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Abstract. We develop a model of the feedback between mutual fund outflows and asset illiquidity. Following a market shock, alert investors anticipate the impact on a fund's net asset value (NAV) of other investors' redemptions and exit first at favorable prices. This first-mover advantage may lead to fund failure through a cycle of falling prices and increasing redemptions. Our analysis shows that (i) the first-mover advantage introduces a nonlinear dependence between a market shock and the aggregate impact of redemptions on the fund's NAV; (ii) as a consequence, there is a critical magnitude of the shock beyond which redemptions brings down the fund; (iii) properly designed *swing pricing* transfers liquidation costs from the fund to redeeming investors and, by removing the nonlinearity stemming from the first-mover advantage, it reduces these costs and prevents fund failure. Achieving these objectives requires a larger swing factor at larger levels of outflows. The swing factor for one fund may also depend on policies followed by other funds.

1. Introduction

The size of the open-end mutual fund industry has increased substantially in recent years. In the United States, the total assets managed by open-end mutual funds grew by $6.8 trillion over the last decade.[1] In particular, fixed income mutual funds posted significant net inflows: 16.3% of outstanding corporate bonds held in the United States are owned by mutual funds as of 2017, up from 3.5% in 1990.[2]

Liquidity management by funds has attracted regulators' attention because of the structural *liquidity mismatch* in open-end mutual funds: funds price their shares daily, but the assets they hold may not be as easy to sell on short notice, such as in the case of corporate and emerging market bond funds. To meet investor redemptions, a fund may be forced to sell assets at reduced prices, but investors' redeemed shares are paid at the end-of-day net asset value (NAV), which may not account for the total liquidation costs incurred in subsequent days.[3] This liquidity mismatch creates an incentive for investors to redeem their shares early as they anticipate that the cost of other investors' redemptions will be reflected in the future NAV of the fund.

In extreme stress scenarios, this *first-mover advantage* can trigger a spiral of redemptions that brings down the fund. A prominent example is the junk-bond fund Third Avenue Focused Credit. Impacted by heavy redemptions, from July to December 2015, the fund lost more than half of its market value, falling below $1 billion from an initial value of $2.1 billion. Third Avenue suspended redemptions and began liquidating the fund because it could not meet withdrawal requests by selling shares of its assets at "rational" prices. In its application to the SEC for the approval of the redemption block, Third Avenue wrote

> If the relief is not granted, and the Fund is unable to suspend redemptions, the institutional investors would likely be best positioned to take advantage of any redemption opportunity, to the detriment of those investors—most likely, retail investors—who remain in the Fund. These remaining investors would suffer a rapidly declining net asset value and an even further diminished liquidity of the Fund's securities portfolio. The relief would help avoid such an outcome.

In October 2016, the Securities and Exchange Commission announced the adoption of amendments to Rule 22c-1 to promote liquidity risk management in the open-end investment company industry. The rule, effective on November 19, 2018, allows open-end funds to use "swing pricing" under certain circumstances.

Swing pricing allows a fund to adjust (swing) its net asset value per share to effectively pass on the costs stemming from shareholder purchase or redemption activity to the shareholders associated with that activity; see Securities and Exchange Commission (2016).

We develop a theoretical framework for the analysis of this rule and its implications for financial stability. Our analysis shows the following. (i) The first-mover advantage magnifies fire sale effects and introduces a crucial nonlinear dependence between the aggregate price impact due to redemptions and an initial market shock. (ii) There is a critical threshold for the market shock beyond which the fire sale–driven amplification leads to the failure of the fund in the sense that the fund is unable to repay shares of redeeming investors at the promised NAV. (iii) Swing pricing, under an ideal implementation, transfers the cost of liquidation from the fund to the redeeming investors and—importantly—reduces this cost by removing the nonlinear amplification stemming from the first-mover advantage. (iv) Swing pricing as currently applied in practice may not achieve these objectives because funds apply a fixed adjustment instead of an adjustment that increases with the number of investors' redemptions. (v) In an economy with multiple funds that all adopt swing pricing, the NAV adjustment required to remove all cross-fund externalities would be lower than in the case in which some funds do not apply swing pricing while others impose a swing price that removes only their own fund's externalities.

Our analysis builds on empirical work exploring the connection between market liquidity, mutual fund performance, and investor flows. Several studies, including Chordia (1996), Ferson and Schadt (1996), Sirri and Tufano (1998), and Warther (1995) have documented relationships between investor flows and fund performance; Edelen (1999), in particular, finds that negative abnormal returns in open-end mutual funds can be explained by the liquidation costs induced by investor flows. Other important contributions include Chen et al. (2010) and Goldstein et al. (2017), which study the sensitivity of outflows to underperformance in the context of equity and fixed income funds, respectively. Goldstein et al. (2017) compare the flow-to-performance relation of funds holding liquid assets with that of funds holding illiquid assets.[4] They find that funds holding illiquid assets are more sensitive to bad performance and exhibit a greater first-mover advantage.

Few other works have explored the theoretical underpinnings of the interactions between asset illiquidity, market stress, and redemption flows. In Chen et al. (2010), the authors present a model that explains why only some investors redeem in response to a fund's bad performance. They attribute this behavior to informational asymmetries: investors receive different signals about the fund's future performance; some investors believe that improved future performance can compensate for the costs of liquidation in the face of an immediate redemption, and others believe the opposite. Chordia (1996) studies the use of load fees to discourage redemptions in a model with redemptions driven by investor liquidity shocks rather than by fund performance; load fees are fixed and, unlike swing pricing, do not respond to the level of redemptions. Lewrick and Schanz (2017b) develop an equilibrium model that yields the welfare-optimal swing price and discuss its dependence on trading costs and investors' liquidity needs.

In contrast to these models, which build on the foundational work on bank runs by Diamond and Dybvig (1983), our study considers an ex post scenario in which the shock has already occurred and investors withdraw in response to it. This allows us to quantify the total amount of redemptions and the critical level of the initial shock that brings down the fund. In Diamond and Dybvig (1983), withdrawing decisions are made ex ante—depositors strategically decide whether to withdraw their deposits without being prompted by an exogenous shock.

Zeng (2017) develops a dynamic model of an open-end mutual fund that holds illiquid assets and manages its cash buffer over time. He argues that, even if redeeming investors were internalizing the liquidation costs they create, there would still be a negative externality imposed on the fund because the fund would need to rebuild its cash position at a later date by selling illiquid assets, a costly operation. Whereas the focus of Zeng (2017) is on the cash management policy and its dynamic relation with shareholder redemptions, our focus is on how the feedback between market and liquidity shocks is reinforced through the first-mover advantage and stopped by an appropriate swing-pricing rule. Unlike in Zeng (2017), the redemption mechanism in our study is triggered by an exogenous market shock that not only reduces the value of a fund share, but also exerts downward pressure on the price of the asset and, in extreme scenarios, brings the fund down. Morris et al. (2017) study, both theoretically and empirically, how asset managers manage liquidity when they interact with redeeming investors. They analyze the trade-off between cash hoarding and pecking-order liquidity management. They find that, if the costs of future fire sales are high relative to the liquidity discount that applies to instantaneous liquidation, funds hoard cash and liquidate more assets than necessary to meet current redemptions.

Our study is complementary to Bernardo and Welch (2004). In their paper, investors decide on a liquidation strategy in anticipation of a potential idiosyncratic liquidity shock, and the proportion of otherwise identical investors who sell their shares

early is endogenously determined in equilibrium. Our investors sell in response to a market price shock, and we assume two types of investors, including a fixed proportion of sophisticated investors who sell early. For example, these could be the institutional investors cited in the Third Avenue application quoted previously. These first-mover investors react to an initial market price shock, accounting for the shock size and the liquidation costs imposed by other investors in the fund. Because our focus is on financial fragility in the mutual fund industry, we assess the consequences of an optimally designed swing-pricing rule as a remedy to this fragility. In contrast to the solutions proposed in Bernardo and Welch (2004), no government intervention is required in our setting because swing pricing can be implemented autonomously by the fund. The design of such a swing-pricing formula and its implications on the reduction of fire-sale externalities are the main focus of our paper. Bernardo and Welch (2013) study the feedback between asset sales and asset price declines in a model in which financial firms decide on both their leverage and the timing of asset sales. Firms may decide to sell assets preemptively to avoid the price knocked-down effects caused by other deleveraging firms. Lagos et al. (2011) study the relation between asset sales and price declines in over-the-counter markets using a dealership model in which search frictions prohibit outside investors to trade continuously with dealers.

Our paper is also related to the literature studying the asset pricing implications of forced sales by leveraged financial institutions (e.g., banks), which need to comply with prescribed balance sheet requirements (e.g., Adrian and Shin 2010). The typical mechanism works as follows. After an initial market shock, leverage ratios may deviate from their targets, prompting the institutions to sell illiquid assets to return to their targets. The aggregate impact of asset liquidation on prices is linear in the size of the exogenous market shock (see Capponi and Larsson 2015, Greenwood et al. 2015, Braouezec and Wagalath 2018, and Duarte and Eisenbach 2018).

The mechanism of fire sales triggered by redemptions of mutual funds, however, is different due to their distinct institutional structure: because of the first-mover advantage, the value of a fund share and the price of an asset share depend *nonlinearly* on the initial market shock. A larger shock creates a stronger incentive to redeem early, forcing the fund to liquidate superlinearly with respect to the size of the shock. Our model shows that only in an idealized setting without first movers (or, equivalently, with an appropriate swing price) is the impact of redemptions on prices linear. These findings imply that treating the mutual fund structure like that of a bank and ignoring institutional features of the first-mover advantage

would underestimate the effects.[5] The asset pricing implications of investor redemptions may be significant, especially in periods of market stress or if the fund is managing illiquid assets, such as high-yield or emerging market corporate debt.

We build an analytically tractable model that mimics the redemption mechanism identified by the empirical literature on mutual fund flows and use it to explain the effects of the liquidity mismatch in open-end mutual funds. Our model features a continuum of investors who are sensitive to the fund's performance: a decrease in the fund's NAV leads to an increasing fraction of investors exiting the fund, consistent with the empirical studies of Chen et al. (2010) and Goldstein et al. (2017). The first movers anticipate this drop in the NAV and sell before it materializes, thus imposing an even larger externality on the fund.

We show that, if the initial market shock exceeds a certain critical threshold, the incentive to sell early and the number of early redemptions become so large that the fund is unable to repay investors at the nominal NAV because of the significant price drop of its asset shares. For brevity, we refer to this outcome as a fund *failure*.[6] Our model can, thus, be adopted as a reverse stress-testing tool: after calibrating it to fund flow data via econometric specifications proposed in the empirical literature (e.g., Ellul et al. 2011 and Goldstein et al. 2017), we can find the critical shock size that triggers the fund failure for each given level of asset illiquidity. Notably, our model can be used to design stress-testing scenarios that explicitly incorporate the risk of spiraling redemptions. This, in turn, enables the positive analysis of regulatory measures targeting fund stability and liquidity management, such as minimum cash requirements and adoption of swing pricing.

We propose a formal definition of swing pricing that captures the adjustment to the end-of-day NAV required to remove the first-mover advantage. Although stylized, our definition embodies the salient features of the amended SEC 22c-1 rule. We show that, to eliminate the first-mover advantage, the swing price should be linear in the size of redemptions with a slope determined by the illiquidity of the asset. This linear specification makes swing pricing effective even under scenarios of extreme market stress. In fact, swing pricing turns the one-sided first-mover advantage into a trade-off: by redeeming early, investors avoid the costs imposed by their redemptions on the fund's future NAV; on the other hand, a crowding of redemptions results in a larger swing price for redeemers. The major benefit of swing pricing stems from the reduction in the magnitude of early redemptions: with the first-mover advantage removed, a smaller number of investors exit the fund, and the fund is required to sell fewer of its assets at a discount. Swing pricing results not only in a transfer of the liquidation cost,

but also—and more importantly—in a reduction of this cost. It removes the incentive to exit early that can lead to fund failure.

Many European mutual funds adopt a flat swing price when redemptions hit a certain threshold (Lewrick and Schanz 2017a). The empirical results in Lewrick and Schanz (2017a) show that such a swing price is effective in normal times. However, in periods of heavy outflows, such as during the 2013 U.S. "taper tantrum," funds appear not to have benefited from the adoption of the swing-price rule. These empirical observations are consistent with our theoretical predictions: to be effective in periods of intense market stress, the swing price should be strictly increasing in the amount of redemptions. The empirical studies by Chernenko and Sunderam (2016) and Jiang et al. (2017) discuss more traditional liquidity management policies followed by mutual funds, such as cash buffering and cost-effective liquidation strategies.

Our study sheds some light on how open-end mutual funds may pose a threat to financial stability. As argued by Feroli et al. (2014), the absence of leverage is not enough to dismiss potential financial risks: in a downturn scenario, intermediaries that exhibit a procyclical behavior exert an additional adverse pressure on the market. Empirical evidence (Chen et al. 2010) indicates that, when returns are negative, mutual funds tend to liquidate assets, thus magnifying market shocks as opposed to absorbing them. Portfolio commonality exposes funds to similar market risks, and hence, large capital outflows often occur simultaneously at several funds. This exacerbates the impact of redemptions on the fund and asset performance (Coval and Stafford 2007, Koch et al. 2016). In some less liquid markets in which the presence of the mutual fund industry is prominent (for instance, U.S. corporate bonds), financial distress can escalate and lead to market turbulence with negative consequences for the real economy. Feroli et al. (2014) discuss a model in which funds' sales are triggered by relative performance concerns. Our study instead analyzes the fire-sale amplifications driven by the first-mover advantage.

In an extension of our baseline model to multiple funds, we show how portfolio commonality and simultaneous redemptions generate cross-fund externalities and exacerbate the price pressure from mutual funds' asset sales. A fund's swing-pricing rule should, therefore, not only account for the externalities imposed on the fund by its own redeeming investors, but also for those imposed by redeeming investors of other funds. Interestingly, we show that, if a fund sets a swing price that accounts for the externalities imposed by all funds' first movers and all other funds do the same, then the required NAV adjustment would be smaller than it would be if other funds did not adopt swing pricing even if the swing price set by the fund

were to account only for the externalities imposed by its own first movers. The intuition underlying this phenomenon is that no amplification resulting from first movers' redemptions occurs when all funds apply swing pricing. If some of these funds were not to apply swing pricing, then their first movers' redemptions would amplify the pressure on prices imposed by first movers of other funds that did apply swing pricing, hence requiring a larger adjustment to the end-of-day NAV.

The rest of the paper is organized as follows. We present the model in Section 2 and solve it in Section 3. Section 4 introduces swing pricing and analyzes its preventive role against fund failure. We study how first-mover advantage gets amplified in the presence of multiple funds in Section 5. Section 6 discusses operational challenges related to the implementation of a swing pricing rule. Section 7 concludes the paper. Additional discussions and proofs of technical results are deferred to the appendix.

2. The Model

An open-end mutual fund holds Q_0 units of an asset; each unit of the asset can be thought of as a unit of the portfolio managed by the fund. The market price at time 0 of an asset share is P_0. Investors hold N_0 mutual-fund shares, which they may redeem (sell back to the fund) at any time. If the fund holds C_0 in cash, then dividing the fund's total assets by the number of fund shares yields the fund's share price (the fund's net asset value or NAV) of $S_0 = (Q_0 \cdot P_0 + C_0)/N_0$. For simplicity, we focus on the case $C_0 = 0$ of a mutual fund with a zero cash buffer. The inclusion of a cash buffer does not alter our main findings and is studied in the online appendix. We also assume that, initially, the number of shares issued by the fund equals the number of asset shares, so $N_0 = Q_0$. This assumption does not qualitatively impact our conclusions but leads to simpler expressions.

The asset is illiquid in the sense that selling shares of the asset impacts its price by an amount

$$\Delta P = \varphi(\Delta Q),$$

where $\Delta Q < 0$ records the number of shares sold, ΔP is the resulting price change, and $\varphi(\cdot)$ is the price impact function, which is assumed to be increasing and continuous with $\varphi(0) = 0$. In most examples we assume linear price impact, $\varphi(\Delta Q) = \gamma \Delta Q$, where γ is a measure of the asset's illiquidity; when $\gamma = 0$, the asset is perfectly liquid.[7]

Investors redeem fund shares in response to bad short-term performance of the fund: the number R of redeemed fund shares is assumed to be proportional to the drop $\Delta S < 0$ in value of a fund share,

$$R = -\beta \Delta S, \tag{1}$$

where β represents the sensitivity of investors to bad performance.[8] We focus on negative market shocks and take $\beta > 0$.[9]

At the end of each business day, the fund calculates the value of a fund share (its NAV) based on the current market prices of the fund's assets. Investors who redeem shares of the fund receive a payoff per share equal to the first NAV calculated after their redemption order is submitted.[10] If the fund is unable to sell shares of its asset to repay redeeming investors before the NAV is set, the cost of asset liquidation is borne by the fund and not by the redeeming investors. This creates a liquidity mismatch between the liquidity of the asset held by the fund and the liquidity provided by the fund to its investors. Moreover, an investor who promptly redeems a share of the fund receives a payoff that is higher than the payoff obtained by an identical investor who directly holds shares of the asset and sells them because the direct investor bears the cost of the price impact. We refer to this feature of the mutual fund structure as the first-mover advantage.

Our model has two types of redeeming investors: first movers and second movers. Both types redeem in response to declines in the fund's NAV, as in (1), but first movers anticipate and react to the final change in the fund's NAV, and second movers react only to the observed change in NAV. In this sense, first movers are forward-looking: they observe a negative shock to the market, anticipate the response of other investors and the impact of their redemptions on the fund's NAV, and immediately react by redeeming shares—selling a greater quantity than they would sell if they held the asset directly.[11] First movers are fast and redeem fund shares before the fund starts to sell asset shares and, thus, before this selling affects the NAV. Second movers redeem more gradually as the NAV falls, so the fund can anticipate their actions and liquidate assets simultaneously with their redemptions. The behavior of first movers is more typical of institutional investors, and the behavior of second movers is closer to that of retail investors. But our categorization of first and second movers pertains only to the behavior of the investors and not to their identity. We assume a continuum of investors and use $\pi \in [0,1]$ to denote the proportion of first movers and $1 - \pi$ for the proportion of second movers.[12]

We illustrate the timeline of the model in Figure 1. Initially, a negative market shock ΔZ decreases the price of the fund's asset, and this translates into a shock $\Delta S_0 = \frac{Q_0}{N_0}\Delta Z = \Delta Z$ to the value of a fund share. First movers redeem immediately after[13] the shock at an NAV of $S_0 + \Delta S_0$. The fund liquidates shares of the asset to repay first movers, driving down the fund's NAV. Second movers respond to the observed change in the fund's NAV, setting off a cycle of further redemptions and price drops.

2.1. Investor Behavior and Model Primitives

To formulate the model precisely, we need to specify the actions of first movers, second movers, the fund, and the market. We begin with second movers.

2.1.1. Second Movers.
When second movers observe a realized decline $\Delta S < 0$ in the fund's NAV, they redeem fund shares according to the performance sensitivity rule in (1). They make up a fraction $1 - \pi$ of investors, so the total number of fund shares they redeem is given by

$$\Delta R^{sm} = -(1 - \pi)\beta\Delta S. \tag{2}$$

2.1.2. The Mutual Fund.
To pay redeeming investors, the fund sells asset shares. If the fund needs to pay $S\Delta R$ to redeeming investors and if it can sell asset shares at a price P, then it needs to trade $\Delta Q < 0$ shares with $-P\Delta Q = S\Delta R$, so

$$\Delta Q = -\Delta R \frac{S}{P}. \tag{3}$$

2.1.3. The Market.
The asset price P in (3) reflects the impact of the ΔQ shares traded by the fund. If $P_0 + \Delta Z$ denotes the asset price before the fund sells, then

$$P = P_0 + \Delta Z + \varphi(\Delta Q). \tag{4}$$

2.1.4. First Movers.
The behavior of first movers is the most distinctive feature of the model. Like second movers, they exhibit the performance sensitivity in (1), but they respond to the anticipated rather than the realized change in NAV and redeem

$$R^{fm} = -\pi\beta\Delta S_{tot}. \tag{5}$$

Here, $\Delta S_{tot} = \Delta S_{tot}^{fm} + \Delta S_{tot}^{sm}$ is the decline in the fund's NAV, combining the decline ΔS_{tot}^{fm} following first movers' redemptions and the further decline ΔS_{tot}^{sm} following the second movers' redemptions. The initial shock ΔS_0 is included in ΔS_{tot}^{fm}.

2.1.5. Solution Concept.
A solution to our model is defined by a set of asset prices, fund share prices, quantities sold by the mutual fund, and quantities redeemed by first and second movers consistent with (2)–(5). In these dynamics, second movers, the mutual fund, and the market all follow fixed rules; only the first movers act strategically; that is, their redemption strategy is a fixed point of a feedback system. The asset price is subject to an exogenous initial market shock ΔZ. The key endogenous quantity is ΔS_{tot}, the decline in the fund's NAV resulting from the exogenous shock ΔZ and the impact of investors' redemptions, which immediately determines the quantity of first movers' redemption by (5); all other quantities follow directly once ΔS_{tot} is known. We need to show that

Figure 1. Timeline of the Model
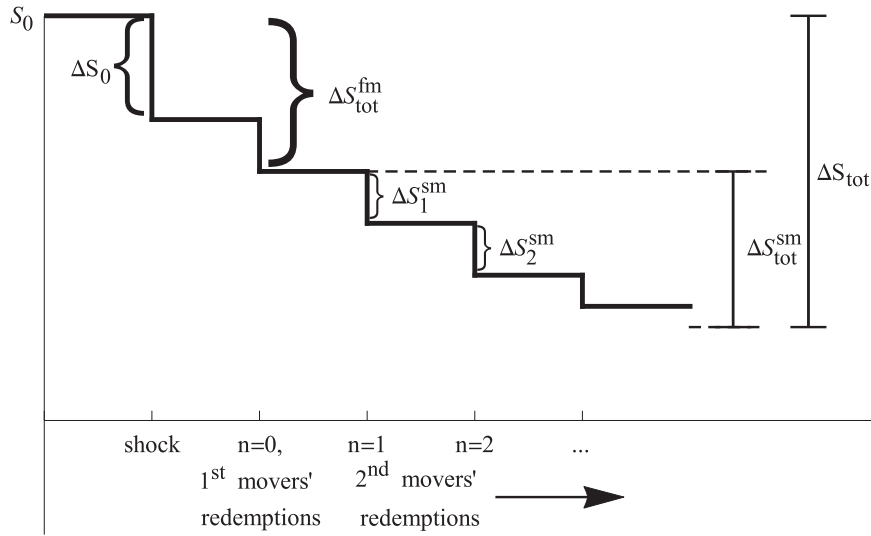
Notes. The initial market shock is $\Delta S_0 = \Delta Z$. The total drop in NAV following first-mover redemptions is ΔS_{tot}^{fm}; this drop triggers an initial round of second-mover redemptions, and each such round leads to further redemptions with further NAV drops ΔS_n^{sm}.

ΔS_{tot} and the rule (5) we have posited for first movers are well defined. We do this by describing the consequences of (2)–(4) for second movers' redemptions and then working backward to derive the first movers' redemptions. The validity of (5) then depends on a fixed-point argument.

2.2. Second Movers' Redemptions

The second movers' redemptions in Figure 1 are triggered by an observed change in the fund's NAV, which we denote by ΔS_0^{sm}. Later, we specify that $\Delta S_0^{sm} = \Delta S_{tot}^{fm}$, the combined effect of the initial market shock and first movers' redemptions, but for now, $\Delta S_0^{sm} < 0$ is arbitrary. Let R_0^{sm}, S_0^{sm}, P_0^{sm}, and Q_0^{sm} denote, respectively, the number of fund shares redeemed, the fund NAV, the asset price, and the number of asset shares held by the fund before any second movers have redeemed but after the first movers have redeemed.

Second movers redeem fund shares according to (2), leading the fund to sell assets according to (3), driving down the asset price according to (4). The decline in the asset price lowers the fund's NAV, leading to a further round of second movers' redemptions as illustrated in Figure 1. In the nth round, second movers observe a change ΔS_n^{sm} in the value of a fund share and redeem an additional amount

$$\Delta R_{n+1}^{sm} = -(1 - \pi)\beta\Delta S_n^{sm} \tag{6}$$

of fund shares. When they redeem fund shares, they are paid at a price of $S_{n+1}^{sm} = S_n^{sm} + \Delta S_{n+1}^{sm}$, where ΔS_{n+1}^{sm} is the drop in the NAV created by their redemptions. In other words, second movers incur the liquidation costs of their redemptions; this is the key property of

second movers. In response to the redemptions (6), the fund, therefore, raises cash by trading a quantity

$$\Delta Q_{n+1}^{sm} = -\Delta R_{n+1}^{sm} \frac{S_{n+1}^{sm}}{P_{n+1}^{sm}} \tag{7}$$

of the asset, where the price P_{n+1}^{sm} depends on the quantity the fund sells. To get the fund's NAV, we divide the value of the fund's assets by the number of fund shares and get

$$S_{n+1}^{sm} = \frac{Q_{n+1}^{sm} P_{n+1}^{sm}}{N_0 - R_{n+1}^{sm}}, \tag{8}$$

where $R_{n+1}^{sm} = R_n^{sm} + \Delta R_{n+1}^{sm}$ is the cumulative number of fund shares redeemed and $Q_{n+1}^{sm} = Q_n^{sm} + \Delta Q_{n+1}^{sm}$ is the quantity of the asset held by the fund. The asset price is given by

$$P_{n+1}^{sm} = P_0 + \Delta Z + \varphi(Q_{n+1}^{sm} - Q_0). \tag{9}$$

Equations (6)–(9) define the actions of second movers, the fund, and the market. We can combine these equations into a mapping

$$\begin{aligned}
&\left(R_{n+1}^{sm}, Q_{n+1}^{sm}, S_{n+1}^{sm}, P_{n+1}^{sm}\right) \\
&\quad = \Phi\left(R_n^{sm}, Q_n^{sm}, S_n^{sm}, P_n^{sm}; \Delta S_n^{sm}\right).
\end{aligned} \tag{10}$$

The mapping Φ is well defined: by substituting (8) in (7), one can evaluate Q_{n+1}^{sm}, then P_{n+1}^{sm}, and then S_{n+1}^{sm}.[14]

Suppose that starting from $(R_0^{sm}, Q_0^{sm}, S_0^{sm}, P_0^{sm}; \Delta S_0^{sm})$ and proceeding through iterative application of Φ, the fund's NAV S_n^{sm} converges to a limit (as we confirm later)

$$S_n^{sm} \to S_{tot} \equiv S_{tot}\left(R_0^{sm}, Q_0^{sm}, S_0^{sm}, P_0^{sm}; \Delta S_0^{sm}\right); \tag{11}$$

the limit S_{tot}, if it exists, depends on the initial values of the variables in the recursion. Let $\Delta S_{tot}^{sm} = S_{tot} - S_0^{sm}$ denote the total NAV impact of second movers' redemptions.

2.3. First Movers' Redemptions

First movers' redemptions are characterized by the equations

$$R_{tot}^{fm} = -\pi\beta\Delta S_{tot}, \tag{12}$$

$$S_{tot}^{fm} = \frac{Q_{tot}^{fm}P_{tot}^{fm}}{N_0 - R_{tot}^{fm}}, \tag{13}$$

$$Q_{tot}^{fm} = Q_0 - R_{tot}^{fm}\frac{S_0 + \Delta S_0}{P_{tot}^{fm}}, \tag{14}$$

$$P_{tot}^{fm} = P_0 + \Delta Z + \varphi\left(Q_{tot}^{fm} - Q_0\right), \tag{15}$$

with (as in Figure 1)

$$\Delta S_{tot} = \Delta S_{tot}^{fm} + \Delta S_{tot}^{sm} = \left(S_{tot}^{fm} - S_0\right) + \Delta S_{tot}^{sm}. \tag{16}$$

The variables on the left side of (12)–(15) measure quantities after first movers redeem and before second movers redeem. These equations differ from the second-mover equations in two important ways. Redemptions in (12) are driven by the anticipated NAV decline ΔS_{tot} and not by an observed decline as in (6). Moreover, first movers redeem at the first day's NAV of $S_0 + \Delta S_0$ regardless of how many shares they redeem; they do not bear the cost of their liquidation, which the fund incurs in subsequent days. Therefore, in (14), the fund uses $S_0 + \Delta S_0$ in determining how much cash it needs to raise, whereas the NAV used in (7) reflects the fact that second movers do bear their liquidation costs. To link first and second movers, we set

$$R_0^{sm} = R_{tot}^{fm}, \quad Q_0^{sm} = Q_{tot}^{fm}, \quad S_0^{sm} = S_{tot}^{fm}, \quad P_0^{sm} = P_{tot}^{fm}, \tag{17}$$

so the second movers begin their redemptions by observing the consequences of the first movers' redemptions. The cumulative NAV impact ΔS_{tot}^{sm} of the second movers depends on these initial conditions as emphasized in (11). By combining (11) and (17), we can write

$$\Delta S_{tot}^{sm} = \Delta S_{tot}^{sm}\left(R_{tot}^{fm}, Q_{tot}^{fm}, S_{tot}^{fm}, P_{tot}^{fm}; \Delta S_{tot}^{fm}\right), \tag{18}$$

so the total NAV impact of second movers depends on all quantifies affected by the first movers' redemptions. But the redemptions of first movers depend on what they anticipate the impact of the second movers to be through (12) and (16). The existence of a solution to (12)–(16) is, thus, a fixed-point problem. Solving the model means finding a solution or showing that none exists.

To provide further insight into the existence of a solution, we can write (14) as

$$\left(Q_{tot}^{fm} - Q_0\right)P_{tot}^{fm} = -R_{tot}^{fm}(S_0 + \Delta S_0). \tag{19}$$

The expression on the right is the cash the fund needs to meet first movers' redemptions, and the expression on the left is the cash the fund raises by selling assets. For a small shock $\Delta S_0 = \Delta Z \approx 0$, the equations always have a solution, and the fund can pay redeeming investors. But the cash the fund can raise is bounded, so for a sufficiently large shock, the equations have no solution, and the fund cannot meet its obligations. In other words, we show that the existence of a solution to (12)–(16) depends on the magnitude of the initial shock, and the failure of the equations to have a solution should be interpreted as the failure of the fund to meet redemptions.

Solving the equations implicitly requires that the total second-mover impact in (11) is well defined when the initial conditions for second movers are determined by the actions of the first movers through (17) because ΔS_{tot} in (12) and (16) simplifies to $S_{tot} - S_0$; we address this point as well. Given a solution, we are particularly interested in $\Delta P_{tot} = \lim_{n\to\infty}(P_n^{sm} - P_0)$, the cumulative impact on the market price of the asset resulting from the combined effects of the market shock and all redemptions.

2.4. Direct Ownership Benchmark

To isolate the effect of the mutual fund structure in amplifying a fire sale, we compare the results of our model against a benchmark in which investors hold the asset directly rather than through a mutual fund. To make the comparison consistent, we assume the same price impact function φ for investors selling the asset as we assumed for the fund. We translate the assumed redemption mechanism $R = -\beta\Delta S$ for a fund investor to the relation $\Delta Q = \beta\Delta P$ for a direct investor. In either case, the sale may be triggered by an investor's financial constraints. For example, if an investor has purchased fund shares on margin, then a substantial drop in the fund's share price could force the investor to sell; if the investor bought shares of the asset itself on margin, a drop in the asset price would again force its sale.

In the case of direct ownership, an initial asset price shock $\Delta P_0 = \Delta Z$ triggers a cycle of forced sales described by the following equations:

$$Q_n = Q_{n-1} + \beta\Delta P_{n-1}, \qquad P_n = P_0 + \Delta Z + \varphi(Q_n - Q_0), \tag{20}$$

with $\Delta P_{n-1} = P_{n-1} - P_{n-2}$. The cumulative change in the asset price is $\Delta P_{tot}^{dir} := \lim_{n\to\infty}(P_n - P_0)$. The institutional structure of the fund amplifies the fire sale if the total price change $|\Delta P_{tot}|$ when the asset is held

through the fund exceeds the price change $|\Delta P_{tot}^{dir}|$ when the asset is held directly.

3. Model Solution

3.1. Convergence of Second Movers' Redemptions

The following lemma provides a sufficient condition for the convergence of the second movers' redemption procedure. Recall that Q_0^{sm} and R_0^{sm} refer to initial conditions observed by second movers before any second-mover redemption.

Lemma 1. *Assume that*

$$(1 - \pi)\beta\left(Q_0^{sm}/(N_0 - R_0^{sm})\right)^2 \lim_{x \to -\infty} \varphi'(x) < 1.$$

Then the sequence defined in (10) *converges.*

The condition required in this lemma is automatically satisfied for a price impact function of the form $\varphi(x) = -\gamma(-x)^\alpha$ on $(-\infty, 0]$ for $\alpha < 1$. Several empirical studies have concluded that price impact is well described by a power function of the traded volume with an exponent smaller than one (e.g., Lillo et al. 2003). If $\varphi(x) = \gamma x$, the condition becomes $(1 - \pi)\beta(Q_0^{sm}/(N_0 - R_0^{sm}))^2\gamma < 1$. According to Lemma B.2, this is equivalent to assuming that second movers' redemptions, as described in Section 2.2, decrease in each round. If this condition does not hold, even a minimal drop in the value of a fund share would trigger a spiral of redemptions and impose a downward price pressure that causes the failure of the fund. Hence, parameters violating this condition are not economically plausible.

3.2. Without First Movers

Next, we show that, in the absence of first movers, the cumulative price impact is the same whether investors hold the asset through a mutual fund or directly. In other words, without first movers, there is no further fire-sale amplification.

To see this, notice that, if $\pi = 0$, then $\Delta S_0^{sm} = \Delta S_0 = \Delta Z$. It follows from Lemma B.1 that

$$S_{n+1}^{sm} - S_0 = P_{n+1}^{sm} - P_0 = \Delta Z + \varphi\left(\beta\left(P_n^{sm} - P_0\right)\right),$$
$$n = 1, 2, \ldots.$$

Similarly, the price evolution under direct ownership resulting from (20) can be summarized as $P_{n+1} - P_0 = \Delta Z + \varphi(\beta(P_n - P_0))$ with $\Delta P_0 = \Delta Z$. In other words, the price impact under direct ownership and ownership through the mutual fund are identical in the absence of first movers.

Using the convergence of the sequence guaranteed by Lemma 1 and the preceding price evolution equation, we obtain a more explicit characterization of the cumulative price decline.

Proposition 1. *Suppose $\pi = 0$ and $\beta \lim_{x \to -\infty} \varphi'(x) < 1$. For any initial market shock $\Delta Z < 0$, the cumulative price declines $\Delta P_{tot}^{sm} = \lim_{n \to \infty} P_n^{sm} - P_0$ and $\Delta S_{tot}^{sm} = \lim_{n \to \infty} S_n^{sm} - S_0$ are well defined with $\Delta P_{tot}^{sm} = \Delta S_{tot}^{sm}$ satisfying*

$$\Delta P_{tot}^{sm} = \Delta Z + \varphi\left(\beta \Delta P_{tot}^{sm}\right).$$

The cumulative price decline under direct ownership $\Delta P_{tot}^{dir} = \lim_{n \to \infty} \Delta P_n$ is also well defined and equals ΔP_{tot}^{sm}. In particular, if $\varphi(x) = \gamma x$, then $\Delta P_{tot}^{dir} = \frac{\Delta Z}{1 - \gamma\beta}$. If $\varphi(x) = -\gamma(-x)^{1/2}$ for $x < 0$, then $\Delta P_{tot}^{dir} = \Delta Z - \frac{1}{2}(\gamma^2\beta + \gamma\sqrt{\gamma^2\beta^2 - 4\beta\Delta Z})$.

Because of the price impact generated by selling shares of the asset, second movers who redeem in round n impose an externality on second movers who redeem later. However, Proposition 1 shows that this effect is not a consequence of the mutual fund structure, but purely a result of the asset's illiquidity; the fire-sale impact is just as large under direct ownership.

Proposition 1 also shows that, in the case $\varphi(x) = \gamma x$ of linear price impact, the cumulative price change grows linearly with the exogenous market shock ΔZ and increases with both the illiquidity of the asset γ and the sensitivity to the fund's performance β. For small values of $\gamma > 0$, the change in value of a fund share ΔS_{tot}^{sm} caused by all second movers' redemptions admits the representation

$$\Delta S_{tot}^{sm} \approx \Delta Z + \gamma\beta\Delta Z + \gamma^2\beta^2\Delta Z + \cdots. \quad (21)$$

Each term of the sum reflects a new round of redemptions. Each round has an impact on the value of a fund share, and the final value is the cumulative effect of the redemption and liquidation process.

3.3. With First Movers

Once we reintroduce first movers, the cumulative price effect is no longer linear in ΔZ. We characterize the changes in asset price and value of a fund share in the presence of first movers for small γ in Proposition 2.

Proposition 2. *Assume $\pi > 0$ and $\varphi(x) = \gamma x$. For small γ, the changes in asset price and value of a fund share after redemptions by first and second movers are*

$$\Delta P_{tot} = \Delta Z + \gamma\beta\Delta Z$$
$$+ \gamma^2\left(\beta^2\Delta Z - \beta\frac{\pi^2\beta^2\Delta Z^2}{N_0 + \pi\beta\Delta Z}\right.$$
$$\left. - \frac{\pi^2\beta^2\Delta Z^2}{P_0 + \Delta Z}\frac{N_0 + \beta\Delta Z}{N_0 + \pi\beta\Delta Z}\right) + o(\gamma^2), \quad (22)$$

$$\Delta S_{tot} = \Delta Z + \gamma\left(\beta\Delta Z - \frac{\pi^2\beta^2\Delta Z^2}{N_0 + \pi\beta\Delta Z}\right) + o(\gamma). \quad (23)$$

To quantify the externality imposed by the first movers on the fund, compare the expression (23) to the expansion (21). As expected, the impact of the liquidation process on the value of a fund share is higher when some investors are first movers (recall that ΔZ is negative) because first movers do not internalize the

costs imposed by their redemptions.[15] As a consequence, a share of the fund is worth less than a share of the asset after first movers' redemptions.

The term $\gamma\pi^2\beta^2\Delta Z^2/(N_0 + \pi\beta\Delta Z)$ is, to first order in γ, the fraction of the liquidation cost resulting from first movers' redemptions that needs to be absorbed by each remaining investor in the fund. This term may be understood as follows. The numerator $\pi^2\beta^2\Delta Z^2$ captures the cost incurred by the fund when it liquidates shares to repay first movers. To first order, first movers redeem $R_{tot}^{fm} \approx \pi\beta\Delta Z$ of fund shares, and the fund trades $\Delta Q^{fm} \approx \pi\beta\Delta Z$ shares of the underlying asset to repay first movers. The price per share of the asset is $P^{fm} = P_0 + \Delta Z + \gamma\Delta Q^{fm}$; hence, the liquidation cost resulting from first movers is $\gamma\Delta Q^{fm} \times \Delta Q^{fm} \approx \gamma\pi^2\beta^2\Delta Z^2$. The cost is quadratic in quantities because price impact per share is linear in quantities. The denominator $N_0 + \pi\beta\Delta Z$ represents the amount of outstanding shares after redemptions by first movers.

Interestingly, the first-mover advantage not only reduces the value of a fund share, but also negatively affects the market price of the asset. However, the asset price in the presence of first movers differs from that in the absence of first movers only at the second order in γ (see Equation (22)). This is because the first-mover advantage affects the asset price only indirectly although it directly impacts the value of a fund share: as more investors exit the fund in response to the NAV drop, the fund needs to further liquidate asset shares, exacerbating price impact.

In more detail, two forces contribute to the cumulative price impact. The first is the higher flow of investors' redemptions: because the NAV drop ΔS_{tot} is greater with first movers, the resulting number of shares redeemed by investors is greater, triggering more asset sales by the fund and leading to a lower market price for the asset as captured by the term $\beta\pi^2\beta^2\Delta Z^2/(N_0 + \pi\beta\Delta Z)$. The second force is the increased amount of asset sales required to meet investors' redemptions: to repay first movers, the fund needs to liquidate an additional number $\gamma\pi^2\beta^2\Delta Z^2/(P_0 + \Delta Z)$ of asset shares on top of the number of redeemed shares R_{tot}^{fm} to cover the liquidation costs ($\Delta Q^{fm} \approx R_{tot}^{fm} + \gamma\pi^2\beta^2\Delta Z^2/(P_0 + \Delta Z)$). This yields a second order effect on market prices. A proportion of this cost is borne by second movers, so it is normalized by $(N_0 + \beta\Delta Z)/(N_0 + \pi\beta\Delta Z)$, which is the number of shares held by the remaining investors in the fund over the number of shares held by remaining investors and second-mover redeemers.

3.4. Redemption Outflows vs. Bank Deleveraging
Prior work has analyzed price linkages arising when financial institutions manage their balance sheets to comply with prescribed leverage requirements.

Greenwood et al. (2015) show that the amplification effects on prices arising when banks liquidate assets to target their leverage are linear in the exogenous shock if one takes into account only the first round of deleveraging. Capponi and Larsson (2015) confirm this linear dependence even if one accounts for higher order effects caused by repeated rounds of deleveraging needed to restore banks' leverage levels to their targets. The bank deleveraging mechanism in these models is essentially equivalent to the redemption mechanism in a mutual fund without first movers: each round of deleveraging has an impact on the price and leads to a successive round of asset liquidation because it depresses prices. In the absence of first movers ($\pi = 0$), if price impact is linear, the cumulative impact of redemptions on prices is still linear (see Proposition 1). The iterative redemption procedure executed by second movers converges to a fixed point if $\gamma\beta < 1$.[16]

The presence of first movers introduces an important structural difference between the fire-sale mechanism imposed by leverage targeting and that triggered by mutual fund redemptions. After accounting for the first-mover advantage, Proposition 2 shows that the cumulative impact of liquidation on asset prices is no longer linear in the exogenous shock. This point is illustrated in Figure 2, which compares the total price drop $-\Delta P_{tot}$ resulting from a shock ΔZ with (solid) and without (dashed) first movers for two different liquidity regimes.[17] Additionally, in the presence of first movers, the condition for the existence of a solution ΔS_{tot} to the model, that is, a fixed point for (18), takes a more complex form and depends crucially on the size of the initial shock (see Proposition 3).

Recent work on leverage-constrained banks includes cases in which the dependence of the cumulative price impact on the initial shock is nonlinear (Cont and Schaanning 2017, Duarte and Eisenbach 2018). But the nonlinearity in these models stems from nonlinear assumptions on the delevering strategy of the banks. In our model, nonlinearity is a consequence of the institutional structure of mutual funds.

4. Fund Failure, Swing Pricing, and Stress Testing
The incentive to redeem early increases with the illiquidity of the asset held by the fund. We show that the first-mover advantage may induce enough early redemptions and asset fire sales to bring down the fund if the fund's asset is sufficiently illiquid. Swing pricing is intended to stop the transfer of liquidation costs from first movers to investors remaining in the fund. In this section, we provide a formal definition of the swing price that achieves this objective, and we show that the same adjustment eliminates the fire-sale amplification created by the mutual fund structure.

Figure 2. (Color online) Aggregate Impact on Market Prices of an Initial Market Shock
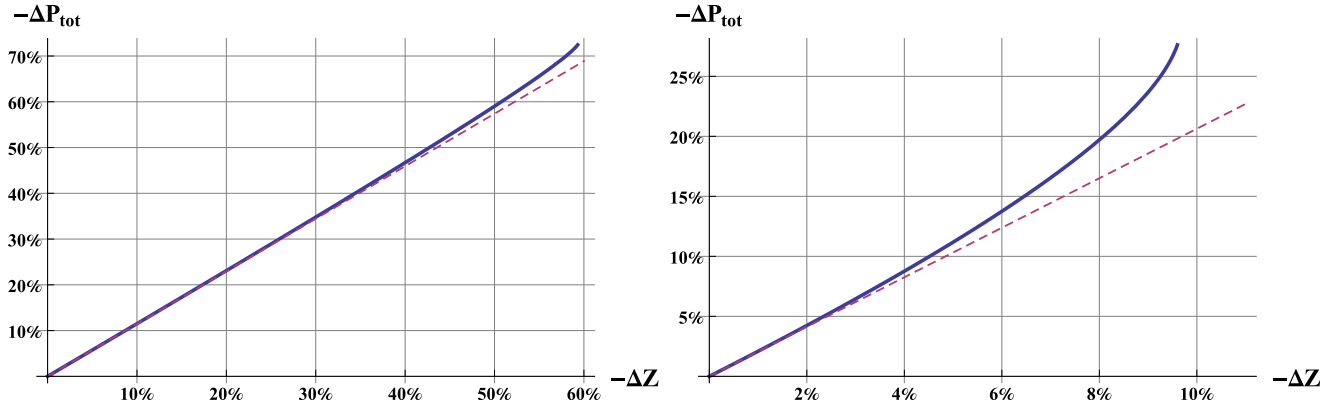
Notes. The graphs show the aggregate impact on market prices of an initial market shock, for $\pi = 75\%$ (solid line) and $\pi = 0$ (dashed line). The asset illiquidity parameter γ is 0.5×10^{-8} (left panel) and 2×10^{-8} (right panel). These values of γ correspond to typical liquidity levels for corporate bonds as reported in the empirical study by Ellul et al. (2011). The flow–performance relation $\frac{\beta}{N_0}$ is chosen to be 0.859, consistently with the estimates in Goldstein et al. (2017). In the presence of first movers, the impact on prices grows superlinearly with the size of the shock if the asset is illiquid.

We use these results to construct a stress-testing scenario to address the potential failure of a fund.

4.1. Redemption Flow and Fund Failure

Consider the case of linear price impact, that is, $\varphi(x) = \gamma x$. In the absence of first movers, the redemption procedure converges if $\gamma \beta < 1$ because this condition ensures that the NAV impacts of subsequent rounds of redemptions decrease geometrically; see (21). If first movers are present, the model may not have a solution even if $\gamma \beta < 1$, and the existence of a solution strongly depends on the size ΔZ of the initial shock. The cumulative price impact fails to converge if a shock of large size forces the failure of the fund.[18]

Proposition 3. *Assume $\pi > 0$ and $\varphi(x) = \gamma x$. There exists a critical value $\Delta Z^* < 0$ such that the fixed point equation (18) has a solution ΔS_{tot} following a price shock $\Delta Z < 0$ if and only if $|\Delta Z| \leq |\Delta Z^*|$. Furthermore, $|\Delta Z^*|$ decreases with the illiquidity parameter γ of the asset.*

If $\pi > 0$ and the exogenous shock ΔZ is sufficiently large, the number of investors that redeem early is so high that the fund becomes unable to repay them. This can be understood as follows. With each additional fund share redeemed by first movers, the marginal cost of liquidation increases, and the NAV that needs to be paid to first movers stays constant. The fund may eventually run short of asset shares or obtain negligible marginal revenue from asset sales.

The graph in the right panel in Figure 3 plots the relation between the critical value ΔZ^* and the asset illiquidity parameter γ. We set $\frac{\beta}{N_0} = 0.859$ in all numerical examples for consistency with the empirical estimates in Goldstein et al. (2017). If the asset is perfectly liquid ($\gamma = 0$), there is no first-mover advantage.

As γ increases, the critical threshold for the shock size that leads to spiraling redemptions and, consequently, to the fund's failure is smaller (in absolute value). Figure 4 illustrates how the fund may become unable to raise enough cash to repay its first movers if the asset is not sufficiently liquid. For any candidate fixed point ΔS, we can compute the cash raised by the fund and the cash owed to its redeeming investors. If γ is large, the cash raised by the fund, as a function of ΔS, does not grow fast enough to ever meet the level of cash owed to first movers.[19]

Despite the temporary nature of price impact, the fund may still be unable to meet investors' redemptions before prices revert to fundamentals. Even if the fund survives the wave of redemptions, its NAV never recovers completely. We refer to the online appendix for a detailed discussion of temporary and permanent NAV losses.

4.2. Swing Pricing

To formalize the notion of swing pricing, let ΔS^{sw} be an adjustment applied to the value of a fund share following a market shock of ΔZ, meaning that investors who redeem shares the day of the shock are paid at an adjusted price of $S_0 + \Delta Z + \Delta S^{sw}$ rather than $S_0 + \Delta Z$.

Definition 1. Let $\Delta S_{tot}^{\pi=0}$ be the cumulative change in value of a fund share in the absence of first movers (that is, with $\pi = 0$). For $\pi > 0$, suppose that, following a market shock of $\Delta Z < 0$, the fund adjusts its share price to $S_0 + \Delta Z + \Delta S^{sw}$. The adjustment ΔS^{sw} is a swing price if the resulting cumulative change in value of a fund share ΔS_{tot} is equal to $\Delta S_{tot}^{\pi=0}$.

Swing pricing is, thus, the adjustment to the value of a fund share that makes the first movers internalize all externalities imposed on the fund.

Figure 3. (Color online) First Movers' Redemptions and Critical Level of the Exogenous Shock
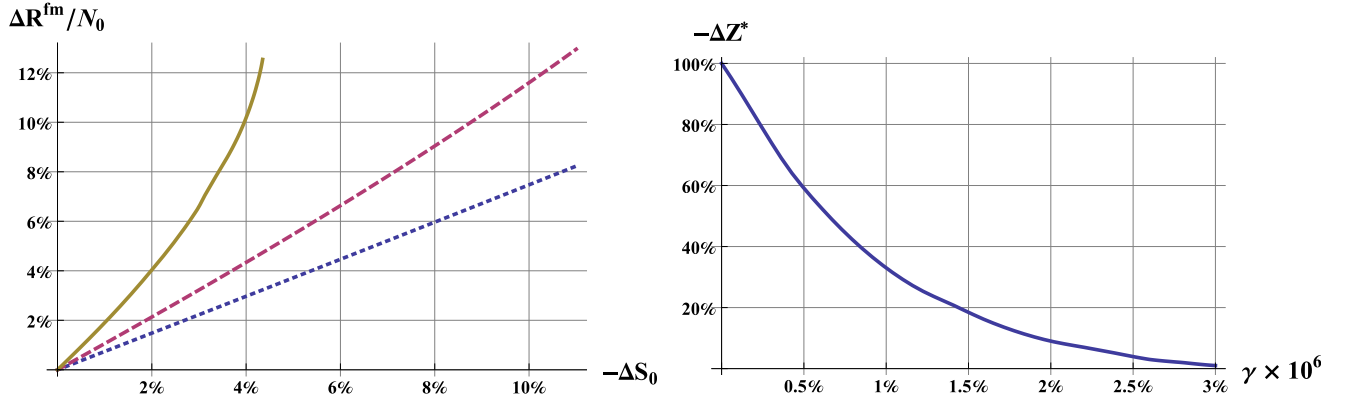
Notes. The graph in the left panel shows the outflow resulting from first movers in response to an exogenous shock on the fund's NAV. The flow-to-performance relation depends on the liquidity of the asset held by the fund: the asset illiquidity parameter γ is 0.5×10^{-8} (dotted line), 1.5×10^{-8} (dashed line), and 2.5×10^{-8} (solid line). The graph in the right panel shows the critical level ΔZ^* as a function of the illiquidity parameter γ. The horizontal axis reports the price impact per $1 million. The proportion π of first movers is 75%.

Proposition 4. *Assume* $\beta \lim_{x \to -\infty} \varphi'(x) < 1$. *The swing price, as specified in Definition 1, is uniquely given by*

$$\Delta S^{sw} = -\varphi(R_{tot}^{fm}). \quad (24)$$

In the special case of a linear price impact function, $\varphi(x) = \gamma x$, *it is given by*

$$\Delta S^{sw} = -\gamma R_{tot}^{fm} = \gamma \frac{\pi \beta \Delta Z}{1 - \beta \gamma}. \quad (25)$$

Let $\Delta P_{tot}^{\pi=0}$ *be the cumulative change in price of an asset share in the absence of first movers. In the presence of swing pricing,* $\Delta P_{tot} = \Delta P_{tot}^{\pi=0}$.
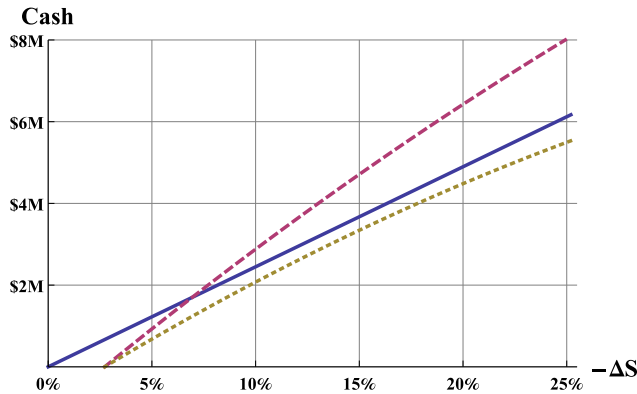
By Definition 1, swing pricing eliminates the first-mover advantage and results in the same impact on

Figure 4. (Color online) Cash Owed to First Movers and Cash Raised

Notes. For each value of ΔS, we compute R_{tot}^{fm} from Equation (12) and Q_{tot}^{fm} from Equation (13). The graph shows the cash owed to first movers, that is, the right-hand side of Equation (19), (solid line) and the cash raised by selling asset shares, that is, the left-hand side of Equation (19), for $\gamma = 1 \times 10^{-8}$ (dashed line), and for $\gamma = 2.5 \times 10^{-8}$ (dotted line). For $\gamma = 2.5 \times 10^{-8}$, the fund is unable to repay its redeeming investors for any value of ΔS.

the fund's NAV following the price shock ΔZ as would result if $\pi = 0$. As a consequence, it eliminates the transfer of liquidation costs from first movers to second movers. But swing pricing has a further systemic benefit. By reducing the NAV impact to what it would be if $\pi = 0$, swing pricing reduces the cumulative price impact ΔP_{tot} to what it would be without first movers. In addition to benefitting second movers, swing pricing eliminates the fire-sale amplification in the market price of the asset caused by the mutual fund structure.[20]

From (25), we see that the swing price adjustment is greater when $\beta \gamma$ is closer to one, which corresponds to greater amplification of the initial shock ΔZ. The numerator $\gamma \pi \beta \Delta Z$ represents the direct price impact that results from the first movers' response to the initial shock; dividing by $1 - \gamma \beta$ yields the amplification created through the response of second movers. The swing price in (25) is, thus, just enough to offset the initial reaction of first movers and stem their flight. If γ is large, the initial amount $\pi \beta \Delta Z$ of shares redeemed may account for only a small fraction of the total first-movers redemptions. In other words, the liquidation cost eliminated by swing pricing may be much larger than the cost that is merely transferred from one set of investors to another.

Equation (24) characterizes the swing price through the first-mover redemptions R_{tot}^{fm}, which depend on ΔZ through (12)–(16). Once a fund adopts ideal swing pricing, it eliminates first-mover redemptions, making R_{tot}^{fm} unobservable. Before implementing swing pricing, a fund could use past data to estimate the relationship between R_{tot}^{fm} and ΔZ and, thus, to estimate a nonparametric swing adjustment. In practice, it may be difficult to distinguish first movers' redemptions from other transactions, but the fund could compare an account's transactions following a market shock with past activity on the account to gauge

how much additional investor flow is driven by a price shock.[21] In the linear case (25), the fund can set the swing price based on the market shock ΔZ using estimates of π, β, and γ.

4.3. Swing Pricing Practices

Starting in November 2018, the amendments to Rule 22c-1 by the Securities and Exchange Commission allow U.S.-based mutual funds to adopt swing pricing. Swing pricing is already used in other jurisdictions, particularly Luxembourg. The vast majority of funds that use swing pricing adopt a rule defined by a redemption threshold and a predetermined swing factor: when net redemptions exceed the threshold, the fund applies a fixed percentage adjustment to its NAV. Such an adjustment differs from the swing-pricing formula in Proposition 4. Therefore, it does not remove the first-mover advantage and cannot guarantee prevention of fund failure; see Figure 5. According to the survey by the Association of the Luxembourg Fund Industry (2015), some asset managers already apply or are considering applying multiple swing factors, depending on the level of redemptions. Our study supports such an implementation. Our analysis also identifies two important features that yield an effective swing price:

1. The adjustment should take into account the dependence of the asset price on traded quantities. As the liquidation cost per traded share increases with the number of liquidated shares, the swing price should also increase with the flow of redemptions. A fixed swing price may have limited efficacy during periods of heavy outflows.

2. Investors should be informed about a fund's swing-pricing mechanism. Liquidation costs are

Figure 5. (Color online) NAV Change under Different Adjustment Rules

Notes. The graph shows the change in value of a fund share using the swing price specified in Proposition 4 (dotted line), without swing price (dashed line), and with a fixed NAV adjustment applied when more than 5% of investors exit the fund (solid line). The proportion π of first movers is 75%.

reduced and not just transferred only when investors understand that the first-mover advantage has been eliminated. In practice, it is unlikely that an investor in a fund with a broad investor base would be able to game a well-designed swing-pricing mechanism. But funds rarely disclose their swing thresholds, and they do not report the specific days on which the NAV was swung.

Some asset managers have expressed concerns that swing pricing may increase the volatility of a fund's NAV; see Securities and Exchange Commission (2016), section III-C. Our analysis shows that an effective swing price alleviates fire sales and prevents NAV dilution, hence mitigating large fluctuations in the fund's NAV, particularly in periods of market stress. For additional information on swing pricing practices, we refer to Investment Company Institute (2016) and Association of the Luxembourg Fund Industry (2015).
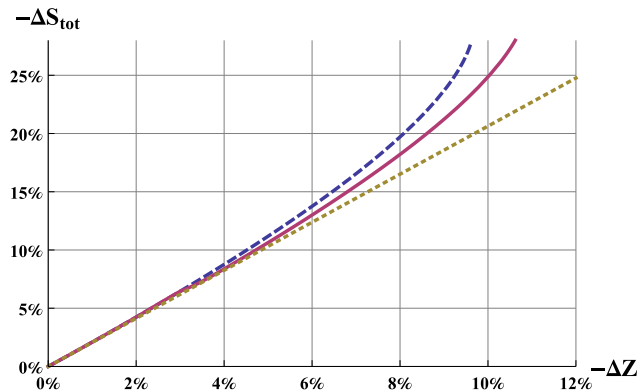
4.4. A Stress-Testing Example

We illustrate how a calibrated version of our model can be used for stress testing. We quantify the first-mover advantage for both high- and low-liquidity regimes, and we compute the threshold on the shock size beyond which redemptions would lead to fund failure.

We calibrate the model parameters using empirical estimates from the literature on fund flows and abnormal returns resulting from fire sales for corporate bond funds. We normalize the initial price of the asset and the value of a fund share to $1 so that $P_0 = S_0 = \$1$. Goldstein et al. (2017) estimate the flow–performance relation for corporate bond mutual funds: in the case of negative fund performance, the value of $\frac{\beta}{N_0}$ is approximately 0.859. This relation is asymmetric in the fund's performance: if the fund performance is positive, the corresponding value is 0.238.

To estimate the illiquidity parameter γ we follow Ellul et al. (2011), who analyze the impact of fire sales in the corporate bond market. To estimate deviations of prices from (unobservable) fundamentals, the authors analyze the temporary drop of bond prices after a downgrade and their rebound to fundamental value. The price impact per $1 million is on the order of 1% (ranging from 0.4% to 1.9% in different years and with different sets of controls). We consider two illiquidity regimes for the asset: a regime of typical liquidity with price impact of 1% per $1 million, and a regime of high illiquidity with price impact of 2.5% per $1 million. We assume that the fund holds $30 million in the asset and apply a market shock that reduces the current asset price by 5%, so $\frac{\Delta Z}{P_0} = -5\%$.

Table 1 decomposes the endogenous shock into contributions from the first, second, and third round of second movers' redemptions in the absence of first

Table 1. Endogenous Shock $\Delta S_{tot} - \Delta Z$ Caused by Fire Sales When There Are No First Movers and Contributions of Successive Rounds of Redemptions to This Shock

Price impact	Endogenous shock	First round	Second round	Third round
1×10^{-8}	−1.74%	−1.29%	−0.33%	−0.09%
2.5×10^{-8}	−9.05%	−3.22%	−2.08%	−1.34%

movers ($\pi = 0$). Recall that the second movers respond to and then contribute to a sequence of price declines. Their cumulative impact generates endogenous shocks of 1.74% and 9.05% for price impact parameters of 1×10^{-8} and 2.5×10^{-8}, respectively. Without first movers, the change in value of a fund share and of the asset are identical.

4.4.1. Impact of First Movers. Figure 6 highlights the additional impact on the value of a fund share triggered by first-mover redemptions with higher (left panel) and lower (right panel) liquidity. In both cases, the endogenous shock grows with the proportion of first movers π. In the right panel, the endogenous shock grows and eventually leads to fund failure because the critical threshold $|\Delta Z^*|$ (which depends on π) becomes smaller than the initial shock $|\Delta Z|$ for $\pi \geq 70\%$. Recall that, when price impact is linear, the cash raised from selling shares of the asset is a quadratic function of the number of shares sold. When the fund sells a large quantity of the asset and the price impact is high, the marginal revenue from the sale might become negative. The right panel of Figure 6 illustrates a situation in which the fund fails to raise sufficient cash to repay its first movers.

4.4.2. Swing Pricing Prevents Spiraling Redemptions. The adoption of swing pricing can substantially reduce the number of redemptions and, therefore, the total liquidation costs. To see this, compare Figures 6

and 7. If $\pi = 60\%$, we see from Figure 6 that the externalities imposed by first movers on the fund when $\gamma = 2.5 \times 10^{-8}$ are 50 times larger than when $\gamma = 1 \times 10^{-8}$. But the swing price in Figure 7 is only five times higher. When the asset is illiquid, the swing-price adjustment is small compared with the enormous costs of spiraling redemptions triggered by first movers.

5. Systemic Amplification of the First-Mover Advantage

We now extend our analysis to consider spillover effects across mutual funds holding the same asset. Early redemptions by first movers of one fund increase the incentive for investors in other funds to redeem early, further depressing the price of the asset. Section 5.1 studies swing pricing in an economy with multiple funds. Section 5.2 analyzes the benefits resulting from the simultaneous application of swing pricing by all funds. To highlight the main economic forces driving cross-fund externalities, we assume price impact to be linear, that is, $\varphi(x) = \gamma x$, throughout this section. We leave supporting details to Appendix A and present only the main results here.

5.1. First-Mover Advantage with Common Asset Ownership

We consider two funds that hold the same illiquid asset; the asset may be thought of as representative of their entire portfolios. Let β_1 and β_2 denote the

Figure 6. (Color online) Impact of First Movers' Redemptions on the Value of a Fund Share
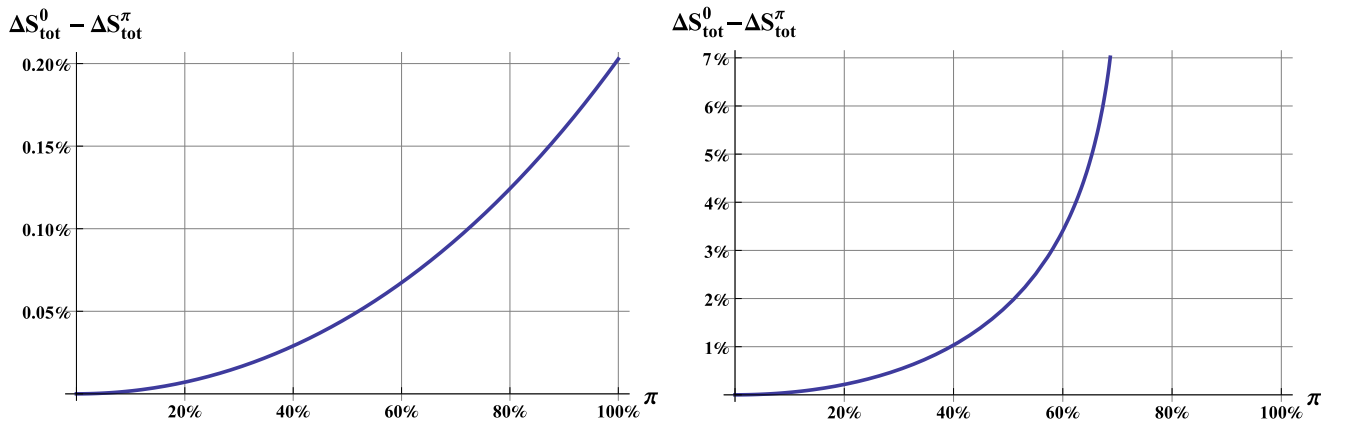
Notes. We set the price impact parameter $\gamma = 1 \times 10^{-8}$ (left panel) and $\gamma = 2.5 \times 10^{-8}$ (right panel). For the larger value of γ, the impact diverges if $\pi \geq 70\%$.

Figure 7. (Color online) Swing Price as a Function of π
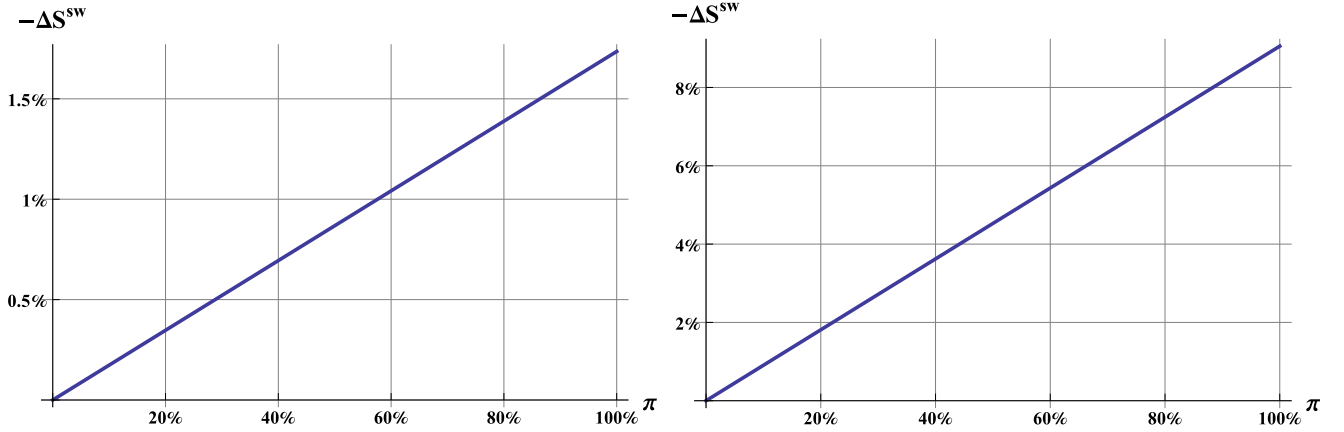
Note. We set the price impact parameter $\gamma = 1 \times 10^{-8}$ (left panel) and $\gamma = 2.5 \times 10^{-8}$ (right panel).

sensitivity to bad performance of investors in funds 1 and 2, respectively. We use π_1 and π_2 to denote the fractions of first movers in funds 1 and 2, respectively. Consistent with previous sections, we make the assumption that the initial number of asset shares equals the initial number of fund shares for each fund: $Q_{0,i} = N_{0,i}$ for $i = 1, 2$. The simplified setting of two funds with common asset ownership allows us to highlight the amplification channel of fire-sale externalities across funds.

For $i = 1, 2$, let $\Delta S_{tot,i}$ be the aggregate change in NAV of fund i caused by all redemptions of both funds 1 and 2. The redemptions by first movers of one fund exacerbate liquidation losses of the other fund, which simultaneously experiences redemptions of its own first movers in response to the same negative market shock of the asset. The total impact of these redemptions on the value of a share of fund 1 is (omitting terms of higher order in γ)

$$\Delta S_{tot,1} \approx \Delta Z + \underbrace{\gamma\left(\beta_1 \Delta Z - \frac{(\beta_1 \pi_1 \Delta Z)^2}{N_{0,1} + \beta_1 \pi_1 \Delta Z}\right)}_{\text{Own Impact}}$$
$$+ \underbrace{\gamma\left(\beta_2 \Delta Z - \frac{(\beta_2 \pi_2 \Delta Z)(\beta_1 \pi_1 \Delta Z)}{N_{0,1} + \beta_1 \pi_1 \Delta Z}\right)}_{\text{Other Fund's Impact}}.$$

In addition to the price impact resulting from an individual fund as in Equation (22), there is a cross-fund price impact that imposes additional negative pressure on the asset price:

$$\Delta P_{tot} \approx \Delta Z + (\text{Impact from Fund 1})$$
$$+ (\text{Impact from Fund 2}) + (\text{Cross-impact}),$$

where the analytical expressions of these price impact terms are given in Remark A.2.

If investors of multiple funds holding overlapping asset portfolios redeem fund shares simultaneously,

the feedback between fund performance, outflow, and asset liquidation is reinforced. Additionally, we assume that the first movers of each fund anticipate the other fund's outflow; as a consequence, they redeem more shares than they would if each fund operated in isolation. To eliminate the first-mover advantage, each fund needs to consider the impact of the other fund. If both funds implement swing pricing, the adjustment is (see Proposition A.3)

$$\Delta S_{both}^{sw} = -\gamma\left(R_{tot,1}^{fm} + R_{tot,2}^{fm}\right) = \gamma\frac{(\pi_1\beta_1 + \pi_2\beta_2)\Delta Z}{1 - (\beta_1 + \beta_2)\gamma}. \quad (26)$$

A comparison of (26) and (25) shows that the swing price required by each of the two funds with common asset ownership is greater than it would be if the funds operated in isolation.

5.2. The Benefits of Cooperative Swing Pricing
Swing pricing may be adopted unevenly across the mutual fund industry. A fund implementing swing pricing must, therefore, choose whether to neutralize only the effect of its own first movers or to anticipate the effect of first movers at other funds as well. If two funds hold the same asset, then heavy redemptions at one fund can drive down the share price of both funds.

If only one fund were to adopt swing pricing, the NAV adjustment it would need to offset the impact of first movers at all funds would be larger than the adjustment required if the fund operated in isolation. If both funds adopt swing pricing, the NAV adjustment required to remove all first-mover externalities would be smaller than in the case in which one fund does not apply swing pricing while the other does; in fact, it is even smaller than the adjustment required for one fund to remove its own first movers' externalities.

These statements are illustrated in Figure 8. To make them precise, suppose only fund 2 adopts swing pricing.

Figure 8. (Color online) Comparison of Different Swing Price Specifications
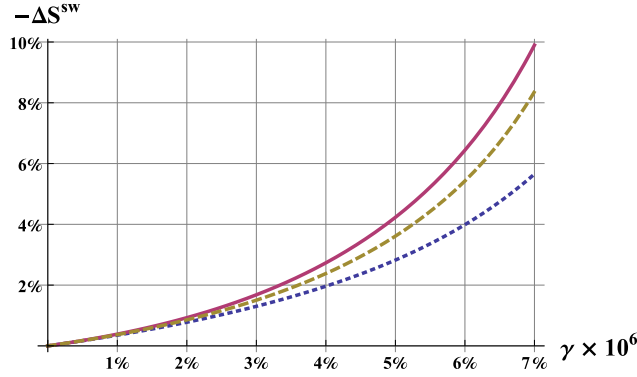
Notes. Comparison of the swing prices ΔS_{both}^{sw} (dotted line), ΔS_{loc}^{sw} (dashed line), and ΔS_{glob}^{sw} (solid line) for different levels of illiquidity γ. The horizontal axis reports the price impact per \$1 million. The proportion of first movers in each fund is $\pi_1 = \pi_2 = 75\%$.

Let ΔS_{loc}^{sw} be the NAV adjustment that makes fund 2's first movers internalize their liquidation costs. This is the swing price leading to the same change in NAV as if $\pi_1 > 0$ and $\pi_2 = 0$. Let ΔS_{glob}^{sw} be the swing price for fund 2 that offsets the effect of first movers at both funds, leading to the same change in NAV as if $\pi_1 = 0, \pi_2 = 0$. (See Appendix A for mathematical details.) We now have the following result:

Proposition 5. *Suppose $\pi_1, \pi_2 > 0$ and that only fund 2 applies swing pricing. For small γ,*

$$\left| \Delta S_{both}^{sw} \right| \leq \left| \Delta S_{loc}^{sw} \right| \leq \left| \Delta S_{glob}^{sw} \right|.$$

The intuition underlying this result is as follows. The externalities imposed on a fund by its first movers are amplified by other funds' first movers. Hence, if only a single fund applies swing pricing, the NAV adjustment required to eliminate these externalities needs to account for the fire-sale amplification driven by other funds' first movers. On the other hand, if each fund were to adopt swing pricing, cross-fund amplification resulting from first movers' redemptions would be eliminated.

A mutual fund that does not adopt swing pricing still benefits from the implementation of swing pricing by other funds because of the reduced selling pressure imposed on it by the other funds' first movers. The presence of mutual funds that do not implement swing pricing imposes a cost on the first movers of funds that do adopt swing pricing because their exit NAV is smaller than in the case in which all funds cooperate in the adoption of swing pricing.

5.3. Cross-Asset Price Impact
Our analysis has assumed that mutual funds hold a single asset representative of their entire portfolio. In an extended framework in which funds holds

multiple assets, the total execution costs would depend on their liquidation strategy. Such a strategy depends not only on the composition of the fund's target portfolio, but also on the cross-asset structure of the price impact function. For example, Tsoukalas et al. (2019) show that over-trading may actually reduce execution costs because of asset correlation and cross-asset price impact. Jotikasthira et al. (2012) find that fund flows triggered by a shock to one asset push the fund to trade other assets, impacting both the price of the shocked asset and that of other assets in the fund's portfolio. Our analysis could potentially be extended to cover cross-asset impacts by replacing the parameter γ with coefficients γ_{ij} measuring the price impact on asset j of selling a share of asset i.

6. Implementation Challenges
An implementation of our proposed swing-pricing rule requires a fund to estimate the liquidity of its underlying portfolio. The relevant price impact function should reflect liquidity following a market shock, when the fund needs to sell assets to meet redemption requests. The fund might choose a conservative estimate of γ, but in setting a swing-pricing policy, a fund faces competing objectives. It wants the adjustment to be sufficiently large to protect investors who remain in the fund from bearing the liquidation costs of those who exit but not so large that it deters investors from entering the fund. Funds in the United States also face an operational challenge because they have a relatively short window (compared with many European funds) between the time they stop accepting orders for the day and the time they calculate their NAV.

An effective swing price should account for redemptions at other funds holding the same assets. Sharing of detailed information between funds would face many obstacles for practical, competitive, and possibly legal reasons. But sharing aggregate data on investor flows might be feasible and could help funds in making the appropriate NAV adjustment. Where swing pricing has been used, the industry has been reluctant to share details with investors to avoid strategic redemptions aimed at gaming the swing-pricing mechanism. The Association Française de la Gestion Financière (AFG) writes,

> The AFG strongly advises member asset management companies not to disclose parameters that are too detailed and recent to avoid reducing the effectiveness of this system. In particular, the management company should not disclose (in writing or verbally) the current levels of the trigger thresholds and should ensure that the internal information channels are limited in order to maintain the confidentiality of this information and avoid any misuse.

We are not aware of any analysis supporting this recommendation. In our analysis, for swing pricing to

be effective, it is crucial that investors understand how the mechanism is applied. This point argues in favor of transparency in a fund's swing-pricing policy. It also suggests that it is more important for the fund and its investors to have consistent views on the liquidity of the fund's portfolio than to have a perfect model of price impact. The liquidity disclosure rules included in the SEC's Rule 22e-4 may help achieve this alignment.

7. Concluding Remarks

Our study models and quantifies the externalities stemming from the liquidity mismatch in open-end mutual funds. By analyzing the interactions between fund performance, net outflows, and asset liquidity, we provide a unified framework that delivers several predictions. First, the first-mover advantage amplifies the effects of fire sales and introduces a nonlinear relation between the aggregate price impact and the magnitude of an exogenous shock. Second, the first-mover advantage may trigger a cascade of redemptions following bad fund performance, leading to asset sales that further drive down prices and generate further redemptions, potentially to a point at which the fund may be unable to repay redeeming investors and, thus, fails. Third, our definition of swing pricing neutralizes the first-mover advantage. It does so by transferring the costs of liquidation to redeeming investors. Importantly, it also reduces these costs by eliminating the incentive for investors to redeem earlier. At large levels of redemptions, the required swing adjustment is larger than the fixed adjustment seen in practice.

The major policy implication of our study is the provision of an ideal yet simple swing-pricing rule, which is based on roughly observable quantities. Funds need to account for the net outflows of first movers and estimate the illiquidity of the asset to decide how much to adjust their NAV. We have assumed a single level of liquidity for all of a fund's holdings. Extrapolating to more general cases, our proposed adjustment suggests a need to partition a fund's portfolio into liquidity buckets. The current SEC 22e-4 rule requires funds to divide their assets into buckets based on time for liquidation, but our analysis points to the importance of distinguishing by liquidation costs as well because a fund may be forced to sell assets quickly to meet redemptions.

The amendments to the SEC 22c-1 rule on swing pricing impose a 2% cap, relative to the fund's NAV, on the swing factor. Such a constraint may limit the efficacy of swing pricing in periods of severe market illiquidity. To deter any misuse of swing pricing, other forms of regulatory oversight, such as appropriate disclosures on the adopted swing-pricing mechanism, should be considered.

Our analysis shows that greater benefits are attained if swing pricing is applied by all mutual funds investing in the same illiquid assets. Under these circumstances, the externalities imposed on the funds are internalized by their first movers at a lower cost compared with the case in which some funds apply swing pricing but others do not.

Appendix A. Multiple Funds and Swing Pricing

Throughout this section, we assume $\varphi(x) = \gamma x$. Consider two funds that hold shares of the same asset. First movers of each fund $i = 1, 2$ redeem $R_{tot,i}^{fm}$ fund shares, and fund i liquidates $\Delta Q_{tot,i}^{fm}$ asset shares to meet these redemption requests, where

$$R_{tot,i}^{fm} = -\beta_i \pi_i \Delta S_{tot,i}, \quad -\Delta Q_{tot,i}^{fm} \times \left(P_0 + \Delta Z + \gamma \left(\Delta Q_{tot,1}^{fm} + \Delta Q_{tot,2}^{fm} \right) \right)$$
$$= R_{tot,i}^{fm}(S_{0,i} + \Delta Z). \tag{A.1}$$

Second movers redeem $\Delta R_{tot,i}^{sm}$ shares, and fund i liquidates $\Delta Q_{tot,i}^{sm}$ asset shares, where

$$\Delta R_{tot,i}^{sm} = -\beta_i(1 - \pi_i) \Delta S_{tot,i}, \quad \Delta Q_{tot,i}^{sm} = -\Delta R_{tot,i}^{sm} \frac{S_0 + \Delta S_{tot,i}}{P_0 + \Delta P_{tot}}. \tag{A.2}$$

The change in value $\Delta S_{tot,i}$ of fund i's share, for $i = 1, 2$, and the change in price of an asset share ΔP_{tot} are given by the solution to the following system of equations:

$$\Delta P_{tot} = \Delta Z + \gamma \left(\Delta Q_{tot,1}^{fm} + \Delta Q_{tot,2}^{fm} + \Delta Q_{tot,1}^{sm} + \Delta Q_{tot,2}^{sm} \right),$$

$$\Delta S_{tot,1} = \frac{\left(Q_0 + \Delta Q_{tot,1}^{fm} + \Delta Q_{tot,1}^{sm} \right)(P_0 + \Delta P_{tot})}{N_0 - R_{tot,1}^{sm}} - S_{0,1},$$

$$\Delta S_{tot,2} = \frac{\left(Q_0 + \Delta Q_{tot,2}^{fm} + \Delta Q_{tot,2}^{sm} \right)(P_0 + \Delta P_{tot})}{N_0 - R_{tot,2}^{sm}} - S_{0,2}. \tag{A.3}$$

Proposition A.1. *Let $R_i := -\beta_i \Delta S_{tot,i}$ be the amount of redeemed shares. Assume that $\pi_1 = \pi_2 = 0$ and that the number of asset*

and fund shares are equal for each fund: $Q_i = N_i$ for $i = 1, 2$. The change in value of fund i's share $\Delta S_{tot,i}$, $i = 1, 2$, and the change in asset price ΔP_{tot} are

$$\Delta S_{tot,i} = \Delta Z + \gamma \frac{E_1 + E_2}{1 - (\beta_1 + \beta_2)\gamma}, \quad \Delta P_{tot} = \Delta Z + \gamma \frac{E_1 + E_2}{1 - (\beta_1 + \beta_2)\gamma},$$

(A.4)

where $E_i = \beta_i \Delta Z$.

Remark A.1. Cross-price impact effects are important. The impact on the funds' share value and the asset price imposed by the simultaneous liquidation procedure of multiple funds is larger than the sum of the impacts of each individual fund without accounting for spillover effects:

$$\Delta P_{tot} \approx \Delta Z + \underbrace{\gamma E_1 + \gamma^2 \beta_1 E_1}_{\text{Fund 1 Impact}} + \underbrace{\gamma E_2 + \gamma^2 \beta_2 E_2}_{\text{Fund 2 Impact}} + \underbrace{\gamma^2 (\beta_1 E_2 + \beta_2 E_1)}_{\text{Cross-impact}}.$$

Proposition A.2. *Assume that $\pi_1, \pi_2 > 0$. Define $E_i = \beta_i \Delta Z$, $E_i^\pi = \beta_i \pi_i \Delta Z$, $Rem_i^\pi = N_i + \beta_i \pi_i \Delta Z$ the number of remaining shares after first-mover redemptions at order 0 in γ, and $Rem_i = N_i + \beta_i \Delta Z$ the number of remaining shares after first and second movers' redemptions at order 0 in γ. For small γ, the change in value of fund i's share is*

$$\Delta S_{tot,i} = \Delta Z + \gamma \left((E_1 + E_2) - \frac{E_i^\pi (E_1^\pi + E_2^\pi)}{Rem_i^\pi} \right) + o(\gamma).$$

For small γ, the change in the asset price is

$$\Delta P_{tot} = \Delta Z + \gamma(E_1 + E_2) + \gamma^2 \bigg((\beta_1 + \beta_2)(E_1 + E_2)$$
$$- \beta_1 E_1^\pi \frac{E_1^\pi + E_2^\pi}{Rem_1^\pi} - \beta_2 E_2^\pi \frac{E_1^\pi + E_2^\pi}{Rem_2^\pi} - E_1^\pi \frac{E_1^\pi + E_2^\pi}{P_0 + \Delta Z} \frac{Rem_1}{Rem_1^\pi}$$
$$- E_2^\pi \frac{E_1^\pi + E_2^\pi}{P_0 + \Delta Z} \frac{Rem_2}{Rem_2^\pi} \bigg) + o(\gamma^2).$$

Remark A.2. The expressions in Proposition A.2 can be restated as

$$\Delta S_{tot,1} \approx \Delta Z + \underbrace{\gamma \left(E_1 - \frac{(E_1^\pi)^2}{Rem_1^\pi} \right)}_{\text{Own Impact}} + \underbrace{\gamma \left(E_2 - \frac{E_2^\pi E_1^\pi}{Rem_1^\pi} \right)}_{\text{Other Fund's Impact}},$$

$$\Delta P_{tot} \approx \Delta Z + \underbrace{\gamma E_1 + \gamma^2 \left(\beta_1 E_1 - \beta_1 \frac{(E_1^\pi)^2}{Rem_1^\pi} - \frac{(E_1^\pi)^2}{P^{\Delta Z}} \frac{Rem_1}{Rem_1^\pi} \right)}_{\text{Impact from Fund 1}}$$

$$+ \underbrace{\gamma E_2 + \gamma^2 \left(\beta_2 E_2 - \beta_2 \frac{(E_2^\pi)^2}{Rem_2^\pi} - \frac{(E_2^\pi)^2}{P^{\Delta Z}} \frac{Rem_2}{Rem_2^\pi} \right)}_{\text{Impact from Fund 2}}$$

$$+ \gamma^2 \underbrace{\left(\begin{array}{c} \beta_1 E_2 + \beta_2 E_1 - \beta_1 \frac{E_1^\pi E_2^\pi}{Rem_1^\pi} - \beta_2 \frac{E_1^\pi E_2^\pi}{Rem_2^\pi} \\ - \frac{E_1^\pi E_2^\pi}{P^{\Delta Z}} \frac{Rem_1}{Rem_1^\pi} - \frac{E_1^\pi E_2^\pi}{P^{\Delta Z}} \frac{Rem_2}{Rem_2^\pi} \end{array} \right)}_{\text{Cross-impact}}.$$

Proposition A.3. *Assume that $\pi_1, \pi_2 > 0$ and that the number of asset shares equals the number of fund shares for each fund: $Q_i = N_i$*

for $i = 1, 2$. Assume both funds 1 and 2 apply swing pricing. The swing price of fund $i = 1, 2$ is

$$\Delta S_{both}^{sw} = \gamma \frac{E_1^\pi + E_2^\pi}{1 - (\beta_1 + \beta_2)\gamma}.$$

Viewed as a function of first movers' redemptions, the swing price takes the form

$$\Delta S_{both}^{sw} = -\gamma \left(R_{tot,1}^{fm} + R_{tot,2}^{fm} \right),$$

(A.5)

where $R_{tot,i}^{fm}$ is the number of shares redeemed by first movers of fund $i = 1, 2$.

Proposition A.4. *Assume $\pi_1, \pi_2 > 0$ and that the number of asset and fund shares for each fund are equal: $Q_i = N_i$, $i = 1, 2$. For small γ, the change in value of fund 2's share $\Delta S_{tot,2}$ is*

$$\Delta S_{tot,2} = \Delta Z + \gamma(\beta_1 + \beta_2)\Delta Z$$
$$+ \gamma^2 \left((\beta_1 + \beta_2)^2 \Delta Z - (E_1^\pi)^2 \frac{Rem_1 + \beta_1(P_0 + \Delta Z)}{(P_0 + \Delta Z)Rem_1^\pi} \right) + o(\gamma^2).$$

(A.6)

If only one fund applies swing pricing, the fund may decide to implement an adjustment that removes either the impact of first movers of both funds or only the impact of its own first movers. Swing price ΔS_{loc}^{sw} is computed such that the fund attains the change in NAV (A.6), and swing price ΔS_{glob}^{sw} is computed such that the fund's NAV change is (A.4).

Proposition A.5. *Assume $\pi_1, \pi_2 > 0$ and that the number of asset and fund shares for each fund: $Q_i = N_i$ for $i = 1, 2$. Assume that only fund 2 applies swing pricing. For small γ,*

$$\Delta S_{loc}^{sw} = \Delta S_{both}^{sw} + \gamma^2 \frac{\begin{array}{c} E_1^\pi \big(\beta_1(P_0 + 2\Delta Z)\big(Rem_2^\pi - \Delta Z \pi_1(\beta_1 \pi_1 + \beta_2 \pi_2) \big) \\ + N_1(N_2 - \beta_1 \Delta Z \pi_1) \big) \end{array}}{(P_0 + \Delta Z)Rem_1^\pi}$$
$$+ o(\gamma^2),$$

$$\Delta S_{glob}^{sw} = \Delta S_{loc}^{sw} + \gamma^2 \frac{\beta_1 \pi_1}{\beta_2 \pi_2} \frac{Rem_2^\pi}{Rem_1^\pi} \frac{E_1^\pi(\beta_1(P_0 + \Delta Z) + Rem_1)}{P_0 + \Delta Z} + o(\gamma^2).$$

Appendix B. Technical Proofs

Define $\Delta \bar{P}_n := P_n^{sm} - P_0^{sm}$ and $\Delta \bar{S}_n := S_n^{sm} - S_0^{sm}$ to be the cumulative price changes over n rounds of second-mover redemptions, and define the function

$$g(x) = \frac{Q_0^{sm}}{N_0 - R_0^{sm}} \bigg(P_0 + \Delta Z - P_0^{sm} + \varphi \big(Q_0^{sm} - Q_0 + (1 - \pi)\beta \frac{Q_0^{sm}}{N_0 - R_0^{sm}} (\Delta S_0^{sm} + x) \big) \bigg).$$

Lemma B.1. *With the preceding notation, we have $\Delta \bar{S}_n = \frac{Q_0^{sm}}{N_0 - R_0^{sm}} \Delta \bar{P}_n = g(\Delta \bar{S}_{n-1})$.*

Proof. Plugging (6) and (8) into (7), we obtain $\Delta Q_{n+1}^{sm} = (1 - \pi)\beta \frac{Q_n^{sm}}{N_0 - R_n^{sm}} \Delta S_n^{sm}$. It follows that

$$\frac{Q_n^{sm} + \Delta Q_{n+1}^{sm}}{N_0 - R_n^{sm} - \Delta R_{n+1}^{sm}} = \frac{Q_n^{sm}}{N_0 - R_n^{sm}} \frac{1 + (1 - \pi)\beta \Delta S_n^{sm}/(N_0 - R_n^{sm})}{1 + (1 - \pi)\beta \Delta S_n^{sm}/(N_0 - R_n^{sm})}$$
$$= \frac{Q_n^{sm}}{N_0 - R_n^{sm}}.$$

Hence, by induction, $\frac{Q_n^{sm}}{N_0 - R_n^{sm}} = \frac{Q_0^{sm}}{N_0 - R_0^{sm}}$. Equation (8) may be rewritten as

$$\frac{S_n^{sm}}{P_n^{sm}} = \frac{Q_n^{sm}}{N_0 - R_n^{sm}} = \frac{Q_0^{sm}}{N_0 - R_0^{sm}}.$$

Hence, $S_n^{sm} = \frac{Q_0^{sm}}{N_0 - R_0^{sm}} P_n^{sm}$, and then $\Delta \bar{S}_n = \frac{Q_0^{sm}}{N_0 - R_0^{sm}} \Delta \bar{P}_n$ as claimed. By rewriting (7) as

$$Q_{n+1}^{sm} + R_{n+1}^{sm} \frac{S_{n+1}^{sm}}{P_{n+1}^{sm}} = Q_n^{sm} + R_n^{sm} \frac{S_{n+1}^{sm}}{P_{n+1}^{sm}},$$

it can be seen that $Q_n^{sm} - Q_0^{sm} = -\frac{Q_0^{sm}}{N_0 - R_0^{sm}}(R_n^{sm} - R_0^{sm})$ for all n. Using Equation (6), we obtain

$$Q_n^{sm} - Q_0^{sm} = -\frac{Q_0^{sm}}{N_0 - R_0^{sm}}(R_n^{sm} - R_0^{sm})$$
$$= (1 - \pi)\beta \frac{Q_0^{sm}}{N_0 - R_0^{sm}}(\Delta S_0^{sm} + \Delta \bar{S}_{n-1}).$$

Together with Equation (9), this concludes the proof.

Lemma B.2. *If $\varphi(x) = \gamma x$, then $\Delta S_n^{sm} = (1 - \pi)\beta\gamma(\frac{Q_0^{sm}}{N_0 - R_0^{sm}})^2 \Delta S_{n-1}^{sm}$.*

Proof. With linear price impact, $P_0^{sm} = P_0 + \Delta Z + \gamma(Q_0^{sm} - Q_0)$, and the recursion in Lemma B.1 simplifies to $\Delta \bar{S}_n = (1 - \pi)\beta\gamma \cdot (Q_0^{sm}/(N_0 - R_0^{sm}))^2(\Delta S_0^{sm} + \Delta \bar{S}_{n-1})$. The result then follows by induction because $\Delta \bar{S}_n = \Delta \bar{S}_{n-1} + \Delta S_n^{sm}$. $\square$

Proof of Lemma 1. The condition in the lemma implies that $\lim_{x \to -\infty} g'(x) < 1$. This, in turn, implies the existence of a finite x_l such that $g(x) > x$ for $x < x_l$. Because $(1 - \pi)\beta(Q_0^{sm}/(N_0 - R_0^{sm}))\Delta S_0^{sm} < 0$ and $P_0^{sm} = P_0 + \Delta Z + \varphi(Q_0^{sm} - Q_0)$, we get $g(0) < 0$. Because $\varphi(\cdot)$ is increasing, $g(\cdot)$ is also increasing. Now consider the sequence $\Delta \bar{S}_{n+1} = g(\Delta \bar{S}_n)$ with $\Delta \bar{S}_0 = 0$. We have $\Delta \bar{S}_1 < \Delta \bar{S}_0$ because $g(0) < 0$. Monotonicity of g then implies $g(\Delta \bar{S}_1) \le g(\Delta \bar{S}_0)$, so $\Delta \bar{S}_2 \le \Delta \bar{S}_1$, and proceeding by induction, we conclude that the $\Delta \bar{S}_n$ form a decreasing sequence. As $g(\Delta \bar{S}_n) \le \Delta \bar{S}_n$, we must have $\Delta \bar{S}_n \ge x_l$, for all n. In other words, the $\Delta \bar{S}_n$ form a decreasing sequence bounded from below and, thus, converge to a limit. Moreover, the continuity of g implies that the limit is a fixed point of g. $\square$

Proof of Proposition 1. If $\pi = 0$, then $Q_0^{sm} = Q_0$ and $R_0^{sm} = 0$. Hence, the condition in Lemma 1 reduces to $\beta \lim_{x \to -\infty} \varphi'(x) < 1$, and convergence is guaranteed. From Lemma B.1, we get $\Delta \bar{S}_n = \Delta \bar{P}_n = \varphi(\beta(\Delta Z + \Delta \bar{P}_{n-1}))$. It follows that $\Delta P_{tot}^{sm} = \Delta Z + \varphi(\beta(\Delta P_{tot}^{sm}))$.

The equations in (20) can be rewritten as $Q_n - Q_0 = \beta(P_{n-1} - P_0)$ and $P_n = P_0 + \Delta Z + \varphi(\beta(P_{n-1} - P_0))$. The argument in Lemma 1 guarantees that the sequence $(P_n)_{n \ge 0}$ converges. The limit $\Delta P_{tot}^{dir} = \lim_{n \to \infty} P_n - P_0$ then satisfies the fixed-point equation $\Delta P_{tot}^{dir} = \Delta Z + \varphi(\beta \Delta P_{tot}^{dir})$. $\square$

Lemma B.3. *Let $\varphi(x) = \gamma x$. Suppose that, after first movers' redemptions, the fund holds $Q_0^{sm} := Q_{tot}^{fm}$ asset shares; the number of outstanding shares of the fund is $N_0 - R_0^{sm}$, where $R_0^{sm} := R_{tot}^{fm}$; the asset price is $P_0^{sm} := P_{tot}^{fm} = P_0 + \Delta Z + \gamma(Q_{tot}^{fm} - Q_0)$; and the value of a fund share is $S_0^{sm} := S_{tot}^{fm} = Q_{tot}^{fm} P_{tot}^{fm} / (N_0 - R_{tot}^{fm})$. Assume that $\beta\gamma(1 - \pi)(Q_{tot}^{fm}/(N_0 - R_{tot}^{fm}))^2 < 1$. The cumulative changes in asset*

price and fund share value after second movers' redemptions are given by

$$\Delta P_{tot}^{fm} + \Delta P_{tot}^{sm} = \Delta P_{tot}^{fm} + \beta\gamma(1 - \pi)\frac{Q_0^{sm}}{N_0 - R_{tot}^{fm}}$$
$$\cdot \frac{\Delta S_0^{sm}}{1 - \beta\gamma(1 - \pi)\left(\frac{Q_0^{sm}}{N_0 - R_{tot}^{fm}}\right)^2},$$

$$\Delta S_{tot}^{fm} + \Delta S_{tot}^{sm} = \frac{\Delta S_0^{sm}}{1 - \beta\gamma(1 - \pi)\left(\frac{Q_0^{sm}}{N_0 - R_{tot}^{fm}}\right)^2}.$$

Proof. From Lemmas B.1 and 1, we get that ΔS_{tot}^{sm} satisfies $\Delta S_{tot}^{sm} = g(\Delta S_{tot}^{sm})$. Specializing to the case of linear φ, solving for ΔS_{tot}^{sm}, and using $P_0^{sm} = P_0 + \Delta Z + \gamma(Q_0^{sm} - Q_0)$ yields the result. The expression for ΔP_{tot}^{sm} follows from the first equality in Lemma B.1. $\square$

Proof of Proposition 2. Using Equation (15), we can rewrite (14) as the quadratic equation

$$\gamma\left(\Delta Q_{tot}^{fm}\right)^2 + (P_0 + \Delta Z)\Delta Q_{tot}^{fm} - R_{tot}^{fm}(S_0 + \Delta S_0) = 0,$$

with $\Delta S_0 = \Delta Z$ because we assume $Q_0 = N_0$. If this quadratic equation in ΔQ_{tot}^{fm} admits a solution, then the smallest number of asset shares the fund has to trade to repay first movers is

$$\Delta Q_{tot}^{fm} = -\frac{P_0 + \Delta Z - \sqrt{(P_0 + \Delta Z)^2 - 4\gamma R_{tot}^{fm}(P_0 + \Delta Z)(Q_0/N_0)}}{2\gamma}.$$

The number of redemptions R_{tot}^{fm} in (12) is strictly decreasing in ΔS_{tot}. Hence, the negative change in holdings ΔQ_{tot}^{fm} is strictly increasing in ΔS_{tot}. Using the assumptions $P_0 = S_0$ and $Q_0 = N_0$, and (15), we can rewrite Equation (14) as

$$\frac{Q_0 + \Delta Q_{tot}^{fm}}{N_0 - R_{tot}^{fm}} = 1 - \frac{\gamma}{P_0 + \Delta Z}\frac{\left(\Delta Q_{tot}^{fm}\right)^2}{N_0 - R_{tot}^{fm}}. \tag{B.1}$$

It follows that the fraction $(Q_0 + \Delta Q_{tot}^{fm})/(N_0 - R_{tot}^{fm})$ is strictly increasing in ΔS_{tot}. The change in value of a fund share resulting from first movers' redemptions is $\Delta S_{tot}^{fm} = Q_{tot}^{fm} P_{tot}^{fm} / (N_0 - R_{tot}^{fm}) - S_0$. Using Lemma B.3, we obtain that, after second-mover redemptions, the change in value of a fund share is

$$\Delta S_{tot} = \frac{\Delta S_{tot}^{fm}}{1 - \beta\gamma(1 - \pi)\left(\frac{Q_0 + \Delta Q_{tot}^{fm}}{N_0 - R_{tot}^{fm}}\right)^2}. \tag{B.2}$$

Because both $\frac{Q_0 + \Delta Q_{tot}^{fm}}{N_0 - R_{tot}^{fm}}$ and P_{tot}^{fm} are strictly increasing in ΔS_{tot}, we obtain that ΔS_{tot}^{fm} and, therefore, also the right-hand side of (B.2) is strictly increasing in ΔS_{tot}. After plugging the expressions for ΔS_{tot}^{fm}, ΔQ_{tot}^{fm}, and ΔR_{tot}^{fm} into the right-hand side of Equation (B.2), we may rewrite (B.2) as

$$\Delta S_{tot} = f_\gamma(\Delta S_{tot}), \tag{B.3}$$

where
$f_\gamma(x)$

$$= \frac{2\beta\pi\Delta Z x + N_0\left(\Delta Z - P_0 + \sqrt{P_0 + \Delta Z}\sqrt{P_0 + \Delta Z + 4\beta\pi\gamma x}\right)}{2(N_0 + \beta\pi x)\left(1 - \frac{\beta(1-\pi)\left(P_0 + \Delta Z - 2\gamma N_0 - \sqrt{P_0 + \Delta Z}\sqrt{P_0 + \Delta Z + 4\beta\pi\gamma x}\right)^2}{4\gamma(N_0 + \beta\pi x)^2}\right)}.$$
(B.4)

We have shown that the right-hand side in (B.2) is strictly increasing in ΔS_{tot}. Equivalently, the function $f_\gamma(x)$ is strictly increasing for $x < 0$. Furthermore, it can be immediately verified that $f_\gamma(\Delta Z) \leq f_\gamma(0) = \frac{\Delta Z}{1 - \beta\gamma(1-\pi)} < \Delta Z$. Because

$$\sqrt{P_0 + \Delta Z + 4\beta\pi\gamma x} = \sqrt{P_0 + \Delta Z} + \frac{2\beta\pi x}{\sqrt{P_0 + \Delta Z}}\gamma + O(\gamma^2),$$

we get that $f_0(x) := \lim_{\gamma \to 0^+} f_\gamma(x) = \Delta Z$. Hence, the initial shock $\Delta S_0 = \Delta Z$ is a fixed point of $f_\gamma(\cdot)$ when $\gamma = 0$. Because $f_0'(x) \neq 1$ for every x and the dependence of both $f_\gamma(\cdot)$ and its derivative on γ is continuous, there exists $\gamma^* > 0$ such that, for $0 < \gamma < \gamma^*$, there exists a solution to the fixed-point equation $x = f_\gamma(x)$.

We can Taylor expand the fixed point $\Delta S_{tot}(\gamma)$ around $\gamma = 0$ to obtain

$$\Delta S_{tot}(\gamma) = \Delta S_{tot}(0) + \gamma \frac{\partial \Delta S_{tot}(\gamma)}{\partial \gamma}\bigg|_{\gamma=0} + o(\gamma).$$

Because $\lim_{\gamma \to 0^+} f_\gamma(\Delta S_{tot}(\gamma)) = \Delta Z$, we get that $\Delta S_{tot}(0) = \Delta Z$. By differentiating both sides of the fixed-point equation $\Delta S_{tot}(\gamma) = f_\gamma(\Delta S_{tot}(\gamma))$ with respect to γ, we obtain

$$\frac{\partial \Delta S_{tot}}{\partial \gamma} = \frac{\partial f_\gamma(\Delta S_{tot})}{\partial \gamma} + \frac{\partial f_\gamma(\Delta S_{tot})}{\partial \Delta S_{tot}}\frac{\partial \Delta S_{tot}}{\partial \gamma}.$$

It can be verified that $\frac{\partial f_\gamma(\Delta S_{tot})}{\partial \Delta S_{tot}}\big|_{\gamma=0} = 0$ and $\frac{\partial f_\gamma(\Delta S_{tot})}{\partial \gamma}\big|_{\gamma=0} = \beta\Delta Z - \frac{\pi^2\beta^2\Delta Z^2}{N_0 + \pi\beta\Delta Z}$. Hence,

$$\Delta S_{tot} = \Delta Z + \gamma\left(\beta\Delta Z - \frac{\pi^2\beta^2\Delta Z^2}{N_0 + \pi\beta\Delta Z}\right) + o(\gamma).$$

From Lemma B.3, we get that

$$\Delta P_{tot} = \Delta Z + \gamma(Q_{tot}^{fm} - Q_0) + \beta\gamma(1-\pi)\frac{Q_{tot}^{fm}}{N_0 - R_{tot}^{fm}}\Delta S_{tot},$$

where both Q_{tot}^{fm} and R_{tot}^{fm} are functions of ΔS_{tot}. Given the Taylor expansion in γ for ΔS_{tot}, we can compute the expansion for ΔP_{tot}:

$$\begin{aligned}
\lim_{\gamma \to 0^+} \Delta P_{tot} &= \Delta Z \lim_{\gamma \to 0^+} \frac{\Delta P_{tot} - \Delta Z}{\gamma} \\
&= \beta\Delta Z \lim_{\gamma \to 0^+} \frac{\Delta P_{tot} - \Delta Z - \gamma\beta\Delta Z}{\gamma^2} \\
&= \beta^2\Delta Z - \beta\frac{\pi^2\beta^2\Delta Z^2}{N_0 + \pi\beta\Delta Z} - \frac{\pi^2\beta^2\Delta Z^2}{P_0 + \Delta Z}\frac{N_0 + \beta\Delta Z}{N_0 + \pi\beta\Delta Z}. \quad\square
\end{aligned}$$

Proof of Proposition 3. In the proof of Proposition 2, we showed that ΔS_{tot} is a fixed point, if it exists, of the function f_γ given in (B.4). It can be verified immediately that, if $\Delta Z = 0$, then $\Delta S_{tot} = 0$ is the unique fixed point of $f_\gamma(\cdot)$.

Notice that the maximum amount of cash the fund can retrieve from asset sales is $\max_{\Delta Q} \Delta Q(P_0 + \Delta Z + \gamma\Delta Q) = (P_0 + \Delta Z)^2/(4\gamma)$. Hence, the fund becomes unable to repay first movers when $R^{fm}(S_0 + \Delta S_0) > (P_0 + \Delta Z)^2/(4\gamma)$, where the left-hand side is the amount of cash the fund owes to first movers. In other words, if first movers redeem $R^{fm} = -\beta\pi\Delta S_{tot}$ in response to an anticipated final change in value of a fund share ΔS_{tot}, this solvency-type condition reads as $\Delta S_{tot} < -(P_0 + \Delta Z)/(4\gamma\pi\beta)$ (recall that $Q_0 = N_0$). Hence, since $\Delta S_{tot} < \Delta S_0$, if $\Delta S_0 = \Delta Z < -P_0/(1 + 4\gamma\beta\pi)$, the fund becomes unable to meet its first movers' redemption requests.

Throughout the proof, we write $f_\gamma(x, \Delta Z)$ to highlight the dependence of $f_\gamma(x)$ on ΔZ. Define the solvency set $\mathscr{S}_\gamma := \{\Delta Z : \exists \Delta S_{tot} < 0 \text{ such that } \Delta S_{tot} = f_\gamma(\Delta S_{tot}, \Delta Z)\}$. This is the set of initial shocks ΔZ such that the fund is able to raise enough cash to repay its first movers. We have already shown that, if $\Delta Z < -P_0/(1 + 4\gamma\beta\pi)$, then ΔZ does not belong to $\mathscr{S}_\gamma$.

With $\Delta Z^* := \inf \mathscr{S}_\gamma$, it follows that $-\frac{P_0}{1 + 4\gamma\beta\pi} \leq \Delta Z^*$. Combining (B.2) and (B.3), we obtain

$$f_\gamma\left(-\frac{R_{tot}^{fm}}{\beta\pi}, \Delta Z\right) = \frac{\Delta S_{tot}^{fm}(\Delta Z)}{1 - \beta\gamma(1-\pi)\left(\frac{Q_0 + \Delta Q_{tot}^{fm}(\Delta Z)}{N_0 - R_{tot}^{fm}}\right)^2}.$$

For a given quantity of first-mover redemptions R_{tot}^{fm}, the amount of asset shares ΔQ_{tot}^{fm} the fund trades to repay first movers is an increasing function of ΔZ. Hence, $\Delta S_{tot}^{fm} := Q_{tot}^{fm}P_{tot}^{fm}/(N_0 - R_{tot}^{fm}) - S_0$ is also an increasing function of ΔZ. It follows that, for any $x \in [-\frac{P_0 + \Delta Z}{4\gamma\pi\beta}, 0]$, $f_\gamma(x, \Delta Z)$ is increasing in ΔZ. Notice that if $0 > \Delta Z_1 \notin \mathscr{S}_\gamma$, because $f_\gamma(0, \Delta Z_1) < 0$, we have $f_\gamma(x, \Delta Z_1) < x$ for all $x \leq 0$. Hence, the monotonicity of $f_\gamma(x, \Delta Z)$ in ΔZ implies that, if $\Delta Z_2 < \Delta Z_1$, then $f_\gamma(x, \Delta Z_2) < f_\gamma(x, \Delta Z_1) < x$ for all $x \leq 0$, and therefore, $\Delta Z_2 \notin \mathscr{S}_\gamma$, which is to say that the fund remains solvent for any $\Delta Z > \Delta Z^*$. Because the set $\mathscr{S}_\gamma$ is closed, the fund remains solvent also for $\Delta Z = \Delta Z^*$.

It can easily be seen that $f_\gamma(x, \Delta Z)$ is decreasing in γ for any $\Delta Z \in [-P_0/(1 + \gamma\beta\pi), 0]$ and any $x \in [-(P_0 + \Delta Z)/(4\gamma\pi\beta), 0]$. Let $\gamma_1 < \gamma_2$. Because $f_\gamma(x, \Delta Z)$ is decreasing in γ and $-(P_0 + \Delta Z)/(4\beta\pi\gamma)$ is increasing in γ, we obtain that $\mathscr{S}_{\gamma_2} \subset \mathscr{S}_{\gamma_1}$. This implies that $\Delta Z^*(\gamma_2) \geq \Delta Z^*(\gamma_1)$ and concludes the proof. $\square$

Proof of Proposition 4. Assume the fund adjusts its NAV by ΔS^{adj} when first movers redeem. The cumulative change in fund share value is the fixed point ΔS_{tot} in (18) after replacing $S_0 + \Delta S_0$ with $S_0 + \Delta S_0 + \Delta S^{adj}$ in (14).

Set $\Delta S^{adj} := \varphi(-R_{tot}^{fm})$. For a fixed ΔS_{tot}, solving Equations (12)–(15) for Q_{tot}^{fm}, we obtain $Q_{tot}^{fm} = Q_0 - R_{tot}^{fm} = Q_0 + \pi\beta\Delta S_{tot}$. Because $Q_0 = N_0$, this yields $Q_{tot}^{fm}/(N_0 - R_{tot}^{fm}) = 1$. From Lemma B.1, we obtain that $\Delta S_0^{sm} + \bar{\Delta}_n = \Delta S_0^{sm} + g(\Delta S_0^{sm} + \bar{\Delta}_{n-1} - \Delta S_0^{sm})$. Hence, ΔS_{tot} is a fixed point of the equation $\Delta S_{tot} = \Delta S_0^{sm} + g(\Delta S_{tot} - \Delta S_0^{sm})$. The function $g(x)$ reduces to $g(x) = \Delta Z - \Delta S_0^{sm} + \varphi(\pi\beta\Delta S_{tot} + (1-\pi)\beta(\Delta S_0^{sm} + x))$. In other terms, ΔS_{tot} is a fixed point of the equation $\Delta S_{tot} = \Delta Z + \varphi(\beta\Delta S_{tot})$. Let $\bar{x} < 0$ be the solution to $\bar{x} = \Delta Z + \varphi(\beta\bar{x})$; the existence of $\bar{x}$ follows from the argument used to prove

Proposition 1. We have shown that, if $\Delta S^{adj} = \varphi(-R_{tot}^{fm})$, then $\Delta S_{tot} = \bar{x}$ and $\Delta P_{tot} = \bar{x}$ solve (18).

If $\varphi(x) = \gamma x$, then $\bar{x} = \Delta Z/(1 - \beta\gamma)$. Since $R_{tot}^{fm} = -\pi\beta\Delta S_{tot} = -\pi\beta\Delta Z/(1 - \beta\gamma)$, we get that $\Delta S^{sw} = -\gamma R_{tot}^{fm} = \gamma\pi\beta\Delta Z/(1 - \beta\gamma)$. Notice that Q_{tot}^{fm} is a strictly decreasing function of ΔS^{adj}. Because ΔS_{tot} increases with Q_{tot}^{fm}, also ΔS_{tot} is a strictly decreasing function of ΔS^{adj}. Hence, the swing price is unique. □

Proof of Proposition 5. The second order terms in the expansion formulas given in Proposition A.5 are strictly negative. The result follows immediately. □

Proof of Proposition A.1. The proof proceeds along similar lines as the proof of Lemma B.3. The change in asset price ΔP_{tot} and the change in fund i's NAV $\Delta S_{tot,i}$, for $i = 1, 2$, are given by the solution of the system of Equations (A.1)–(A.3) with $\pi_i = 0$, and therefore, $R_{tot,i}^{fm} = 0$ and $\Delta Q_{tot,i}^{fm} = 0$ for $i = 1, 2$. A solution to these equations is given by the triplet $(\Delta S_{tot,1}, \Delta S_{tot,2}, \Delta P_{tot})$ defined by $\Delta S_{tot,1} = \Delta S_{tot,2} = \Delta P_{tot} = \Delta Z + \gamma\frac{E_1 + E_2}{1 - (\beta_1 + \beta_2)\gamma}$. □

Proof of Proposition A.2. The proof follows the same lines as the proof of Proposition 2. The change in asset price ΔP_{tot} and the change in fund i's NAV $\Delta S_{tot,i}$, for $i = 1, 2$, are given by the solution of the system of Equations (A.1)–(A.3). We rewrite the equations in the system (A.3) as $\Delta P_{tot} = g_P(\gamma, \Delta P_{tot}, \Delta S_{tot,1}, \Delta S_{tot,2})$, $\Delta S_{tot,1} = g_{S,1}(\gamma, \Delta P_{tot}, \Delta S_{tot,1}, \Delta S_{tot,2})$, and $\Delta S_{tot,2} = g_{S,2}(\gamma, \Delta P_{tot}, \Delta S_{tot,1}, \Delta S_{tot,2})$ for appropriately defined functions g_P, $g_{S,1}$, and $g_{S,2}$. Differentiating both sides of these equations with respect to γ and evaluating them at $\gamma = 0$ yields the coefficients of the expansions

$$\Delta P_{tot}(\gamma) = \Delta P_{tot}(0) + \gamma\frac{\partial\Delta P_{tot}(\gamma)}{\partial\gamma}\bigg|_{\gamma=0} + \frac{\gamma^2}{2}\frac{\partial^2\Delta P_{tot}(\gamma)}{\partial\gamma^2}\bigg|_{\gamma=0} + o(\gamma^2)$$

and

$$\Delta S_{tot,i}(\gamma) = \Delta S_{tot,i}(0) + \gamma\frac{\partial\Delta S_{tot,i}(\gamma)}{\partial\gamma}\bigg|_{\gamma=0} + o(\gamma),$$

for $i = 1, 2$. □

Proof of Proposition A.3. The proof follows the same lines as the proof of Proposition 4. The NAV that fund i's first movers receive is adjusted by an amount ΔS_i^{adj} for $i = 1, 2$. Hence, the change in asset price ΔP_{tot} and the change in fund i's NAV $\Delta S_{tot,i}$ for $i = 1, 2$ are given by the solution of the system of Equations (A.1)–(A.3), where the second line of the system (A.1) gets replaced by $-\Delta Q_{tot,i}^{fm} \times (P_0 + \Delta Z + \gamma(\Delta Q_{tot,1}^{fm} + \Delta Q_{tot,2}^{fm})) = R_{tot,i}^{fm}(S_{0,i} + \Delta Z + \Delta S_i^{adj})$ for $i = 1, 2$. It can be verified that $\Delta S_i^{adj} = \gamma(E_1^\pi + E_2^\pi)/(1 - (\beta_1 + \beta_2)\gamma)$, $\Delta Q_{tot,i}^{fm} = E_i^\pi/(1 - (\beta_1 + \beta_2)\gamma)$, for $i = 1, 2$, and $\Delta S_{tot,1} = \Delta S_{tot,2} = \Delta P_{tot} = \Delta Z + \gamma(E_1 + E_2)/(1 - (\beta_1 + \beta_2)\gamma)$ solve the system of equations. It follows that $\Delta S_{both}^{sw} := \gamma(E_1^\pi + E_2^\pi)/(1 - (\beta_1 + \beta_2)\gamma)$ is the swing adjustment. It also satisfies $\Delta S_{both}^{sw} = -\gamma(R_{tot,1}^{fm} + R_{tot,2}^{fm})$. □

Proof of Proposition A.4. The proof follows the same lines as the proofs of Propositions 2 and A.2. By assumption, only fund 1 has first-mover investors. Hence, the change in asset price ΔP_{tot} and the change in fund i's NAV $\Delta S_{tot,i}$ for $i = 1, 2$ are given by the solution to the system of Equations (A.1)–(A.3) with $\pi_2 = 0$. Therefore, $R_{tot,2}^{fm} = 0$ and $\Delta Q_{tot,2}^{fm} = 0$. The expansions for ΔP_{tot}, $\Delta S_{tot,1}$, and $\Delta S_{tot,2}$ can be computed as in Proposition A.2. □

Proof of Proposition A.5. The NAV that fund 2's first movers receive is adjusted by an amount ΔS^{adj}. Fund 1's NAV does not get adjusted. Hence, the change in asset price ΔP_{tot} and the change in fund i's NAV $\Delta S_{tot,i}$ for $i = 1, 2$ are given by the solution of the system of Equations (A.1)–(A.3), where the second line of the system (A.1) gets replaced by the equations

$$-\Delta Q_{tot,1}^{fm} \times \left(P_0 + \Delta Z + \gamma\left(\Delta Q_{tot,1}^{fm} + \Delta Q_{tot,2}^{fm}\right)\right) = R_{tot,1}^{fm}(S_{0,1} + \Delta Z),$$

$$-\Delta Q_{tot,2}^{fm} \times \left(P_0 + \Delta Z + \gamma\left(\Delta Q_{tot,1}^{fm} + \Delta Q_{tot,2}^{fm}\right)\right)$$
$$= R_{tot,2}^{fm}\left(S_{0,2} + \Delta Z + \Delta S^{adj}\right).$$

We rewrite the equations in the system (A.3) as

$$\Delta P_{tot} = g_P\left(\gamma, \Delta S^{adj}, \Delta P_{tot}, \Delta S_{tot,1}, \Delta S_{tot,2}\right),$$
$$\Delta S_{tot,1} = g_{S,1}\left(\gamma, \Delta S^{adj}, \Delta P_{tot}, \Delta S_{tot,1}, \Delta S_{tot,2}\right), \quad \text{(B.5)}$$
$$\Delta S_{tot,2} = g_{S,2}\left(\gamma, \Delta S^{adj}, \Delta P_{tot}, \Delta S_{tot,1}, \Delta S_{tot,2}\right).$$

Next, we compute the expansion of

$$\Delta S_{glob}^{sw} = \Delta S_{glob}^{sw}(0) + \gamma\frac{\partial\Delta S_{glob}^{sw}(\gamma)}{\partial\gamma}\bigg|_{\gamma=0} + \frac{\gamma}{2}\frac{\partial^2\Delta S_{glob}^{sw}(\gamma)}{\partial\gamma^2}\bigg|_{\gamma=0} + o(\gamma^2).$$

Proposition A.1 states that, for $\pi_1 = \pi_2 = 0$, the change in NAV of fund 2 is

$$\Delta S_{tot,2} = \Delta Z + \gamma(\beta_1 + \beta_2)\Delta Z + \gamma^2(\beta_1 + \beta_2)^2\Delta Z + \cdots. \quad \text{(B.6)}$$

By definition, the adjustment ΔS_{glob}^{sw} is the one that fund 2 needs to apply to guarantee that $\Delta S_{tot,2}$ admits the expansion (B.6). Hence, if $\Delta S^{adj} = \Delta S_{glob}^{sw}$, then

$$\Delta S_{tot,2}(0) = \Delta Z, \frac{\partial\Delta S_{tot,2}(\gamma)}{\partial\gamma}\bigg|_{\gamma=0} = (\beta_1 + \beta_2)\Delta Z, \frac{\partial^2\Delta S_{tot,2}(\gamma)}{\partial\gamma^2}\bigg|_{\gamma=0}$$
$$= 2(\beta_1 + \beta_2)^2\Delta Z.$$

By letting γ decrease to zero on both sides of each equation in (B.5) and using $\Delta S_{tot,2}(0) = \Delta Z$, we get that $\Delta P_{tot}(0) = \Delta S_{tot,1}(0) = \Delta Z$ and $\Delta S^{adj} = 0$. By differentiating both sides of each equation in (B.5) with respect to γ, evaluating the derivatives at $\gamma = 0$, and using the fact that

$$\frac{\partial\Delta S_{tot,2}(\gamma)}{\partial\gamma}\bigg|_{\gamma=0} = (\beta_1 + \beta_2)\Delta Z,$$

we obtain the values for

$$\frac{\partial\Delta P_{tot}(\gamma)}{\partial\gamma}\bigg|_{\gamma=0}, \quad \frac{\partial\Delta S_{tot,1}(\gamma)}{\partial\gamma}\bigg|_{\gamma=0} \quad \text{and} \quad \frac{\partial\Delta S^{adj}(\gamma)}{\partial\gamma}\bigg|_{\gamma=0}.$$

By differentiating the same equations again, we can compute

$$\frac{\partial^2\Delta S^{adj}(\gamma)}{\partial\gamma^2}\bigg|_{\gamma=0}.$$

The resulting values for

$$\Delta S_{tot,2}(0), \quad \frac{\partial \Delta S_{tot,2}(\gamma)}{\partial \gamma}\bigg|_{\gamma=0} \quad \text{and} \quad \frac{\partial^2 \Delta S_{tot,2}(\gamma)}{\partial \gamma^2}\bigg|_{\gamma=0}$$

are the coefficients in the expansion of ΔS^{sw}_{glob}.

If $\pi_1 > 0$ and $\pi_2 = 0$, then fund 2's change in NAV is given by (A.6). Because the adjustment ΔS^{sw}_{loc} is such that $\Delta S_{tot,2}$ satisfies (A.6), in this case, $\Delta S_{tot,2}(0) = \Delta Z$, $\frac{\partial \Delta S_{tot,2}(\gamma)}{\partial \gamma}\big|_{\gamma=0} = (\beta_1 + \beta_2)\Delta Z$, and

$$\frac{\partial^2 \Delta S_{tot,2}(\gamma)}{\partial \gamma^2}\bigg|_{\gamma=0} = 2\left((\beta_1 + \beta_2)^2 \Delta Z - (E_1^\pi)^2 \frac{\text{Rem}_1 + \beta_1(P_0 + Z)}{(P_0 + \Delta Z)\text{Rem}_1^\pi}\right).$$

The coefficients in the expansion of ΔS^{sw}_{loc} can now be found by repeating the procedure used for ΔS^{sw}_{glob}. □

Endnotes

[1] See the Flow of Funds Accounts, Z.1 Financial Accounts of the United States, published by the Federal Reserve Board. Compare the table L.122 for March 2006, reporting that the total value of financial assets held by mutual funds in 2005 is $6.05 trillion, with the table for March 2016, which indicates that the total value of assets held by mutual funds in 2015 is $12.9 trillion.

[2] See table L.213, respectively table L.212, in the Flow of Funds Accounts, Z.1 Financial Accounts of the United States, published by the Federal Reserve Board in September 2017, respectively in September 1996.

[3] The share price at which an investor is repaid is set at the end of the day. The payment itself typically occurs within one to three days and up to a maximum of seven days under U.S. regulations.

[4] We use the term "illiquid" to refer to what might more precisely be referred to as "less liquid" assets. U.S. mutual funds are barred from holding more than 15% of their assets in illiquid securities, but they may hold a greater portion in corporate bonds and other less liquid securities.

[5] Cetorelli et al. (2016a, b) and Fricke and Fricke (2017) quantify the impact of mutual fund fire sales on asset prices and conclude that the funds' aggregate vulnerability of U.S. open-end mutual funds is small (compared with banks). Their analysis, however, does not account for the first-mover advantage.

[6] In explaining its liquidity rules, the SEC frequently refers to "reducing the risk that funds would be unable to meet redemption and other legal obligations." See, for example, *Federal Register*, November 18, 2016, vol. 81, no. 223, p. 82158 and p. 82235.

[7] Any model of fire sales relies on some friction that constrains or deters arbitrageurs from stepping in to buy when an asset price falls below fundamental value. In our setting, the potential buyers include fund investors who choose not to sell. Our model abstracts from the underlying source of market illiquidity and captures these effects in reduced form through the parameter γ and the actions of the first movers. In other words, γ measures price impact net of any buying by bargain shoppers, and the liquidation by first movers anticipates the extent to which second movers sell as the share price falls.

[8] Throughout the paper, we work in an environment in which $P_0 \geq 0$, $-\Delta Q \leq Q_0$, $\Delta R \leq N_0$, and $-\Delta S \leq S_0$. Violation of these conditions implies the failure of the fund as defined in Section 4.1. We provide sufficient conditions for the existence of such an environment in endnote 14.

[9] The linear relation between investors' redemptions and fund performance is driven by the empirical literature (e.g., Goldstein et al. 2017). Our model can also be used to study the effect of positive market shocks and capital inflow. In this case, the parameter β may

depend on the sign of ΔS. In fact, sensitivity of redemptions to past performance is not symmetric: it tends to be convex (see, e.g., Ippolito 1992) for funds specialized in more liquid assets and concave (see, e.g., Goldstein et al. 2017) for funds specializing in more illiquid assets. If investors are rationally inattentive, they may respond more strongly to large changes than small changes; alternative behavioral assumptions could lead to interesting extensions of our model.

[10] We refer to the SEC Rule 22c-1 for more regulatory details and Investment Company Institute (2018) for industry practices.

[11] The forward-looking behavior of some investors provides a possible explanation for the redemption patterns observed empirically in Chen et al. (2010): first movers anticipate that asset liquidation worsens the fund performance, especially if the fund manages illiquid assets; therefore, the amount of shares they redeem grows with the illiquidity of the underlying asset (see Section 4.1 and the left panel in Figure 3).

[12] In revising rules for money market funds, the SEC wrote that "the first investors to redeem from a stable value money market fund that is experiencing a decline in its NAV benefit from a 'first-mover advantage,'" and also comments that "We further believe history shows that, to date, institutional investors have been significantly more likely than retail investors to act on this incentive." See *Federal Register*, August 14, 2014, vol. 79, no. 157, p. 47774.

[13] Here and throughout, we make the simplifying assumption that the fund does not sell assets on the day of the shock in response to first-mover redemptions. This assumption holds, in particular, if the redemptions are submitted late in the day. As long as the amount sold by the fund on the first day is less than the amount required to meet redemptions on the first day, the first movers impose liquidation costs on other investors.

[14] Notice that additional conditions are required to guarantee that $Q_n^{sm} \geq 0$ and $P_n^{sm} \geq 0$ for all n. If price impact is linear, that is, $\varphi(x) = \gamma x$ and $\pi = 0$, these conditions are $P_0 \geq -\frac{\Delta Z}{1-\gamma\beta}$ and $Q_0 \geq -\frac{\beta\Delta Z}{1-\gamma\beta}$, which should be viewed as sensible bounds on the initial shock ΔZ.

[15] Chen et al. (2010) estimate a smaller interaction between performance sensitivity and asset illiquidity in institutionally oriented funds compared with retail-oriented funds, but the difference is not statistically significant. They note that, if a single investor owns a large fraction of the fund's shares, this investor would bear most of the cost of asset liquidation and therefore has less of an incentive to sell early. We have in mind a setting with a large number of investors, each of which owns a small fraction of the fund's shares, so each investor is primarily affected by the redemptions of other investors.

[16] Such a condition is equivalent to assuming that the matrix in equation 4 in Greenwood et al. (2015) (or the systemicness matrix defined in equation 2.2 of Capponi and Larsson 2015) has spectral radius smaller than one. In economic terms, this means that a round of deleveraging causes another round of deleveraging that is smaller than the previous one. In particular, the condition for the convergence of this liquidation procedure is independent of the initial market shock ΔZ.

[17] A value of $\gamma = 10^{-8}$ should be interpreted to mean that selling $1 million of the asset has a price impact of 1%.

[18] The fund may decide to suspend redemptions if it foresees that they would be insufficient to repay exiting investors as happened in the case of the Third Avenue Focused Credit fund discussed in the introduction.

[19] We have assumed a fixed γ for tractability. If a fund were to sell its more liquid assets first, γ would increase as the fund sold more assets, further amplifying the effects in the figures.

[20] Stale prices for assets held by a mutual fund distort the fund's NAV to the benefit of certain investors as in Zitzewitz (2006). Swing pricing can be seen as correcting soon-to-be-stale prices.

[21] SEC rules require a mutual fund to implement a customer identification program, so at a minimum, a fund would be able to distinguish retail and institutional accounts.

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