Lecture 7: Standing Wave and Input Impedance

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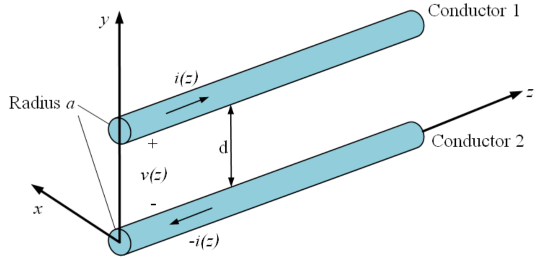
Recall: General solution



telegrapher's equation

$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\,\tilde{I}(z),$$
$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\,\tilde{V}(z).$$

traveling wave solutions

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$
$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

propagation constant

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')} = \sqrt{Z' \cdot Y'}$$

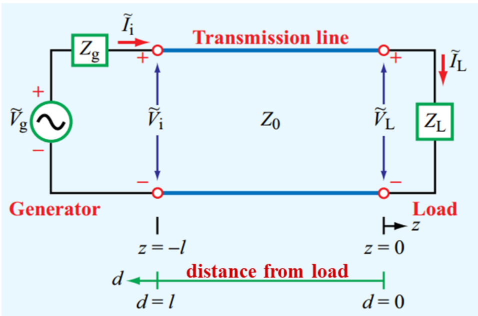
Recall: Reflection coefficient

In order to incorporate the ***source and load condition*** (boundary condition) into the general wave solution, we define a ***voltage reflection coefficient*** $\Gamma(z)$ as the ratio of the reflected and incident voltage waves

$$\Gamma(z) \overset{\text{def}}{=} \frac{V_0^- e^{\gamma z}}{V_0^+ e^{-\gamma z}} = \frac{V_0^-}{V_0^+} e^{2\gamma z}$$

How to evaluate V_0^-/V_0^+ ?

One notices the direction of current: $\Gamma(z) = -\frac{I_0^-}{I_0^+} e^{2\gamma z}$



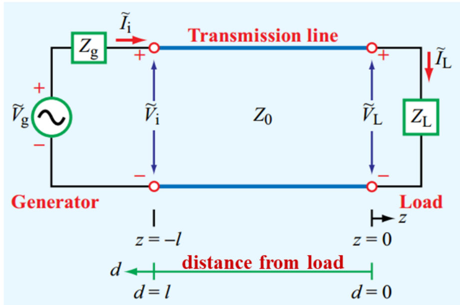
Considering the transmission line in the context of the complete circuit, including:

A generator circuit at its input terminals ($z = -l$)

A l-length TL($l \le z \le 0$)

A load terminated at the output ($z = 0$)

Recall: Reflection coefficient



load impedance

$$Z_L = \frac{\tilde{V}(0)}{\tilde{I}(0)} = \frac{V_0^+ + V_0^-}{\frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}} = \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} Z_0$$

$$\frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Therefore, the reflection coefficient along the line is:

$$\Gamma(z) = \frac{V_0^-}{V_0^+} e^{2\gamma z} = \frac{Z_L - Z_0}{Z_L + Z_0} e^{2\gamma z} = \Gamma_L e^{2\gamma z}$$

Especially, the reflection coefficient at the load :

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Recall: Γ_L for various types of load

In general, Z_L is a complex quantity $Z_L = R + jX$, therefore, $\Gamma_L = |\Gamma_L| e^{j\theta_r}$

Load	$\lvert\Gamma\rvert$	θ_r
Z_0, $Z_L = (r + jx)Z_0$	$\left[\dfrac{(r-1)^2 + x^2}{(r+1)^2 + x^2}\right]^{1/2}$	$\tan^{-1}\left(\dfrac{x}{r-1}\right) - \tan^{-1}\left(\dfrac{x}{r+1}\right)$
Z_0, Z_0	0 (no reflection)	irrelevant
Z_0 (short)	1	$\pm 180°$ (phase opposition)
Z_0 (open)	1	0 (in-phase)
Z_0, $jX = j\omega L$	1	$\pm 180° - 2\tan^{-1} x$
Z_0, $jX = \dfrac{-j}{\omega C}$	1	$\pm 180° + 2\tan^{-1} x$

$z_L = Z_L/Z_0 = (R + jX)/Z_0 = r + jx$

$r = R/Z_0$ and $x = X/Z_0$ are the real and imaginary parts of z_L, respectively.

Magnitude of voltage on the line

Substitute $V_0^- = \Gamma_L V_0^+$ in traveling wave equations, one can obtain

$$\tilde{V}(z) = V_0^+ (e^{-\gamma z} + \Gamma_L e^{\gamma z})$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} (e^{-\gamma z} - \Gamma_L e^{\gamma z})$$

Only one unknown V_0^+ to be determined

For a complex load, $\Gamma_L = |\Gamma_L| e^{j\theta_r}$, one can have the magnitude of $\tilde{V}(z)$.

$$|\tilde{V}(z)| = [\tilde{V}(z) \cdot \tilde{V}^*(z)]^{1/2}$$

complex conjugate of $\tilde{V}(z)$

$$= \{[V_0^+(e^{-\gamma z} + |\Gamma_L| e^{j\theta_r} e^{\gamma z})] \cdot [V_0^{+*}(e^{\gamma z} + |\Gamma_L| e^{-j\theta_r} e^{-\gamma z})]\}^{1/2}$$

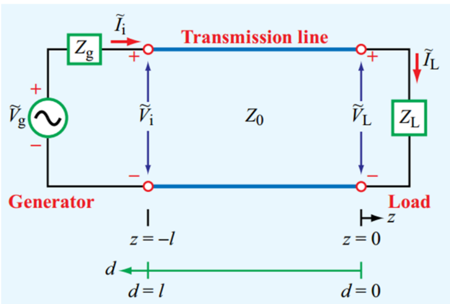
$$= |V_0^+| [1 + |\Gamma_L|^2 + |\Gamma_L| (e^{2\gamma z + j\theta_r} + e^{-(2\gamma z + j\theta_r)})]^{1/2}$$

$$= |V_0^+| [1 + |\Gamma_L|^2 + 2|\Gamma_L| \cos(-2j\gamma z + \theta_r)]^{1/2}$$

$e^{jx} + e^{-jx} = 2\cos x$

lossless transmission lines, $\alpha = 0, \gamma = j\beta$

$$|\tilde{V}(z)| = |V_0^+| [1 + |\Gamma_L|^2 + 2|\Gamma_L| \cos(2\beta z + \theta_r)]^{1/2}$$



From view of the load, we replace z with $-d$ to express $|\tilde{V}(z)|$ as a function of d.

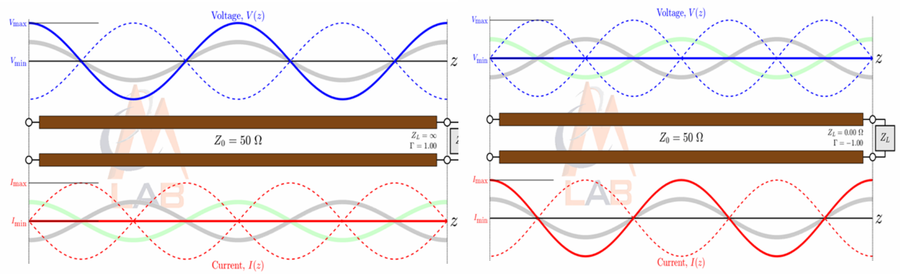
$$|\tilde{V}(d)| = |V_0^+| [1 + |\Gamma_L|^2 + 2|\Gamma_L| \cos(2\beta d - \theta_r)]^{1/2}$$

By applying the same steps to $\tilde{I}(z)$, a similar expression can be derived for $|\tilde{I}(z)|$

$$|\tilde{I}(d)| = \frac{|V_0^+|}{Z_0} [1 + |\Gamma_L|^2 - 2|\Gamma_L| \cos(2\beta d - \theta_r)]^{1/2}$$

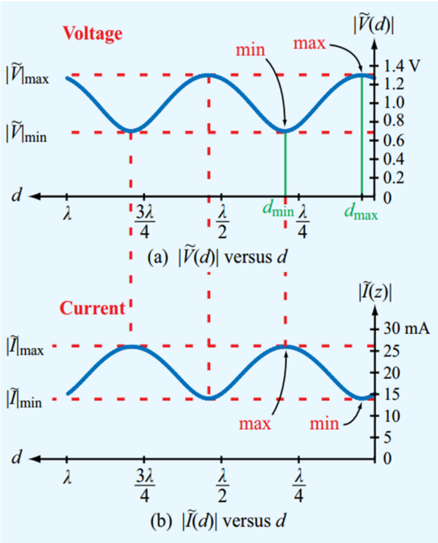
Standing wave pattern

Standing wave is a wave which oscillates in time but whose ***peak amplitude profile does not move in space***. This phenomenon are caused by the ***interference*** of the two traveling waves traveling in ***opposite directions*** (incident and reflected waves).



Open circuit　　Short circuit

Standing wave pattern



standing wave patterns for a lossless TL

- characteristic impedance: $Z_0 = 50\Omega$
  reflection coefficient: $\Gamma_L = 0.3e^{j\pi/6}$
  magnitude of incident wave: $|V_0^+| = 1\text{V}$
- The **max** $|\tilde{V}(d)|$ corresponds to the position at which the incident and reflected waves are in-phase

$$2\beta d - \theta_r = 2n\pi$$

$$\text{Max}|\tilde{V}(d)| = |V_0^+|(1+\Gamma_L) = 1.3V$$

- The **min** $|\tilde{V}(d)|$ corresponds to the position at which the incident and reflected waves are opposite in-phase

$$2\beta d - \theta_r = (2n+1)\pi$$

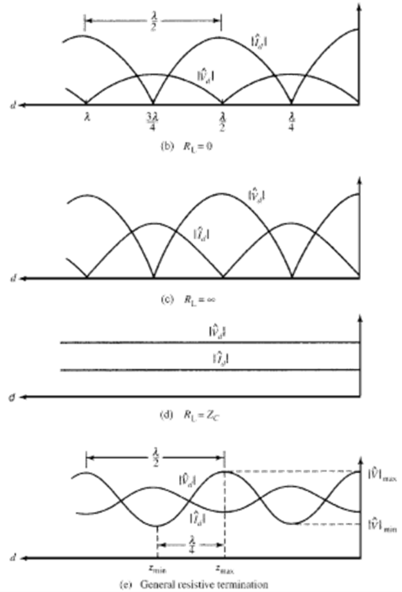
$$\text{Min}|\tilde{V}(d)| = |V_0^+|(1-\Gamma_L) = 0.7V$$

- The repetition period of standing waves is $\lambda/2$

$$\cos(2\beta d - \theta_r) = \cos(4\pi d/\lambda - \theta_r)$$

$$2\lambda = 2\pi/\beta$$

Voltage Standing-Wave Ratio (VSWR)



In the cases of a short-circuit or an open-circuit load (***severely mismatched***), the *maximum and the adjacent minimum are separated by* ***1/4 wavelength****, and* locations on the line corresponding to ***voltage maxima*** correspond to ***current minima***, Vice versa.

$\lambda/4$ difference of $\tilde{V}$ and $\tilde{I}$ is undesirable!

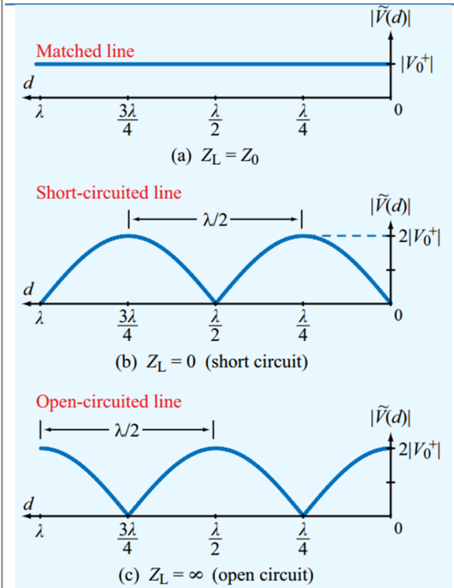
It will cause:

1. Large voltages can occur that may damage insulation;
2. Load voltages and currents are radically different from the input voltage to the line.

We want a matched load where there is no effect on the signal transmission (other than a phase shift of time delay.)

How to evaluate the degree of "match"?

Voltage Standing-Wave Ratio (VSWR)



$$|\tilde{V}|_{max} = |V_0^+|(1+|\Gamma_L|)$$

$$|\tilde{V}|_{min} = |V_0^+|(1-|\Gamma_L|)$$

The ratio of maximum $\tilde{V}$ to minimum $\tilde{V}$ is called the Voltage standing-wave ratio (VSWR)

$$S = \frac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}$$

A quantity measure "match":

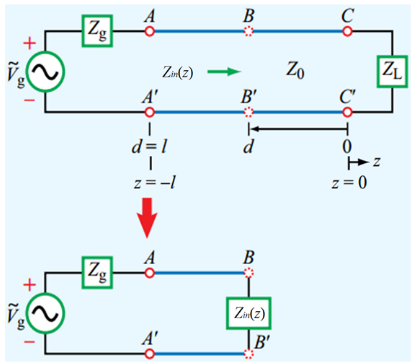
matched ($Z_L = Z_0$) ,$\Gamma_L = 0$, $S = 1$
short circuit ($Z_L = 0$) ,$\Gamma_L = -1$, $S = \infty$
open circuit ($Z_L = \infty$) ,$\Gamma_L = 1$, $S = \infty$

Input Impedance

Recall:
$$\tilde{V}(z) = V_0^+(e^{-\gamma z} + \Gamma_L e^{\gamma z})$$
$$\tilde{I}(z) = \frac{V_0^+}{Z_0}(e^{-\gamma z} - \Gamma_L e^{\gamma z}) \qquad \Gamma(z) = \Gamma_L e^{2\gamma z}$$

Define the ***input impedance*** as voltage to current ratio at a point on a line:

$$Z_{in}(z) \stackrel{\text{def}}{=} \frac{\tilde{V}(z)}{\tilde{I}(z)} = Z_0\frac{e^{-\gamma z} + \Gamma_L e^{\gamma z}}{e^{-\gamma z} - \Gamma_L e^{\gamma z}} = Z_0\frac{1+\Gamma_L e^{2\gamma z}}{1-\Gamma_L e^{2\gamma z}} = Z_0\frac{1+\Gamma(z)}{1-\Gamma(z)}$$



Consider input ports B&B' as two ends of a resistor, then input impedance is the resistance of this resistor.

Input Impedance

Characteristic impedance Zc:

The ratio of the voltage and current in the traveling waves.

▶ It should be noted that Z_0 is equal to the ratio of the voltage amplitude to the current amplitude for each of the traveling waves individually (with an additional minus sign in the case of the $-z$ propagating wave), but it is not equal to the ratio of the total voltage $\tilde{V}(z)$ to the total current $\tilde{I}(z)$, unless one of the two waves is absent. ◀

A ***matched line*** is one in which the load impedance is equal to the characteristic impedance, $Z_L=Z_C$.

The input impedance to a matched line is equal to the characteristic impedance at all points along the line.

For a lossless line, Z_C is a real number. Hence, in order to match a lossless line the load impedance can only be purely resistive.

Thank you!