Physics C161; Problem Set #7

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due Friday, 3/08, at midnight

Problem 1: Dark Energy, the Accelerating Universe and the Cosmic Event Horizon

Observations have indicated that not only is our universe expanding, but that this expansion is *accelerating*. This can be explained by invoking a new kind of substance which has negative pressure. One such example is the cosmological constant, which has pressure $P = -\epsilon$ (which implies equation of state parameter $w = P/\epsilon = -1$).

1a) Show that the energy density of a substance with $w = -1$ remains constant even as the scale factor of the universe, $a(t)$, changes.

comment: Given the constancy of ϵ over cosmic expansion, energy with $w = -1$ is sometimes called a "cosmological constant" and given the label Λ.

1b) A substance with $w = -1$ will act to produce cosmic acceleration. To demonstrate this graphically, write the Friedmann equation in "effective potential" form

$$\frac{1}{2}\dot{a}^2 + V_{\text{eff}}(a) = E$$

where E is some constant. Sketch the shape of V_{eff}, then argue that universes containing only energy with $w = -1$ should accelerate.

1c) Any substance that acts to accelerate the expansion of the universe is called "dark energy". Argue (for example, using the potential energy approach) that any substance with $w < -1/3$ classifies as dark energy.

1d) Consider a flat universe consisting only of a cosmological constant with $w = -1$. Solve for the scale factor $a(t)$. Apply the condition $a(t_0) = 1$.

comment: Note that this solution does not have a "big bang" moment where $a \to 0$ at $t = 0$. A flat universe with nothing but Λ, while not strictly static, has a kind of steady-state quality to it, as the rate of expansion $H(t) = \dot{a}/a$ does not change over time. This spacetime is called "de-Sitter space"[1].

[1] An associated kind of spacetime considers a universe with a substance with *negative* energy density ($\Omega_{0,\Lambda} < 0$) and $w = -1$. The solution of $a(t)$ in this case gives an exponentially *shrinking* scale factor. This is called *anti-de Sitter space*. While it is not of much interest in cosmological models, it is of great current interest in studies of quantum gravity and black holes.

If we consider a universe that includes both ordinary (pressureless) matter and Λ, we would find that at early times the matter content of the universe dominates. When matter dominates, the scale factor should roughly follow the solution for a matter dominated universe, $a(t) \propto t^{2/3}$ which has a singularity at $t = 0$.

1e) In this flat universe filled with a cosmological constant, consider a galaxy that emits at comoving coordinate r that sends a signal to us now (i.e., at $t = t_0$). At what time in the future will this signal reach us? Write your answer in terms of ct_H (where $t_H = 1/H_0$ is the Hubble time).

1f) Show that there is comoving coordinate $r_{\rm hor}$, such that any galaxy further away than this ($r > r_{\rm hor}$) can no longer be observed. In other words, if a source located at $r > r_{\rm hor}$ sends us a signal to us now, it will never reach us.

comment: The value $r_{\rm hor}$ you calculated for de-Sitter space is often called the cosmic "event horizon" – it plays an analogous role as the event horizon in a black hole. Light signals emitted from behind the event horizon of a black hole will never reach an observer outside. Similarly, signals emitted from beyond the cosmic event horizon in deSitter space will never reach us. The cosmic event horizon is a finite bound of the observable universe.

1g) Consider a galaxy located just within the cosmic event horizon (i.e., at a coordinate $r < r_{\rm hor}$) that emits a signal towards us right now ($t = t_0$). Calculate the redshift of this light when finally observed on earth at some future time[2]. Show that the redshift is larger for a source closer to $r_{\rm hor}$ and goes to infinity for a source at the cosmic event horizon.

[2] Be careful; most often when calculating a redshift we are considering the case where light was emitted in the past and is observed now. But here we are considering light that is emitted now and observed in the future. Make sure to use the right expression for the redshift z in this case.

comment: The redshifting of light near the cosmic horizon is also analogous to what we discussed for a black hole. Light emitted from outside the event horizon of a black hole is gravitationally redshifted when seen by an observer far away. As the light source approaches the event horizon, the gravitational redshift goes to infinity.

We make this connection between the deSitter universe and black holes more explicit. We can change coordinates to make the deSitter metric look like a *static* spacetime. If we redefine the zeropoint of our time coordinate to call the present day $t_0 = 0$, then the familiar FRW metric is for deSitter spacetime, where $a(t) = e^{H_0 t}$

$$ds^2 = -c^2 dt^2 + e^{2H_0 t}\left[dr^2 + r^2 d\Omega^2\right]$$

(since deSitter space is flat, we used $S_k(r) = r$). Now we define a new set of coordinates (t', r') called *static coordinates*, related to the old ones by

$$r' = re^{H_0 t} \qquad t' = t - \frac{1}{2H_0} \ln\left[re^{2H_0 t} - \frac{c^2}{H_0^2}\right]$$

The θ and ϕ coordinates remain the same. After doing some math to transform the metric above into these static coordinates, and replacing H_0 with the horizon $r_{\rm hor} = c/H_0$, we would find that deSitter spacetime can be expressed as

$$ds^2 = -\left(1 - \frac{r^2}{r_{\rm hor}^2}\right) c^2 dt'^2 + \frac{dr'^2}{(1 - r'^2/r_{\rm hor}^2)} + r'^2 d\Omega^2$$

This looks *almost* like the Schwarzchild metric! There is a coordinate singularity at $r' = r_{\rm hor}$, which is an indication of the cosmic event horizon. However there is no true singularity (this metric remains well behaved at $r' = 0$, unlike the Schwarzchild metric). We could play the same games we did with the Schwarzchild metric to consider the effects of "gravitational" time dilation and "gravitational" redshift, and to calculate the paths of light. Our description of events using static coordinates will *seem* different than when using the FRW coordinates, but every physical prediction we make must be the same, since all we did was relabel our coordinates.

But if deSitter space is exponentially expanding, how is it possible that we can express it in *time-independent* static coordinates? Because in relativity there is no absolute sense of what is moving and what is not moving – it just depends on your coordinate frame. In deSitter space the expansion is always at the same rate (the Hubble parameter $H(t)$ is a constant H_0), and so the static coordinates just switch to a coordinate mesh that "moves along" with the accelerating spacetime. According to the equivalence principle, gravity and acceleration are different ways of looking at the same thing. So we can interpret deSitter space as is either accelerating (the usual FRW coordinates) or static but with a gravitational force (static coordinates).

Problem 2: Phantom Energy and the Big Rip

WE HAVE SEEN THAT dark energy with pressure $P < -\epsilon/3$ (i.e., $w < -1/3$) produces a repulsive sort of gravitational effect, which can drive an accelerated expansion of the Universe. Distant galaxies are being driven further and further away. But will this repulsive effect of dark energy ultimately rip apart our own Milky Way Galaxy? Will it rip apart our solar system, or... ourselves?

The answer is no, *unless* the equation of state parameter of the dark is $w < -1$. A substance with the weird property $w < -1$ has been dubbed "phantom energy"[3] and leads to a catastrophic "cosmic doomsday" in our future. You can read one of the original and highly cited papers on the topic (Caldwell et al (2003) . From the material we have covered so far in C161 you should find this paper to be largely understandable; and in this problem you will derive the core results of that work[4].

[3] The intial papers on phantom energy came out a few years after the Star Wars prequel *The Phantom Menance*, which maybe was an inspiration.

[4] We will do the problem choosing a specific value of $w = -5/3$, but is not too much harder to work out the general case of arbitrary w. You'll find that the Caldwell paper quotes the results for arbitrary w but leaves the derivation to you!.

2a) Show that if $w < -1$ the energy density of the dark energy *increases* as the universe expands.

comment: In an expanding universe, the energy within some volume is not necessarily constant. We can accept this by saying that global energy conservation does not necessarily hold in general relativity – an uncomfortable position, perhaps, but one that reflects the mathematical freedom available in the theory. Phantom energy is even more uncomfortable; not only does the global energy in the universe increase with a, the *local* energy density at one point in space also increases, as if energy were being created locally out of nowhere.

As discussed at the end of this problem, there are reasons to think phantom energy is unphysical. Nevertheless, current observations do not rule out dark energy with $w < -1$ and so it is interesting to explore the consequences if this were the case.

2b) Consider a flat universe with matter ($\Omega_{m,0}$) and "phantom" dark energy ($\Omega_{w,0}$) with $w = -5/3$. Calculate the evolution of the scale factor, $a(t)$, at late times, applying the condition that $a(t_0) = 1$ at the present time t_0. Since phantom energy will increasingly dominate over matter as $a(t)$ increases, you can ignore[5] the matter term when solving the Friedmann equation (i.e., approximate $\Omega_{m,0}a^{-3} \approx 0$)

[5] Because we are ignoring the $\Omega_{m,0}$ term, your solution will not be accurate at small $a(t)$, when matter dominates the phantom energy. For example, your solution will not have $a(0) = 0$. That is OK; while it would be straightforward to calculate the full solution numerically, all we are interested in here is the later time behavior when $a(t)$ is large.

2c) Show that the scale factor *diverges* (i.e., blows up to infinity) at some finite time we can call t_{rip}. Find the expression for t_{rip} in terms of $\Omega_{w,0}$, t_0 and the Hubble time $t_H = 1/H_0$.

comment: The divergence of the scale factor is called the *big rip*. As we will see below, if our universe is filled with phantom energy we will not be able to avoid a fate in which everything in the universe is torn to pieces as we approach the time t_{rip}.

2d) Observations suggest that the universe is flat with a current matter density of $\Omega_{m,0} \approx 0.3$ and a Hubble constant about $H_0 = 70$ km s^{-1} Mpc^{-1}. If the remaining energy density in the universe is phantom energy with $w = -5/3$ calculate how many years in the future we will experience the big rip.

THE BIG RIP IS SO CALLED because it will eventually tear apart all bound structures. For example, consider the solar system. The earth is held bound in its orbit by the gravity of a body of mass M (the Sun). How strong would phantom energy have to become to rip the earth and Sun apart? Well, the "acceleration equation" (or second Friedmann equation)[6] is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) = -\frac{4\pi G}{3c^2}\epsilon(1 + 3w) \tag{1}$$

[6] This equation can be derived from the First Friedmann equation and the fluid equation. See the Ryden textbook or the discussion in class.

We can see again that substances with $w < -1/3$ have a repulsive effect (i.e., produce positive accelerations, $\ddot{a} > 0$). If the distance between the earth and sun is R, the effective acceleration pulling them apart due to phantom energy is $R(\ddot{a}/a)$. This can be compared to the acceleration on the earth towards the earth produced by the gravitational pull of the sun, which is GM/R^2 (where M is the mass of the sun.) Thus we can expect phantom energy to overpower the gravitational binding of the earth and sun when

$$\frac{\ddot{a}}{a}R > \frac{GM}{R^2}$$

2e) Show that for $w = -5/3$, the condition above for the earth to become unbound from the sun will just be reached at a time

$$t = t_{\rm rip} - \frac{\sqrt{2}}{2\pi}T \tag{2}$$

where T is the period of the earth's orbit (equal to 1 year)[7].

[7] The period of the earth's orbit, assumed to be circular is

$$T = \frac{2\pi R}{v}$$

where $v = \sqrt{GM/R}$.

comment: Given that $T = 1$ year, we find that our planet' will be torn from the Sun a little less than 3 months before the big rip. A larger structure like the Milky Way has a rotational period of around 180 million years, and so would be unbound earlier, about 60 million years before the big rip.

As we approach the big rip, everything will start to get torn apart. Using the mass M and radius R of the earth in the calculation, we find that our planet will be torn apart about 30 minutes before the big rip. A very similar calculation shows that even individual atoms are torn apart about 10^{-19} seconds before the big rip[8]

[8] Atoms are held together by electric forces, not gravity. We could carry out the calculation using a simple classical model of atom where an electron of mass m_e and charge e orbits around a proton. . Since the electric force is so much stronger than the gravitational force, and the radius of atoms are small, it will only be very close to the big rip that atoms are torn apart.

2f) More likely we live in a universe with a cosmological constant (with $w = -1$) and not phantom energy (with $w < -1$). Show that for flat $w = -1$ universe (deSitter spacetime), the condition for whether the expansion rips things apart

$$\frac{\ddot{a}}{a}R > \frac{GM}{R^2}$$

Does not change with time. Therefore is a system is bound now, it will always remain bound.

comment: If the dark energy in our Universe is a cosmological constant, we expect the earth, the Milky way, and all other bound structures to remain intact for all of time. The difference with phantom energy is that the energy density $\epsilon(t)$ increases as the universe expands. Then we inevitably reach a time when the phantom energy inside any volume overpowers whatever attractive force was holding a structure together.

comment: There are reasons to think that such phantom energy is unphysical. In general relativity, mass/energy is described by the energy-momentum tensor $T^{\mu\nu}$. Mathematically, we have considerable freedom in defining $T^{\mu\nu}$, so physicists try try to define certain *energy conditions*, which restrict what $T^{\mu\nu}$ may actually be physically realizable. It is not obvious what the energy conditions should be, but some suggested ones are:

The *weak energy condition* states that the energy-density ϵ (described by the T^{00} component) must be non-negative. This seems like a reasonable physical requirement, and we have no experimental evidence that it is violated.

The *strong energy condition* states that mass/energy must always be gravitational attractive (i.e., $w > -1/3$). This may initially seem like a reasonable restriction, but it can be shown to be violated by quantum vacuum energy and certain scalar quantum fields, and we have experimental evidence that dark energy exists and is driving the acceleration of the universe. So as far as we can tell, the strong energy condition is *not* always obeyed.

The *dominate energy condition* states that the pressure $|P| < \epsilon$ or equivalently $|w| < 1$. This condition assures that mass-energy never flows faster than the speed of light. Phantom energy would violate this condition, opening up some worrisome possibilities of causality violation and perhaps leading to certain unpleasant instabilities (see e.g., this post). Many people dismiss phantom energy for its violation of the DEC, but a considerable amount of theoretical work has tried to find realizations of phantom energy that would avoid the unpleasant consequences of $w < -1$.