Computational Higher Type Theory (CHTT)

Robert Harper

Lecture Notes of Week 2 by Stephanie Balzer and Yue Niu

1 Setting the Scene

Last week, we explored *closed term computation* for the simply typed lambda calculus (STLC) with the goal to prove that all well-typed STLC programs terminate. Our exploration lead us to redescover Tait's 1967 *hereditary termination* $\mathsf{HT}_A(M)$ of a term M at type A and *hereditary terminating substitution* $\mathsf{HT}_\Gamma(\gamma)$ of closing substitution mapping γ at context Γ to strengthen our inductive hypothesis. The predicates $\mathsf{HT}_A(M)$ and $\mathsf{HT}_\Gamma(\gamma)$ amount to *behavioral* invariants, indexed by a type and typing context, respectively. Depending on the source, different terms are used in the literature to refer to methods based on such behavioral invariants. Most commonly used is the term *hereditary*. Tait himself used the term *computability*, whereas some people refer to such methods as *Tait's Method*. Statman 1985 introduced the term *logical relation*, which underlines the idea of a behavioral invariant. Moreover, the term *Girard's method* can be found in the literature as well as the term *Reducibility Method*, which is most likely due to Girard himself.

In this week's lectures, we are going to generalize the results of the previous week to *open term computation*. Before doing so, however, we first have to finish our proof of hereditary termination.

2 Closed Term Computation Continued

Our main result of last week is the proof or Theorem 1, which states that well-typed terms are hereditarily terminating under substitution by hereditarily terminating maps:

Theorem 1 (Hereditary termination)**.** *If* $\Gamma \vdash M : A$ *and* $\gamma : \cdot \to \Gamma$ *so that* $\mathsf{HT}_\Gamma(\gamma)$ *then* $\mathsf{HT}_A(\hat{\gamma}(M))$.

Given this result, we should move on to proving that hereditary termination actually implies termination. Before doing so, however, we first have to finish our proof Theorem 1, which relies on a lemma known as *head expansion* or *reverse execution*, which we are going to prove next:

Lemma 2 (Head expansion)**.** *If* $\mathsf{HT}_A(M')$ *and* $M \mapsto M'$ *then* $\mathsf{HT}_A(M)$.

Proof. By induction on the structure of A.

- $A = b$
  Suppose $\mathsf{HT}_b(M')$ and $M \mapsto M'$. We need to show $\mathsf{HT}_b(M)$. By the definition of hereditary termination, it suffices to show that $M \Downarrow$. This is clear since $M' \Downarrow$ and $M \mapsto M'$.

- $A = A_1 \to A_2$
  Suppose $\mathsf{HT}_{A_1 \to A_2}(M')$ and $M \mapsto M'$. We need to show $\mathsf{HT}_{A_1 \to A_2}(M)$. Now suppose $\mathsf{HT}_{A_1}(N)$, and we need to show $\mathsf{HT}_{A_2}(M\ N)$. By the definition of hereditary termination and by our assumption, we have $\mathsf{HT}_{A_2}(M'\ N)$. Note that $M\ N \mapsto M'\ N$ by our assumption and the compuation rule. Now applying the IH[1], we have $\mathsf{HT}_{A_2}(M\ N)$. □

Let's now return to proving our ultimate goal, namely that hereditary termination implies termination of terms in the STLC:

Theorem 3 (Termination)**.** *If* $\mathsf{HT}_A(M)$ *then* $M\ \mathsf{term}_\beta$.

[1]Note that the IH can be applied because the induction is on the structure of A, not on terms. Whereas the terms in the hereditary termination predicate get bigger, the types get smaller.

Proof Attempt. Induction on A.

- $A = b$
  Suppose $\mathsf{HT}_b(M)$. Done by definition of hereditary termination.
- $A = A_1 \to A_2$
  Suppose $\mathsf{HT}_{A_1 \to A_2}(M)$. We need to show $M\ \mathsf{term}_\beta$. We are stuck here.

X

The question is what we should do at this point. One possibility is to basically "give up" and weaken the theorem to expect termination to hold only for observables, i.e., for terms of the base type. Alternatively, we could strengthen the induction hypothesis even further by changing the defintion of $\mathsf{HT}_A(M)$ to also insist on $M\ \mathsf{term}_\beta$.

Another option is to change the definition of hereditary termination entirely and use a *positive* formulation rather than a *negative* one. Whereas our current negative formulation is phrased in terms of an implication, the positive formulation requires the term to be an actual lambda. *Positive hereditary termination* $\mathsf{HT}^+_A(M)$ of a term M at type A is defined as follows:

$$\begin{aligned}
\mathsf{HT}^+_b(M) &\triangleq M \mapsto^* c \quad (\exists c : b) \\
\mathsf{HT}^+_{A \to B}(M) &\triangleq M \mapsto^* \lambda x{:}A.\, M_2 \text{ and } \forall M_1.\ \mathsf{HT}^+_A(M_1) \implies \mathsf{HT}^+_{A_2}([M_1/x]M_2)
\end{aligned}$$

The positive formulation is also known as the *method of canonical forms*. Using a positive formulation of hereditary termination, the proof of the introductory rule is straightforward, whereas the elimination rule takes some work, relying on head expansion, in particular.

We can rephrase the above definition of $\mathsf{HT}^+_A(M)$ by introducing the notion of a *hereditary value* $\mathsf{HV}_A(V)$ of a value V at type A

$$\mathsf{HT}^+_b(M) \triangleq M \mapsto^* V\ \mathsf{value} \text{ such that } \mathsf{HV}_A(V)$$

and then give a definition of a hereditary value that differs depending on whether a *call-by-value* (cbv) or *call-by-name* (cbn) semantics is used:

$$\begin{aligned}
\mathsf{HV}_b(V) &\triangleq V = c : b \\
\mathsf{HV}^{\mathsf{cbv}}_{A_1 \to A_2}(V) &\triangleq V = \lambda x{:}A.\, M_2 \text{ and } \forall V_1.\mathsf{HV}^{\mathsf{cbv}}_{A_1}(V_1) \implies \mathsf{HT}_{A_2}([V_1/x]M_2) \\
\mathsf{HV}^{\mathsf{cbn}}_{A_1 \to A_2}(V) &\triangleq V = \lambda x{:}A.\, M_2 \text{ and } \forall M_1.\mathsf{HT}_{A_1}(M_1) \implies \mathsf{HT}_{A_2}([M_1/x]M_2)
\end{aligned}$$

In both interpretations a hereditary value at a base type is simply a value, whereas at a function type the insistance on a value as an argument is only made for a call-by value semantics, but not for a call-by-name semantics. From this perspective, the negative formulation of hereditary termination only really makes sense in the context of a call-by-name semantics. Only then, function arguments can be unevaluated computations that permit the weakened termination theorem discussed earlier that insists on termination for base types only. For a call-by-value semantics, on the other hand, a positive formulation of hereditary termination is required for termination, as this formulation is phrased in terms of values.

In this case study of closed term computation we have assumed a call-by-name evaluation semantics. For a call-by-value evaluation semantics, our main fundamental Theorem 1 needs to be phrased in terms of hereditary values, with $\mathsf{HV}^{\mathsf{cbv}}_\Gamma(\gamma)$ correspondingly defined:

Theorem 4 (Call-by-value hereditary termination)**.** *If* $\Gamma \vdash M : A$ *and* $\gamma : \cdot \to \Gamma$ *so that* $\mathsf{HV}^{\mathsf{cbv}}_\Gamma(\gamma)$ *then* $\mathsf{HT}^+_A(\hat{\gamma}(M))$.

2.1 Summary

The introduction of behavioral invariants has allowed us to express the *semantic* property of *syntax*. Specifically, we have shown that well-type terms terminate. To talk about the semantic properties of terms, we introduce the following notation:

A type	iff	$A, B ::= b \mid A \to B$
$\gamma \in \Gamma$	iff	$\mathsf{HT}_\Gamma(\gamma)$
$M \in A$	iff	$\mathsf{HT}_A(M)$
$\Gamma \gg M \in A$	iff	$\gamma \in \Gamma \implies \hat{\gamma}(M) \in A$

Note that the "membership" relation has a computational flavor; it is a behavorial condition on M (or γ), which says that M (or γ) satisfies the specification A (or Γ).

Now, we can state our main fundamental Theorem 1 as follows:

Theorem 5. *If* $\Gamma \vdash M : A$*, then* $\Gamma \gg M \in A$*.*

This is in some sense a soundness theorem (in the language of formal logics), which means that the formally derivable terms are actually true (according to the computational specification). Therefore, we can think of formal systems as a way of *accessing* the *truth*. Note that we make no claims that $\Gamma \gg M \in A$ be decidable, as it is fruitless to expect the truth to be decidable in general. [2]

3 Open Term Computation

We can also interpret variables in a different sense, where they stand for open terms - they are indeterminates. Then the judgment $\Gamma \vdash M : A$ could be interpreted as a mapping M from open terms to the type A.

Recall the beta-contraction relation, $P \ \mathrm{contr}_\beta \ P'$:

$$(\lambda x{:}A.\, M)\ N \ \mathrm{contr}_\beta \ [N/x]M$$

With this, we formulate beta-reduction:

$$\frac{M \ \mathrm{contr}_\beta \ N}{M \mapsto_\beta N} \qquad \frac{M \mapsto_\beta M'}{M\ N \mapsto_\beta M'\ N} \qquad \frac{N \mapsto_\beta N'}{M\ N \mapsto_\beta M\ N'} \qquad \frac{M \mapsto_\beta M'}{\lambda x{:}A.\, M \mapsto_\beta \lambda x{:}A.\, M'}$$

Further, we define beta-normal form, $N \ \mathrm{nf}_\beta$:

$$N \ \mathrm{nf}_\beta \triangleq N \not\mapsto_\beta$$

Where $N \not\mapsto_\beta$ means that no beta-reduction rule applies to N. This informal notation can be formalized.

Finally, we can define when a term is beta-normalizing, $M \ \mathrm{norm}_\beta$:

$$M \ \mathrm{norm}_\beta \triangleq \exists N. M \mapsto_\beta^* N \text{ and } N \ \mathrm{nf}_\beta$$

Now we can state the fundamental theorem for normalization:

Theorem 6. *If* $\Gamma \vdash M : A$ *then* M *norm*$_\beta$*.*

[2] See Derek Dreyer's Milner Award Lecture for further discussion.

Following the proof for termination, we introduce a stronger notion for normalization, *hereditary normalization*:

$$\mathsf{HN}_b^\Delta(M) \triangleq M \text{ norm}_\beta$$
$$\mathsf{HN}_{A\to B}^\Delta(M) \triangleq \forall N.\ \mathsf{HN}_A^\Delta(N) \implies \mathsf{HN}_B^\Delta(M\ N)$$

As before, we define $\mathsf{HN}_\Gamma^\Delta(\gamma)$ to be $\forall x : A \in \Gamma.\ \mathsf{HN}_A^\Delta(\gamma(x))$

Some notes about this definition:

- This predicate is defined on terms M s.t. $\Delta \vdash M : A$. The extra argument to the hereditary normalization (compared to termination) Δ reflects this fact, since normalization is defined on open terms.
- We formulate hereditary normalization in the negative way; turns out this is the only way
- What can we say about positive types (sums)?

The proof can be divided into three lemmas:

Lemma 7. *If* $\Gamma \vdash M : A$ *and* $\mathsf{HN}_\Gamma^\Delta(\gamma)$*, then* $\mathsf{HN}_A^\Delta(\hat\gamma(M))$*.*

Lemma 8. *If* $\mathsf{HN}_A^\Delta(M)$ *then* M *norm*$_\beta$*.*

Lemma 9. $\mathsf{HN}_\Gamma^\Gamma(\mathsf{id}(\Gamma))$*.*

3.1 Lemma 7

Lemma. *If* $\Gamma \vdash M : A$ *and* $\mathsf{HN}_\Gamma^\Delta(\gamma)$*, then* $\mathsf{HN}_A^\Delta(\hat\gamma(M))$*.*

Proof. Induction on typing.

- $\Gamma', x : A \vdash x : A$
  Suppose $\mathsf{HN}_\Gamma^\Delta(\gamma)$. We need to show $\mathsf{HN}_A^\Delta(\hat\gamma(x))$, which follows directly by our assumption.
- $\Gamma \vdash c : b$
  Suppose $\mathsf{HN}_\Gamma^\Delta(\gamma)$. We need to show that $\mathsf{HN}_b^\Delta(c)$, which is to show that c norm$_\beta$. Since c nf$_\beta$, we have c normalizing in 0 steps.
- $$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}$$
  Applying the IH, we have $\mathsf{HN}_{A\to B}^\Delta(\hat\gamma(M))$ and $\mathsf{HN}_A^\Delta(\hat\gamma(N))$. By definition of hereditary normalization, we have $\mathsf{HN}_B^\Delta(\hat\gamma(M)\ \hat\gamma(N))$. Since $\hat\gamma(M\ N) = \hat\gamma(M)\ \hat\gamma(N)$, we have $\mathsf{HN}_B^\Delta(\hat\gamma(M\ N))$.
- $$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x{:}A.\, M : A \to B}$$
  Suppose $\mathsf{HN}_\Gamma^\Delta(\gamma)$. We need to show that $\mathsf{HN}_{A\to B}^\Delta(\hat\gamma(\lambda x{:}A.\, M))$. Thus, suppose $\mathsf{HN}_A^\Delta(N)$. It suffices to show that $\mathsf{HN}_B^\Delta(\hat\gamma(\lambda x{:}A.\, M)\ N)$. Notice that with N, we can extend γ to be $\gamma' = \gamma[x \mapsto N]$, and since N is hereditarily normalizing in A, it follows that $\mathsf{HN}_{\Gamma,x:A}^\Delta(\gamma')$. Applying the IH, we have that $\mathsf{HN}_B^\Delta(\hat{\gamma'}(M))$. Since substitution is commutative, $\hat{\gamma'}(M) = [N/x]\hat\gamma(M)$, and $\hat\gamma(\lambda x{:}A.\, M)\ N = \lambda x{:}A.\, \hat\gamma(M)\ N$. Further, $\lambda x{:}A.\, \hat\gamma(M)\ N \mapsto [N/x]\hat\gamma(M)$ by the computation rule. The result then follows from head expansion. □

The proof for the next 2 lemmas will be mutual induction. However, this is not enough. We will need to further strengthen Lemma 9, but let's see where we fail.

3.2 Lemma 8

Lemma. *If* $\mathsf{HN}_A^\Delta(M)$ *then* M *norm*$_\beta$*.*

Proof Attempt. Induction on A.

- $A = b$
  Immediate from definition of hereditary normalization.

- $A = A_1 \to A_2$
  Suppose $\mathsf{HN}^{\Delta}_{A}(M)$. We need to show M norm_β. Now we are in a bind. If we can somehow use Lemma 9 to obtain a hereditarily normalizing input of type A_1, then we can proceed with induction to obtain the result. For now, we will assume (*) that gives us a Γ' such that $x : A_1 \in \Gamma'$. By Lemma 9 we have $\mathsf{HN}^{\Gamma'}_{\Gamma'}(\mathsf{id}(\Gamma'))$, in particular, $\mathsf{HN}^{\Gamma'}_{A_1}(x)$. By the definition of hereditary normalization, we further have that $\mathsf{HN}^{\Gamma'}_{A_2}(M\ x)$. Now applying the IH, we see that $M\ x$ norm_β, and by Lemma 11 we have M norm_β.

X

Postponing the issues with this proof, we will move on to proving Lemma 9, and resolve everything after presenting the two proofs.

3.3 Lemma 9

Lemma. $\mathsf{HN}^{\Gamma}_{\Gamma}(\mathsf{id}(\Gamma))$.

Proof Attempt. Let $x : A \in \Gamma$ be arbitrary. Proceed with induction on A.

- $A = b$
  We need to show $\mathsf{HN}^{\Gamma}_{b}(x)$, which holds since x nf_β.

- $A = A_1 \to A_2$
  We need to show $\mathsf{HN}^{\Gamma}_{A_1 \to A_2}(x)$. Suppose $\mathsf{HN}^{\Gamma}_{A_1}(M_1)$, and it suffices to show that $\mathsf{HN}^{\Gamma}_{A_2}(x\ M_1)$. Now it would be nice to apply the IH to complete the proof, but which is not strong enough.

X

Notice that for every variable of function type, we want them to be hereditarily normalizing when applied to beta-normal terms. We will add this to the lemma:

Lemma 10. *For all k, if $x : A_1 \to \cdots \to A_k \to A \in \Gamma$ and for each $i \le k$, $\Gamma \vdash M_i : A_i$ and M_i norm_β, then $\mathsf{HN}^{\Gamma}_{A}(x\ M_1\ \ldots M_k)$.*

Note that we need each argument to be *normalizing* instead of merely *hereditarily normalizing*. Before proving Lemma 10, we shall see how to fix Lemma 8. The crucial fact is the (*) that gave us a context Γ' out of thin air, which allowed us to supply the argument to the function term. To do this, we will make a change to the definition of hereditary normalization, which will involve the notion of Kripke semantics.

3.4 Kripke semantics

Kripke semantics, or presheafs, are related to/gives the solution to the following:

- allocating a "fresh variable" in a context
- why should hereditary normalization be stable under context extensions? (In other words, why should context extensions be admissible for hereditary normalization?)

If we think of the typing context as a model of the "world", then Kripke semantics suggests that we can stipulate hereditary normalization to hold in all context extensions, or "future words". Viewed logically, this would be the admissibility of weakening with respect to hereditary normalization.

Thus for our definition, we change the higher order terms to expect "world extensions":

$$\mathsf{HN}^{\Delta}_{A_1 \to A_2}(M) \triangleq \text{ if } \Delta' \ge \Delta \text{ and } \mathsf{HN}^{\Delta'}_{A_1}(M_1) \text{ then } \mathsf{HN}^{\Delta'}_{A_2}(M\ M_1)$$

Note that $\geq$ is a pre-order on contexts Δ, and is reflexive and transitive. Now we can fix both lemmas.

3.5 Fix Lemmas

Lemma. *If* $\mathsf{HN}_A^\Delta(M)$ *then* M *norm*$_\beta$.

Proof. Induction on A.

- $A = b$
  Immediate from definition of hereditary normalization.

- $A = A_1 \to A_2$
  Suppose $\mathsf{HN}_A^\Delta(M)$. We need to show M norm$_\beta$. Let $\Delta' = \Delta, x : A_1$. By IH on Lemma 10 (with $k = 0$), we have $\mathsf{HN}_{A_1}^{\Delta'}(x)$. By the definition of hereditary termination, we have $\mathsf{HN}_{A_2}^{\Delta'}(M\ x)$. Applying the IH, we have $M\ x$ norm$_\beta$, and by Lemma 12 we have M norm$_\beta$. □

Lemma. *For all* k, *if* $x : A_1 \to \cdots \to A_k \to A \in \Gamma$ *and for each* $i \leq k$, $\Gamma \vdash M_i : A_i$ *and* M_i *norm*$_\beta$, *then* $\mathsf{HN}_A^\Gamma(x\ M_1\ \ldots M_k)$.

Proof. Proceed with induction on A.

- $A = b$
  We need to show $x\ M_1\ \ldots M_k$ norm$_\beta$, which holds since each M_i normalizing, and thus the term does not beta-reduce.

- $A = A_{k+1} \to A_{k+2}$
  We need to show $\mathsf{HN}_A^\Gamma(x\ M_1\ \ldots M_k)$. Let $\Gamma' \geq \Gamma$ and $\mathsf{HN}_{A_{k+1}}^{\Gamma'}(M_{k+1})$. It suffices to show that $\mathsf{HN}_{A_{k+2}}^{\Gamma'}(x\ M_1\ \ldots M_k\ M_{k+1})$. By IH on Lemma 8, we have that M_{k+1} norm$_\beta$, and we obtain the result by applying the IH with $k + 1$.

□

3.6 Lemma 10 with evaluation context

Recall the definition of evaluation contexts:

$$\mathcal{E} ::= \cdot \mid \mathcal{E}\ M$$

We now further characterize evaluation contexts as mappings between types[3]:

$$\frac{}{\cdot : (\Gamma \rhd A) \rightsquigarrow (\Gamma \rhd A)} \qquad \frac{\mathcal{E} : (\Gamma \rhd A) \rightsquigarrow (\Gamma' \rhd A_1 \to A_2) \qquad \Gamma \vdash M : A_1}{\mathcal{E}\ M : (\Gamma \rhd A) \rightsquigarrow (\Gamma' \rhd A_2)}$$

Where $\mathcal{E} : (\Gamma \rhd A) \rightsquigarrow (\Gamma' \rhd A')$ can be read as if $\Gamma \vdash M : A$, then $\Gamma' \vdash \mathcal{E}\{M\} : A'$.

Further, we can define when evalutaion contexts are beta-normalizing:

$$\frac{}{\cdot\ \text{norm}_\beta} \qquad \frac{M\ \text{norm}_\beta \qquad \mathcal{E}\ \text{norm}_\beta}{\mathcal{E}\ M\ \text{norm}_\beta}$$

Now, Lemma 10 can be formulated as follows:

Lemma 11. *If* $\mathcal{E} : (\Gamma \rhd C) \rightsquigarrow (\Gamma \rhd A)$ *and* $\mathcal{E}$ *norm*$_\beta$, *then* $\mathsf{HN}_A^{\Gamma, x:C}(\mathcal{E}\{x\})$

Proof. Proceed with induction on the structure of A.

[3] more details to be found in chapter 46 of PFPL

- $A = b$
  Need to show that $\mathsf{HN}_b^\Gamma(x)$, which follows since variables are beta-normal.

- $A = A_1 \to A_2$
  Proceed with nested induction on context typing and normalization.
  - $\mathcal{E} = \cdot$
    Let $\Gamma' \geq \Gamma, x : A$. Suppose $\mathsf{HN}_{A_1}^{\Gamma'}(M)$. It suffices to show $\mathsf{HN}_{A_2}^{\Gamma'}(x\ M)$. Note M norm_β from IH on Lemma 8. Now apply the outer IH with $\cdot\ M : (\Gamma \rhd A) \rightsquigarrow (\Gamma \rhd A_2)$ to obtain $\mathsf{HN}_A^{\Gamma, x:A}(\cdot\ M\{x\})$, and result follows from monotonicity of hereditary normalization.

  - $\mathcal{E} = \mathcal{E}'\ M$ because $\mathcal{E}' : (\Gamma \rhd C) \rightsquigarrow (\Gamma \rhd A' \to A), \Gamma \vdash M : A'$ and M norm_β, $\mathcal{E}'$ norm_β
    We need to show that $\mathsf{HN}_A^{\Gamma, x:C}(\mathcal{E}\{x\})$. Let $\Gamma' \geq \Gamma, x : C$, and suppose $\mathsf{HN}_{A_1}^{\Gamma'}(N)$. We need to show $\mathsf{HN}_{A_2}^{\Gamma'}(\mathcal{E}\{x\}\ N)$. Since $\mathcal{E}\{x\}\ N = \mathcal{E}\ N\{x\}$, it suffices to show $\mathsf{HN}_{A_2}^{\Gamma'}(\mathcal{E}\ N\{x\})$. By the definition of context typing, we have $\mathcal{E}\ N : (\Gamma \rhd C) \rightsquigarrow (\Gamma \rhd A_2)$. In addition, by IH on Lemma 8, we have N norm_β, consequently $\mathcal{E}\ N$ norm_β. Lastly, we obtain the result by applying the outer IH on $\mathcal{E}\ N$.

□

Note that the proof is an induction over the lexicographical ordering on the structures of $(A,\ \mathcal{E} : (\Gamma \rhd C) \rightsquigarrow (\Gamma \rhd C),\ \mathcal{E}\ \mathrm{norm}_\beta)$. In particular, when appealing to the inductive hypothesis, the size of structurese relating to $\mathcal{E}$ increases in some cases, but only when A is correspondingly decreasing. We can recover Lemma 10 by instantiating Lemma 11 with the empty evaluation context.

We have one last lemma to prove, namely:

Lemma 12. *If $M\ x$ norm_β, then M norm_β.*

Proof. Let $M\ x \mapsto_\beta^* N$ and N nf_β. Proceed with induction on the length of the sequence to beta-normal form.

- $M\ x \mapsto_\beta^{(0)} N$
  Then $M\ x$ nf_β, and M nf_β, and M also beta-normalizes in 0 steps.

- $M\ x \mapsto_\beta^{(k+1)} N$
  Proceed by cases on the first step.
  - $\lambda x{:}a.\, M'\ x \mapsto_\beta M'$
    Then M' beta-normalizes in k steps. The same k steps will also normalize M performed under the lambda.

  - $M\ x \mapsto_\beta M'\ x$
    By IH, M' norm_β, and it follows that M norm_β.

  - $M\ x \mapsto_\beta M'\ N$
    Impossible.

  - $\lambda x{:}A.\, M' \mapsto_\beta \lambda x{:}A.\, M''$
    Impossible.

□

4 Strong Normalization

References

Richard Statman. Logical relations and the typed λ-calculus. *Information and Control*, 65(2/3):85–97, 1985.

William W. Tait. Intensional Interpretations of Functionals of Finite Type I. *Journal of Symbolic Logic*, 1967.