Physics C161; Problem Set #5

prof. kasen

due Friday, 2/23, at midnight

Problem 1: Hawking Radiation

Using a mixture of general relativity and quantum field theory, Stephen Hawking showed that black holes may radiate. While we won't go into the technical details, we can make some simple and interesting estimates of the physical consequences. Hawking radiation turns out to be thermal (i.e., a blackbody) in which case the characteristic energy of the emitted particles is $E = kT$, where T is the Hawking temperature and k is the Boltzmann constant. The only quantity with dimensions of length for a black hole is the Swarzchild radius, r_s, so let's guess that Hawking radiation comes in the form of photons with a typical wavelength $\lambda \sim r_s$ (which turns out to be roughly correct).

1a) According to quantum mechanics, the energy of a photon of frequency ν is $E = h\nu$, where h is Planck's constant. Given the above assumptions, and the fact that $\nu = c/\lambda$ for photons, give an order of magnitude[1] expression for the blackbody Hawking temperature T of a black hole in terms of its mass M.

[1] i.e., in this problem feel free to drop numerical factors like 2 or π if you want to.

1b) What (roughly) is the temperature T (in Kelvins) of a one solar mass black hole?

comment: Think for a second about how your expression for T scales with the black hole mass M. Typically when you add energy to matter, it gets hotter. But you have found here that if you dump mass/energy into a black hole, M increases and the black hole *cools off*. Similarly, we will see that as a black hole emits Hawking radiation, it loses mass/energy and *heats up*. Unlike most normal terrestrial materials (but similar to many stars) black holes have a *negative* heat capacity.

1c) The luminosity emitted from the surface of a blackbody with surface area A is

$$L = A\sigma_{\rm sb}T^4 \tag{1}$$

where $\sigma_{\rm sb}$ is the Stefan-Boltzmann constant. Assuming the radiating surface of the black hole is a sphere at r_s, write an expression for the Hawking luminosity, L, of the black hole in terms of the mass M and physical constants.

comment: Again, note the scaling with M (everything else is just physical constants). Bigger black holes radiate *less* power than smaller ones (because they are cooler). If you go ahead and plug in values[2] you'll see that the luminosity of Hawking radiation is tiny for a solar mass black hole.

1d) The energy carried away by the Hawking radiation must cause the mass/energy of the black hole ($E = Mc^2$) to decrease according to

$$\frac{d(Mc^2)}{dt} = -L \tag{2}$$

Solve this differential equation to show that the mass of black with initial mass M_0 evolves[3]

$$M(t) = M_0 \left(1 - \frac{t}{t_e}\right)^{1/3} \tag{3}$$

and give an expression for t_e, which you can see is the time for a black hole to evaporate (i.e., the mass reaches $M(t) = 0$ at $t = t_e$).

1e) Calculate the value (in years) for a solar mass black hole to evaporate (i.e., reach $M = 0$) due to Hawking radiation.[4]

1f) Derive the expressions for the Hawking luminosity, $L(t)$ and temperature $T(t)$ of a black hole as a function of time, given that the black hole has an initial mass M_0 at time $t = 0$.

Could we ever see the Hawking radiation from a black hole? Your estimates suggest that for a solar mass black hole the Hawking luminosity and temperature would be quite small and essentially undetectable. However, as the black hole mass gets small, the Hawking luminosity rises and the evaporation timescale decreases.

In the 1960's, astronomers discovered *gamma-ray bursts*, which are brief flashes of gamma-rays, which may last from seconds to hours. The source of these flashes was initially a mystery. Among many early theories was the possibility that the gamma-ray bursts were Hawking radiation from evaporating black holes. We can work out the basic theory.

1g) What mass (in grams) would a black hole need to be to radiate gamma-ray photons with a typical energy of 1 GeV?

1h) How long (in seconds) would a black hole of the mass you just calculated last before evaporating?

comment: The mass of the black hole needed to produce GeV gamma-rays is much smaller than a solar mass. Such a small black hole could not be formed in any stellar like phenomenon that we know about, but it is has been suggested that the dark matter in the universe may be comprised of numerous low mass black holes born early on the universe. Such so-called *primordial black holes* could potentially evaporate on short time scales, producing bursts of energetic photons.

[2] A nice website that calculates most of the relevant quantities for Hawking radiation is here .

[3] This can be compared to Equation 7 of Hawking's paper ,

[4] You will find it takes a *long* time for a solar mass black hole to evaporate by Hawking radiation. But in the real world it is actually longer than this. We will see later in the class that the current temperature of the universe (or more exactly, the cosmic microwave background (CMB) radiation field that pervades the universe) is 2.7 K. The temperature of a stellar mass black hole is much less than this, and since net heat flows from hot bodies to cold bodies, a solar mass black hole in the universe today would actually *absorb* more energy from the CMB than it emits by Hawking radiation. So currently black holes are gaining in mass, and will not actually start to evaporate until the universe has cooled down enough that the background CMB temperature is less than the black hole temperature. In a few weeks, we will be able to calculate when this occurs in our expanding, cooling universe.

The idea was explored by Hawking himself , but doesn't work for explaining the gamma-ray bursts (GRBs) observed. Besides issues with explaining the observed time-scales and spectrum of GRB, when the distance to GRBs was measured it was determined that they were very far away, such that their luminosity was significantly larger than predicted by Hawking radiation from an evaporating black hole. The favored model of GRBs today still involves black holes, but stellar mass sized ones that are accreting material and turning that energy into jets of material moving near the speed of light.

Problem 2: Gravitational Wave Inspiral

THE FIRST GRAVITATIONAL WAVE source ever detected with GW1501914, interpreted to be the merger of two black holes. The spiraling around of these two objects caused ripples in spacetime that propagated across the universe. Far from the gravitational wave (GW) source, the metric is nearly flat and (for a wave propagating in the z direction) is

$$ds^2 = -c^2 dt^2 + [1 + h(t)]dx^2 + [1 - h(t)]dy^2 + dz^2 \tag{4}$$

The dimensionless quantity $h(t)$ is called the *strain* and describes how much the GW stretches and squeezes spacetime[5]. Far from the source, GWs are understood to be small perturbations to the metric, and so $h(t) \ll 1$.

Figure 1 shows the first GW data published by the LIGO collaboration (the measured strain $h(t)$ over time). We can see that the amplitude and frequency of the signal initially increases over the course of tens of miliseconds. The signal is called a "chirp", since a sharp increase of frequency would sound like a bird's chirp. With a little work we can derive the expression for the chirp and analyze the data to infer some of the properties of the merging black holes. We will use (but not prove) the result from General Relativity that the luminosity emitted in gravity waves by a binary star system is

$$L = \frac{32}{5} \frac{G^4}{c^5} \frac{M m_1^2 m_2^2}{a^5} \tag{5}$$

As the system radiates gravitational wave emission, the orbit loses energy and the separation distance shrinks.

We'll adopt a Newtonian approach, which should be a reasonable approximation when the stars are not too close together[6]. In the Newtonian two body problem, the energy of two stars of mass m_1 and m_2 in a circular orbit separated by a distance a is

$$E = -\frac{G m_1 m_2}{2a} \tag{6}$$

[5] Recall that the terms in front of dx^2 and dy^2 in the metric are essentially the "scale bars" that tell us how to convert coordinate distances on our "map" to physical values. We see that scale bars change with time and out of phase (i.e., opposite sign), with the wave stretching in the x direction when squeezing in the y direction.

Gravitational waves can be written in this transverse form, where the perturbations to the metric only occur in the dimenions perpendicular to the direction of propagation. This form of the metric describes a gravitational wave in the so called + polarization state. There is another possible polarization state of the gravitational wave that stretches and squeezes spacetime along the diagonals, and is called the x (or cross) polarization state.

[6] For a more detailed approach you could treat the behavior of orbits in Schwarzchild metric, but this will not change the main points we consider here in a significant way.

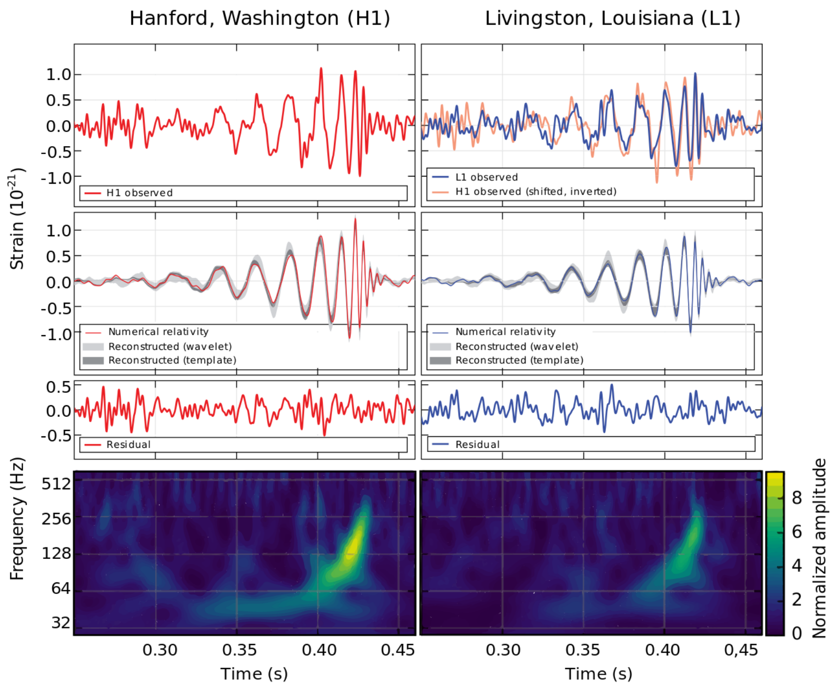

Figure 1: Detection of the gravitational wave source GW150914 by the two LIGO interfermeters in Hanford and Livingston. The top plots show the waveform, with the y-axis being the strain (or the fractional change in length of the arms of the detector). That two detectors at different points in the US measured the same signal at nearly the same time is strong evidence that the source is astrophysical and not just some local noise (e.g., someone in the lab dropping a hammer). The bottom two plots show how the spectrum of the wave changes with time; the light green/yellow regions show the frequency of the wave which increases over time as the black holes spiral together. Original paper is here .

and the orbital period is

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM}} \tag{7}$$

where $M = m_1 + m_2$ is the total mass of the system.

2a) Find an equation[7] for the time dependence of the orbital separation $a(t)$. Take the orbit to start at a distance a_0 at $t = 0$.

2b) Find an expression for the time, t_m, it will take for the binary orbit to shrink from an initial separation a_0 to merger[8] at $a(t_m) = 0$.

2c) The frequency of the gravitational waves is $f = 2/T$ (twice that of the orbital frequency). Show that the frequency has a time dependence

$$f(t) = \frac{f_0}{(1 - t/t_m)^{3/8}} \tag{8}$$

where $f_0 = \sqrt{GM}/\pi a_0^{3/2}$ is the initial frequency of the orbit when the separation is a_0

2d) Make a sketch of $f(t)$ and compare by eye to the spectrogram shown in the bottom panels of Figure 1.

comment: The formula you derived for $f(t)$ should have the same shape as that seen in the actual GW source. You can play around and find what values of m_1 and m_2 give you a frequencies and timescales

[7] The luminosity emitted in gravitational waves tells us the rate at which the orbital energy changes

$$\frac{dE}{dt} = -L$$

Use the expression for E and the chain rule to turn this into an expression for da/dt and solve the differential equation.

[8] The merger will actually happen earlier when the separation is of order the innermost stable circular orbit, $a \approx r_{\rm isco}$, which we previously found was at $r_{\rm isco} = 3r_s$ for a non-spinning black hole. But typically $a_0 \gg r_{\rm insco}$ so we can make an approximation that the final separation is at $a = 0$.

in the same range as the observed event. This is essentially how GW signals are analyzed. The next (optional) part of the problem goes on to derive an explicit expression for the masses in the binary given measurements f and the rate of change of f.

2e) optional (not graded) From your expression for $f(t)$ calculate the rate of change df/dt and use your relations to show that

$$\mathcal{M} = \frac{c^3}{G}\left[\frac{5}{96}\pi^{-8/3} f^{-11/3} \frac{df}{dt}\right]^{3/5} \tag{9}$$

where the *chirp mass* is defined as

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} \tag{10}$$

This is Eq. 1 of the original LIGO paper .

comment: You have derived an expression $f(t)$ for the chirp signal seen in the LIGO data of Figure 1. You can use the function to fit the data and infer properties of the binary. Looking at the figure we see that a typical frequency is around $f \approx 100$ Hz, and that it changes by about 100 Hz in about 0.01 seconds, so $df/dt \approx 100/0.01 \approx 10^4$. Plugging into Eq. 9 we calculate that the chirp mass is of order $30\ M_\odot$. So this system consisted of some good sized black holes!

The signal provides a strong constraint on the chirp mass, which is a combination of m_1 and m_2. More detailed modeling of the signal allows for some less tight constraints of the individual masses, as well as their spins. While our Newtonian approximation is OK when the stars are far apart, closer in one should use perturbation theory to include the leading order corrections from general relativity. When the stars are close to or in the act of merging, one must solve the full equations of GR, which has to be done numerically on computers.

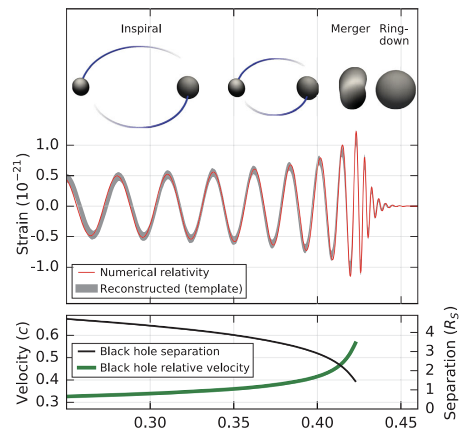

Figure 2: Illustration of how the inspiral of a black hole binary leads to a GW waveform that increases in frequency over time.

Problem 3: Galaxy Counting

Most modern models in cosmology adopt the *cosmological principle*, i.e., that the distribution of matter in the universe is uniform. One of the simplest observational tests of this assumption is to look out in the sky and count the number of galaxies that you see.

Let us make the following assumptions to be able to develop this into a rigorous observational test:

1) All of the galaxies in the Universe have the same luminosity, L_0.

2) Spacetime is Euclidean and nearly static, such that the volume out to some distance D is simply $V = 4\pi D^3/3$.

3) The number of galaxies, N, in some volume V is $N = nV$ where the number density n is a constant[9].

[9] All of these assumptions are suspect – Galaxies generally *don't* all have the same luminosity and our universe is *not* static. Still, this gives us a simple starting point from which we may be able to relax the assumptions.

The experiment is simple – you take deep images of the sky which include many galaxies, and then count how many galaxies have an observed brightness above a certain flux F. Call this number $N(> F)$

3a) Show that $N(> F)$, the number of galaxies one observes above a given flux[10] , is proportional to $F^{-3/2}$

[10] The inverse square law says that flux observed from an object of luminosity L and distance D is $F = L/4\pi D^2$, since the emitted light spreads out over a sphere.

ASTRONOMERS DESCRIBE THE BRIGHTNESS of sources using the *magnitude system*. For a source with flux F on earth, the apparent magnitude is defined as

$$m = -2.5 \log_{10}(F) + \text{constant} \tag{11}$$

The magnitude system has its roots in the ancient Greek method of classifying stars. We still use it today because... well, I don't know why we do, but you'll have to get used it[11]. Because of the minus sign in the definition, *dimmer sources have larger magnitudes*.

[11] The Greeks called the brightest stars in the sky magnitude 1. The next brightest stars, which were about a factor of two (or more closely 2.5) times dimmer were called magnitude 2; magnitude 3 stars were a factor of 2.5 times dimmer than that, and so on. Mathematically, taking the log of the flux and multiplying by 2.5 roughly captures this behavior, as our eyes have a nearly logarithmic sensitivity to light. The minus sign is there in the definition of magnitude so that brighter sources have lower magnitude as in the Greek system.

3b) Show that for n constant, the number of galaxies observed with an apparent magnitude less than m is

$$N(< m) \propto 10^{0.6m} \tag{12}$$

comment: Figure 3 shows a compilation of data from the literature for $N(< m)$. For $m < 21$, the data closely follow $N \propto 10^{\alpha m}$ with $\alpha \approx 0.6$, in agreement with your analysis. For larger m (i.e., more distant galaxies) the curve flattens and $\alpha \approx 0.4$. This behavior at larger m could be a result of one or more of our 3 assumptions above breaking down. For example, we can consider a breakdown in assumption (1) in which the luminosity L of galaxies is not constant.

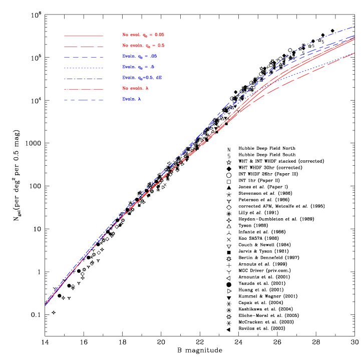

Figure 3: Compilation of galaxy counts above a certain apparent magnitude (these observed magnitude are in B-band, which means observing light at blue wavelengths). The y-axis is logarithmically spaced, and so a power law $N(M < m) \propto 10^{\alpha m}$ should appear as a straight line with slope α.

3c) Imagine now that galaxies do not all have the same luminosity, but that more distant ones are systematically dimmer. For simplicity, we'll model the trend as a power-law[12] so that galaxies at distance D from earth have a luminosity

$$L(D) = L_0 \left(\frac{D}{D_0}\right)^{-b} \tag{13}$$

where D_0 is some characteristic distance and b is some constant (you can assume $b < 3$). Show that

$$N(< m) \propto 10^{\alpha m} \tag{14}$$

and find the value of α in terms of b.

[12] More distant galaxies could be intrinsically less luminous because of *galaxy evolution*. When we look at distant galaxies we are seeing them earlier in time since it took time for the light to reach us (see the problem below for more details). It seems plausible that galaxies in the past may have been less luminous, since they have had less time to evolve and form stars. Lacking an detailed model of galaxy evolution, we just choose here perhaps the simplest expression we can imagine for how the luminosity has changed with time – a power law. You could try using more realistic functions for $L(D)$, but the power-law is relatively easy to do calculations with and as long as the real evolution function $L(D)$ is pretty smooth and monotonic, a power-law approximation with some exponent b is probably not a bad approximation.

comment: You see that galaxy evolution may be able to account for the turnover in the galaxy counts at higher m seen in Figure 3 (if $b \approx 1$ for distance galaxies). Later on we can include the effects of cosmological expansion which can have a similar effect. If we could distinguish these two effects we could learn something both about the history of cosmic expansion and the evolution of galaxies over time. The colored lines in the Figure are attempts to do so using models along the lines you calculated here. In practice, evolution effects and cosmological effects are fairly degenerate, so it is challenging to disentangle them using this data set.

Problem 4: An Empty Universe

THE METRIC OF THE EXPANDING UNIVERSE is the Robertson-Walker metric, which can be written as[13]

$$ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + S_k^2(r)d\Omega^2\right] \tag{15}$$

where r is a *comoving coordinate*, which we define such that galaxies remain at a fixed r coordinate over time. We account for the expansion of the universe by the *scale factor*, $a(t)$, which describes how much bigger or smaller the universe is at time t relative to today .

We will soon learn how to calculate $a(t)$, and see that it depends on the amount and type of matter/energy is in the universe. But in this problem we will play with a simple expansion where we assume the size of the universe increases linearly with time

$$a(t) = \frac{t}{t_0} \qquad \text{(empty universe)} \tag{16}$$

where t is the time since the big bang (taken to occur at $t = 0$) and t_0 is the time now[14]. It turns out that this $a(t)$ describes an empty universe, where there is no gravitational attraction to slow down the expansion (or dark energy to speed it up).

The **Hubble parameter** describes how fast the universe is expanding *relative to its current scale*, and so is defined by

$$H(t) = \frac{\dot{a}}{a} \tag{17}$$

where the dot means a time derivative. The Hubble constant is just the Hubble parameter at the current time $H_0 = H(t_0)$.

4a) What is the Hubble parameter $H(t)$ in the empty universe, in terms of t and t_0?

comment: Although this empty universe expands at a constant rate (da/dt =constant), the Hubble parameter decreases with time since H is defined as the rate of change of the scale factor *relative to* the value of the scale factor at that time, $H(t) = \dot{a}/a$.

4b) What is the Hubble constant in this empty universe, in terms of t_0, the time since the big bang?

[13] The function $S_k(r)$ accounts for the curvature of the universe, which we will discuss further in the class, but won't need to worry about in this problem.

[14] By definition, we always have $a(t_0)$ equal to 1 at the present time, such that $a(t)$ tells us how big or small the universe scale is relative to now.

comment: The quantity $t_H = 1/H_0$ is called the *Hubble time*, and sets the scale of the age of the universe. In an empty universe, we see that the Hubble time is *exactly* the time since the big bang. In other universes that don't expand at a constant rate, the age of the universe will differ from t_H by some numerical factor which we can calculate.

4c) Say a galaxy at comoving radial coordinate r emits a photon that travels radially through an empty universe[15] that is observed on earth (at $r = 0$) at the current time t_0. Determine how much time, Δt, it took the photon to reach us, in terms of r, t_0, and constants.

[15] We've already done problems solving for light paths within a given metric (e.g., see the "Shapiro Delay" problem on a previous homework), so we know the procedure to handle these sorts of problems.
Obviously if there are sources and photons and an earth in this universe it can not strictly be "empty". However, if the mass/energy density of the universe is very small we can approximate it as an empty universe.

4d) Show that for nearby sources ($r \ll ct_0$) the time it took the light to get to us is $\Delta t \approx r/c$ as we might expect. For more distant sources, the light travel time can be shorter than r/c. That is because this is an expanding universe, and when the light was emitted the universe was actually smaller than it is now.

4e) The *proper distance*, d_p, is the spacetime distance s when measured at fixed time ($dt = 0$) – i.e., it is the "measuring-tape distance" where we read the ends of the measuring tape at the same time. Find an expression for the proper distance measured now, $d_p(t_0)$ to the galaxy at coordinate r.

4f) Find an expression for the proper distance, $d(t_e)$, to the same galaxy (at coordinate r) when the photon was originally emitted at time t_e. Compare this to the proper distance to the galaxy now, $d(t_0)$.

comment: If you look at a far away galaxy, the proper distance to that galaxy is different now then when the light was emitted (because the universe expanded while the light was on its way to us). And both of these distances are different than the distance the light traveled to get to us ($= c\Delta t$). In cosmology, we have to be careful when we talk about "the distance to a galaxy" and specify exactly what we are talking about.