Lecture 10: Solutions for Telegrapher's equations excited by Lumped Source : Chain Parameter Matrix

Yan-zhao XIE

Xi'an Jiaotong University

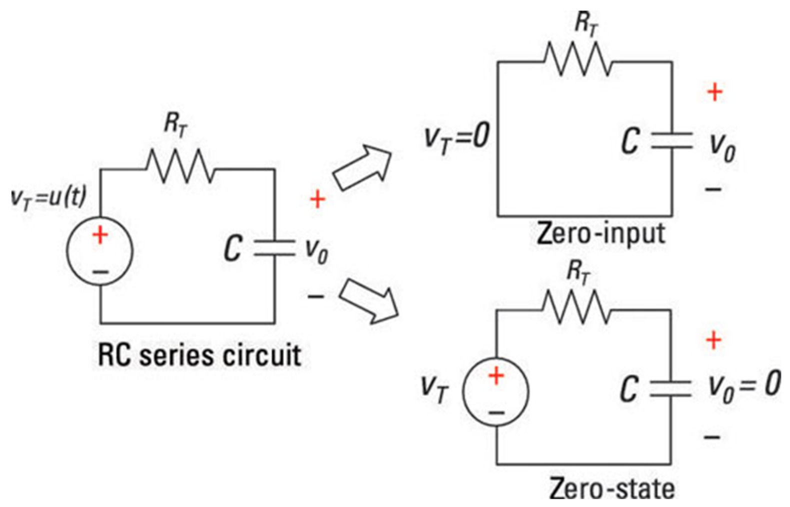
2020.09.29

■ System response

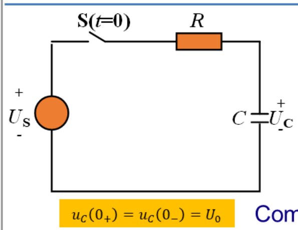
– Natural + Forced

– Transient + Steady-state

– Zero-input + Zero-state



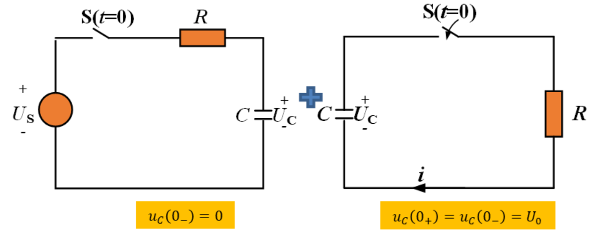
Recall: Response of first-order differential circuit



$$RC\frac{du_C(t)}{dt}+u_C(t)=U_s$$ first-order nonhomogeneous differential equation

$$\boxed{u_C(t)}=U_s+[U_0-U_s]e^{-\frac{t}{\tau}}=\boxed{U_s\left(1-e^{-\frac{t}{\tau}}\right)}+\boxed{U_0e^{-\frac{t}{\tau}}}$$

Complete response Zero state response Zero input response



The zero state response (ZSR) is the behavior or response of a circuit with initial state of zero.

The **zero input response** (**ZIR**) is the behavior or response of a circuit without external excitation source (forcing source).

https://www.dummies.com/education/science/science-electronics/find-the-zero-input-and-zero-state-responses-of-a-series-rc-circuit/

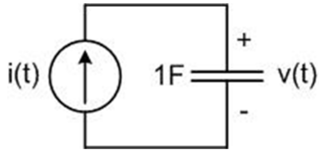
Two-port representation of TL

Similarly, when a function is put into a ***linear time-invariant (LTI)*** system, an output can be characterized by a ***superposition or sum*** of the ***zero input response*** and the ***zero state response***.

$f(t) \longrightarrow$ system $\longrightarrow$ $y(t) = y(t_0) + \cancel{\int_{t_0}^{t} f(\tau)d\tau}$

Zero input response　Zero state response

input　output

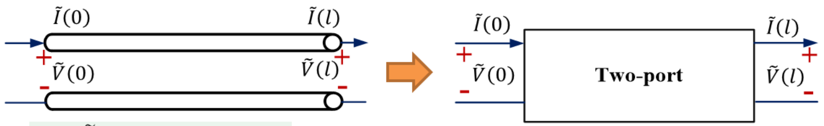


$$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau$$

Two-port representation of TL

The independent variable is time t for a ***linear time-invariant (LTI)*** system whereas the independent variable now is the line axis variable z.

Since we are generally not interested in the line voltages and currents at points along the line other than at the ***terminations***, we can regard the line as a two-port circuit.



$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\,\tilde{I}(z),$$
$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\,\tilde{V}(z).$$

$$\frac{d}{dz}\hat{\mathbf{X}}(z) = \hat{\mathbf{A}}\,\hat{\mathbf{X}}(z)$$

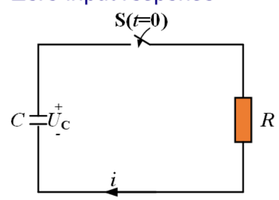
$Y = G' + j\omega C'$　$Z = R' + j\omega L'$

$$\hat{\mathbf{X}}(z) = \begin{bmatrix}\hat{\mathbf{V}}(z)\\ \hat{\mathbf{I}}(z)\end{bmatrix} \qquad \hat{\mathbf{A}} = \begin{bmatrix}\mathbf{0} & -\hat{\mathbf{Z}}\\ -\hat{\mathbf{Y}} & \mathbf{0}\end{bmatrix}$$

first-order homogeneous differential equation

Two-port representation of TL

Zero input response



$$RC\frac{du_C(t)}{dt} + u_C(t) = \boxed{0}$$

$$u_C(t) = u_C(0_+)e^{-\frac{1}{RC}t} = U_0 e^{-\frac{1}{RC}t}$$

The compact matrix form of the Telegrapher's equations are identical in form to the state-variable equations that describe some linear systems but the independent variable is the line axis variable z instead of time t.

$$\frac{d}{dz}\hat{\mathbf{X}}(z) = \hat{\mathbf{A}}\,\hat{\mathbf{X}}(z)$$

$$\begin{bmatrix}\tilde{V}(l)\\ \tilde{I}(l)\end{bmatrix} = \begin{bmatrix}\phi_{11}(l) & \phi_{12}(l)\\ \phi_{21}(l) & \phi_{22}(l)\end{bmatrix}\begin{bmatrix}\tilde{V}(0)\\ \tilde{I}(0)\end{bmatrix}$$

$Z = R' + j\omega L'$　$Y = G' + j\omega C'$

$$\begin{bmatrix}\tilde{V}(l)\\ \tilde{I}(l)\end{bmatrix} = \underbrace{\begin{bmatrix}\tilde{\phi}_{11}(l) & \tilde{\phi}_{12}(l)\\ \tilde{\phi}_{21}(l) & \tilde{\phi}_{22}(l)\end{bmatrix}}_{\tilde{\Phi}(l)}\begin{bmatrix}\tilde{V}(0)\\ \tilde{I}(0)\end{bmatrix}$$

Chain-parameter matrix

Chain-parameter matrix

To obtain chain-parameter matrix, calculating the general solution at $z=0$ and $z=l$.

$$\tilde V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$\tilde I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

$$\begin{bmatrix}\tilde V(0)\\ \tilde I(0)\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1/Z_0 & -1/Z_0\end{bmatrix}\begin{bmatrix}V_0^+\\ V_0^-\end{bmatrix} \quad \text{(a)}$$

$$\begin{bmatrix}\tilde V(l)\\ \tilde I(l)\end{bmatrix} = \begin{bmatrix}e^{-\gamma l} & e^{\gamma l}\\ e^{-\gamma l}/Z_0 & -e^{\gamma l}/Z_0\end{bmatrix}\begin{bmatrix}V_0^+\\ V_0^-\end{bmatrix} \quad \text{(b)}$$

Substituting (a) into (b):

$$\begin{bmatrix}\tilde V(l)\\ \tilde I(l)\end{bmatrix} = \begin{bmatrix}e^{-\gamma l} & e^{\gamma l}\\ e^{-\gamma l}/Z_0 & -e^{\gamma l}/Z_0\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & Z_0\\ 1 & -Z_0\end{bmatrix}\begin{bmatrix}\tilde V(0)\\ \tilde I(0)\end{bmatrix}$$

$$= \begin{bmatrix}\dfrac{e^{\gamma l}+e^{-\gamma l}}{2} & -Z_0\dfrac{e^{\gamma l}-e^{-\gamma l}}{2}\\ -\dfrac{1}{Z_0}\dfrac{e^{\gamma l}-e^{-\gamma l}}{2} & \dfrac{e^{\gamma l}+e^{-\gamma l}}{2}\end{bmatrix}\begin{bmatrix}\tilde V(0)\\ \tilde I(0)\end{bmatrix}$$

The Chain-parameter is

$$\tilde\phi_{11}(l) = \frac{e^{\gamma l}+e^{-\gamma l}}{2} = cos\gamma l \qquad \tilde\phi_{12}(l) = -Z_0\frac{e^{\gamma l}-e^{-\gamma l}}{2} = -jZ_0 sin\gamma l$$

$$\tilde\phi_{21}(l) = -\frac{1}{Z_0}\frac{e^{\gamma l}-e^{-\gamma l}}{2} = -j\frac{sin\gamma l}{Z_0} \qquad \tilde\phi_{22}(l) = \frac{e^{\gamma l}+e^{-\gamma l}}{2} = cos\gamma l$$

It can also be deduced that: $\tilde\Phi(l)\tilde\Phi^{-1}(l) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \tilde\Phi(l)\tilde\Phi(-l)$

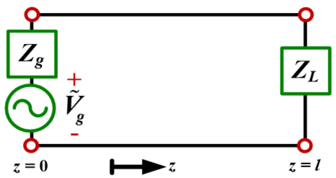
$$\tilde\Phi^{-1}(l) = \tilde\Phi(-l) \Rightarrow \tilde\Phi^{-1}(l) = \tilde\Phi(-l) = \begin{bmatrix}\tilde\phi_{11}(l) & -\tilde\phi_{12}(l)\\ -\tilde\phi_{21}(l) & \tilde\phi_{22}(l)\end{bmatrix}$$

We can get the inverse relationship:

$$\begin{bmatrix}\tilde V(0)\\ \tilde I(0)\end{bmatrix} = \tilde\Phi(-l)\begin{bmatrix}\tilde V(l)\\ \tilde I(l)\end{bmatrix}$$

It is a simple reversal of the line axis scale (replacing z with -z) similar to the reversal in time for the state-transition matrix of lumped systems .

*The chain parameters only relate the voltage and current **at one end of the line** to the voltage and current **at the other end of the line**. They do not explicitly determine those voltages and currents until we incorporate the terminals conditions.*



boundary conditions:

$$\tilde V(0) = \tilde V_g - Z_g\tilde I(0) \qquad \tilde V(l) = Z_L\tilde I(l)$$

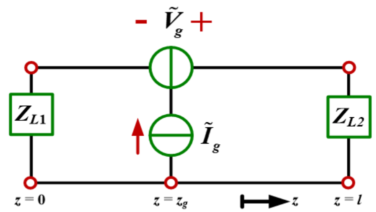
$$\tilde V(l) = Z_L\tilde I(l) = \tilde\phi_{11}[\tilde V_g - Z_g\tilde I(0)] + \tilde\phi_{12}\tilde I(0)$$

$$\tilde I(l) = \tilde\phi_{21}[\tilde V_g - Z_g\tilde I(0)] + \tilde\phi_{22}\tilde I(0)$$

Then one can get:

$$\tilde I(0) = \frac{Z_L\tilde\phi_{21} - \tilde\phi_{11}}{\tilde\phi_{12} - \tilde\phi_{11}Z_g - Z_L\tilde\phi_{22} + Z_L\tilde\phi_{21}Z_g}\tilde V_g$$

$$\tilde I(l) = \tilde\phi_{21}\tilde V_g + \frac{[\tilde\phi_{22} - \tilde\phi_{21}Z_g][Z_L\tilde\phi_{21} - \tilde\phi_{11}]}{\tilde\phi_{12} - \tilde\phi_{11}Z_g - Z_L\tilde\phi_{22} + Z_L\tilde\phi_{21}Z_g}\tilde V_g$$

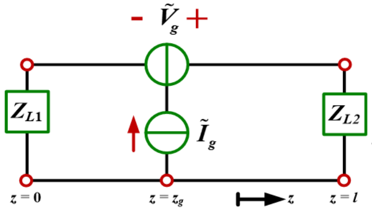


For $z < z_g$, one renames $\tilde V$, $\tilde I$ as $\tilde V_1$, $\tilde I_1$

$$\begin{bmatrix}\tilde V_1(z)\\ \tilde I_1(z)\end{bmatrix} = \tilde\Phi(z)\begin{bmatrix}\tilde V_1(0)\\ \tilde I_1(0)\end{bmatrix} \qquad \tilde\Phi(z)=\begin{bmatrix}\tilde\phi_{11}(z) & \tilde\phi_{12}(z)\\ \tilde\phi_{21}(z) & \tilde\phi_{22}(z)\end{bmatrix}$$

For $z > z_g$, one renames $\tilde V$, $\tilde I$ as $\tilde V_2$, $\tilde I_2$

$$\begin{bmatrix}\tilde V_2(z)\\ \tilde I_2(z)\end{bmatrix} = \tilde\Phi(z-l)\begin{bmatrix}\tilde V_2(l)\\ \tilde I_2(l)\end{bmatrix} \qquad \tilde\Phi(z-l) = \begin{bmatrix}\tilde\phi_{11}(z-l) & \tilde\phi_{12}(z-l)\\ \tilde\phi_{21}(z-l) & \tilde\phi_{22}(z-l)\end{bmatrix}$$



Terminal 1: load Z_{L1} $\quad \tilde V_1(0) = -\tilde I_1(0)\cdot Z_{L1}$

Terminal 2: load Z_{L2} $\quad \tilde V_2(l) = \tilde I_2(l)\cdot Z_{L2}$

Voltage difference: source $\tilde V_g$ $\quad \tilde V_1(z_g) + \tilde V_g = \tilde V_2(z_g)$

Current difference: source $\tilde I_g$ $\quad \tilde I_1(z_g) + \tilde I_g = \tilde I_2(z_g)$

Then one can get:

$$(\tilde\phi_{11}(z_g) - \tilde\phi_{12}(z_g)/Z_{L1})\tilde V_1(0) + \tilde V_g = (\tilde\phi_{11}(z_g - l) + \tilde\phi_{12}(z_g - l)/Z_{L2})\,\tilde V_2(l)$$

$$(\tilde\phi_{21}(z_g) - \tilde\phi_{22}(z_g)/Z_{L1})\tilde V_1(0) + \tilde I_g = (\tilde\phi_{21}(z_g - l) + \tilde\phi_{22}(z_g - l)/Z_{L2})\,\tilde V_2(l)$$

For $z = l$,

$$\tilde V_2(l) = \frac{(1+\Gamma_{L2})e^{-\gamma l}}{2(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left[(e^{\gamma z_g} - \Gamma_{L1}e^{-\gamma z_g})\tilde V_g + (e^{\gamma z_g} + \Gamma_{L1}e^{-\gamma z_g})Z_0\tilde I_g\right]$$

$$\tilde I_2(l) = \frac{(1-\Gamma_{L2})e^{-\gamma l}}{2Z_0(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left[(e^{\gamma z_g} - \Gamma_{L1}e^{-\gamma z_g})\tilde V_g + (e^{\gamma z_g} + \Gamma_{L1}e^{-\gamma z_g})Z_0\tilde I_g\right]$$

For $z = 0$,

$$\tilde V_1(0) = \frac{(1+\Gamma_{L1})e^{-\gamma l}}{2(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left\{-\left[e^{\gamma(l-z_g)} - \Gamma_{L2}e^{-\gamma(l-z_g)}\right]\tilde V_g + \left[e^{\gamma(l-z_g)} + \Gamma_{L2}e^{-\gamma(l-z_g)}\right]Z_0\tilde I_g\right\}$$

$$\tilde I_1(0) = \frac{(1-\Gamma_{L1})e^{-\gamma l}}{2Z_0(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left\{-\left[e^{\gamma(l-z_g)} - \Gamma_{L2}e^{-\gamma(l-z_g)}\right]\tilde V_g + \left[e^{\gamma(l-z_g)} + \Gamma_{L2}e^{-\gamma(l-z_g)}\right]Z_0\tilde I_g\right\}$$

One can obtain $\tilde V(z)$, $\tilde I(z)$ by $\begin{bmatrix}\tilde V_1(z)\\ \tilde I_1(z)\end{bmatrix} = \tilde\Phi(z)\begin{bmatrix}\tilde V_1(0)\\ \tilde I_1(0)\end{bmatrix}$ $\quad \begin{bmatrix}\tilde V_2(z)\\ \tilde I_2(z)\end{bmatrix} = \tilde\Phi(z-l)\begin{bmatrix}\tilde V_2(l)\\ \tilde I_2(l)\end{bmatrix}$

For $z > z_g$,

$$\tilde V_2(z) = \frac{e^{-\gamma z} + \Gamma_{L2}e^{\gamma(z-2l)}}{2(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left[(e^{\gamma z_g} - \Gamma_{L1}e^{-\gamma z_g})\tilde V_g + (e^{\gamma z_g} + \Gamma_{L1}e^{-\gamma z_g})Z_0\tilde I_g\right]$$

$$\tilde I_2(z) = \frac{e^{-\gamma z} - \Gamma_{L2}e^{\gamma(z-2l)}}{2Z_0(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left[(e^{\gamma z_g} - \Gamma_{L1}e^{-\gamma z_g})\tilde V_g + (e^{\gamma z_g} + \Gamma_{L1}e^{-\gamma z_g})Z_0\tilde I_g\right]$$

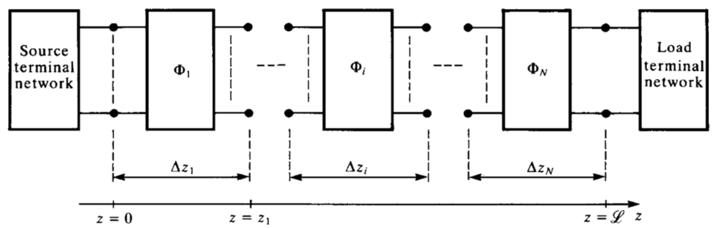
For $z < z_g$,

$$\tilde V_1(z) = \frac{e^{\gamma(z-l)} + \Gamma_{L1}e^{-\gamma(z+l)}}{2(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left\{-\left[e^{\gamma(l-z_g)} - \Gamma_{L2}e^{-\gamma(l-z_g)}\right]\tilde V_g + \left[e^{\gamma(l-z_g)} + \Gamma_{L2}e^{-\gamma(l-z_g)}\right]Z_0\tilde I_g\right\}$$

$$\tilde I_1(z) = \frac{e^{\gamma(z-l)} - \Gamma_{L1}e^{-\gamma(z+l)}}{2Z_0(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left\{-\left[e^{\gamma(l-z_g)} - \Gamma_{L2}e^{-\gamma(l-z_g)}\right]\tilde V_g + \left[e^{\gamma(l-z_g)} + \Gamma_{L2}e^{-\gamma(l-z_g)}\right]Z_0\tilde I_g\right\}$$

Product of Chain-parameter matrices

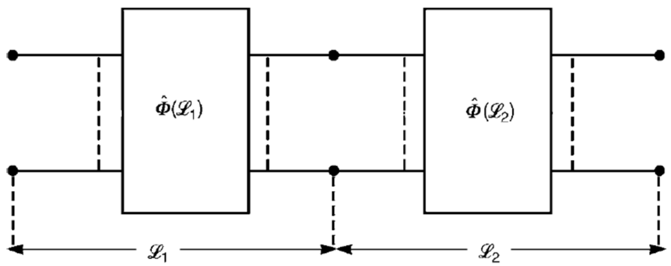
The overall chain-parameter matrix of several such lines that are *cascaded* in series can be obtained as the product of the chain-parameter matrices of the sections.



$$\begin{bmatrix}\hat V(z_{k+1})\\ \hat I(z_{k+1})\end{bmatrix} = \hat\Phi_{k+1}(\Delta z_{k+1})\begin{bmatrix}\hat V(z_k)\\ \hat I(z_k)\end{bmatrix} \qquad \begin{bmatrix}\hat V(z_k)\\ \hat I(z_k)\end{bmatrix} = \hat\Phi_k(\Delta z_k)\begin{bmatrix}\hat V(z_{k-1})\\ \hat I(z_{k-1})\end{bmatrix}$$

$$\hat\Phi(\mathscr{L}) = \hat\Phi_N(\Delta z_N)\times\cdots\times\hat\Phi_i(\Delta z_i)\times\cdots\times\hat\Phi_1(\Delta z_1) = \prod_{k=1}^{N}\hat\Phi_{N-k+1}(\Delta z_{N-k+1})$$

That's why we call "chain"!



$$\hat{\Phi}(\mathscr{L}_1 + \mathscr{L}_2) = \hat{\Phi}(\mathscr{L}_2)\hat{\Phi}(\mathscr{L}_1)$$

$$\begin{aligned}\hat{\Phi}(\mathscr{L})\hat{\Phi}(-\mathscr{L}) &= \hat{\Phi}\left(\mathscr{L} + (-\mathscr{L})\right) \\ &= \hat{\Phi}(0) \\ &= \mathbf{1}_{2n}\end{aligned}$$

?

quote

"History will be kind to me for I intend to write it."

\- Winston Churchill

Thank you again!