Design and Analysis of Algorithms

Sorting

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The problem of sorting

多考虑升序

Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Example:

Input: 8 2 4 9 3 6

Output: 2 3 4 6 8 9

Overview

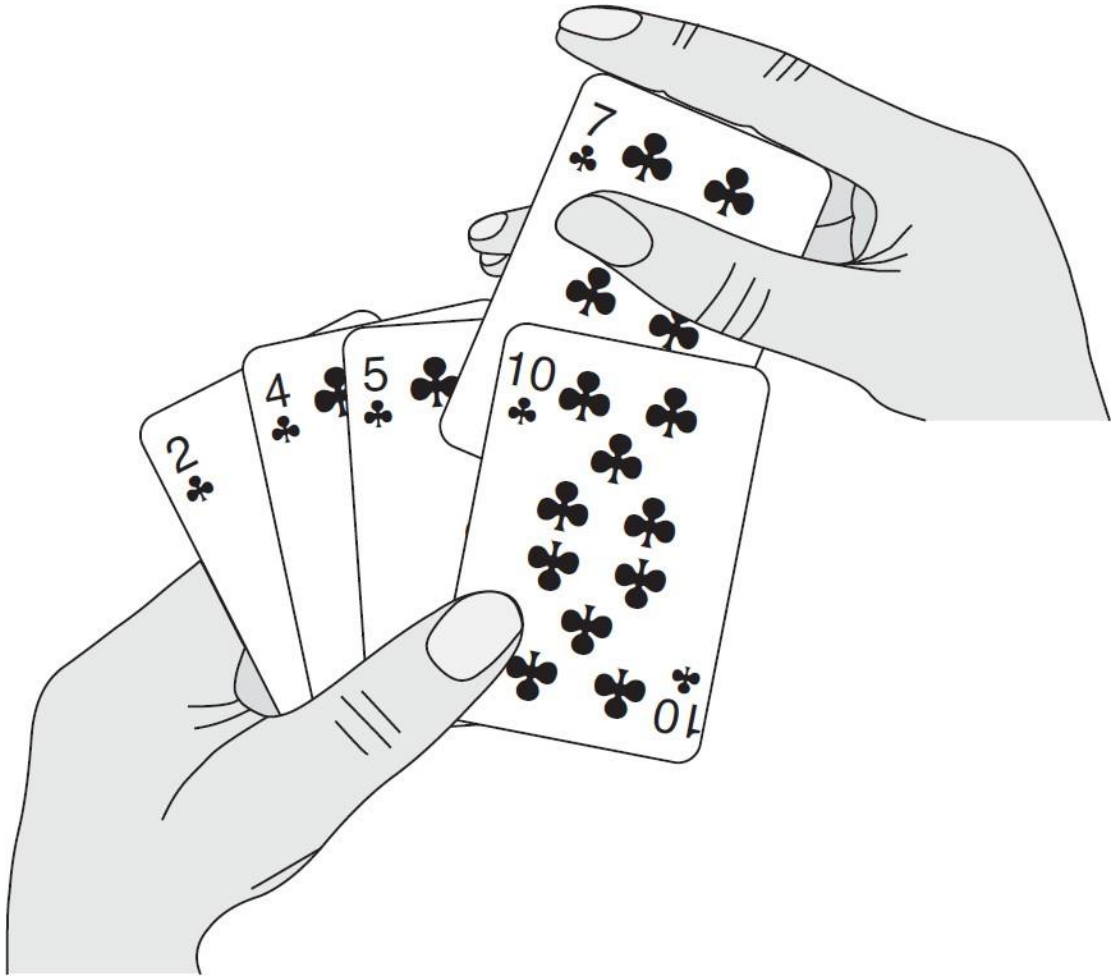
- ***Goals:***

  Start using frameworks for describing and analyzing algorithms.

  - See how to describe algorithms in pseudocode.
  - Begin using asymptotic notation to express running-time analysis.
  - Learn the technique of "divide and conquer" in the context of merge-sort.
  - Examine two algorithms for sorting: insertion-sort and merge-sort.

Insertion Sort

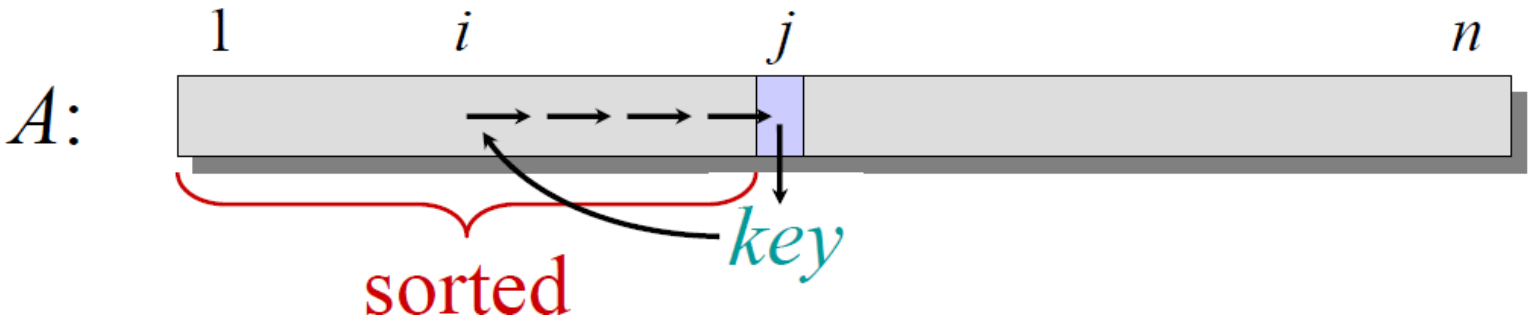
- Sorting a hand of cards using insertion sort.



Insertion sort

"pseudocode"

```
INSERTION-SORT (A, n)        ▷ A[1 . . n]
    for j ← 2 to n
        do key ← A[j]
           i ← j – 1
           while i > 0 and A[i] > key
               do A[i+1] ← A[i]
                  i ← i – 1
           A[i+1] = key
```



Example of insertion sort

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Example of insertion sort

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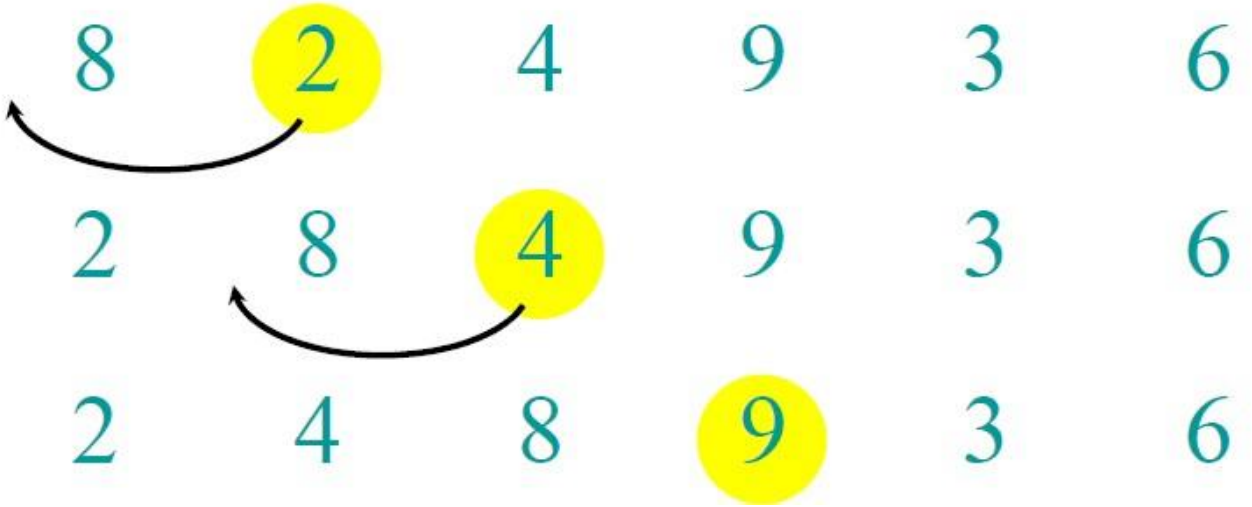

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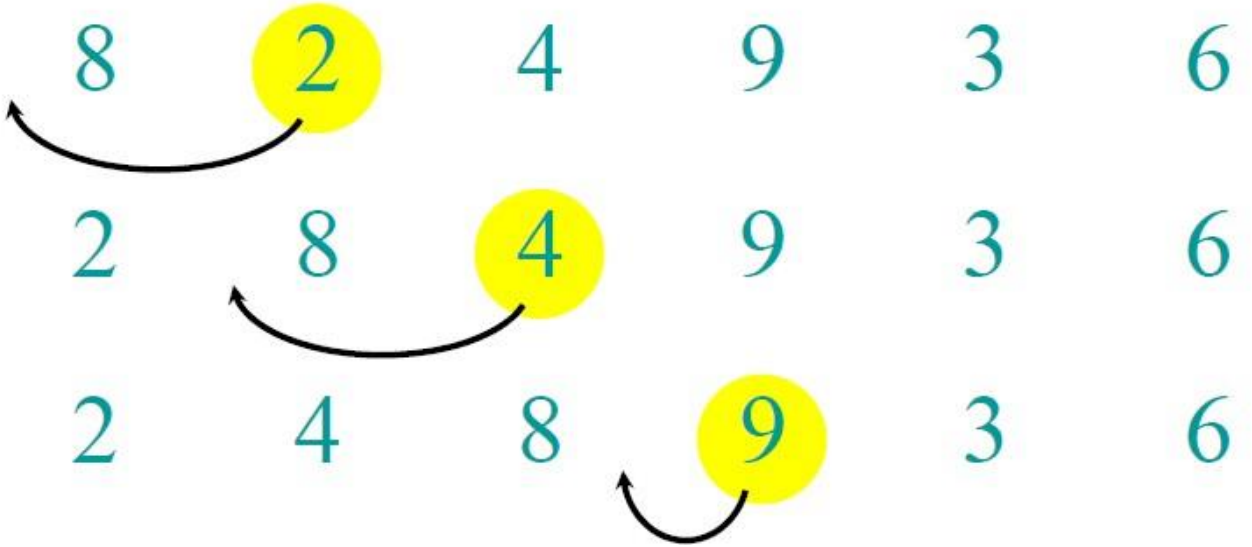

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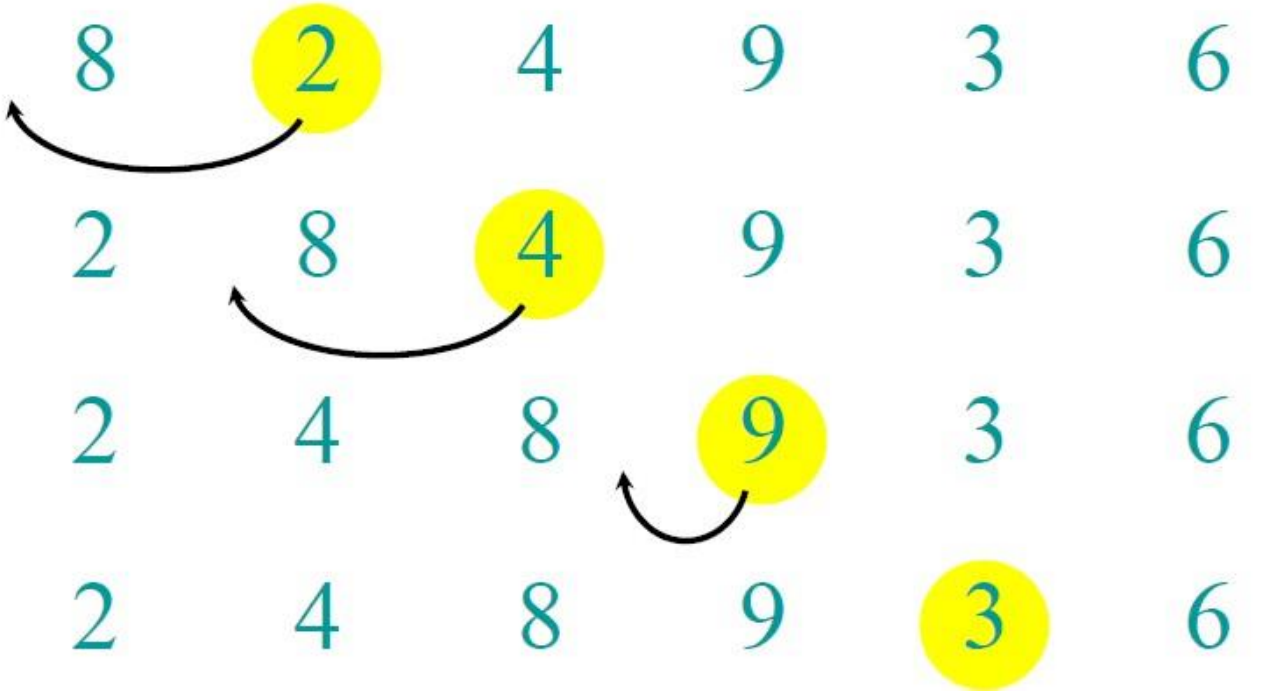
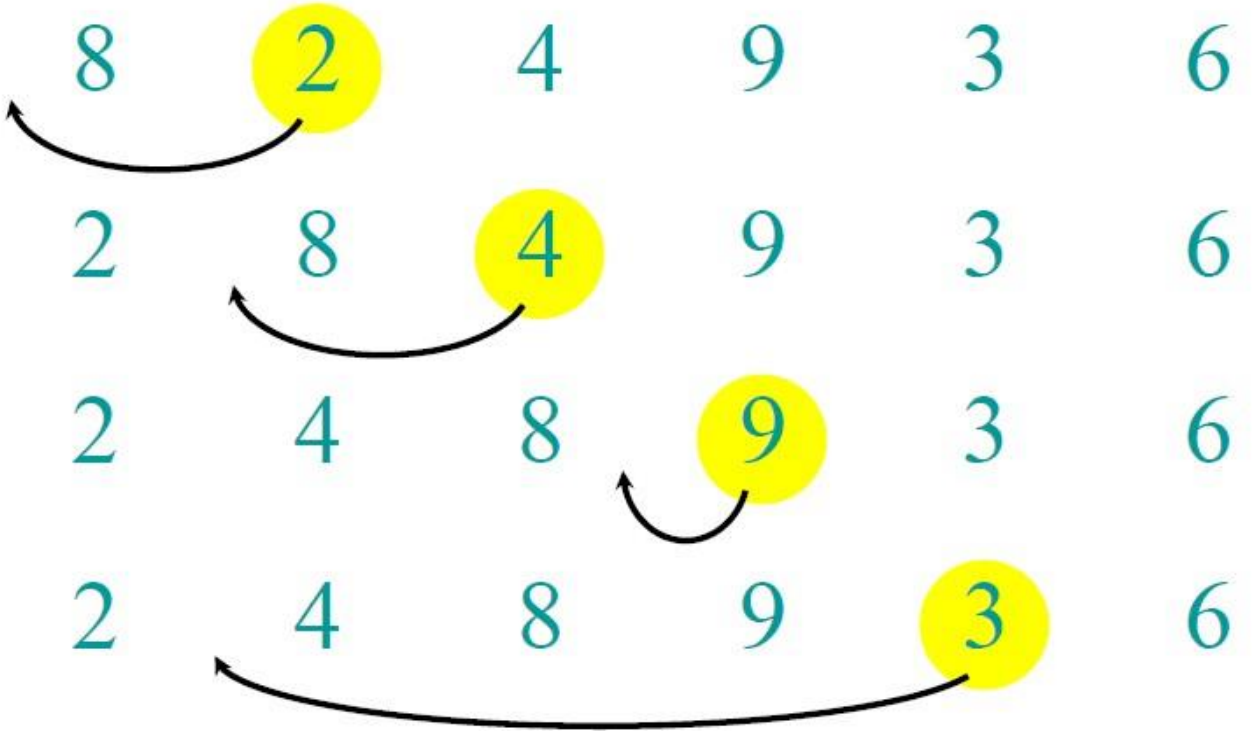
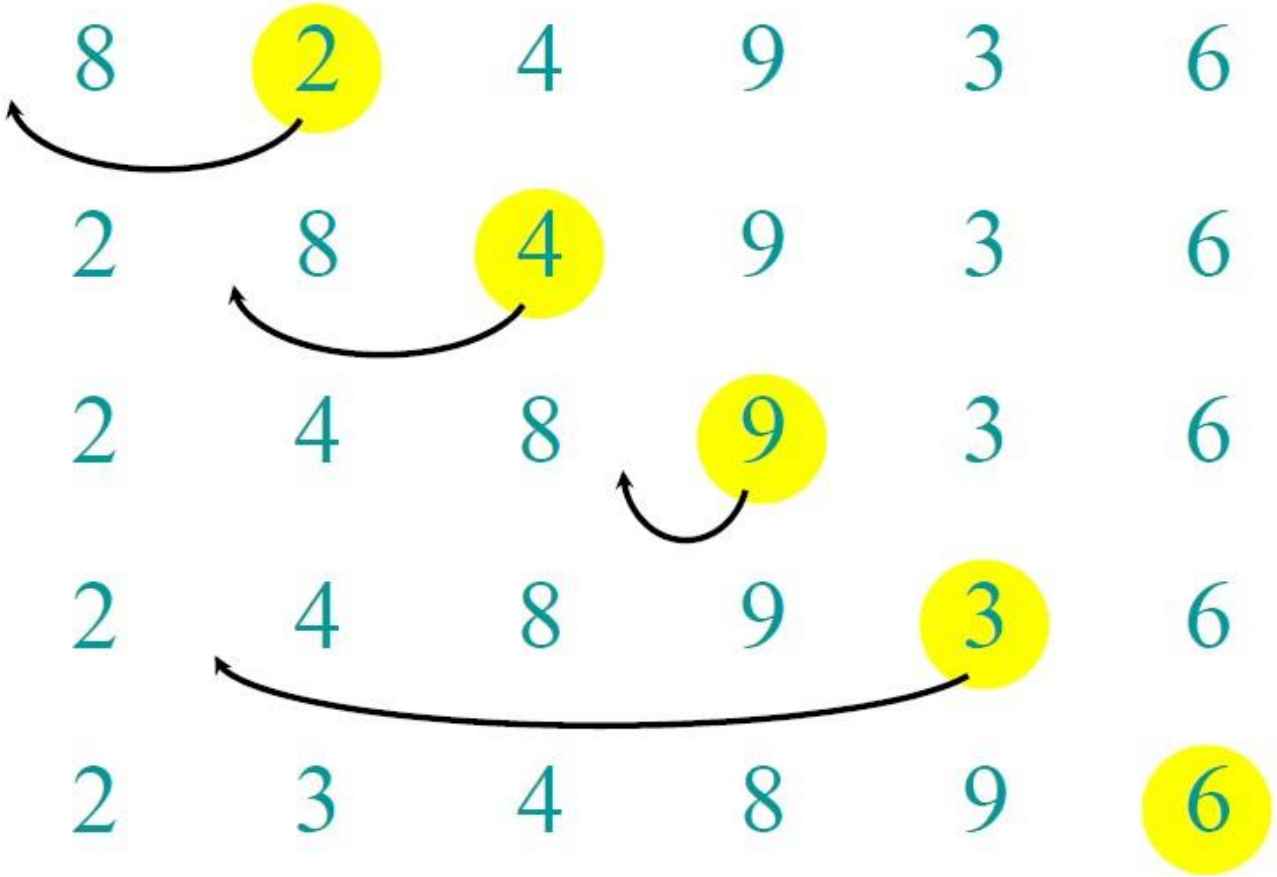


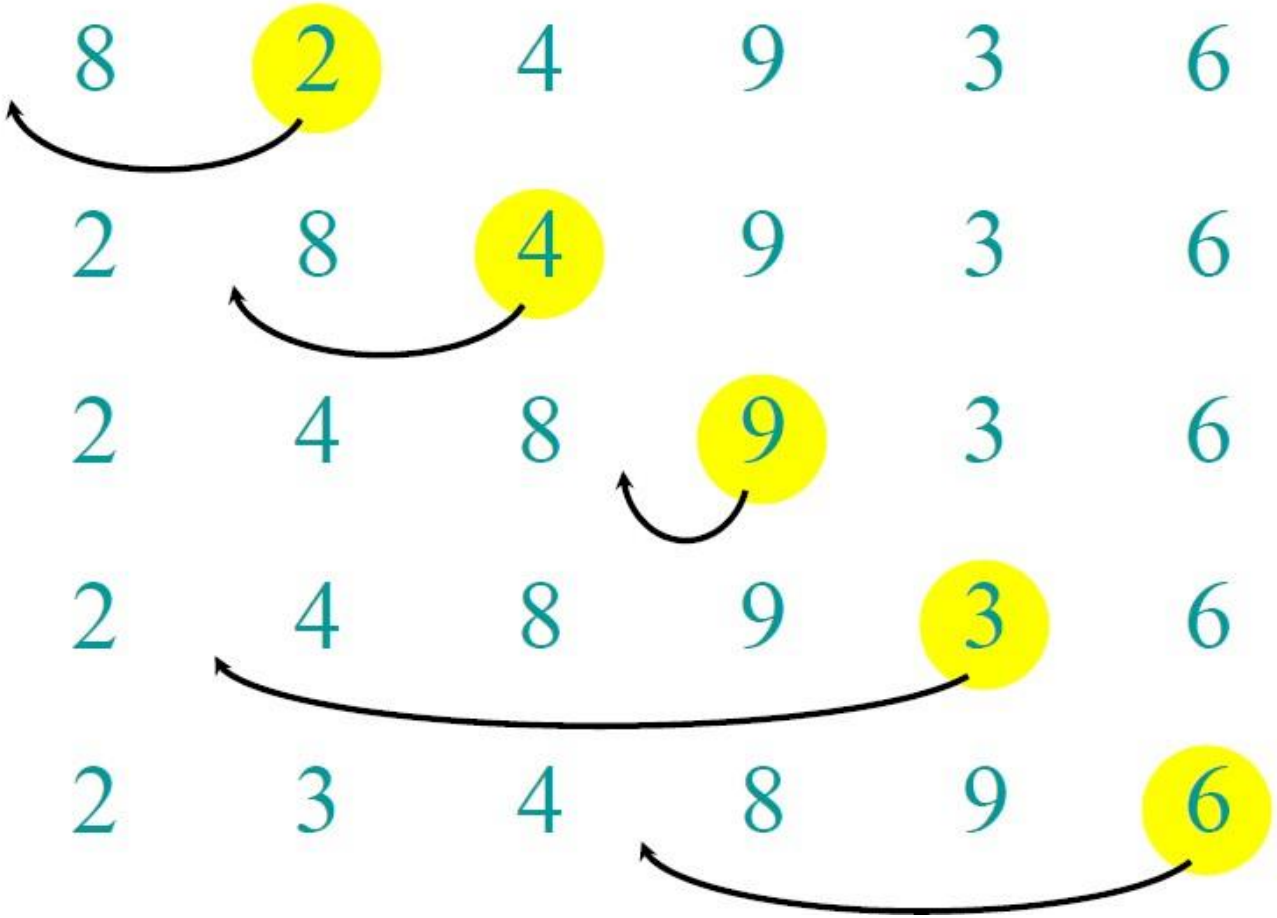
Example of insertion sort



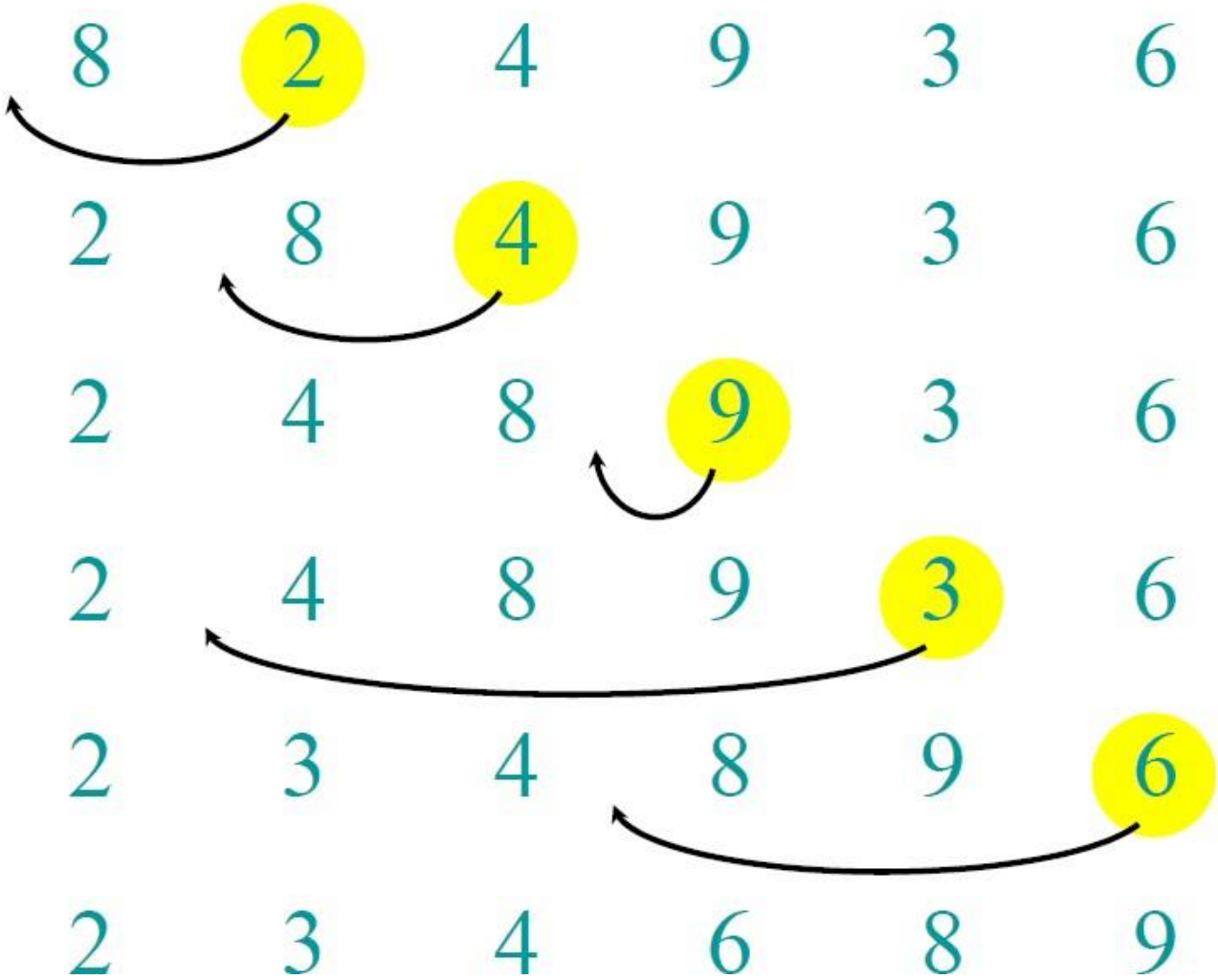
Example of insertion sort



Example of insertion sort

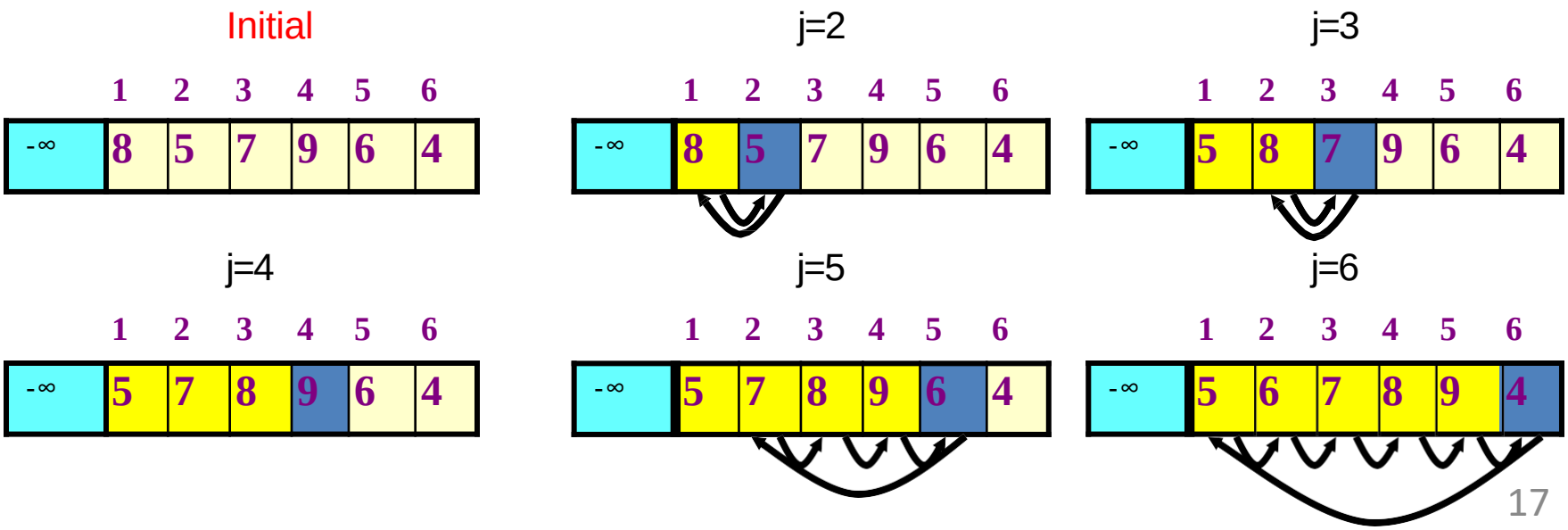


Example of insertion sort



Insertion Sort (another example)

```
INSERTION-SORT (A, n) ▷ A[1 . . n]
1  for j ← 2 to n
2       do key ← A[j ]
3          i ← j – 1
4          while i > 0 and A[i] > key
5                do A[i + 1] ← A[i]
6                   i ← i – 1
7          A[i + 1] = key
```



Running time

- The running time depends on the input: an already sorted sequence is easier to sort.

- Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.

- Generally, we seek upper bounds on the running time, because everybody likes a guarantee.

Θ-notation

Math:

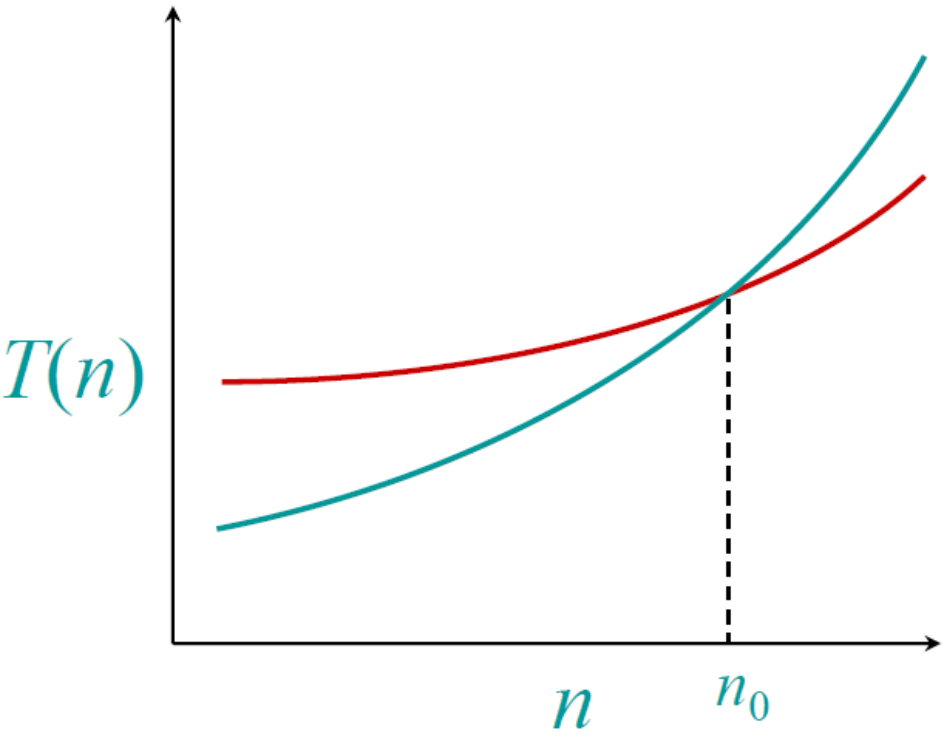
$\Theta(g(n)) = \{ f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0 \}$

Engineering:

- Drop low-order terms; ignore leading constants.
- Example: $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

Asymptotic performance

When n gets large enough, a $\Theta(n^2)$ algorithm *always* beats a $\Theta(n^3)$ algorithm.



- We shouldn't ignore asymptotically slower algorithms, however.
- Real-world design situations often call for a careful balancing of engineering objectives.
- Asymptotic analysis is a useful tool to help to structure our thinking.

Analysis of INSERTION-SORT

	INSERTION-SORT (A, n) ▷ A[1 . . n]	cost	times
1	for j ← 2 to n	c_1	n - 1
2	do key ← A[j]		
3	i ← j – 1		
4	while i > 0 and A[i] > key		
5	do A[i + 1] ← A[i]		
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Analysis of INSERTION-SORT

	INSERTION-SORT (A, n) ▷ A[1 . . n]	cost	times
1	for j ← 2 to n	c_1	n - 1
2	do key ← A[j]	c_2	n - 1
3	i ← j – 1		
4	while i > 0 and A[i] > key		
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7	A[i + 1] = key		

Analysis of INSERTION-SORT

	INSERTION-SORT (A, n) ⊳ A[1 . . n]	cost	times
1	for j ← 2 to n	c_1	n - 1
2	do key ← A[j]	c_2	n - 1
3	i ← j − 1	c_3	n - 1
4	while i > 0 and A[i] > key		
5	do A[i + 1] ← A[i]		
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3	i ← j – 1	c_3	n –1
4	while i > 0 and A[i] > key	c_4	$\sum_{j=2}^{n} t_j$
5	do A[i + 1] ← A[i]		
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4	while i > 0 and A[i] > key	c_4	$\sum_{j=2}^{n} t_j$
5	do A[i + 1] ← A[i]	c_5	$\sum_{j=2}^{n} (t_j - 1)$
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5	do A[i + 1] ← A[i]	c_5	$\sum_{j=2}^{n} (t_j - 1)$
6	i ← i – 1	c_6	$\sum_{j=2}^{n} (t_j - 1)$
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Analysis of INSERTION-SORT

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7	A[i + 1] = key	c_7	n –1

Let $T(n)$ = running time of **INSERTION-SORT**.

$$T(n) = c_1 (n-1) + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7(n-1)$$

	INSERTION-SORT (A, n) ▷ A[1 . . n]	cost	times
1	for j ← 2 to n	c_1	n-1
2	do key ← A[j]	c_2	$n-1$
3	i ← j − 1	c_3	$n-1$
4	while i > 0 and A[i] > key	c_4	$\sum_{j=2}^{n} t_j$
5	do A[i + 1] ← A[i]	c_5	$\sum_{j=2}^{n} (t_j - 1)$
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7	A[i + 1] = key	c_7	$n-1$

$$T(n) = c_1 (n-1) + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7(n-1)$$

Best-case: The array is already sorted.

- Always find that **A[i] ≤ key** upon the first time the while loop test is run (when i = j − 1).
- All t_j are 1.
- Running time is

$$T(n) = c_1(n-1) + c_2(n-1) + c_3(n-1) + c_4(n-1) + c_7(n-1)$$
$$= (c_1 + c_2 + c_3 + c_4 + c_7)n - (c_1 + c_2 + c_3 + c_4 + c_7)$$

- Can express $T(n)$ as $an + b$ for constants a and b (that depend on the statement costs c_i) ⇒ $T(n)$ is a linear function of n. ⇒ $T(n) = \Theta(n)$

	INSERTION-SORT (A, n) ▷ A[1 . . n]	cost	times
1	for j ← 2 to n	c_1	n-1
2	do key ← A[j]	c_2	n-1
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7	A[i + 1] = key	c_7	$n - 1$

$$T(n) = c_1(n\text{-}1) + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7(n-1)$$

Worst-case: The array is in reverse sorted order.

- Always find that ***A[i] > key*** in while loop test.

	INSERTION-SORT (A, n) ▷ A[1 . . n]	cost	times
1	for j ← 2 to n	c_1	n-1
2	do key ← A[j]	c_2	n-1
3	i ← j – 1	c_3	n-1
4	while i > 0 and A[i] > key	c_4	$\sum_{j=2}^{n} t_j$
5	do A[i + 1] ← A[i]	c_5	$\sum_{j=2}^{n} (t_j - 1)$
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7	A[i + 1] = key	c_7	$n - 1$

$$T(n) = c_1(n\text{-}1) + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7(n-1)$$

Worst-case: The array is in reverse sorted order.

- Have to compare ***key*** with all elements to the left of the *j*th position ⇒ compare with *j* – *1* elements.

	INSERTION-SORT (A, n) ▷ A[1 . . n]	cost	times
1	for j ← 2 to n	c_1	n-1
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$$T(n) = c_1(n-1) + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7(n-1)$$

Worst-case: The array is in reverse sorted order.

- Since the while loop exits because i reaches 0, there's one additional test after the $j - 1$ tests $\Rightarrow t_j = j$.

	INSERTION-SORT (A, n) ▷ A[1 . . n]	cost	times
1	for j ← 2 to n	c_1	n-1
2	do key ← A[j]	c_2	n-1
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7	A[i + 1] = key	c_7	$n - 1$

$$T(n) = c_1(n-1) + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7(n-1)$$

Worst-case: The array is in reverse sorted order.

- $\sum_{j=2}^{n}(t_j - 1) = \sum_{j=2}^{n}(j - 1)$

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$$T(n) = c_1(n-1) + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7(n-1)$$

Worst-case: The array is in reverse sorted order. Running time:

- $T(n) = c_1(n-1) + c_2(n-1) + c_3(n-1) + c_4\left(\frac{n(n+1)}{2} - 1\right) + c_5\left(\frac{n(n-1)}{2} - 1\right) + c_6\left(\frac{n(n-1)}{2} - 1\right) + c_7(n-1)$

$$= \underbrace{\left(\frac{c_4}{2} + \frac{c_5}{2} + \frac{c_6}{2}\right)}_{a} n^2 + \underbrace{\left(c_1 + c_2 + c_3 - \left(\frac{c_4}{2} + \frac{c_5}{2} + \frac{c_6}{2}\right) + c_7\right)}_{b} n \underbrace{- (c_1 + c_3 + c_4 + c_7)}_{c}$$

	INSERTION-SORT (A, n) ▷ A[1 . . n]	cost	times
1	for j ← 2 to n	c_1	n-1
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$$T(n) = c_1(n-1) + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7(n-1)$$

Worst-case: The array is in reverse sorted order.

- *Can express* $T(n)$ *as* $an^2 + bn + c$ *for constants* a, b, c
- $T(n)$ *is a quadratic function of* n. $\Rightarrow T(n) = \Theta(n^2)$

Order of Growth

We will only consider order of growth of running time:

- We can ignore the lower-order terms, since they are relatively insignificant for very large n.
- We can also ignore leading term's constant coefficients, since they are not as important for the rate of growth in computational efficiency for very large n.
- For the insertion-sort algorithm, we just said that best case was linear to *n* and worst/average case quadratic to *n*.

Designing Algorithms

- **We discussed insertion sort**

  - Can we design better than n^2 sorting algorithms?

  - We will do so using one of the most powerful algorithm design techniques.

Divide-and-Conquer

- **To solve problem P:**
  - Divide P into smaller problems P_1, P_2, …, P_k.
  - Conquer by solving the (smaller) subproblems recursively.
  - Combine the solutions to P_1, P_2, …, P_k into the solution for P.

Merge-Sort Algorithm

- Using divide-and-conquer, we can obtain the ***Merge-Sort*** algorithm
  - Divide: Divide the n elements into two subsequences of n/2 elements each.
  - Conquer: Sort the two subsequences recursively.
  - Combine: Merge the two sorted subsequences to produce the sorted answer.

Merge-Sort (A, p, r)

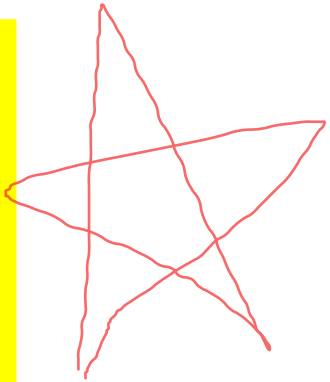
- **INPUT: a sequence of n numbers stored in array A**

- **OUTPUT: an ordered sequence of n numbers**

```
MERGE-SORT (A, p, r)
1   if p < r
2       then q ← ⌊(p + r) / 2⌋
3            MERGE-SORT(A, p, q)
4            MERGE-SORT(A, q + 1, r)
5            MERGE(A, p, q, r)
```

Merge (A, p, q, r)

```
MERGE (A, p, q, r)
1   n1 ← q – p + 1
2   n2 ← r – q
3   create arrays L[1..n1 + 1] and R[1..n2 + 1]
4   for i ← 1 to n1
5         do L[i] ← A[p + i - 1]
6   for j ← 1 to n2
7         do R[j] ← A[q + j]
8   L[n1 + 1] ← ∞
9   L[n2 + 1] ← ∞
10  i ← 1
11  j ← 1
12  for k ← p to r
13        do if L[i] ≤ R[j]
14              then A[k] ← L[i]
15                   i ← i + 1
16              else A[k] ← R[j]
17                   j ← j + 1
```



20	12
13	11
7	9
2	1

Merging two sorted arrays

20	12
13	11
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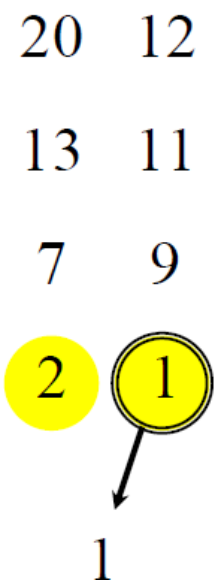
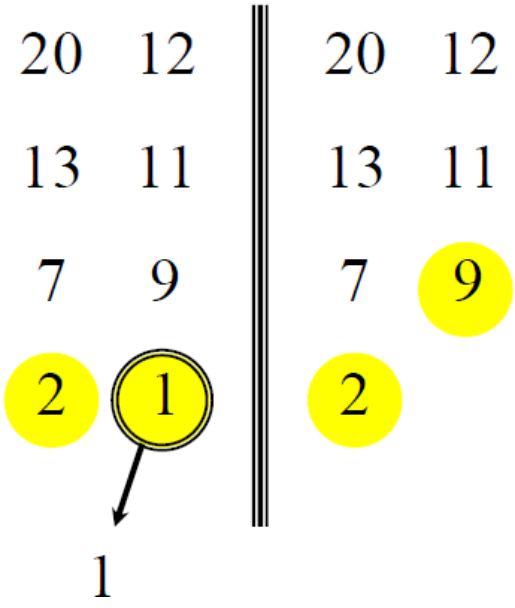
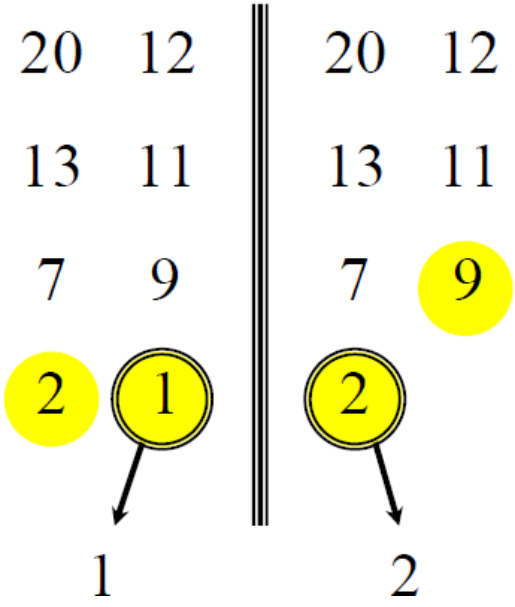
Merging two sorted arrays



Merging two sorted arrays



Merging two sorted arrays



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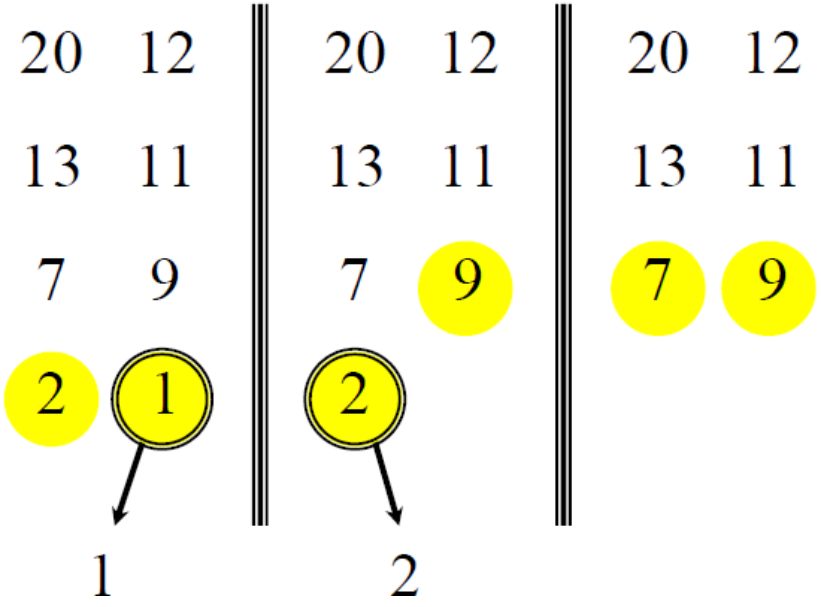

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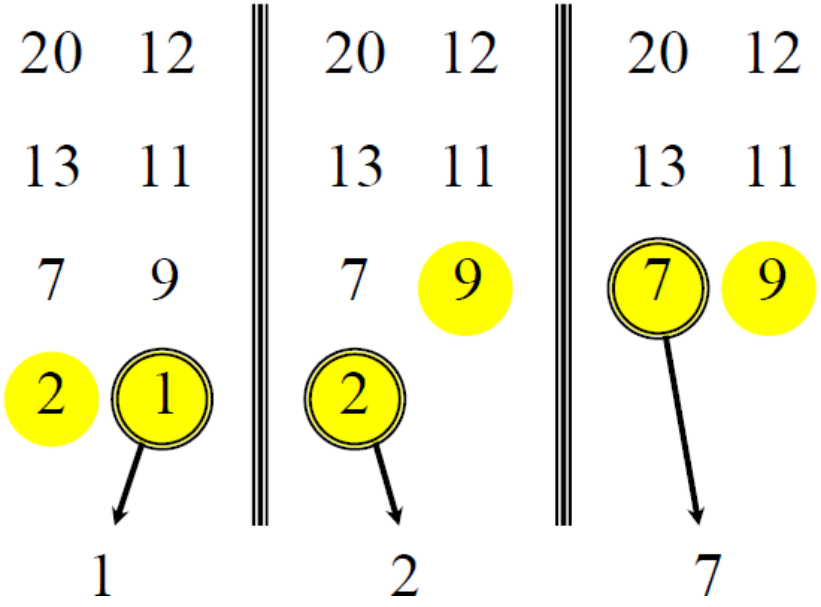

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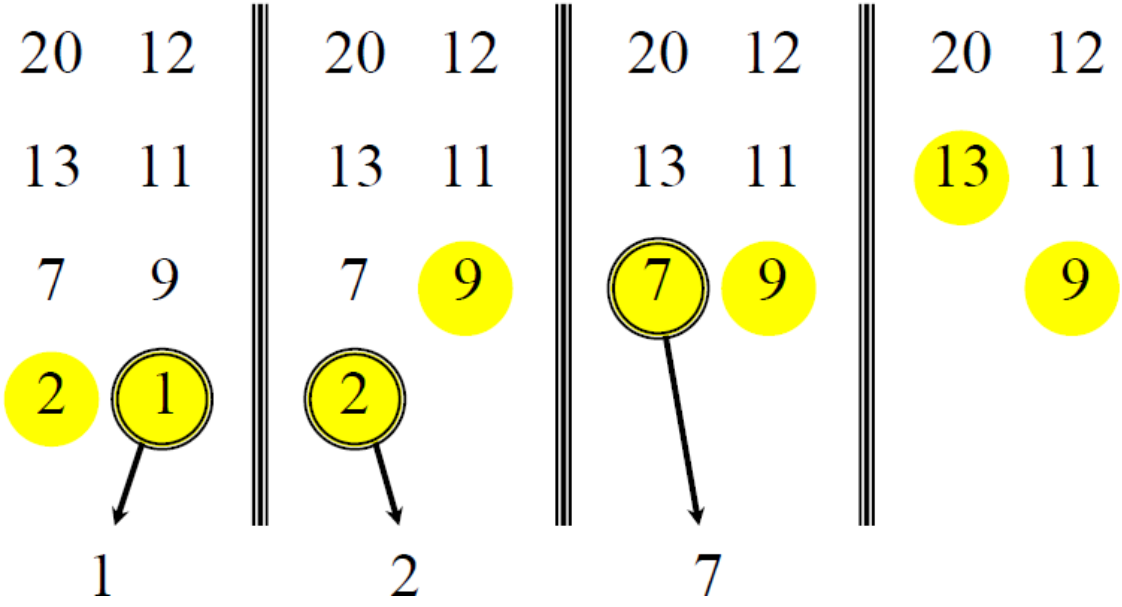

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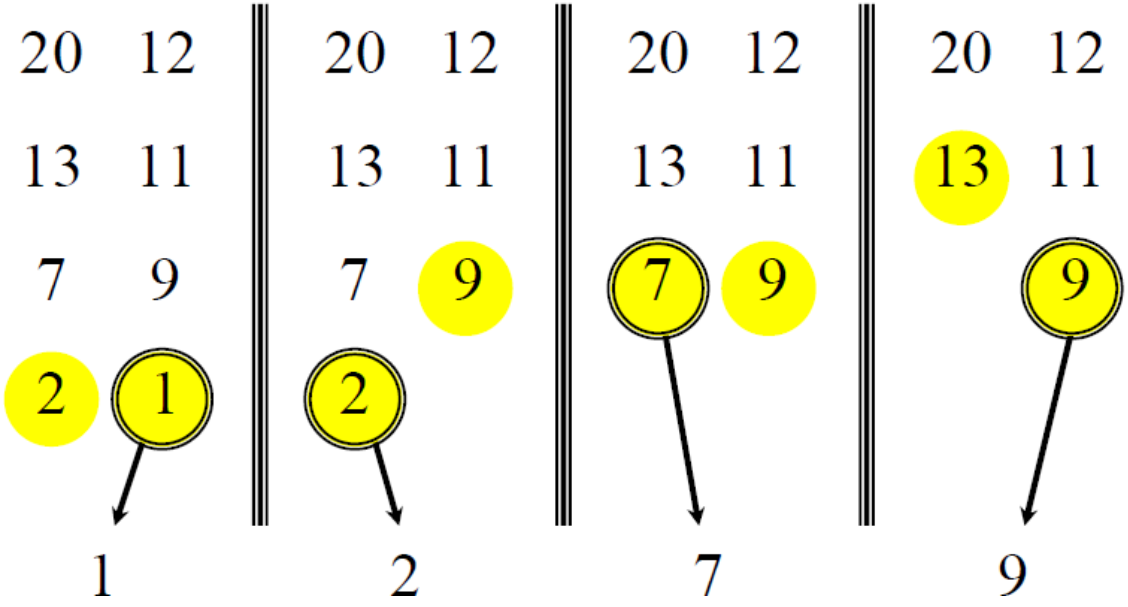

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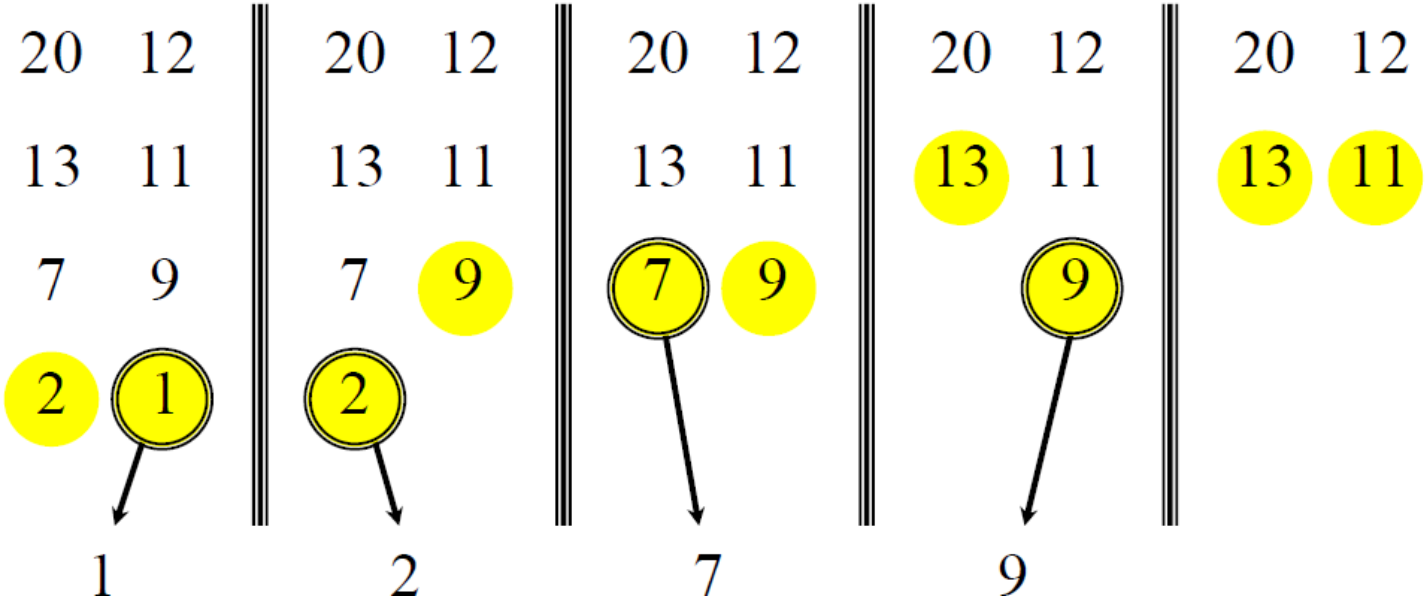
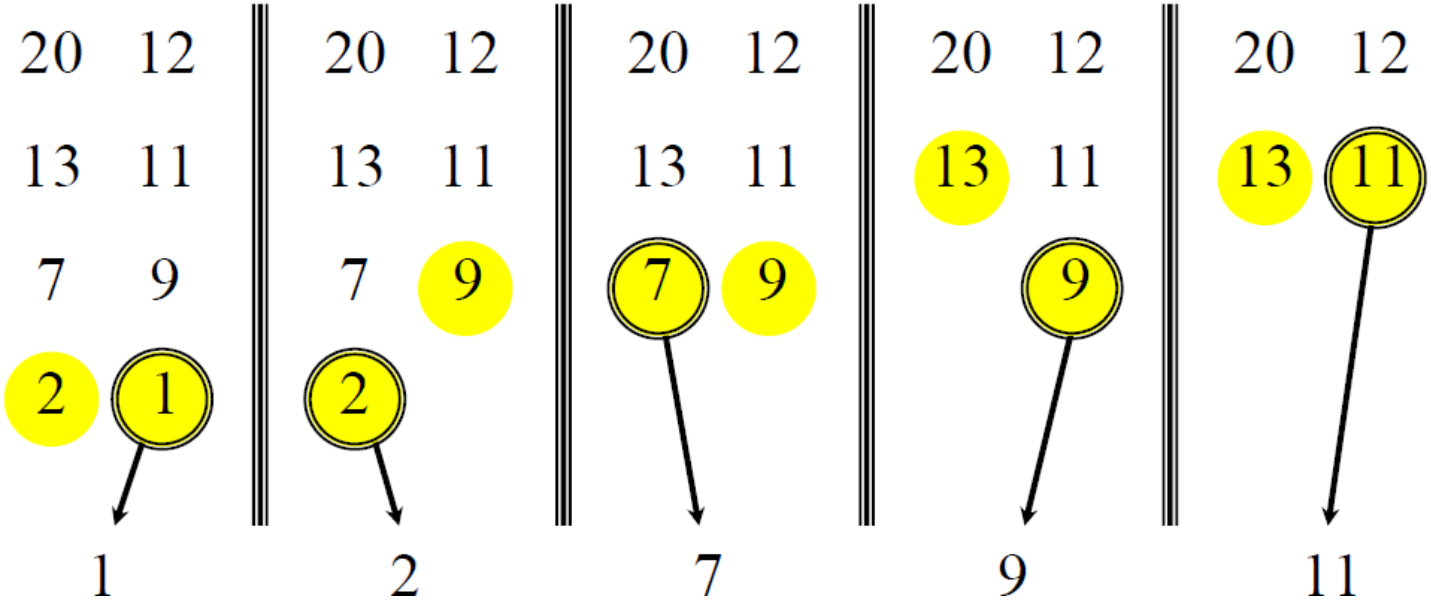


Merging two sorted arrays



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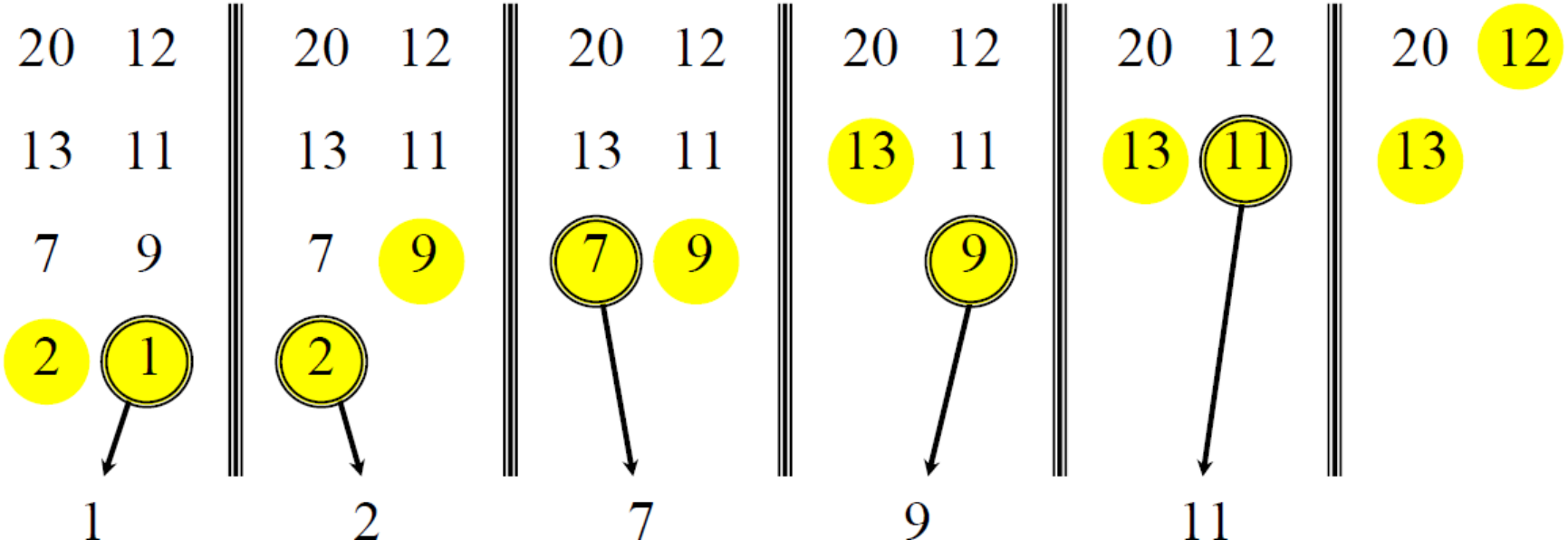

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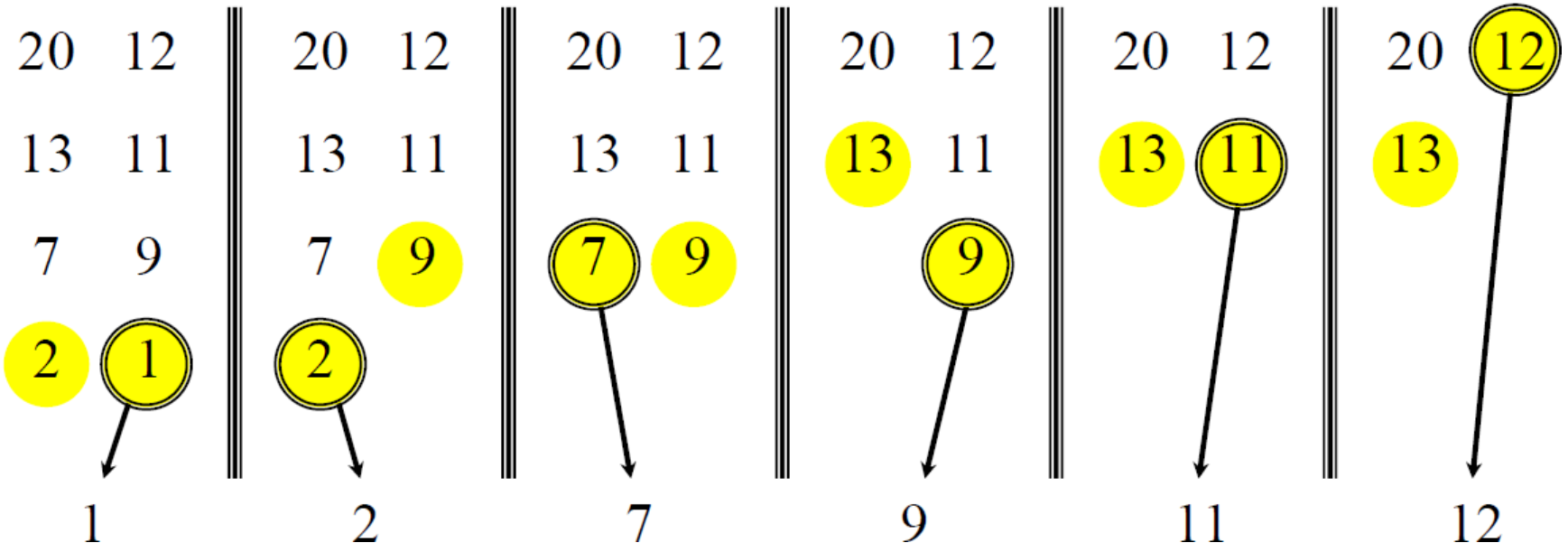
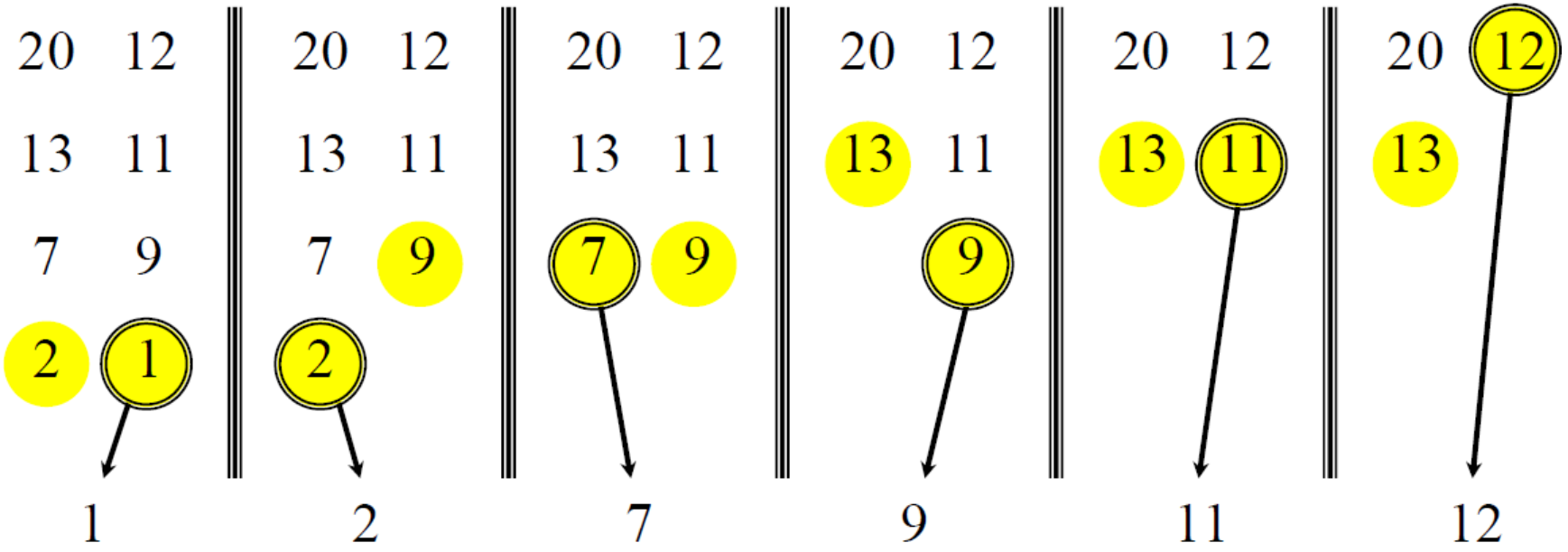


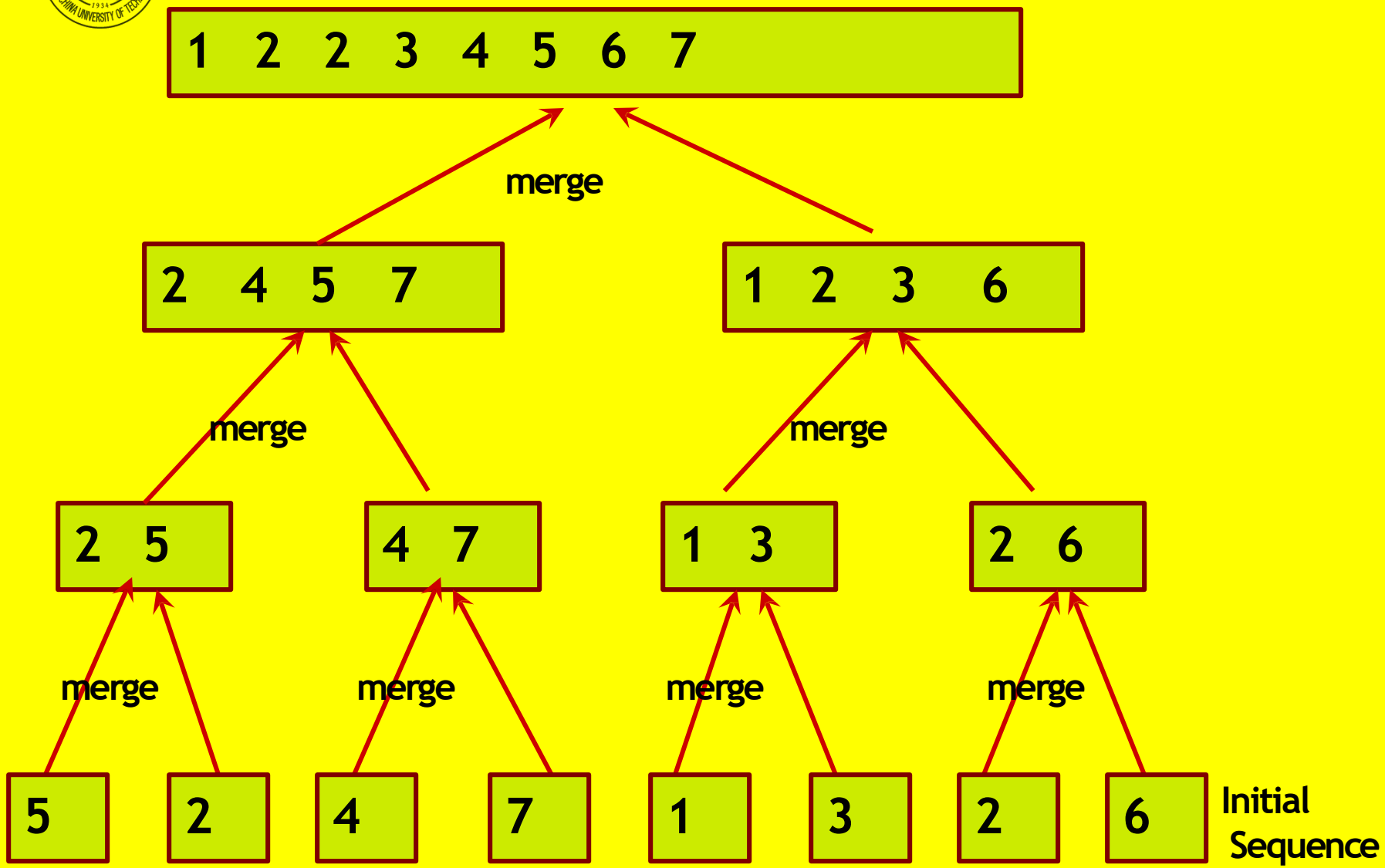
Time?

Merging two sorted arrays



Time = $\Theta(n)$ to merge a total of n elements (linear time).

Action of Merge Sort



Analyzing Merge-Sort

- **How long does merge-sort take?**
  - **-- Bottleneck = merging (and copying).**
    - **>> merging two files of size *n/2* requires *n* comparisons**
  - **-- *T(n)* = comparisons to merge sort *n* elements.**
    - **>>to make analysis cleaner, assume n is a power of 2**

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{Sorting both halves}} + \underbrace{\Theta(n)}_{\text{merging}} & \text{otherwise} \end{cases}$$

排序 + 合并

- **Claim. $T(n) = n \log_2 n$**
  - **-- Note: same number of comparisons for ANY file.**
    - **>> even already sorted**

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

Recursion tree

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$T(n)$

Recursion tree

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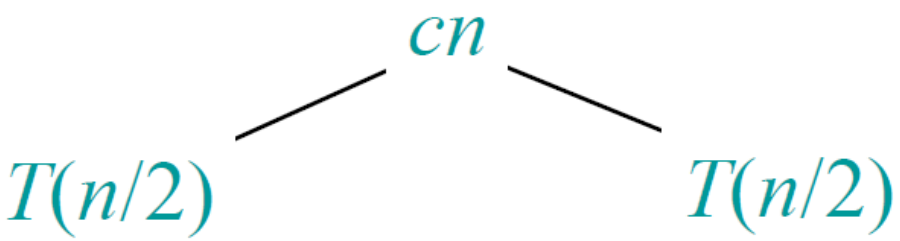

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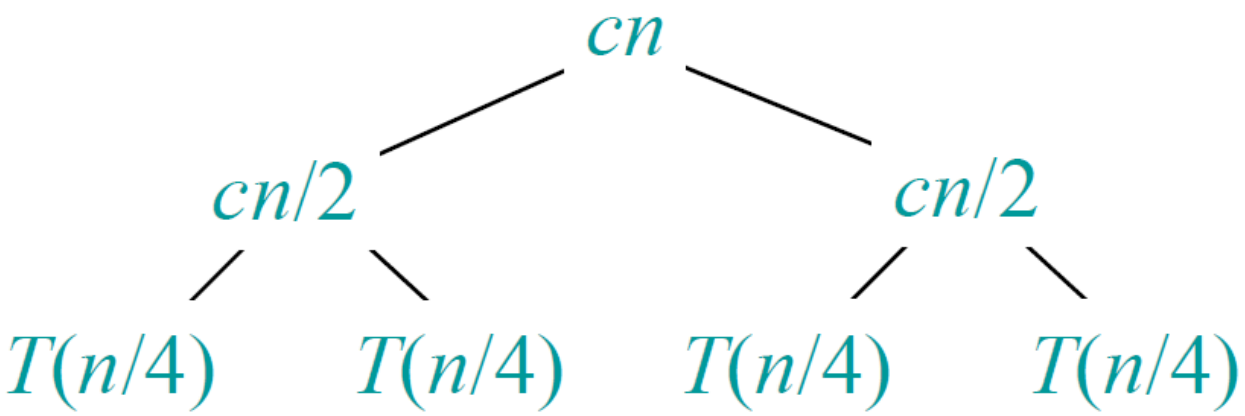

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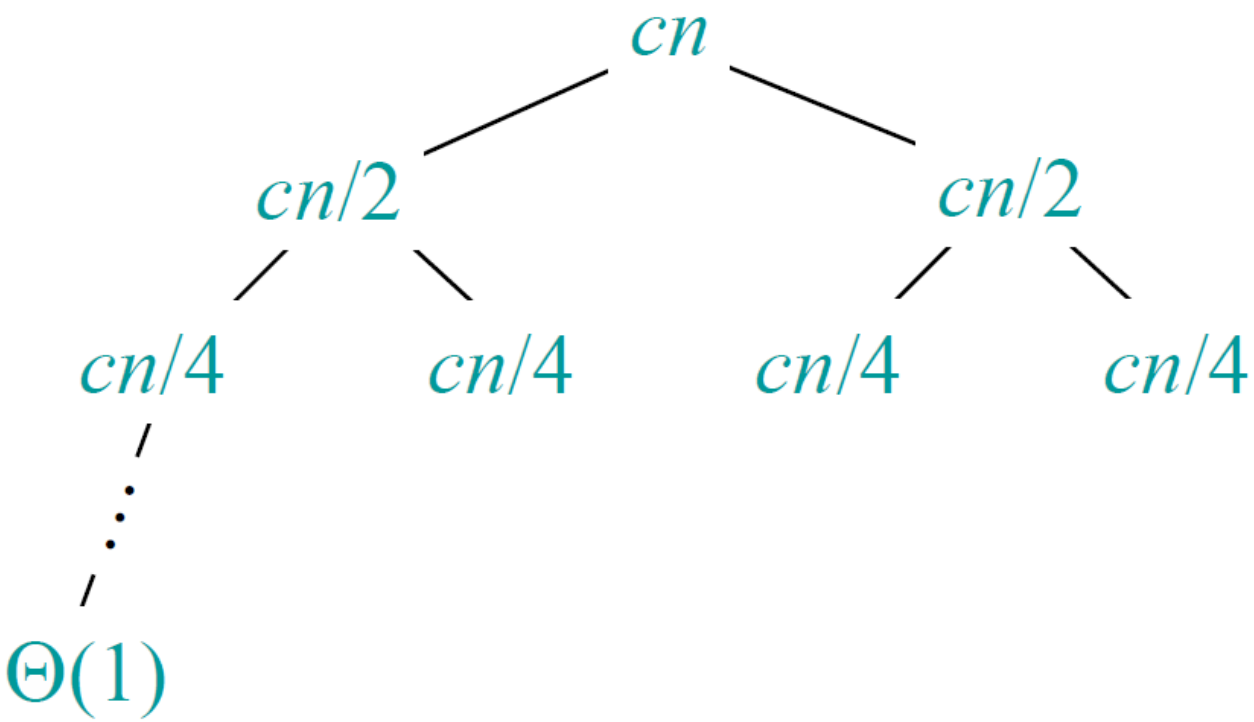

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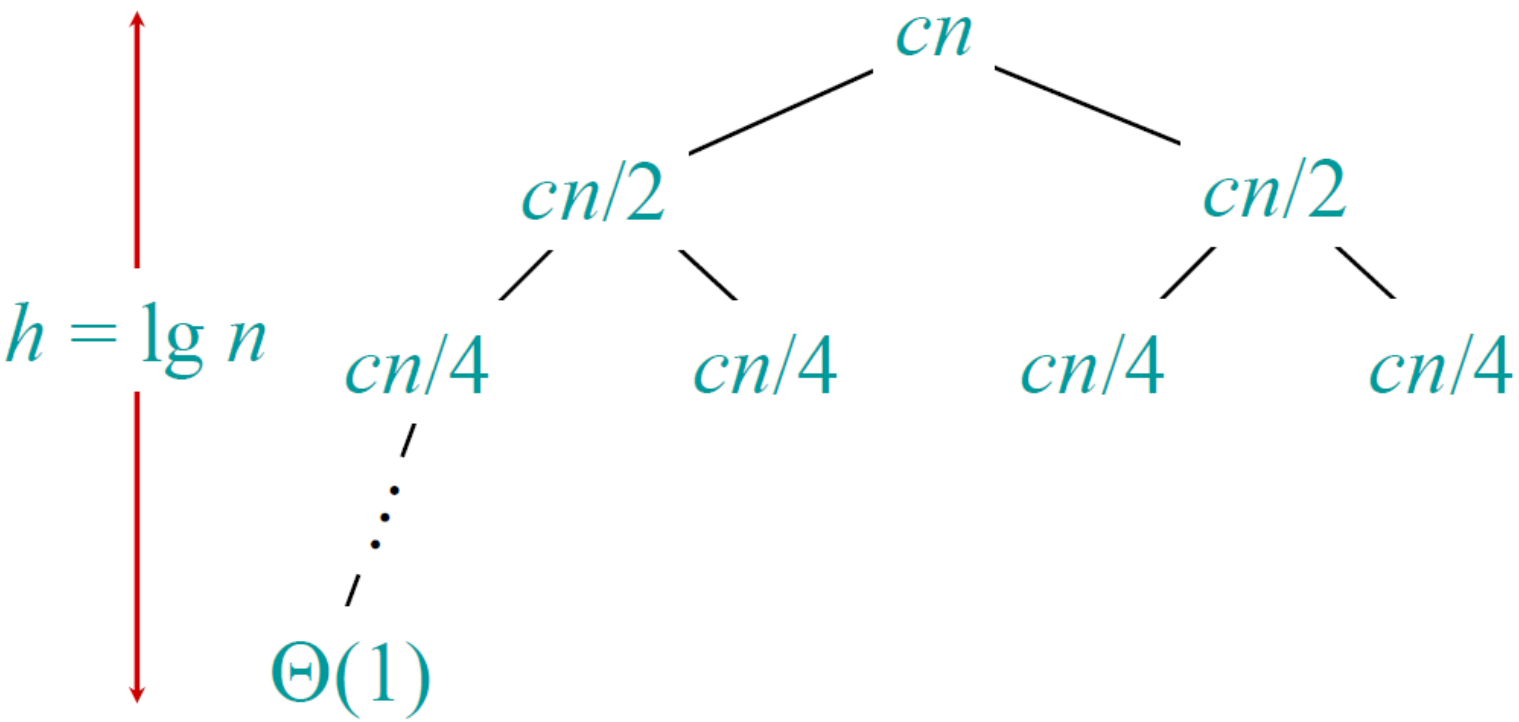

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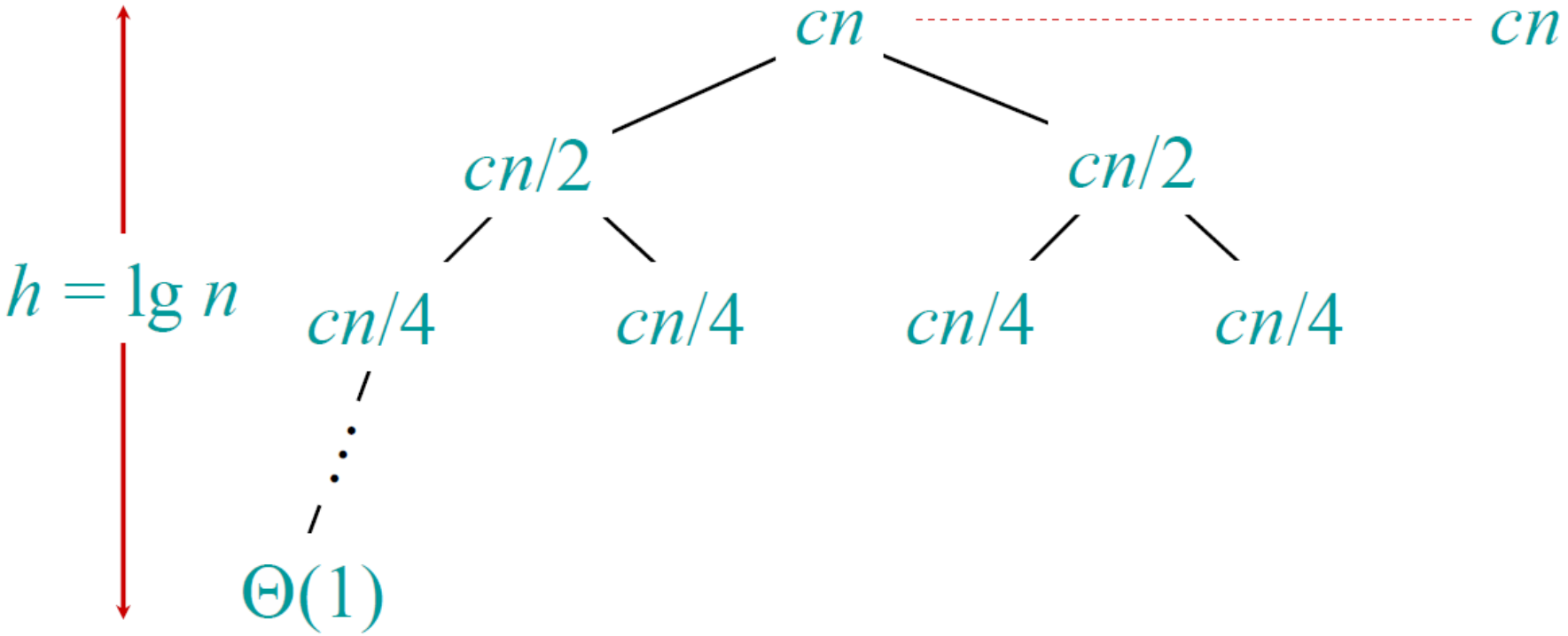

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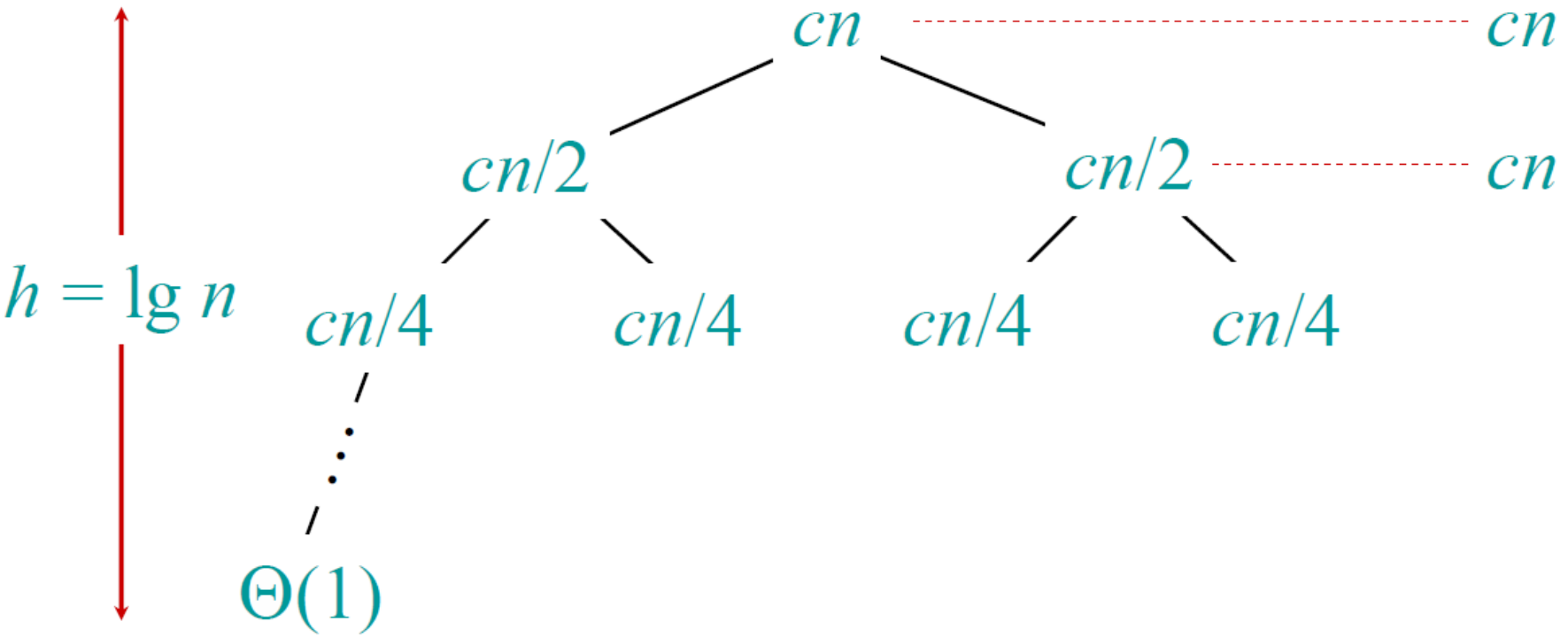

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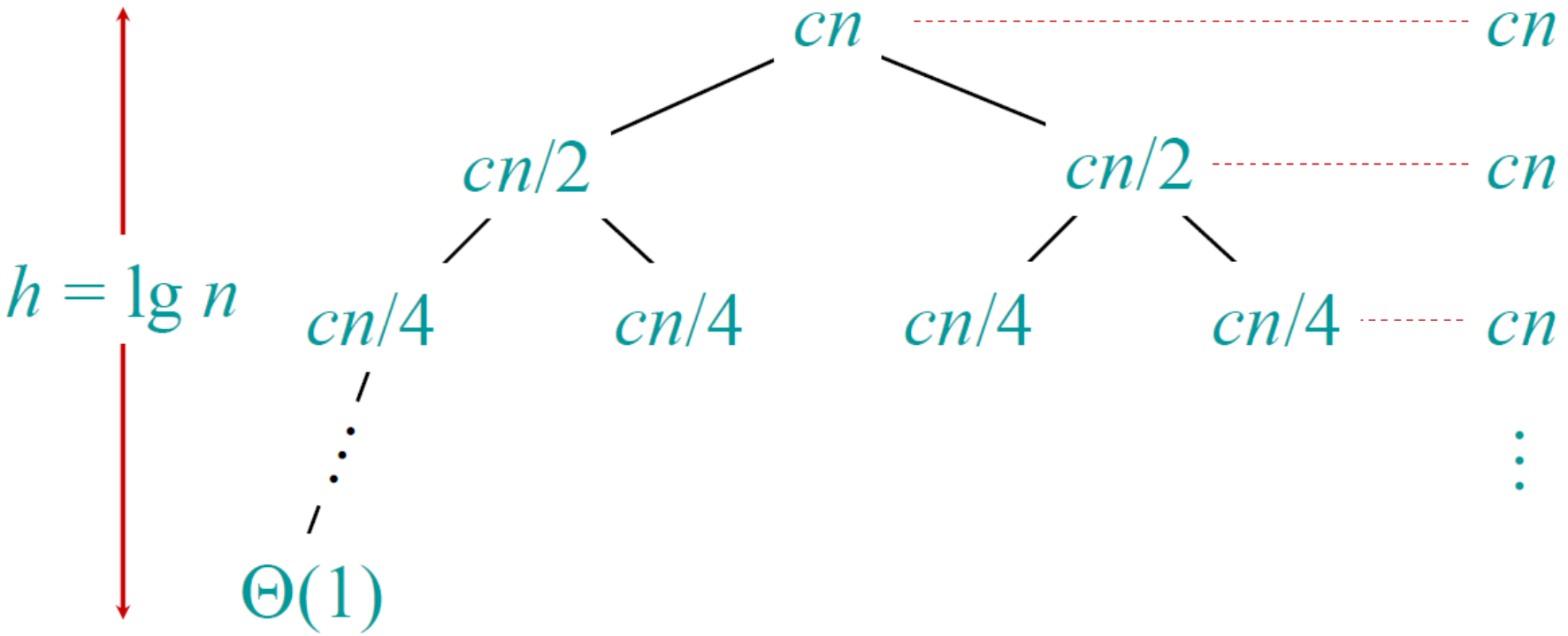

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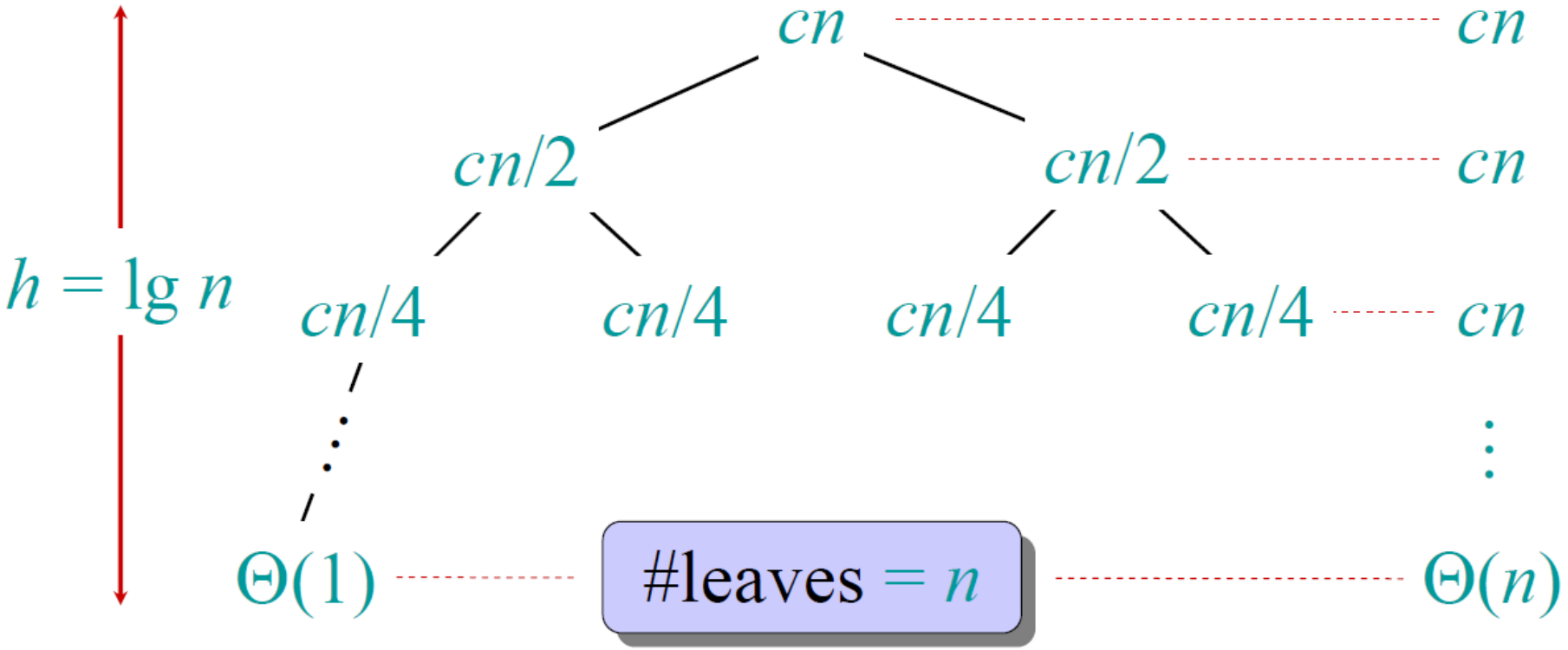

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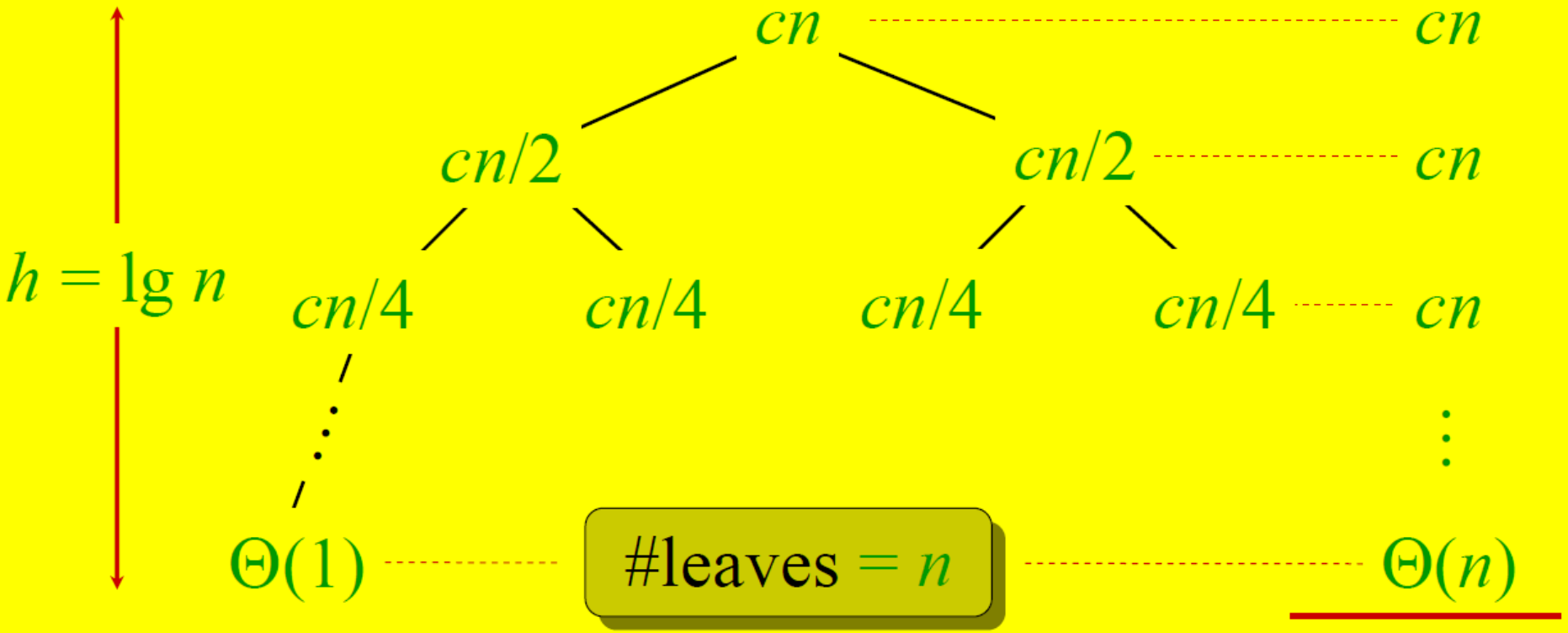

Total = $\Theta(n \lg n)$

Conclusions

- $\Theta(nlgn)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge-sort asymptotically beats insertion-sort in the worst case.
- In practice, merge-sort beats insertion-sort for $n > 30$.