Problem 1.

1a).

$$ds^2 = -(1-\frac{r_s}{r})c^2dt^2 + \frac{dr^2}{1-\frac{r_s}{r}} + r^2 d\Omega = 0, \quad d\Omega = 0$$

$$\Rightarrow \frac{dr}{dt} = (1-\frac{r_s}{r})c$$

1b). $$\Delta t = 2\int dt = 2\int_{r_V}^{r_E} \frac{dr}{(1-\frac{r_s}{r})c} \overset{x=\frac{r}{r_s}}{=\!=\!=} \frac{2r_s}{c}\int_{\frac{r_V}{r_s}}^{\frac{r_E}{r_s}} (1+\frac{1}{x-1})dx = \frac{2r_s}{c}\cdot(\frac{r_E-r_V}{r_s} + \ln\frac{r_E-r_s}{r_V-r_s})$$

$$\therefore \Delta t = \frac{2(r_E-r_V)}{c} + \frac{2r_s}{c}\cdot\ln\frac{r_E-r_s}{r_V-r_s}$$

1c) $$\Delta t_{shap} = \frac{2r_s}{c}\ln\frac{r_E-r_s}{r_V-r_s} = \frac{2\times10^3}{3\times10^8}\ln\left(\frac{1-\frac{3}{1.496\times10^8}}{0.7-\frac{3}{1.496\times10^8}}\right)$$

$$\approx \frac{2r_s}{c}\ln\frac{r_E}{r_V} = 7.133\times10^{-6}\text{ s}$$

Prob 2.

2a) for both θ_2, θ_E are small, we have:

$$\tan\theta_2 \approx \theta_2 = \frac{b}{D_{LS}}, \quad \tan\theta_E \approx \theta_E = \frac{b}{D_L}$$

$$\alpha = \theta_2 + \theta_E = \frac{4GM}{c^2 b} = \frac{b}{D_{LS}} + \frac{b}{D_L} = \theta_E(1+\frac{D_L}{D_{LS}})$$

$$b^2 = \frac{4GM}{c^2}\cdot\frac{D_{LS}D_L}{D_S}$$

$$\therefore \theta_E = \frac{4GM}{c^2 b}\frac{D_{LS}}{D_S} = \frac{4GM}{c^2}\frac{D_{LS}}{D_S}\cdot\sqrt{\frac{c^2\cdot D_S}{4GM D_{LS}D_L}}$$

$$= \sqrt{\frac{4GM}{c^2}\cdot\frac{D_{LS}}{D_S\cdot D_L}} \qquad \square.$$

Prob 3.

3a). $\frac{dt}{d\tau} = \frac{1}{d\tau/dt} = \frac{1}{\sqrt{\frac{dt^2 - dx^2/c^2}{dt^2}}} = \frac{1}{\sqrt{1-v^2/c^2}} = \gamma, \quad v = \frac{dx}{dt}$

3b). $\frac{dt}{d\tau} = (1 - v^2/c^2)^{-1/2} = 1 + (-\frac{1}{2}) \cdot \frac{1}{1!} \cdot (-\frac{v^2}{c^2}) + \cdots = 1 + \frac{1}{2} \cdot \frac{v^2}{c^2} + \cdots$

Prob 4.

4a). $\frac{d\tau^2}{d\tau^2} = -\frac{ds^2/c^2}{d\tau^2} = (1-\frac{r_s}{r})\frac{dt^2}{d\tau^2} - \frac{1}{1-\frac{r_s}{r}} \cdot \frac{1}{c^2} \cdot \frac{dr^2}{d\tau^2} = 1$

$\therefore \quad \frac{dr^2}{d\tau^2} = c^2 (1-\frac{r_s}{r})^2 \frac{dt^2}{d\tau^2} - (1-\frac{r_s}{r}) \cdot c^2$

$\therefore \quad \frac{1}{2} m (\frac{dr}{d\tau})^2 - \frac{GMm}{r}$

$= \frac{1}{2} m c^2 \cdot \left[(1-\frac{r_s}{r})^2 (\frac{dt}{d\tau})^2 - (1-\frac{r_s}{r})\right] - mc^2 \cdot \frac{r_s}{2r}$

since $\frac{E}{m} = (1 - \frac{r_s}{r}) c^2 \frac{dt}{d\tau}$

$\therefore \frac{1}{2} m (\frac{dr}{d\tau})^2 - \frac{GMm}{r} = \frac{1}{2} mc^2 (\frac{E}{mc^2})^2 - \frac{1}{2} mc^2 = $ const of E, m, c $\square$.

4b).

$-ds^2/c^2 = d\tau^2 = (1-\frac{r_s}{r}) dt^2 - \frac{dr^2}{(1-\frac{r_s}{r})c^2}$

$\Rightarrow \quad \frac{dt^2}{d\tau^2} = \frac{1}{1-\frac{r_s}{r}} + \frac{1}{(1-\frac{r_s}{r})^2 c^2} \frac{dr^2}{d\tau^2} \Rightarrow \frac{dt}{d\tau} = \sqrt{\frac{1}{1-\frac{r_s}{r}} + \frac{1}{(1-\frac{r_s}{r})^2 c^2} \cdot \frac{dr^2}{d\tau^2}}$

$\therefore E = mc^2 \cdot \sqrt{1 - \frac{r_s}{r} + \frac{1}{c^2} \frac{dr^2}{d\tau^2}}$

$\sqrt{1 - \frac{r_s}{r} + \frac{1}{c^2} \frac{dr^2}{d\tau^2}} \approx 1 + \frac{1}{2} \cdot \frac{1}{1!} (-\frac{r_s}{r} + \frac{1}{c^2} \cdot \frac{dr^2}{d\tau^2}) = 1 - \frac{r_s}{2r} + \frac{1}{2c^2} \frac{dr^2}{d\tau^2}$

$\therefore E = mc^2 - \frac{mc^2 \cdot r_s}{2r} + \frac{1}{2} m \frac{dr^2}{d\tau^2},$

$\therefore E_n = E - mc^2 = \frac{1}{2} m (\frac{dr}{d\tau})^2 - \frac{GMm}{r}$ $\square$.

4c). if we take $E \approx mc^2$ then $\epsilon = 0$.

$$\therefore \left(\frac{dr}{d\tau}\right)^2 = \frac{2GM}{r} = \frac{r_s}{r}\cdot c^2$$

$$\Rightarrow \quad d\tau = \sqrt{\frac{r}{r_s}}\cdot\frac{1}{c}\,dr \qquad \therefore \tau = \frac{1}{c}\int_{r_1}^{r_2}\sqrt{\frac{r}{r_s}}\,dr = \frac{2}{3}\cdot\frac{r_s}{c}\left[\left(\frac{r_2}{r_s}\right)^{3/2} - \left(\frac{r_1}{r_s}\right)^{3/2}\right]$$

4d)

$$\tau = \frac{2}{3}\frac{r_s}{c} \approx 6.67\times10^{-6}\,s$$

4e) $\because \left(\frac{dr}{d\tau}\right)^2 = \frac{2\epsilon}{m} + \frac{2GM}{r} = \frac{2GM}{r} = \frac{r_s}{r}\cdot c$

$$\therefore \left|\frac{dr}{dt}\frac{dt}{d\tau}\right| = c\cdot\sqrt{\frac{r_s}{r}}$$

since $\frac{E}{m} = g_{tt}\cdot\frac{dt}{d\tau} \Rightarrow \frac{dt}{d\tau} = \frac{E}{m}\frac{1}{g_{tt}} = \frac{E}{mc^2}\frac{1}{1-\frac{r_s}{r}}$, $E = mc^2$

$$\therefore \left|\frac{dr}{dt}\right| = \left|\frac{d\tau}{dt}\right|\cdot c\sqrt{\frac{r_s}{r}} = \frac{1}{\frac{E}{mc^2}\cdot\frac{1}{1-\frac{r_s}{r}}}\cdot c\sqrt{\frac{r_s}{r}} = c\sqrt{\frac{r_s}{r}}\cdot\left(1-\frac{r_s}{r}\right) \quad \square.$$