Physics C161; Problem Set #2

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due Friday, 2/2, at midnight

Problem 1: Shortcut to Tomorrow

It is Tuesday 4 PM and as C161 class goes on, you think, "I can't believe I am going to have to wait 2 days for the next class! Why can't it come sooner?". After class ends, you remain in your chair, waiting motionless until the next lecture starts two days later.

Your impatient friend, on the other hand, realizes that she can take a *shortcut through time*. At 4 PM she jumps in her car and drives away at constant speed v. One day later (by your clock), she turns around and drives home at the same speed, arriving back to the Physics building at 4 PM on Thursday.

1a) Draw a spacetime diagram in your rest frame coordinate system (x, ct) that shows the paths of you and your friend from event A (Tuesday's class at 4PM) to event B (Thursday's class at 4 PM)

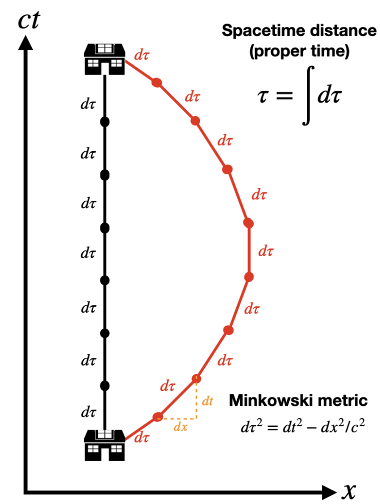

Figure 1: Paths in a spacetime diagram illustrate the location of things at different times. We can sum up the spacetime "length" of these paths by breaking it up into little segments and summing up the value of $d\tau$ (given by the metric) for each. For timelike paths, the spacetime length τ is the time experienced by someone on that path.

You and your friend began and ended at the same points in spacetimes, but you took different paths there, and as such traversed different *spacetime distances*. While in your drawing it may look like your friends path is longer, in fact the opposite is true. That is because the rule for calculating distances – the *metric* – is for flat (Minkowski) spacetime

$$ds^2 = -c^2 dt^2 + dx^2 \tag{1}$$

Physical objects can only move on *timelike* intervals[1] with $ds^2 < 0$. For timelike intervals, is convenient to multiply ds^2 by minus one so that we can work with a positive number, so we define

$$d\tau^2 = \frac{-ds^2}{c^2} = dt^2 - \frac{dx^2}{c^2} \tag{2}$$

where τ is a positive number that we call the *proper time* [2] (we divide by c^2 to get units of time). Whatever you want to call it, the important thing is that τ is just another way of writing s, the *invariant* measure of the spacetime "length" of some path.

This metric gives the spacetime distance along a tiny segment of a path. To get the total spacetime distance along a path, we integrate up ds (or equivalently $d\tau$) along that path (see Figure 1)

$$\tau = \int d\tau = \int \sqrt{dt^2 - \frac{1}{c^2}dx^2} \tag{3}$$

Where the integral is over the spacetime path.

[1] Requiring an object's path to be timelike is equivalent to saying that it cannot travel faster than the speed of light. In the Minkowski metric

$$-c^2dt^2 + dx^2 < 0 \implies dx^2 < c^2dt^2$$

and so

$$dx < cdt \implies \frac{dx}{dt} < c$$

[2] The reason it is called a proper *time* is because someone who moves along the path would choose a coordinate system where they remain at rest at the origin, so $x' = 0$ and $dx' = 0$. For such a coordinate system, the spacetime distance only has a time contribution, $ds^2 = -c^2dt'^2$. Hence the proper-time is the same as the time measured on clock moving along the path, $d\tau = dt'$.

For example, say we know the position, $x(t)$, of an object as a function of time in some coordinate system[3] . Rearranging Eq. 3 by pulling out a factor of dt^2 lets us calculate the proper time as an integral over t

$$\tau = \int d\tau = \int_0^{t_{\text{end}}} dt \sqrt{1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2} \tag{4}$$

where dx/dt is just the the velocity $v(t)$ along the path.

1b) Assuming your friend moves at a speed (measured in your rest frame) of $v = \sqrt{99/100}\, c$, calculate the proper time τ_{me} along your path from A to B and the proper time, τ_{her}, for your friend's[4] path.

1c) Consider a different case where your friend does not drive at a constant speed, but accelerates such that her position as a function of time is[5]

$$x(t) = \frac{ct_0}{\pi} \sin\left(\frac{\pi t}{t_0}\right) \tag{5}$$

where $t_0 = 2$ days is the total time of her round trip in your rest frame. Make a sketch of the spacetime diagram of your friend's wordline (again, in your rest frame coordinate system).

1d) Calculate the proper time of your friend's path Eq. 5 through spacetime.

comment: While you and your friend started and ended at the same point in spacetime, your friend took a "shorter" path and experienced less proper time. While this may seem strange, the analogous behavior in space is very familiar (see Figure 2) Say you live 1 mile due North of campus. If you drive straight home, your odometer will register $l = 1$ mile traveled. If you *don't* drive straight home, but veer off to the East then turn back West, your odometer will register a distance $l > 1$ mile. Because we live in a universe with more than 1 spatial dimension, there are an infinite number of paths joining any two points in space, and the distance along these paths can differ.

Similarly, since we live in a 4-dimensional *spacetime manifold* there are an infinite number of possible paths between two spacetime events. In this problem, you traveled a "straight" path while your friend veered off in another dimension, and so her "time odometer" racked up a different proper time. While on the spacetime diagram Figure 2 it may *look* like your friend's path is longer, this is a result of our Euclidean bias. In the Minkowski metric, the closer an interval is to the diagonal, the smaller it is, so we realize that our friend's path is indeed a shortcut in time whereas you took the "longway to tomorrow".

Problem 2: Black Hole Surveyor

[3] Since the spacetime distance is an *invariant*, it doesn't matter what coordinate system we use. We will always get the same result.

[4] For your friend's path, you will have to calculate the spacetime distance of each segment of her path (outgoing and incoming) and add them together.

[5] You may notice that this path gives a speed $dx/dt = c$ for $t = 0$ and $t = 2t_0$. While this is technically impossible (since only massless objects can ever move at the speed of light) it does not harm the end result of the problem, and so for simplicity we use Eq. 5 as an approximation to the realistic case

$$x(t) = \frac{vt_0}{\pi} \sin\left(\frac{\pi t}{t_0}\right)$$

where v is *very* close to c.

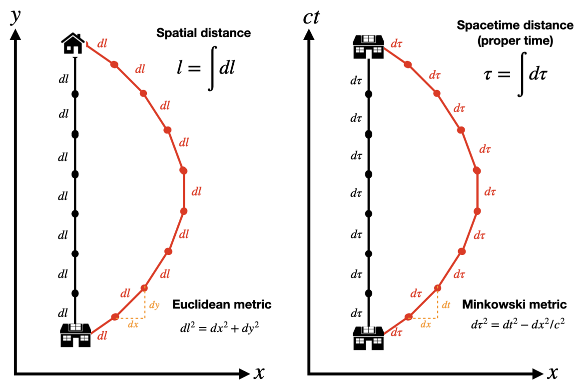

Figure 2: The left panel shows that two curves connecting the same two points in space can have different lengths, since instead of going just forward in the y-direction (black line) we can veer off into the x direction (red line). The red path is longer since the dx^2 and dy^2 increments add in the Euclidean metric.

The right panel shows that the same thing holds for spacetime – instead of going only forward in time (black line) we could also veer off in space (by moving, red line). While the length of the red path *looks* longer than the black line in the picture, it is actually shorter when we apply the proper Minkowski metric, since dt^2 and dx^2 subtract rather than add.

THE MINKOWSKI METRIC IS WRITTEN in 4D Cartesian coordinates as

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{6}$$

In many cases, it is convenient to write this in spherical polar coordinates, which. can be show to be

$$ds^2 = -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7}$$

Whatever coordinate system we choose, this metric describes flat (uncovered spacetime).

The Schwarzchild metric describes the spacetime around a central object of mass M, such as a black hole, where spacetime *is* curved. The expression for this metric is

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)c^2dt^2 + \frac{dr^2}{(1 - \frac{r_s}{r})} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{8}$$

where $r_s = 2GM/c^2$ is the Schwarzchild radius. The coordinates used here (t, r, θ, ϕ) are known as *Schwarzchild coordinates*. They are just one choice for labeling the 4D spacetime. In some other coordinate system, the metric equation will look different but the spacetime will be the same.

Schwarzchild coordinates slice up the spacetime into spherical shells, where θ and ϕ describe the location on a spherical shell (analogous to the angles used in ordinary spherical coordinates). Each spherical shell is labeled by the radial coordinate r. However, unlike in ordinary (flat spacetime) spherical coordinates, the value of r does *not* equal the distance from the center.

Let's use the Schwarzchild metric to make some (theoretical) measurements around a black hole. We call a measurement made at a fixed time (i.e., with $dt = 0$) a *proper distance* (or "measuring tape distance") and can write $dl = \sqrt{ds^2}$, where l is a proper spatial distance. We call a measurement made at a fixed position (i.e.,

$dr = d\theta = d\phi = 0$) a *proper time* (or "wristwatch time" of someone at that location) and write $d\tau = \sqrt{-ds^2}$ where τ is the proper time

2a) Imagine laying out a measuring tape in the radial direction and measuring the spacetime interval at fixed time. Integrate[6] the metric to calculate the radial distance from a location $r = 2r_s$ to $r = 3r_s$.

[6] Since we are measuring a path only along the r direction, we can take $d\theta = 0, d\phi = 0$ and since it is at a fixed time $dt = 0$. Since the integral is fairly difficult, you can use a resource like Wolfram Alpha . For example, to integrate x^2 from $x = 0$ to $x = 1$ you would type

$$\texttt{Integrate } x^2 \texttt{ from } x = 0 \texttt{ to } 1$$

2b) What is the proper distance from $r = 1.1r_s$ to $r_2 = 2.1R_s$?

comment: You see that the proper distance between to r coordinates a distance $\Delta r = r_s$ apart is not equal to r_s. Instead, it depends on the distance from r_s, becoming becomes larger as we get closer to the event horizon at $r = r_s$. In other words, our r coordinate flags are not equally spaced in proper distance.

While you can do the the indefinite integral to get a general expression for the proper length between two arbitrary radial points r_1 and r_2, the result is not too insightful. Instead, let's consider the limit of large r and use an expansion to see how the proper distance differs from the coordinate distance.

2c) Consider measuring distances at locations $r \gg r_s$. Use the binomial approximation[7] to simplify the expression for the metric. Carry out the integral to get an expression of the proper distance between two arbitrary points r_1 and $r_2 > r_1$ both much larger than r_s.

[7] The binomial approximation can be used when we have some small quantity x, where $x \ll 1$. Then the approximation is

$$(1+x)^\alpha \approx 1 + \alpha x + ...$$

where the ... represents higher order terms. Here you need only keep the leading order term in the expansion of the small quantity r_s/r.

2d) Imagine measuring the circumference of a circle around the black hole by wrapping a measuring tape around the equatorial region $\theta = \pi/2$ at a fixed coordinate r (so $dr = 0$). Calculate the proper distance around the ϕ direction to relate the circumference to r.

comment: We see that $C = 2\pi r$. So the coordinate r is a *circumferential coordinate* – it is *defined* by measuring the circumference of a circle about the center, and then setting the coordinate $r = C/2\pi$. The coordinate $r = 100$ km, for example, does not label a point 100 km from the center, but rather labels the point where the circumference of a circle is $2\pi \times 100$ km. Of course there are other choices you could make for labeling a radial coordinate, but this is a convenient one. [8]

[8] Be careful here: when we say that $C = 2\pi r$ in this metric, this does *not* suggest that the space is Euclidean (i.e., flat). That is because r is *not* the radius of the circle – you just showed above that the r coordinate does not correspond to distances in the radial direction. The Schwarzchild metric describes a non-Euclidean space, and in general if you draw some circle with a proper radius R, the circumference will not be equal to $2\pi R$.

Problem 3: Embeddings and Wormholes

WE HAVE VISUALIZED A CURVED SPACE by imagining ants crawling on the surface of a sphere. This world has only two accessible spatial dimensions to the ant. The 3rd spatial dimension does is simply a trick to help us visualize the effects of curvature by *embedding* a 2D space in a 3rd dimension.

Let's consider embeddings that will help visualize the curvature of space around a black hole. Imagine ants that live along the 1D string shown in Figure 3. The string is embedded in 2D space (the $x - z$ plane) where it is curved into the shape of a parabola given by the equation

$$x = \frac{z^2}{4R_c} + R_c \tag{9}$$

where R_c is some characteristic length scale.

Let's use the x coordinate to label points on the line. As can be seen in the figure, the distance between x coordinates are not all the same. To calculate a distance, we need a metric, which in flat 2D space (staying at constant y, so $dy = 0$, and at a fixed time, so $dt = 0$) is

$$dl^2 = dx^2 + dz^2 \tag{10}$$

Since the string is a 1D space, we should be able to eliminate the z coordinate and write the metric in terms of the one coordinate x.

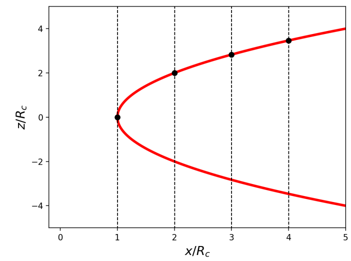

Figure 3: A 1D string curved like a parabola in a 2D space. Points on the top half of the string are labeled by coordinate x. The physical distance between x coordinates along the string is not the constant due to the curve of the string.

3a) Use the equation of the string shape to eliminate z and show that the metric for spatial distances can be written

$$dl^2 = \frac{dx^2}{(1 - R_c/x)} \tag{11}$$

comment: We notice a few interesting things. First, $dl^2 \neq dx^2$, which indicates that for each coordinate increment of dx, the physical length dl is larger by a factor of $1/(1 - R_c/x)$. We see that the distance between two points, dl, separated by dx becomes larger as x gets smaller. The dependence is exactly like the term $dr^2/(1 - R_c/r)$ seen in the Schwartzchild metric.

Second, we see that this metric has a singularity at $x = R_c$. Assuming our string loops around, we see that there is nothing physical weird at this point – i.e., the string is not torn and the curvature is not infinite at $r = R_c$. An ant could walk through $x = R_c$ without any trouble. This is merely a *coordinate singularity* that arises because our choice of x coordinate mathematically breaks down at the point where the string becomes vertical. We could avoid this breakdown by choosing some other coordinate to describe positions on the string.

Finally consider ants confined to the 2D surface of a paraboloid embedded in 3D space, shown in Figure 4 The equation of this paraboloid (which is just the parabola rotated around the z-axis) is is

$$r = \frac{z^2}{4R_c} + R_c \tag{12}$$

where $r = \sqrt{x^2 + y^2}$ and we can use cylindrical coordinates where $x = r\cos\phi$ and $y = r\sin\phi$.

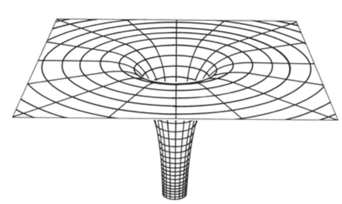

Figure 4: The 2D surface of a paraboloid, gotten by rotating a shifted parabola about the vertical axis.

3b) Find the metric on the surface of a paraboloid by starting with the metric for flat 3D space

$$dl^2 = dx^2 + dy^2 + dz^2 \tag{13}$$

and using the equation of a paraboloid to eliminating dz (and using r, ϕ coordinates)

$$dl^2 = \frac{dr^2}{(1 - R_c/r)} + r^2 d\phi^2 \tag{14}$$

comment: Identifying R_c with the Schwarzchild radius r_s, we see that this is exactly the Schwarzchild metric for a slice along the equator (where $\theta = 90^o$ and $d\theta = 0$). In other words, to visualize the spatial curvature of a 2D slice of the black hole we can embed the 2D surface[9] as a *paraboloid*. You have probably seen pictures of such embeddings in popular science description of black holes – now you have calculated the exact parabolic shape of the embedding. Remember, the z-dimension of these figures is not a real dimension, but just a trick to help visualize how the distance between r coordinates is not constant due to the curvature.

[9] You might compare the metric here to the one you derived previously for the surface of the sphere. There you showed using a similar embedding that for a sphere of radius a

$$dl^2 = \frac{dr^2}{1 - r^2/a^2} + r^2 d\phi^2$$

which is similar, but different than what you just found for the paraboloid. So the curvature around a black hole is different from that of a sphere.

The Schwarzchild metric has a singularity at $r = r_s$, but as in the above example of the curved string, this singularity is just a coordinate singularity, and not a point where the space actually is torn or infinitely curved.

comment: If we were to rotate the parabola in Figure 3 that loops back around, we get a 2D surface that looks like Figure 5. This is known as *Flamm's paraboloid*. The region in the top part of the figure describes the space outside a black hole. In this configuration, however, matter that enters the black hole does not fall into a singularity but opens up into a new region of spacetime at the bottom of the figure. This bottom half of spacetime in sometimes called a *white hole* – while the black hole sucks matter in, the white hole would expel matter out.

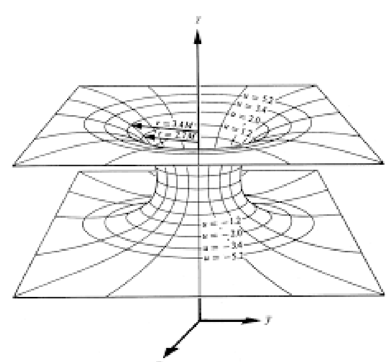

Figure 5: The 2D surface of a paraboloid gotten by rotating the looped string in Figure **??** about the vertical axis.

It has been speculated that such a geometry could allow black holes to serve as *wormhole* that would allow for rapid passage to a different point in the Universe or even another point in time (See Figure 6). The tunnel between the two regions is called an *Einstein-Rosen bridge*. There are a few problems with this idea, though. First, it is not clear how such a wormhole would ever be created. When a massive star collapses to a black hole, it can be shown that the interior of the black hole forms a singularity at the center, rather than opening up into white hole. Even if you imagined that somehow a wormhole was created, one can show that General Relativity predicts that this configuration is unstable, and the bridge between the black and white holes would rapidly close and pinch off into a singularity. Thus our paraboloid embedding really only applies for $r > r_s$.

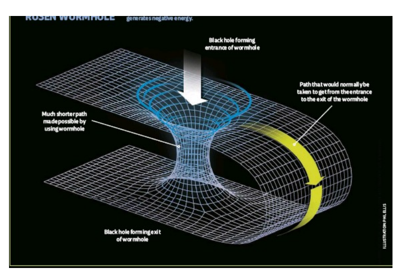

Figure 6: Visualization of how the spacetime of the paraboloid could produce a wormhole that allows for quick passage to another part of the Universe, or another point in time.

Physicists have studied more complicated configurations, where the black hole is rapidly spinning and some new, exotic form of energy is imagined to exist inside the bridge to keeps it stable against closing. In these scenarios, a wormhole could in principle be traversable, however it is very unlikely that this situation actually is realizable in our Universe. Still, it is cool to imagine if it was...