Probl. 1a).

$$d\tau^2 = -\frac{ds^2}{c^2}$$

$$\frac{ds^2}{d\tau^2} = -c^2 = -(1-\frac{r_s}{r})c^2(\frac{dt}{d\tau})^2 + \frac{1}{1-\frac{r_s}{r}}\cdot(\frac{dr}{d\tau})^2 + r^2\cdot(\frac{d\phi}{d\tau})^2$$

$$\text{since}\begin{cases}\frac{L}{m} = r^2\frac{d\phi}{d\tau}\\ \frac{E}{m} = (1-\frac{r_s}{r})c^2\cdot\frac{dt}{d\tau}\end{cases}$$

$$\therefore\quad -c^2 = -\frac{1}{1-\frac{r_s}{r}}\cdot\frac{1}{c^2}\cdot(\frac{E}{m})^2 + \frac{1}{1-\frac{r_s}{r}}\cdot(\frac{dr}{d\tau})^2 + \frac{1}{r^2}\cdot(\frac{L}{m})^2$$

$$\Rightarrow (\frac{dr}{d\tau})^2 = -(1-\frac{r_s}{r})c^2 + \frac{1}{c^2}(\frac{E}{m})^2 - (1-\frac{r_s}{r})\frac{1}{r^2}\cdot(\frac{L}{m})^2$$

$$= \frac{E^2}{m^2c^2} - c^2 + \frac{2GM}{r} - (1-\frac{2GM}{rc^2})\frac{1}{r^2}(\frac{L}{m})^2$$

$$\therefore\quad \frac{1}{2}m(\frac{dr}{d\tau})^2 - \frac{GMm}{r} + \frac{L^2}{2mr^2} - \frac{L^2GM}{mc^2r^3}$$

$$= \frac{1}{2}m\cdot\left[\frac{E^2}{m^2c^2} - c^2 + \frac{2GM}{r} - \frac{L^2}{m^2r^3} + \frac{2GM}{r^3c^2}\cdot\frac{L^2}{m^2}\right] - \frac{GMm}{r} + \frac{L^2}{2mr^3} - \frac{L^2GM}{mc^2r^3}$$

$$= \left[\frac{E^2}{2mc^2} - \frac{mc^2}{2} + \underline{\frac{GMm}{r}} - \underline{\frac{L^2}{2mr^3} + \frac{GML^2}{mc^2r^3}}\right] - \underline{\frac{GMm}{r}} + \underline{\frac{L^2}{2mr^3} - \frac{L^2GM}{mc^2r^3}}$$

$$= \frac{E^2}{2mc^2} - \frac{mc^2}{2} \Rightarrow \text{const of } E, m, c \qquad \text{proofed.}$$

1b).

$$V_{eff} = -\frac{GMm}{r} + \frac{L^2}{2mr^2} - \frac{L^2GM}{mc^2r^3}$$

$$\left.\frac{dV_{eff}}{dr}\right|_{r=r_c} = \frac{GMm}{r^2} - \frac{L^2}{mr^3} + \frac{3L^2GM}{mc^2r^4} = \left.\frac{1}{r^4}(GMm\cdot r^2 - \frac{L^2}{m}\cdot r + \frac{3L^2GM}{mc^2})\right|_{r=r_c} = 0$$

$$r_c = \frac{\frac{L^2}{m} \pm \sqrt{\frac{L^4}{m^2} - \frac{12L^2G^2M^2}{c^2}}}{2GMm} = (\frac{L}{m})^2\cdot\frac{1 \pm \sqrt{m^2 - \frac{12G^2M^2m^4}{L^2c^2}}}{2GM}$$

1c)

$$\Delta = m^2 - \frac{12G^2M^2m^4}{L^2c^2} \geq 0$$

$$\Rightarrow L^2 \geq \frac{12G^2M^2m^2}{c^2} \Rightarrow L \geq \sqrt{12}\cdot\frac{GMm}{c} \quad \text{when } L = \frac{2\sqrt{3}GMm}{c},$$

$$r_{stable} = (\frac{L}{m})^2\cdot\frac{1}{2GM} = \frac{1}{2GMm^2}\cdot\frac{12G^2M^2m^2}{c^2} = \frac{6GM}{c^2} = 3r_s$$

Prob 2.

2a) Schwarzchild metric: $ds^2 = -(1-\frac{r_s}{r})c^2dt^2 + \frac{dr^2}{1-\frac{r_s}{r}} + r^2 d\phi^2$

$$\frac{ds^2}{d\tau^2} = -(1-\frac{r_s}{r})c^2 \cdot \frac{dt^2}{d\tau^2} + \frac{1}{1-\frac{r_s}{r}} \cdot (\frac{dr}{d\tau})^2 + r^2 \cdot (\frac{d\phi}{d\tau})^2 = 0$$

since $\begin{cases} e = -(1-\frac{r_s}{r})c^2 \cdot \frac{dt}{d\tau} \\ l = r^2 \cdot \frac{d\phi}{d\tau} \end{cases}$

$$\therefore -\frac{1}{(1-\frac{r_s}{r})c^2} \cdot e^2 + \frac{1}{1-\frac{r_s}{r}} \cdot (\frac{dr}{d\tau})^2 + \frac{1}{r^2} l^2 = 0$$

$$\frac{e^2}{c^2} = (\frac{dr}{d\tau})^2 + \frac{1-\frac{r_s}{r}}{r^2} \cdot l^2 \quad , V_{eff}(r) = \frac{r-r_s}{r^3} \cdot l^2$$

2b). $\frac{\partial V_{eff}}{\partial r} = (-\frac{2}{r^3} + \frac{3r_s}{r^4}) l^2 = \frac{3r_s - 2r}{r^4} \cdot l^2$

$\because \frac{\partial V_{eff}}{\partial r}\Big|_{r=r_p} = 0 \qquad \therefore r_p = \frac{3}{2} r_s$

2c). $\frac{\partial^2 V_{eff}}{\partial r^2}\Big|_{r=r_p} = (-\frac{12r_s}{r^5} + \frac{6}{r^4}) l^2 \Big|_{r=r_p} = \frac{6r_p - 12r_s}{r^5} \cdot l^2 < 0$

$\therefore$ it is not stable when $r = r_p = \frac{3}{2} r_s$

2d).

Prob 3.

3a). $\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 - \frac{GMm}{r} + \frac{L^2}{2mr^2} - \frac{L^2GM}{mc^2r^3} = \frac{E^2}{2mc^2} - \frac{mc^2}{2}$

$\Rightarrow E^2 = m^2c^2\left(\frac{dr}{d\tau}\right)^2 - \frac{2GMm^2c^2}{r} + \frac{L^2}{r^2}c^2 - \frac{2L^2GM}{r^3} + m^2c^4$

for $\frac{dr}{d\tau} = 0$, we have:

$E = \sqrt{m^2c^4 - \frac{2GMm^2}{r}\cdot c^2 + \frac{L^2c^2}{r^2} - \frac{2L^2GM}{r^3}}$

$E_{isco} = \sqrt{m^2c^4 - \frac{2GMm^2}{3\cdot\frac{2GM}{c^2}}\cdot c^2 + \frac{L^2c^2}{9\cdot 4G^2M^2}\cdot c^4 - \frac{2L^2GM}{27\cdot 8G^3M^3}c^6}$

$= \sqrt{\frac{2}{3}m^2c^4 + \frac{1}{36}\frac{L^2c^6}{G^2M^2} - \frac{1}{108}\frac{L^2c^6}{G^2M^2}}$, $L^2 = \frac{12G^2M^2m^2}{c^2}$

$= \frac{2\sqrt{2}}{3}mc^2$

3b) at $r \gg r_s$:

$E \approx mc^2$

at $r = r_s$:

$E = \frac{2\sqrt{2}}{3}mc^2$ $\qquad \therefore \Delta E = \left(1 - \sqrt{\frac{8}{9}}\right)mc^2$

Prob 4.

4a) $g_{rr} = \frac{r^2+a^2\cos^2\theta}{r^2-r_s\cdot r+a^2}$ when $g_{rr} \to \infty$ we have: $r^2-r_s\cdot r+a^2=0$

$$r = \frac{r_s \pm \sqrt{r_s^2-4a^2}}{2}$$

$$r_1 = \frac{GM}{c^2} - \sqrt{\frac{G^2M^2}{c^4} - \frac{J^2}{M^2c^2}} \quad , \quad r_2 = \frac{GM}{c^2} + \sqrt{\frac{G^2M^2}{c^4} - \frac{J^2}{M^2c^2}}$$

4b). consider $\sqrt{r_s^2-4a^2}$, when $r_s^2-4a^2 \le 0$, then r_1, r_2 has imaginary part.

$$\Rightarrow a \geq \frac{1}{2}r_s = \frac{GM}{c^2}$$

$\therefore$ when $a \geq \frac{GM}{c^2} = a_{max}$, r will be imaginary number.

4c) now $r_1 = r_2 = \frac{r_s}{2} = \frac{GM}{c^2}$

4d) $\because dr = d\theta = d\phi = 0$

$\therefore ds^2 = -\left(1 - \frac{r_s\cdot r}{r^2+a^2\cos^2\theta}\right)c^2dt^2$, to be timelike, we need $ds^2<0$

$\therefore 1 - \frac{r_s\cdot r}{r^2+a^2\cos^2\theta} > 0 \Rightarrow r^2 - r_s\cdot r + a^2\cos^2\theta > 0$

$r > \frac{r_s + \sqrt{r_s^2-4a^2\cos^2\theta}}{2}$ if we want $ds^2<0$.