# Lecture 14: Numerical Time-domain Solutions of Transmission Lines: FDTD

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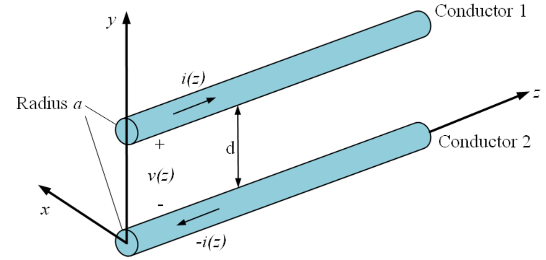
## Recall: telegrapher's equations




- Telegrapher's Equations in time domain (for lossless TLs)

$$\frac{\partial V(z,t)}{\partial z} = -L' \frac{\partial I(z,t)}{\partial t}$$
$$\frac{\partial I(z,t)}{\partial z} = -C' \frac{\partial V(z,t)}{\partial t}$$

$$V(z,t) = V^{+}\left(t-\frac{z}{v}\right) + V^{-}\left(t+\frac{z}{v}\right)$$

$$I(z,t) = I^{+}\left(t-\frac{z}{v}\right) + I^{-}\left(t+\frac{z}{v}\right) = \frac{1}{Z_C} V^{+}\left(t-\frac{z}{v}\right) - \frac{1}{Z_C} V^{-}\left(t+\frac{z}{v}\right)$$

## Finite-Difference, Time-Domain (FDTD)

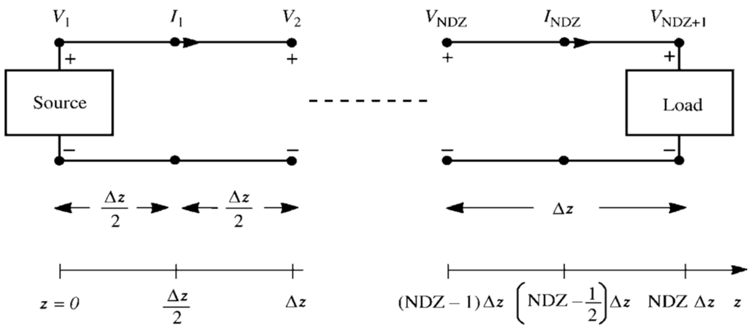
**FDTD (finite-difference, time-domain)** is a numerical method which can approximately determine the time-domain solution. The derivatives in the telegrapher's equations are discretized and approximated with various finite differences.

In this method, the position variable $z$ is discretized as $\Delta z$ and the time variable $t$ is discretized as $\Delta t$.

## FDTD method

In order to ensure stability of the discretization and to ensure second-order accuracy.

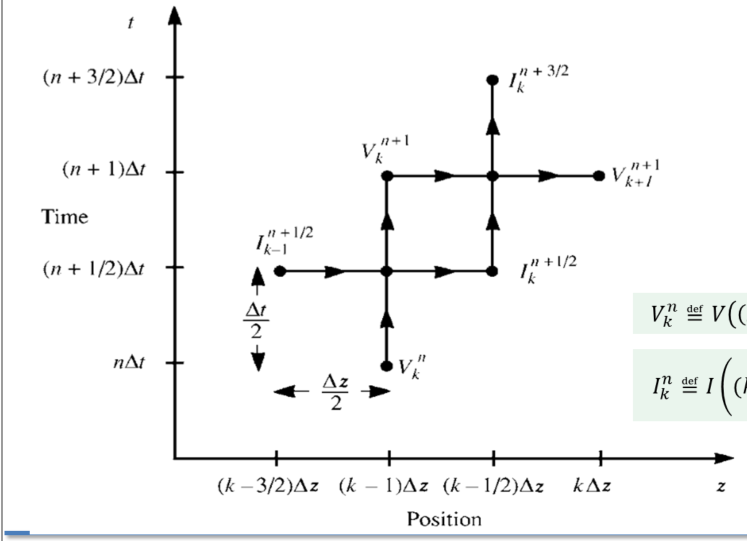
Divide the line into $NDZ$ sections and divide total solution time into $NDT$ sections. Voltage points and current points are interlaced with distance interval of $\Delta z/2$ and time interval of $\Delta t/2$.




$$V_k^n \stackrel{\text{def}}{=} V((k-1)\Delta z, n\Delta t)$$

$$I_k^n \stackrel{\text{def}}{=} I\left(\left(k-\frac{1}{2}\right)\Delta z, n\Delta t\right)$$

Interlacing the current and voltage solutions in space and time for the FDTD analysis




$V_k^n \stackrel{\text{def}}{=} V((k-1)\Delta z, n\Delta t)$

$I_k^n \stackrel{\text{def}}{=} I\left((k-\frac{1}{2})\Delta z, n\Delta t\right)$

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Using *first derivative central difference* approximation,

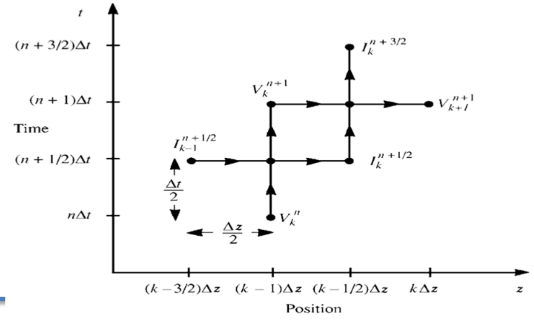
$$f'(t_0) \cong \frac{f(t_0+\Delta t)-f(t_0-\Delta t)}{2\Delta t}$$

$V_k^n \stackrel{\text{def}}{=} V((k-1)\Delta z, n\Delta t)$

$I_k^n \stackrel{\text{def}}{=} I\left((k-\frac{1}{2})\Delta z, n\Delta t\right)$

$$\frac{\partial V(z,t)}{\partial z} = -L'\frac{\partial I(z,t)}{\partial t} \Rightarrow \frac{V_{k+1}^{n+1}-V_k^{n+1}}{\Delta z} + L'\frac{I_k^{n+3/2}-I_k^{n+1/2}}{\Delta t} = 0 \quad k = 1,2,\dots\dots,NDZ$$

$$\frac{\partial I(z,t)}{\partial z} = -C'\frac{\partial V(z,t)}{\partial t} \Rightarrow \frac{I_k^{n+1/2}-I_{k-1}^{n+1/2}}{\Delta z} + C'\frac{V_k^{n+1}-V_k^n}{\Delta t} = 0 \quad k = 2,\dots\dots,NDZ$$

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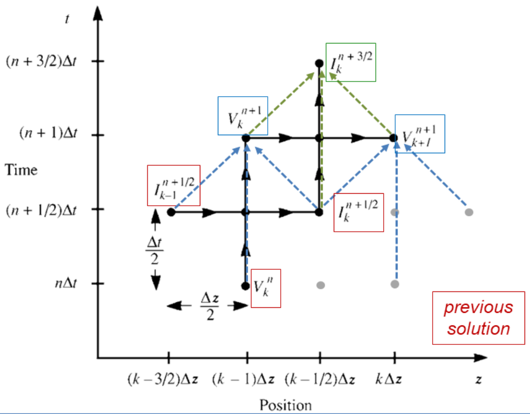
# Leapfrog diagram

We can get the recursion relations for the interior points on the line.

$$I_k^{n+3/2} = I_k^{n+1/2} - \frac{\Delta t}{\Delta z}\frac{1}{L'}\left(V_{k+1}^{n+1}-V_k^{n+1}\right) \quad \text{(a)} \quad k = 1,2,\dots\dots,NDZ$$

$$V_k^{n+1} = V_k^n - \frac{\Delta t}{\Delta z}\frac{1}{C'}\left(I_k^{n+1/2}-I_{k-1}^{n+1/2}\right) \quad \text{(b)} \quad k = 2,\dots\dots,NDZ$$




previous solution $t \le (n+1/2)\Delta t$

↓ (b)

voltages along the line $t = (n+1)\Delta t$

↓ (a)

current along the line $t = (n+3/2)\Delta t$

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# Incorporation of terminal conditions

FDTD voltages and currents at each end of the line, $V_1$, $I_1$, and $V_{NDZ-1}$, $I_{NDZ-1}$, are not collocated in space or time, whereas the terminal conditions relate the voltage and current at the same position and at the same time.

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# Incorporation of terminal conditions

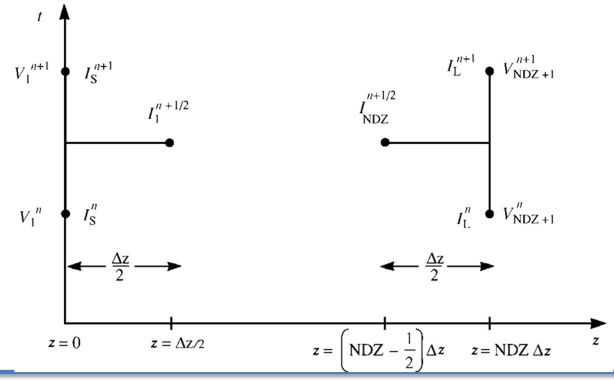
Discretize the transmission line equation at the source by averaging the source current $I_S$ and at the load by averaging the load current $I_L$, respectively.

$$\frac{1}{\Delta z/2}\left(I_1^{n+1/2} - \frac{I_S^{n+1}+I_S^n}{2}\right) + \frac{C'}{\Delta t}\left(V_1^{n+1}-V_1^n\right) = 0$$

$$\frac{1}{\Delta z/2}\left(\frac{I_L^{n+1}+I_L^n}{2} - I_{NDZ}^{n+1/2}\right) + \frac{C'}{\Delta t}\left(V_{NDZ+1}^{n+1}-V_{NDZ+1}^n\right) = 0$$

$$\frac{\partial V(z,t)}{\partial z} = -L'\frac{\partial I(z,t)}{\partial t}$$

$$\frac{\partial I(z,t)}{\partial z} = -C'\frac{\partial V(z,t)}{\partial t}$$




The recursion relations are:

$$V_1^{n+1} = V_1^n - \frac{2\Delta t}{\Delta z}\frac{1}{C'}I_1^{n+\frac{1}{2}} + \frac{\Delta t}{\Delta z}\frac{1}{C'}\left(I_S^{n+1}+I_S^n\right) = 0$$

$$V_{NDZ+1}^{n+1} = V_{NDZ+1}^n + \frac{2\Delta t}{\Delta z}\frac{1}{C'}I_{NDZ+1}^{n+\frac{1}{2}} - \frac{\Delta t}{\Delta z}\frac{1}{C'}\left(I_L^{n+1}+I_L^n\right) = 0$$

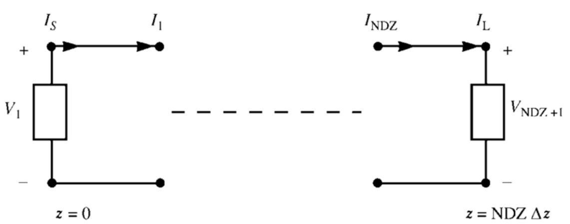
By Thevenin equivalent relations, we have

$$V_1 = V_S - Z_S I_S \qquad V_{NDZ+1} = V_L + Z_L I_L$$

By substituting, the recursion relations of terminal points are given

$$V_1^{n+1} = \left(\frac{\Delta z}{\Delta t}Z_S C' + 1\right)^{-1}\left[\left(\frac{\Delta z}{\Delta t}Z_S C' - 1\right)V_1^n - 2Z_S I_1^{n+\frac{1}{2}} + \left(V_S^{n+1}+V_S^n\right)\right] \quad \text{(a)}$$

$$V_{NDZ+1}^{n+1} = \left(\frac{\Delta z}{\Delta t}Z_L C' + 1\right)^{-1}\left[\left(\frac{\Delta z}{\Delta t}Z_L C' - 1\right)V_{NDZ+1}^n + 2Z_L I_{NDZ}^{n+\frac{1}{2}} + \left(V_L^{n+1}+V_L^n\right)\right] \quad \text{(b)}$$



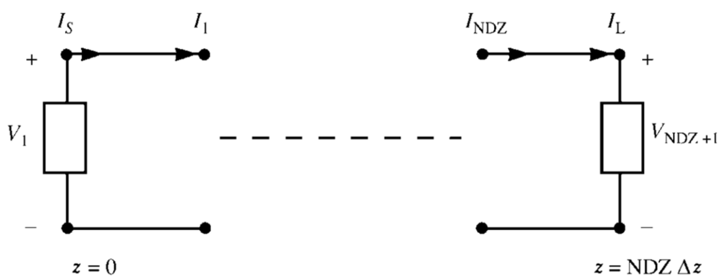

Recall: the recursion relations of interior points

$$V_k^{n+1} = V_k^n - \frac{\Delta t}{\Delta z}\frac{1}{C'}\left(I_k^{n+1/2} - I_{k-1}^{n+1/2}\right) \quad \text{(c)}$$

$k = 2,3,\ldots\ldots,NDZ$

$$I_k^{n+3/2} = I_k^{n+1/2} - \frac{\Delta t}{\Delta z}\frac{1}{L'}\left(V_{k+1}^{n+1} - V_k^{n+1}\right) \quad \text{(d)}$$

$k = 1,2,\ldots\ldots,NDZ$



For stability of the solution, the position and time discretizations must satisfy the Courant condition

$$\Delta t \leq \frac{\Delta z}{v}$$

The Courant condition provides that for stability of the solution the time step must be greater than the propagation time over each cell.

Thank you again!