Physics C161; Problem Set #6

prof. kasen

due Friday, 3/01, at midnight

Problem 1: The Edge of the Universe

Even if the Universe were infinite in size, the size of the *observable universe* may be finite. If the universe began a finite time ago (as is the case in many interesting cosmological models), some parts of the universe will be so far away that the light from them has not yet had time enough to reach us. The "edge" of the observable universe (i.e., the farthest thing we can see today) is called the *cosmological horizon* (or sometimes the *particle horizon*).

1a) Consider light traveling through the FRW metric for a universe that is static, so $a(t)$ = constant. Calculate the current proper distance[1] to the cosmological horizon (from which light emitted at $t = 0$ will just be reaching us now) You answer will be in terms of t_0 (the current age of the universe).

[1] Recall how proper distance is related to the comoving coordinate r that appears in the in the FRW metric.

1b) Consider now light traveling through a universe that expands[2] as $a(t) \propto t^{2/3}$. Calculate the current proper distance to the cosmological horizon in this expanding model.

[2] You will see in the next problem that this expansion law corresponds to a flat universe filled only with matter. For now, you can figure out the proportionality constant by using the condition on $a(t)$ at the current time t_0.

1c) How does the cosmological horizon you find for this expanding universe compare to what you found in part a) for a static universe? Give a brief (couple sentence) reason why this makes sense.

1d) Say that in this same universe (that expands as $a(t) \propto t^{2/3}$) we observe a galaxy with a redshift $z = 3$. What is the current proper distance to this galaxy?

1e) Argue that the redshift of a galaxy increases with its proper distance, and that the redshift becomes infinite for a galaxy at the cosmological horizon.

comment: In an expanding universe, galaxies with larger redshift z are at a greater distance. Since measuring redshift is fairly easy (while measuring distances is hard) we often use z as a proxy for how far away a galaxy is. But note that the relationship between proper distance and z is not linear. And the exact relationship will depend on the behavior of $a(t)$.

1f) For this universe that expands forever as $a(t) \propto t^{2/3}$, find the time at which light emitted *now* from a source at coordinate r_{source} will reach us in the future.

1g) From your work on part d), argue that, if wait long enough, eventually *every* source of light in the universe will become visible to us.

comment: In this cosmological model (where cosmic expansion proceeds forever as $a(t) \propto t^{2/3}$) the observable universe continues to grow with time without bound. But later on we will see that there are other model universes where the cosmic expansion is so fast that eventually the light from far away sources will never be able to reach us. In such a universe, distant galaxies will eventually fall out of view, and astronomers living billions of years from now will have a much more boring night sky. It appears that the fate of our own universe may be exactly this.

Problem 2: A Matter Filled Universe

THE DYNAMICAL EVOLUTION OF a homogenous, isotropic Universe is described by two equations; the *Friedman equation* describes the evolution of the scale factor[3] $a(t)$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{\kappa c^2}{R_0^2 a^2} \tag{1}$$

[3] By convention, we set the scale factor equal to unity at the present time, $a(t_0) = 1$. At any other time $a(t)$ tells us by what factor the Universe is smaller or bigger than it is today.

and the *fluid equation* describes the evolution of the energy density of components inside the Universe

$$\frac{d\epsilon}{dt} = -3\frac{\dot{a}}{a}(\epsilon + P) \tag{2}$$

To these must be added a third equation – the *equation of state* – relating the pressure, P, to the energy density, ϵ,

$$P = w\epsilon \tag{3}$$

We will usually consider cases where the equation of state parameter, w, is a dimensionless constant[4].

[4] More generally, w, need not be constant and may be a function of ϵ. This does not really change the approach we use for calculating for the dynamics of the Universe, but the integrations become more difficult and generally have to be done numerically.

2a) Show that the energy density varies with the scale factor as

$$\epsilon = \epsilon_0 a^{-3(1+w)} \tag{4}$$

where we use the standard notation is that all quantities with a subscript 0 are evaluated at the present time, t_0, i.e., $\epsilon_0 = \epsilon(t_0)$.

2b) Show that the curvature[5] of the Universe at the present time is given by

$$\frac{\kappa}{R_0^2} = -\frac{H_0^2}{c^2}(1 - \Omega_0) \tag{5}$$

[5] Recall that we use notation that splits apart the degree of curvature and its sign. Here κ is equal to either $0, -1$, or $+1$, and tells us the sign of the curvature, with 0 indicating flat (no curvature), +1 indicating positive (i.e., "spherically") curvature and -1 indicating negatively (i.e., "hyperbolically") curvature. The quantity R_0 is the current "radius of the universe", and the Gaussian curvature is $1/R_0^2$.

where the Ω_0 is the energy density scaled by the critical energy density

$$\Omega_0 = \frac{\epsilon_0}{\epsilon_{c,0}} \qquad \text{where} \;\; \epsilon_{c,0} = \frac{3c^2 H_0^2}{8\pi G} \tag{6}$$

and the Hubble constant is defined by

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} \tag{7}$$

comment: This result shows that if we measure the current energy density of the Universe, Ω_0 and the current rate of expansion, H_0, we can determine the curvature R_0 and κ. When the energy density is at the critical density, $\Omega_0 = 1$, the Universe is flat. For $\Omega_0 > 1$ the Universe is positively curved (and called a *closed* Universe). For $\Omega_0 < 1$ the Universe is negatively curved (and called an *open* Universe).

2c) Combine the equations to derive a particular useful equation for the dynamical Universe

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left[\frac{\Omega_0}{a^{3(1+w)}} + \frac{1-\Omega_0}{a^2} \right] \tag{8}$$

comment: We define the Hubble parameter as $H = \dot{a}/a$ so the equation you derived is often written

$$\frac{H^2}{H_0^2} = \frac{\Omega_0}{a^{3(1+w)}} + \frac{1-\Omega_0}{a^2} \tag{9}$$

This equation is for a Universe with only one component of energy density within it. The actual Universe is filled with different kinds of energy density (matter, radiation, dark energy,...) each with a different equation of state parameter w. For a multiple component universe, the generalization is simply

$$\frac{H^2}{H_0^2} = \sum_i \frac{\Omega_{i,0}}{a^{3(1+w_i)}} + \frac{1-\Omega_0}{a^2} \tag{10}$$

where $\Omega_{i,0}$ and w_i are the scaled energy density and equation of state parameter for each of the components. The symbol Ω_0 here is the sum of all of the components, $\Omega_0 = \sum_i \Omega_{i,0}$.

2d) Consider a flat universe ($\Omega_0 = 1$) with only cold, pressureless matter ($w = 0$) within it. Solve for the time-dependent scale factor $a(t)$.

2e) What is the age of the Universe, t_0, for the flat, matter-dominated Universe? Use the fact that $a(t_0) = 1$.

2f) Show that the flat, matter-dominated Universe is always expanding and decelerating[6].

[6] By expanding we mean that $a(t)$ is increasing, and by decelerating we mean that the rate of expansion, $\dot{a}(t)$, is decreasing.

Problem 3: Radiation Universe

OUR UNIVERSE ALSO contains a homogenous, isotropic distribution of *radiation* called the Cosmic Microwave Background. Radiation refers to particles with zero rest mass like photons[7]. The pressure of radiation is $p = \epsilon/3$ so $w = 1/3$.

[7] Neutrinos have a rest mass but it is very small, so they too act as a background of radiation.

3a) Consider a flat universe $(\Omega_0 = 1)$ with only radiation $(w = 1/3)$ within it. Solve for the time-dependent scale factor $a(t)$.

3b) Find an expression for the current age of the Universe for a flat, radiation dominated Universe. Is it greater than or less than t_0 for the flat matter-dominated case?

3c) The flat universe ($\Omega_0 = 1$) will keep expanding forever, but a closed ($\Omega_0 > 1$) radiation dominated Universe will at some point stop expanding and turn around, and then start shrinking. For a closed radiation dominated Universe find an expression for the maximum size that the scale factor ever reaches (as a function of Ω_0, the scaled radiation density at the present time)[8].

[8] You do not need to integrate to solve for $a(t)$ here; you can use the dynamical equation before integration to find the turnaround point of the Universe. Think about the speed at which $a(t)$ is changing at this turnaround point.

3d) Optional; not graded The solution for $a(t)$ for the case $\Omega_0 \neq 1$ is harder to work out, but can be done in a straightforward fashion and put in a relatively simple form. Show that for an open ($\Omega_0 < 1$) radiation dominated Universe

$$a(t) = \left[(H_0 t + \sqrt{\Omega_0})^2 - \Omega_0\right]^{1/2} \tag{11}$$