Lecture 20: Frequency-domain analysis of multiconductor lines

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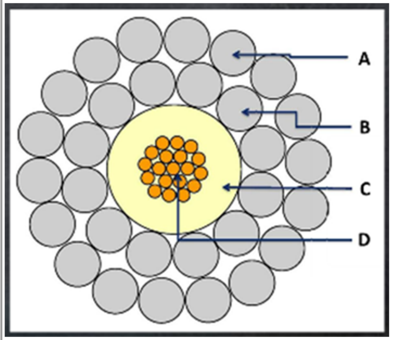

Review of the MTL equation in time-domain

$$\boxed{\frac{\partial}{\partial z}\mathbf{V}(z,t) = -\mathbf{R}\mathbf{I}(z,t) - \mathbf{L}\frac{\partial}{\partial t}\mathbf{I}(z,t)}$$

$$\boxed{\frac{\partial}{\partial z}\mathbf{I}(z,t) = -\mathbf{G}\,\mathbf{V}(z,t) - \mathbf{C}\frac{\partial}{\partial t}\mathbf{V}(z,t)}$$

$$\mathbf{V}(z,t) = \begin{bmatrix} V_1(z,t) \\ \vdots \\ V_i(z,t) \\ \vdots \\ V_n(z,t) \end{bmatrix} \qquad \mathbf{I}(z,t) = \begin{bmatrix} I_1(z,t) \\ \vdots \\ I_i(z,t) \\ \vdots \\ I_n(z,t) \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} (r_1+r_0) & r_0 & \cdots & r_0 \\ r_0 & (r_2+r_0) & \cdots & r_0 \\ \vdots & \vdots & \ddots & \vdots \\ r_0 & r_0 & \cdots & (r_n+r_0) \end{bmatrix} = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{bmatrix} + \begin{bmatrix} r_0 & r_0 & \cdots & r_0 \\ r_0 & r_0 & \cdots & r_0 \\ \vdots & \vdots & \ddots & \vdots \\ r_0 & r_0 & \cdots & r_0 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{12} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{1n} & l_{2n} & \cdots & l_{nn} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \sum_{k=1}^{n} c_{1k} & -c_{12} & \cdots & -c_{1n} \\ -c_{12} & \sum_{k=1}^{n} c_{2k} & \cdots & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{1n} & -c_{2n} & \cdots & \sum_{k=1}^{n} c_{nk} \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \sum_{k=1}^{n} g_{1k} & -g_{12} & \cdots & -g_{1n} \\ -g_{12} & \sum_{k=1}^{n} g_{2k} & \cdots & -g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -g_{1n} & -g_{2n} & \cdots & \sum_{k=1}^{n} g_{nk} \end{bmatrix}$$

The MTL equation in frequency-domain

Replacing all **time derivatives** in the time-domain MTL equations with ***jω***, the frequency-domain (phasor) MTL equations are given in matrix form

$$\boxed{\frac{\partial}{\partial z}\mathbf{V}(z,t) = -\mathbf{R}\mathbf{I}(z,t) - \mathbf{L}\frac{\partial}{\partial t}\mathbf{I}(z,t)}$$

$$\boxed{\frac{\partial}{\partial z}\mathbf{I}(z,t) = -\mathbf{G}\,\mathbf{V}(z,t) - \mathbf{C}\frac{\partial}{\partial t}\mathbf{V}(z,t)}$$

$$\Rightarrow \boxed{\begin{aligned}\frac{d}{dz}\hat{\mathbf{V}}(z) &= -\hat{\mathbf{Z}}\,\hat{\mathbf{I}}(z)\\ \frac{d}{dz}\hat{\mathbf{I}}(z) &= -\hat{\mathbf{Y}}\,\hat{\mathbf{V}}(z)\end{aligned}}$$

where

$$\hat{\mathbf{V}}(z) = \begin{bmatrix}\hat{V}_1(z)\\ \vdots \\ \hat{V}_i(z) \\ \vdots \\ \hat{V}_n(z)\end{bmatrix} \qquad \hat{\mathbf{I}}(z) = \begin{bmatrix}\hat{I}_1(z)\\ \vdots \\ \hat{I}_i(z) \\ \vdots \\ \hat{I}_n(z)\end{bmatrix}$$

$$V_i(z,t) = \mathrm{Re}\{\hat{V}_i(z)\,e^{j\omega t}\}$$

$$I_i(z,t) = \mathrm{Re}\{\hat{I}_i(z)\,e^{j\omega t}\}$$

The p.u.l. parameter matrices

The $n \times n$ per-unit-length ***impedance*** and ***admittance*** matrices are given by

$$\boxed{\begin{aligned}\frac{d}{dz}\hat{\mathbf{V}}(z) &= -\hat{\mathbf{Z}}\,\hat{\mathbf{I}}(z)\\ \frac{d}{dz}\hat{\mathbf{I}}(z) &= -\hat{\mathbf{Y}}\,\hat{\mathbf{V}}(z)\end{aligned}}$$

$$\hat{\mathbf{Z}} = \mathbf{R} + j\omega\mathbf{L}$$

$$\hat{\mathbf{Y}} = \mathbf{G} + j\omega\mathbf{C}$$

These matrices contain the $n \times n$ per-unit-length resistance **R**, inductance (containing both internal and external inductance) $\mathbf{L} = \mathbf{L}_i + \mathbf{L}_e$, conductance **G**, and capacitance **C** matrices.
Since **R**, **L**, **C**, and **G** are symmetric matrices, the ***impedance* and *admittance* matrices** are also symmetric.

The p.u.l. parameter matrices

Perfect Conductors in Lossless, Homogeneous Media
Consider the case of perfect conductors for which

$$\mathbf{R} = \mathbf{0}$$

If the surrounding medium is *homogeneous* with parameters σ, ε, and μ, then we have the important identities:

$$\mathbf{CL} = \mathbf{LC} = \mu\varepsilon\,\mathbf{1}_n$$

$$\mathbf{GL} = \mathbf{LG} = \mu\sigma\,\mathbf{1}_n$$

Where $\mathbf{1}_n$ is the $n \times n$ identity matrix.

Note:
If the surrounding medium is inhomogeneous, these identities don't apply.

Second-order MTL equations

The coupled, first-order MTL equations can be placed in the form of uncoupled, second-order ordinary differential equations by differentiating one with respect to line position z and substituting the other, and vice versa, to yield.

$$\frac{d^2}{dz^2}\hat{\mathbf{V}}(z) = \hat{\mathbf{Z}}\hat{\mathbf{Y}}\,\hat{\mathbf{V}}(z)$$

$$\frac{d^2}{dz^2}\hat{\mathbf{I}}(z) = \hat{\mathbf{Y}}\hat{\mathbf{Z}}\,\hat{\mathbf{I}}(z)$$

Note:
the per-unit-length parameter matrices do not commute, that is

$$\mathbf{ZY} \neq \mathbf{YZ}$$

the equations are coupled together because **ZY** and **YZ** are full matrices related by

$$(\hat{\mathbf{Z}}\hat{\mathbf{Y}})^T = \hat{\mathbf{Y}}^T\hat{\mathbf{Z}}^T = \hat{\mathbf{Y}}\hat{\mathbf{Z}}$$

Decoupling the MTL equations

We will use a ***change of variables*** to decouple the second-order differential equations by putting them into the form of *n* separate equations describing *n* isolated two-conductor lines.

The $n \times n$ complex matrices $\hat{\mathbf{T}}_V$ and $\hat{\mathbf{T}}_I$ define a change of variables between the actual phasor line voltages and currents, $\hat{\mathbf{V}}$ and $\hat{\mathbf{I}}$, and the *mode* voltages and currents, $\hat{\mathbf{V}}_{\mathrm{m}}$ and $\hat{\mathbf{I}}_{\mathrm{m}}$.

$$\hat{\mathbf{V}}(z) = \hat{\mathbf{T}}_V \, \hat{\mathbf{V}}_{\mathrm{m}}(z)$$

$$\hat{\mathbf{I}}(z) = \hat{\mathbf{T}}_I \, \hat{\mathbf{I}}_{\mathrm{m}}(z)$$

In order for this to be valid, these $n \times n$ matrices must be nonsingular, that is,

$$\hat{\mathbf{T}}_V^{-1} \qquad \hat{\mathbf{T}}_I^{-1}$$

must exist.