Lecture 18: Solutions of the three-conductor MTL equations under the weak-coupling assumption

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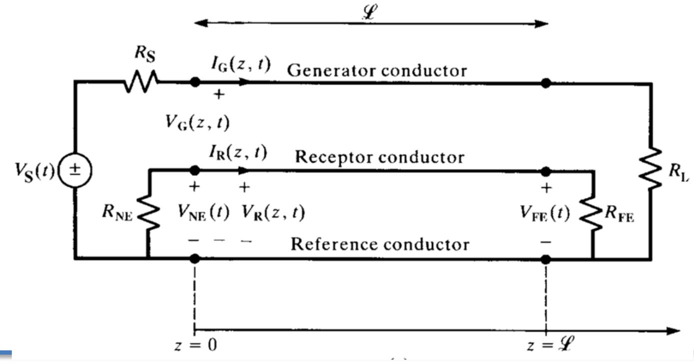
The purpose of this lecture

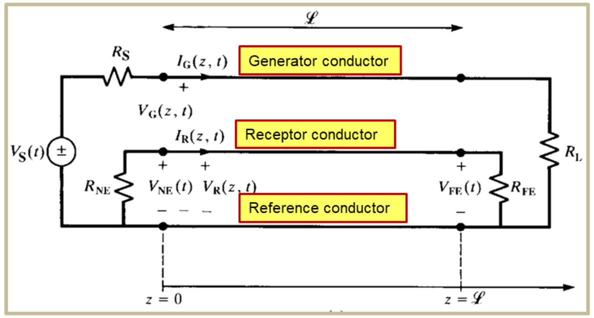
This lecture is to derive the literal or symbolic solution of the MTL equations for a three-conductor **under the assumption of weak coupling**.

The three-conductor line is the simplest wring structure providing the possibility of generating interference between the circuits attached to the ends of the line conductors resulting from crosstalk.



The three-conductor MTL circuit

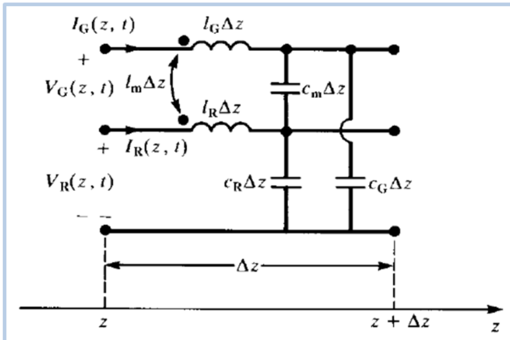
line dimensions and terminal characterization



The line consists of three perfect conductors immersed in a lossless medium.
The **generator circuit** is composed of a **generator conductor** with the **reference conductor**.
The **receptor circuit** is composed of a **receptor conductor** and the **reference conductor**.
S-source; **G**-generator conductor; **R**-receptor conductor; **L**-load; **NE**-near end (wrt the signal source); **FE**-far end (wrt the signal source)

The three-conductor MTL circuit

per-unit-length (p.u.l.) equivalent circuit



Subscripts **G** denote quantities associated with the generator circuit, whereas subscripts **R** denote quantities associated with the receptor circuit. The subscript **m** denotes the mutual quantity between the two circuits.

The three conductor MTL equation can be derived

$$\frac{\partial}{\partial z}\mathbf{V}(z,t) = -\mathbf{L}\frac{\partial}{\partial t}\mathbf{I}(z,t)$$

$$\frac{\partial}{\partial z}\mathbf{I}(z,t) = -\mathbf{C}\frac{\partial}{\partial t}\mathbf{V}(z,t)$$

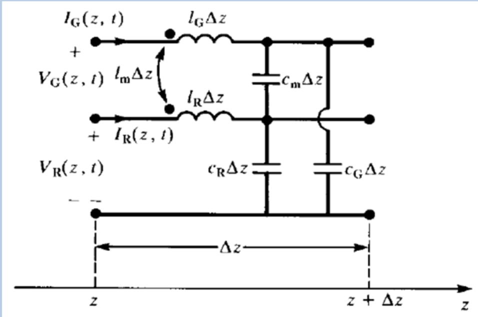
Where,

$$\mathbf{V}(z,t) = \begin{bmatrix} V_G(z,t) \\ V_R(z,t) \end{bmatrix}$$

$$\mathbf{I}(z,t) = \begin{bmatrix} I_G(z,t) \\ I_R(z,t) \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} l_G & l_m \\ l_m & l_R \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} c_G + c_m & -c_m \\ -c_m & c_R + c_m \end{bmatrix}$$



$$\frac{\partial}{\partial z}\mathbf{V}(z,t) = -\mathbf{L}\frac{\partial}{\partial t}\mathbf{I}(z,t)$$

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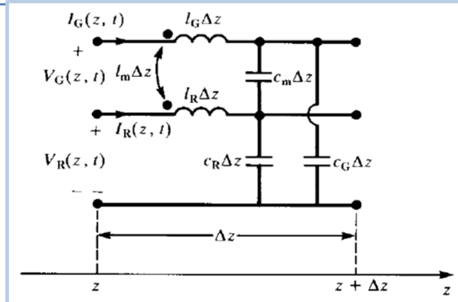


For the generator circuit we have

$$\frac{\partial V_G(z,t)}{\partial z} + l_G\frac{\partial i_G(z,t)}{\partial t} = -l_m\frac{\partial i_R(z,t)}{\partial t}$$
$$\frac{\partial i_G(z,t)}{\partial z} + (c_G + c_m)\frac{\partial v_G(z,t)}{\partial t} = c_m\frac{\partial v_R(z,t)}{\partial t}$$

For the receptor circuit we have

$$\frac{\partial V_R(z,t)}{\partial z} + l_R\frac{\partial i_R(z,t)}{\partial t} = -l_m\frac{\partial i_G(z,t)}{\partial t}$$
$$\frac{\partial i_R(z,t)}{\partial z} + (c_R + c_m)\frac{\partial v_R(z,t)}{\partial t} = c_m\frac{\partial v_G(z,t)}{\partial t}$$

Relations between the p.u.l. parameters

Approximations:
the surrounding medium is *homogeneous* and characterized by permittivity ε and permeability μ.

$$\mathbf{LC} = \mu\varepsilon\,\mathbf{1}_2 = \frac{1}{v^2}\mathbf{1}_2$$

by multiplying both sides by $\mathbf{L}^{-1}$,

$$\underbrace{\begin{bmatrix} (c_G + c_m) & -c_m \\ -c_m & (c_R + c_m) \end{bmatrix}}_{\mathbf{C}} = \frac{1}{v^2}\underbrace{\frac{1}{(l_G l_R - l_m^2)}\begin{bmatrix} l_R & -l_m \\ -l_m & l_G \end{bmatrix}}_{\mathbf{L}^{-1}}$$

where v = $1/(\mu\varepsilon)^{1/2}$ is the velocity of propagation of the waves in the homogeneous medium.

Relations between the p.u.l. parameters

$$\underbrace{\begin{bmatrix} (c_G + c_m) & -c_m \\ -c_m & (c_R + c_m) \end{bmatrix}}_{\mathbf{C}} = \frac{1}{v^2}\underbrace{\frac{1}{(l_G l_R - l_m^2)}\begin{bmatrix} l_R & -l_m \\ -l_m & l_G \end{bmatrix}}_{\mathbf{L}^{-1}}$$

This identity gives the following relations between the per-unit-length parameters:

$$l_G(c_G + c_m) = l_R(c_R + c_m)$$

$$l_m(c_G + c_m) = l_R c_m$$

$$l_m(c_R + c_m) = l_G c_m$$

$$(c_G + c_m) = \frac{l_R}{v^2\,(l_G l_R - l_m^2)}$$

$$(c_R + c_m) = \frac{l_G}{v^2\,(l_G l_R - l_m^2)}$$

$$c_m = \frac{l_m}{v^2\,(l_G l_R - l_m^2)}$$

The computation of p.u.l. parameters

$$l_G = \frac{\mu_0}{2\pi}\ln\left(\frac{d_G}{r_{wG}}\right) + \frac{\mu_0}{2\pi}\ln\left(\frac{d_G}{r_{w0}}\right) = \frac{\mu_0}{2\pi}\ln\left(\frac{d_G^2}{r_{wG} r_{w0}}\right)$$

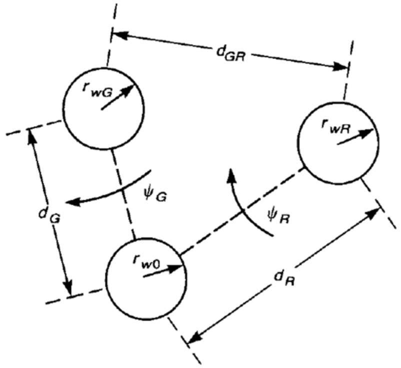
$$l_m = \frac{\mu_0}{2\pi}\ln\left(\frac{d_G}{d_{GR}}\right) + \frac{\mu_0}{2\pi}\ln\left(\frac{d_R}{r_{w0}}\right) = \frac{\mu_0}{2\pi}\ln\left(\frac{d_G d_R}{d_{GR} r_{w0}}\right)$$

$$l_R = \frac{\mu_0}{2\pi}\ln\left(\frac{d_R^2}{r_{wR} r_{w0}}\right)$$

$$c_m = \frac{l_m}{v^2(l_G l_R - l_m^2)}$$

$$c_G + c_m = \frac{l_R}{v^2(l_G l_R - l_m^2)}$$

$$c_R + c_m = \frac{l_G}{v^2(l_G l_R - l_m^2)}$$



The weak-coupling assumption

The weak-coupling assumption is reasonable for many cases of practical interest and means the following. The voltage and current of the generator circuit produce electric and magnetic fields that interact with the receptor circuit and induce voltages and currents in that circuit. This inducement of voltages and currents in the receptor circuit are through the terms:

$$-l_m \frac{\partial i_G(z,t)}{\partial t} \quad \text{and} \quad c_m \frac{\partial v_G(z,t)}{\partial t}$$

These induced voltages and currents, in turn, produce electric and magnetic fields that provide a **back-interaction** or **second-order effect** by inducing voltages and currents in the generator circuit. This **back-interaction** is symbolized by the terms

$$-l_m \frac{\partial i_R(z,t)}{\partial t} \quad \text{and} \quad c_m \frac{\partial v_R(z,t)}{\partial t}$$

The weak-coupling assumption

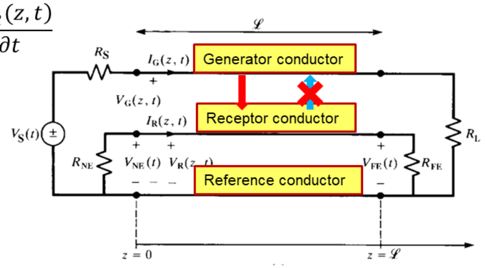
The weak-coupling assumption assumes that this back interaction is negligible so that

$$\frac{\partial V_G(z,t)}{\partial z} + l_G \frac{\partial i_G(z,t)}{\partial t} = -l_m \frac{\partial i_R(z,t)}{\partial t}$$

$$\frac{\partial i_G(z,t)}{\partial z} + (c_G + c_m)\frac{\partial v_G(z,t)}{\partial t} = c_m \frac{\partial v_R(z,t)}{\partial t}$$

$$\frac{\partial V_G(z,t)}{\partial z} + l_G \frac{\partial i_G(z,t)}{\partial t} = 0$$

$$\frac{\partial i_G(z,t)}{\partial z} + (c_G + c_m)\frac{\partial v_G(z,t)}{\partial t} = 0$$



The weak-coupling assumption

The circuit characteristic impedances are defined by

$$Z_{CG} = \sqrt{\frac{l_G}{(c_G + c_m)}} = v l_G \sqrt{1-k^2} = \frac{1}{v(c_G + c_m)\sqrt{1-k^2}}$$

$$Z_{CR} = \sqrt{\frac{l_R}{(c_R + c_m)}} = v l_R \sqrt{1-k^2} = \frac{1}{v(c_R + c_m)\sqrt{1-k^2}}$$

Where the **coupling coefficient between the two circuits** is defined by

$$k = \frac{l_m}{\sqrt{l_G l_R}} = \frac{c_m}{\sqrt{(c_G + c_m)(c_R + c_m)}}$$

Hence, we would expect that **weak coupling** would be a reasonable assumption so long as

$$\sqrt{1-k^2} \cong 1$$

The weak-coupling assumption

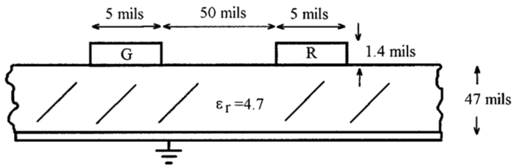
Observe that these definitions of each characteristic impedance do not simply involve just the self capacitances of each circuit, c_G and c_R, but also contain the mutual capacitance cm as (c_G + c_m) and (c_R + c_m).
For weakly coupled lines, these do indeed reduce to the scalar characteristic impedances of the isolated lines:

$$
\begin{aligned}
Z_{CG} &= \sqrt{\frac{l_G}{(c_G + c_m)}} \\
&= vl_G\sqrt{1-k^2} \\
&= \frac{1}{v(c_G + c_m)\sqrt{1-k^2}}
\end{aligned}
\qquad \xrightarrow{\sqrt{1-k^2} \cong 1} \qquad Z_{CG} \cong vl_G
$$

$$
\begin{aligned}
Z_{CR} &= \sqrt{\frac{l_R}{(c_R + c_m)}} \\
&= vl_R\sqrt{1-k^2} \\
&= \frac{1}{v(c_R + c_m)\sqrt{1-k^2}}
\end{aligned}
\qquad \qquad Z_{CR} \cong vl_R
$$

An example of a numerical solution

This is an example of a coupled microstrip which resembles a PCB. Two 5-mil lands are on one side of a glass-epoxy board ($\varepsilon_r = 4.7$) of thickness 47 mils. The lands are 5 mils in width and 1.4-mil thickness. They are separated edge-to-edge by 50 mils. A ground plane (the reference conductor) is on the other side of the board. (1mil=1/1000inch=0.0254mm)



The per-unit-length parameters can be determined a numerical program.

$$
\begin{aligned}
l_G = l_R &= 0.8655\mu\text{H/m} \\
l_m &= 0.1369\mu\text{H/m} \\
(c_G + c_m) = (c_R + c_m) &= 40.1065\text{pF/m} \\
c_m &= 4.1527\text{pF/m}
\end{aligned}
$$

Using these parameters, we calculate $k = 0.158$ and $\sqrt{1-k^2} = 0.987$.
Hence, we would expect that, for this separation, the **weak-coupling assumption would be valid**.

Thank you!