Prob 1

(1a) $\because \epsilon = \epsilon_0 a^{-3(1+w)}$, for matters $w=0$, for radiation $w=\frac{1}{3}$

$$\frac{\epsilon_{m,0}}{\epsilon_{r,0}} = \frac{\Omega_{m,0}}{\Omega_{r,0}} = \frac{0.3}{8\times10^{-5}}$$

set to when radiation/matter equal:

$$1 = \frac{\epsilon_m(t)}{\epsilon_r(t)} = \frac{\Omega_{m,0}\cdot a^{-3}}{\Omega_{r,0}\cdot a^{-4}} \Rightarrow \frac{1}{a} = \frac{\Omega_{m,0}}{\Omega_{r,0}} = 1+z_{eq}$$

$\therefore z_{eq} \approx 3749$

(1b) CMB temperature for today: $T_0 = 2.7K$

$\because T_{CMB} = \frac{T}{a} = T_0(1+z)$

$\therefore$ when radiation/matter equal;

$$T_{CMB} = T_0 \cdot 3750 = 10125\ K$$

(1c) $\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \cdot \frac{\Omega_{r,0}}{a^4}$, $\Omega_{r,0}=1 \Rightarrow \dot{a} = \frac{H_0}{a}$

$\therefore a(t) = \sqrt{2H_0 t}$

$$a(t_{eq}) = \frac{1}{1+z_{eq}} = \sqrt{2H_0 t_{eq}} \Rightarrow t_{eq} = \frac{t_H}{2\cdot 3750^2} \approx 487.8 \text{ years}$$

(1d) $\frac{n_p n_e}{n_H} = \frac{1}{\lambda_{e,T}^3} e^{-Q/k_B T}$, $n_p = x_{ion}\cdot n_b$,

$$\therefore \frac{x_{ion}^2}{1-x_{ion}} \cdot n_b = \frac{1}{\lambda_{e,T}^3} \cdot e^{-Q/k_B T}$$

$$\Rightarrow T_{rec} = \frac{Q}{k_B} \cdot \frac{1}{\ln\left(\frac{x_{ion}^2}{1-x_{ion}} \cdot n_b \cdot \lambda_{e,T}^3\right)^{-1}} = \frac{Q}{k_B} \cdot \frac{1}{\ln\left(\frac{1-x_{ion}}{x_{ion}^2 \cdot n_b \cdot \lambda_{e,T}^3}\right)}$$

if $x_{ion} = \frac{1}{2}$, then $T_{rec} = \frac{Q}{k_B} \cdot \frac{1}{\ln\left(\frac{2}{n_b \cdot \lambda_{e,T}^3}\right)}$

(1e). $T_{rec} = T_0(1+z_{rec}) \Rightarrow z_{rec} = \frac{T_{rec}}{T_0} - 1 \approx 1388$

(1f). take $\Omega_{r,0} = \Omega_{\Lambda,0} = 0$, $\Omega_{m,0} = 0.3$;

$$\frac{\dot{a}^2}{a^2} = H_0^2 \cdot \frac{0.3}{a^3} \Rightarrow a(t) = \left(\frac{3}{2}\sqrt{\frac{3}{10}} H_0 t\right)^{2/3}$$

$$\frac{1}{a(t_{rec})} = 1 + z_{rec} \Rightarrow t_{rec} \approx 1,265,000 \text{ years}$$

(1g) $H^2 = H_0^2\left(\frac{\Omega_{m,0}}{a^3}\right)$, $\Gamma_c \approx x_{ion} n_b \cdot c\,\sigma_T = H$

$$a = \left(\frac{H_0^2 \cdot \Omega_{m,0}}{H^2}\right)^{\frac{1}{3}} \approx \left(\frac{0.3}{x_{ion}^2 \cdot n_b^2 \cdot c^2 \cdot \sigma_T^2 \cdot t_H^2}\right)^{\frac{1}{3}}$$

$$\therefore z_{dec} = \frac{1}{a} - 1 \approx x_{ion}^{\frac{2}{3}} \cdot (10^{-5})^{-\frac{1}{3}} - 1 \approx 46 \cdot x_{ion}^{\frac{2}{3}} - 1$$