Lecture 9: Solutions for Telegrapher's equations excited by Lumped Source : BLT Equation

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Recall: Combined Voltage Wave

Combined voltage wave

$$\boxed{\tilde{W}(z)_{\pm} = \tilde{V}(z) \pm Z_0 \cdot \tilde{I}(z)}$$

$\tilde{W}(z)_{+} = \tilde{V}(z) + Z_0 \cdot \tilde{I}(z)$ forward traveling combined voltage wave

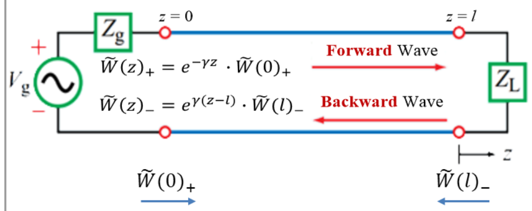
$\tilde{W}(z)_{-} = \tilde{V}(z) - Z_0 \cdot \tilde{I}(z)$ backward traveling combined voltage wave

$$\boxed{\tilde{W}(z)_{\pm} = e^{\mp\gamma(z-z0)} \cdot \tilde{W}(z0)_{\pm}}$$

z_0: a reference point of z

$\tilde{W}(z)_{+} = e^{-\gamma z} \cdot \tilde{W}(0)_{+}$ forward traveling combined voltage wave

$\tilde{W}(z)_{-} = e^{\gamma(z-l)} \cdot \tilde{W}(l)_{-}$ backward traveling combined voltage wave



$$\tilde{V}(z) = \frac{1}{2}[\tilde{W}(z)_{+} + \tilde{W}(z)_{-}]$$

$$\tilde{I}(z) = \frac{1}{2}[\tilde{W}(z)_{+} - \tilde{W}(z)_{-}]/Z_0$$

Combined voltage differential equation

If one considers the excitation source, the telegrapher's equation is:

$$\frac{d\tilde{V}(z)}{dz} = -Z' \cdot \tilde{I}(z) + \tilde{V}_g(z)$$

$$\frac{d\tilde{I}(z)}{dz} = -Y' \cdot \tilde{V}(z) + \tilde{I}_g(z)$$

One can also define the combined excitation source, the rationality of definition can refer to lecture 5.

$$\tilde{W}^{(s)}(z)_{\pm} = \tilde{V}_g(z) \pm Z_0 \cdot \tilde{I}_g(z)$$

Thus, one can get the combined voltage differential equation:

$$\frac{d\tilde{W}(z)_{\pm}}{dz} \mp \gamma \tilde{W}(z)_{\pm} = \tilde{W}^{(s)}(z)_{\pm}$$

First-order inhomogeneous linear differential equation

Solution of differential equation

The ***general form*** of first-order linear differential equation :

$$\boxed{\frac{dy(z)}{dz} + P(z)y(z) = Q(z)}$$

When $Q(z) = 0$: *homogeneous differential equation*

$$\frac{dy(z)}{dz} + P(z)y(z) = 0$$

general solution

$$y = C^{-\int P(x)dx}$$

When $Q(z) \neq 0$: *inhomogeneous differential equation*

$$\frac{dy(z)}{dz} + P(z)y(z) = Q(z)$$

$$y = e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx + C\right] = \boxed{Ce^{-\int P(x)dx}} + \boxed{e^{-\int P(x)dx} \cdot \int Q(x)e^{\int P(x)dx}dx}$$

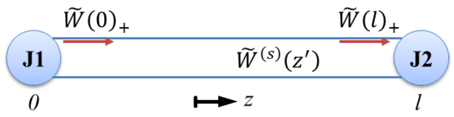
General solution　　special solution

Solution of combined voltage differential equation:

$$\widetilde{W}(z)_{\pm}=e^{\mp\gamma(z-z_0)}\cdot\widetilde{W}(z_0)_{\pm}+\int_{z_0}^{z}e^{\mp\gamma(z-z')}\cdot\widetilde{W}^{(s)}(z')_{\pm}dz'$$

For a TL with length l, one define:

- Outgoing combined voltage wave (W) which leaves the TL junction: $\widetilde{W}(0)_{+}$
- Incoming combined voltage wave (W) which enters the TL junction: $\widetilde{W}(l)_{+}$



One can obtain following equation:

$$\boxed{\widetilde{W}(l)_{+}}=\boxed{e^{-\gamma l}}\cdot\boxed{\widetilde{W}(0)_{+}}+\boxed{\int_{0}^{l}e^{-\gamma(l-z')}\cdot\widetilde{W}^{(s)}(z')_{+}dz'}$$

incoming combined voltage wave

propagation parameter

outgoing combined voltage wave

combined excitation source

Propagation and scattering equations

One can rewrite into matrix form:

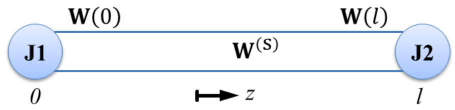
propagation equation $[\mathbf{W}(l)]=[\mathbf{\Gamma}]\cdot[\mathbf{W}(0)]+[\mathbf{W}^{(S)}]$

incoming W wave vector

propagation matrix

Outgoing W wave vector

excitation source vector



Incoming wave equals to the superposition of outgoing wave after attenuation of propagation and excitation source on TL.

In addition, outgoing W wave and incoming W wave can be related by ***scattering*** at the junction

scattering equation $[\mathbf{W}(0)]=[\mathbf{S}]\cdot[\mathbf{W}(l)]$

scattering matrix

BLT equations

One can associate propagation equation and scattering equation:

$$[\mathbf{W}(0)]=([\mathbf{1}]-[\mathbf{S}][\mathbf{\Gamma}])^{-1}\cdot[\mathbf{S}]\cdot[\mathbf{W}^{(S)}]$$

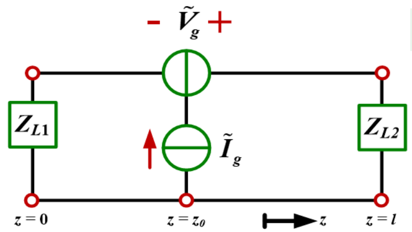
BLT equation by C.E. Baum, T.K. Liu and F. M. Tesche

BLT equation is widely applied in transmission line network. It is an efficient method while it can only calculate the voltage and current at the terminals.

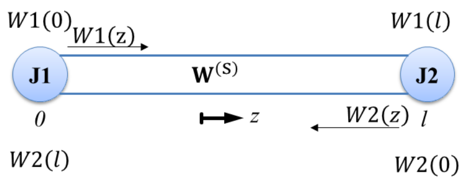
培根、生菜、西红柿

a **bacon**, **lettuce**, and **tomato** sandwich

Examples:



$$[\mathbf{W}(0)]=([\mathbf{1}]-[\mathbf{S}][\mathbf{\Gamma}])^{-1}\cdot[\mathbf{S}]\cdot[\mathbf{W}^{(S)}]$$



$[\mathbf{W}(0)]=\begin{bmatrix}\widetilde{W}1(0)\\ \widetilde{W}2(0)\end{bmatrix}$ outgoing W wave vector

$[\mathbf{\Gamma}]=\begin{bmatrix}e^{-\gamma l} & 0\\ 0 & e^{-\gamma l}\end{bmatrix}$ propagation matrix

$$W-W=\begin{array}{c|cc} & W_1 & W_2\\ \hline W_1 & 0 & 1\\ W_2 & 1 & 0\end{array}\quad\Rightarrow\quad[\mathbf{S}]=\begin{bmatrix}0 & \Gamma_{L1}\\ \Gamma_{L2} & 0\end{bmatrix}$$

scattering matrix

For junction 2, W1 enters where W2 leaves. Vice versa.

combined excitation source

$$[\mathbf{W}^{(S)}(l)]=\begin{bmatrix}\widetilde{W}1^{(s)}(l)\\ \widetilde{W}2^{(s)}(l)\end{bmatrix}\quad \begin{aligned}\widetilde{W}1^{(s)} &= \int_0^l e^{-\gamma(l-z')}\cdot(\tilde{V}_g+Z_0\tilde{I}_g)\delta(z_g)dz' = e^{-\gamma(l-z_g)}(\tilde{V}_g+Z_0\tilde{I}_g)\\ \widetilde{W}2^{(s)} &= \int_0^l e^{-\gamma(l-z')}\cdot(-\tilde{V}_g+Z_0\tilde{I}_g)\delta(l-z_g)dz' = e^{-\gamma z_g}(-\tilde{V}_g+Z_0\tilde{I}_g)\end{aligned}$$

$$[\mathbf{W}(0)] = ([\mathbf{1}]-[\mathbf{S}][\mathbf{\Gamma}])^{-1}\cdot[\mathbf{S}]\cdot[\mathbf{W}^{(S)}]$$

Substituting each element, one can have:

$$\begin{bmatrix}\widetilde{W}1(0)\\ \widetilde{W}2(0)\end{bmatrix} = \left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}0 & \Gamma_{L1}\\ \Gamma_{L2} & 0\end{bmatrix}\begin{bmatrix}e^{-\gamma l} & 0\\ 0 & e^{-\gamma l}\end{bmatrix}\right)^{-1}\begin{bmatrix}0 & \Gamma_{L1}\\ \Gamma_{L2} & 0\end{bmatrix}\begin{bmatrix}e^{-\gamma(l-z_g)}(\tilde{V}_g+Z_0\tilde{I}_g)\\ e^{-\gamma z_g}(-\tilde{V}_g+Z_0\tilde{I}_g)\end{bmatrix}$$

$$[\mathbf{W}(0)] = [\mathbf{S}]\cdot[\mathbf{W}(l)]$$

$$\begin{bmatrix}\widetilde{W}1(0)\\ \widetilde{W}2(0)\end{bmatrix} = \begin{bmatrix}0 & \Gamma_{L1}\\ \Gamma_{L2} & 0\end{bmatrix}\begin{bmatrix}\widetilde{W}1(l)\\ \widetilde{W}2(l)\end{bmatrix}$$

$$\tilde{V}(0) = \frac{1}{2}[\widetilde{W}1(0)+\widetilde{W}2(l)] \qquad \tilde{V}(l) = \frac{1}{2}[\widetilde{W}2(0)+\widetilde{W}1(l)]$$

$$\tilde{I}(0) = \frac{1}{2}[\widetilde{W}2(l)-\widetilde{W}1(0)]/Z_0 \qquad \tilde{I}(l) = \frac{1}{2}[\widetilde{W}1(l)-\widetilde{W}2(0)]/Z_0$$

For $z = l$,

$$\tilde{V}(l) = \frac{(1+\Gamma_{L2})e^{-\gamma l}}{2(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}[(e^{\gamma z_g}-\Gamma_{L1}e^{-\gamma z_g})\tilde{V}_g+(e^{\gamma z_g}+\Gamma_{L1}e^{-\gamma z_g})Z_0\tilde{I}_g]$$

$$\tilde{I}(l) = \frac{(1-\Gamma_{L2})e^{-\gamma l}}{2Z_0(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}[(e^{\gamma z_g}-\Gamma_{L1}e^{-\gamma z_g})\tilde{V}_g+(e^{\gamma z_g}+\Gamma_{L1}e^{-\gamma z_g})Z_0\tilde{I}_g]$$

For $z = 0$,

$$\tilde{V}(0) = \frac{(1+\Gamma_{L1})e^{-\gamma l}}{2(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left\{-\left[e^{\gamma(l-z_g)}-\Gamma_{L2}e^{-\gamma(l-z_g)}\right]\tilde{V}_g+\left[e^{\gamma(l-z_g)}+\Gamma_{L2}e^{-\gamma(l-z_g)}\right]Z_0\tilde{I}_g\right\}$$

$$\tilde{I}(0) = \frac{(1-\Gamma_{L1})e^{-\gamma l}}{2Z_0(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left\{-\left[e^{\gamma(l-z_g)}-\Gamma_{L2}e^{-\gamma(l-z_g)}\right]\tilde{V}_g+\left[e^{\gamma(l-z_g)}+\Gamma_{L2}e^{-\gamma(l-z_g)}\right]Z_0\tilde{I}_g\right\}$$

One can compare the result of BLT equation with Green function’s.

Thank you!