Design and Analysis of Algorithms
Linear Programming

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Topics

- An Example
- Standard Form
- Geometry
- Linear Algebra
- Simplex Algorithm

Linear Programming

Linear programming. Optimize a linear function subject to linear inequalities.

$$\max \sum_{j=1}^{n} c_j x_j$$
$$s.t. \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad 1 \leq i \leq m$$
$$x_j \geq 0 \quad 1 \leq j \leq n$$

Ranked among most important scientific advances of 20th century.

Linear Programming

Linear programming. Optimize a linear function subject to linear inequalities.

Generalizes: AX=B, 2-person zero-sum games, shortest path, max flow, assignment problem, …

Brewery Problem

Small brewery produces ale and beer.

- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.

Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

How can brewer maximize profit?

- Devote all resources to ale: 34 barrels of ale -> $442
- Devote all resources to beer: 32 barrels of beer -> $736
- 7.5 barrels of ale, 29.5 barrels of beer -> $ 776
- 12 barrels of ale, 28 barrels of beer -> $800

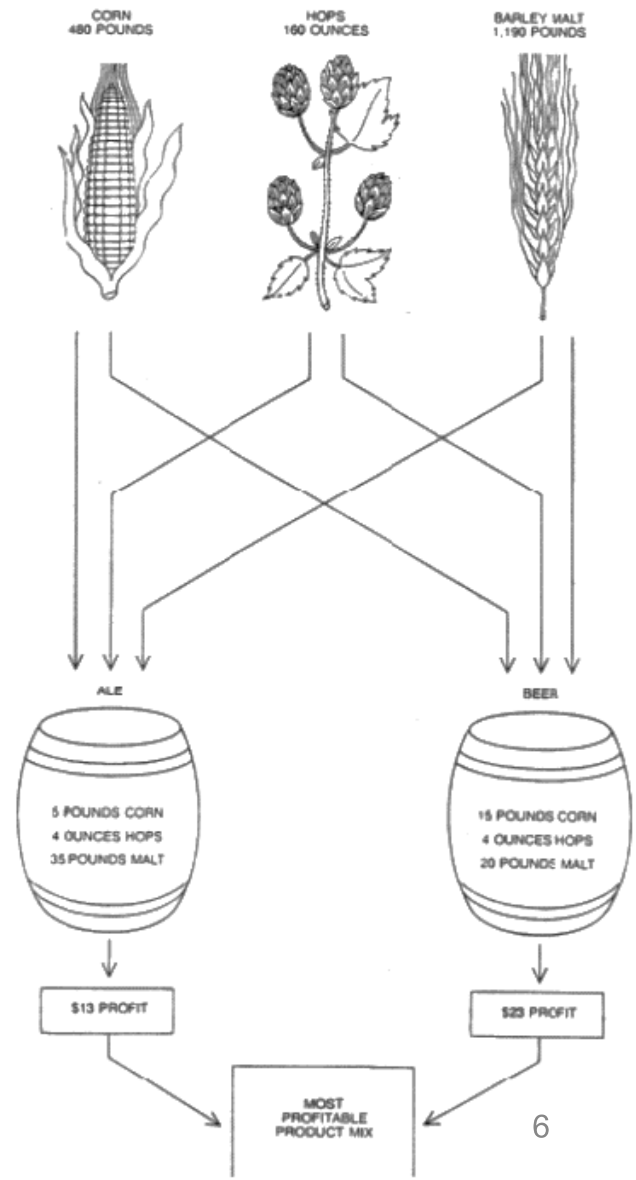
Brewery Problem

objective function

$$
\begin{array}{rcccccl}
 & \text{Ale} & & \text{Beer} & & & \\
\max & 13A & + & 23B & & & \text{Profit} \\
\text{s.t.} & 5A & + & 15B & \le & 480 & \text{Corn} \\
 & 4A & + & 4B & \le & 160 & \text{Hops} \\
 & 35A & + & 20B & \le & 1190 & \text{Malt} \\
 & A & , & B & \ge & 0 &
\end{array}
$$

constraint

decision variable



Standard Form

用等号

"Standard form" of a linear program.

- Input: real numbers a_{ij}, c_j, b_i.
- Output: real numbers x_j.
- $n = \#$ decision variables, $m = \#$ constraints.
- Maximize linear objective function subject to linear equalities.

$$\max \sum_{j=1}^{n} c_j x_j$$
$$s.t. \sum_{j=1}^{n} a_{ij} x_j = b_i \quad 1 \le i \le m$$
$$x_j \ge 0 \quad 1 \le j \le n$$

Brewery Problem: Converting to Standard Form

Original input.

$$
\begin{array}{lrcrcr}
\max & 13A & + & 23B & & \\
\text{s.t.} & 5A & + & 15B & \leq & 480 \\
 & 4A & + & 4B & \leq & 160 \\
 & 35A & + & 20B & \leq & 1190 \\
 & A & , & B & \geq & 0
\end{array}
$$

Standard form？

- Add slack variable for each inequality.
- Now a 5-dimensional problem.

$$
\begin{array}{lrcrcrcrcrcr}
\max & 13A & + & 23B & & & & & & & & \\
\text{s.t.} & 5A & + & 15B & + & S_C & & & & & = & 480 \\
 & 4A & + & 4B & & & + & S_H & & & = & 160 \\
 & 35A & + & 20B & & & & & + & S_M & = & 1190 \\
 & A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0
\end{array}
$$

Basic and Non-basic Variables

Basic variables are selected arbitrarily with the restriction that there will be as many basic variables as the equations. The remaining variables are non-basic variables.

$$\begin{aligned} x_1 + 2x_2 + s_1 \quad\quad &= 32 \\ 3x_1 + 4x_2 \quad + s_2 &= 84 \end{aligned}$$

This system has two equations, we can select any two of the four variables as basic variables. The remaining two variables are then non-basic variables. A solution found by setting the two non-basic variables equal to 0 and solving for the two basic variables is a basic solution. If a basic solution has no negative values, it is a basic feasible solution.

Equivalent Forms

Easy to convert variants to standard form.

$$\begin{aligned} &\max c^T x \\ &s.t. \quad Ax = b \\ &\qquad\quad x \geq 0 \end{aligned}$$

- Less than to equality.

$$x + 2y - 3z \leq 17$$

- Greater than to equality.

$$x + 2y - 3z \geq 17$$

- Min to max.

$$\min x + 2y - 3z$$

- Unrestricted to nonnegative.

x unrestricted

Equivalent Forms

Easy to convert variants to standard form.

$$\max c^T x$$
$$s.t.\ \ Ax = b$$
$$x \geq 0$$

- Less than to equality.

$$x + 2y - 3z \leq 17 \ \rightarrow x + 2y - 3z + s = 17, s \geq 0$$

- Greater than to equality.

$$x + 2y - 3z \geq 17 \ \rightarrow x + 2y - 3z - s = 17, s \geq 0$$

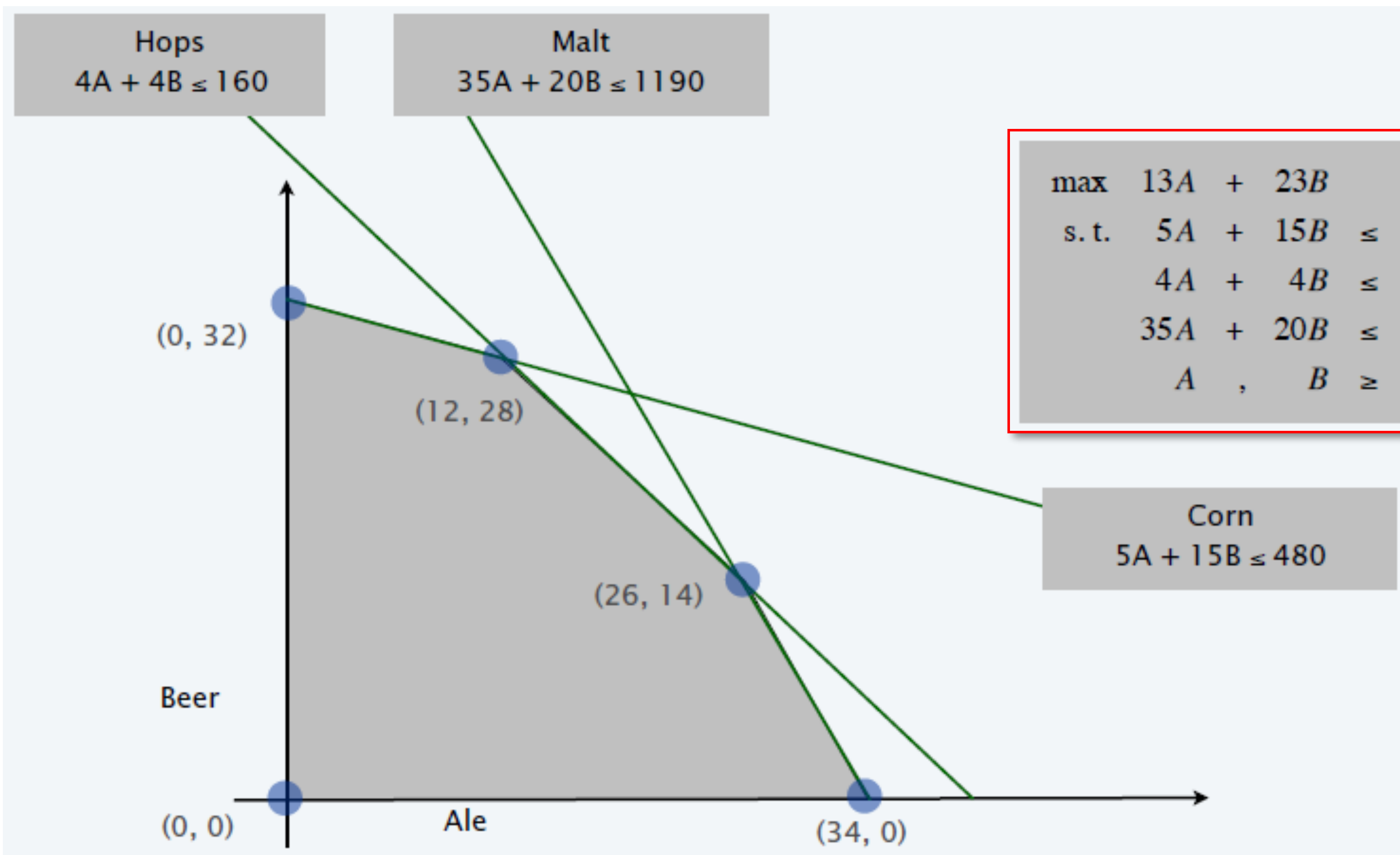
- Min to max.

$$\min x + 2y - 3z \ \rightarrow \max -x - 2y + 3z$$

- Unrestricted to nonnegative.

$$x \text{ unrestricted} \rightarrow x = x^+ - x^-, x^+ \geq 0, x^- \geq 0$$

Brewery Problem: Feasible Region

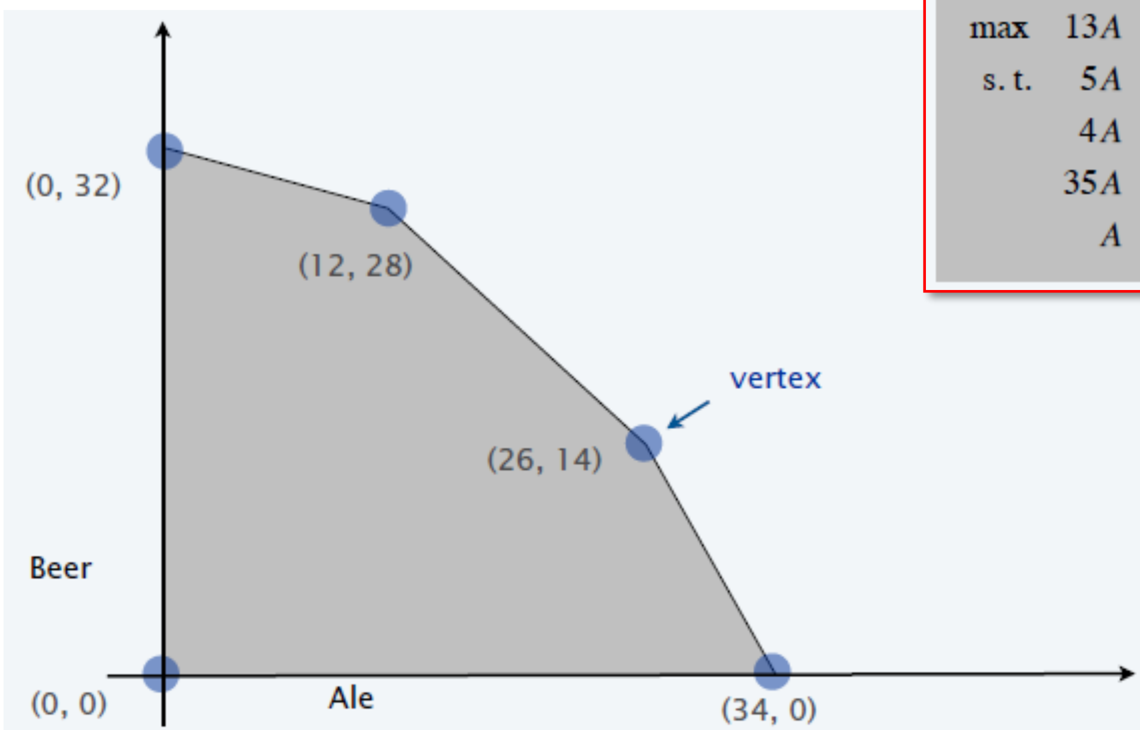


$$
\begin{array}{lrcrcr}
\max & 13A & + & 23B & & \\
\text{s. t.} & 5A & + & 15B & \leq & 480 \\
 & 4A & + & 4B & \leq & 160 \\
 & 35A & + & 20B & \leq & 1190 \\
 & A & , & B & \geq & 0
\end{array}
$$

Brewery Problem: Geometry

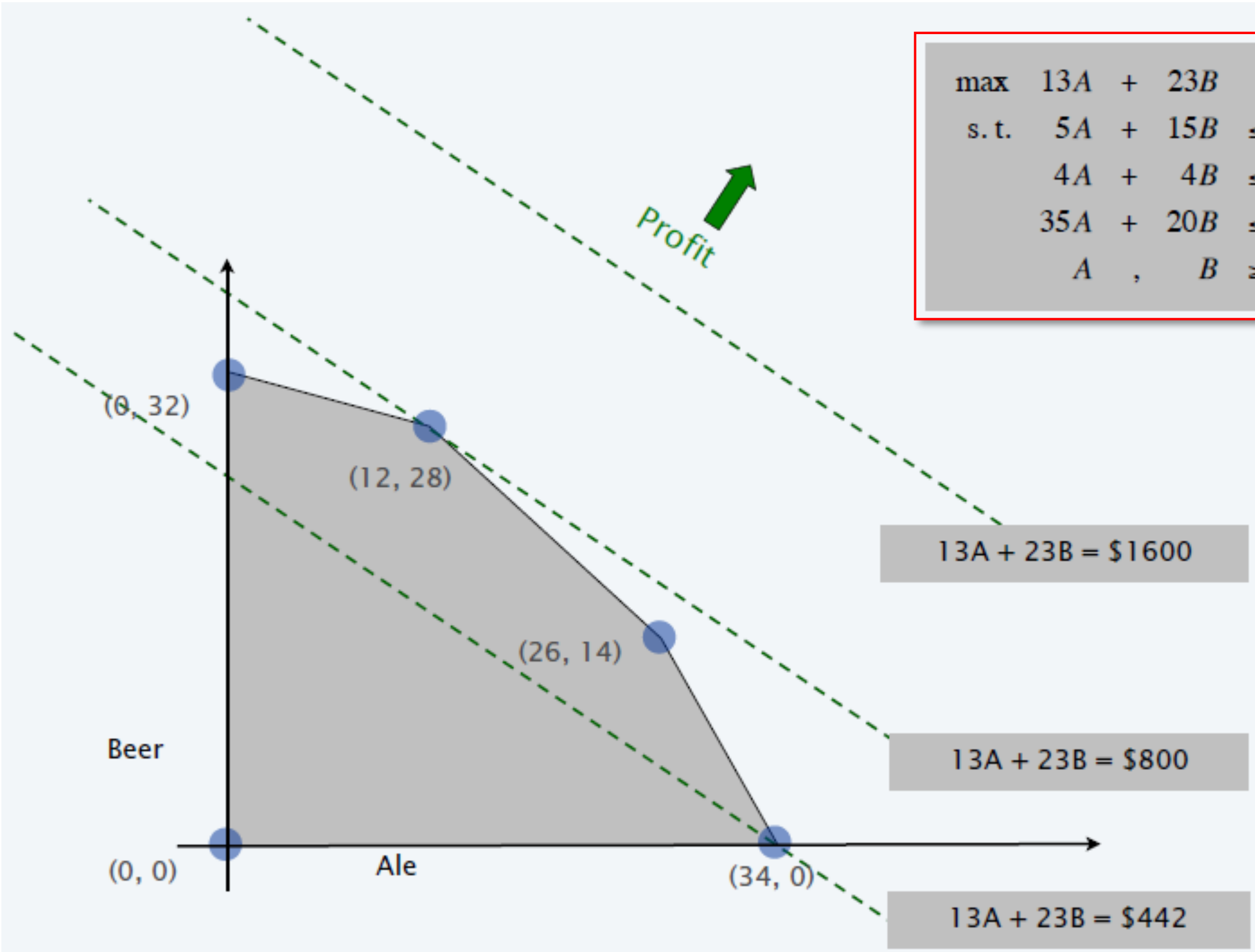
Brewery problem observation. Regardless of objective function coefficients, an optimal solution occurs at a vertex.

$$
\begin{array}{lrcrcr}
\max & 13A & + & 23B & & \\
\text{s. t.} & 5A & + & 15B & \leq & 480 \\
 & 4A & + & 4B & \leq & 160 \\
 & 35A & + & 20B & \leq & 1190 \\
 & A & , & B & \geq & 0
\end{array}
$$



Brewery Problem: Objective Function

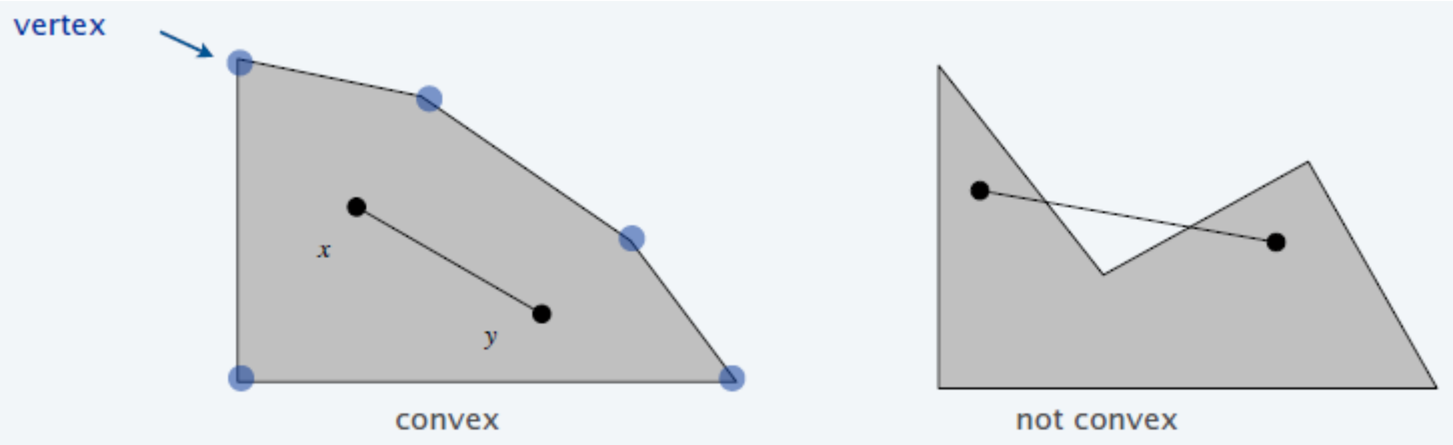
$$
\begin{array}{lrcrcr}
\max & 13A & + & 23B & & \\
\text{s. t.} & 5A & + & 15B & \le & 480 \\
 & 4A & + & 4B & \le & 160 \\
 & 35A & + & 20B & \le & 1190 \\
 & A & , & B & \ge & 0
\end{array}
$$



Convexity

Convex set. If two points x and y are in the set, then so is $\lambda x + (1 - \lambda)y$ for $0 \leq \lambda \leq 1$.

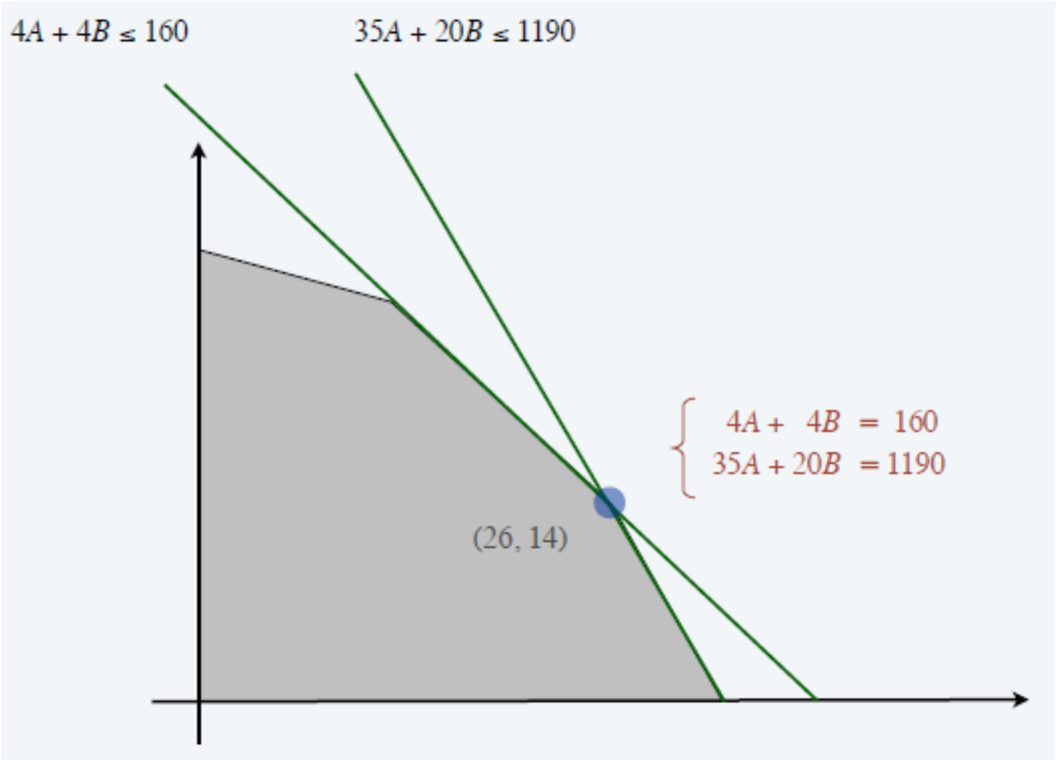
Vertex. A point x in the set that can't be written as a strict convex combination of two distinct points in the set.



Observation. LP feasible region is a convex set.

Vertex

Intuition. A vertex in R^m is uniquely specified by m linearly independent equations.

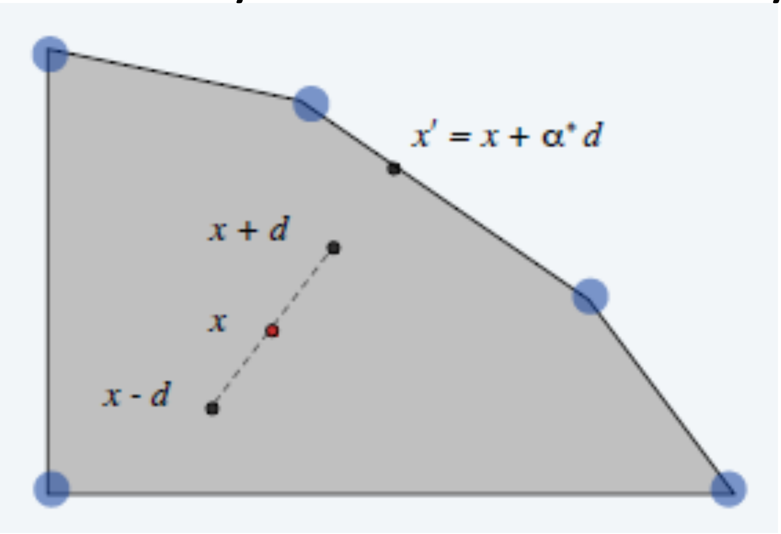


Vertex

Theorem. If there exists an optimal solution to (P), then there exists one that is a vertex.

$$\begin{aligned} &\max c^T x \\ &s.t. \quad Ax = b \\ &\qquad\quad x \geq 0 \end{aligned}$$

Intuition. If the optimum is not a vertex, move in a non-decreasing direction until you reach a boundary.



Vertex

Theorem. If there exists an optimal solution to (P), then there exists one that is a vertex.

Pf.

Since there exists an optimal solution, there exists an optimal solution x with *a minimal number of non-zero components*.

Suppose x is not a vertex, so that

$$x = \lambda u + (1 - \lambda)v,$$

for some $u \neq v$, $\lambda \in (0,1)$.

Vertex

Theorem. If there exists an optimal solution to (P), then there exists one that is a vertex.

Since x is optimal, $c^T u \leq c^T x$ and $c^T v \leq c^T x$.
But also $c^T x = \lambda c^T u + (1-\lambda) c^T v$ so in fact $c^T u = c^T v = c^T x$.
Now consider the line defined by

$$x(\epsilon) = x + \epsilon(u - v)$$

Then

- $Ax = Au = Av = b$ so $Ax(\epsilon) = b$ for all ϵ,
- $c^T x(\epsilon) = c^T x$ for all ϵ,
- If $x_i = 0$ then $u_i = v_i = 0$, which implies $x(\epsilon)_i = 0$ for all ϵ,
- If $x_i > 0$ then $x(0)_i > 0$, and $x(\epsilon)_i$ is continuous in ϵ.

Vertex

Theorem. If there exists an optimal solution to (P), then there exists one that is a vertex.

So we can increase ϵ from zero, in a positive or a negative direction as appropriate, until at least one extra component of $x(\epsilon)$ becomes zero.

This gives an optimal solution $x(\epsilon)$ with fewer non-zero components than x.

So x must be a vertex.

Basic Feasible Solution

Theorem. Let $P = \{x: Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j:\ x_j > 0\}$. Then, x is a vertex iff A_B has linearly independent columns.

Notation. Let $B =$ set of column indices. Define A_B to be the subset of columns of A indexed by B.

Ex.

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 7 & 3 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 16 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad B = \{1, 3\}, \quad A_B = \begin{bmatrix} 2 & 3 \\ 7 & 2 \\ 0 & 0 \end{bmatrix}$$

Basic Feasible Solution

Theorem. Let $P = \{x: Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j:\ x_j > 0\}$. Then, x is a vertex iff A_B has linearly independent columns.

Pf. Assume x is not a vertex.

- There exist direction $d \neq 0$ such that $x \pm d \in P$.
- $Ad = 0$ because $A(x \pm d) = b$.
- Define $B' = \{j:\ d_j \neq 0\}$.
- $A_{B'}$ has linearly dependent columns since $d \neq 0$.
- Moreover, $d_j = 0$ whenever $x_j = 0$ because $x \pm d \geq 0$.
- Thus $B' \subseteq B$, so $A_{B'}$ is a submatrix of A_B.
- Therefore, A_B has linearly dependent columns.

Basic Feasible Solution

Theorem. Let $P = \{x: Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j:\ x_j > 0\}$. Then, x is a vertex iff A_B has linearly independent columns.

Pf. Assume A_B has linearly dependent columns.

- There exist $d \neq 0$ such that $A_B d = 0$.
- Extend d to R^n by adding 0 components.
- Now, $Ad = 0$ and $d_j = 0$ whenever $x_j = 0$.
- For sufficiently small λ, $x \pm \lambda d \in P\ \rightarrow x$ is not a vertex.

Basic Feasible Solution

Theorem. Given $P = \{x: Ax = b, x \geq 0\}$, x is a vertex iff there exists $B \subseteq \{1, \ldots, n\}$ such $|B| = m$ and:

- A_B is nonsingular.
- $x_B = A_B^{-1} b \geq 0$ (basic feasible solution).
- $x_N = 0$.

Pf. Augment A_B with linearly independent columns (if needed).

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 7 & 3 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 16 \\ 0 \end{bmatrix}$$

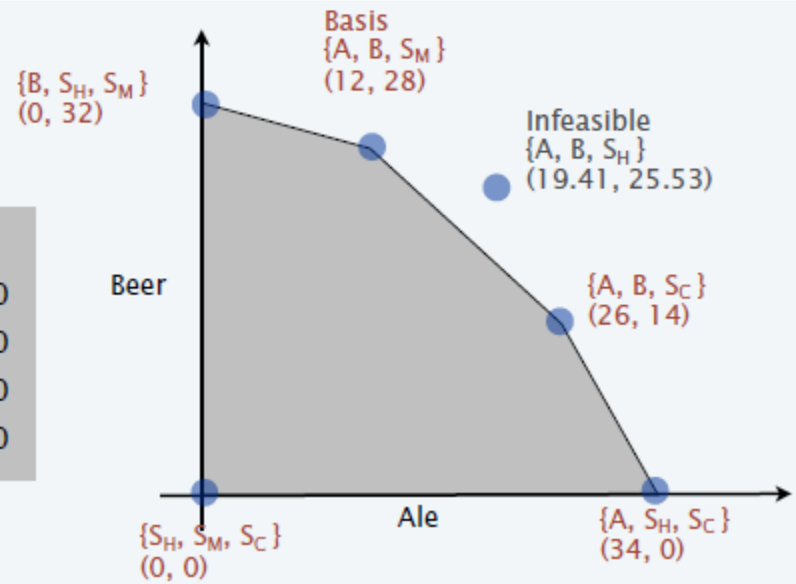
$$x = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad B = \{1, 3, 4\}, \quad A_B = \begin{bmatrix} 2 & 3 & 0 \\ 7 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

Assumption. $A \in R^{m \times n}$ has full row rank.

Basic Feasible Solution: Example

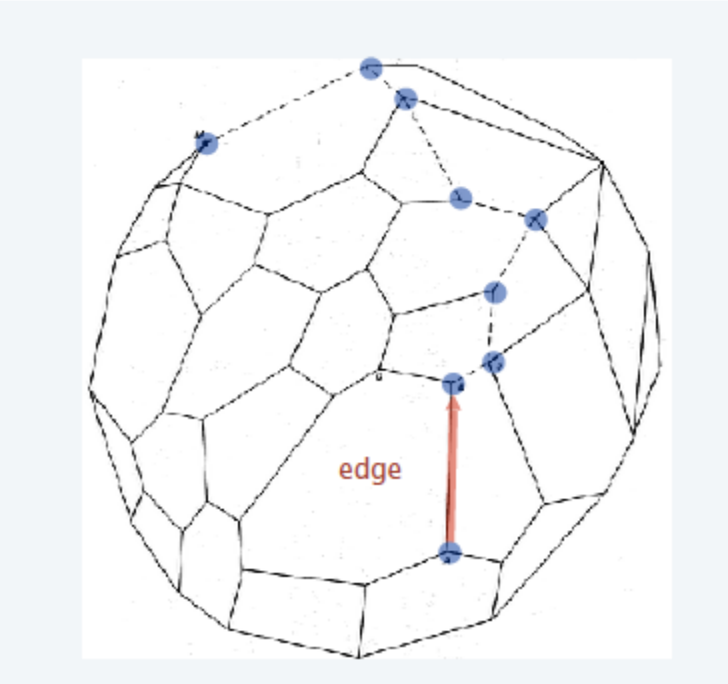
Basic feasible solutions.

$$
\begin{array}{lrcrcccccl}
\max & 13A & + & 23B & & & & & & \\
\text{s.t.} & 5A & + & 15B & + & S_C & & & = & 480 \\
 & 4A & + & 4B & & & + \; S_H & & = & 160 \\
 & 35A & + & 20B & & & & + \; S_M & = & 1190 \\
 & A & , & B & , & S_C & , \; S_H & , \; S_M & \geq & 0
\end{array}
$$



Simplex Algorithm: Intuition

Simplex algorithm. Move from BFS (Basic Feasible Solution) to adjacent BFS, without decreasing objective function (replace one basic variable with another).



Greedy property. BFS optimal iff no adjacent BFS is better.

Simplex Algorithm: Initialization

max Z subject to

$$
\begin{array}{rcrcccccccccr}
13A & + & 23B & & & & & & & - & Z & = & 0 \\
\hline
5A & + & 15B & + & S_C & & & & & & & = & 480 \\
4A & + & 4B & & & + & S_H & & & & & = & 160 \\
35A & + & 20B & & & & & + & S_M & & & = & 1190 \\
A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0
\end{array}
$$

Basis = $\{S_C, S_H, S_M\}$

$A = B = 0$

$Z = 0$

$S_C = 480$

$S_H = 160$

$S_M = 1190$

Simplex Algorithm: Pivot 1

$$
\begin{array}{l}
\max Z \text{ subject to} \\
\begin{array}{rcrcrcrcrcrcr}
13A & + & 23B & & & & & & & - & Z & = & 0 \\
\hline
5A & + & 15B & + & S_C & & & & & & & = & 480 \\
4A & + & 4B & & & + & S_H & & & & & = & 160 \\
35A & + & 20B & & & & & + & S_M & & & = & 1190 \\
A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0
\end{array}
\end{array}
$$

正的系数max

Basis = $\{S_C, S_H, S_M\}$
$A = B = 0$
$Z = 0$
$S_C = 480$
$S_H = 160$
$S_M = 1190$

Substitute: $B = 1/15\,(480 - 5A - S_C)$

将选中的式子中选中的变量替换
同时对于该变量 $\left\{\begin{array}{l}\text{选中式子的 coefficient=1} \\ \text{other function's coefficient=0}\end{array}\right.$
(消掉)

$$
\begin{array}{l}
\max Z \text{ subject to} \\
\begin{array}{rcrcrcrcrcrcr}
\frac{16}{3}A & & & - & \frac{23}{15}S_C & & & & & - & Z & = & -736 \\
\hline
\frac{1}{3}A & + & B & + & \frac{1}{15}S_C & & & & & & & = & 32 \\
\frac{8}{3}A & & & - & \frac{4}{15}S_C & + & S_H & & & & & = & 32 \\
\frac{85}{3}A & & & - & \frac{4}{3}S_C & & & + & S_M & & & = & 550 \\
A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0
\end{array}
\end{array}
$$

Basis = $\{B, S_H, S_M\}$
$A = S_C = 0$
$Z = 736$
$B = 32$
$S_H = 32$
$S_M = 550$

Simplex Algorithm: Pivot 1

max Z subject to

$$
\begin{array}{rcrcccccl}
13A & + & 23B & & & & - & Z & = 0 \\
\hline
5A & + & 15B & + & S_C & & & & = 480 \\
4A & + & 4B & & & + S_H & & & = 160 \\
35A & + & 20B & & & & + S_M & & = 1190 \\
A & , & B & , & S_C & , S_H & , S_M & & \geq 0
\end{array}
$$

Basis = $\{S_C, S_H, S_M\}$
$A = B = 0$
$Z = 0$
$S_C = 480$
$S_H = 160$
$S_M = 1190$

Q. Why pivot on column 2 (or 1)? 系数大的(非基变量中)
A. Each unit increase in B increases objective value by \$23.

Q. Why pivot on row 2.
A. Preserves feasibility by ensuring *RHS (Right Hand Side)* ≥ 0.
(min ratio rule: min{480/15, 160/4, 1190/20})

Simplex Algorithm: Pivot 2

要换掉的变量的coefficient must > 0

max Z subject to

$$
\begin{array}{rcrcrcrcrcrcr}
\frac{16}{3}A & & & - & \frac{23}{15}S_C & & & & & - & Z & = & -736 \\
\hline
\frac{1}{3}A & + & B & + & \frac{1}{15}S_C & & & & & & & = & 32 \\
\frac{8}{3}A & & & - & \frac{4}{15}S_C & + & S_H & & & & & = & 32 \\
\frac{85}{3}A & & & - & \frac{4}{3}S_C & & & + & S_M & & & = & 550 \\
A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0
\end{array}
$$

Basis = $\{B, S_H, S_M\}$
$A = S_C = 0$
$Z = 736$
$B = 32$
$S_H = 32$
$S_M = 550$

Substitute: $A = 3/8\,(32 + 4/15\,S_C - S_H)$

if all of coefficient ≤ 0, then over!

max Z subject to

$$
\begin{array}{rcrcrcrcrcrcr}
 & & & - & S_C & - & 2\,S_H & & & - & Z & = & -800 \\
\hline
 & & B & + & \frac{1}{10}S_C & + & \frac{1}{8}S_H & & & & & = & 28 \\
A & & & - & \frac{1}{10}S_C & + & \frac{3}{8}S_H & & & & & = & 12 \\
 & & & - & \frac{25}{6}S_C & - & \frac{85}{8}S_H & + & S_M & & & = & 110 \\
A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0
\end{array}
$$

Basis = $\{A, B, S_M\}$
$S_C = S_H = 0$
$Z = 800$
$B = 28$
$A = 12$
$S_M = 110$

Simplex Algorithm: Optimality

Q. When to stop pivoting?

A. When all coefficients in top row are non-positive.

Q. Why is the resulting solution optimal?

A. Any feasible solution satisfies systems of equations in tableau.

- In particular: $Z = 800 - S_C - 2S_H$, $S_C \geq 0$, $S_H \geq 0$.
- Thus, optimal objective value $Z^* \leq 800$.
- Current BFS has value 800 -> optimal.

max Z subject to

$$
\begin{array}{rrrrrrrrrrrr}
 & & & - & S_C & - & 2\,S_H & & - & Z & = & -800 \\
\hline
 & & B & + & \frac{1}{10}\,S_C & + & \frac{1}{8}\,S_H & & & & = & 28 \\
A & & & - & \frac{1}{10}\,S_C & + & \frac{3}{8}\,S_H & & & & = & 12 \\
 & & & - & \frac{25}{6}\,S_C & - & \frac{85}{8}\,S_H & + & S_M & & = & 110 \\
A & , & B & , & S_C & , & S_H & , & S_M & & \geq & 0
\end{array}
$$

Basis = $\{A, B, S_M\}$

$S_C = S_H = 0$

$Z = 800$

$B = 28$

$A = 12$

$S_M = 110$

Variant Tableau

The constraints are a linear system including m equations and n variables. m of the variables can be evaluated in terms of the other $n-m$ variables

$$x_1 = b_1 - a_{1,m+1}x_{m+1} - \cdots - a_{1,n}x_n$$
$$x_2 = b_2 - a_{2,m+1}x_{m+1} - \cdots - a_{2,n}x_n$$
$$\ldots\ldots$$
$$x_m = b_m - a_{m,m+1}x_{m+1} - \cdots - a_{m,n}x_n$$

Objective function $z = \sum_{j=1}^{n} c_j x_j$

$$= \sum_{i=1}^{m} c_i b_i + \sum_{j=m+1}^{n} (c_j - \sum_{i=1}^{m} c_i a_{ij}) x_j .$$

Let $z^0 = \sum_{i=1}^{m} c_i b_i$, $\sigma_j = c_j - \sum_{i=1}^{m} c_i a_{ij}$, and we have

$$z = z^0 + \sum_{j=m+1}^{n} \boxed{\sigma_j} x_j$$

indicator

Variant Tableau

c_j		c_1	c_2	$\ldots$	c_m	c_{m+1}	$\ldots$	c_n	b	θ
C_B	X_B	x_1	x_2	$\ldots$	x_m	x_{m+1}	$\ldots$	x_n		
c_1	x_1	1	0	$\ldots$	0	$a'_{1,m+1}$	$\ldots$	a'_{1n}	b'_1	
c_2	x_2	0	1	$\ldots$	0	$a'_{2,m+1}$	$\ldots$	a'_{2n}	b'_2	
$\ldots$	$\ldots$				$\ldots\ldots$				$\ldots$	
c_m	x_m	0	0	$\ldots$	1	$a'_{m,m+1}$	$\ldots$	a'_{mn}	b'_m	
	σ_j	0	0	$\ldots$	0	$c_{m+1} - \sum_{i=1}^{m} c_i a'_{i,m+1}$				

Variant Tableau

To solve a linear programming problem, use the following steps:

1. Convert each inequality in the set of constraints to an equation by adding slack variables. 加松驰变量
2. Create the initial simplex tableau.
3. Select the pivot column *(The column with the "most positive value" element in the last row).*
4. Select the pivot row 比例min *(The row with the smallest non-negative result when the last element in the row is divided by the corresponding in the pivot column).*
5. Use elementary row operations calculate new values for the pivot row so that the pivot is 1.
6. Use elementary row operations to make all numbers in the pivot column equal to 0 except for the pivot.
7. If all entries in the bottom row are non-positive, this the final tableau. If not, go back to Step 3.

这之后有点看不懂, try again!

Variant Tableau: An Example

$$\max \quad z = 2x_1 + 3x_2$$

$$s.t.\begin{cases} 2x_1 + x_2 \le 4 \\ x_1 + 2x_2 \le 5 \\ x_1, x_2 \ge 0 \end{cases}$$

$$\Rightarrow$$

$$\max \quad z = 2x_1 + 3x_2$$

$$s.t.\begin{cases} 2x_1 + x_2 + x_3 \qquad = 4 \\ x_1 + 2x_2 \qquad + x_4 = 5 \\ x_1, x_2, x_3, x_4 \ge 0 \end{cases}$$

Variant Tableau: An Example

Pivot column. The column of the tableau representing the variable to be entered into the solution mix.

Pivot row. The row of the tableau representing the variable to be replaced in the solution mix.

Basic variable. Variables in the solution mix.

Initial tableau

Pivot column

Min ratio rule

Pivot row

c_j		2	3	0	0	b	θ
C_B	X_B	x_1	x_2	x_3	x_4		
0	x_3	2	1	1	0	4	4/1
0	x_4	1	2	0	1	5	5/2
σ_j		2	3	0	0		

Variant Tableau: An Example

c_j		2	3	0	0	b	θ
C_B	X_B	x_1	x_2	x_3	x_4		
0	x_3	2	1	1	0	4	4/1
0	x_4	1	2	0	1	5	5/2
	σ_j	2	**3**	0	0		

- Since the entry 3 is the most positive entry in the last row of the tableau, the second column in the tableau is the pivot column.
- Divide each positive number of the pivot column into the corresponding entry in the column of constants. The ratio 5/2 is less then the ratio 4/1, so row 2 is the pivot row.

Variant Tableau: An Example

c_j		2	3	0	0	b	θ
C_B	X_B	x_1	x_2	x_3	x_4		
0	x_3	3/2	0	1	-1/2	3/2	1
3	x_2	1/2	1	0	1/2	5/2	5
σ_j		**1/2**	0	0	-3/2		

- Since the entry 1/2 is the most positive entry in the last row of the tableau, the first column in the tableau is the pivot column.
- Divide each positive number of the pivot column into the corresponding entry in the column of constants. The ratio 3/2 is less then the ratio 5/2, so row 1 is the pivot row.

Variant Tableau: An Example

c_j		2	3	0	0	b	θ
C_B	X_B	x_1	x_2	x_3	x_4		
2	x_1	1	0	2/3	-1/3	1	
3	x_2	0	1	-1/3	2/3	2	
σ_j		0	0	-1/3	-4/3		

- The last row of the tableau contains no positive numbers, so an optimal solution has been reached.

Matrix Form

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Initial simplex tableau

$$
\begin{array}{rcrcl}
c_B^T x_B & + & c_N^T x_N & = & Z \\
A_B x_B & + & A_N x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

Simplex tableau corresponding to basis B.

$$
\begin{array}{rcrcll}
 & & (c_N^T - c_B^T A_B^{-1} A_N)\, x_N & = & Z - c_B^T A_B^{-1} b & \leftarrow \text{subtract } c_B{}^T A_B{}^{-1} \text{ times constraints} \\
I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b & \leftarrow \text{multiply by } A_B{}^{-1} \\
x_B & , & x_N & \geq & 0 &
\end{array}
$$

$x_B = A_B{}^{-1} b \geq 0$ $x_N = 0$	$c_N{}^T - c_B{}^T A_B{}^{-1} A_N \leq 0$
basic feasible solution	optimal basis

Matrix Form

Standard form:

$$\max Z = C^T X$$
$$s.t. \;\; AX = b$$
$$X \geq 0$$

Let $A = [A_B, A_N]$, $X = \begin{bmatrix} X_B \\ X_N \end{bmatrix}$, $C = \begin{bmatrix} C_B \\ C_N \end{bmatrix}$, we have

$$A_B X_B + A_N X_N = b$$
$$\rightarrow \; X_B = A_B^{-1} b - A_B^{-1} A_N X_N$$

For the basis B,

$$Z = C^T X = [C_B^T, C_N^T] \begin{bmatrix} X_B \\ X_N \end{bmatrix} = C_B^T X_B + C_N^T X_N$$
$$= C_B^T (A_B^{-1} b - A_B^{-1} A_N X_N) + C_N^T X_N$$
$$= C_B^T A_B^{-1} b + (C_N^T - C_B^T A_B^{-1} A_N) X_N$$

Matrix Form: Variant Tableau

	$\boldsymbol{C_B^T}$	$\boldsymbol{C_N^T}$	
	$\boldsymbol{X_B^T}$	$\boldsymbol{X_N^T}$	
$\boldsymbol{C_B}$ $\boldsymbol{X_B}$	$\boldsymbol{I}$	$\boldsymbol{A_B^{-1}A_N}$	$\boldsymbol{A_B^{-1}b}$
Indicator	$\boldsymbol{0}$	$\boldsymbol{C_N^T - C_B^T A_B^{-1} A_N}$	