Prob 1.

1a). $E = \frac{hc}{\lambda} \sim \frac{hc}{r_s} \sim \frac{hc^3}{2GM} = k_B T$

$$\Rightarrow T \sim \frac{hc^3}{2k_B G M}$$

1b). take $M \sim 2\times10^{30}$ kg, $G \sim 6.67\times10^{-11}\,m^3/(kg\cdot s^2)$, $k_B \sim 1.38\times10^{-23}$ J/k

$$T \sim \frac{6.62\times10^{-34}\,J s \cdot (3\times10^{8} m/s)^3}{2\times1.38\times10^{-23}\,J/k \cdot 6.67\times10^{-11} \times 2\times10^{30}\,m^3/s^2} \sim 5\times10^{-6}\,K$$

1c) $A = 4\pi r_s^2$

$$\therefore L = 4\pi r_s^2 \cdot \sigma_{sb} \left(\frac{hc}{r_s}\right)^4$$

$$= 4\pi \sigma_{sb} \frac{h^4 c^4}{r_s^2} = \pi \sigma_{sb} \frac{h^4 c^8}{G^2 M^2}$$

1d) $\frac{d(Mc^2)}{dt} = c^2 \frac{dM}{dt} = -L = -K \cdot \frac{1}{M^2}$, $K = \frac{\pi \sigma_{sb} h^4 c^8}{G^2}$

$$\Rightarrow M^2 dM = k \cdot dt, \quad k = -\frac{K}{c^2}$$

integral: $\frac{1}{3} M^3 \Big|_{M_0}^{M(t)} = kt\Big|_0^t$

$$\Rightarrow M^3(t) - M_0^3 = 3kt$$

$\because t = t_e,\ M(t) = 0 \quad \therefore t_e = -\frac{M_0^3}{3k}$

$$M^3(t) = M_0^3 + 3kt = M_0^3\left(1 - \frac{t}{t_e}\right) \Rightarrow M(t) = M_0\left(1 - \frac{t}{t_e}\right)^{1/3}$$

1e) $t_e = \frac{M^3}{3 \cdot \frac{\pi \sigma_{sb} h^4 \cdot c^6}{G^2}} \sim \frac{M^3 G^2}{3\pi \sigma_{sb} h^4 c^6} \sim 4.7\times10^{158}$ s

1f) $T(t) \sim \frac{hc^3}{2k_B G} \cdot \frac{1}{M_0} \left(1 - \frac{t}{t_e}\right)^{-1/3}$ $\quad L(t) \sim \pi \sigma_{sb} h^4 c^8 G^{-2} \cdot M_0^{-2} \cdot \left(1 - \frac{t}{t_e}\right)^{-2/3}$

1g) $L \sim 1\,GeV \Rightarrow M_0 \sim \left[\frac{1\,GeV}{\pi \sigma_{sb} h^4 c^8 G^{-2}}\right]^{-\frac{1}{2}} \sim 2.7\times10^{-22}\,kg \sim 2.7\times10^{-19}\,g$

1h) $t_e \sim 4.7\times10^{158}\,s \cdot \left(\frac{2.7\times10^{-22}\,kg}{2\times10^{30}\,kg}\right)^3 \sim 1.16\times10^{3}\,s$

Prob 2.

2a). $\frac{dE}{dt} = \frac{dE}{da}\frac{da}{dt} = \frac{G m_1 m_2}{2a^2} \cdot \frac{da}{dt} = -\frac{32}{5}\frac{G^4}{c^5}\frac{M m_1^2 m_2^2}{a^5}$

$$\therefore \quad a^3 da = -\frac{64}{5}\frac{G^3 m_1 m_2}{c^5} dt$$

$$\frac{1}{4} a^4 \Big|_{a_0}^{a(t)} = -\frac{64}{5}\frac{G^3 m_1 m_2}{c^5} \cdot t\Big|_0^t$$

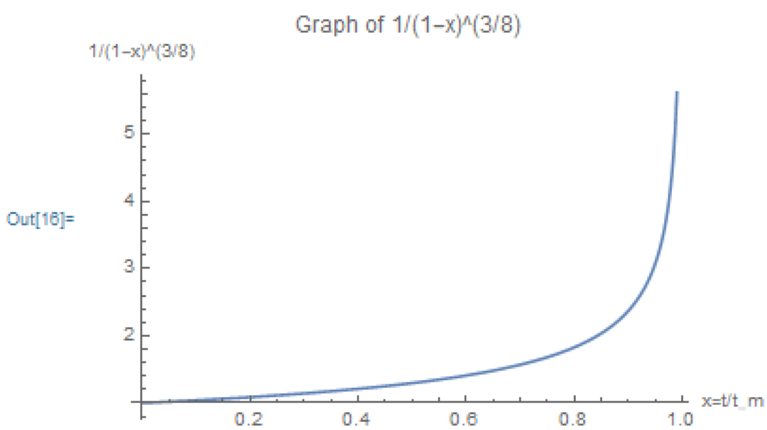
$$a(t) = a_0\left(1 - \frac{t}{t_m}\right)^{\frac{1}{4}}, \quad \text{here } t_m = a_0^4 \cdot \frac{5}{64} \cdot \frac{c^5}{G^3 m_1 m_2}$$

2b). $t_m = \frac{5}{64} \cdot \frac{c^5 \cdot a_0^4}{G^3 m_1 m_2}$

2c). $T(t) = \frac{2\pi}{\sqrt{GM}} a^{3/2}(t) = \frac{2\pi a_0^{3/2}}{\sqrt{GM}} \left(1 - \frac{t}{t_m}\right)^{3/8}$

$$\therefore \quad f(t) = \frac{2}{T(t)} = \frac{\sqrt{GM}}{\pi a^{3/2}} \cdot \frac{1}{(1 - t/t_m)^{3/8}} = \frac{f_0}{(1 - t/t_m)^{3/8}}$$

2d).



it is similar to fig1.

Prob 3.

3a). $F = \frac{L}{4\pi D^2} \Rightarrow D \sim F^{-1/2}$ distance less than D the brightness will be greater than F

$N(>F) \sim n \cdot V \sim D^3 \sim F^{-3/2}$

3b). suppose $m'(F')$ is less than m, we need $F' > F$

$N(<m) = N(>F) \propto F^{-3/2}$

$\because m = -2.5\log_{10}(F) + c \quad \therefore F = 10^{-\frac{2}{5}(m-c)}$

$N(<m) \propto \left[10^{-\frac{2}{5}(m-c)}\right]^{-3/2} \propto 10^{\frac{3}{5}m} = 10^{0.6m}$

3c) $\therefore m = -2.5\log_{10}\left(\frac{L_0}{4\pi}\left(\frac{D}{D_0}\right)^{-b} \cdot D^{-2}\right) + \text{constant}$

$m = -2.5\log_{10}\left(D^{-(b+2)}\right) + \text{constant}$

$N(<m) = N(<D) \propto D^3, \quad D^{-(b+2)} \sim 10^{-\frac{2}{5}m}$

$\therefore N(<m) \propto \left[10^{\frac{2m}{5(b+2)}}\right]^3 = 10^{\frac{6}{5(b+2)} \cdot m}$

Prob 4.

4a). $H(t) = \frac{\dot{a}}{a} = \frac{\frac{1}{t_0}}{t/t_0} = \frac{1}{t}$

4b). Hubble constant: $H_0 = H(t_0) = \frac{1}{t_0}$

4c). for photon, $ds^2 = 0 = -c^2dt^2 + a^2(t)\left[dr^2 + S_k^2(r)d\Omega^2\right]$, $d\Omega = 0$.

$\therefore c^2dt^2 = (\frac{t}{t_0})^2 dr^2 \Rightarrow \frac{dr}{dt} = c \cdot \frac{t_0}{t}$

$\int dr = ct_0 \int \frac{dt}{t} \Rightarrow \Delta r = ct_0 \ln\frac{t_2}{t_1} = r_2 - r_1$

here we take $r_1 = r$, $r_2 = 0$, $t_2 = t_0$, $t_1 = t_0 - \Delta t$

$\therefore r = ct_0 \ln\frac{t_0}{t_0 - \Delta t} \Rightarrow \frac{t_0}{t_0 - \Delta t} = e^{r/ct_0}$

$\Delta t = t_0(1 - e^{-\frac{r}{ct_0}})$

4d). since $r << ct_0$, we have

$\Delta t \approx t_0\left[1 - (1 - \frac{r}{ct_0})\right] = \frac{r}{c}$

4e). $dt = 0$, $t = t_0$

$ds^2 = (\frac{t_0}{t_0})^2 \cdot dr^2 = dr^2$

$\therefore d_p(t_0) = r.$

4f). $ds^2 = (\frac{t_e}{t_0})^2 dr^2$, $ds = \frac{t_e}{t_0} dr$

$\therefore d_p(t_e) = \int \frac{t_e}{t_0} dr = \int_0^r e^{-\frac{r'}{ct_0}} dr' = ct_0 \cdot (1 - e^{-\frac{r}{ct_0}})$

$\because e^{-\frac{r}{ct_0}} > 1 - \frac{r}{ct_0}$ $\therefore 1 - e^{-\frac{r}{ct_0}} < \frac{r}{ct_0}$

$\therefore d_p(t_e) < r = d_p(t_0)$