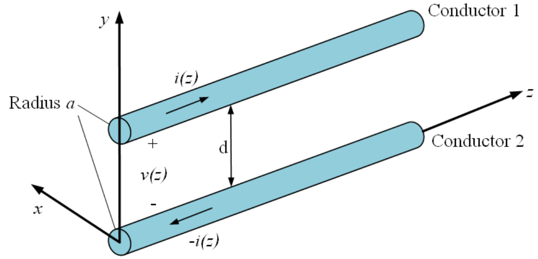
Lecture 12: Time-domain Analysis of Transmission Lines

Yan-zhao XIE

Xi'an Jiaotong University

2020.10.10

Recall: telegrapher's equations



- Telegrapher's Equations in time domain

$$-\frac{\partial V(z,t)}{\partial z} = R'I(z,t) + L'\frac{\partial I(z,t)}{\partial t}$$
$$-\frac{\partial I(z,t)}{\partial z} = G'V(z,t) + C'\frac{\partial V(z,t)}{\partial t}$$

- Telegrapher's Equations in frequency domain

$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\,\tilde{I}(z),$$
$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\,\tilde{V}(z).$$

From frequency domain to time domain

We already dealt with the solution of telegrapher's equations in the *frequency domain*, that is, for the case of *sinusoidal steady-state excitation* of the line. The sources are sinusoids at a single frequency and are assumed to have been applied for a sufficiently long time such that all transients have decayed to zero leaving only the steady-state solution.

We will examine the solution of the telegrapher's equations for sources that have any general *time variation*. This will include both the *transient and the steady-state components* of the solution and represents the solution in the *time domain*.

$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\,\tilde{I}(z),$$
$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\,\tilde{V}(z).$$

$\Rightarrow$

$$-\frac{\partial V(z,t)}{\partial z} = R'I(z,t) + L'\frac{\partial I(z,t)}{\partial t}$$
$$-\frac{\partial I(z,t)}{\partial z} = G'V(z,t) + C'\frac{\partial V(z,t)}{\partial t}$$

V and I vary in z and ω.

V and I vary in z and t.

Time-domain solutions for lossless TLs

The transmission line can be designed to exhibit low ohmic losses by selecting conductors with very high conductivities and dielectric materials (separating the conductors) with negligible conductivities.

Especially for lossless TLs, we set $R' \approx 0$ and $G' \approx 0$.

$$\frac{\partial V(z,t)}{\partial z} = -L'\frac{\partial I(z,t)}{\partial t}$$
$$\frac{\partial I(z,t)}{\partial z} = -C'\frac{\partial V(z,t)}{\partial t}$$

Differentiating one equation with respect to z and the other with respect to t and substituting yields the uncoupled second-order differential equations, which is a *scalar wave equation in one space dimension*.

$$\frac{\partial^2 V(z,t)}{\partial z^2} = L'C'\frac{\partial^2 V(z,t)}{\partial t^2}$$
$$\frac{\partial^2 I(z,t)}{\partial z^2} = L'C'\frac{\partial^2 I(z,t)}{\partial t^2}$$

The solution† of the wave equation is:

$$V(z,t) = \underbrace{V^+\left(t-\frac{z}{v}\right)}_{\text{forward } +z \text{ traveling wave}} + \underbrace{V^-\left(t+\frac{z}{v}\right)}_{\text{backward } -z \text{ traveling wave}}$$

$$I(z,t) = I^+\left(t-\frac{z}{v}\right) + I^-\left(t+\frac{z}{v}\right)$$
$$= \frac{1}{Z_C}V^+\left(t-\frac{z}{v}\right) - \frac{1}{Z_C}V^-\left(t+\frac{z}{v}\right)$$

Where the *characteristic impedance* is

$$Z_C = \sqrt{L'/C'}$$

The *velocity of propagation* of forward traveling wave and backward traveling wave is

$$v = \frac{1}{\sqrt{L'C'}}$$

† reference of the solution: https://en.wikipedia.org/wiki/Wave_equation

Note that:

1. The characteristic impedance, ***Z_C*, is a real (not complex) number**. Hence it would be more properly called the characteristic resistance. However, it has become an industry standard to refer to Z_C as the characteristic impedance, as we will continue to do here.
2. The general forms of the solution are in terms of the functions $V^+(t - z/v)$ and $V^-(t + z/v)$.
   The precise forms of these functions will be determined by the functional time-domain form of the excitation source, $V_S(t)$.
   Nevertheless, they show that time and position must be related as $t - z/v$ and the $t + z/v$ in these forms.
3. The property **$L'C' = \mu\varepsilon$ holds** as long as the medium surrounding the line conductors is homogeneous.

time domain solution

$$V(z,t) = V^+\left(t-\frac{z}{v}\right) + V^-\left(t+\frac{z}{v}\right)$$
$$I(z,t) = I^+\left(t-\frac{z}{v}\right) + I^-\left(t+\frac{z}{v}\right)$$

frequency domain solution

$$\tilde{V}(z) = V_0^+e^{-\gamma z} + V_0^-e^{\gamma z}$$
$$\tilde{I}(z) = I_0^+e^{-\gamma z} + I_0^-e^{\gamma z}$$

forward $+z$

backward $-z$

The waves V^+ and I^+ are traveling in the $+z$ direction. *As t increases, z must also increase in order to track a point on the waveform.* Hence, they are called forward-traveling waves.

The characteristic impedance relates the voltage and current in the forward-traveling wave and in the backward-traveling wave as

$$V^+\left(t-\frac{z}{v}\right) = Z_C I^+\left(t-\frac{z}{v}\right)$$
$$V^-\left(t+\frac{z}{v}\right) = -Z_C I^-\left(t+\frac{z}{v}\right)$$

So far, $V^+(t,z)$ and $V^-(t,z)$ are yet unknown but have time and position related ***only as*** $t \pm z/v$.

Reflection

Assume resistive loads R_S and R_L. For lines with total length l, the forward and backward traveling waves are related by the *load reflection coefficient*.

At the load $\boldsymbol{z = l}$

$$V(l,t) = R_L I_L(l,t)$$

……

$$\Gamma_L = \frac{V^-\left(t+\frac{l}{v}\right)}{V^+\left(t-\frac{l}{v}\right)} = \frac{R_L - Z_C}{R_L + Z_C}$$

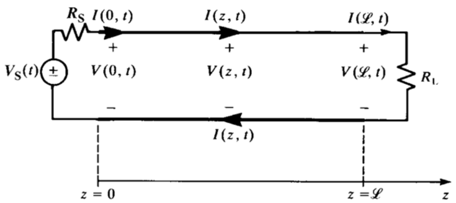
Reflection

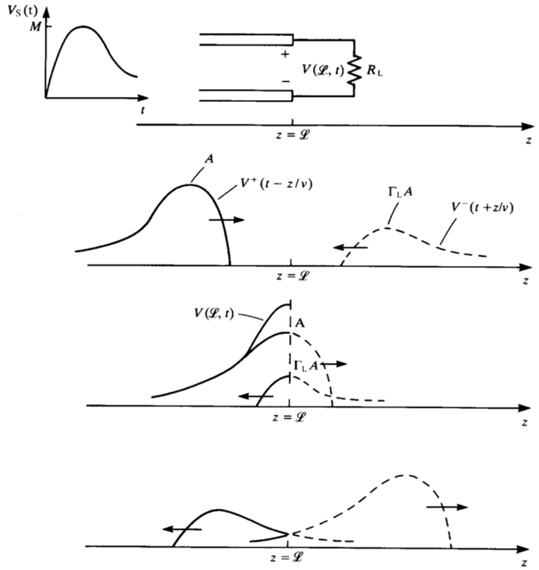
Therefore, the reflected *voltage wave* at the load can be found from the incident wave using the load reflection coefficient as

$$V^-\left(t+\frac{l}{v}\right)=\Gamma_L V^+\left(t-\frac{l}{v}\right) \quad \Longrightarrow \quad \Gamma^V=\Gamma_L$$

For the reflected *current wave,*

$$I^-\left(t+\frac{l}{v}\right)=-\Gamma_L I^+\left(t-\frac{l}{v}\right) \quad \Longrightarrow \quad \Gamma^I=-\Gamma_L=-\Gamma^V$$





The reflection process can be viewed as a mirror that produces, as a reflected V^-, a replica of V^+ that is "flipped around," and all points on the V^- waveform are the corresponding points on the V^+ waveform multiplied by Γ_L.

$$V(l,t)=V^+\left(t-\frac{l}{v}\right)+V^-\left(t+\frac{l}{v}\right)$$
$$=(1+\Gamma_L)V^+\left(t-\frac{l}{v}\right)$$

When we initially connect the source to the line, we reason that a forward-traveling wave will be propagated down the line. A backward-traveling wave won't appear on the line until this initial forward-traveling wave has reached the load, a time delay of

$$T_D=l/v$$

The portion of incident wave that is reflected at the load will require an additional time delay T_D to move back to the source. For $0\le t\le 2T_D$, no backward-traveling waves will appear at $z=0$. Then total voltage and current *at $z=0$* will consist only of forward-traveling waves V^+ and I^+.

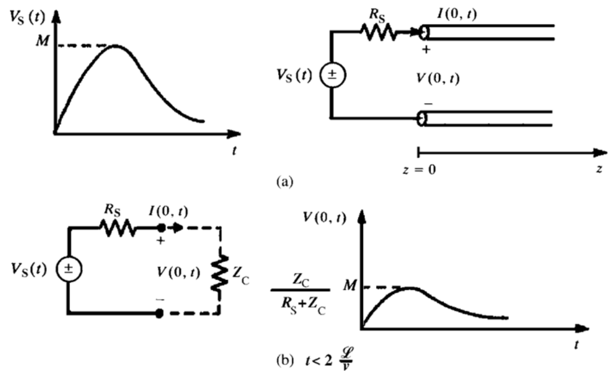
$$V(0,t)=V^+\left(t-\frac{0}{v}\right)$$

$$I(0,t)=I^+\left(t-\frac{0}{v}\right)=\frac{1}{Z_C}V^+\left(t-\frac{0}{v}\right)$$

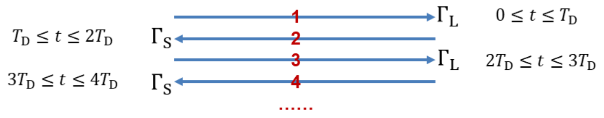
Using voltage division, we can get

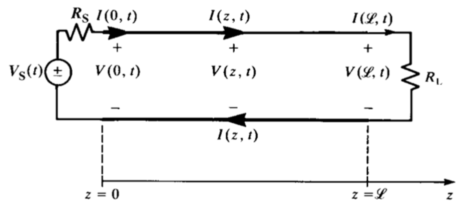
$$V(0,t)=\frac{Z_C}{R_S+Z_C}V_S(t) \qquad \text{for } 0\le t\le 2T_D$$

$$I(0,t)=\frac{1}{R_S+Z_C}V_S(t) \qquad \text{for } 0\le t\le 2T_D$$



The initially launched wave travels toward the load requiring a time delay T_D for the leading edge of the pulse to reach the load. After this pulse reaches the load and reflects, it will reach the source in another T_D.





For ***t* >2*T*$_D$**, at ***z*=0**,

$$V(0,t) = -R_S I(0,t) + V_S(t)$$

$$V^+ + V^- = -R_S(I^+ + I^-) + V_S$$

$$V^+ + V^- = -R_S\left(\frac{V^+}{Z_C} - \frac{V^-}{Z_C}\right) + V_S$$

$$V^+\left(1 + \frac{R_S}{Z_C}\right) = V^-\left(\frac{R_S}{Z_C} - 1\right) + V_S$$

$$V^+ = \underbrace{\frac{R_S - Z_C}{R_S + Z_C}}_{\Gamma_s} V^- + \underbrace{\frac{Z_C}{R_S + Z_C}}_{T_S} V_S$$

Define the ***reflection coefficient*** at the source as $\Gamma_S = \dfrac{R_S - Z_C}{R_S + Z_C}$

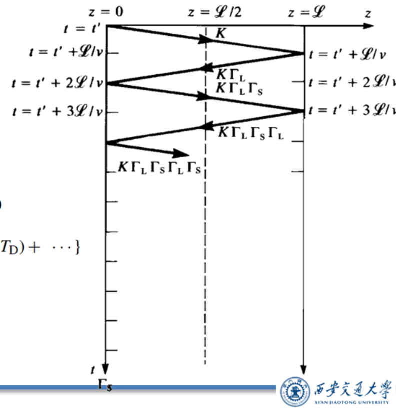
Define the ***transmission coefficient*** as $T_S = \dfrac{Z_C}{R_S + Z_C}$

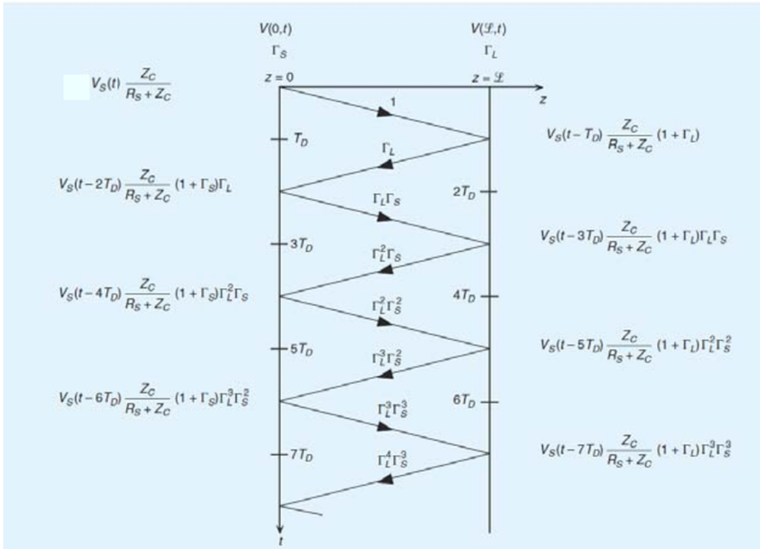
The process of propagation and reflection is repeated with factor of Γ_S and Γ_L. At any time, the total voltage (current) at any point on the line is the **sum** of all the individual voltage (current) waves existing on the line at that point and time.

- The *horizontal axis represents position* along the line
- The *vertical axis denotes time*
- The *bounce diagram consists of a zigzag line* indicating the progress of the voltage wave on the line

$$V(0,t) = \frac{Z_C}{Z_C + R_S}\{V_S(t) + (1+\Gamma_S)\Gamma_L V_S(t-2T_D) + (1+\Gamma_S)(\Gamma_S\Gamma_L)\Gamma_L V_S(t-4T_D) + (1+\Gamma_S)(\Gamma_S\Gamma_L)^2\Gamma_L V_S(t-6T_D) + \cdots\}$$

$$\begin{aligned} V(\mathscr{L},t) &= \frac{Z_C}{Z_C + R_S}\{(1+\Gamma_L)V_S(t-T_D) + (1+\Gamma_L)\Gamma_S\Gamma_L V_S(t-3T_D) \\ &\quad + (1+\Gamma_L)(\Gamma_S\Gamma_L)^2 V_S(t-5T_D) + (1+\Gamma_L)(\Gamma_S\Gamma_L)^3 V_S(t-7T_D) + \cdots\} \\ &= \frac{Z_C}{Z_C + R_S}(1+\Gamma_L)\{V_S(t-T_D) + \Gamma_S\Gamma_L V_S(t-3T_D) \\ &\quad + (\Gamma_S\Gamma_L)^2 V_S(t-5T_D) + (\Gamma_S\Gamma_L)^3 V_S(t-7T_D) + \cdots\} \end{aligned}$$





Matched Line

If the line is matched at the load end,

$$V(0,t) = \frac{Z_C}{R_S + Z_C} V_S(t), \qquad R_L = ZC$$

$$V(\boldsymbol{l},t) = \frac{Z_C}{R_S + Z_C} V_S(t - \boldsymbol{TD}), \qquad R_L = ZC$$

Note that:

- The only effect of the line is to impose *a time delay.*
- The *input and output voltage waveforms* of the line are identical.

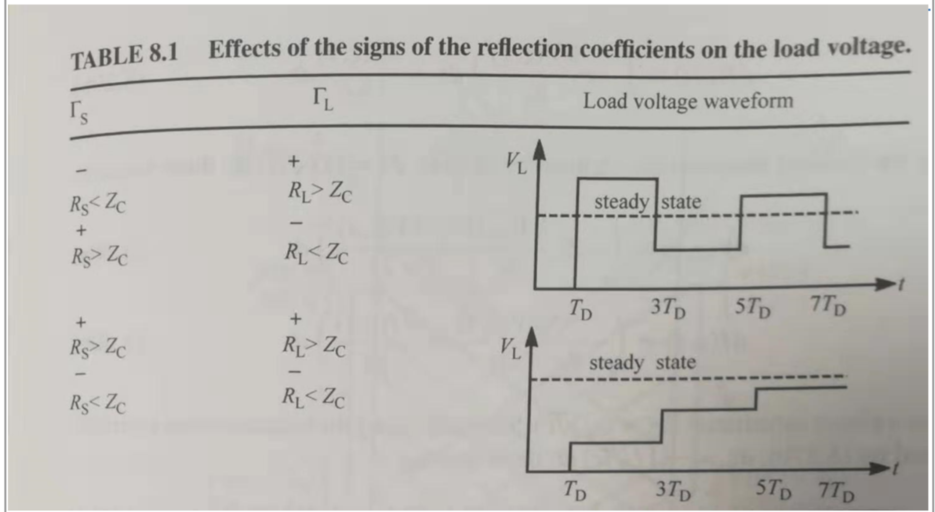


TABLE 8.1 Effects of the signs of the reflection coefficients on the load voltage.

Γ_S	Γ_L	Load voltage waveform
− $R_S < Z_C$	+ $R_L > Z_C$	V_L; steady state; T_D, $3T_D$, $5T_D$, $7T_D$, t
+ $R_S > Z_C$	− $R_L < Z_C$	
+ $R_S > Z_C$	+ $R_L > Z_C$	V_L; steady state; T_D, $3T_D$, $5T_D$, $7T_D$, t
− $R_S < Z_C$	− $R_L < Z_C$	

Brainy Quote

Continuous effort - not strength or intelligence - is the key to unlocking our potential.

Thank you!