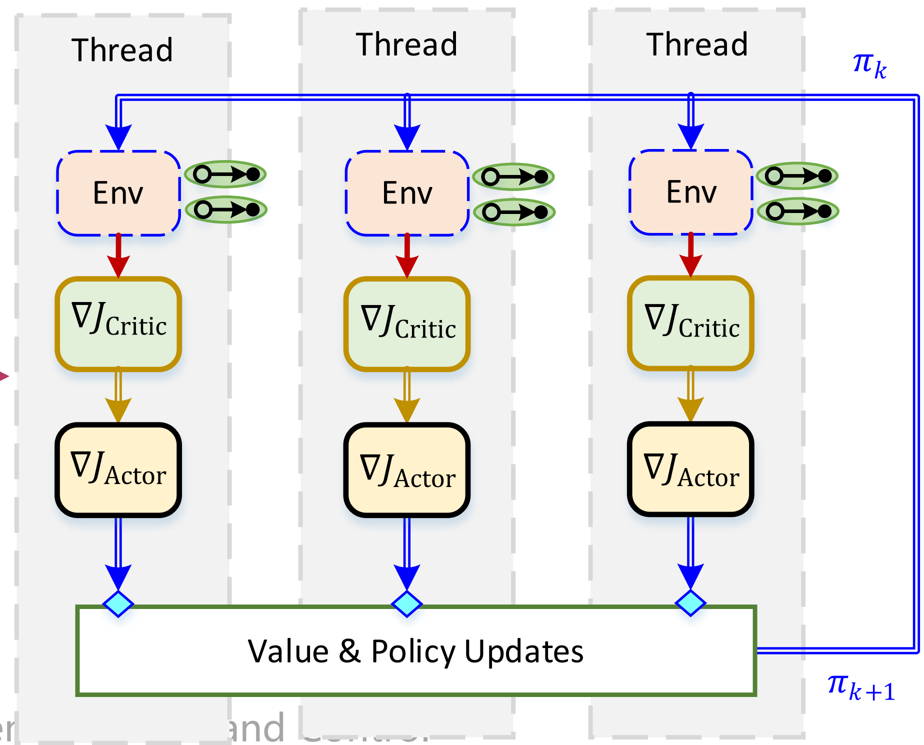
Off-policy AC with Advantage Function (A2C)

$$\nabla_w J_{\text{Critic}} \leftarrow \frac{1}{|\mathcal{B}|}\sum_{\mathcal{B}} \rho \cdot \left(r + \gamma V(s';w) - V(s;w)\right)\frac{\partial V(s;w)}{\partial w}$$

$$\nabla_\theta J_{\text{Actor}} \leftarrow \frac{1}{|\mathcal{B}|}\sum_{\mathcal{B}} \rho \cdot \nabla_\theta \log \pi(a|s;\theta)\left(r + \gamma V(s';w) - \zeta(s)\right)$$

$\mathcal{B}$: mini-batch

- **A3C** : Asynchronous advantage actor-critic
- **IMPALA**: A3C with importance sampling technique (Google Deepmind)



Actor-Critic RL

Deterministic Policy Gradient (DPG)

$$J_{\text{Actor}}(\theta) = \mathbb{E}_{s\sim d(s)}\{q^{\pi_\theta}(s, \boldsymbol{\pi_\theta(s)})\}$$

DPG

$$\nabla_\theta J_{\text{Actor}}(\theta) \approx \mathbb{E}_{s\sim d_b / s\sim d_\pi}\{\nabla_\theta \pi_\theta(s) \nabla_a q^{\pi_\theta}(s,a)|_{a=\pi_\theta(s)}\}$$

Off-policy Deterministic Actor-Critic

Critic gradient

$$\nabla_w J_{\text{Critic}} \leftarrow \frac{1}{|\mathcal{B}|}\sum_{\mathcal{B}} \rho\big(r + \gamma Q(s', a'; w) - Q(s, a; w)\big)\frac{\partial Q(s, a; w)}{\partial w}$$

(1) $\rho = 1$: s' are from behavior policy and $a' \sim \pi(s')$ is from target policy
(2) $\rho = \rho_{t+1}$: s, a, s', a' are from behavior policy

Actor gradient

$$\nabla_\theta J_{\text{Actor}} \leftarrow \frac{1}{|\mathcal{B}|}\sum_{\mathcal{B}} \nabla_\theta \pi(s;\theta)\nabla_a Q(s,a;w)$$

$\mathcal{B}$: mini-batch

State-of-the-art of AC Algorithms

Algorithm	Policy	Value	Critic Update	Actor Update	On/Off policy
DDPG	D	Q	TD-based	Vanilla PG	Off
TRPO	S	V	TD-based	Natural PG	On
PPO	S	V	TD-based	Clipped PG	On
TD3	D	Q	Clipped Double Q-learning	Vanilla PG	Off
D4PG	D	Q	Discrete Distributional Q-TD	Vanilla PG	Off
ACKTR	S	V	TD-based	Natural PG	On
A2C/A3C	S	V	TD-based	Vanilla PG	On
Off-PAC	S	V	TD-based	Vanilla PG	Off
ACER	S	Q	TD-based	Vanilla PG	Off
IMPALA	S	V	TD-based	Vanilla PG	Off
Soft Q-learning	S	Q	Soft Q-iteration	Soft PG	Off
SAC	S	Q	Clipped Double-Q	Soft PG	Off
DSAC	S	Q	Continuous Distributional Q-TD	Soft PG	Off

Design using N-step TD

N-step TD error

$$\delta_V^{\mathrm{TD}(n)}(s_t) \stackrel{\text{def}}{=} \underbrace{G_{t:t+n-1} + \gamma^n V^\pi(s_{t+n})}_{n\text{−step TD target}} - V^\pi(s_t)$$

- For off-policy critic update

V	$J_{\mathrm{Critic}} = \mathbb{E}_s\left\{\left(\rho_{t:t+n-1}R^{(n)} - V(s_t;w)\right)^2\right\}, R^{(n)} = G_{t:t+n-1} + \gamma^n V(s_{t+n};w)$
Q	$J_{\mathrm{Critic}} = \mathbb{E}_{s,a}\left\{\left(\rho_{t+1:t+n-1}R^{(n)} - Q(s_t,a_t;w)\right)^2\right\}, R^{(n)} = G_{t:t+n-1} + \gamma^n Q(s_{t+n},a_{t+n};w)$

- For off-policy actor update

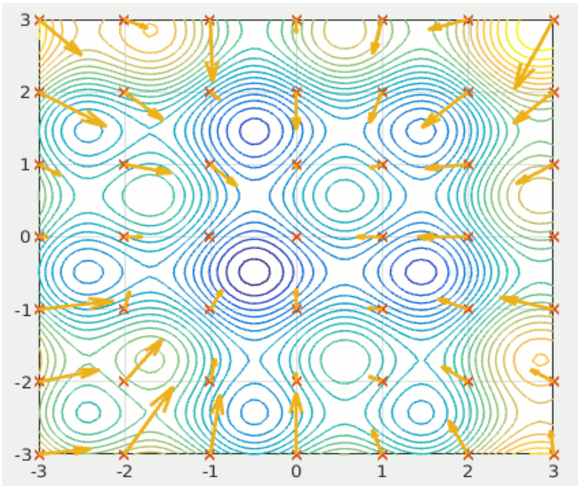
	Stochastic	Deterministic
V	$\nabla_\theta J_{\mathrm{Actor}} = \mathbb{E}_b\left\{\rho_{t:t+n-1}\delta_V^{\mathrm{TD}(n)}\nabla_\theta \log\pi_\theta(a\vert s)\right\}$	
Q	$\nabla_\theta J_{\mathrm{Actor}} = \mathbb{E}_{s\sim d_b,a\sim b}\left\{\frac{\pi_\theta(a\vert s)}{b(a\vert s)}Q(s,a)\nabla_\theta \log\pi_\theta(a\vert s)\right\}$	$\nabla_\theta J_{\mathrm{Actor}} = \mathbb{E}_{s\sim d_b}\{\nabla_\theta\pi_\theta(s)\nabla_a Q(s,a)\}$

Outline

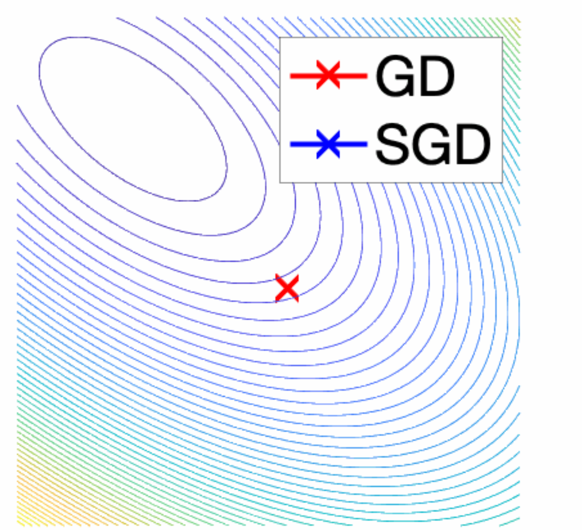
1. Indirect RL vs Direct RL
2. Likelihood Ratio Gradient
3. AC from Direct RL
4. **Optimization Viewpoint**

Types of Stochastic Optimization

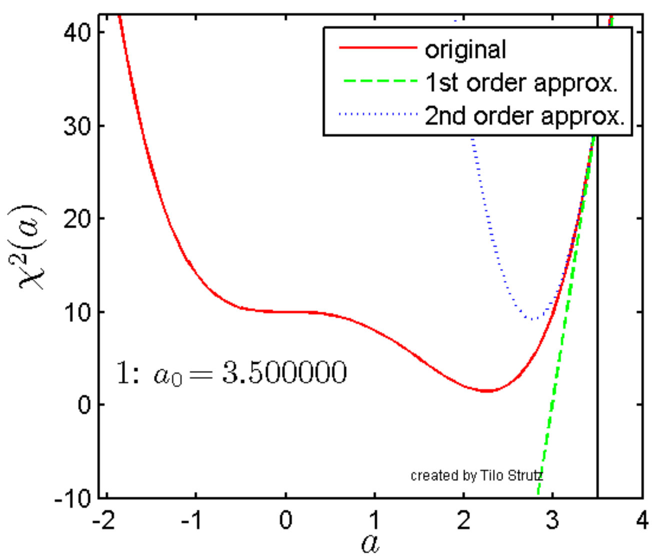
- **Derivative-free optimization**
  - Evolutionary method
  - Bayesian optimization
- **First-order optimization**
  - Stochastic gradient descent
- **Second-order optimization**
  - Newton-Raphson method



Zero-order



First-order



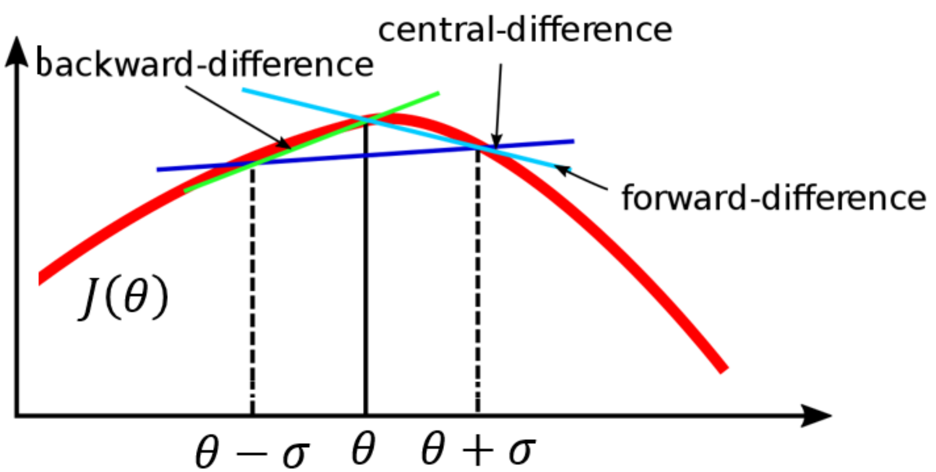
Second-order

Derivative-free Optimization

Derivative-free optimization

- Only zeroth-order information (i.e., function value) is available
- Finite difference method
- Simplest form $\widehat{\nabla J}(\theta) = \frac{1}{n}\sum_{i=1}^{n} \frac{J(\theta + \sigma\epsilon_i) - J(\theta)}{\sigma}\epsilon_i$

Forward difference

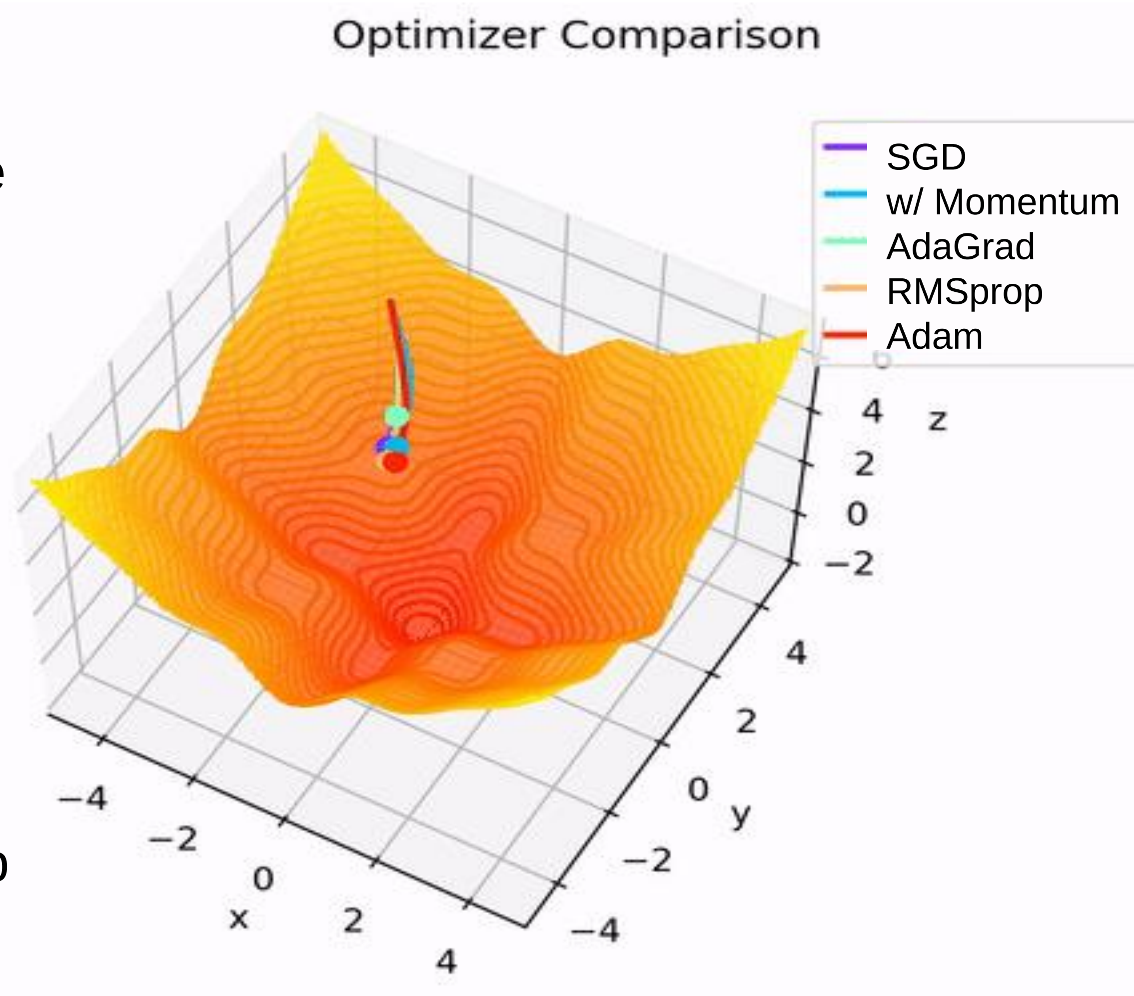


- It is essentially a simple "gradient" estimator
- Scale poorly with the dimension of parameter space

First-order Optimization

Accelerating technique for SGD

- (1) w/ Momentum: accumulate the gradient of past steps to determine the direction to go
- (2) RMSProp: automatically adjust the learning rate and choose a different learning rate for each parameter
- (3) Adam: combination of Momentum and RMSProp



Minorize-maximization optimization

- Primal objective function

$$\max_{\theta} f(\theta)$$

- The lower bound or surrogate function $g(\theta|\theta_k)$ is

$$g(\theta|\theta_k) \le f(\theta), \forall\theta$$
$$g(\theta_k|\theta_k) = f(\theta_k)$$

- Optimize surrogate function $g(\theta|\theta_k)$ at the k-th step

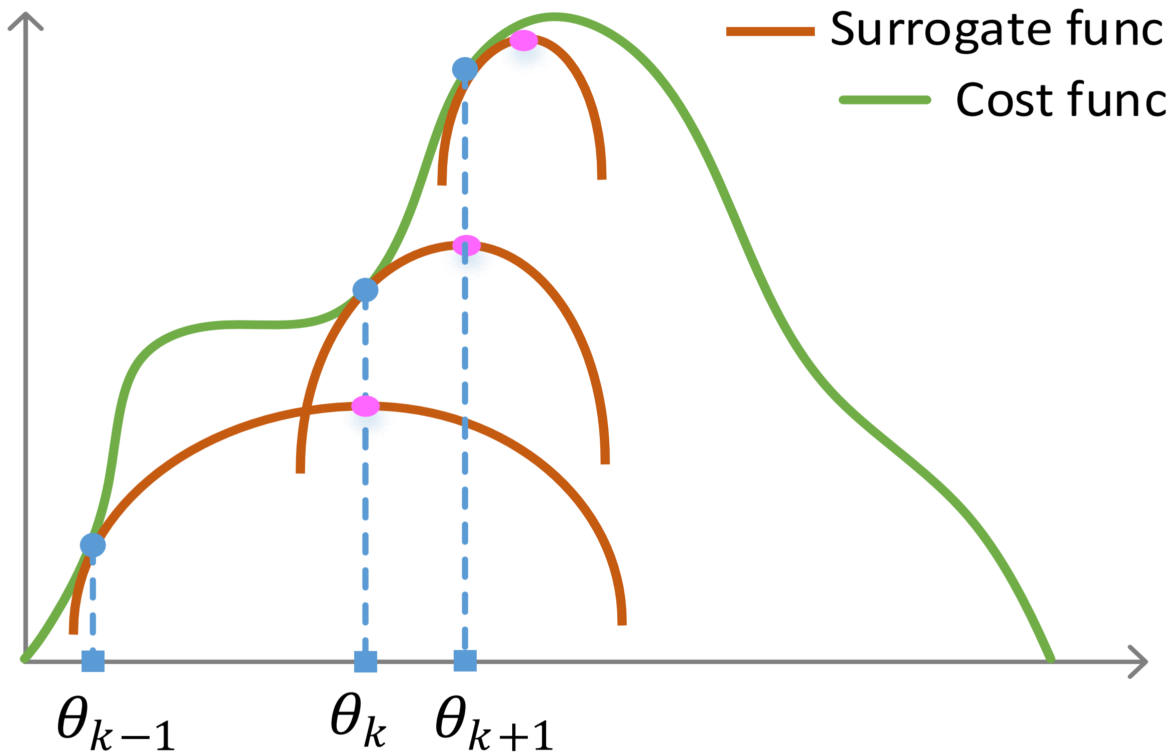
$$\theta_{k+1} = \arg\max_{\theta} g(\theta|\theta_k)$$

This iteration will guarantee convergence to the optimum

$$f(\theta_{k+1}) \ge g(\theta_{k+1}|\theta_k) \ge g(\theta_k|\theta_k) = f(\theta_k)$$

First-order Optimization

Minorize-maximization optimization



Surrogate function = The lower bound of primal objective function

First-order Optimization

Natural Policy Gradient

- Surrogate function for RL objective function

$$J(\pi) \geq L_{\pi_{\text{old}}}(\pi) - C \cdot D_{\text{KL}}^{\max}(\pi_{\text{old}}, \pi)$$

  - C-penalty coefficient, $L_{\pi_{\text{old}}}(\pi)$ - local approximate function

$$L_{\pi_{\text{old}}}(\pi) = J(\pi_{\text{old}}) + \sum_s d_{\pi_{\text{old}}}^{\gamma}(s) \sum_a \pi(a|s) A^{\pi_{\text{old}}}(s, a)$$

MM Optimization

$$\max_{\pi} \left\{ L_{\pi_{\text{old}}}(\pi) - C \cdot D_{\text{KL}}^{\max}(\pi_{\text{old}}, \pi) \right\}$$

First-order Optimization

Natural Policy Gradient

- Consider penalty coefficient C as a Lagrange multiplier

$$\max_{\theta} L_{\pi_{\text{old}}}(\pi_\theta)$$

Subj. to

$$D_{\text{KL}}^{\max}(\pi_{\text{old}}, \pi_\theta) \leq \delta$$

- Replace max operator with average operator

$$\max_{\theta} L_{\pi_{\text{old}}}(\pi_\theta) = \max_{\theta} \mathbb{E}_{\pi_{\text{old}}} \left\{ \frac{\pi_\theta(a|s)}{\pi_{\text{old}}(a|s)} A^{\pi_{\text{old}}}(s,a) \right\}$$

$$D_{\text{KL}}^{\max}(\pi_{\text{old}}, \pi_\theta) \approx \bar{D}_{\text{KL}}(\pi_{\text{old}}, \pi_\theta) = \mathbb{E}_{s \sim d_{\pi_{\text{old}}}} \{ D_{\text{KL}}(\pi_{\text{old}}(\cdot\,|s), \pi_\theta(\cdot\,|s)) \}$$

- Trust Region Policy Optimization (TRPO)

$$\max_{\theta} \mathbb{E}_{\pi_{\text{old}}} \left\{ \frac{\pi_\theta(a|s)}{\pi_{\text{old}}(a|s)} A^{\pi_{\text{old}}}(s,a) \right\}$$

Subject to

$$\bar{D}_{\text{KL}}(\pi_{\text{old}}, \pi_\theta) \leq \delta$$

⇨ Natural policy gradient

Second-order Optimization

Newton Method

- Approximating RL objective function by second-order Taylor's expansion

$$\max_{\Delta\theta} g^{\mathrm{T}}\Delta\theta + \frac{1}{2}\Delta\theta^{\mathrm{T}}F\Delta\theta$$

$g = \nabla_\theta J(\theta)$ first-order derivative

$F = \nabla_\theta^2 J(\theta)$ second-order derivative (i.e., Hessian matrix)

- Analytical solution

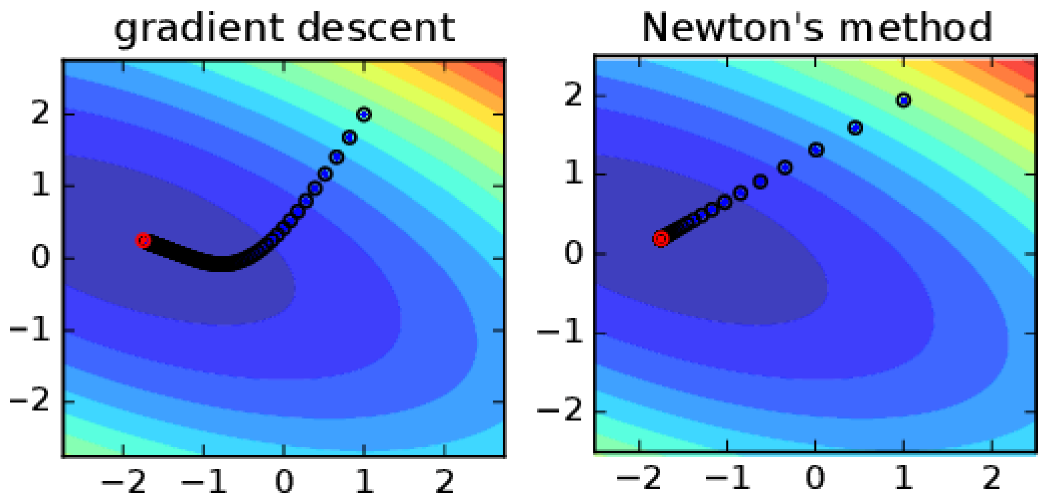
$$\Delta\theta^* = F^{-1}g = \left[\nabla_\theta^2 J(\theta)\right]^{-1}\nabla_\theta J(\theta)$$

- Updating rule $\theta \leftarrow \theta + \Delta\theta^*$

The key is how to
efficiently and accurately compute Hessian and its inverse matrix

Second-order Optimization

Convergence: super-linear rate



Disadvantage

- High cost of computing inverse Hessian matrix
  - Quasi-Newton method: BFGS (Broyden-Fletcher-Goldfarb-Shanno) algorithm
- Poor performance in non-convex optimization
  - Decreasing step size: $\theta \leftarrow \theta + \alpha_n \Delta\theta^*$

The End!