Physics C161; Problem Set #4

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due Friday, 2/16, at midnight

Problem 1: Schwarzchild Orbits

The last problem considered the purely radial motion of somebody plunging directly into a black hole. Let's consider now the orbits of objects around a spherical mass M, which could be a black hole or a star. We will consider orbits that lie in the equatorial plane, so $\theta = \pi/2$ and $d\theta = 0$.

Let us first recall (as reviewed in problem set #1) that in Newtonian physics the energy equation for an orbiting object is

$$E = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{1}{2} m r^2 \left(\frac{d\phi}{dt}\right)^2 - \frac{GMm}{r} \tag{1}$$

The first terms on the right hand side are the radial and angular kinetic energy. We have two constants of motion in Newtonian orbits, the energy E and the angular momentum

$$L = mr \left(r \frac{d\phi}{dt} \right) \tag{2}$$

Using this defintition of L the energy equation becomes

$$E = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} \tag{3}$$

which we can rewrite

$$E = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + V_{\rm eff}(r) \tag{4}$$

where $V_{\rm eff}$ is an *effective potential* that includes both the gravitational potential term $-GMm/r$ (which tends to pull objects to the center) and the centrifugal term $L/2mr^2$ (which tends to push objects out). With this equation we have isolated the radial motion of the orbit.

As the margin figure shows, there is a minimum in the effective where a mass can stably orbit in a circle. This minimum occurs at radius

$$r_c = \frac{L^2}{GMm^2} \tag{5}$$

which will be the radius of a circular for a mass of angular momentum L. Circular orbits of smaller radius have smaller L, and in Newtonian physics there are stable orbits as close to the mass as you want. This will change in general relativity.

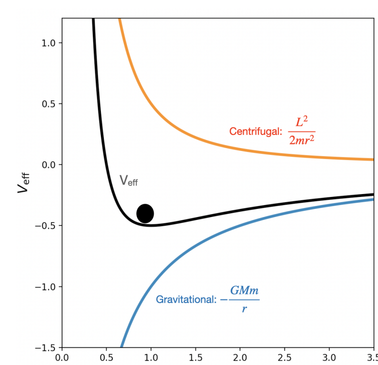


Figure 1: Plot of the effective potential of an orbit in Newtonian gravity.

In general relativity we can study orbits using the Schwarzchild metric, which for the equatorial plane ($\theta = \pi/2$) is

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \frac{dr^2}{(1 - r_s/r)} + r^2 d\phi^2 \tag{6}$$

In addition to the energy constant of motion, E/m there is a constant of motion related to the fact that metric is independent of the coordinate ϕ. This constant we call the *angular momentum* (per unit mass) L/m

$$\frac{L}{m} = g_\phi \frac{d\phi}{d\tau} = r^2 \frac{d\phi}{d\tau} = \text{ constant} \tag{7}$$

1a) Rewrite the metric in terms of the constants of motion to find

$$\epsilon = \frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 - \frac{GMm}{r} + \frac{L^2}{2mr^2} - \frac{L^2 GM}{mc^2 r^3} \tag{8}$$

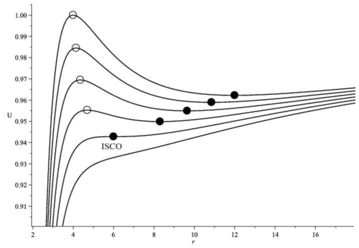


Figure 2: Plots of the effective potential for Schwarzchild orbits of different angular momentum L/m. Compared to the Newtonian result (Figure 1) the drop in V_{eff} at small radius is a new feature of GR. There are two equilibrium points, but only one is stable (the one marked with a solid black dot). As L/m is decreased, the stable orbit moves inward, but below some minimum value there is no longer any stable circular orbits.

comment: The Schwarzchild orbit equation looks very similar to the Newtonian one, with the notable exception that there is an extra term in the effective potential that goes like $-1/r^3$. This is sometimes called the "pit in the potential", because it causes the effective potential to turn over and become negative at small r.

1b) Use the condition $dV_{\text{eff}}/dr = 0$ to determine the radius, r_c, of a circular Schwarzchild orbit for an object with angular momentum per unit mass L/m.

comment: Unlike Newtonian orbits, you will find *two* circular orbit positions in Schwarzchild orbits. As can be seen in the margin figure, the outer position corresponds to a stable orbit, while the inner orbit is unstable[1] (any small perturbation would make the object fly off that circular orbit either into or away from the central mass).

[1] You can determine for yourself (if you want) which orbits are stable or unstable by calculating d^2V_{eff}/dr^2 and evaluating it at each possible value of r_c. The stable orbits have $d^2V_{\text{eff}}/dr^2 > 0$ and the unstable ones $d^2V_{\text{eff}}/dr^2 < 0$.

1c) Note that if L becomes too small, there is no solution for r_c (that is a real number at least). Find this critical value of L and use it to show that the minimum radius of a stable circular orbit is

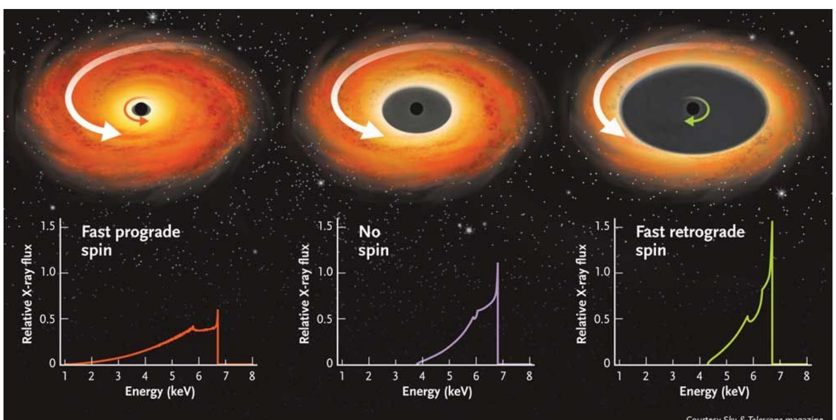
$$r_{\text{isco}} = 3r_s \tag{9}$$

comment: Black holes have an "innermost stable circular orbit" (or "ISCO" for short) that is a few times r_s. No object can stably orbit interior to this radius, and will instead plunge into the black hole. If we consider, say, two black holes orbiting each other, the binary would remain stable as long as the separation between the black holes was $> r_{\text{isco}}$. If the orbit were to shrink below r_{isco} the two black holes would coalesce together as one (which is what we see in many gravitational wave sources).

While we are not able to observe black holes directly, what we do generally see is gas orbiting around the black hole in the form of a disk. As the disk swirls around, viscosity (i.e., friction between the disk layers) converts some of the orbital energy into heat, which can then be partially radiated away as light before the disk material accretes into the black hole.

The fact that black holes have an innermost stable orbit at $r_{\rm isco}$ is an important factor in the light we observe from the disk. We expect the accretion disks of black holes to only extend down to $r_{\rm isco}$, and not all the way to r_s. There thus will be a "gap" region between the inner edge of the disk and the event horizon (see figure below) and this gap influences the spectrum we see from the disk.

For a spinning black hole, the metric is not the Schwarzchild metric but the Kerr metric , which is quite a bit more complicated. By an analogous calculation you would find that the radius of the innermost stable orbit depends upon the black hole spin (and whether the disk spins in the same direction as the black hole, or in the opposite (retrograde) direction). Thus, one way to measure how fast a black hole is spinning is to try measure $r_{\rm isco}$ by studying the emission from the closest gas orbiting the black hole.



A black hole's spin determines how close the swirling accretion disk can come to the event horizon. If the black hole and disk spin the same direction (left), the disk will hug more closely than if the black hole doesn't spin (center) or if it spins the opposite direction (right). This information can appear in emission from the object (simulated spectra shown).

Problem 2: The Photon Sphere

THE FAMOUS IMAGE TAKEN BY the Event Horizon Telescope – often described as "a picture of a black hole" – is of course instead showing light emitted by gas orbiting around a supermassive black hole at the center of a nearby galaxy. The edge of the dark circle in the image is not actually the event horizon r_s. It is not exactly $r_{\rm isco}$ either. While the emitting gas only extends down to $r_{\rm isco}$, some of the light it emits can circle around due to lensing, which produces an inner edge somewhere between r_s and r_{isco}.

For light, the metric interval has $ds^2 = 0$ (called a light-like, or null geodesic) and so the proper time of any light like path is $\tau = 0$. We thus cannot use proper-time as a parameter to describe a lightlike geodesic. We can use, however, some other parameter (call it σ) that marks off positions along the photon trajectory. It will not be important exactly how we define σ here. Simply note that the constants of motion[2] will be as before but with $d\tau$ replaced by $d\sigma$

$$e = g_t \frac{dt}{d\sigma} \quad \text{if metric independent of } t$$

$$l = g_\phi \frac{d\phi}{d\sigma} \quad \text{if metric independent of } \phi$$

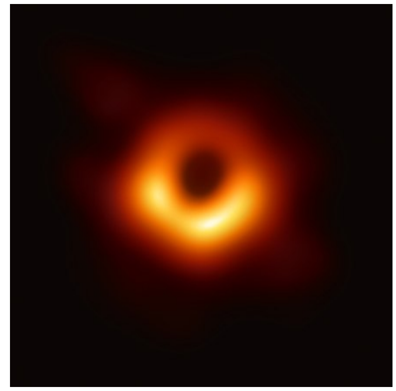


Figure 3: Image of gas around a supermassive black hole, as taken by the Event Horizon Telescope, a set of radio observatories.

[2] Since a photon has no mass ($m = 0$), instead of writing the constants of motion as E/m and L/m we will write them as e ("specific energy") and l ("specific angular momentum").

2a) Consider a photon orbiting a non-spinning black hole in the equatorial plane ($\theta = \pi/2$ and $d\theta = 0$). Divide the appropriate metric through by $d\sigma^2$, and use the constants of motions to find

$$\frac{e^2}{c^2} = \left(\frac{dr}{d\sigma}\right)^2 + V_{\text{eff}}(r) \tag{10}$$

and find an expression for the effective potential V_{eff} for photons.

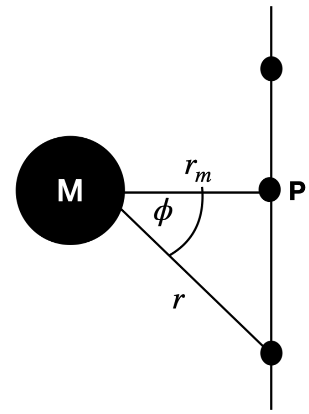
2b) Show that there is a single radius, $r_p = (3/2)r_s$, at which there is a circular orbit for light.

comment: The value $r_p = (3/2)r_s$ is known as the *photon sphere*. It is the location where light can orbit in a circle around the black hole. If you stood at r_p and looked ahead, you would see the back of your head! This is not a stable orbit, though, so light could not orbit around indefinitely at this radius. Eventually the light would either fall into the black hole or out to infinity.

2c) Show that the circular orbit for photons at r_p is an unstable orbit.

comment: The edge of the dark circle we see from the Event Horizon Telescope image most closely tracks the photon sphere. Light emitted from gas around the black that approaches this radius will swing around in a circle, and eventually will either fall into the black hole or fly off to be observed. Light that goes below r_p will rapidly fall into the black hole. This produces the so-called "shadow" of the black hole.

By measuring the size of the imaged dark circle, we can estimate the mass of the black hole. There are a few complexities though: if the black hole is spinning (as it most likely is) the metric is different than the Schwarzchild one, and the location of the photon sphere and $r_{\rm isco}$ are modified. In addition the inclination of the disk from our viewing angle also affects the image. To do a careful analysis, astrophysicists integrate the trajectories of photons to generate simulated images. If you want, you can optionally continue the problem to derive expressions for how to do such integrals for the photon trajectories.



To extend this formalism to calculate the trajectories of gravitational lensed light rays, consider a photon in the $\theta = \pi/2$ plane that passes by a mass M. Since e and l are constants of motion, the ratio e/l for such a photon is also a constant of motion. When the photon is at the radial coordinate of closest approach $r = r_m$ (point P in the margin figure) it is moving tangentially to the black hole so $dr/d\sigma = 0$.

2d) **Optional; not graded:** Use the metric to show that for the photon

$$\frac{e}{l} = \frac{c}{b} \quad \text{where } b = r_m \left(1 - \frac{r_s}{r_m}\right)^{-1/2} \tag{11}$$

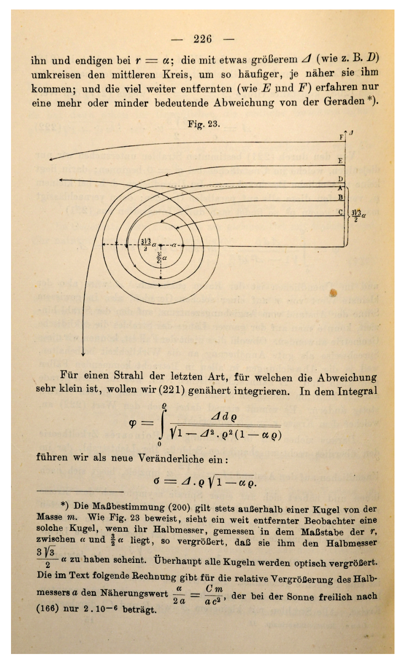
The quantity b is called the *impact parameter* and in General Relativity differs slightly from r_m.

2e) **Optional; not graded:** We would like to calculate the trajectory of the photon, that is determine the function $r(\phi)$. Return to the metric and use the chain rule

$$\frac{dr}{d\sigma} = \frac{dr}{d\phi}\frac{d\phi}{d\sigma} \tag{12}$$

and the constants of motion to derive an equation for $dr/d\phi$ and integrate it to show

$$\phi = \int \left[\frac{r^2}{b^2} - \left(1 - \frac{r_s}{r}\right)\right]^{-1/2} \frac{dr}{r}$$



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ihn und endigen bei $r = \alpha$; die mit etwas größerem $\varDelta$ (wie z. B. D) umkreisen den mittleren Kreis, um so häufiger, je näher sie ihm kommen; und die viel weiter entfernten (wie E und F) erfahren nur eine mehr oder minder bedeutende Abweichung von der Geraden *).

Fig. 23.

Für einen Strahl der letzten Art, für welchen die Abweichung sehr klein ist, wollen wir (221) genähert integrieren. In dem Integral

$$\varphi = \int_0^{\varrho} \frac{\varDelta\, d\varrho}{\sqrt{1 - \varDelta^2 . \varrho^2 (1 - \alpha\varrho)}}$$

führen wir als neue Veränderliche ein:

$$\sigma = \varDelta . \varrho \sqrt{1 - \alpha\varrho}.$$

*) Die Maßbestimmung (200) gilt stets außerhalb einer Kugel von der Masse m. Wie Fig. 23 beweist, sieht ein weit entfernter Beobachter eine solche Kugel, wenn ihr Halbmesser, gemessen in dem Maßstabe der r, zwischen α und $\frac{3}{2}\alpha$ liegt, so vergrößert, daß sie ihm den Halbmesser $\frac{3\sqrt{3}}{2}\alpha$ zu haben scheint. Überhaupt alle Kugeln werden optisch vergrößert. Die im Text folgende Rechnung gibt für die relative Vergrößerung des Halbmessers a den Näherungswert $\frac{\alpha}{2a} = \frac{C m}{a c^2}$, der bei der Sonne freilich nach (166) nur $2 . 10^{-6}$ beträgt.

comment: The integral over r can not be easily done on pen and paper, but it would be fairly straightforward to integrate it with a numerical code to determine the trajectories of light rays undergoing gravitational lensing. This is effectively what was done, for example, to construct the images of light around a black hole in the Interstellar movie poster. The margin figure also shows an attractive example that predates computers, from a 1920 relativity text by Van Laue. In the limit of small angle lensing, with $b \gg r_s$, one can expand out the term in brackets and do the integral to find an angular deviation of $\Delta\phi = 2r_s/b$, a result we found in class using Huygens principle.

Problem 3: Black Hole Accretion

BLACK HOLES ARE PERHAPS the most efficient energy generation sources in the universe. When matter accretes onto a black hole, some fraction of its rest mass energy can be tapped and radiated as light. This is the process we think powers some of the most luminous sources in the Universe, such as quasars.

Gas falling into a black hole will often have some angular momentum and so form an accretion disk that circles the black hole in a nearly circular orbit. If there were no friction within the disk, such gas could orbit around indefinitely. However, the viscosity and shearing in the disk releases heat and slows down the inner layers, which fall closer to the black hole. Any gas that drops below $r_{\rm isco} = 3r_s$ can no longer orbit stably and will fall quickly below the event horizon.

3a) Consider a bit of mass m in a disk around a Schwarzchild black hole. Determine the energy $E_{\rm isco}$ of this mass[3] assuming it is in a circular orbit at the innermost stable orbit, $r_{\rm isco}$

[3] Here E is your constant of motion that you worked with in a previous problem. For circular orbits $dr/d\tau = 0$, so look at the energy for the L and r at the $r_{\rm isco}$.

3b) Assume the initial energy of this piece of mass was $E = mc^2$ (i.e., it started out at very large r moving very slowly, so that its energy was essentially just its rest mass energy). Show that energy it must have lost in going from the initial radius (at $r \gg r_s$) to its final circular orbit at $r_{\rm isco}$ is

$$\Delta E = \left(1 - \sqrt{\frac{8}{9}}\right) mc^2 \approx 0.052\ mc^2 \tag{13}$$

comment: You have shown that for a bit of disk mass to move from a circular orbit at infinity to one at $r_{\rm isco}$ it must lose about 5% of its initial energy. It can do this by radiating away the energy as light. As different layers of the disk slide and shear against each other, viscosity turns some of the orbital kinetic energy into heat, and the hot gas can then emit light. The remaining 95% or so of the mass/energy winds up being eaten by the black hole once the gas falls below $r_{\rm isco}$.

We can compare this energy release to other physical processes. Typical chemical processes (e.g., burning coal) involve changes to atomic/molecular structure (i.e., energy of order eV). The rest mass of atoms are of order 1 GeV (the mass of a proton), so chemical processes release around eV/GeV $\sim 10^{-9}$ of the rest mass. Nuclear reactions (e.g., in burning in stars) release about an MeV per nucleus, so the efficiency is around 10^{-3}. These are all well below the possible energy release efficiency of black hole accretion.

For rotating black holes, $r_{\rm isco}$ is smaller, reaching $r_{\rm isco} = r_s$ for maximal spin (assuming the disk rotates in the same direction as the black hole). In this case, even greater energy release is possible, reaching around 12% of mc^2. But accretion does not always lead to a large energy release. If gas lacks angular momentum, it may plunge directly into the black hole (rather than remaining on circular orbits) and only radiate a small amount of its energy before getting eaten. This is the case with the supermassive black hole in our own Galaxy, and the one in the M87 galaxy (famously imaged by the Event Horizon Telescope), both of which radiate quite inefficiently.

Problem 4: A Rotating Black Hole

We have studied the Schwarzchild metric which gives a description of the geometry of spacetime outside a non-rotating spherical mass M. For a *rotating* spherically symmetric mass M and angular momentum J, the spacetime is given by the **Kerr metric**. Using a specific coordinate system (called "Boyer–Lindquist coordinates"), the metric line element is[4]

$$ds^2 = -\left(1 - \frac{r_s r}{r^2 + a^2\cos^2\theta}\right) c^2 dt^2 + \left(\frac{r^2 + a^2\cos^2\theta}{r^2 - r_s r + a^2}\right) dr^2 + (\text{angular terms})$$

[4] We won't write out the angular terms with $d\theta$ and $d\phi$ since the expression gets rather long (and we won't need them in this problem) but you can find them in many references, e.g., the Wikipedia page .

where $r_s = 2GM/c^2$ and $a = J/Mc \geq 0$ is an angular momentum per unit mass often called the *spin parameter*. This a bit more complicated than the Schwarzchild metric, but still manageable and we can approach it with the same mathematical tools. The physics of rotating black holes, however, is much richer than that of non-rotating black holes, and remains the subject of much ongoing research.

4a) The Schwarzchild metric had a coordinate singularity at $r = r_s$ that we associated with the event horizon. Find the singularities in the Kerr metric by finding the values of r where the dr^2 term in the metric blows up to infinity. You will see that there are *two* event horizons for a rotating black hole. Are these event horizons larger or smaller than that of a non-rotating black hole (i.e., the Schwarzchild radius)? Double-check that your expression makes sense for the case $a = 0$.

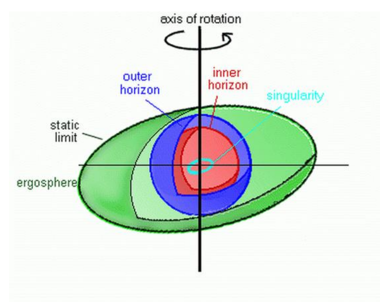

Figure 4: Illustration of the interesting surfaces in the Kerr metric of a rotating black hole. We will quantify them in this problem.

comment: As in the Schwarzchild metric, the metric singularities at these event horizons are only coordinate singularities, and can be removed by changing the coordinate system. The Kerr metric does have a *true* singularity at $r = 0$, where the curvature really does blow up and the metric breaks down completely.

4b) Show that when a is above a certain value, your expression for the location of the event horizons gives an *imaginary* number. In this case, since there are no real values for the location of the event horizons, there *are no* event horizons.

comment: Your result shows that if the spin parameter is greater than $a_{\max} = GM/c^2$, there are no event horizons. That means that the black hole would have a true singularity at $r = 0$ that is not hidden by any event horizon. This is called a "naked singularity". There is a hypothesis called that "cosmic censorship hypothesis" that naked singularities cannot exist. If that is true, than there is a maximal spin of a black hole, $a = a_{\max}$, corresponding to an angular momentum of $J = GM^2/c$. Black holes can't spin any faster.

4c) What are the locations (i.e., r coordinate values) of the event horizons for a black hole spinning at the maximal rate $a_{\max}$? (often called an *extremal* black hole?

4d) Imagine an observer tries to hover at a steady position (r, θ and ϕ are all constant) above an extremal black hole (with $a = a_{\max}$). Find the condition on r such that such a constant r path through spacetime is timelike (i.e., $ds^2 < 0$). (Your solution of a quadratic equation will give you two values of r; consider the larger value).

comment: In relativity we make the restriction that any object needs to move along a timelike path (otherwise there can be violations of causality). Below the value of r that you found, there are *no* timelike paths for which r (and θ and ϕ) are all constant. That means that an object simply cannot remain stationary in this region, and it must at least orbit around the black hole. You can think of the black hole as "dragging" the spacetime around it, so that anything in this region. must co-rotate.

The r coordinate you found is called the "static limit", for obvious reasons. The region in between the two values of r you found is called the *ergosphere*. Looking at your expression, you can see that the shape of it is like a flattened sphere – i.e., the value of r is smaller at the pole (at $\theta = 0$) and larger at the equator (at $\theta = \pi/2$).

We are now starting to understand the structure of rotating black holes shown in Figure 4. Note that the outer edge of ergosphere is outside the outer event horizon. With further analysis, it can be shown that it is possible to extract energy from the spin of the black hole by sending particles through the ergosphere. This is known as the *Penrose process*. Some form of the Penrose process is likely responsible for powering the jets that emerge from black holes that we see in active galactic nuclei and gamma-ray bursts[5].

[5] The Penrose process offers a source of energy in black hole accretion that is distinct from the accretion energy you considered on a previous problem. That accretion energy came from gas in the disk falling towards the black hole and getting heated up, but in the Penrose process the energy is extracted from the spin of the black hole itself.