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Welcome to this conference!

Lecture 4: Telegrapher's Equation derived from Maxwell's Equations

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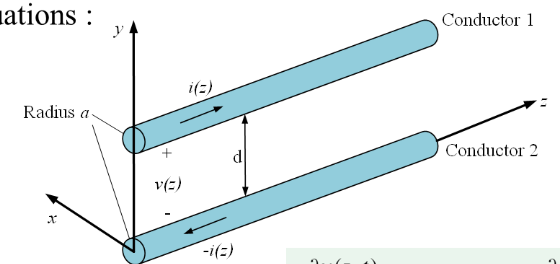
Review

In previous course, we get Telegrapher's Equation from circuit equations :



- Telegrapher's Equations in time domain

$$-\frac{\partial v(z,t)}{\partial z} = R'\, i(z,t) + L'\,\frac{\partial i(z,t)}{\partial t}$$
$$-\frac{\partial i(z,t)}{\partial z} = G'\, v(z,t) + C'\,\frac{\partial v(z,t)}{\partial t}$$

- Telegrapher's Equations in frequency domain

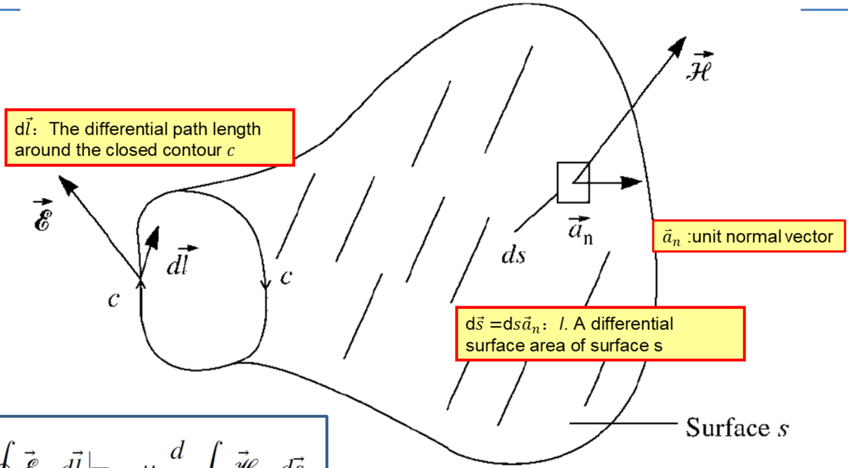
$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\,\tilde{I}(z),$$
$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\,\tilde{V}(z).$$

Maxwell equations

Name	Integral equations	Differential equations	Meaning
Maxwell–Faraday equation (Faraday's law of induction)	$\oint_{\partial\Sigma} \mathbf{E}\cdot d\boldsymbol{\ell} = -\frac{d}{dt}\iint_{\Sigma} \mathbf{B}\cdot d\mathbf{S}$	$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	The voltage induced in a closed circuit is proportional to the rate of change of the magnetic flux it encloses.
Ampère's circuital law (with Maxwell's addition)	$\oint_{\partial\Sigma} \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 \iint_{\Sigma} \mathbf{J}\cdot d\mathbf{S} + \mu_0\varepsilon_0 \frac{d}{dt}\iint_{\Sigma} \mathbf{E}\cdot d\mathbf{S}$	$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial \mathbf{E}}{\partial t}\right)$	The magnetic field induced around a closed loop is proportional to the electric current plus displacement current (rate of change of electric field) it encloses.

- Ω is any fixed volume with closed boundary surface $\partial\Omega$, and
- Σ is any fixed surface with closed boundary curve $\partial\Sigma$,

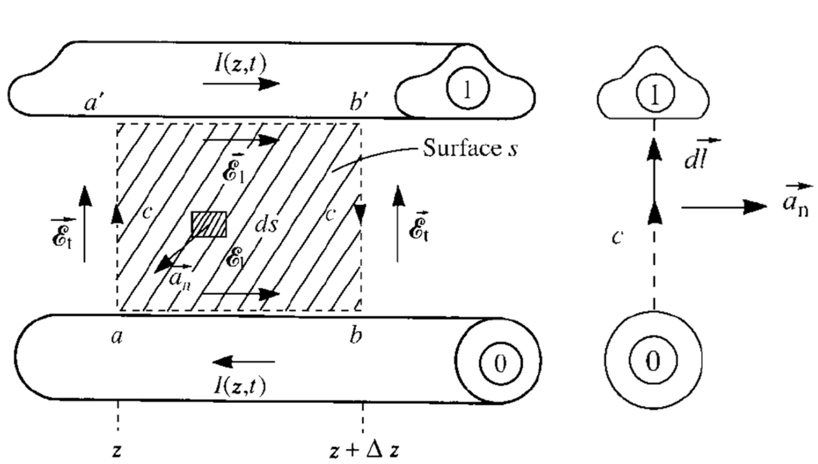
Recall Faraday's law in integral



$$\oint_c \vec{\mathscr{E}} \cdot d\vec{l} = -\mu \frac{d}{dt} \int_s \vec{\mathscr{H}} \cdot d\vec{s}$$

Faraday equation: a time-varying magnetic field will always accompany a spatially varying, non-conservative electric field, and vice versa.

The First Telegrapher's Equation



Definition of the contour c and associated surface s for derivation of the first transmission-line equation

The First Telegrapher's Equation

Faraday's law around s

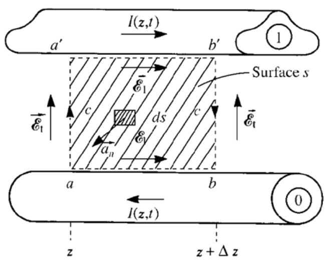
$$\int_a^{a'} \vec{\mathscr{E}}_t \cdot d\vec{l} + \int_{a'}^{b'} \vec{\mathscr{E}}_l \cdot d\vec{l} + \int_{b'}^{b} \vec{\mathscr{E}}_t \cdot d\vec{l} + \int_b^a \vec{\mathscr{E}}_l \cdot d\vec{l} = \mu \frac{d}{dt} \int_s \vec{\mathscr{H}}_t \cdot \vec{a}_n \, ds$$

The voltage between two conductors

$$V(z,t) = -\int_a^{a'} \vec{\mathscr{E}}_t(x,y,z,t) \cdot d\vec{l}$$

$$V(z+\Delta z,t) = -\int_b^{b'} \vec{\mathscr{E}}_t(x,y,z+\Delta z,t) \cdot d\vec{l}$$

Define the per-unit-length conductor resistance of each conductor as r_1 Ω/m and r_0 Ω/m.



$$-\int_{a'}^{b'} \vec{\mathscr{E}}_l \cdot d\vec{l} = -r_1 \, \Delta z \, I(z,t)$$

$$-\int_b^a \vec{\mathscr{E}}_l \cdot d\vec{l} = -r_0 \, \Delta z \, I(z,t)$$

The First Telegrapher's Equation

The current for TEM field structure as $I(z,t) = \oint_{c'} \vec{\mathscr{H}}_t \cdot d\vec{l}'$

Faraday's law equation becomes

$$-V(z,t) + r_1 \, \Delta z \, I(z,t) + V(z+\Delta z,t) + r_0 \Delta z \; I(z,t) = \mu \frac{d}{dt} \int_s \vec{\mathscr{H}}_t \cdot \vec{a}_n \, ds$$

Dividing both sides by z and rearranging gives

$$\frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = -r_1 I(z,t) - r_0 I(z,t) + \mu \frac{1}{\Delta z} \frac{d}{dt} \int_s \vec{\mathscr{H}}_t \cdot \vec{a}_n \, ds$$

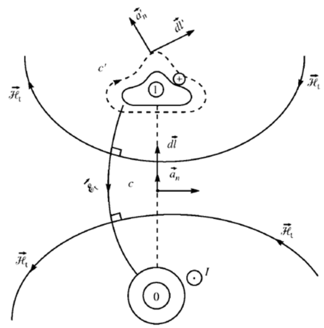
The magnetic flux penetrating the surface per unit of line length

$$\psi = -\mu \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_s \vec{\mathscr{H}}_t \cdot \vec{a}_n \, ds$$

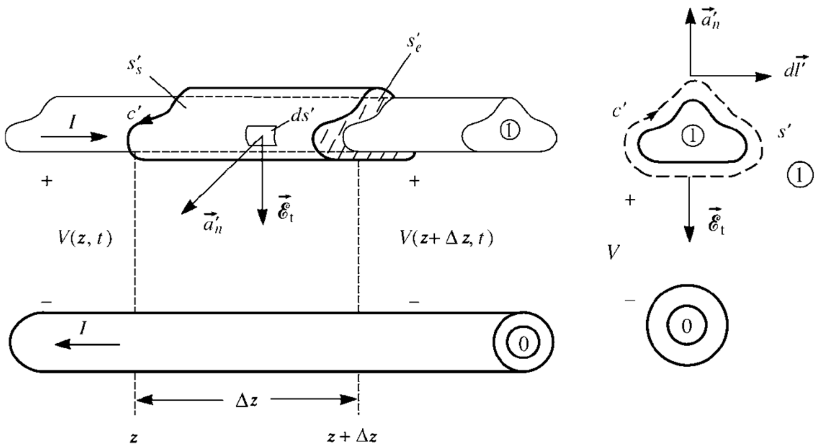
$$= l \; I(z,t)$$

$\Delta z \to 0$

$$\boxed{\frac{\partial V(z,t)}{\partial z} = -r \, I(z,t) - l \, \frac{\partial I(z,t)}{\partial t}}$$



The Second Telegrapher's Equation



Definition of the contour c' and surface s' for derivation of the second MTL equation

The Second Telegrapher's Equation

The equation of conservation of charge: the change in the amount of electric charge in any volume of space is exactly equal to the amount of charge flowing into the volume minus the amount of charge flowing out of the volume

$$\oint_{s'} \vec{\mathscr{J}} \cdot d\vec{s}' = -\frac{d}{dt} Q_{\text{enc}}$$

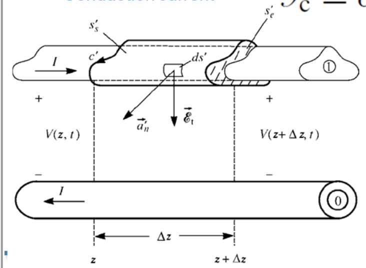
Q_{enc} is the total charge in the closed surface

Over the end caps
$$\iint_{s'_e} \vec{\mathscr{J}} \cdot d\vec{s}' = I(z+\Delta z, t) - I(z, t)$$

Over the sides of the surface

Conduction current $\vec{\mathscr{J}}_c = \sigma \vec{\mathscr{E}}_t$

Displacement current $\vec{\mathscr{J}}_d = \varepsilon(\partial \vec{\mathscr{E}}_t / \partial t)$



The transverse conduction current

$$\iint_{s'_s} \vec{\mathscr{J}}_c \cdot d\vec{s}' = \sigma \iint_{s'_s} \vec{\mathscr{E}}_t \cdot d\vec{s}'$$

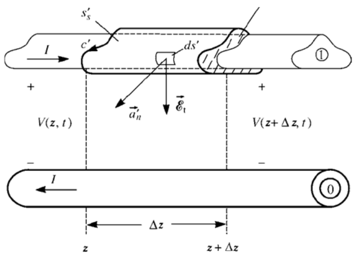
The Second Telegrapher's Equation

Considering per-unit-length conductance $gV(z,t) = \sigma \lim_{\Delta z \to 0} \frac{1}{\Delta z} \iint_{s'_s} \vec{\mathscr{E}}_t \cdot d\vec{s}'$

The charge enclosed by the surface $Q_{\text{enc}} = \varepsilon \iint_{s'_s} \vec{\mathscr{E}}_t \cdot d\vec{s}'$

the per-unit-length capacitance c $c\,V(z,t) = \varepsilon \lim_{\Delta z \to 0} \frac{1}{\Delta z} \iint_{s'_s} \vec{\mathscr{E}}_t \cdot d\vec{s}'$

$$\frac{I(z+\Delta z, t) - I(z,t)}{\Delta z} + \sigma \frac{1}{\Delta z} \iint_{s'_s} \vec{\mathscr{E}}_t \cdot d\vec{s}' = -\varepsilon \frac{1}{\Delta z} \frac{d}{dt} \iint_{s'_s} \vec{\mathscr{E}}_t \cdot d\vec{s}'$$



Let $\Delta z \to 0$

$$\boxed{\frac{\partial I(z,t)}{\partial z} = -g\,V(z,t) - c\,\frac{\partial V(z,t)}{\partial t}}$$

Telegrapher's Equation derived from different views

EM Field

$$\frac{\partial V(z,t)}{\partial z} = -rI(z,t) - l\frac{\partial I(z,t)}{\partial t}$$

$$\frac{\partial I(z,t)}{\partial z} = -gV(z,t) - c\frac{\partial V(z,t)}{\partial t}$$

Circuit

$$-\frac{\partial v(z,t)}{\partial z} = R'\,i(z,t) + L'\,\frac{\partial i(z,t)}{\partial t}$$

$$-\frac{\partial i(z,t)}{\partial z} = G'\,v(z,t) + C'\,\frac{\partial v(z,t)}{\partial t}$$

Success is not final, failure is not fatal: it is the courage to continue that counts.

---Winston Churchill

Properties of the Per-Unit-Length Parameters

■ Quick question.

derive the following parameters satisfy the relations:

$$lc = \mu\varepsilon \qquad gl = \sigma\mu$$

$$\frac{g}{c} = \frac{\sigma}{\varepsilon}$$

Properties of the Per-Unit-Length Parameters

The first transmission-line equation: $\frac{\partial V(z,t)}{\partial z} = -l\frac{\partial I(z,t)}{\partial t}$

The second transmission-line equati $\frac{\partial I(z,t)}{\partial z} = -gV(z,t) - c\frac{\partial V(z,t)}{\partial t}$

$$\frac{\partial^2 V(z,t)}{\partial z^2} = gl\frac{\partial V(z,t)}{\partial t} + lc\frac{\partial^2 V(z,t)}{\partial t^2}$$

$$\frac{\partial^2 I(z,t)}{\partial z^2} = gl\frac{\partial I(z,t)}{\partial t} + lc\frac{\partial^2 I(z,t)}{\partial t^2}$$

$$gl = \mu\sigma$$

Compare

$$lc = \mu\varepsilon$$

The transverse field vect satisfy the equation

$$\frac{\partial^2 \vec{\mathscr{E}}_t}{\partial z^2} = \mu\sigma\frac{\partial \vec{\mathscr{E}}_t}{\partial t} + \mu\varepsilon\frac{\partial^2 \vec{\mathscr{E}}_t}{\partial t^2}$$

$$\frac{\partial^2 \vec{\mathscr{H}}_t}{\partial z^2} = \mu\sigma\frac{\partial \vec{\mathscr{H}}_t}{\partial t} + \mu\varepsilon\frac{\partial^2 \vec{\mathscr{H}}_t}{\partial t^2}$$

Integrate

$$I(z,t) = \oint_{c'} \vec{\mathscr{H}}_t \cdot d\vec{l}'$$

$$V(z,t) = -\int_{c} \vec{\mathscr{E}}_t \cdot d\vec{l}$$

$$\frac{\partial^2 V(z,t)}{\partial z^2} = \mu\sigma\frac{\partial V(z,t)}{\partial t} + \mu\varepsilon\frac{\partial^2 V(z,t)}{\partial t^2}$$

$$\frac{\partial^2 I(z,t)}{\partial z^2} = \mu\sigma\frac{\partial I(z,t)}{\partial t} + \mu\varepsilon\frac{\partial^2 I(z,t)}{\partial t^2}$$

Properties of the Per-Unit-Length Parameters

$$c = \varepsilon\frac{\oint_{c'} \vec{\mathscr{E}}_t \cdot \vec{a}'_n dl'}{-\int_{c} \vec{\mathscr{E}}_t \cdot d\vec{l}}$$

$$l = -\mu\frac{\int_{c} \vec{\mathscr{H}}_t \cdot \vec{a}_n dl}{\oint_{c'} \vec{\mathscr{H}}_t \cdot d\vec{l}'}$$

$$lc = -\mu\frac{\int_{c} \vec{\mathscr{H}}_t \cdot \vec{a}_n dl}{\oint_{c'} \vec{\mathscr{H}}_t \cdot d\vec{l}'}\varepsilon\frac{\oint_{c'} \vec{\mathscr{E}}_t \cdot \vec{a}'_n dl'}{-\int_{c} \vec{\mathscr{E}}_t \cdot d\vec{l}}$$

subscribe

$$\vec{\mathscr{E}}_t^{\pm} = \mp\eta(\vec{a}_z \times \vec{\mathscr{H}}_t^{\pm})$$

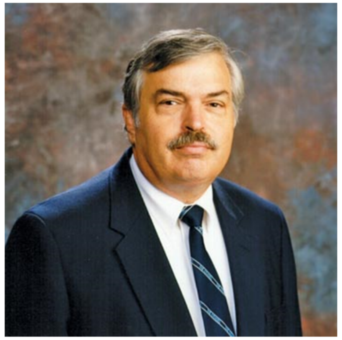
$$\vec{\mathscr{H}}_t^{\pm} = \pm\frac{1}{\eta}(\vec{a}_z \times \vec{\mathscr{E}}_t^{\pm})$$

$$lc = -\mu\frac{\int_{c} \pm\frac{1}{\eta}(\vec{a}_z \times \vec{\mathscr{E}}_t^{\pm}) \cdot \vec{a}_n dl}{\oint_{c'} \vec{\mathscr{H}}_t^{\pm} \cdot d\vec{l}'}\varepsilon\frac{\oint_{c'} \mp\eta(\vec{a}_z \times \vec{\mathscr{H}}_t^{\pm}) \cdot \vec{a}'_n dl'}{-\int_{c} \vec{\mathscr{E}}_t^{\pm} \cdot d\vec{l}}$$

$$lc = -\mu\frac{\pm\int_{c} \vec{\mathscr{E}}_t^{\pm} \cdot d\vec{l}}{\oint_{c'} \vec{\mathscr{H}}_t^{\pm} \cdot d\vec{l}'}\varepsilon\frac{\pm\oint_{c'} \vec{\mathscr{H}}_t^{\pm} \cdot d\vec{l}'}{-\int_{c} \vec{\mathscr{E}}_t^{\pm} \cdot d\vec{l}}$$

$$= \mu\varepsilon$$

In Memoriam - Clayton R. Paul



Clayton R. Paul (1941-2012)

The EMC community has lost an icon – educator and author, Dr. Clayton R. Paul. Clayton passed away June 27, 2012 at the age of 70. He was born September 6, 1941 in Macon, GA. He received his BSEE at the Citadel in 1963, his MSEE at Georgia Tech in 1964 and PhD in EE from Purdue in 1970. After serving on a Post Doctoral Fellowship at Rome Air Development Center (1970-1971), he went on to teach at the University of Kentucky, where he eventually retired in 1998. He moved on to Mercer University School of Engineering, where he again retired in 2012.

Through his 49 year career teaching engineering, he published 19 textbooks; He also published over 200 technical papers. He's a Life Fellow of the IEEE, and Honorary Life Member of the IEEE EMC Society and was awarded the prestigious 2005 IEEE Electromagnetics Award. He was also award the IEEE EMC Society's Hall of Fame Award in 2011.

Thank you !