Geodesic under Different Metric

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1 What is geodesic?

In geometry, a geodesic is a curve representing in some sense the shortest path between two points in a surface. It is a generalization of the notion of a "straight line".

In general relativity, a geodesic generalizes the notion of a "straight line" to curved spacetime. Importantly, the world line of a particle free from all external, non-gravitational forces is a particular type of geodesic. In other words, a freely moving or falling particle always moves along a geodesic.

A geodesic is a curve along which the tangent vector is parallel-transported, since it is computationally much more straightforward. The tangent vector to a path $x^\mu(\lambda)$ is $\frac{dx^\mu}{d\lambda}$. The condition that it be parallel transported is thus

$$\frac{D}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = 0, \tag{1}$$

or alternatively

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0. \tag{2}$$

where $\Gamma^\mu_{\rho\sigma}$ are Christoffel symbols and λ is a scalar parameter of motion[1]

So how do we drive the geodesic equation?

One way to get the geodesic equation is to vary the action. First we know the action can be written in:

$$S = \int \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda$$

We can now go ahead and vary this action with respect to the curve x^μ. By the principle of least action we get:

$$0 = \delta S = \delta\int\left(\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\right)d\lambda$$

Using the product rule we get:

$$\begin{aligned}
0 &= \int\left(\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\delta g_{\mu\nu} + g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda}\right)d\lambda \\
&= \int\left(\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\partial_\alpha g_{\mu\nu}\delta x^\alpha + 2g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)d\lambda
\end{aligned}$$

Integrating by-parts the last term and dropping the total derivative (which equals to zero at the boundaries) we get that:

$$0 = \int \left(\frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \partial_\alpha g_{\mu\nu} \delta x^\alpha - 2 \frac{dx^\mu}{d\lambda} \frac{d}{d\lambda} \left(g_{\mu\nu} \frac{dx^\nu}{d\lambda} \right) \right) d\lambda$$

Integrating by-parts the last term and dropping the total derivative (which equals to zero at the boundaries) we get that:

$$0 = \int \left(\frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \partial_\alpha g_{\mu\nu} \delta x^\alpha - 2 \frac{dx^\mu}{d\lambda} \partial_\alpha g_{\mu\nu} \frac{dx^\nu}{d\lambda} - 2 \frac{dx^\mu}{d\lambda} g_{\mu\nu} \frac{d^2 x^\nu}{d\lambda^2} \right) d\lambda$$

Simplifying a bit we see that:

$$0 = \int \left(-2 \frac{d}{d\lambda} \left(g_{\mu\nu} \frac{dx^\nu}{d\lambda} \right) + \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \right) \delta x^\mu d\lambda$$

so,

$$0 = \int \left(g^{\mu\nu} \left(\frac{d^2 x_\nu}{d\lambda^2} + \Gamma^\rho_{\sigma\nu} \frac{dx^\sigma}{d\lambda} \frac{dx^\rho}{d\lambda} \right) \delta x_\mu \right) d\lambda$$

So by Hamilton's principle we find that the Euler–Lagrange equation is

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$$

with the Christoffel symbol defined in terms of the metric tensor as

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\nu\mu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta} \right)$$

2 Geodesics Around the Black Hole

2.1 the Schwarzschild Metric

The Schwarzschild metric describes the spacetime curvature around static massive objects. Examples of such an object is a non-rotating star or a static black hole.

The Schwarzschild metric is given by:

$$c^2 d\tau^2 = c^2 (1 - \frac{2M}{r}) dt^2 + \frac{1}{1 - 2M/r} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \tag{2.1.1}$$

The Lagrangian for a geodesic is given by

$$L = \frac{1}{2} g_{\mu\nu} \dot{x^\mu} \dot{x^\nu} \tag{2.1.2}$$

where the dot means the derivative to some affine parameter λ.

The Lagrangian for the Schwarzschild metric is:

$$L = \frac{1}{2} \left[\left(1 - \frac{2M}{r} \right) \dot{t}^2 - \frac{1}{1 - 2M/r} \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right] \tag{2.1.3}$$

We can derive geodesic equation from Euler-Lagrange's equation,so we are going to introduce some common concepts we need to solve Euler-Lagrange's equation.

The corresponding conjugated momenta ($p_\mu = \frac{\partial}{\partial \dot{x^\mu}} L$) are:

$$\begin{aligned}
p_t &= \left(1 - \frac{2M}{r}\right)\dot{t} \\
p_r &= -\frac{1}{1-2M/r}\dot{r} \\
p_\theta &= -r^2\dot{\theta} \\
p_\phi &= -(r^2\sin^2\theta)\dot{\phi} \qquad (2.1.4)
\end{aligned}$$

From Euler-Lagrange equation

$$\frac{\partial L}{\partial x^\mu} - \frac{\partial}{\partial \lambda}\frac{\partial L}{\partial \dot{x^\mu}} = 0$$

we know that p_t, p_ϕ are constants of motion:

$$\begin{aligned}
p_t = E = \left(1 - \frac{2m}{r}\right)\dot{t} \\
p_\phi = -L = -r^2\dot{\phi} \qquad (2.1.5)
\end{aligned}$$

We can get Hamiltonian from conjugate momenta:

$$\begin{aligned}
H = p_t\dot{t} + p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - L = L = \text{constant} \\
\Rightarrow\ 2L = \frac{E^2}{1-2M/r} - \frac{\dot{r}^2}{1-2m/r} - \frac{L^2\sin^2\theta}{r^2} - r^2\dot{\theta}^2 = \delta \qquad (2.1.6)
\end{aligned}$$

where δ equals to -1 or 0 for respectively timelike or null geodesics. We often set $\theta = \frac{\pi}{2}$ (in the equatorial plane) for a better form:

$$\delta = \frac{E^2}{1-2M/r} - \frac{\dot{r}^2}{1-2m/r} - \frac{L^2}{r^2} \qquad (2.1.7)$$

2.1.1 Timelike Geodesics

For the case of a timelike geodesic, (2.1.7) becomes:

$$\left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2m}{r}\right)\left(\delta + \frac{L^2}{r^2}\right) = E^2 \qquad (2.1.8)$$

since $L = r^2\dot{\phi}$, we change variables with $u \equiv 1/r$ and the above equation can be expressed as

$$\left(\frac{du}{d\phi}\right)^2 = 2Mu^3 - u^2 + \frac{2M}{L^2}u - \frac{1-E^2}{L^2} \qquad (2.1.9)$$

The equation describe the shape of the geodesic in one plane. Using the constants of motion (2.1.5) we can solve the equation.[2]

Besides, from equation (2.1.8) we can derive the integral form of the coordinate time:

$$t = \int^r \frac{Edr}{(1-2M/r)[E^2 - (1-2M/r)(1+L^2/r^2)]^{1/2}} \qquad (2.1.10)$$

2.1.2 Null geodesic

For null geodesic we just need to set $\delta = 0$ in equation (2.1.8):

$$\left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{2m}{r}\right)\frac{L^2}{r^2} = E^2 \tag{2.1.11}$$

Similarly , using $u \equiv 1/r$ to get an equation which describes the shape of geodesic in a plane:

$$\left(\frac{du}{d\phi}\right)^2 = 2Mu^3 - u^2 + \frac{E^2}{L^2} \tag{2.1.12}$$

Besides, a radical geodesic ($L = 0$) can be described by

$$\frac{dr}{d\lambda} = \pm E = \pm\left(1 - \frac{2M}{r}\right) \tag{2.1.13}$$

Integrating it:

$$t = \pm\left[r + 2m\log\left(\frac{r}{2m} - 1\right)\right] + C \tag{2.1.14}$$

where C is a constant.

2.2 the Kerr Metric

The solution to the Einstein's equation for a spinning,rotating massive objects without charge, is given by the Kerr-metric.

Before talking about the detail form of the Kerr metric,first we need to make sure what coordinate system are we going to use. The most common coordinate system used is Boyes-Lindquist coordinate system. The most significant advantage of using this coordinate system is that it is written in spherical coordinates, which is easy for us to have a direct image of the whole picture. But it also may lead to coordinate singularity(which we won't mention too much in this paper) at the event horizon. We can remove it by using another coordinate system:Kerr coordinate,so we will not pay much attention to this trouble.

The Kerr metric in Boyes-Lindquist coordinates is given by[3]:

$$\begin{aligned} c^2 d\tau^2 &= -(1 - \frac{2Mr}{\sum^2})c^2dt^2 - \frac{4Mcra\sin^2\theta}{\sum^2}dtd\phi + \frac{\sum^2}{\Delta}dr^2 \\ &\sum^2 d\theta^2 + \left[(r^2 + a^2)\sin^2\theta + \frac{2Mra^2\sin^2\theta}{\sum^2}\right]d\phi^2 \end{aligned} \tag{2.2.1}$$

In which:

$$\sum^2 \equiv r^2 + a^2\cos^2\theta \tag{2.2.2}$$

$$\Delta \equiv r^2 + a^2 - 2Mr \tag{2.2.3}$$

where M, a are constants in the following form (These two constants are the mass and angular momentum per unit mass of the black hole) :

$$M \equiv Gm/c^2 \tag{2.2.4}$$

$$a = J/mc \tag{2.2.5}$$

m is the mass of black hole and G is gravitational constant.

For parameter a, a positive value indicates a clockwise rotation of the object, a negative value indicates a counterclockwise rotation. For convenience, we choose special units so that $c = 1$ and $G = 1$.

Lagrangian for free test-particle under a given metric is $L = \frac{1}{2}g_{\mu\nu}\dot{x^{\mu}}\dot{x^{\nu}}$,with the derivative to affine parameterλ(For time-like geodesic this parameter is proper time,and for null geodesic it is a chosen "good" parameter for the nice analysis form).So

$$\begin{aligned} L(x^{\mu}, \dot{x^{\mu}}) \ &= \ \frac{1}{2}g_{\mu\nu}\dot{x^{\mu}}\dot{x^{\nu}} \ = \ -\frac{1}{2}\left(1 - \frac{2Mr}{\sum^2}\right)\dot{t}^2 - \frac{2Mra\sin^2\theta}{\sum^2}\dot{t}\dot{\phi} + \frac{\sum^2}{2\Delta}\dot{r}^2 \\ &\quad \frac{\sum^2}{2}\dot{\theta}^2 + \frac{1}{2}\left[(r^2 + a^2)\sin^2\theta + \frac{2Mra^2\sin^2\theta}{\sum^2}\right]\dot{\phi}^2 \end{aligned} \tag{2.2.6}$$

The reason why I mention how to express the Lagrangian L here is that we will use it in the next subsection.

And now the Euler-Lagrange's equation can be expressed as:

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x^{\mu}}} = \frac{\partial L}{\partial x^{\mu}} \tag{2.2.7}$$

Note that if the metric does not depend on a given coordinate x^{μ}, then we can define a constant of motion to simplify our process to get the geodesic equation.(due to Euler-Lagrange's equation) The Kerr metric is independent of t, ϕ,therefore, geodesic motion in Kerr geometry is characterized by two constants of motion,

$$\begin{aligned} E \ &\equiv \ -k^{\mu}\frac{dx^{\mu}}{d\lambda} = -\frac{dx^{t}}{d\lambda} \quad \text{constant along geodesics} \\ L \ &\equiv \ m^{\mu}\frac{dx^{\mu}}{d\lambda} = \frac{dx^{\phi}}{d\lambda} \quad \text{constant along geodesics} \end{aligned} \tag{2.2.8}$$

where $k^{\mu} = (1, 0, 0, 0), m^{\mu} = (0, 0, 0, 1)$ are the Killing vectors, which describe the symmetry of the metric.

Besides, we can choose the suitable affine parameter λ to express the algebraic relations in a better way:

$$g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = \kappa \tag{2.2.9}$$

where

$$\begin{aligned} \kappa = -1 &\quad \text{for timelike geodesics} \\ \kappa = 1 &\quad \text{for spacelike geodesics} \\ \kappa = 0 &\quad \text{for null geodesics} \end{aligned} \tag{2.2.10}$$

2.2.1 Four constants of motion

To find expressions for the geodesics in the spacetime, we need four constants of motion. As analysis mechanics, we can use Lagrangian L and Euler-Lagrange equation to derive the restrains.

The first three constants are easy to get, and we will use the Hamilton-Jacobi approach to derive the fourth one.[6]

The Hamiltonian is given by:

$$H(x^\mu, p_\mu) - p_\mu \dot{x^\mu} - L(x^\mu, \dot{x^\mu}) \tag{2.2.11}$$

For free particle, the Lagrangian L is

$$L(x^\mu, \dot{x^\mu}, t) = \frac{1}{2} g_{\mu\nu} \dot{x^\mu} \dot{x^\nu} \tag{2.2.12}$$

The conjugate momenta p_μ(not confused with the fourth-momentum of the particle P^μ) is defined as

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x^\mu}} = g_{\mu\nu} \dot{x^\nu} \tag{2.2.13}$$

Inverting the conjugate momenta we can get

$$\dot{x^\mu} = g^{\mu\nu} p_\nu \tag{2.2.14}$$

Now insert the formulas we can get

$$H = g_{\mu\nu} \dot{x^\mu} \dot{x^\nu} - \frac{1}{2} g_{\mu\nu} \dot{x^\mu} \dot{x^\nu} = \frac{1}{2} g_{\mu\nu} \dot{x^\mu} \dot{x^\nu} = L \tag{2.2.15}$$

Using (2.2.9) and (2.2.10) we gives the following expression for the Hamilton

$$H = \frac{1}{2} g_{\mu\nu} \dot{x^\mu} \dot{x^\nu} = \frac{1}{2} p_\mu \dot{x^\mu} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \tag{2.2.16}$$

Geodesic equations are equivalent to the Euler-Lagrange equations for the Lagrangian functional, which are equivalent to the Hamilton equations for the Hamiltonia functional:

$$\begin{aligned} \dot{x^\mu} &= \frac{\partial H}{\partial p_\mu} \\ \dot{p_\mu} &= -\frac{\partial H}{\partial x^\mu} \end{aligned} \tag{2.2.17}$$

It is as hard to solve eqs.(2.2.17) as solving Euler-Lagrange's equation. However, in the Hamilton-Jacobi approach, it gives a easier way. [5]

In the Hamilton-Jacobi approach, we look for a function of the coordinates and of the curve parameter λ,

$$S = S(x^\mu, \lambda) \tag{2.2.18}$$

which is the solution of the Hamilton-Jacobi equation

$$H\left(x^\mu, \frac{\partial S}{\partial x^\mu}\right) + \frac{\partial S}{\partial \lambda} = 0 \tag{2.2.19}$$

And for a solution S to the Hamilton-Jacobi equation,we have

$$\frac{\partial S}{\partial x^\mu} = p_\mu \tag{2.2.20}$$

In general such solution depends on four integration constants.
We have already know that:

$$\begin{aligned} H &= \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = \frac{1}{2}\kappa \\ p_t &= -E \text{ constant} \\ p_\phi &= L \text{ constant} \end{aligned} \tag{2.2.21}$$

So the S is required to be :

$$S = -\frac{1}{2}\kappa\lambda - Et + L\phi + S^{(r\theta)}(r,\theta) \tag{2.2.22}$$

where $S^{(r\theta)}(r,\theta)$ is a function we are looking for.
We can make the ansatz that $S^{(r\theta)}$ can separate variables

$$S = -\frac{1}{2}\kappa\lambda - Et + L\phi + S^{(r)}(r) + S^{(\theta)}(\theta) \tag{2.2.23}$$

Take (2.2.23) into Hamilton-Jacobi equation(2.2.19) and though lots of work,we will get

$$\begin{aligned} &\Delta\left(\frac{dS^{(r)}}{dr}\right)^2 - \kappa r^2 - \frac{(r^2+a^2)^2}{\Delta}E^2 + \frac{4Mra}{\Delta}EL - \frac{a^2}{\Delta}L^2 + a^2E^2 + L^2 \\ &= -\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 + \kappa a^2\cos^2\theta + a^2\cos^2\theta E^2 - \frac{\cos^2\theta}{\sin^2\theta}L^2. \end{aligned} \tag{2.2.24}$$

In equation (2.2.24), the left-hand side does not depend on θ, and is equal to the right-hand side which does not depend on r, therefore this quantity must be a constant C:

$$\begin{aligned} &\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 - \cos^2\theta\left[(\kappa+E^2)a^2 - \frac{1}{\sin^2\theta}L^2\right] = C \\ &\Delta\left(\frac{dS^{(r)}}{dr}\right)^2 - \kappa r^2 - \frac{(r^2+a^2)^2}{\Delta}E^2 + \frac{4Mra}{\Delta}EL - \frac{a^2}{\Delta}L^2 + a^2E^2 + L^2 \\ &= \Delta\left(\frac{dS^{(r)}}{dr}\right)^2 - \kappa r^2 + (L-aE)^2 - \frac{1}{\Delta}[E(r^2+a^2) - La]^2 = -C \end{aligned} \tag{2.2.25}$$

where $\Delta = r^2 + a^2 - 2Mr$, and C is the Carter's constant.
So we can define the functions $R(r)$ and $\Theta(\theta)$ as

$$\begin{aligned} \Theta(\theta) &\equiv C + \cos^2\theta\left[(\kappa+E^2)a^2 - \frac{1}{\sin^2\theta}L^2\right] \\ R(r) &\equiv \Delta\left[-C + \kappa r^2 - (L-aE)^2\right] + \left[E(r^2+a^2) - La\right]^2 \end{aligned} \tag{2.2.26}$$

Then

$$\begin{aligned} \left(\frac{dS^{(\theta)}}{d\theta}\right)^2 &= \Theta \\ \left(\frac{dS^{(r)}}{dr}\right)^2 &= \frac{R}{\Delta^2} \end{aligned} \tag{2.2.27}$$

and the solution of the Hamilton-Jacobi equation is

$$S = -\frac{1}{2}\kappa\lambda - Et_L\phi + \int \frac{\sqrt{R}}{\Delta}dr + \int \sqrt{\Theta}d\theta \tag{2.2.28}$$

Now we have the solution of Hamilton-Jacobi equation,depending on four constants (κ, E, L, C). Using (2.2.13) and (2.2.20) to express conjugate momenta

$$\begin{aligned} p_\theta^2 &= (\sum \dot{\theta})^2 = \Theta(\theta) \\ p_r^2 &= \left(\frac{\sum}{\Delta}\dot{r}\right)^2 = \frac{R(r)}{\Delta^2} \end{aligned} \tag{2.2.29}$$

where $\sum = r^2 + a^2\cos^2\theta$. Therefore

$$\begin{aligned} \dot{\theta} &= \pm\frac{1}{\sum}\sqrt{\Theta} \\ \dot{r} &= \pm\frac{1}{\sum}\sqrt{R} \end{aligned} \tag{2.2.30}$$

The Left two conjugate momenta are:

$$\begin{aligned} p_t &= -\left(1 - \frac{2Mr}{\sum^2}\right)\dot{t} - \frac{2Mra\sin^2\theta}{\sum^2}\dot{\phi} = -E \\ p_\phi &= -\frac{2Mra\sin^2\theta}{\sum^2}\dot{t} - \left[(r^2+a^2)\sin^2\theta + \frac{2Mra^2\sin^2\theta}{\sum^2}\right]\dot{\phi} = L \end{aligned} \tag{2.2.31}$$

It takes a little calculation to get the similar form of (2.1.30),I will put this in the next section.

2.2.2 Equatorial geodesics

In this section we study motion in the equatorial plane,i.e. geodesics with $\theta = \frac{\pi}{2}$.On the equatorial plane,$\sum = r^2$,therefore

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2M}{r}\right) \\ g_{t\phi} &= -\frac{2Ma}{r} \\ g_{rr} &= \frac{r^2}{\Delta} \\ g_{\phi\phi} &= r^2 + a^2 + \frac{2Ma}{r} \end{aligned} \tag{2.2.32}$$

Consider equations(2.2.31) in the equatorial plane:

$$E = \left(1 - \frac{2M}{r}\right)\dot{r} + \frac{2Ma}{r}\dot{\phi} \tag{2.2.33}$$

$$L = -\frac{2Ma}{r}\dot{t} + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\dot{\phi} \tag{2.2.34}$$

To solve these two equations (2.2.33),(2.2.34), we define

$$\begin{aligned} A &\equiv 1 - \frac{2M}{r} \\ B &\equiv \frac{2Ma}{r} \\ C &\equiv r^2 + a^2 + \frac{2Ma^2}{r} \end{aligned} \tag{2.2.35}$$

and write (2.2.33),(2.2.34) as

$$E = A\dot{t} + B\dot{\phi} \tag{2.2.36}$$

$$L = -B\dot{t} + C\dot{\phi} \tag{2.2.37}$$

Therefore,

$$\begin{aligned} CE - BL &= [AC + B^2]\dot{t} = \Delta\dot{t} \\ AL + BE &= [AC + B^2]\dot{\phi} = \Delta\dot{\phi} \end{aligned} \tag{2.2.38}$$

i.e.

$$\boxed{\begin{aligned} \dot{t} = &\frac{1}{\Delta}\left[\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)E - \frac{2Ma}{r}L\right] \\ \dot{\phi} = &\frac{1}{\Delta}\left[\left(1 - \frac{2M}{r}\right)L + \frac{2Ma}{r}E\right] \end{aligned}} \tag{2.2.39}$$

Now we consider the radial component of the the four-velocity. We will use the algebraic relations (2.2.9) to derive:

$$\begin{aligned} g_{\mu\nu} &= \kappa \\ &= -A\dot{t}^2 - 2B\dot{t}\dot{\phi} + C\dot{\phi}^2 + \frac{r^2}{\Delta}\dot{r}^2 \end{aligned} \tag{2.2.40}$$

Therefore,

$$\dot{r}^2 = \frac{1}{r^2}[CE^2 - 2BLE - AL^2] + \frac{\kappa\Delta}{r^2} \tag{2.2.41}$$

The polynomial $[CE^2 - 2BLE - AL^2]$ will be zero at

$$V_{\pm} = \frac{BL \pm \sqrt{B^2L^2 + ACL^2}}{C} = \frac{L}{C}[B \pm \sqrt{\Delta}] \tag{2.2.42}$$

Using this, equation (2.2.41) can be written as

$$\boxed{\dot{r}^2 = \frac{(r^2 + a^2)^2 - a^2\Delta}{r^4}(E - V_+)(E - V_-) + \frac{\kappa\Delta}{r^2}} \tag{2.2.43}$$

where

$$\boxed{V_{\pm} = \frac{2Mar \pm r^2\sqrt{\Delta}}{(r^2 + a^2)^2 - a^2\Delta}L} \tag{2.2.44}$$

2.2.3 Geodesic equation derived from Christoffel symbols

The previous sections discussed how to obtain four constants of motion through the Hamilton-Jacobi approach, thereby deriving the geodesic equation. Now, we will start from the Christoffel symbols and directly solve the geodesic equation to obtain the expression for the four-dimensional vector $x^\mu = (t, r, \theta, \phi)$.

We are going to do the work under metric

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & 0 & 0 & g_{t\phi} \\ 0 & g_{rr} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ g_{\phi t} & 0 & 0 & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} -(1-\frac{2Mr}{\sum^2}) & 0 & 0 & -\frac{2Mar\sin^2\theta}{\sum} \\ 0 & \sum^2/\Delta & 0 & 0 \\ 0 & 0 & \sum^2 & 0 \\ -\frac{2Mar\sin^2\theta}{\sum^2} & 0 & 0 & (r^2+a^2+\frac{2Ma^2r\sin^2\theta}{\sum})\sin^2\theta \end{pmatrix}$$

$$g^{\mu\nu} = g^{-1}_{\mu\nu} = \begin{pmatrix} \frac{g_{\phi\phi}}{g_{tt}g_{\phi\phi}-g^2_{t\phi}} & 0 & 0 & -\frac{g_{t\phi}}{g_{tt}g_{\phi\phi}-g^2_{t\phi}} \\ 0 & \frac{1}{g_{rr}} & 0 & 0 \\ 0 & 0 & \frac{1}{g_{\theta\theta}} & 0 \\ -\frac{g_{t\phi}}{g_{tt}g_{\phi\phi}-g^2_{t\phi}} & 0 & 0 & \frac{g_{tt}}{g_{tt}g_{\phi\phi}-g^2_{t\phi}} \end{pmatrix} = \begin{pmatrix} \frac{r^2+a^2-a^2\Delta\sin^2\theta}{\sum^2\Delta} & 0 & 0 & \frac{2Mar}{\sum^2\Delta} \\ 0 & -\frac{\Delta}{\sum^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sum^2} & 0 \\ \frac{2Mar}{\sum^2\Delta} & 0 & 0 & -\frac{\Delta-a^2\sin^2\theta}{\sum^2\Delta\sin^2\theta} \end{pmatrix}$$

using the Christoffel equation,

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\nu\mu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}) \tag{2.2.45}$$

thus

$$\Gamma^0_{01} = \frac{2M(r^2+a^2)(r^2-a^2\cos^2\theta)}{2\sum^2\Delta}, \qquad \Gamma^0_{02} = -\frac{2Ma^2r\sin\theta\cos\theta}{\sum^2},$$

$$\Gamma^0_{23} = \frac{2Ma^3r\sin^3\theta\cos\theta}{\sum^2}, \qquad \Gamma^0_{13} = \frac{2Ma\sin^2\theta[a^2\cos^2\theta(a^2-r^2)-r^2(a^2+3r^2)]}{2\sum^2\Delta},$$

$$\Gamma^1_{00} = \frac{2M\Delta(r^2-a^2\cos^2\theta)}{2\sum^3}, \qquad \Gamma^1_{03} = -\frac{2M\Delta a\sin^2\theta(r^2-a^2\cos^2\theta)}{2\sum^3},$$

$$\Gamma^1_{11} = \frac{2ra^2\sin^2\theta-2M(r^2-a^2\cos^2\theta)}{2\sum\Delta}, \qquad \Gamma^1_{12} = -\frac{a^2\sin\theta\cos\theta}{\sum},$$

$$\Gamma^1_{22} = -\frac{r\Delta}{\sum}, \qquad \Gamma^1_{33} = \frac{\Delta\sin^2\theta}{2\sum^3}[-2r\sum^2 + 2Ma^2\sin^2\theta(r^2-a^2\cos^2\theta)],$$

$$\Gamma^2_{00} = -\frac{2Ma^2r\sin\theta\cos\theta}{\sum^3}, \qquad \Gamma^2_{03} = \frac{2Mar(r^2+a^2)\sin\theta\cos\theta}{\sum^3},$$

$$\Gamma^2_{11} = \frac{a^2\sin\theta\cos\theta}{\sum\Delta}, \qquad \Gamma^2_{12} = \frac{r}{\sum},$$

$$\Gamma^2_{22} = -\frac{a^2\sin\theta\cos\theta}{\sum}, \qquad \Gamma^2_{33} = -\frac{\sin\theta\cos\theta}{\sum^3}[A\sum + (r^2+a^2)2Ma^2r\sin^2\theta],$$

$$\Gamma^3_{01} = \frac{2Ma(r^2-a^2\cos^2\theta)}{2\sum^2\Delta}, \qquad \Gamma^3_{02} = -\frac{2Mar\cot\theta}{\sum^2},$$

$$\Gamma^3_{23} = \frac{\cot\theta}{\sum^2}[\sum^2 + 2Ma^2r\sin^2\theta], \qquad \Gamma^3_{13} = \frac{2r\sum^2 + 2M[a^4\sin^2\theta\cos^2\theta - r^2(\sum + r^2 + a^2)]}{2\sum^2\Delta}.$$

where

$$A = [(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta]$$

due to geodesic equations we have

$$\begin{aligned}
\ddot{t} &= \Gamma^0_{01} \dot{t} \dot{r} + \Gamma^0_{02} \dot{t} \dot{\theta} + \Gamma^0_{13} \dot{r} \dot{\phi} + \Gamma^0_{23} \dot{\theta} \dot{\phi} \\
\ddot{r} &= \Gamma^1_{00} \dot{t}^2 + \Gamma^1_{03} \dot{t} \dot{\phi} + \Gamma^1_{11} \dot{r}^2 + \Gamma^1_{12} \dot{r} \dot{\theta} + \Gamma^1_{22} \dot{\theta}^2 + \Gamma^1_{33} \dot{\phi}^2 \\
\ddot{\theta} &= \Gamma^2_{00} \dot{t}^2 + \Gamma^2_{03} \dot{t} \dot{\phi} + \Gamma^2_{11} \dot{r}^2 + \Gamma^2_{12} \dot{r} \dot{\theta} + \Gamma^2_{22} \dot{\theta}^2 + \Gamma^2_{33} \dot{\phi}^2 \\
\ddot{\phi} &= \Gamma^3_{01} \dot{t} \dot{r} + \Gamma^3_{02} \dot{t} \dot{\theta} + \Gamma^3_{13} \dot{r} \dot{\phi} + \Gamma^3_{23} \dot{\theta} \dot{\phi}
\end{aligned} \tag{2.2.46}$$

To solve these equations we need to separate variables and set some constants:[5]

$$\begin{aligned}
E &= -p_t = -\frac{dx^t}{d\lambda} \\
L &= p_\phi = \frac{dx^\phi}{d\lambda}
\end{aligned} \tag{2.2.47}$$

These two are similar to what we have done in previous section.
The third constant of motion is the particle's rest mass(zero for null geodesic):

$$\mu = (-g_{\alpha\beta} p_\alpha p_\beta)^{1/2}, \quad p_\nu = \frac{dx^\nu}{d\lambda} \tag{2.2.48}$$

The fourth is Carter's constant:

$$C = p_\theta^2 + \cos^2 \theta [a^2(\mu^2 - E^2) + \frac{L^2}{\sin^2 \theta}] \tag{2.2.49}$$

By combining equations (2.2.47)-(2.2.49) with equations (2.2.46), the result is

$$\begin{aligned}
\sum \frac{d\theta}{d\lambda} &= \sqrt{\Theta}, \\
\sum \frac{dr}{d\lambda} &= \sqrt{R}, \\
\sum \frac{d\phi}{d\lambda} &= -\left(aE - \frac{L}{\sin^2 \theta} \right) + \left(\frac{a}{\Delta} \right) P, \\
\sum \frac{dt}{d\lambda} &= -a(aE \sin^2 \theta - L) + \frac{(r^2 + a^2)}{\Delta} P
\end{aligned}$$

where

$$\begin{gathered}
\Theta = C - \cos^2 \theta [a^2(\mu^2 - E^2) + \frac{L^2}{\sin^2 \theta}], \\
P = E(r^2 + a^2) - La, \\
R = P^2 - \Delta[\mu^2 r^2 + (L - aE)^2 + C].
\end{gathered}$$

At last, if we set $a = 0$, what we derive under Kerr metric will turn to what we discovered in Schwarzschild metric.

3 Geodesic In The Expanding Universe

In the expanding universe, we often use the FLRW metric to describe the space.

The Friedmann–Lemaître–Robertson–Walker metric (FLRW is a metric based on an exact solution of the Einstein field equations of general relativity. The metric describes a homogeneous, isotropic, expanding (or otherwise, contracting) universe that is path-connected, but not necessarily simply connected.[7]

We can write the FLRW metric in [4]

$$ds^2 = -c^2dt^2 + R^2(t)\left[d\chi^2 + S_k^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right],$$

where

$$S_k(\chi) = \begin{cases} \sin\chi, & k = +1, \quad \text{closed}, \\ \chi, & k = 0, \quad \text{flat}, \\ \sinh\chi, & k = -1, \quad \text{open}, \end{cases}$$

Here we only consider the case of $k = 0$ (flat universe), which is a reduced-circumference polar coordinates. And to be specific, we can let S_k to be r^2,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right],$$

Then we are going to calculate the Christoffel symbol of this metric

$$g_{\mu\nu} \rightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2\sin^2\theta \end{pmatrix}$$

$$g^{\mu\nu} \rightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-kr^2}{a^2(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{a^2(t)r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{a^2(t)r^2\sin^2\theta} \end{pmatrix}$$

using the Christoffel symbol equation,

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\nu\mu}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\right)$$

we get

$$\Gamma^{0}_{\alpha\beta} = \frac{1}{2}g^{00}\left(\partial_\alpha g_{0\beta} + \partial_\beta g_{0\alpha} - \partial_0 g_{\alpha\beta}\right)$$

$$\Gamma^{1}_{\alpha\beta} = \frac{1}{2}g^{11}\left(\partial_\alpha g_{1\beta} + \partial_\beta g_{1\alpha} - \partial_1 g_{\alpha\beta}\right)$$

$$\Gamma^{2}_{\alpha\beta} = \frac{1}{2}g^{22}\left(\partial_\alpha g_{2\beta} + \partial_\beta g_{2\alpha} - \partial_2 g_{\alpha\beta}\right)$$

$$\Gamma^{3}_{\alpha\beta} = \frac{1}{2}g^{33}\left(\partial_\alpha g_{3\beta} + \partial_\beta g_{3\alpha} - \partial_3 g_{\alpha\beta}\right)$$

$$
\begin{array}{llll}
\Gamma^0_{11} = \dfrac{a\dot{a}}{1-kr^2} & \Gamma^0_{22} = a\dot{a}r^2 & \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta & \\
\Gamma^1_{11} = \dfrac{kr}{1-kr^2} & \Gamma^1_{22} = -r(1-kr^2) & \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta & \Gamma^1_{01} = \dfrac{\dot{a}}{a} \\
\Gamma^2_{02} = \dfrac{\dot{a}}{a} & \Gamma^2_{12} = \dfrac{1}{r} & \Gamma^2_{33} = -\sin\theta\cos\theta & \\
\Gamma^3_{13} = \dfrac{1}{r} & \Gamma^3_{03} = \dfrac{\dot{a}}{a} & \Gamma^3_{23} = \cot\theta, &
\end{array}
$$

Here we specialize to orbits in the $\theta = \pi/2$ plane. Plug the Christoffel symbol into the geodesic equation,

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0,$$

Thus, the geodesic equation is

$$\frac{d^2t}{d\lambda^2} + a\dot{a}\left[\frac{1}{1-kr^2}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2\right] = 0,$$

$$\frac{d^2r}{d\lambda^2} + \frac{kr}{1-kr^2}\left(\frac{dr}{d\lambda}\right)^2 - r(1-kr^2)\left(\frac{d\phi}{d\lambda}\right)^2 + \frac{\dot{a}}{a}\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0,$$

$$\frac{d^2\phi}{d\lambda^2} + \frac{2\dot{a}}{a}\frac{d\phi}{d\lambda}\frac{dt}{d\lambda} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} = 0.$$

Defining $\frac{d}{dl} \equiv a^2\frac{d}{d\lambda}$ along the geodesics, we can obtain

$$\frac{d^2t}{d\lambda^2} + a\dot{a}\left[\frac{1}{1-kr^2}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2\right] = 0,$$

$$\frac{d^2r}{dl^2} + \frac{kr}{1-kr^2}\left(\frac{dr}{dl}\right)^2 - r(1-kr^2)\left(\frac{d\phi}{dl}\right)^2 = 0,$$

$$\frac{d^2\phi}{dl^2} + \frac{2}{r}\frac{d\phi}{dl}\frac{dr}{dl} = 0,$$

we can get one solution as

$$\frac{d\phi}{dt} = 0, \quad dl = \frac{dr}{\sqrt{1-kr^2}}.$$

if $d\phi/dl \neq 0$, the general equation can be expressed as

$$\frac{d^2r}{d\phi^2} - \frac{2-3kr^2}{r(1-kr^2)}\left(\frac{dr}{d\phi}\right)^2 - r(1-kr^2) = 0.$$

Defining $u \equiv r^{-2} - k$, this becomes

$$\frac{d^2u}{d\phi^2} - \frac{1}{2u}\left(\frac{du}{d\phi}\right)^2 + 2u = 0,$$

from which it follows that

$$\frac{d}{d\phi}\left[u^{-1}\left(\frac{du}{d\phi}\right)^2 + 4u\right] = 0.$$

Therefore there is an integration constant c_1 for which

$$u^{-1}\left(\frac{du}{d\phi}\right)^2 + 4u = c_1.$$

Now defining $w \equiv u - \frac{c_1}{8}$ we find that

$$\left(\frac{dw}{d\phi}\right)^2 + 4w^2 = \frac{c_1^2}{16}.$$

Now differentiating the last equation with respect to ϕ we obtain

$$\frac{d^2w}{d\phi^2} + 4w = 0.$$

The general solution to this is

$$w = A\cos(2(\phi + \beta))$$

where A is a nonnegative constant, and β is a constant. Now from above we find that the integration constant is given by $c_1 = 8A$ (where we have also used the fact that c_1 is nonnegative). It then follows from the definition of w that

$$u = 2A\cos^2(\phi + \beta),$$

And thus from the definition of u that

$$r = \frac{1}{\sqrt{k + 2A\cos^2(\phi + \beta)}}.$$

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