Lecture 8: Solutions for Telegrapher's equations excited by Lumped Source: Green Function

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Recall:

telegrapher's equations

$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\,\tilde{I}(z),$$
$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\,\tilde{V}(z).$$

traveling wave solutions

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$
$$\tilde{I}(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

reflection coefficient at the load

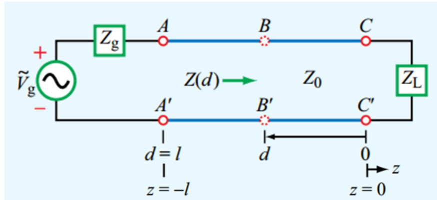
$$\Gamma_L = \frac{V_0^-}{V_0^+} e^{2\gamma z}\Big|_{z=0} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

reflection coefficient along the line

$$\Gamma(z) = \Gamma_L e^{2\gamma z}$$

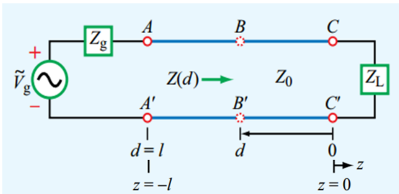
input impedance

$$Z_{in}(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)} = Z_0 \frac{1+\Gamma(z)}{1-\Gamma(z)}$$



For a point on the line, one can obtain:

$$\tilde{V}(z) = V_0^+ (e^{-\gamma z} + \Gamma_L e^{\gamma z})$$
$$\tilde{I}(z) = \frac{V_0^+}{Z_0} (e^{-\gamma z} - \Gamma_L e^{\gamma z})$$



If we can determine the undetermined constant V_0^+, then we can evaluate $\tilde{V}(z)$. Knowing boundary conditions, one can specify the exact solutions.

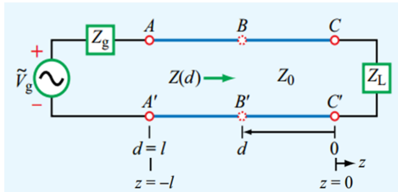
at the source terminal $\tilde{V}(-l) = \tilde{V}_g - Z_g \tilde{I}(-l)$

Substituting the boundary conditions into $\tilde{V}(z)$ and $\tilde{I}(z)$:

$$V_0^+ (e^{\gamma l} + \Gamma_L e^{-\gamma l}) = \tilde{V}_g - Z_g \frac{V_0^+}{Z_0} (e^{\gamma l} - \Gamma_L e^{-\gamma l})$$

One can define the reflection coefficient at the source terminal:

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0}$$



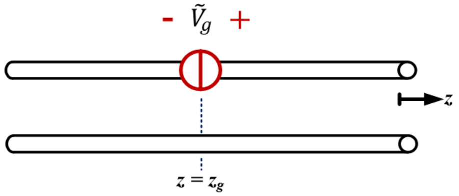
$$V_0^+ = \tilde{V}_g \frac{\dfrac{Z_0}{Z_g + Z_0} e^{-\gamma l}}{1 - \dfrac{Z_g - Z_0}{Z_g + Z_0}\Gamma_L e^{-2\gamma l}} = \tilde{V}_g \frac{\dfrac{1+\Gamma_g}{2} e^{-\gamma l}}{1 - \Gamma_g \Gamma_L e^{-2\gamma l}}$$

Finally, we can evaluate the exact solutions of $\tilde{V}(z)$ and $\tilde{I}(z)$ while the source is at the terminal of TL:

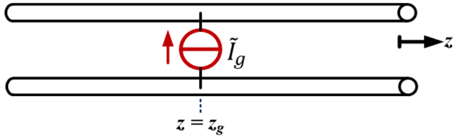
$$\tilde{V}(z) = \frac{1}{2} \frac{(1+\Gamma_g) e^{-\gamma l}}{1 - \Gamma_g \Gamma_L e^{-2\gamma l}} (e^{-\gamma z} + \Gamma_L e^{\gamma z}) \tilde{V}_g$$

$$\tilde{I}(z) = \frac{1}{2Z_0} \frac{(1+\Gamma_g) e^{-\gamma l}}{1 - \Gamma_g \Gamma_L e^{-2\gamma l}} (e^{-\gamma z} - \Gamma_L e^{\gamma z}) \tilde{V}_g$$

So far, we assume that a lumped source is on a terminal of TL. In fact a lumped source could locate at arbitrary position along TL.



Infinite length

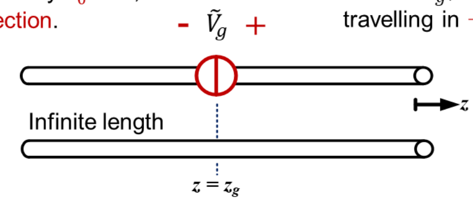


Infinite length

TL excited by a lumped voltage source

For $z<z_g$, there is only $V_0^- e^{\gamma z}$, travelling in $-z$ direction.

For $z>z_g$, there is only $V_0^+ e^{-\gamma z}$, travelling in $+z$ direction.



$$\tilde V(z)=V_0^+e^{-\gamma z}+V_0^-e^{\gamma z}$$

$$\tilde I(z)=\frac{V_0^+}{Z_0}e^{-\gamma z}-\frac{V_0^-}{Z_0}e^{\gamma z}$$

At $z=z_g$, current is continuous while there is a voltage difference $\tilde V_g$.

$$V_0^-e^{\gamma z_g}+\tilde V_g=V_0^+e^{-\gamma z_g}$$

$$-\frac{V_0^-}{Z_0}e^{\gamma z_g}=\frac{V_0^+}{Z_0}e^{-\gamma z_g}$$

$$\Rightarrow$$

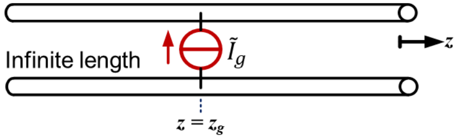
$$V_0^+=\frac{\tilde V_g}{2}e^{\gamma z_g}$$

$$V_0^-=-\frac{\tilde V_g}{2}e^{-\gamma z_g}$$

For $z>z_g$, $\quad \tilde V(z)=\frac{\tilde V_g}{2}e^{-\gamma(z-z_g)} \quad \tilde I(z)=\frac{\tilde V_g}{2Z_0}e^{-\gamma(z-z_g)}$

For $z<z_g$, $\quad \tilde V(z)=-\frac{\tilde V_g}{2}e^{\gamma(z-z_g)} \quad \tilde I(z)=\frac{\tilde V_g}{2Z_0}e^{\gamma(z-z_g)}$

TL excited by a lumped current source



At $z=z_g$, voltage is continuous while there is a current difference $\tilde I_g$.

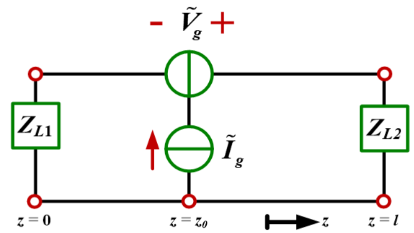
Similarly, one can obtain:

For $z>z_g$, $\quad \tilde V(z)=\frac{\tilde I_g Z_0}{2}e^{-\gamma(z-z_g)} \quad \tilde I(z)=\frac{\tilde I_g}{2}e^{-\gamma(z-z_g)}$

For $z<z_g$, $\quad \tilde V(z)=\frac{\tilde I_g Z_0}{2}e^{\gamma(z-z_g)} \quad \tilde I(z)=-\frac{\tilde I_g}{2}e^{\gamma(z-z_g)}$

For $z<z_g$, we rename $\tilde V$, $\tilde I$, V_0^- and V_0^+ as $\tilde V_1$, $\tilde I_1$, V_1^- and V_1^+

For $z>z_g$, we rename $\tilde V$, $\tilde I$, V_0^- and V_0^+ as $\tilde V_2$, $\tilde I_2$, V_2^- and V_2^+



reflection coefficient at two load

$$\Gamma_{L1}=\frac{Z_{L1}-Z_0}{Z_{L1}+Z_0}$$

$$\Gamma_{L2}=\frac{Z_{L2}-Z_0}{Z_{L2}+Z_0}$$

Terminal 1: load Z_{L1}	$\tilde V_1(0)=-\tilde I_1(0)\cdot Z_{L1}$	$\Rightarrow$	V_1^-
Terminal 2: load Z_{L2}	$\tilde V_2(l)=\tilde I_2(l)\cdot Z_{L2}$	$\Rightarrow$	V_1^+
Voltage difference: source $\tilde V_g$	$\tilde V_1(z_0)+\tilde V_g=\tilde V_2(z_0)$	$\Rightarrow$	V_2^-
Current difference: source $\tilde I_g$	$\tilde I_1(z_0)+\tilde I_g=\tilde I_2(z_0)$	$\Rightarrow$	V_2^+

For $z > z_g$,

$$\tilde V(z) = \frac{e^{-\gamma z} + \Gamma_{L2} e^{\gamma(z-2l)}}{2(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left[(e^{\gamma z_g} - \Gamma_{L1}e^{-\gamma z_g})\tilde V_g + (e^{\gamma z_g}+\Gamma_{L1}e^{-\gamma z_g})Z_0\tilde I_g\right]$$

$$\tilde I(z) = \frac{e^{-\gamma z} - \Gamma_{L2} e^{\gamma(z-2l)}}{2Z_0(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left[(e^{\gamma z_g} - \Gamma_{L1}e^{-\gamma z_g})\tilde V_g + (e^{\gamma z_g}+\Gamma_{L1}e^{-\gamma z_g})Z_0\tilde I_g\right]$$

For $z < z_g$,

$$\tilde V(z) = \frac{e^{\gamma(z-l)} + \Gamma_{L1} e^{-\gamma(z+l)}}{2(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left\{-\left[e^{\gamma(l-z_g)} - \Gamma_{L2}e^{-\gamma(l-z_g)}\right]\tilde V_g + \left[e^{\gamma(l-z_g)} + \Gamma_{L2}e^{-\gamma(l-z_g)}\right]Z_0\tilde I_g\right\}$$

$$\tilde I(z) = \frac{e^{\gamma(z-l)} - \Gamma_{L1} e^{-\gamma(z+l)}}{2Z_0(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left\{-\left[e^{\gamma(l-z_g)} - \Gamma_{L2}e^{-\gamma(l-z_g)}\right]\tilde V_g + \left[e^{\gamma(l-z_g)} + \Gamma_{L2}e^{-\gamma(l-z_g)}\right]Z_0\tilde I_g\right\}$$

For $z > z_g$,

$$\tilde G_{VV}(z) = \frac{e^{-\gamma z} + \Gamma_{L2} e^{\gamma(z-2l)}}{2(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}(e^{\gamma z_g} - \Gamma_{L1}e^{-\gamma z_g})$$

$$\tilde G_{IV}(z) = \frac{e^{-\gamma z} + \Gamma_{L2} e^{\gamma(z-2l)}}{2(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}(e^{\gamma z_g} + \Gamma_{L1}e^{-\gamma z_g})Z_0$$

$$\tilde G_{VI}(z) = \frac{e^{-\gamma z} - \Gamma_{L2} e^{\gamma(z-2l)}}{2Z_0(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}(e^{\gamma z_g} - \Gamma_{L1}e^{-\gamma z_g})$$

$$\tilde G_{II}(z) = \frac{e^{-\gamma z} - \Gamma_{L2} e^{\gamma(z-2l)}}{2Z_0(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}(e^{\gamma z_g} + \Gamma_{L1}e^{-\gamma z_g})Z_0$$

For $z < z_g$,

$$\tilde G_{VV}(z) = -\frac{e^{\gamma(z-l)} + \Gamma_{L1} e^{-\gamma(z+l)}}{2(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left[e^{\gamma(l-z_g)} - \Gamma_{L2}e^{-\gamma(l-z_g)}\right]$$

$$\tilde G_{IV}(z) = \frac{e^{\gamma(z-l)} + \Gamma_{L1} e^{-\gamma(z+l)}}{2(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left[e^{\gamma(l-z_g)} + \Gamma_{L2}e^{-\gamma(l-z_g)}\right]Z_0$$

$$\tilde G_{VI}(z) = -\frac{e^{\gamma(z-l)} - \Gamma_{L1} e^{-\gamma(z+l)}}{2Z_0(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left[e^{\gamma(l-z_g)} - \Gamma_{L2}e^{-\gamma(l-z_g)}\right]$$

$$\tilde G_{II}(z) = \frac{e^{\gamma(z-l)} - \Gamma_{L1} e^{-\gamma(z+l)}}{2Z_0(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left[e^{\gamma(l-z_g)} + \Gamma_{L2}e^{-\gamma(l-z_g)}\right]Z_0$$

$$\tilde V(z) = \tilde G_{VV}(z)\tilde V_g + \tilde G_{IV}(z)\tilde I_g$$

$$\tilde I(z) = \tilde G_{VI}(z)\tilde V_g + \tilde G_{II}(z)\tilde I_g$$

Especially, one can obtain the voltage and current at terminals.

For $z = 0$,

$$\tilde V(0) = \frac{(1+\Gamma_{L2})e^{-\gamma l}}{2(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left[(e^{\gamma z_g} - \Gamma_{L1}e^{-\gamma z_g})\tilde V_g + (e^{\gamma z_g}+\Gamma_{L1}e^{-\gamma z_g})Z_0\tilde I_g\right]$$

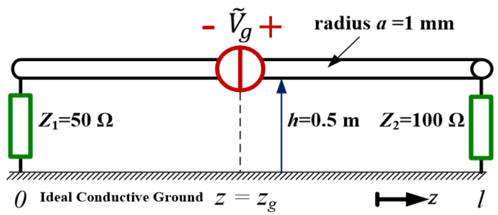
$$\tilde I(0) = \frac{(1-\Gamma_{L2})e^{-\gamma l}}{2Z_0(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left[(e^{\gamma z_g} - \Gamma_{L1}e^{-\gamma z_g})\tilde V_g + (e^{\gamma z_g}+\Gamma_{L1}e^{-\gamma z_g})Z_0\tilde I_g\right]$$

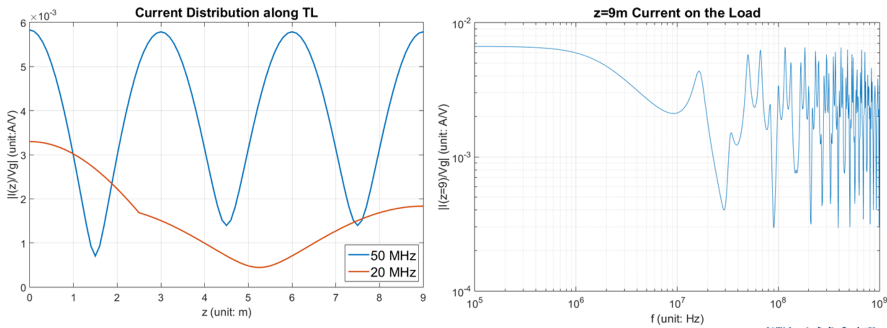
For $z = l$,

$$\tilde V(l) = \frac{(1+\Gamma_{L1})e^{-\gamma l}}{2(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left\{-\left[e^{\gamma(l-z_g)} - \Gamma_{L2}e^{-\gamma(l-z_g)}\right]\tilde V_g + \left[e^{\gamma(l-z_g)} + \Gamma_{L2}e^{-\gamma(l-z_g)}\right]Z_0\tilde I_g\right\}$$

$$\tilde I(l) = \frac{(1-\Gamma_{L1})e^{-\gamma l}}{2Z_0(1-\Gamma_{L1}\Gamma_{L2}e^{-2\gamma l})}\left\{-\left[e^{\gamma(l-z_g)} - \Gamma_{L2}e^{-\gamma(l-z_g)}\right]\tilde V_g + \left[e^{\gamma(l-z_g)} + \Gamma_{L2}e^{-\gamma(l-z_g)}\right]Z_0\tilde I_g\right\}$$

Example:





quote

*We **make a living** by what we get.*
*We **make a life** by what we give.*

Thank you again!