Lecture 5: General Solution of Telegrapher's Equations: Wave Propagation

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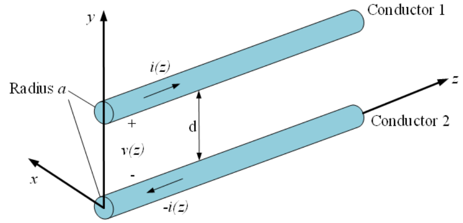
Recall: Time Domain Telegrapher's Equations



$$-\frac{\partial v(z,t)}{\partial z} = R'\, i(z,t) + L' \frac{\partial i(z,t)}{\partial t}$$

$$-\frac{\partial i(z,t)}{\partial z} = G'\, v(z,t) + C' \frac{\partial v(z,t)}{\partial t}$$

For the solution of the ***two-conductor*** transmission-line equations,

- The excitation sources for the line are ***single frequency sinusoidal waveforms*** (in steady state), such as $x(t) = X\cos(\omega t + \theta X)$
- The analysis method is the ***phasor technique*** of electric circuit theory

$$x(t) \Leftrightarrow \hat{X}(j\omega)$$

$$\hat{X}(j\omega) = X\angle\theta_X = X e^{j\theta_X}$$

Recall: Time Domain Telegrapher's Equations

Substituting above equations into Time Domain Telegrapher's Equations, and utilizing

$$\frac{\partial}{\partial t} \Leftrightarrow j\omega$$

We obtain ***Telegrapher's Equations in frequency domain***

$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\,\tilde{I}(z),$$

$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\,\tilde{V}(z).$$

- Observe that the frequency-domain transmission-line equations become ***ordinary differential*** equations
- Only one variable, the line axis variable z
- The solution of the transmission-line equations becomes ***simpler***

Solution 1:

Traveling wave

The two first order **coupled** equations

$$-\frac{d\boxed{\tilde{V}(z)}}{dz}=(R'+j\omega L')\boxed{\tilde{I}(z)} \qquad \textbf{(a)}$$

$$-\frac{d\boxed{\tilde{I}(z)}}{dz}=(G'+j\omega C')\boxed{\tilde{V}(z)} \qquad \textbf{(b)}$$

How to decouple the 1st order coupled equation?

Differentiate both sides of **(a)** with respect to z.

$$-\frac{d^2\tilde{V}(z)}{dz^2}=(R'+j\omega L')\frac{d\tilde{I}(z)}{dz}$$

Substitute **(b)** for $d\tilde{I}(z)/dz$,

$$\frac{d^2\tilde{V}(z)}{dz^2}-(R'+j\omega L')(G'+j\omega C')\tilde{V}(z)=0$$

Wave equations

Second order **uncoupled** equations : wave equations

$$\frac{d^2\tilde{V}(z)}{dz^2}-\gamma^2\tilde{V}(z)=0, \qquad \frac{d^2\tilde{I}(z)}{dz^2}-\gamma^2\tilde{I}(z)=0.$$

(wave equation for $\tilde{V}(z)$) **(wave equation for $\tilde{I}(z)$)**

$$\gamma=\sqrt{(R'+j\omega L')(G'+j\omega C')}=\sqrt{Z'\cdot Y'}$$

(propagation constant)

$Z'=R'+j\omega L'$

$Y'=G'+j\omega C'$

Complex propagation constant

$$\gamma=\alpha+j\beta$$

$\alpha=\Re\mathfrak{e}(\gamma)$ Unit: Np/m

$$=\Re\mathfrak{e}\left(\sqrt{(R'+j\omega L')(G'+j\omega C')}\right)$$

(attenuation constant)

$\beta=\Im\mathfrak{m}(\gamma)$ Unit: rad/m

$$=\Im\mathfrak{m}\left(\sqrt{(R'+j\omega L')(G'+j\omega C')}\right)$$

(phase constant)

lossless transmission line: $\alpha=0$ lossy transmission line: $\alpha\neq 0$

Traveling wave solutions

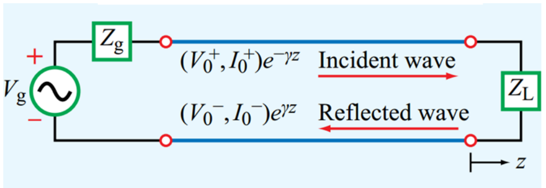
$$\frac{d^2\tilde{V}(z)}{dz^2}-\gamma^2\tilde{V}(z)=0, \qquad \frac{d^2\tilde{I}(z)}{dz^2}-\gamma^2\tilde{I}(z)=0.$$

(wave equation for $\tilde{V}(z)$) **(wave equation for $\tilde{I}(z)$)**

The general solution (traveling wave solutions) to the above coupled second-order equations is

$$\tilde{V}(z)=V_0^+e^{-\gamma z}+V_0^-e^{\gamma z} \qquad \text{(V)},$$

$$\tilde{I}(z)=I_0^+e^{-\gamma z}+I_0^-e^{\gamma z} \qquad \text{(A)}.$$



Two traveling wave
- Incident wave $+z$ direction: $e^{-\gamma z}$ amplitudes: (V_0^+, I_0^+)
- Reflected wave $-z$ direction: $e^{\gamma z}$ amplitudes: (V_0^-, I_0^-)

Characteristic impedance

$$-\frac{d\tilde{V}(z)}{dz}=(R'+j\omega L')\tilde{I}(z)$$

$$\tilde{V}(z)=V_0^+e^{-\gamma z}+V_0^-e^{\gamma z}$$

$$\Rightarrow \quad \tilde{I}(z)=\frac{\gamma}{R'+j\omega L'}[V_0^+e^{-\gamma z}-V_0^-e^{\gamma z}]$$

Comparing each term with the corresponding term in $\tilde{I}(z)=I_0^+e^{-\gamma z}+I_0^-e^{\gamma z}$

$$\frac{V_0^+}{I_0^+}=Z_0=\frac{-V_0^-}{I_0^-} \quad \text{where} \quad Z_0=\frac{R'+j\omega L'}{\gamma}=\sqrt{\frac{R'+j\omega L'}{G'+j\omega C'}} \quad (\Omega)$$

characteristic impedance

▶ It should be noted that Z_0 is equal to the ratio of the voltage amplitude to the current amplitude for each of the traveling waves individually (with an additional minus sign in the case of the $-z$ propagating wave), but it is not equal to the ratio of the total voltage $\tilde{V}(z)$ to the total current $\tilde{I}(z)$, unless one of the two waves is absent. ◀

traveling wave solutions

$$\tilde{V}(z)=V_0^+e^{-\gamma z}+V_0^-e^{\gamma z}$$

$$\tilde{I}(z)=\frac{V_0^+}{Z_0}e^{-\gamma z}-\frac{V_0^-}{Z_0}e^{\gamma z}$$

Z_0 is not equal to $\tilde{V}(z)/\tilde{I}(z)$!

Characteristic impedance

Hence,

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

Only two unknowns!

In later sections, we apply ***boundary conditions*** at the ***source*** and ***load ends*** of the transmission line to obtain expressions for the amplitudes V_0^+ and V_0^-. In general, each is a ***complex quantity*** characterized by a magnitude and a phase angle:

$$V_0^+ = |V_0^+| e^{j\phi^+}$$

$$V_0^- = |V_0^-| e^{j\phi^-}$$

we can convert back to the ***time domain*** to obtain an expression for *v(z, t)*

$$\begin{aligned} v(z,t) &= \mathfrak{Re}(\tilde{V}(z) e^{j\omega t}) \\ &= \mathfrak{Re}\left[\left(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}\right) e^{j\omega t}\right] \\ &= \mathfrak{Re}[|V_0^+| e^{j\phi^+} e^{j\omega t} e^{-(\alpha+j\beta)z} \\ &\quad + |V_0^-| e^{j\phi^-} e^{j\omega t} e^{(\alpha+j\beta)z}] \\ &= |V_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \phi^+) \\ &\quad + |V_0^-| e^{\alpha z} \cos(\omega t + \beta z + \phi^-). \end{aligned}$$

Solution 2:

Combined voltage wave

Combined voltage wave

We define ***Combined voltage wave*** as

$$\boxed{\tilde{W}(z)_\pm = \tilde{V}(z) \pm Z_0 \cdot \tilde{I}(z)}$$

forward traveling combined voltage wave

$$\tilde{W}(z)_+ = \tilde{V}(z) + Z_0 \cdot \tilde{I}(z)$$

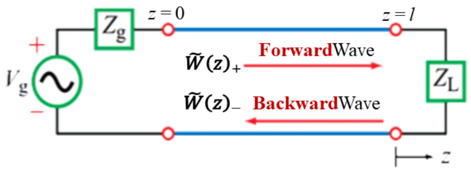
backward traveling combined voltage wave

$$\tilde{W}(z)_- = \tilde{V}(z) - Z_0 \cdot \tilde{I}(z)$$

One can obtain the following relation:

$$\tilde{V}(z) = \frac{1}{2}[\tilde{W}(z)_+ + \tilde{W}(z)_-]$$

$$Z_0 \cdot \tilde{I}(z) = \frac{1}{2}[\tilde{W}(z)_+ - \tilde{W}(z)_-]$$



It allows one to easily separate the voltage and current vectors into ***forward*** and ***backward*** waves and easily reconstruct the voltage and current vectors from the waves.

Combined voltage wave equation

We define: $\tilde{W}(z)_\pm = \tilde{V}(z) \pm Z_0 \cdot \tilde{I}(z)$

$$\frac{d\tilde{W}(z)_\pm}{dz} = \frac{d\tilde{V}(z)}{dz} \pm Z_0 \cdot \frac{d\tilde{I}(z)}{dz} = -Z' \cdot \tilde{I}(z) \mp Z_0 \cdot Y' \cdot \tilde{V}(z)$$

telegrapher's equation:

$$\frac{d\tilde{V}(z)}{dz} = -Z' \cdot \tilde{I}(z) \qquad \frac{d\tilde{I}(z)}{dz} = -Y' \cdot \tilde{V}(z)$$

$$Z' = R' + j\omega L' \qquad Y' = G' + j\omega C'$$

we also have $Z_0 \stackrel{\text{def}}{=} \sqrt{Z'/Y'}$, $\gamma = \sqrt{Z' \cdot Y'}$

Hence, we can obtain $\dfrac{d\tilde{W}(z)_\pm}{dz} = -Z_0 \cdot \gamma \cdot \tilde{I}(z) \mp \gamma \cdot \tilde{V}(z) = -\gamma \tilde{W}(z)_\pm$

$$\boxed{\frac{d\tilde{W}(z)_\pm}{dz} = -\gamma \tilde{W}(z)_\pm}$$

Combined voltage wave solutions

$$\frac{d\tilde{W}(z)_\pm}{dz} = -\gamma \tilde{W}(z)_\pm$$

Now we only need to solve 1st order equation.

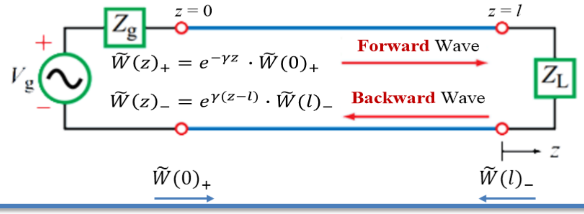
One can obtain solution of $\tilde{W}(z)_\pm$ along z

$$\tilde{W}(z)_\pm = e^{\mp\gamma(z-z_0)} \cdot \tilde{W}(z_0)_\pm$$

z_0: a reference point of z along the TL

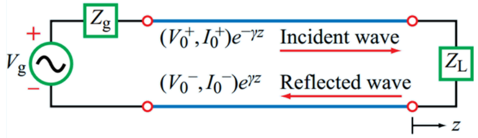
If the length of the TL is l

$$\tilde{W}(z)_+ = e^{-\gamma z} \cdot \tilde{W}(0)_+ \quad \tilde{W}(z)_- = e^{\gamma(z-l)} \cdot \tilde{W}(l)_-$$



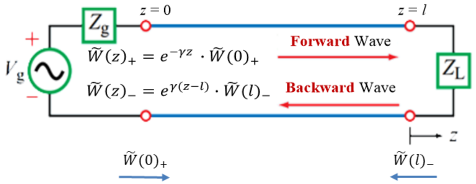
Comparison of two wave solutions

One can compare combined voltage wave solution to traveling wave solution:



$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

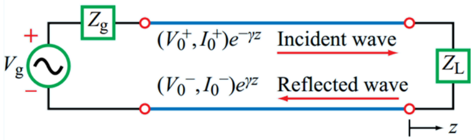
$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$



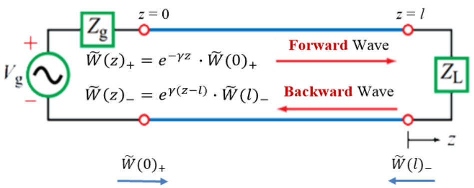
$$\tilde{V}(z) = \frac{1}{2}[\tilde{W}(z)_+ + \tilde{W}(z)_-]$$

$$\tilde{I}(z) = \frac{1}{2}[\tilde{W}(z)_+ - \tilde{W}(z)_-]/Z_0$$

Comparison of two wave solutions



$$\tilde{V}(z) = \underline{V_0^+} \cdot e^{-\gamma z} + \underline{V_0^-} \cdot e^{\gamma z}$$



$$\begin{aligned}
\tilde{V}(z) &= \frac{1}{2}[\tilde{W}(z)_+ + \tilde{W}(z)_-] \\
&= \frac{1}{2}[e^{-\gamma z} \cdot \tilde{W}(0)_+ + e^{\gamma(z-l)} \cdot \tilde{W}(l)_-] \\
&= \underline{\frac{1}{2}\tilde{W}(0)_+} \cdot e^{-\gamma z} + \underline{\frac{1}{2}e^{-\gamma l} \cdot \tilde{W}(l)_-} \cdot e^{\gamma z}
\end{aligned}$$

Thank you!