Physics C161; Problem Set #8

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due Friday, 4/12, at midnight

Problem 1: The Glow from the Early Universe

The universe is filled with a background of cosmic radiation. It was born in the early stages of the universe, where interactions of charged matter particles produced photons. Such photons were reabsorbed and scattered by the matter particles repeatedly such that the matter and radiation reached a *thermodynamic equilibrium* or a maximum entropy state. In thermodynamic equilibrium we do not need to calculate all of the detailed interactions of matter and radiation. Instead, statistical mechanics tells us that the radiation will be in its maximum entropy distribution, which for photons is the blackbody (or Planck) distribution

$$\epsilon_\nu(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_BT}-1}d\nu \tag{1}$$

The above expression gives the energy density of photons with frequencies within a tiny region between ν and $\nu + d\nu$. If we integrate over all frequencies we get the total energy density of photons

$$\epsilon = \int_0^\infty \epsilon_\nu(\nu)d\nu = a_R T^4 \tag{2}$$

where T is the temperature and a_R is the radiation constant[1].

Observations of the cosmic photon radiation – called the cosmic microwave background radiation (or CMB) – have shown that it is today a blackbody at a temperature $T_0 = 2.7$ K. At earlier times in the universe, the radiation was hotter. As the radiation energy density scales with the scale factor of the universe as $\epsilon_R \propto a^{-4}$ and the radiation density scales with temperature as $\epsilon_R \propto T^4$, it follows that the radiation temperature scales as $T \propto 1/a$. Thus the CMB temperature at other times in the universe is

$$T_{\rm CMB} = \frac{T_0}{a} = T_0(1+z) \tag{3}$$

where we used the relation between scale factor and redshift $1 + z = 1/a$.

To understand the basics of the CMB, we must study three important and distinct physical events occurring in its cosmic evolution:

Radiation/Matter equality: This is the epoch at which the energy density of radiation was equal to the energy density of matter.

[1] We can determine the radiation by direct integration of the blackbody expression Eq. 1. Changing variables to $x = h\nu/k_BT$ we can write the integral over all frequencies as

$$\epsilon = \frac{8\pi h}{c^3}\frac{k_B^4 T^4}{h^4}\int_0^\infty \frac{x^3}{e^x-1}dx$$

Looking up the integral (or typing it into Wolfram-alpha online) we find it is equal to $\pi^5/15$ so the result is

$$\epsilon = \frac{8\pi^5 k_B^4}{15c^3h^3}T^4$$

So we find that radiation constant can be expressed in terms of fundamental constants as

$$a_R = \frac{8\pi^5 k_B^4}{15c^3h^3}$$

The radiation constant is related to the Stefan-Boltzmann constant, $\sigma_{\rm sb}$ by $a_R = 4\sigma_{\rm sb}/c$.

In addition to photons, the universe is also filled with neutrinos, which (because of their very low mass) also behave like radiation (i.e., have kinetic energy much greater than their rest mass energy). Because neutrinos are fermions, the expression for the blackbody distribution and energy density of neutrinos is similar to, but slightly different than that of photons (which are bosons).

Recombination: This is the epoch at which the temperature of the universe cooled to the point where protons and neutrons began to significantly recombine into neutral hydrogen atoms. This greatly reduced the scattering cross-section of matter and helped it to become transparent.

Radiation/Matter Decoupling: The epoch at which photons stopped interacting regularly with matter, and were able to freely stream through the now nearly transparent universe.

Let us consider each of these phenomena, one by one

Radiation/Matter Equality

1a) Calculate the redshift, z_{eq} at which radiation/matter equality occurred. Give a numerical value for z_{eq} given the current values for matter $\Omega_{0,m} = 0.3$ and radiation[2] $\Omega_{r,0} \approx 8 \times 10^{-5}$.

[2] This value of $\Omega_{r,0}$ includes both neutrinos and CMB photons; if we were to only consider CMB photons, the value would be $\Omega_{CMB,0} \approx 5 \times 10^{-5}$

1b) What was the temperature (in Kelvin) of the CMB at the time of matter/radiation equality?

1c) In the "benchmark" model, the universe is flat and the Friedman equation is

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left[\Omega_{r,0} a^{-4} + \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0} \right] \tag{4}$$

For redshifts $z > z_{\text{eq}}$ the radiation term in this equation dominates and we can neglect the $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ terms on the right hand side. Making this approximation and assuming a flat universe, solve for $a(t)$ and determine the time[3]. after the big bang (in years) at which radiation/matter equality occurred.

[3] Because the conventional units of the Hubble constant (km/s/Mpc) are a bit awkward, it is convenient to remember instead the Hubble time, $t_H = 1/H_0$ which is about $t_H \approx 14$ billion years

Recombination

In early epochs of our universe, the ordinary matter (i.e., not the dark matter) consisted largely of unbound protons, neutrons and electrons. There were equal numbers of protons and electrons (to maintain net charge neutrality) but they were free (i.e., not bound to each other) until the temperature of the universe cooled to the point that it became favorable for protons and electrons to combine into neutral hydrogen atoms. As the universe became more neutral, interactions with the CMB and matter became less frequent.

According to statistical mechanics, the number density of ionized protons, n_p, free electrons n_e, and neutral hydrogen atoms (i.e., protons with a bound electron) n_H are related by the Saha equation

$$\frac{n_p n_e}{n_H} = \frac{1}{\lambda_{e,T}^3} e^{-Q/k_B T} \tag{5}$$

where $Q = 13.6$ eV is the binding energy of a hydrogen atom and

$$\lambda_{e,T} = \left[\frac{h^2}{2\pi m_e k_B T}\right]^{1/2} \tag{6}$$

is the thermal Debroglie wavelength of the electrons[4].

Say the total number density of baryons (ionized or recombined) is n_b. We can define the ionization fraction $x_{\rm ion}$ as the fraction of those baryons that are ionized, free protons, $n_p = x_{\rm ion} n_b$. Then the number of neutral hydrogen atoms is $n_H = (1 - x_{\rm ion}) n_b$. Since for every free proton there is a free electron, $n_e = n_p$.

[4] The factor of $e^{-Q/k_B T}$ in the Saha equation is the standard Boltzmann factor that reflects the fact that, in thermodynamic equilibrium when energy, the higher energy ionized state is less likely to be populated than the more bound neutral state. The factor of $1/n_e \lambda_{e,T}^3$ can be understood as a *statistical weight* that reflects the fact that there are a greater number of ways to configure a free electron in the ionized state compared to when the electron is bound in the neutral hydrogen atom.

1d) Write the Saha equation in terms of $x_{\rm ion}$. Then, assuming that $\lambda_{e,T}$ is a constant, show that the temperature for which the universe is halfway recombined (i.e., $x_{\rm ion} = 1/2$) is

$$T_{\rm rec} = \frac{Q}{k_B}\frac{1}{\zeta} \tag{7}$$

where ζ is a logarithmic factor that depends on n_b and $\lambda_{e,T}$.

comment: The solution you worked out is not actually an explicit solution for $T_{\rm rec}$ since the thermal deBroglie wavelength $\lambda_{e,T}$ itself depends on temperature. However, because the factor of $\lambda_{e,T}$ appears in a natural log, the result is not too sensitive to its exact value.

We can take away the key physical point about hydrogen recombination. Since $k_B T$ is roughly the average kinetic energy of free electrons, we might expect on energetic grounds that the electrons would get captured and recombine with protons once $k_B T \lesssim Q$, which would give a recombination temperature of $T_{\rm rec} \approx 15{,}000$ K. However, we need to also account for the fact that there are a lot more ways to configure a free electron in space than a bound one, and this increases the probability that electrons will be in an ionized state. The factor of ζ accounts for just this. For typical baryon densities in the early universe, we find $\zeta \approx 40$ and $T_{\rm rec} \approx 3750$ K.

1e) Given $T_{\rm rec} \approx 3750$ K, calculate the redshift of $z_{\rm rec}$ of recombination.

comment: As we have defined it, $T_{\rm rec}$ marks the point when $x_{\rm ion} = 1/2$, i.e., when the universe was half ionized and half neutral. But if you were to ask at what temperature the universe became almost completely neutral (say $x_{\rm ion} = 10^{-2}$) you would find that it occurs at just a slightly lower temperature than 3750 K, given that the factor of $x_{\rm ion}$ would appear in the log in Equation 7. Thus once the universe cools to the point where $T \approx T_{\rm rec}$ it becomes almost completely neutral quite quickly thereafter.

1f) In the "benchmark" model, the universe is flat and the Friedman equation is

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left[\Omega_{r,0} a^{-4} + \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0}\right] \tag{8}$$

For redshifts $z_{\rm rec} < z < z_{\rm eq}$ the matter term in this equation dominates and we can (approximately) neglect the $\Omega_{r,0}$ and $\Omega_{\Lambda,0}$ terms on the right hand side. Making this approximation, and using $\Omega_{m,0} = 0.3$, how many years after radiation/matter equality did recombination occur?

comment: A more careful calculation finds a value[5] of of $t_{\rm rec}$ around 240,000 years. Your calculation will be somewhat off because we have made a rough approximation for the expansion history of the universe, solving for only radiation for $z > z_{\rm eq}$ and only matter for $z < z_{\rm eq}$. A careful calculation which integrated the Friedmann equation including all terms would give a more accurate answer. It is not so hard to write a code to do the integration, or one can also refer to various online cosmology calculators .

[5] Different texts you read may give slightly different values for quantities like $t_{\rm rec}$, $z_{\rm rec}$ and the like. This may because they have taken different values of the cosmological parameters, or defined the time of recombination differently, or made different approximations than we did here.

Decoupling

THE TIME OF DECOUPLING denotes the epoch when photons stop interacting regularly with matter – i.e., when the universe becomes transparent to the CMB. At this point photons can stream freely through space, rarely bumping into anything. Decoupling occurs when the time for a photon to collide with matter becomes longer than the time for the universe to double in size. In lecture, we showed that the rate of collisions (i.e., average number of collisions of a particle per second) is

$$\Gamma_c \approx n v \sigma \tag{9}$$

where n is the number density of target particles, σ is the cross-section of the interaction, and v is the speed of light. In the case of CMB photons, the most important interaction is *electron scattering*, where the target particle is a free electron, and the cross-section is the Thomson cross-section, $\sigma_T = 6.6 \times 10^{-29}$ m^2. Since photons move at the speed of light, $v = c$ and $\Gamma_c = n_e c \sigma_T$.

Decoupling happens when the rate of collisions becomes slower than the expansion of rate of the universe. The critical condition for decoupling to occur is then

$$\Gamma_c \approx H \tag{10}$$

where H is the Hubble parameter, defined by $H = \dot{a}/a$

1g) Using Eq. 8 for H and the assumption that universe is in a matter dominated phase (i.e.,neglecting the $\Omega_{r,0}$ and $\Omega_{\Lambda,0}$ terms) derive an expression for the redshift of decoupling, z_{dec} in terms of the ionization fraction x_{ion}. Plug in the values for the physical constants and the cosmological parameters $\Omega_{m,0} = 0.3$, $n_{b,0} \approx 0.2\ \text{m}^{-3}$ and $t_H = 1/H_0 = 14$ billion years (but leave x_{ion} as a variable).

comment: Your result shows that if the universe remains ionized, $x_{\text{ion}} = 1$, CMB photons would not decouple until a redshift of $z_{\text{dec}} \approx 45$. However this is incorrect, she we showed above the recombination begins to set in about $z_{\text{rec}} \approx 1350$ and x_{ion} drops below 1 rather rapidly thereafter. By redshift $z \approx 1100$ the ionization fraction has dropped to around $x_{\text{ion}} \approx 0.008$ and you see from your result that $z_{\text{dec}} \approx 1100$, indicating that the condition for decoupling is then met ($\Gamma_c \approx H$). The CMB photons will stop interacting significantly with the matter at this point, and freely stream away.

Thus, recombination and decoupling are two distinct, but closely related physical phenomena. *Recombination* is the process of protons and electrons combining into neutral atoms, and it is precipitated by the temperature cooling below the recombination temperature $T_{\text{rec}} \approx 3570$ K. *Decoupling* is the process of the matter becoming transparent to photons, and it is *precipitated by* recombination, which binds up the electrons and removes the particles that scatter the photons. While we have spoken here of a "time of recombination" and a "time of decoupling", of course these are in actually gradual processes that occur over a period of time. If you are an interested in this subject, there is are plenty of interesting calculations to work out to more carefully model the evolution of the CMB.