Physics C161; Problem Set #3

prof. kasen

due Friday, 2/9, at midnight

Problem 1: Shapiro Delay

Massless particles like photons move on lightlike curves with $ds^2 = 0$ (also called "null geodesics"). Say you shoot a a ray of light (i.e., beam of photons) towards the sun. You can assume the sun is a perfect sphere of mass M such that the Schwarzchild metric applies.

1a) Write an expression for the coordinate velocity dr/dt of light moving radially towards or away from the sun.

comment: The speed of light appears to deviate from c, but this is only a *coordinate velocity* that measures how many r labels a beam of light passes in some unit of time. This velocity is just a relative quantity, and depends on the particular coordinates we choose. The *invariant* property of light is that it moves on light-like or null geodesics with $ds^2 = 0$.

Say you shoot a laser beam radially towards the sun. It travels from the Earth and hits Venus, which happens to be along the radial path to the sun. After reflecting off of Venus, the laser beam returns radially to Earth, where you detect it a time Δt after you sent it.

1b) Calculate[1] the total time, Δt, that it takes the light to make the round trip from Earth to Venus and back. Write your expression in terms of the radial coordinates of Earth (r_E) and Venus (r_V) and the mass of the sun M.

1c) In flat spacetime, the light travel time from Earth to Venus and back is $2(r_E - r_V)/c$. You have found that it takes light longer to travel through curved spacetime, and there is an extra term

$$\Delta t = \frac{2(r_E - r_V)}{c} + \Delta t_{\text{shap}}$$

The Schwarzchild radius of the sun is $r_s \approx 3$ km, while the distance[2] of Venus is on average around $r_V \approx 0.7$ AU. Calculate the value of the "*Shapiro delay*", Δt_{shap} for this problem.

comment: Irwin Shapiro suggested just this kind of experiment (bouncing light off of Venus) to test general relativity. The extra time taken on the trip (called the *Shapiro delay*) is pretty small, but experimentally measurable. The effect will be even larger if Venus is behind the sun, so that the light beam travels closer by the sun on its

[1] The technique here take your expression for dr/dt (which describes a radial lightlight trajectory) and separate the two variables r and t onto different sides of the equation. The you can integrate one side over dr and one over dt, using the appropriate limits for the stopping and starting coordinates of the light's path.

(Don't forget that when taking a square root you get a $\pm$. Choosing the minus sign means that for each increment dt the radial coordinate changes by a value $dr < 0$ (so light moves inward). Choosing the plus sign is just the opposite.)

By the way, we are ignoring the mass of Earth and Venus since they are much less than that of the sun, and just using the Schwarzchild metric for the sun of mass M.

[2] The Schwarzchild coordinates r_E and r_V are not strictly the same as the distance from the Sun to Earth and Venus (see your previous calculation of proper distances in the metric). However, for $r \gg r_s$ the radial coordinate is approximately equal to distance.

passage. By bouncing radio waves off of Venus, scientists has verified the time dilation effect.

The Shapiro delay has also been used to measure the mass of stars in binary systems, by looking when the light from one star (usually a pulsar) passes by its companion on its way to us. Once can then measure the Shapiro delay to infer the mass of the companion. This has been used to measure the mass of neutron stars, showing that some can be surprisingly massive.

Problem 2: Einstein Ring

In class we showed that a light beam passing by a spherical mass M at an impact parameter b undergoes a deflection[3] by an angle

$$\alpha = \frac{2R_s}{b} = \frac{4GM}{c^2 b} \tag{1}$$

If a source lies directly behind a lens along the observer line of sight, the light rays will be bent symmetrically around the lens and form a beautiful Einstein ring.

The figure shows the geometry for lensing that produces an Einstein ring, from which we can calculate the size of this ring.

[3] This formula was derived in the limit that the ray passes far away from the mass, $b \gg r_s$, which is the limit of a weak deflection by an angle $\alpha \ll 1$.

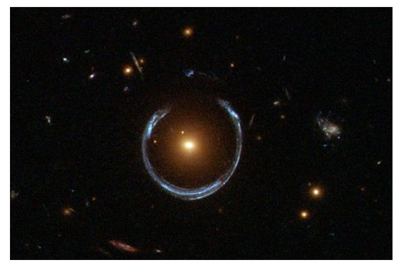
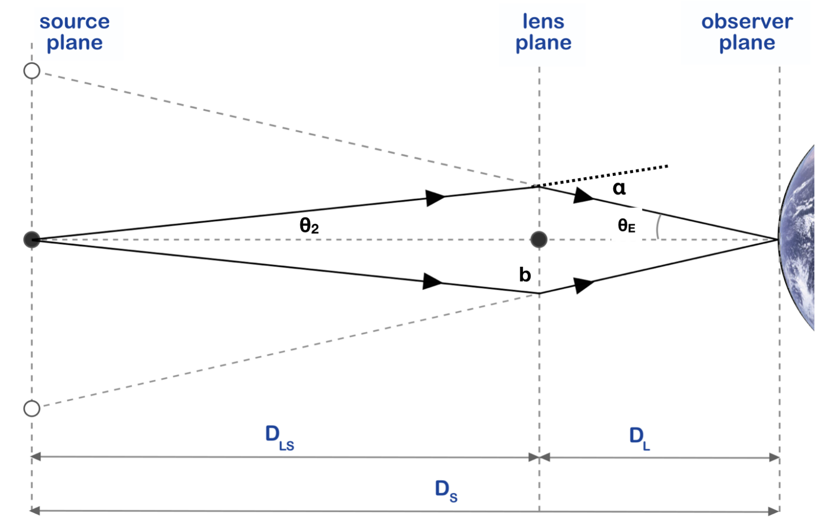


Figure 1: Image of an Einstein ring produced by lensing.

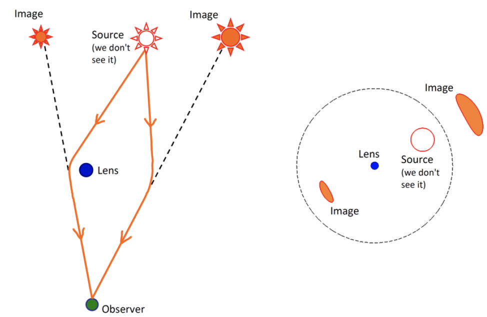


2a) From the geometry of the figure, derive that the angular size of the Einstein ring

$$\theta_E = \sqrt{\frac{4GM}{c^2} \frac{D_{LS}}{D_L D_S}} \tag{2}$$

where D_L is the distance to the lens, D_S the distance to the source, and D_{LS} the distance between the lens and source[4]

[4] These distances are all *angular diameter distances*. Later when we talk about the expanding Universe, we will see there are various ways to define a distance.



comment: If we observe an Einstein ring in the sky, and know the distances to the source and lens, we could use the measured ring angular size θ_E to constrain the mass of the lens. This is one way of applying gravitational lensing to infer the mass of objects, in particular those composed primarily of dark matter that could not be fully probed otherwise.

We only see a full Einstein ring when a source is located almost exactly behind the lens. If the source is a bit offset, you have the geometry shown in the margin figure where you may see multiple images of the same source. By doing a bit more geometry, you could figure out where there images appear and how to analyze them if you see them in your data.

Notice that gravitational lensing takes rays of light that would have gone off an in other directions and bends them into our view. The net effect is that the Einstein ring is *brighter* than the source itself – the lensing produces *magnification*. We can take advantage of the magnification to study far away sources that would be too dim to observe if it weren't for the lensing.

If the lens creating the Einstein ring is a galaxy of mass $M \approx 10^{12}\ M_\odot$, and the source and lens are at cosmological distances ($D \approx 10^9$ light years), your equation gives θ_E to be about few arcseconds, which is small but resolvable with good telescopes. If instead the lens is a stellar mass black hole ($M \approx M_\odot$) lensing a star in our Galaxy ($D \approx 10,000$ light years) the Einstein ring is more like 5×10^{-4} arcseconds. This would be very difficult to resolve, and so for stellar objects in the Galaxy we typically only detected the magnification due to lensing, and not a ring or multiple images. People have even looked for planets lensing their host stars, which can cause a tiny magnification when the geometrical alignment is correct. The lensing can tell us about the mass and orbit of the planet.

Problem 3: Constants of Motion and $E = mc^2$

THE "NATURAL" MOTION OF AN object in spacetime (i.e., the motion it will follow if not acted upon by outside forces) is along a *geodesic*, which is a path that maximizes proper time. There are a few different ways to calculate geodesics. In this problem set we use *constants of motion* to describe the paths.

Consider first the metric of ordinary flat (Minkowski) spacetime is (considering for simplicity only 1 spatial dimension)

$$ds^2 = -c^2 dt^2 + dx^2 \tag{3}$$

Since we will be interested in timelike paths, it makes sense to use the proper time interval $d\tau$ which will be positive. Recall that by definition $d\tau^2 = -ds^2/c^2$ so the Minkowski metric can be written

$$d\tau^2 = dt^2 - \frac{dx^2}{c^2} \tag{4}$$

The natural path of an object in flat spacetime is to move at constant velocity, which is a straight line in the spacetime diagram. As Figure 2 shows, such geodesics have constant slope, and so we can identify constants of motion (i.e., quantities that have the same value at all points on the path)

$$\frac{dt}{d\tau} = \text{ constant} \qquad \text{(in flat space)} \tag{5}$$
$$\frac{dx}{d\tau} = \text{ constant} \qquad \text{(in flat space)} \tag{6}$$

If you want to prove mathematically that these are indeed constants of motion, you can work out the optional problem at the end of this set. Otherwise, we will simply note that from the figure it is apparent that such slopes are constant along the straight line geodesic. Let's get a better feel from the constant of motion given by $dt/d\tau$.

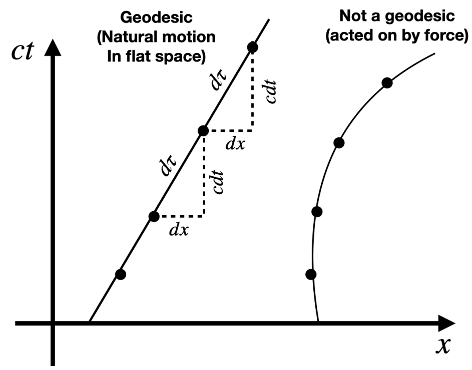


Figure 2: Paths through flat (Minkowski spacetime). The worldline on the left shows motion along a geodesic, which in flat space are straight lines (i.e., moving with constant velocity). The geodesic is seen to have constant slope, such that $dx/d\tau$ and dt/τ have the same value all along the path. The worldine on the right shows motion that is not along a geodesic, and so some external force must have accelerated the object.

3a) Use the Minkowski metric for $d\tau$ (Eq. 4) to replace $d\tau$ and show that with some manipulation

$$\frac{dt}{d\tau} = \gamma \tag{7}$$

where the Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$ and $v = dx/dt$.

3b) The constant of motion $dt/d\tau$ will be related to the energy of an object. To see this consider the non-relativistic limit $v \ll c$ and do a binomial expansion to show that

$$\frac{dt}{d\tau} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + ... \tag{8}$$

comment: Multiplying both sides of the above equation by the rest mass, m, of an object (where m is a constant) and by c^2 we have

$$mc^2\frac{dt}{d\tau} = mc^2 + \frac{1}{2}mv^2 + ... \tag{9}$$

We see that in the limit of $v \ll c$ our constant of motion is related to the Newtonian kinetic energy $mv^2/2$, plus another term, mc^2, which represents a rest mass energy. So we can associate the $dt/d\tau$ constant of motion with the *energy* of an object. For an object at rest ($v = 0$ and $\gamma = 1$) we have $E = mc^2$. You have arrived at the most famous equation in physics by simply considering the consequences of straight line motion in 4D spacetime!

The other constant of motion, $dx/d\tau$ can be associated with the momentum p_x of the object. We thus give our constants of motion suggestive notation

$$\frac{E}{mc^2} = \frac{dt}{d\tau} = \text{ constant} \qquad \text{(in flat space)} \tag{10}$$
$$\frac{p_x}{m} = \frac{dx}{d\tau} = \text{ constant} \qquad \text{(in flat space)} \tag{11}$$
$$\tag{12}$$

This illustrates how the relativistic momentum and energy are related to geometrical properties of paths through spacetime. We will next see how these definitions change in curved spacetimes.

Problem 4: Falling into a Black Hole

THE SPACETIME ABOVE A SPHERICALLY SYMMETRIC mass M (e.g., a black hole, or above the surface of the earth) is given by the Schwarzchild metric

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)c^2dt^2 + \frac{dr^2}{(1 - r_s/r)} + r^2 d\Omega^2 \tag{13}$$

In this problem we will explore the paths of objects moving in this metric. The "natural" motion of an object (i.e., the motion it will follow if not acted upon by outside forces) in a curved spacetime is along a *geodesic*, which is a path that maximizes proper time.

In the previous problem, we saw that there were constants of motion in flat spacetime, which we associated with energy and momentum. For a curved spacetime, like the Schwarzchild metric, the more general constants of motion are the slope of the spacetime path *multiplied by a metric term*. As discussed in class (and you can work out yourself if you want in the optional problem at the end of this problem set), the constants of motion in a metric are

$$\frac{E}{m} = g_{tt}\,\frac{dt}{d\tau} = \text{ constant} \qquad \text{(if } g \text{ is independent of } t\text{)} \tag{14}$$

$$\frac{p_r}{m} = g_{rr}\frac{dr}{d\tau} = \text{ constant} \qquad \text{(if } g \text{ is independent of } r\text{)} \tag{15}$$

Here g_{tt} refers to the metric term in front of the dt^2 term in the metric. For example, the Schwarzchild metric is independent[5] of coordinate t so a constant of motion is

$$\frac{E}{m} = g_{tt}\,\frac{dt}{d\tau} = \left(1 - \frac{r_s}{r}\right)c^2\,\frac{dt}{d\tau} \tag{16}$$

On the other hand, the Schwarzchild metric is not independent of r (as factors of $(1 - r_s/r)$ appear in it) so the radial momentum

$$\frac{p_r}{m} = g_{rr}\frac{dx}{d\tau} = \frac{1}{(1-\frac{r_s}{r})}\frac{dr}{d\tau} \tag{17}$$

[5] By "independent of a coordinate" we mean that that coordinate does not appear itself anywhere in the metric. For example, the coordinate t does not itself appear anywhere in the Schwarzchild metric (the dt differential doesn't count) so the Schwarzchild metric is "independent of t". Another way of saying the same thing is that the metric has a symmetry (i.e., is invariant) under a time translation transformation $t' = t + C$ where C is a constant. To see why this requirement is needed to have the constant of motion, see the optional problem at the end of the set. The reason is related to the famous Noether Theorem which says that for every symmetry there is an associated conserved quantity.

is *not* necessarily a constant of motion.

Using just the constants of motion and the metric, we can calculate the path of objects through spacetime. Let's consider someone falling radially in the Schwarzchild metric.

4a) Consider purely radial motion ($d\Omega^2 = 0$). Write the Schwarzchild metric in terms of proper time $d\tau^2 = -ds^2/c^2$. Then divide the metric Eq. 13 through by $d\tau^2$ and use the constant of motion Eq. 16 to show, after some rearranging

$$\epsilon = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 - \frac{GMm}{r} \tag{18}$$

where ϵ is a constant that depends on E, m and c.

4b) Your result above (Eq. 18) looks very similar to the Newtonian expression for the energy of a mass m in a gravitational field. We can make this identification explicit. As the Newtonian energy, E_N, does not include rest mass energy, we write $E_N = E - mc^2$. In the non-relativistic limit (where $v \ll c$ and hence $E_N \ll mc^2$) use a bionomial expansion to show that $\epsilon = E_N$.

comment: In the Newtonian limit where all speeds are $\ll c$ we also have $d\tau = dt$ (i.e., proper time is equal to coordinate time since time dilation is neglible). Therefore our equation becomes

$$E_n = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 - \frac{GMm}{r} \tag{19}$$

We have derived the Newtonian energy equation for an object in gravitational field using just the Schwarzchild metric and the principle of maximum proper time. Newtonian gravity arises as a limit of the motion of particles in the curved spacetime of GR! In Newtonian physics, we think of Eq. 19 and the sum of the kinetic energy and potential energy. In GR there is no real distinction between kinetic and gravitational potential energy. We instead have a constant of motion E/m that incorporates both.

WE CAN NOW REARRANGE Eq. 18 to

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2\epsilon}{m} + \frac{2GM}{r} \tag{20}$$

and solve this differential equation to determine the radial coordinate r of a mass as a function of its proper time τ.

4c) Imagine that you free fall into a black hole, starting from rest very far away (where your energy is just your rest mass energy, $E \approx$

mc^2). Solve Eq. 20 to show that the proper time you experience when falling from a radius r_2 to a radius r_1 is

$$\tau = \frac{2}{3}\frac{r_s}{c}\left[\left(\frac{r_2}{r_s}\right)^{3/2} - \left(\frac{r_1}{r_s}\right)^{3/2}\right] \tag{21}$$

4d) For the freefall just described, what is the expression for the proper time that it takes for you to fall from the event horizon r_s to the black hole singularity at $r = 0$?

comment: Your result shows that an object in freefall will pass through the event horizon of a black hole and reach the center in a finite proper time. For a stellar mass black hole, $r_s \approx 3$ km, the time to fall from the event horizon to the singularity is only about 6 microseconds. For a supermassive black hole of $M = 10^9 M_\odot$, the time is a bit less than 2 hours. Recall that this freefall is the path of *maximum proper time* so if you try to fight your inevitable path to the singularity (e.g., by firing some rocket thrusters), you will only reach the singularity sooner. You might as well relax and use your 2 hours to watch a Netflix movie[6].

[6] Assuming you already downloaded it, since you won't be able to connect to the internet from inside the event horizon.

4e) Consider a friend watching you fall into the black hole. The friend uses Schwarzchild coordinate time t (i.e., "far away" time) instead of your proper time τ. Use your constant of motion to show that your friend observes you to move with radial speed

$$\left|\frac{dr}{dt}\right| = c\sqrt{\frac{r_s}{r}}\left(1 - \frac{r_s}{r}\right) \tag{22}$$

comment: Starting from $r \gg r_s$, your friend would initially measure your speed to increase as you fell into the black hole, until at some point (you can show it is at $r = 3r_s$) they would measure your speed start to *decrease* as you fell, approaching $dr/dt = 0$ at the event horizon. From your friends point of view, you never actually get to r_s. Of course, we have shown above that from your point of view you do indeed fall through the event and reach $r = 0$ in finite time. The friend does not see this because any signal (e.g., light beams) sent by your will take longer and longer to get to them, with the time taken approaching infinity at r_s.

Optional Problem (not graded) 5: Geodesics and Constants of Motion

In flat Euclidean space, you can draw an infinite number of paths connecting two points. Of these, one path – a straight line – is singled out as having the *minimum* distance along it. In Newtonian physics, a mass undergoing "natural" motion (i.e., not being acted on by any force) moves in a straight line. We generalize this idea to curved spacetime by saying that the "natural" motion of a mass is the *geodesic* where the spacetime distance is an extremum.

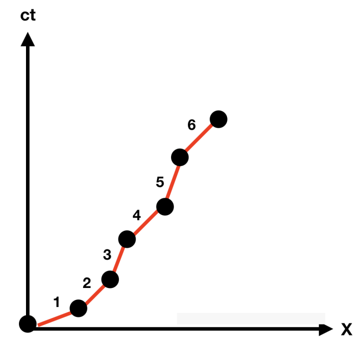


Figure 3: An arbitrary path through spacetime could be broken down into many small straight line segments. Each pair of segments follow the relations you derived

To calculate the spacetime time distance of a path, we break it up into numerous straight line segments (see Figure 3). The metric determines the spacetime distance ds^2 of each segment. Because massive objects are restricted to follow a *time-like* path[7], where $ds^2 <$ 0, it is convenient to write ds^2 as a proper-time, $d\tau^2 = -ds^2/c^2$, which is positive. We can add up all the segments to get the total proper-time of the path

$$\tau = \int d\tau = \int \sqrt{\frac{-ds^2}{c^2}} \tag{23}$$

[7] If the spacetime interval is space-like ($ds^2 > 0$) the object could move ahead of light, which would cause problems with causality. Thus we require that every step of a massive object's path be time-like.

The τ of a path is an invariant (i.e., has the same value in every coordinate system). In particular, in the coordinate system of someone moving along the path, position does not change ($dx' = 0$) and so τ is just the time elapsed in that frame (hence the name "proper time").

We want to find the path through spacetime that *maximizes*[8] the proper time τ. To do so in a simple way[9], let's analyze just two line segments, as shown in Figure 4. The path begins at an initial spacetime point P_i with coordinates $(t, x) = (0, 0)$ and moves to a final point P_f at $(\Delta t, \Delta x)$, passing through a middle point P_m along the way. We fix the end-points P_i and P_f and ask: "How should we pick P_m to maximize the proper time'?'

[8] To be precise, time-like geodesics correspond to *extrema* of proper time. But in basically all cases we will consider, the extrema are maxima, not minima.

[9] There is a field mathematical analysis called Calculus of variations which provides a general framework for finding an extremum of a path. Using it allows one to derive direct equations for the geodesics. We won't go down that road, but instead develop of the key points by finding constants of motion along the geodesic.

5a) Using the Minkowski metric, write down the proper time, $\Delta\tau_1$, for the first path segment (from P_i to P_m) and the proper time, $\Delta\tau_2$ of the second segm.ent (from P_m to P_f) as functions of the variables $\Delta t, \Delta t_1, \Delta x, \Delta x_1$

5b) Maximize the total proper time $\Delta\tau = \Delta\tau_1 + \Delta\tau_2$ with respect to Δx_1 (while holding Δt_1 and the endpoint coordinates Δt and Δx fixed) and show that

$$\frac{\Delta x_1}{\Delta\tau_1} = \frac{\Delta x_2}{\Delta\tau_2} \tag{24}$$

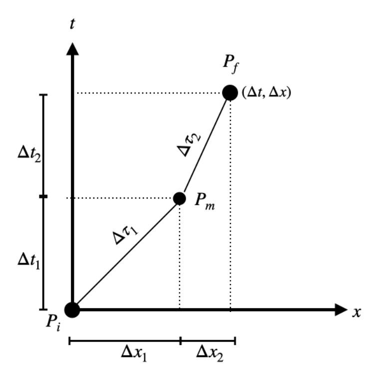


Figure 4: Spacetime path consisting of just two segments. The path starts at point P_i and moves to point P_m and then point P_f. Note that $\Delta x_1 + \Delta x_2 = \Delta x$ and $\Delta t_1 + \Delta t_2 = \Delta t$

comment: As seen in Figure 3, a path through spacetime can be built up by putting together many small straight line segments. We can apply the same calculation above to line segments 2 and 3, and every subsequent pair thereafter, which implies

$$\frac{\Delta x_1}{\Delta \tau_1} = \frac{\Delta x_2}{\Delta \tau_2} = \frac{\Delta x_3}{\Delta \tau_3} = \frac{\Delta x_4}{\Delta \tau_4} = \ldots \tag{25}$$

That is, on the geodesic, $\Delta x/\Delta \tau$ stays the same for all segments of the path.. We can take the limit that the path segments are infinitesimally small, which we notate by changing Δ to d and see that

$$\frac{dx}{d\tau} = \text{ constant} \tag{26}$$

We have found a "constant of the motion".

5c) Show that maximizing the proper time $\Delta\tau$ with respect to Δt_1 (while holding Δx_1 and the endpoint coordinates Δt and Δx fixed) leads to

$$\frac{\Delta t_1}{\Delta \tau_1} = \frac{\Delta t_2}{\Delta \tau_2} \tag{27}$$

comment: Similar to above, we can generalize the argument to multiple segments composing an arbitrary path, and conclude.

$$\frac{dt}{d\tau} = \text{ constant} \tag{28}$$

5d) Using the metric, show that your two constants of motion can be written as

$$\frac{dx}{d\tau} = v\gamma \qquad \frac{dt}{d\tau} = \gamma \tag{29}$$

where $v = dx/dt$ is the coordinate velocity and $\gamma = (1 - v^2/c^2)^{-1/2}$

5e) We often multiply the constant of motion $dt/d\tau$ by mc^2 and notate it with the symbol E

$$E = mc^2 \frac{dt}{d\tau} = mc^2\gamma$$

Show in the limit $v \ll c$ the binomial approximation gives

$$E = mc^2 + \frac{1}{2}mv^2 + \ldots$$

So that this constant of motion is a generalization of the Newtonian kinetic energy *plus* a rest-mass energy mc^2.

comment: Analogous arguments apply for motion in the y and z directions. We can write all of these results together as

$$\frac{d}{d\tau}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \gamma \frac{d}{dt}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c\gamma \\ \gamma u_x \\ \gamma u_y \\ \gamma u_z \end{pmatrix} = \text{ constant} \tag{30}$$

The above is just the definition of the **four velocity vector** U^{μ}, in special relativity, and multiplying it by the rest mass of an object (which is an invariant quantity) gives the **energy-momentum four vector** $p^{\mu} = mU^{\mu}$. This problem shows that our definitions of relativistic momentum and energy as constants of motion arise from the principle of extremal proper time.

Now let's generalize the above arguments to the curved spacetime of general relativity. We write the metric as

$$ds^2 = -g_t\, dt^2 + g_x\, dx^2 \tag{31}$$

where g_t and g_x are components of some general metric[10] and encode the curvature of the spacetime. We have defined g_t and g_x such that for flat Minkowski space $g_t = c^2$, $g_x = 1$. In a curved space these terms will be different and may be functions of the coordinates, $g_t = g_t(t,x)$ and $g_x = g_x(t,x)$.

[10] In general the metric can have mixed terms like $dxdt$

$$ds^2 = -g_t\, dt^2 + g_x\, dx^2 + 2g_{tx}dxdt$$

However, if we use orthogonal coordinates, these metric terms will be $g_{tx} = 0$. We won't bother with these terms here.

5f) Show, using an analogous argument to that above, that if the metric components g_t, g_x are independent of x (and hence Δx_1) that

$$g_x \frac{\Delta x_1}{\Delta \tau_1} = g_x \frac{\Delta x_2}{\Delta \tau_2} \tag{32}$$

comment: Generalizing as above to multiple segments along an arbitrary path, we conclude that $g_x\, dx/d\tau$ is a constant of the motion. Similarly, if the metric is independent of t we can carry out the analagous argument to show that $g_t`dt/d\tau$ is a constant of the motion. We can multiply these expressions by mass m to get the generalization of linear momentum and energy in a curved spacetime

$$p_x = g_x\, m\frac{dx}{d\tau} = \text{ constant (if } g \text{ is independent of } x) \tag{33}$$

$$E = g_t\, m\frac{dt}{d\tau} = \text{ constant (if } g \text{ is independent of } t) \tag{34}$$

For Minkowski space, the metric components are $g_t = c^2$ and $g_x = 1$.

comment: The constant of motion that arise here are related to *Noether's Theorem*, an important result that says under certain conditions each symmetry of a system results in a corresponding conserved quantity. A metric that is independent of x has space translational symmetry (as a transformation that shifts x by a constant value, $x' = x - D$ where D is a constant .does not change the metric). The resulting constant of motion for space translational symmetry is the momentum. A metric that independent of t has time translational symmetry and the associated constant of motion is the energy.