Prob 1.

1a). $ds^2 = -c^2dt^2 + a^2(t)[dr^2 + S_k^2(r)d\Omega^2]$, take $dt=0$, we have

$\Rightarrow ds = a\,dr$

for photon, $ds^2 = 0 = -c^2dt^2 + a^2 \cdot dr^2$

$\Rightarrow dt = \frac{a}{c}dr \Rightarrow \int dt = t_0 = \frac{a}{c} \cdot r$

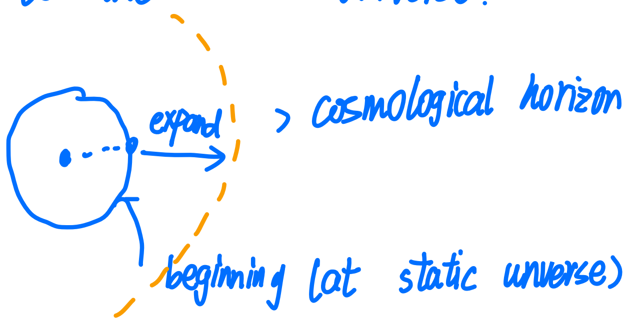
co-distance $\Rightarrow r = \frac{ct_0}{a}$

proper distance: $\int ds = a \int dr = ct_0$

1b). for proper distance: $ds = At^{2/3}dr$, $a = (\frac{t}{t_0})^{2/3}$

$d_H = a(t_0)\int_0^{t_0} c\, a(t)^{-1}dt = \int c \cdot (\frac{t}{t_0})^{-2/3}dt = 3ct_0$

1c). in the universe that expands, the horizon we can see is moving out forward, thus when the light reaches us, it means a further boundary compare to the static universe.



1d) $z = \frac{a(t_0)}{a(t_e)} - 1 \Rightarrow \frac{t_0^{2/3}}{t_e^{2/3}} = 4 \Rightarrow (\frac{t_0}{t_e}) = 8$, $t_e = \frac{t_0}{8}$

$\therefore d_H = a(t_0)\int_{t_e}^{t_0} c \cdot a(t)^{-1}dt = c \cdot 3t^{1/3}\Big|_{\frac{t_0}{8}}^{t_0} \cdot t_0^{-1/3} = \frac{3ct_0}{2}$

1e).

$z = \frac{R(t_0)}{R(t_e)} - 1 \approx \frac{R(t_0)}{R(t_0)\,(1+(t_e-t_0)H(t_0))} - 1 \approx (t_0 - t_e)H(t_0)$

here we take $t_0 - t_e \approx \frac{D}{c}$, so that $z \approx \frac{D}{c}H(t_0)$, it grows up when D become large.

1f) $dt = \frac{a}{c}dr \Rightarrow c\int_{t_0}^{t_0+\Delta t}(\frac{t_0}{t})^{2/3}dt = \int dr \Rightarrow (t_0+\Delta t)^{\frac{1}{3}} - t_0^{\frac{1}{3}} = \frac{r_{source}}{3c\,t_0^{2/3}}$

$t_{arrive} = t_0 + \Delta t = [\frac{r_{source}}{3c\,t_0^{2/3}} + t_0^{\frac{1}{3}}]^3 = t_0 \cdot (\frac{r_{source}}{3ct_0} + 1)^3$

1g). if we wait long enough so that we can take: $t_0 \to \infty$

so that $d_H \propto t_0 \to \infty$, every source of light is visible to us.

Prob 2.

2a) take (3) into (2):

$$\dot{\epsilon} = -3\frac{\dot{a}}{a}\epsilon(1+w) \Rightarrow \frac{\dot{\epsilon}}{\epsilon} = -3(1+w)\frac{\dot{a}}{a}$$

$$\frac{d\ln\epsilon}{dt} = -3(1+w)\frac{d\ln a}{dt} \Rightarrow \epsilon = \epsilon_0 a^{-3(1+w)}$$

2b) Friedman equation at $t=t_0$:

$$\frac{8\pi G}{3c^2}\epsilon(t_0) - H_0^2 = \frac{kc^2}{R_0^2}, \text{ take } \epsilon_{c,0} = \frac{3c^2H_0^2}{8\pi G}.$$

then $$\left(\frac{\epsilon_0}{\epsilon_{c,0}} - 1\right)H_0^2 = \frac{kc^2}{R_0^2} \Rightarrow \frac{k}{R_0^2} = -\frac{H_0^2}{c^2}(1-\Omega_0), \ \Omega_0 = \frac{\epsilon_0}{\epsilon_{c,0}}$$

2c). $$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\cdot\epsilon - \frac{kc^2}{R_0^2}$$

$$= H_0^2\frac{\epsilon_0}{\epsilon_{c,0}}\cdot\frac{\epsilon}{\epsilon_0} + \frac{H_0^2}{c^2}(1-\Omega_0) = H_0^2\left[\Omega_0 \cdot a^{-3(1+w)} + \frac{1-\Omega_0}{a^2}\right]$$

2d). $\epsilon = \epsilon_0 a^{-3}$

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\cdot\left[\Omega_0 a^{-3} + \frac{1-\Omega_0}{a^2}\right]$$

$$\Rightarrow \dot{a}^2 = H_0^2\left[\frac{\Omega_0}{a} + 1-\Omega_0\right] \Rightarrow \left(\frac{da}{dt}\right)^2 = \frac{H_0^2}{a}$$

$$\frac{da}{dt} = H_0\cdot a^{-1/2} \Rightarrow a^{1/2}da = H_0 dt \Rightarrow da^{3/2} = \frac{3H_0}{2}dt$$

$$a(t) = \left(\frac{3H_0}{2}t\right)^{2/3} \propto t^{2/3}$$

2e) $$a(t_0) = \left(\frac{3H_0t_0}{2}\right)^{2/3} = 1, \quad \frac{3H_0t_0}{2} = 1 \Rightarrow t_0 = \frac{2}{3H_0}$$

2f) $a(t) \propto t^{2/3}$, $\dot{a}(t) \propto t^{-1/3} > 0$ $\therefore$ it always expands

$\ddot{a}(t) = -\frac{1}{3}a^{-4/3} < 0$ $\therefore$ it is decelerating.

Prob 3.

3a). $\frac{H^2}{H_0^2} = \frac{1}{a^4} \Rightarrow \frac{da}{dt} = \frac{H_0}{a} \Rightarrow \frac{1}{2} da^2 = H_0 dt$

$a(t) = (2H_0 t)^{1/2}$

3b). $t = \frac{1}{2H_0}$, $t_0 = \frac{2}{3H_0}$ is the age of matter-dominated universe.

$t < t_0$ it is less than the age of matter-dominated universe.

3c) Friedman equation:

$\frac{kc^2}{H^2 R^2} = -\frac{H_0^2}{H^2} \frac{1-\Omega_0}{a^2} = \Omega_r(t) - 1$, take $t = t_0$.

$\Rightarrow \frac{kc^2}{H_0^2 R_0^2} = -(1-\Omega_0) = \Omega_r(t_0) - 1$

$\Rightarrow \Omega_r(t_0) = \Omega_0$

$\because (\frac{\dot{a}}{a})^2 = H_0^2 [\frac{\Omega_0}{a^{-3(1+w)}} + \frac{1-\Omega_0}{a^2}]$

$\therefore$ when a is max, $H^2 = 0$

$\Rightarrow \frac{\Omega_0}{a^4} + \frac{1-\Omega_0}{a^2} = 0 \Rightarrow a_{max} = \sqrt{\frac{\Omega_0}{\Omega_0 - 1}}$