Lecture 15: The Transmission-Line Equations for Multiconductor Lines

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Multiconductor transmission-line (MTL)

A multiconductor transmission line (MTL) is a wiring structure composed of a set of (n+1) parallel conductor line. It consists of n conductors and a reference conductor (denoted as the zeroth conductor) to which the n line voltages will be referenced. This choice of the reference conductor is not unique.

Example (N = 6):

MTL equation derivation from the integral form of Maxwell's equation

Name	Differential form	Integral form
Gauss's law	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$	$\oiint_{\partial V} \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q(V)}{\varepsilon_0}$
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oiint_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\partial \Phi_{B,S}}{\partial t}$
Ampère's circuital law (with Maxwell's correction)	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_S + \mu_0 \varepsilon_0 \frac{\partial \Phi_{E,S}}{\partial t}$

- **Gauss's law:** Electric chctric field. The electric flux across a closed surface is proportion
- **Gauss's law for magnetism:** There are no magnetic monopoles. The magnetic flux across a closed surface is zero.
- **Faraday's law:** Time-varying magnetic fields produce an electric field.
- **Ampère's law:** Steady currents and time-varying electric fields (the latter due to Maxwell's correction) produce a magnetic field.

Derivation of the 1st MTL equation

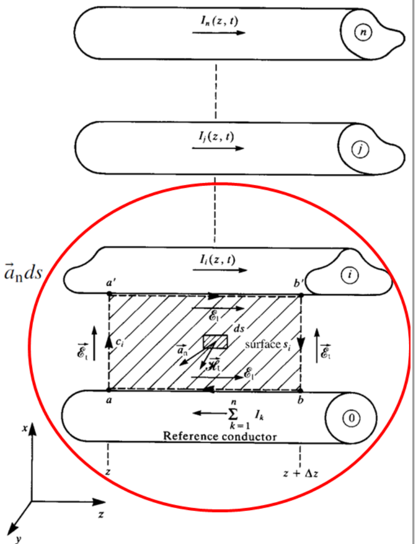
$$\oint_c \vec{\mathscr{E}}\cdot d\vec{l} = -\mu\frac{d}{dt}\int_s \vec{\mathscr{H}}\cdot d\vec{s}$$

Applying **Faraday's law** to the contour c_i that encloses surface s_i shown between the reference conductor and the ith conductor gives

$$\int_a^{a'}\vec{\mathscr{E}}_t\cdot d\vec{l}+\int_{a'}^{b'}\vec{\mathscr{E}}_l\cdot d\vec{l}+\int_{b'}^{b}\vec{\mathscr{E}}_t\cdot d\vec{l}+\int_{b}^{a}\vec{\mathscr{E}}_l\cdot d\vec{l}=\mu\frac{d}{dt}\int_{s_i}\vec{\mathscr{H}}_t\cdot\vec{a}_n ds$$

$\vec{\mathscr{E}}_t$ - the transverse electric field (in the x–y cross-sectional plane)

$\vec{\mathscr{E}}_l$ - the *longitudinal* or *z-directed* electric field (along the surfaces of the conductors)



Derivation of the 1st MTL equation

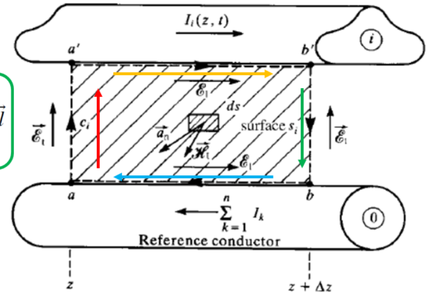
$$\boxed{\int_a^{a'}\vec{\mathscr{E}}_t\cdot d\vec{l}}+\boxed{\int_{a'}^{b'}\vec{\mathscr{E}}_l\cdot d\vec{l}}+\boxed{\int_{b'}^{b}\vec{\mathscr{E}}_t\cdot d\vec{l}}+\boxed{\int_{b}^{a}\vec{\mathscr{E}}_l\cdot d\vec{l}}=\mu\frac{d}{dt}\int_{s_i}\vec{\mathscr{H}}_t\cdot\vec{a}_n ds$$

$$V_i(z,t) = -\boxed{\int_a^{a'}\vec{\mathscr{E}}_t(x,y,z,t)\cdot d\vec{l}}$$

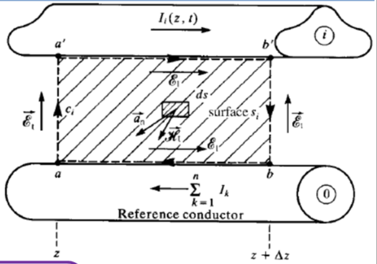
$$V_i(z+\Delta z,t) = -\boxed{\int_b^{b'}\vec{\mathscr{E}}_t(x,y,z+\Delta z,t)\cdot d\vec{l}}$$

$$-\boxed{\int_{a'}^{b'}\vec{\mathscr{E}}_l\cdot d\vec{l}} = -r_i\Delta z I_i(z,t)$$

$$-\boxed{\int_{b}^{a}\vec{\mathscr{E}}_l\cdot d\vec{l}} = -r_0\Delta z\sum_{k=1}^{n} I_k(z,t)$$



Derivation of the 1st MTL equation



$$\boxed{\int_a^{a'}\vec{\mathscr{E}}_t\cdot d\vec{l}}+\boxed{\int_{a'}^{b'}\vec{\mathscr{E}}_l\cdot d\vec{l}}+\boxed{\int_{b'}^{b}\vec{\mathscr{E}}_t\cdot d\vec{l}}+\boxed{\int_{b}^{a}\vec{\mathscr{E}}_l\cdot d\vec{l}}=\mu\frac{d}{dt}\int_{s_i}\vec{\mathscr{H}}_t\cdot\vec{a}_n ds$$

$$-V_i(z,t)+r_i\Delta z I_i(z,t)+V_i(z+\Delta z,t)+\boxed{r_0\Delta z\sum_{k=1}^{n}I_k(z,t)}=\mu\frac{d}{dt}\int_{s_i}\vec{\mathscr{H}}_t\cdot\vec{a}_n ds$$

Dividing both sides by Δz and rearranging gives

$$\frac{V_i(z+\Delta z,t)-V_i(z,t)}{\Delta z} = -r_0 I_1 - r_0 I_2 - \cdots - (r_0+r_i)I_i - \cdots - r_0 I_n + \mu\frac{1}{\Delta z}\frac{d}{dt}\int_{s_i}\vec{\mathscr{H}}_t\cdot\vec{a}_n ds$$

Derivation of the 1st MTL equation

Applying **Ampere's law** to the contour c'_i, the current of the ith conductor is

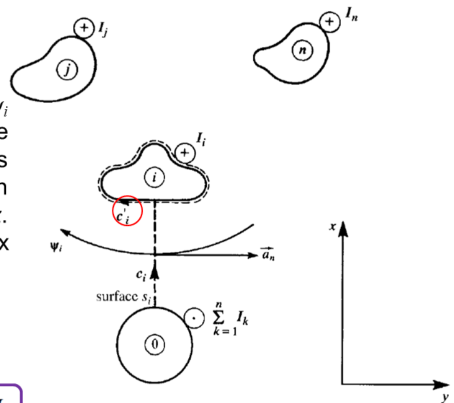
$$I_i(z,t) = \oint_{c'_i}\vec{\mathscr{H}}_t\cdot d\vec{l}'$$

The per-unit-length magnetic flux ψ_i penetrating the surface s_i between the reference conductor and the ith conductor is defined to be in this clockwise direction when looking in the direction of increasing z. Therefore, this per-unit-length magnetic flux penetrating surface s_i can be written as

$$\psi_i = -\mu\lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_{s_i}\vec{\mathscr{H}}_t\cdot\vec{a}_n ds$$

$$=\boxed{l_{i1}I_1 + l_{i2}I_2 + \cdots + l_{ii}I_i + \cdots + l_{in}I_n}$$

The total magnetic flux is a linear combination of the fluxes due to the currents on all the conductors



cross-sectional view of the line looking in the direction of increasing z

Derivation of the 1st MTL equation

$$\begin{cases} \dfrac{V_i(z+\Delta z,t)-V_i(z,t)}{\Delta z} = -r_0 I_1 - r_0 I_2 - \cdots - (r_0+r_i)I_i - \cdots - r_0 I_n \\ \qquad + \mu \dfrac{1}{\Delta z}\dfrac{d}{dt}\displaystyle\int_{s_i} \vec{\mathscr{H}}_t\cdot \vec{a}_n ds \\ \psi_i = -\mu \displaystyle\lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_{s_i}\vec{\mathscr{H}}_t\cdot\vec{a}_n ds \\ \quad = l_{i1}I_1 + l_{i2}I_2+\cdots+l_{ii}I_i+\cdots+l_{in}I_n\end{cases}$$

$\Delta z \to 0$

$$\frac{\partial V_i(z,t)}{\partial z} = -r_0 I_1(z,t) - r_0 I_2(z,t) - \cdots - (r_0+r_i)I_i(z,t) - \cdots$$
$$-r_0 I_n(z,t) - l_{i1}\frac{\partial I_1(z,t)}{\partial t} - l_{i2}\frac{\partial I_2(z,t)}{\partial t} - \cdots - l_{ii}\frac{\partial I_i(z,t)}{\partial t} - \cdots$$
$$-l_{in}\frac{\partial I_n(z,t)}{\partial t}$$

This **first MTL equation** can be written in a compact form using matrix notation as

$$\boxed{\frac{\partial}{\partial z}\mathbf{V}(z,t) = -\mathbf{R}\mathbf{I}(z,t) - \mathbf{L}\frac{\partial}{\partial t}\mathbf{I}(z,t)}$$

Derivation of the 1st MTL equation

The first MTL equation can be written in a compact form using matrix notation as

$$\boxed{\frac{\partial}{\partial z}\mathbf{V}(z,t) = -\mathbf{R}\mathbf{I}(z,t) - \mathbf{L}\frac{\partial}{\partial t}\mathbf{I}(z,t)}$$

where the n × 1 voltage and current vectors are defined as

$$\mathbf{V}(z,t) = \begin{bmatrix} V_1(z,t) \\ \vdots \\ V_i(z,t) \\ \vdots \\ V_n(z,t)\end{bmatrix} \qquad \mathbf{I}(z,t) = \begin{bmatrix} I_1(z,t) \\ \vdots \\ I_i(z,t) \\ \vdots \\ I_n(z,t)\end{bmatrix}$$

The per-unit-length inductance matrix $\mathbf{L}$ contains the individual per-unit-length self-inductances, l_{ii}, of the circuits and the per-unit-length mutual inductances between the circuits, l_{ij}. $\mathbf{L}$ can be shown to be a symmetric matrix. Similarly, we define the *per-unit-length resistance matrix* as $\mathbf{R}$.

$$\mathbf{L} = \begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{12} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{1n} & l_{2n} & \cdots & l_{nn}\end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} (r_1+r_0) & r_0 & \cdots & r_0 \\ r_0 & (r_2+r_0) & \cdots & r_0 \\ \vdots & \vdots & \ddots & \vdots \\ r_0 & r_0 & \cdots & (r_n+r_0)\end{bmatrix} = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n\end{bmatrix} + \begin{bmatrix} r_0 & r_0 & \cdots & r_0 \\ r_0 & r_0 & \cdots & r_0 \\ \vdots & \vdots & \ddots & \vdots \\ r_0 & r_0 & \cdots & r_0\end{bmatrix}$$

Derivation of the 2nd MTL equation

To place a closed surface s' around the ith conductor. The portion of the surface over the end caps is denoted as s'_e, whereas the portion over the sides is denoted as s'_s.

$$\boxed{\oint_{s'} \vec{\mathscr{I}}\cdot d\vec{s'} = -\frac{d}{dt}Q_{\text{enc}}}$$

continuity equation or equation of conservation of charge

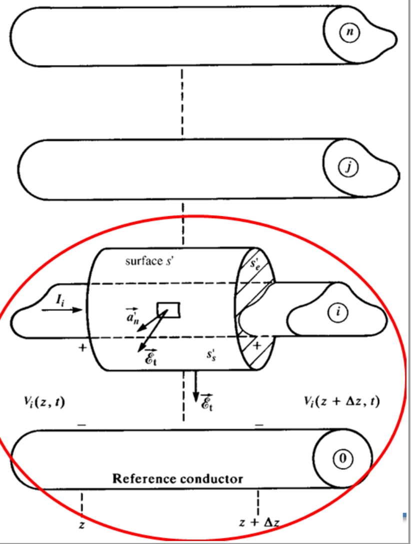
Over the end caps, we have

$$\iint_{s'_e} \vec{\mathscr{I}}\cdot d\vec{s'} = I_i(z+\Delta z,t) - I_i(z,t)$$

Over the sides of the surface,

$$\underbrace{I_t(z,t)}_{\text{transverse current}} = \underbrace{I_g(z,t)}_{\text{conduction current}} + \underbrace{I_c(z,t)}_{\text{displacement current}}$$

- **conduction current** $\vec{\mathscr{I}}_c = \sigma\vec{\mathscr{E}}_t$
- **displacement current** $\vec{\mathscr{I}}_d = \varepsilon\dfrac{\partial\vec{\mathscr{E}}_t}{\partial t}$

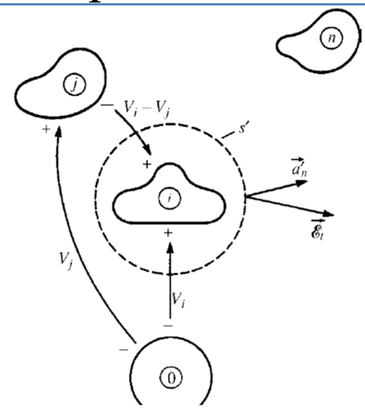


Derivation of the 2nd MTL equation

- **conduction current** $\vec{\mathscr{I}}_c = \sigma\vec{\mathscr{E}}_t$

$$\iint_{s'_s} \vec{\mathscr{I}}_c\cdot d\vec{s'} = \sigma\iint_{s'_s}\vec{\mathscr{E}}_t\cdot d\vec{s'}$$

Defining per-unit-length conductance g_{ij} S/m between each pair of conductors as the ratio of conduction current flowing between the two conductors in the transverse plane to the voltage between the two conductors.



$$\sigma\lim_{\Delta z\to 0}\frac{1}{\Delta z}\iint_{s'_s}\vec{\mathscr{E}}_t\cdot d\vec{s'} = g_{i1}(V_i - V_1) + \cdots + g_{ii}V_i + \cdots + g_{in}(V_i - V_n)$$
$$= -g_{i1}V_1(z,t) - g_{i2}V_2(z,t) - \cdots + \sum_{k=1}^{n} g_{ik}V_i(z,t)$$
$$-\cdots - g_{in}V_n(z,t)$$

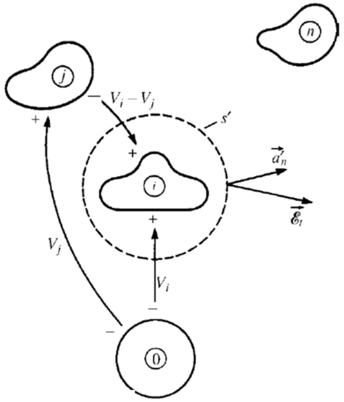
Derivation of the 2nd MTL equation

- **displacement current** $\vec{\mathscr{J}}_{\mathrm{d}} = \varepsilon \dfrac{\partial \vec{\mathscr{E}}_{\mathrm{t}}}{\partial t}$

The charge enclosed by the surface (residing on the conductor surface) is, by Gauss' law,

$$Q_{\mathrm{enc}} = \varepsilon \iint_{s'_s} \vec{\mathscr{E}}_{\mathrm{t}} \cdot d\vec{s'} \qquad \boxed{\oint_{s'} \vec{\mathscr{J}} \cdot d\vec{s'} = -\frac{d}{dt} Q_{\mathrm{enc}}}$$

The charge per unit of line length can be defined in terms of the per-unit-length capacitances c_{ij} between each pair of conductors.



$$\varepsilon \lim_{\Delta z \to 0} \frac{1}{\Delta z} \iint_{s'_s} \vec{\mathscr{E}}_{\mathrm{t}} \cdot d\vec{s'} = c_{i1}(V_i - V_1) + \cdots + c_{ii} V_i + \cdots + c_{in}(V_i - V_n)$$

$$= -c_{i1} V_1(z,t) - c_{i2} V_2(z,t) - \cdots + \sum_{k=1}^{n} c_{ik} V_i(z,t) - \cdots - c_{in} V_n(z,t)$$

Derivation of the 2nd MTL equation

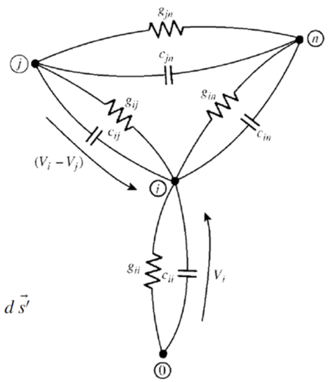
$$\left.\begin{aligned}
&\boxed{\oint_{s'} \vec{\mathscr{J}} \cdot d\vec{s'} = -\frac{d}{dt} Q_{\mathrm{enc}}} \\
&\iint_{s'_e} \vec{\mathscr{J}} \cdot d\vec{s'} = I_i(z+\Delta z, t) - I_i(z,t) \\
&\iint_{s'_s} \vec{\mathscr{J}}_{\mathrm{c}} \cdot d\vec{s'} = \sigma \iint_{s'_s} \vec{\mathscr{E}}_{\mathrm{t}} \cdot d\vec{s'} \\
&Q_{\mathrm{enc}} = \varepsilon \iint_{s'_s} \vec{\mathscr{E}}_{\mathrm{t}} \cdot d\vec{s'}
\end{aligned}\right\}$$

$$\frac{I_i(z+\Delta z, t) - I_i(z,t)}{\Delta z} + \sigma \frac{1}{\Delta z} \iint_{s'_s} \vec{\mathscr{E}}_{\mathrm{t}} \cdot d\vec{s'} = -\varepsilon \frac{1}{\Delta z} \frac{d}{dt} \iint_{s'_s} \vec{\mathscr{E}}_{\mathrm{t}} \cdot d\vec{s'}$$

Derivation of the 2nd MTL equation

$$\left\{\begin{aligned}
&\sigma \lim_{\Delta z \to 0} \frac{1}{\Delta z} \iint_{s'_s} \vec{\mathscr{E}}_{\mathrm{t}} \cdot d\vec{s'} = g_{i1}(V_i - V_1) + \cdots + g_{ii} V_i + \cdots + g_{in}(V_i - V_n) \\
&\qquad = -g_{i1} V_1(z,t) - g_{i2} V_2(z,t) - \cdots + \sum_{k=1}^{n} g_{ik} V_i(z,t) - \cdots - g_{in} V_n(z,t) \\
&\varepsilon \lim_{\Delta z \to 0} \frac{1}{\Delta z} \iint_{s'_s} \vec{\mathscr{E}}_{\mathrm{t}} \cdot d\vec{s'} = c_{i1}(V_i - V_1) + \cdots + c_{ii} V_i + \cdots + c_{in}(V_i - V_n) \\
&\qquad = -c_{i1} V_1(z,t) - c_{i2} V_2(z,t) - \cdots + \sum_{k=1}^{n} c_{ik} V_i(z,t) - \cdots - c_{in} V_n(z,t)
\end{aligned}\right.$$

$$\frac{I_i(z+\Delta z, t) - I_i(z,t)}{\Delta z} + \sigma \frac{1}{\Delta z} \iint_{s'_s} \vec{\mathscr{E}}_{\mathrm{t}} \cdot d\vec{s'} = -\varepsilon \frac{1}{\Delta z} \frac{d}{dt} \iint_{s'_s} \vec{\mathscr{E}}_{\mathrm{t}} \cdot d\vec{s'}$$



$\triangle z \to 0$

$$\frac{\partial I_i(z,t)}{\partial z} = g_{i1} V_1(z,t) + g_{i2} V_2(z,t) + \cdots - \sum_{k=1}^{n} g_{ik} V_i(z,t) + \cdots + g_{in} V_n(z,t) + c_{i1} \frac{\partial}{\partial t} V_1(z,t) + c_{i2} \frac{\partial}{\partial t} V_2(z,t) + \cdots - \sum_{k=1}^{n} c_{ik} \frac{\partial}{\partial t} V_i(z,t) + \cdots + c_{in} \frac{\partial}{\partial t} V_n(z,t)$$

This **second MTL equation** can be written in a compact form using matrix notation as

$$\boxed{\frac{\partial}{\partial z} \mathbf{I}(z,t) = -\mathbf{G}\,\mathbf{V}(z,t) - \mathbf{C} \frac{\partial}{\partial t} \mathbf{V}(z,t)}$$

Derivation of the 2nd MTL equation

The second MTL equation can be written in a compact form using matrix notation as

$$\boxed{\frac{\partial}{\partial z} \mathbf{I}(z,t) = -\mathbf{G}\,\mathbf{V}(z,t) - \mathbf{C} \frac{\partial}{\partial t} \mathbf{V}(z,t)}$$

where the n × 1 voltage and current vectors are defined as

$$\mathbf{V}(z,t) = \begin{bmatrix} V_1(z,t) \\ \vdots \\ V_i(z,t) \\ \vdots \\ V_n(z,t) \end{bmatrix} \qquad \mathbf{I}(z,t) = \begin{bmatrix} I_1(z,t) \\ \vdots \\ I_i(z,t) \\ \vdots \\ I_n(z,t) \end{bmatrix}$$

The per-unit-length conductance matrix $\mathbf{G}$ represents the conduction current flowing between the conductors in the transverse plane and is defined as

$$\mathbf{G} = \begin{bmatrix} \sum_{k=1}^{n} g_{1k} & -g_{12} & \cdots & -g_{1n} \\ -g_{12} & \sum_{k=1}^{n} g_{2k} & \cdots & -g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -g_{1n} & -g_{2n} & \cdots & \sum_{k=1}^{n} g_{nk} \end{bmatrix}$$

The per-unit-length capacitance matrix $\mathbf{C}$ represents the displacement current flowing between the conductors in the transverse plane and is defined as

$$\mathbf{C} = \begin{bmatrix} \sum_{k=1}^{n} c_{1k} & -c_{12} & \cdots & -c_{1n} \\ -c_{12} & \sum_{k=1}^{n} c_{2k} & \cdots & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{1n} & -c_{2n} & \cdots & \sum_{k=1}^{n} c_{nk} \end{bmatrix}$$

Derivation of the MTL equation and its simplify

$$\boxed{\frac{\partial}{\partial z}\mathbf{V}(z,t) = -\mathbf{R}\mathbf{I}(z,t) - \mathbf{L}\frac{\partial}{\partial t}\mathbf{I}(z,t)}$$

$$\boxed{\frac{\partial}{\partial z}\mathbf{I}(z,t) = -\mathbf{G}\,\mathbf{V}(z,t) - \mathbf{C}\frac{\partial}{\partial t}\mathbf{V}(z,t)}$$

If the conductors are perfect conductors, then $\mathbf{R} = \mathbf{0}$, whereas if the surrounding medium is lossless ($\sigma = 0$), then $\mathbf{G} = \mathbf{0}$. The line is said to be lossless if both the conductors and the medium are lossless in which case the MTL equations simplify to

$$\frac{\partial}{\partial z}\begin{bmatrix}\mathbf{V}(z,t)\\ \mathbf{I}(z,t)\end{bmatrix} = -\begin{bmatrix}\mathbf{0} & \mathbf{L}\\ \mathbf{C} & \mathbf{0}\end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{V}(z,t)\\ \mathbf{I}(z,t)\end{bmatrix}$$

Derivation of the uncoupled MTL equation

$$\boxed{\frac{\partial}{\partial z}\mathbf{V}(z,t) = -\mathbf{R}\mathbf{I}(z,t) - \mathbf{L}\frac{\partial}{\partial t}\mathbf{I}(z,t)} \quad \text{(a)}$$

$$\boxed{\frac{\partial}{\partial z}\mathbf{I}(z,t) = -\mathbf{G}\,\mathbf{V}(z,t) - \mathbf{C}\frac{\partial}{\partial t}\mathbf{V}(z,t)} \quad \text{(b)}$$

The first-order, coupled forms can be placed in the form of *second-order, uncoupled* equations by differentiating (a) with respect to *z* and differentiating (b) with respect to *t* to yield

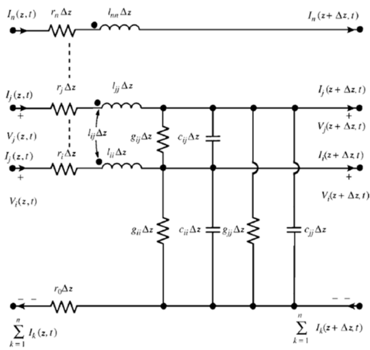
The uncoupled, second-order equations

$$\left\{\begin{aligned}\frac{\partial^2}{\partial z^2}\mathbf{V}(z,t) &= -\mathbf{R}\frac{\partial}{\partial z}\mathbf{I}(z,t) - \mathbf{L}\frac{\partial^2}{\partial z\partial t}\mathbf{I}(z,t)\\ \frac{\partial^2}{\partial z\partial t}\mathbf{I}(z,t) &= -\mathbf{G}\frac{\partial}{\partial t}\mathbf{V}(z,t) - \mathbf{C}\frac{\partial^2}{\partial t^2}\mathbf{V}(z,t)\end{aligned}\right. \Rightarrow \left\{\begin{aligned}\frac{\partial^2}{\partial z^2}\mathbf{V}(z,t) &= [\mathbf{RG}]\,\mathbf{V}(z,t) + [\mathbf{RC}+\mathbf{LG}]\frac{\partial}{\partial t}\mathbf{V}(z,t) + \mathbf{LC}\frac{\partial^2}{\partial t^2}\mathbf{V}(z,t)\\ \frac{\partial^2}{\partial z^2}\mathbf{I}(z,t) &= [\mathbf{GR}]\,\mathbf{I}(z,t) + [\mathbf{CR}+\mathbf{GL}]\frac{\partial}{\partial t}\mathbf{I}(z,t) + \mathbf{CL}\frac{\partial^2}{\partial t^2}\mathbf{I}(z,t)\end{aligned}\right.$$

MTL equation Derivation from the per-unit-length equivalent circuit

Derivation of the 1st MTL equation

As an alternative method, we derive the MTL equations from the per-unit-length equivalent circuit shown in right figure.



Writing Kirchhoff's voltage law around the ith circuit consisting of the ith conductor and the reference conductor yields.

$$-V_i(z,t) + r_i\Delta z I_i(z,t) + V_i(z+\Delta z,t) + r_0\Delta z\sum_{k=1}^{n} I_k(z,t) = -l_{i1}\Delta z\frac{\partial I_1(z,t)}{\partial t}$$
$$-l_{i2}\Delta z\frac{\partial I_2(z,t)}{\partial t} - \cdots - l_{ii}\Delta z\frac{\partial I_i(z,t)}{\partial t} - \cdots - l_{in}\Delta z\frac{\partial I_n(z,t)}{\partial t}$$

$\triangle z \to 0$

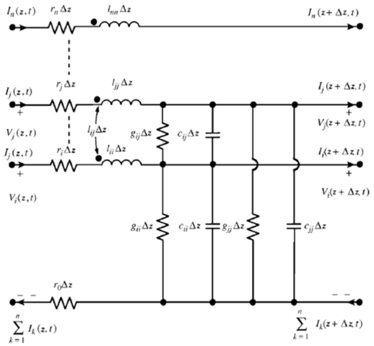
$$\frac{\partial V_i(z,t)}{\partial z} = -r_0 I_1(z,t) - r_0 I_2(z,t) - \cdots - (r_0 + r_i) I_i(z,t) - \cdots$$
$$-r_0 I_n(z,t) - l_{i1}\frac{\partial I_1(z,t)}{\partial t} - l_{i2}\frac{\partial I_2(z,t)}{\partial t} - \cdots - l_{ii}\frac{\partial I_i(z,t)}{\partial t} - \cdots$$
$$-l_{in}\frac{\partial I_n(z,t)}{\partial t}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial z}\mathbf{V}(z,t) = -\mathbf{R}\mathbf{I}(z,t) - \mathbf{L}\frac{\partial}{\partial t}\mathbf{I}(z,t)}$$

Derivation of the 2nd MTL equation

Similarly, the second MTL equation can be obtained by applying Kirchhoff's current law to the ith conductor in the per-unit-length equivalent circuit.

$$I_i(z+\Delta z,t) - I_i(z,t) = -g_{i1}\Delta z\,(V_i - V_1) - \cdots - g_{ii}\Delta z V_i - \cdots$$
$$-g_{in}\Delta z\,(V_i - V_n) - c_{i1}\Delta z\,\frac{\partial}{\partial t}(V_i - V_1) - \cdots - c_{ii}\Delta z\,\frac{\partial}{\partial t}V_i - \cdots$$
$$-c_{in}\Delta z\,\frac{\partial}{\partial t}(V_i - V_n)$$



$\triangle z \to 0$

$$\frac{\partial I_i(z,t)}{\partial z} = g_{i1}V_1(z,t) + g_{i2}V_2(z,t) + \cdots - \sum_{k=1}^{n} g_{ik}V_i(z,t) + \cdots$$
$$+g_{in}V_n(z,t) + c_{i1}\frac{\partial}{\partial t}V_1(z,t) + c_{i2}\frac{\partial}{\partial t}V_2(z,t) + \cdots$$
$$-\sum_{k=1}^{n} c_{ik}\frac{\partial}{\partial t}V_i(z,t) + \cdots + c_{in}\frac{\partial}{\partial t}V_n(z,t)$$

$$\boxed{\frac{\partial}{\partial z}\mathbf{I}(z,t) = -\mathbf{G}\,\mathbf{V}(z,t) - \mathbf{C}\,\frac{\partial}{\partial t}\mathbf{V}(z,t)}$$

Thank you!