Upper Bound:

Let $(S, \preccurlyeq)$ be a poset and let $A \subseteq S$. If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$ then u is an *upper bound* of A.

Least Upper Bound:

An element x that is an upper bound on a subset A and is less than all other upper bounds on A is called the *least upper bound* on A. We abbreviate "lub".

Note: The least upper bound of $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{S}$ is denoted as: a $\oplus$ b (or) a + b (or) a $\vee$ b and read as Join or sum of a and b.

Lower Bound:

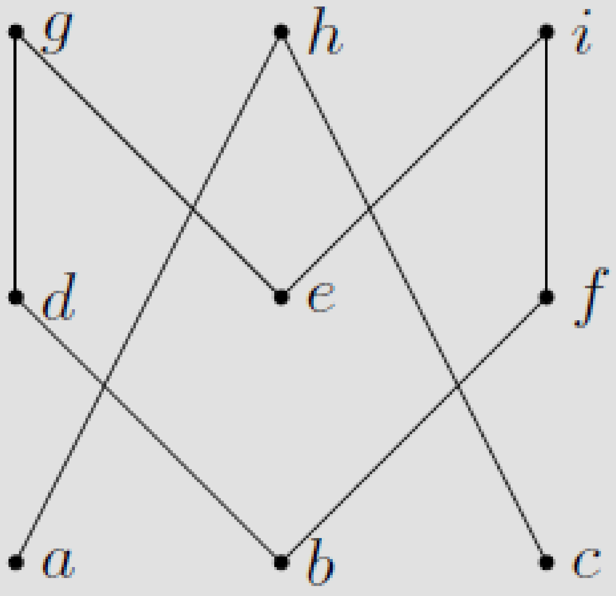
Let $(S, \preccurlyeq)$ be a poset and let $A \subseteq S$. If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$ then l is a *lower bound* of A.

Greatest Lower Bound:

An element x that is a lower bound on a subset A and is greater than all other lower bounds on A is called the *greatest lower bound* on A. We abbreviate "glb".

Note: The greatest lower bound of $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{S}$ is denoted as: a ⋆ b (or) a . b (or) a ∧ b and read as meet or product of a and b.

Example



What are the lower/upper bounds and glb/lub of the sets $\{d, e, f\}$, $\{a, c\}$ and $\{b, d\}$

$\{d, e, f\}$

- Lower Bounds: $\emptyset$, thus no glb either.
- Upper Bounds: $\emptyset$, thus no lub either.

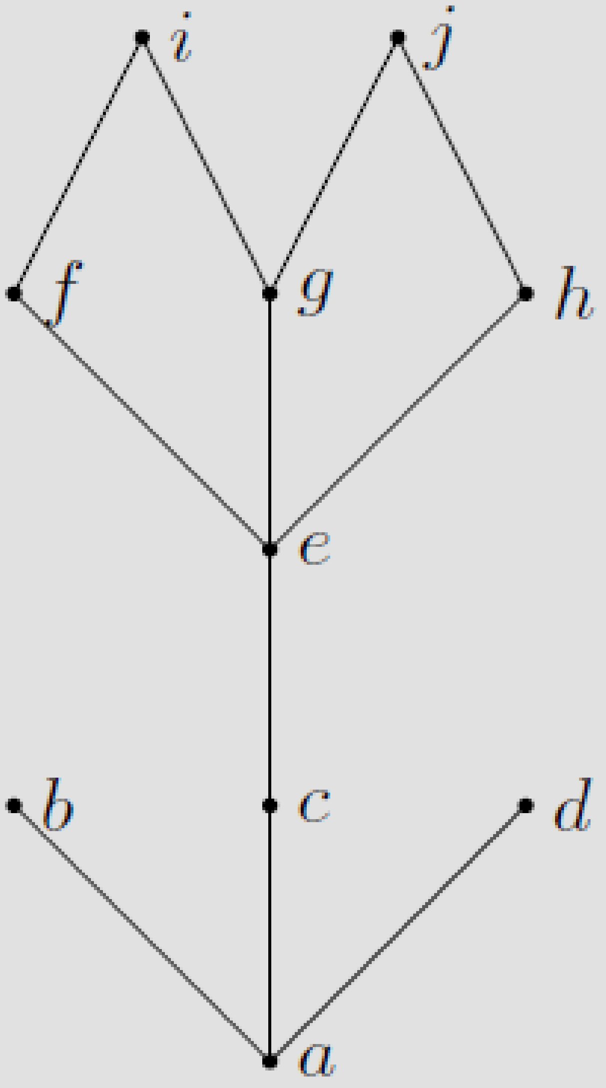
$\{a, c\}$

- Lower Bounds: $\emptyset$, thus no glb either.
- Upper Bounds: $\{h\}$, since its unique, lub is also h.

$\{b, d\}$

- Lower Bounds: $\{b\}$ and so also glb.
- Upper Bounds: $\{d, g\}$ and since $d \prec g$, the lub is d.

Example



Minimal/Maximal elements?

- Minimal & Minimum Element: a.
- Maximal Elements: b, d, i, j.

Bounds, glb, lub of $\{c, e\}$?

- Lower Bounds: $\{a, c\}$, thus glb is c.
- Upper Bounds: $\{e, f, g, h, i.j\}$ thus lub is e

Bounds, glb, lub of $\{b, i\}$?

- Lower Bounds: $\{a\}$, thus glb is a.
- Upper Bounds: $\emptyset$, thus lub DNE.

Ex: Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.

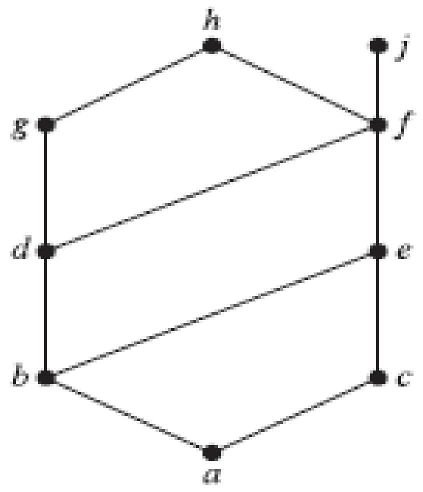


FIGURE 7 **The Hasse Diagram of a Poset.**

Solution: The upper bounds of $\{a, b, c\}$ are e, f, j, and h, and its only lower bound is a. There are no upper bounds of $\{j, h\}$, and its lower bounds are a, b, c, d, e, and f. The upper bounds of $\{a, c, d, f\}$ are f, h, and j, and its lower bound is a. ◀

Find the greatest lower bound and the least upper bound of $\{b, d, g\}$, if they exist, in the poset shown in Figure 7.

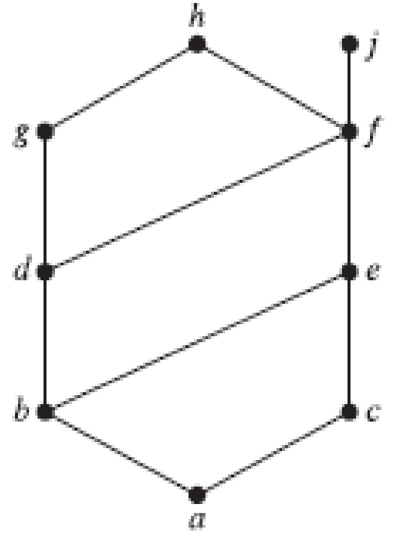


FIGURE 7 **The Hasse Diagram of a Poset.**

Solution: The upper bounds of $\{b, d, g\}$ are g and h. Because $g \prec h$, g is the least upper bound. The lower bounds of $\{b, d, g\}$ are a and b. Because $a \prec b$, b is the greatest lower bound. ◀

LATTICES:

Definition:

- A lattice is a partially ordered set $< L, \leq >$ in which every pair of elements $a, b \in L$ has a greatest lower bound and least upper bound.

i.e. A Poset $(L,\leq)$ is said to be lattice if for every pair of elements $a, b \in L$, $a \oplus b$ and $a \star b$ exists.

Note:

Since the lattices has two binary operations $\oplus$ and $\star$, the lattice can be denoted as a triple ordered set $(L, \oplus, \star)$ or $(L, \vee, \wedge)$.

EX: Determine whether the posets represented by each of the Hasse diagrams in Figure 8 are lattices.

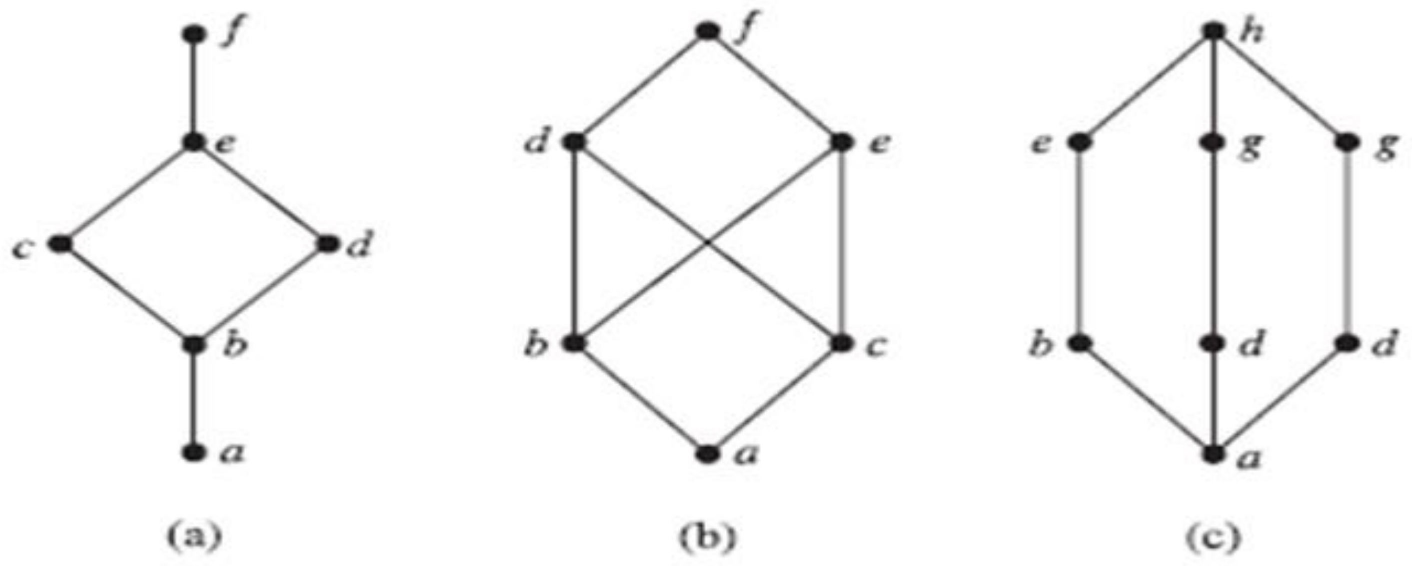
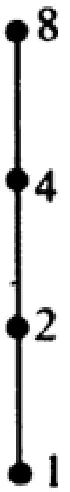


FIGURE 8 **Hasse Diagrams of Three Posets.**

Solution: The posets represented by the Hasse diagrams in (a) and (c) are both lattices because in each poset every pair of elements has both a least upper bound and a greatest lower bound,

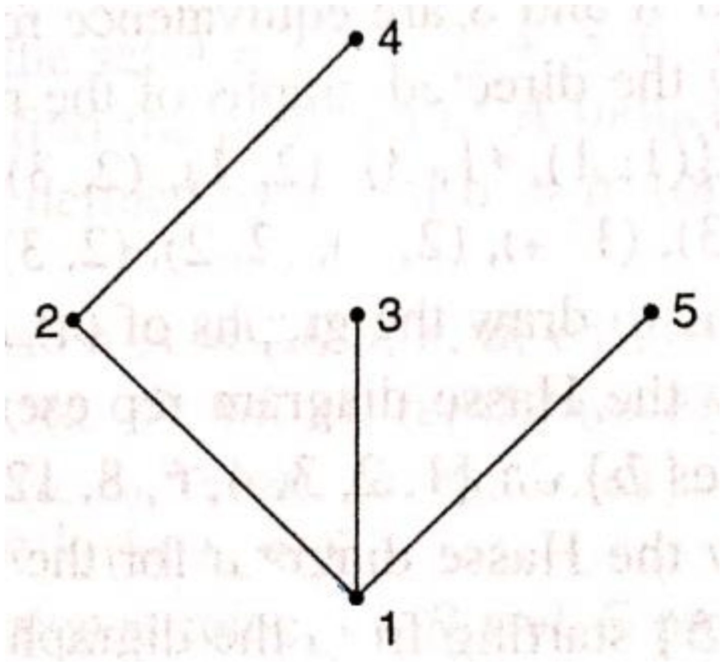
On the other hand, the poset with the Hasse diagram shown in (b) is not a lattice, because the elements b and c have no least upper bound. To see this, note that each of the elements d, e, and f is an upper bound, but none of these three elements precedes the other two with respect to the ordering of this poset. ◀

Example: $(\{1, 2, 4, 8\}, |)$, where | means 'divisor of '. The hasse diagram



LUB= 8,   GLB= 1

So, it is a lattice.



Example: $(\{1, 2, 3, 4, 5\}, |)$

It is not a lattice, since LUB of the pair (2, 3) and (3, 5) do Not exist.

Example . Consider the poset $A = \{a, b, c, d, e, f, g, h\}$, whose Hasse diagram is shown in Figure 7.24. Find all upper and lower bounds of the following subsets of A: (a) $B_1 = \{a, b\}$; (b) $B_2 = \{c, d, e\}$.

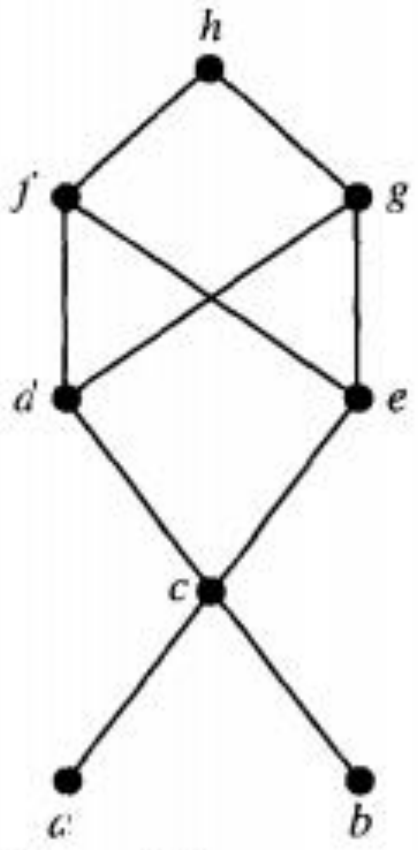


Figure 7.24

Solution

(a) B_1 has no lower bounds; its upper bounds are c, d, e, f, g, and h.

(b) The upper bounds of B_2 are f, g, and h; its lower bounds are c, a, and b.

◆

Example: Is the Poset $(Z^+, |)$ a lattice?

Solution: Let $a, b \in Z^+$, now, LUB of these two integers is the LCM (Least Common multiple) and GLB is the GCD (Greatest Common Divisor).

Example:

- Let n be a positive integer and s_n be the set of all divisors of n
  For example:

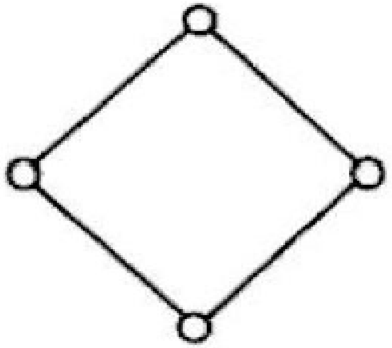
  $n=6,$

  $S_6=\{1,2,3,6\}$ and

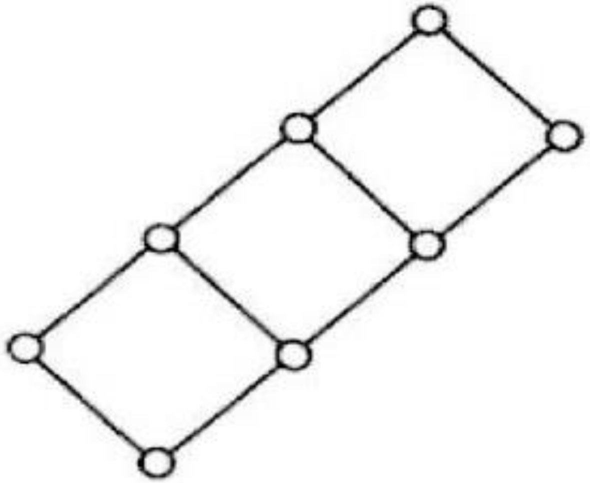
  $n=24$

  $S_{24}=\{1,2,3,4,6,8,12,24\}$

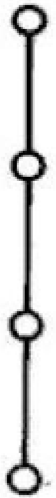
  Let D denote the relation of "division".
- The lattices $<S_6,D>$, $<S_{24},D>$, $<S_8,D>$ and $<S_{30},D>$ are given in lattices fig (b),(g),(a) and (f).



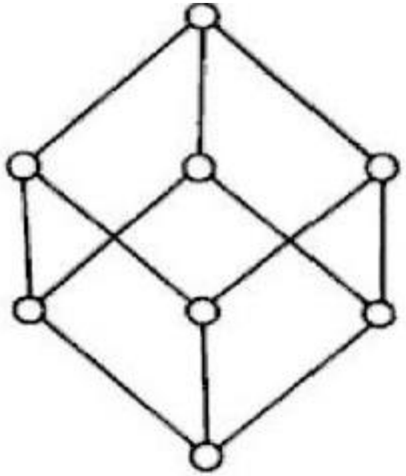
(b)



(g)



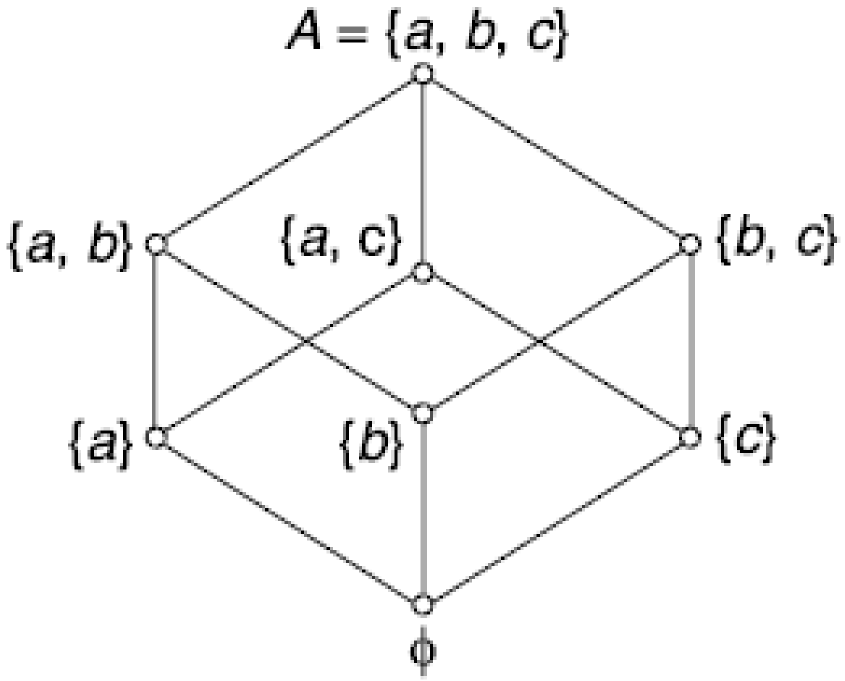
(a)



(f)

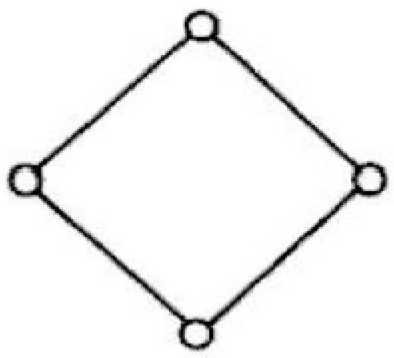
Example: In the case of power set $P(S)$ of any set S, $(P(S), \subseteq)$ is a lattice

Here always $LUB = A \cup B$ and $GLB = A \cap B$, where A and B are any subsets of $P(S)$.

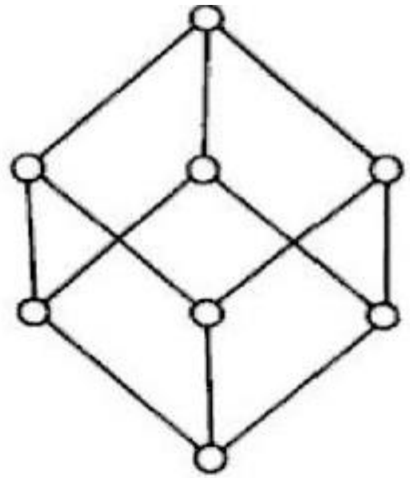


Example:

- Let S be the set and $\rho(s)$ be its power set.
- The partially ordered set $< \rho(s), \subseteq >$ is a lattice in which the meet and join are the same as the operations $\wedge$ and $\vee$ respectively.
- In particular when s has a single element, the corresponding lattices are a chain containing two elements.
- When s has two and three elements, the diagrams of the corresponding lattices are as shown in fig (b) and (f).



(b)



(f)

EX:

- Partially ordered sets which are not lattices: