Chapter 2 Configuration Space

Chapter 3 Rigid-Body Motions

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Important concepts, symbols, and equations

- A configuration can be represented by **exponential coordinates** $\mathcal{S}\theta \in \mathbb{R}^6$: a screw axis $\mathcal{S}$ multiplied by the distance θ it is followed. (Equivalently, $\mathcal{V}t$: a twist $\mathcal{V}$ and a time t it is followed.)

- As with rotations, we can define a matrix exponential and its inverse, the matrix log. The exponential "integrates a twist" for time 1, and the log finds the constant twist needed to achieve the displacement in time 1.

$$
\begin{array}{llcll}
\exp: & [\mathcal{S}]\theta \in se(3) & \rightarrow & T \in SE(3) & \\
\log: & T \in SE(3) & \rightarrow & [\mathcal{S}]\theta \in se(3) & \theta \in [0, \pi]
\end{array}
$$

Important concepts, symbols, and equations

For $S = (\omega, v)$, either

- $\|\omega\| = 1$:

$$e^{[S]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}$$

- or $\omega = 0$ and $\|v\| = 1$:

$$e^{[S]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$$

Important concepts, symbols, and equations (cont.)

- A **wrench** is $\mathcal{F} = (m, f) \in \mathbb{R}^6$. A linear force $f \in \mathbb{R}^3$ at r creates a moment $m = r \times f$.

- The dot product of a wrench and a twist is power: $P = \mathcal{V}^{\mathrm{T}} \mathcal{F}$.

- The same wrench can be expressed in $\{\mathrm{a}\}$ and $\{\mathrm{b}\}$ as $\mathcal{F}_a$ and $\mathcal{F}_b$.

- Changing the frame of representation (power better be independent of the frame we use to represent twists and wrenches!):

$$
\begin{aligned}
\mathcal{V}_b^{\mathrm{T}} \mathcal{F}_b &= \mathcal{V}_a^{\mathrm{T}} \mathcal{F}_a \\
\mathcal{V}_b^{\mathrm{T}} \mathcal{F}_b &= ([\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b)^{\mathrm{T}} \mathcal{F}_a \\
&= \mathcal{V}_b^{\mathrm{T}} [\mathrm{Ad}_{T_{ab}}]^{\mathrm{T}} \mathcal{F}_a.
\end{aligned}
$$

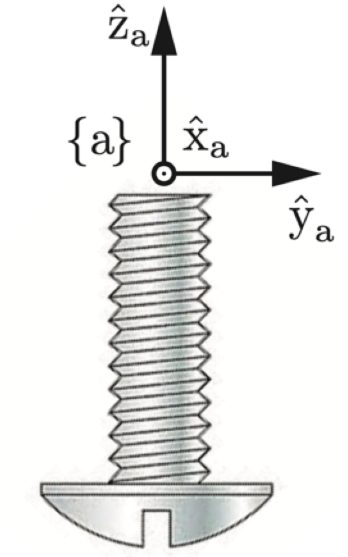
$$\boxed{\mathcal{F}_b = [\mathrm{Ad}_{T_{ab}}]^{\mathrm{T}} \mathcal{F}_a}$$

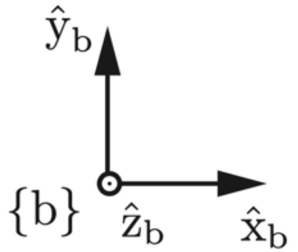
Rotations	**Rigid-Body Motions**
$R \in SO(3) : 3 \times 3$ matrices $R^{\mathrm{T}} R = I, \det R = 1$	$T \in SE(3) : 4 \times 4$ matrices $T = \left[\begin{array}{cc} R & p \\ 0 & 1 \end{array} \right],$ where $R \in SO(3), p \in \mathbb{R}^3$
$R^{-1} = R^{\mathrm{T}}$	$T^{-1} = \left[\begin{array}{cc} R^{\mathrm{T}} & -R^{\mathrm{T}} p \\ 0 & 1 \end{array} \right]$
change of coordinate frame: $R_{ab} R_{bc} = R_{ac}, \;\; R_{ab} p_b = p_a$	change of coordinate frame: $T_{ab} T_{bc} = T_{ac}, \;\; T_{ab} p_b = p_a$

rotating a frame {b}:	displacing a frame {b}:
$R = \mathrm{Rot}(\hat{\omega}, \theta)$	$T = \begin{bmatrix} \mathrm{Rot}(\hat{\omega}, \theta) & p \\ 0 & 1 \end{bmatrix}$
$R_{sb'} = RR_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ $R_{sb''} = R_{sb}R$: rotate θ about $\hat{\omega}_b = \hat{\omega}$	$T_{sb'} = TT_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ (moves {b} origin), translate p in {s} $T_{sb''} = T_{sb}T$: translate p in {b}, rotate θ about $\hat{\omega}$ in new body frame
unit rotation axis is $\hat{\omega} \in \mathbb{R}^3$, where $\|\hat{\omega}\| = 1$	"unit" screw axis is $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\|\omega\| = 1$ or (ii) $\omega = 0$ and $\|v\| = 1$
	for a screw axis $\{q, \hat{s}, h\}$ with finite h, $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$
angular velocity is $\omega = \hat{\omega}\dot{\theta}$	twist is $\mathcal{V} = \mathcal{S}\dot{\theta}$

for any 3-vector, e.g., $\omega \in \mathbb{R}^3$, $$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$$ identities, $\omega, x \in \mathbb{R}^3, R \in SO(3)$: $[\omega] = -[\omega]^{\mathrm{T}}, [\omega]x = -[x]\omega,$ $[\omega][x] = ([x][\omega])^{\mathrm{T}}, R[\omega]R^{\mathrm{T}} = [R\omega]$	for $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, $$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$ (the pair (ω, v) can be a twist $\mathcal{V}$ or a "unit" screw axis $\mathcal{S}$, depending on the context)
$\dot{R}R^{-1} = [\omega_s], \;\; R^{-1}\dot{R} = [\omega_b]$	$\dot{T}T^{-1} = [\mathcal{V}_s], \;\; T^{-1}\dot{T} = [\mathcal{V}_b]$
	$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$ identities: $[\mathrm{Ad}_T]^{-1} = [\mathrm{Ad}_{T^{-1}}]$, $[\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}] = [\mathrm{Ad}_{T_1 T_2}]$

change of coordinate frame: $\hat{\omega}_a = R_{ab}\hat{\omega}_b, \;\; \omega_a = R_{ab}\omega_b$	change of coordinate frame: $\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b, \;\; \mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b$
exp coords for $R \in SO(3)$: $\hat{\omega}\theta \in \mathbb{R}^3$	exp coords for $T \in SE(3)$: $\mathcal{S}\theta \in \mathbb{R}^6$
$\exp : [\hat{\omega}]\theta \in so(3) \to R \in SO(3)$ $R = \mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} =$ $I + \sin\theta[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2$	$\exp : [\mathcal{S}]\theta \in se(3) \to T \in SE(3)$ $T = e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & * \\ 0 & 1 \end{bmatrix}$ where $* =$ $(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v$
$\log : R \in SO(3) \to [\hat{\omega}]\theta \in so(3)$ algorithm in Section 3.2.3.3	$\log : T \in SE(3) \to [\mathcal{S}]\theta \in se(3)$ algorithm in Section 3.3.3.2
moment change of coord frame: $m_a = R_{ab} m_b$	wrench change of coord frame: $\mathcal{F}_a = (m_a, f_a) = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}}\mathcal{F}_b$





What is the screw S_a? S_b?

$$S_a = (0, 0, 1, 0, 0, 5)$$

$$S_b = (0, 1, 0, 0, 5, -4)$$

If $\{b\}$ follows the screw a distance θ, what is the mathematical expression for the final configuration $T_{ab'}$?

$$T_{ab'} = T_{ab} \exp([S_b]\theta) \quad \text{or} \quad T_{ab'} = \exp([S_a]\theta) T_{ab}$$

A screw axis is defined by the screw image (positive motion drives the screw upward), and the pitch is 5 mm/rad. The origin of $\{b\}$ is at $(0,4,-2)$ mm in $\{a\}$.

If $\theta = \pi$, give the numerical entries of $T_{ab'}$.

$$T_{ab'} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 5\pi-2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{by inspection}$$

What is T_{ab}?

$$T_{ab} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Given frames $\{a\}$, $\{b\}$, and $\{c\}$, and their representations relative to each other T_{ab} and T_{ac}, write the twist needed to move $\{b\}$ to $\{c\}$ in t seconds in the $se(3)$ form $[\mathcal{V}_a]$.

$$T_{ac} = \exp([\mathcal{V}_a]t)\, T_{ab}$$

$$T_{ac} T_{ab}^{-1} = \exp([\mathcal{V}_a]t)$$

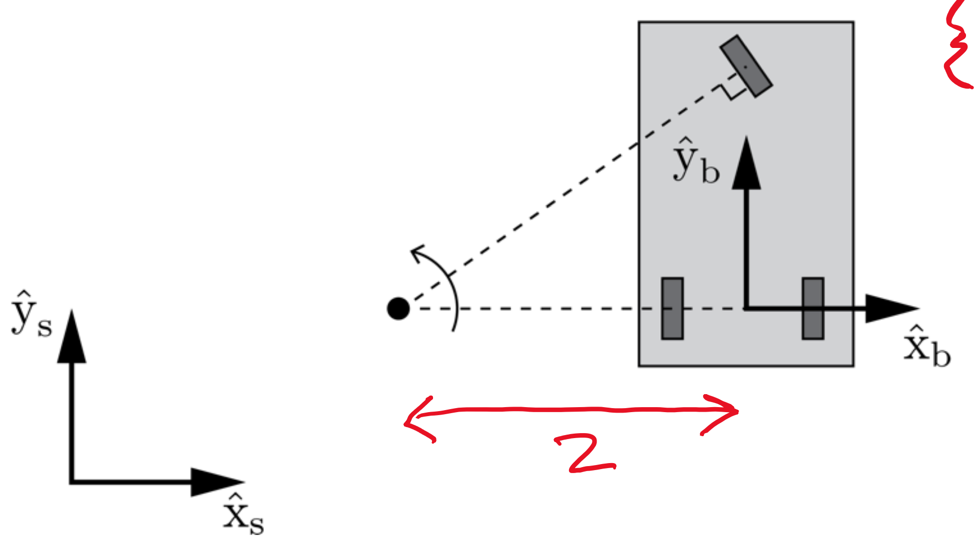
$$\log(T_{ac} T_{ba}) = [\mathcal{V}_a]t$$

$$\frac{1}{t}\log(T_{ac} T_{ba}) = [\mathcal{V}_a]$$

Car $\{b\}$ frame origin is initially at $(4,1,0)$ in $\{s\}$ and it drives at a constant steering angle with a turning radius of 2. What is the screw axis $(q, \hat{s}, h)$ expressed in $\{b\}$? $\{s\}$?

$\{b\}$: $q=(-2,0,0)$ $\hat{s}=(0,0,1)$ $h=0$

$\{s\}$: $q=(2,1,0)$ $\hat{s}=(0,0,1)$ $h=0$



What is the screw $\mathcal{S}_b$? $\mathcal{S}_s$?

$\mathcal{S}_b = (0,0,1,0,2,0)$

$\mathcal{S}_s = (0,0,1,1,-2,0)$

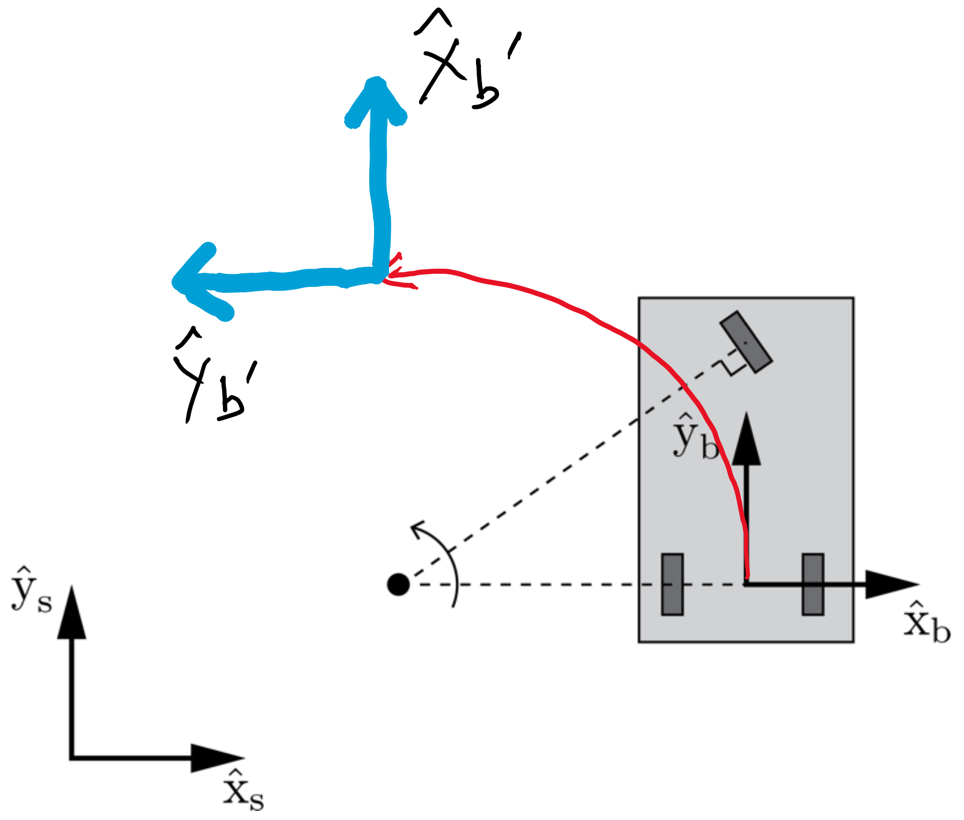
If the car's forward speed is 4, what is $\mathcal{V}_b$? $\mathcal{V}_s$?

linear speed $= r\dot{\theta}$

$4 = 2\dot{\theta}$ $\quad \dot{\theta} = 2$

$\mathcal{V}_b = 2\mathcal{S}_b$

$\mathcal{V}_s = 2\mathcal{S}_s$



If the car completes a quarter of a rotation, what are the exponential coordinates $S_b\theta$? $S_s\theta$?

$\theta = \frac{\pi}{2}$, so $\frac{\pi}{2} \mathcal{S}_b$ and $\frac{\pi}{2} \mathcal{S}_s$

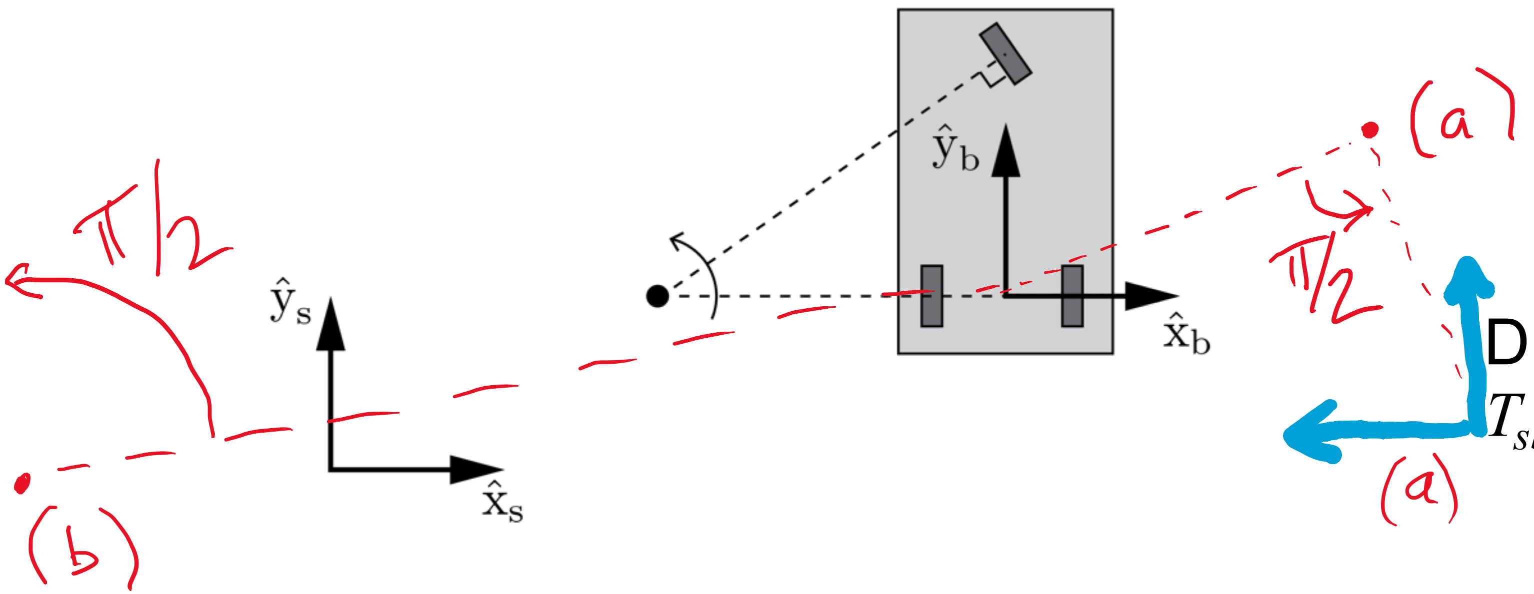
Where does the car end up? Draw a picture.

Express this final configuration mathematically, in terms of T_{sb} (as shown in the figure) and (1) the matrix exponential of $[S_b\theta]$ or (2) the matrix exponential of $[S_s\theta]$.

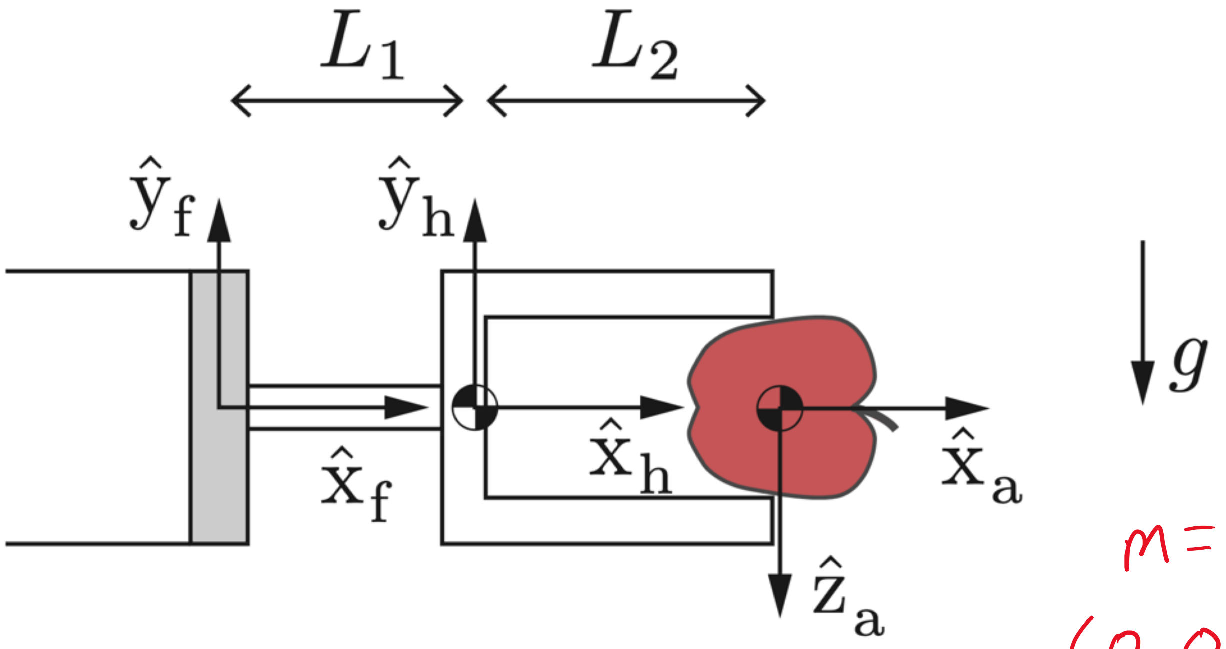
$T_{sb'} = \exp([\mathcal{S}_s]\theta) T_{sb}$ or

$T_{sb'} = T_{sb} \exp([\mathcal{S}_b]\theta)$

(a) Draw the $\{b'\}$ frame if
$T_{sb'} = T_{sb}\exp([S_s\theta])$.



Draw the $\{b'\}$ frame if (b)
$T_{sb'} = \exp([S_b\theta])\,T_{sb}$.



$f = (0, -3, 0)$

$r = (L_1 + L_2, 0, 0)$

$m = r \times f =$
$(0, 0, -3(L_1 + L_2))$

If gravity acting on the apple causes a downward force of 3 N, what is the wrench $\mathcal{F}_f$ felt at the force-torque sensor due to the apple?

$\mathcal{F}_f = (0, 0, -3(L_1 + L_2), 0, -3, 0)$