Prob 1

1a). state equation: $P = w\rho c^2 = -\rho c^2 = -\epsilon$

fluid equation: $\frac{d\epsilon}{dt} = -3\frac{\dot{a}}{a}(\epsilon + P) = 0$

$\therefore \dot{\epsilon} = 0$, $\epsilon = \text{constant}$

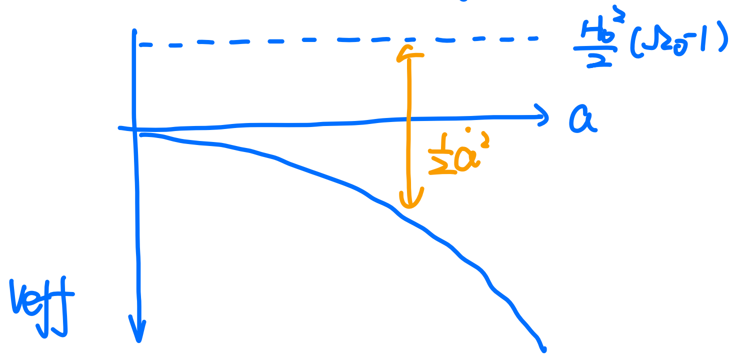
(b) friedmann equation: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\left[\frac{\epsilon_{m,0}}{a^3} + \frac{\epsilon_{r,0}}{a^4} + \epsilon_\Lambda\right] - \frac{kc^2}{R^2}$

$= H_0^2\left[\frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{r,0}}{a^4} + \Omega_\Lambda\right] - H_0^2 \cdot \frac{1-\Omega_0}{a^2}$

$$\Rightarrow \frac{1}{2}\dot{a}^2 \underbrace{- H_0^2\Omega_\Lambda a^2 - \frac{H_0^2}{2}\left(\frac{\Omega_{m,0}}{a} + \frac{\Omega_{r,0}}{a^2}\right)}_{V_{eff}(a)} = \frac{H_0^2}{2}\cdot(\Omega_0 - 1)$$

$\Omega_{m,0} = \frac{\epsilon_{m,0}}{\epsilon_{c,0}}$, $\Omega_{r,0} = \frac{\epsilon_{r,0}}{\epsilon_{c,0}}$, $\Omega_\Lambda = \frac{\epsilon_\Lambda}{\epsilon_c}$, $\epsilon_{c,0} = \frac{3c^2H_0^2}{8\pi G}$

for universe only containing energy with $w=-1$:



$\dot{a}^2$ increase as a get larger.

1c). $\epsilon = \epsilon_0 a^{-3(1+w)}$ with $w < -\frac{1}{3}$

$\therefore \epsilon > \epsilon_0 a^{-2}$

$\therefore \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\frac{\Omega_{sub,0}}{a^{-3(1+w)}}\right] - H_0^2 \cdot \frac{1-\Omega_0}{a^2}$

$\Rightarrow \frac{1}{2}\dot{a}^2 + (-H_0^2)\cdot\frac{\Omega_{sub,0}}{2a^{-1-3w}} = \frac{H_0^2}{2}\cdot(\Omega_0 - 1)$

$\because w < -\frac{1}{3}$ $\therefore -1-3w > 0$ $\therefore V_{eff}(a) \propto -a^k$, $k \geq 0 \Rightarrow$ accelerate expanding.

1d).

$$\dot{a} = \sqrt{\frac{8\pi G}{3}\rho_\Lambda} \Rightarrow a(t) = e^{\sqrt{\frac{8\pi G}{3}\rho_\Lambda}\cdot(t-t_0)}$$

$$\therefore H(t) = \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G\rho_\Lambda}{3}} \quad \therefore a(t) = \exp(H_0(t-t_0))$$

1e). $ds^2 = c^2dt^2 + a^2(t)\cdot dr^2 = 0$

$$\Rightarrow \int_{t_0}^{t_{reach}} \frac{c\,dt}{a(t)} = \int_0^r dr = r \quad , a(t) = e^{H_0(t-t_0)}$$

$$\frac{r}{c} = -\frac{1}{H_0} e^{-H_0(t-t_0)}\Big|_{t_0}^{t_{reach}} = -\frac{1}{H_0}\left[e^{-H_0(t_{reach}-t_0)} - 1\right]$$

$$\frac{rH_0}{c} = 1 - e^{-H_0\cdot\Delta t} \Rightarrow \Delta t = -\frac{1}{H_0}\ln\left(1-\frac{H_0 r}{c}\right)$$

$$t_{reach} = t_0 - t_H\cdot\ln\left(1-\frac{r}{ct_H}\right)$$

1f) if we want $t_{reach} \to \infty$, we need $\frac{r}{ct_H} \to 1$

$$\therefore r_{hor} = ct_H = \frac{c}{H_0}$$

1g) $\because \frac{f_{ob}}{f_{em}} = \frac{a(t_{em})}{a(t_{ob})} \quad \therefore z = \frac{a(t_{ob})}{a(t_{em})} - 1$

$$z = \frac{a(t_0 - t_H\ln(1-\frac{r}{ct_H}))}{a(t_0)} - 1 = \frac{1}{1-\frac{r}{ct_H}} - 1$$

$z'(r) = \frac{-1}{(1-\frac{r}{ct_H})^2}\cdot\left(-\frac{1}{ct_H}\right) > 0 \Rightarrow z$ is increasing as r getting closer to r_{hor}

when $r = r_{hor} = ct_H$, $z(r) = $ infinity.

Prob 2.

2a) $\because \epsilon = \epsilon_0 a^{-3(1+w)}$, $w<-1$ $\therefore \epsilon \propto a^k, k>0$

as a getting larger, ϵ becomes larger.

2b): $\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\frac{\Omega_{m,0}}{a^3} + \Omega_{w,0}\cdot a^2 + \frac{1-\Omega_0}{a^2}\right]$

$\dot{a}^2 = H_0^2\left[\frac{\Omega_{m,0}}{a} + (1-\Omega_0) + \Omega_{w,0}a^4\right]$

for flat space and neglect $\Omega_{m,0}/a^3$, we have:

$\dot{a} = H_0\sqrt{\Omega_{w,0}}\cdot a^2 \Rightarrow a(t) = \frac{1}{1-H_0\sqrt{\Omega_{w,0}}\cdot(t-t_0)}$

2c) $1 = H_0\sqrt{\Omega_{w,0}}(t-t_0) \Rightarrow t_{rip} = t_H\cdot\frac{1}{\sqrt{\Omega_{w,0}}} + t_0$

2d) $\frac{\Omega_{m,0}}{a^3(t_0)} + \Omega_{w,0}\cdot a^2(t_0) = 1 \Rightarrow \Omega_{w,0} = 0.7$

$\Delta t = t_{rip} - t_0 = \frac{1}{\sqrt{\Omega_{w,0}}\cdot H_0} \approx 5.12\times10^{9}$ years

2e) $\frac{\ddot{a}}{a}R = -\frac{4\pi G R}{3c^2}(-4)\,\epsilon_0 a^2 = \frac{16\pi G R}{3c^2}\epsilon_0 a^2 > \frac{GM}{R^2}$

$\Rightarrow a^2 > \frac{3Mc^2}{16\pi\epsilon_{w,0}R^3}$

suppose $t = t_{rip} - x$, then:

$a^2(t) = \frac{1}{(1-H_0\sqrt{\Omega_{w,0}}(t-t_0))^2} > \frac{3Mc^2}{16\pi\epsilon_{w,0}R^3}$, $\Omega_{w,0} = \frac{\epsilon_{w,0}}{\epsilon_{c,0}}$

$\Rightarrow H_0\sqrt{\Omega_{w,0}}\cdot x = x\cdot\sqrt{\frac{\epsilon_{w,0}}{3c^2/(8\pi G)}} < \sqrt{\frac{16\pi\epsilon_{w,0}R^3}{3Mc^2}}$

$\Rightarrow x < \sqrt{\frac{2R^3}{MG}} = \frac{\sqrt{2}}{2\pi}\cdot\frac{2\pi R}{\sqrt{GM/R}} = \frac{\sqrt{2}}{2\pi}\cdot T$

$\therefore t = t_{rip} - \frac{\sqrt{2}}{2\pi}T$

2f) $\frac{\ddot{a}}{a}R = -\frac{4\pi G}{3c^2} R\epsilon(1-3) = \frac{8\pi G}{3c^2} R\epsilon$

$\because \epsilon = \epsilon_0 a^{-3(1+w)} = \epsilon_0$

$\therefore \frac{\ddot{a}}{a}R = \text{constant}$

$\therefore$ a system is bound now, it will always remain bound.