Design and Analysis of Algorithms
Network Flow

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Topics

- **Image Segmentation**

（**https://www.v7labs.com/blog/semantic-segmentation-guide**）

- **Bipartite Matching**
- **Disjoint Paths**

Image Segmentation

Image segmentation.

- Central problem in image processing.
- Divide image into coherent regions.

Ex. Three people standing in front of complex background scene. Identify each person as a coherent object.

Semantic segmentation

liver and hepatic vascularization segmentation

Image Segmentation

Foreground/background segmentation.

- Label each pixel in picture as belonging to foreground or background.
- V = set of pixels, E = pairs of neighboring pixels.
- $a_i \geq 0$ is likelihood pixel i in foreground.
- $b_i \geq 0$ is likelihood pixel i in background.
- $p_{ij} \geq 0$ is separation penalty for labeling one of i and j as foreground, and the other as background.

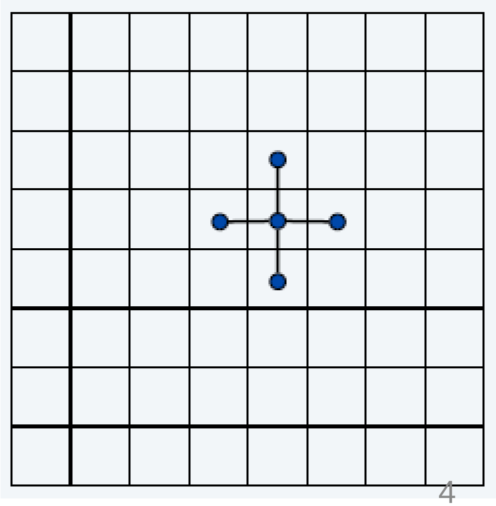

Image Segmentation

Foreground/background segmentation.

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Goals.

- Accuracy: if $a_i > b_i$ in isolation, prefer to label i in foreground.
- Smoothness: if many neighbors of i are labeled foreground, we should be inclined to label i as foreground.
- Find partition (A, B) that maximizes:

$$\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{(i,j) \in E, |A \cap \{i,j\}|=1} p_{ij}$$

Image Segmentation

Formulate as min-cut problem.

- Maximization
- No source or sink.
- Undirected graph.

Turn into minimization problem.

- Maximizing $\sum_{i\in A} a_i + \sum_{j\in B} b_j - \sum_{(i,j)\in E, |A\cap\{i,j\}|=1} p_{ij}$
- Is equivalent to minimizing

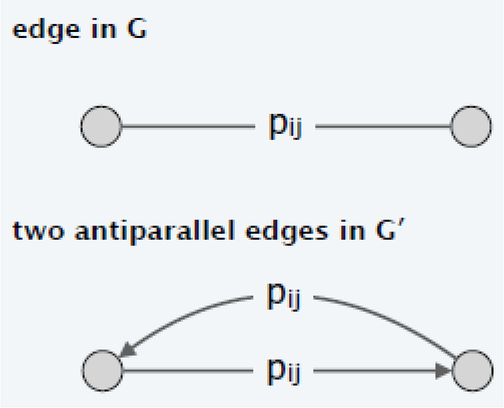
$$\sum_{i\in V} a_i + \sum_{j\in V} b_j - \sum_{i\in A} a_i - \sum_{j\in B} b_j + \sum_{(i,j)\in E, |A\cap\{i,j\}|=1} p_{ij}$$

- Or alternatively $\sum_{j\in B} a_j + \sum_{i\in A} b_i + \sum_{(i,j)\in E, |A\cap\{i,j\}|=1} p_{ij}$

Image Segmentation

Formulate as min-cut problem $G' = (V', E')$.

- Include node for each pixel.
- Use two antiparallel edges instead of undirected edge.
- Add source s to correspond to foreground.
- Add sink t to correspond to background.



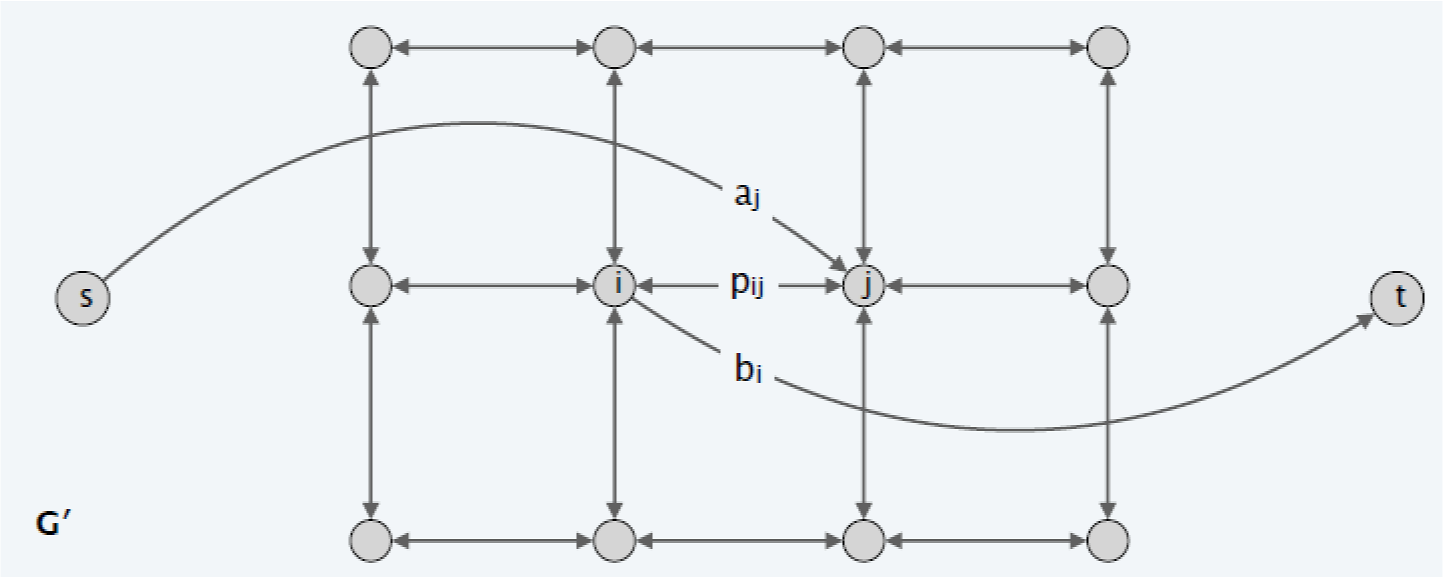


Image Segmentation

Consider min cut (A, B) in G'.

- A = foreground.

$$cap(A,B) = \sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{(i,j) \in E, i \in A, j \in B} p_{ij}$$

- The quantity we want to minimize.

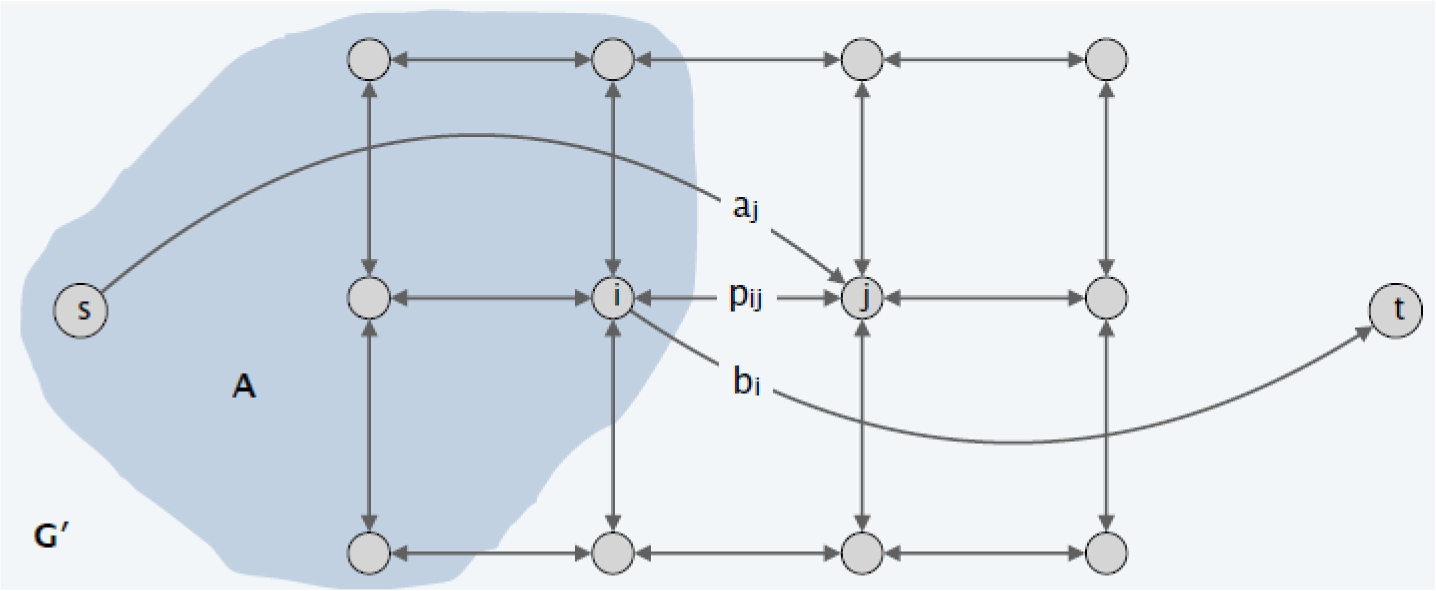
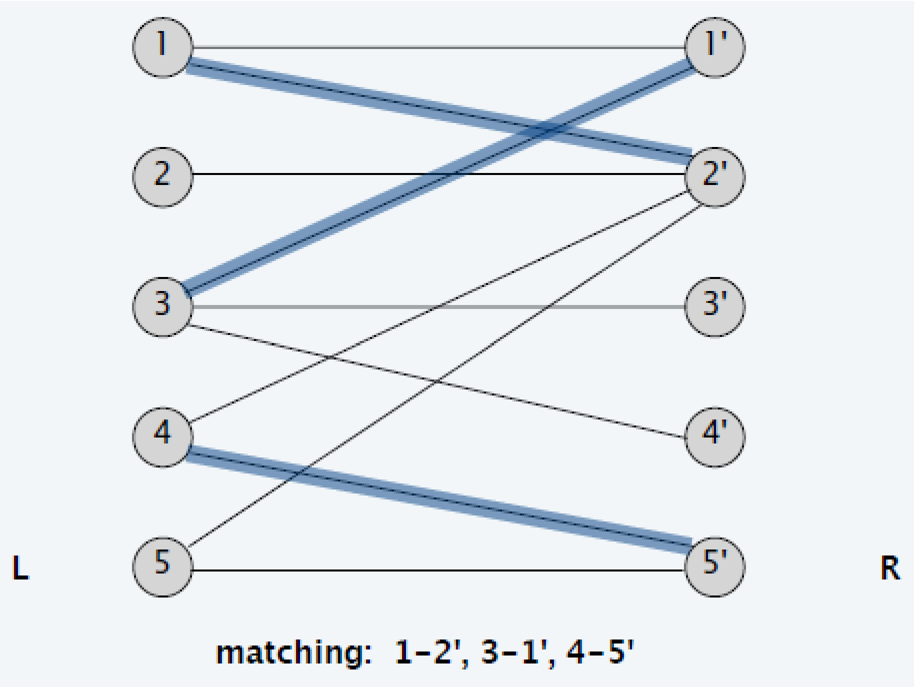


Image Segmentation

Bipartite Matching

Def. A graph G is bipartite if the nodes can be partitioned into two subsets L and R such that every edge connects a node in L to one in R .

Bipartite matching. Given a bipartite graph $G = (L \cup R, E)$, find a max-cardinality matching.



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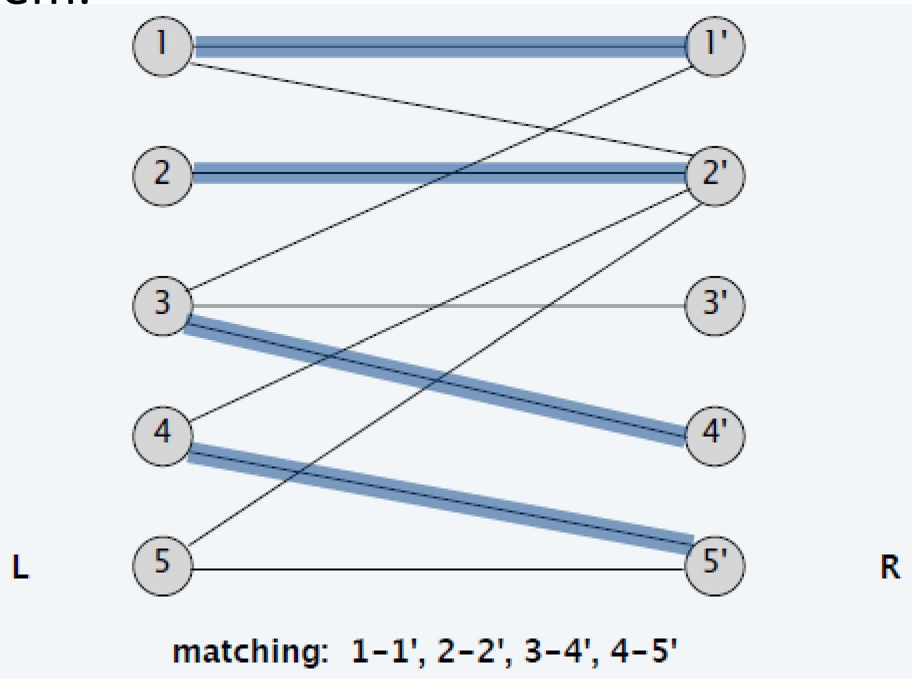
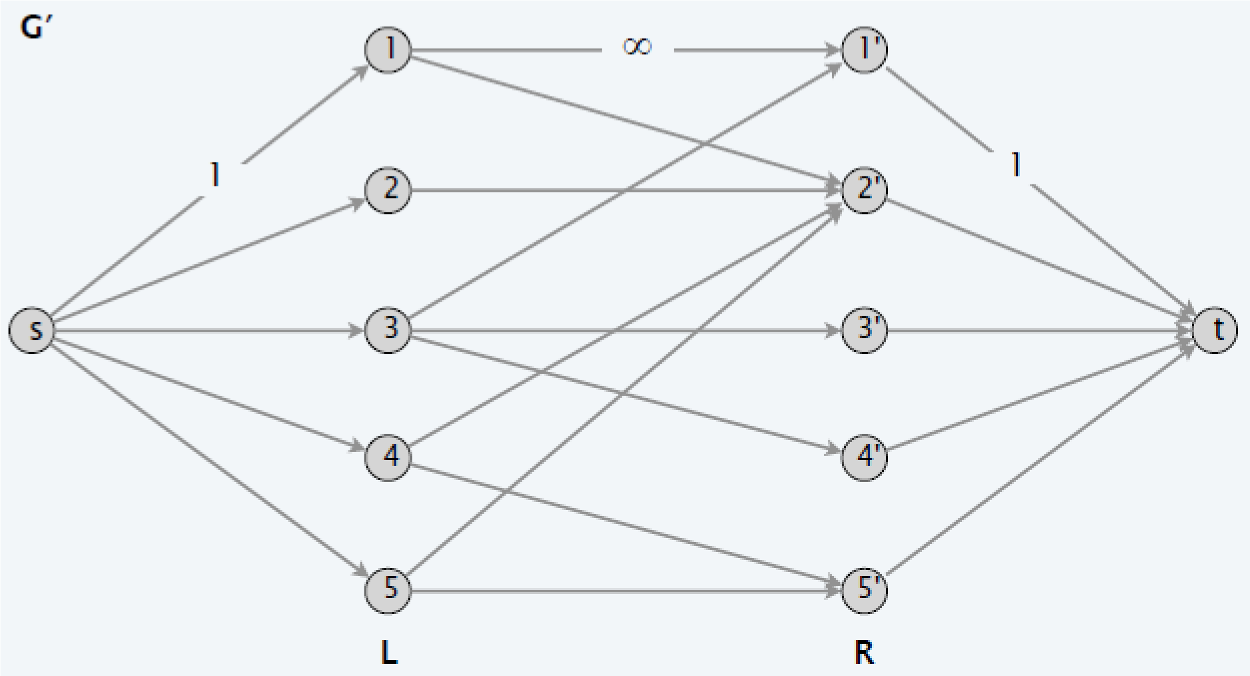
The Ford-Fulkerson algorithm can be implemented to solve the bipartite matching problem.



Bipartite Matching: Max-Flow Formulation

- Create digraph $G' = (L \cup R \cup \{s,t\}, E')$.
- Direct all edges from L to R, and assign infinite (or unit) capacity.
- Add source s, and unit-capacity edges from s to each node in L.
- Add sink t, and unit-capacity edges from each node in R to t.

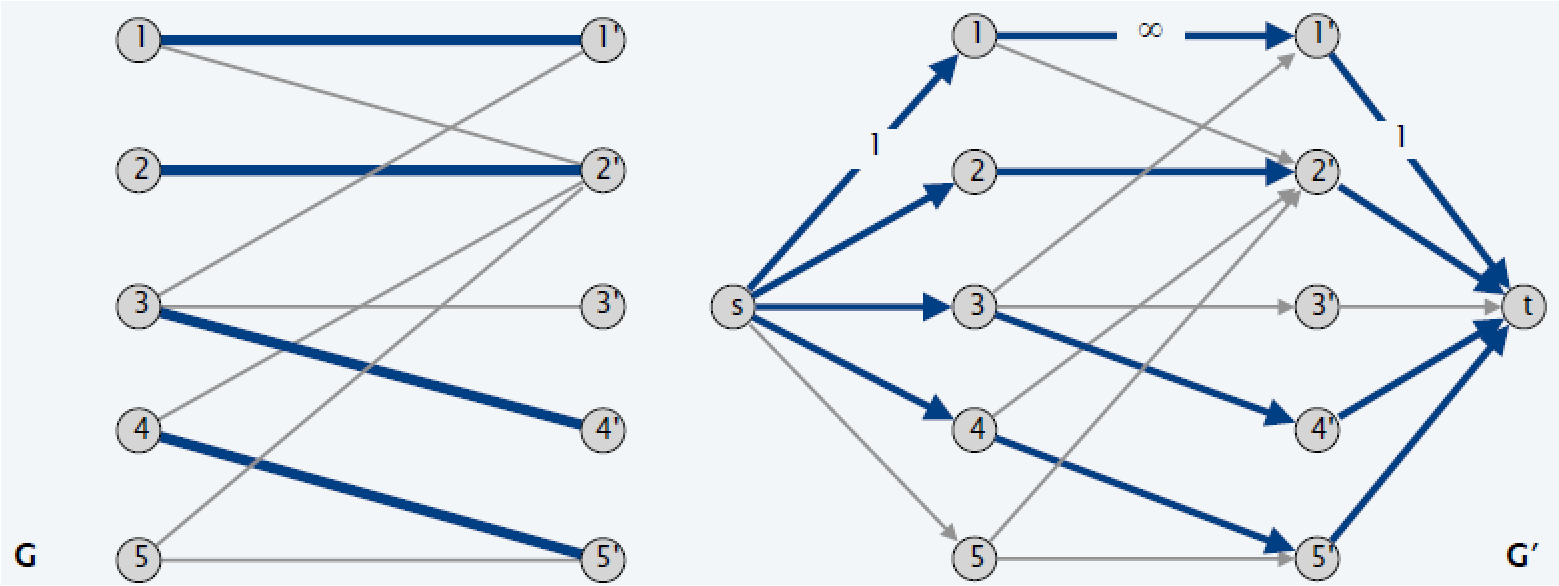


Max-Flow Formulation: Proof of Correctness

Theorem. Max cardinality of a matching in G = value of max flow in G'.

Pf. $\leq$

- Given a max matching M of cardinality k.
- Consider flow f that sends 1 unit along each of k paths.
- f is a flow, and has value k.



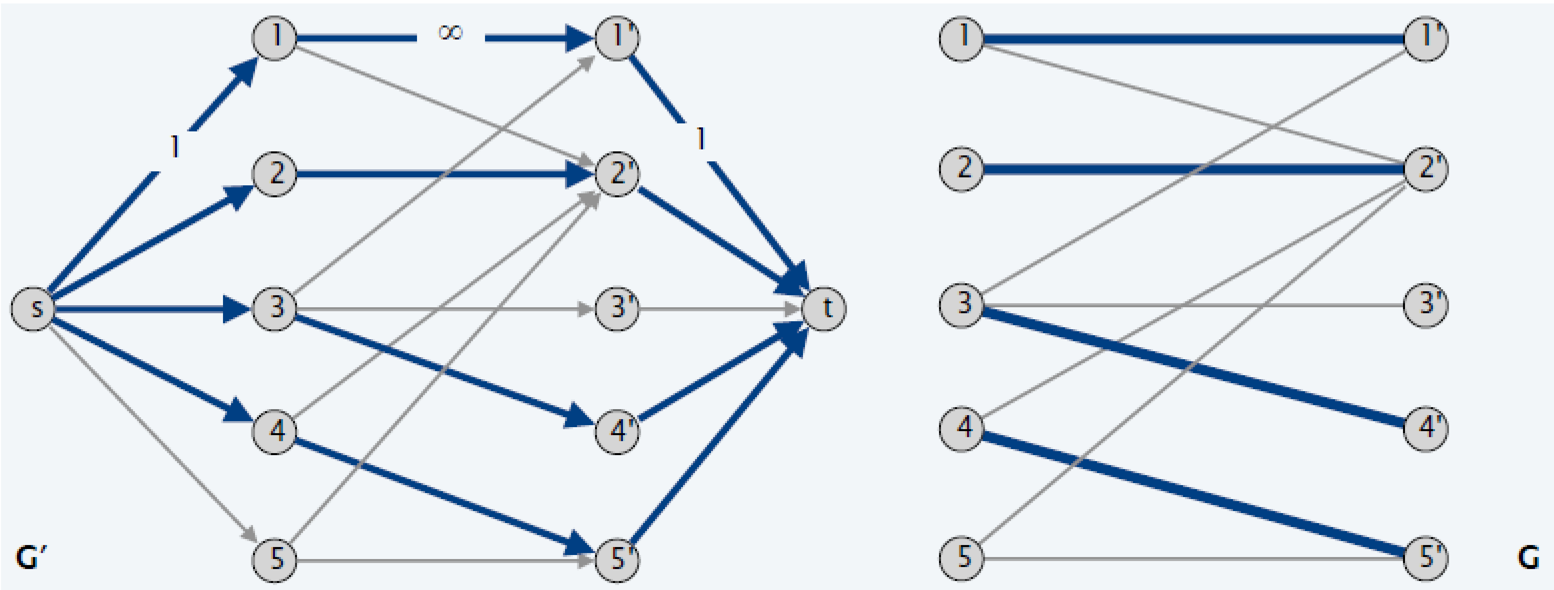
Max-Flow Formulation: Proof of Correctness

Theorem. Max cardinality of a matching in G = value of max flow in G'.

Pf. $\geq$

- Let f be a max flow in G' of value k.
- Integrality theorem $\Longrightarrow k$ is integral and can assume f is 0-1.
- Consider M = set of edges from L to R with $f(e) = 1$.

- Each node in L and R participates in at most one edge in M
- $|M| = k$: consider cut $(L \cup \{s\}, R \cup \{t\})$.



Perfect Matching in a Bipartite graph

Def. Given a graph $G = (V, E)$, a subset of edges $M \subseteq E$ is a perfect matching if each node appears in exactly one edge in M.

Q. When does a bipartite graph have a perfect matching?

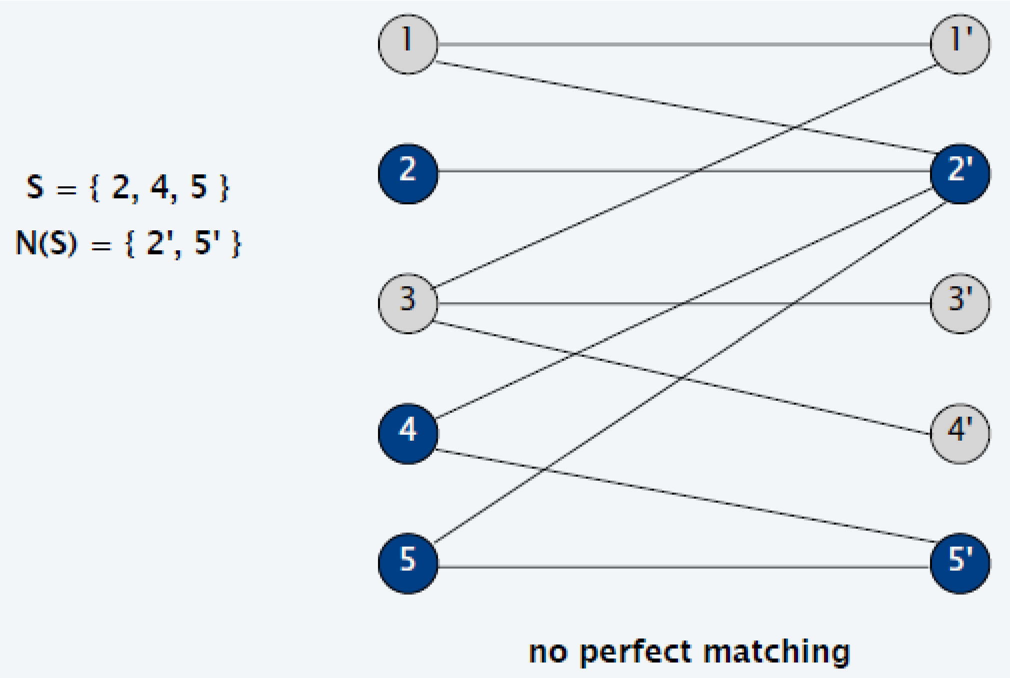
Structure of bipartite graphs with perfect matching.
- Clearly we must have $|L| = |R|$.
- What other conditions are necessary?
- What conditions are sufficient?

Perfect Matching in a Bipartite graph

Notation. Let S be a subset of nodes, and let $N(S)$ be the set of nodes adjacent to nodes in S.

Observation. If a bipartite graph $G = (L \cup R, E)$ has a perfect matching, then $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

Pf. Each node in S has to be matched to a different node in $N(S)$.



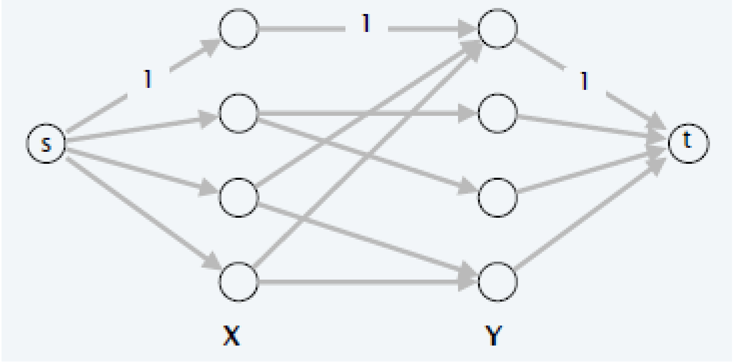
Bipartite Matching

Bipartite matching. Can solve via reduction to maximum flow.

Flow. During Ford-Fulkerson, all residual capacities and flows are 0-1; flow corresponds to edges in a matching M.

Residual graph G_M simplifies to:

- If $(x, y) \notin M$, then (x, y) is in G_M.
- If $(x, y) \in M$, then (y, x) is in G_M.



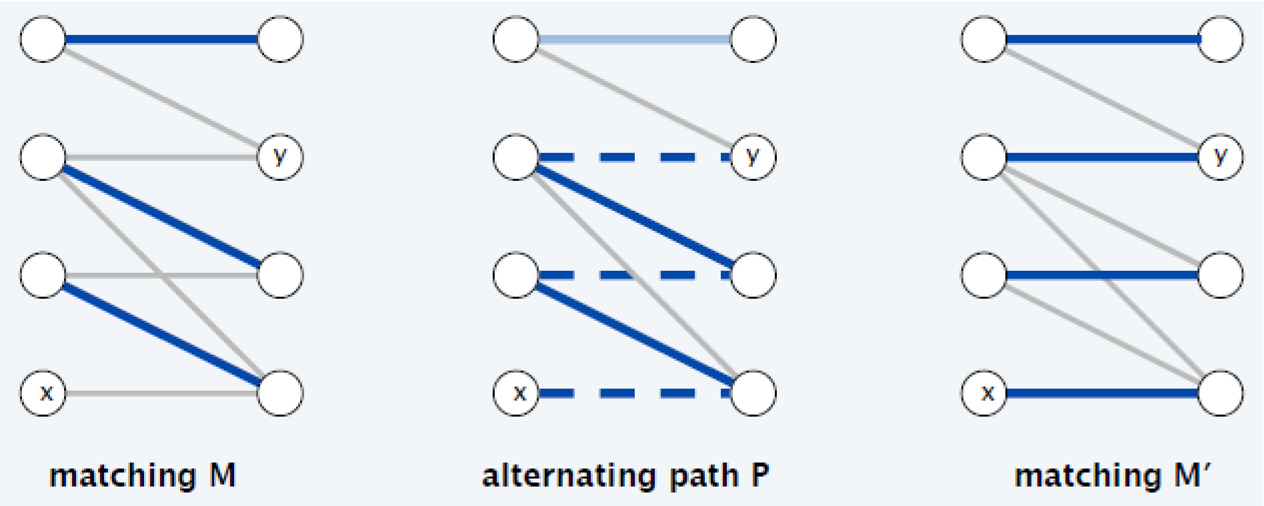
Augmenting path simplifies to:

- Edge from s to an unmatched node $x \in X$,
- Alternating sequence of unmatched and matched edges,
- Edge from unmatched node $y \in Y$ to t.

Alternating Path

Def. An alternating path P with respect to a matching M is an alternating sequence of unmatched and matched edges, starting from an unmatched node $x \in X$ and going to an unmatched node $y \in Y$.

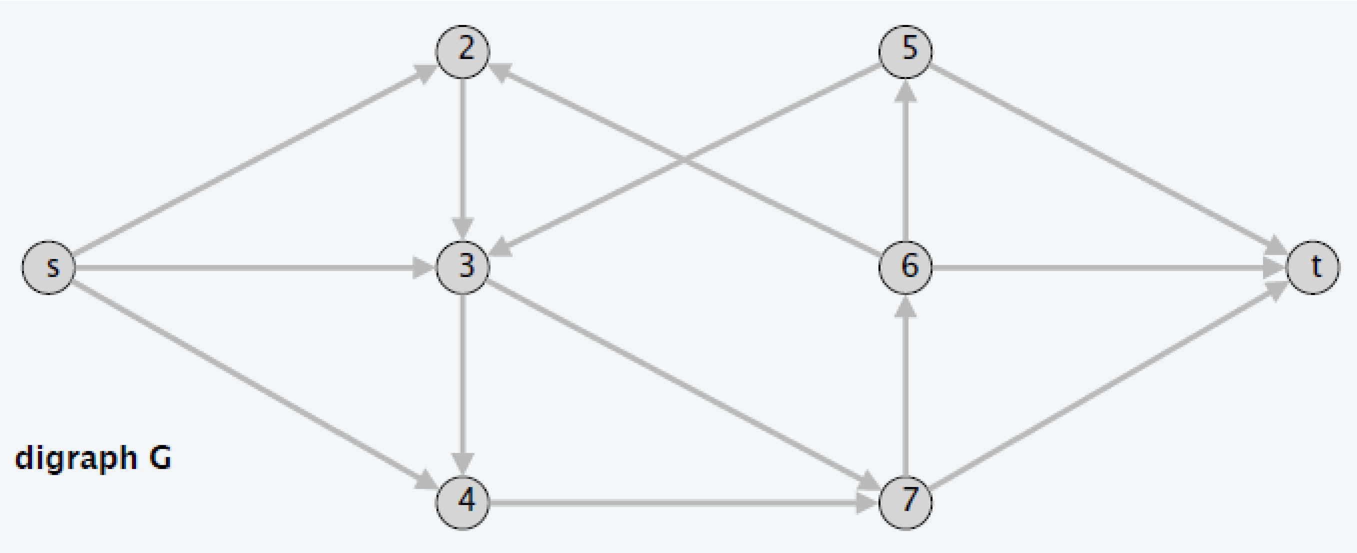
Key property. Can use P to increase by one the cardinality of the matching.



Edge-Disjoint Paths

Def. Two paths are edge-disjoint if they have no edge in common.

Disjoint path problem. Given a digraph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint $s \to t$ paths.



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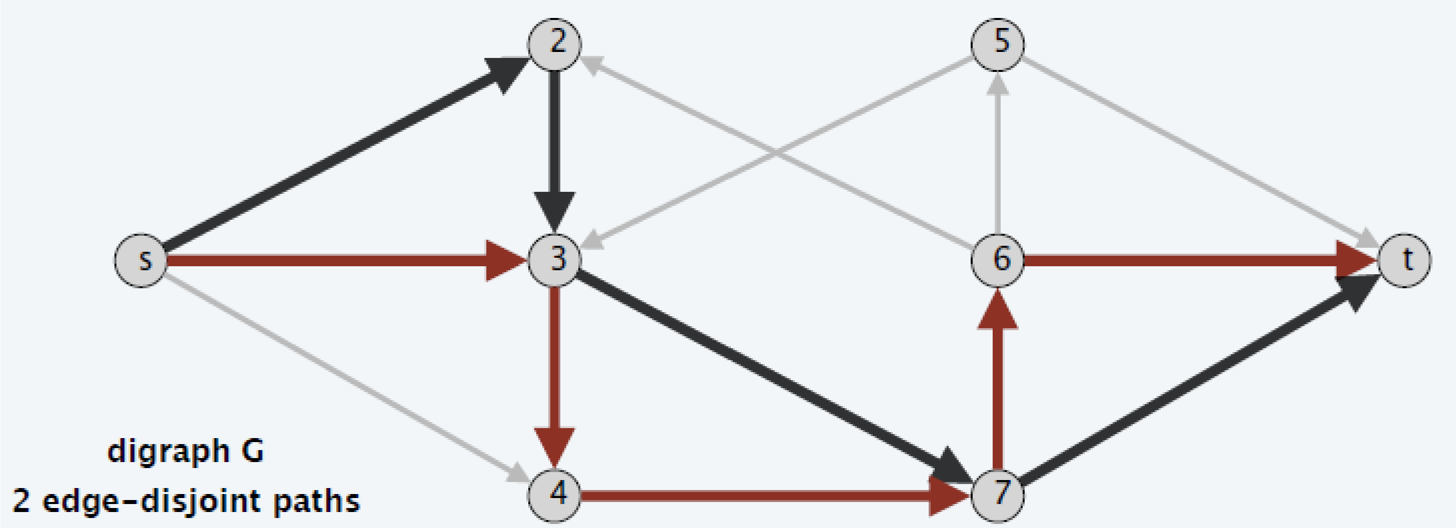


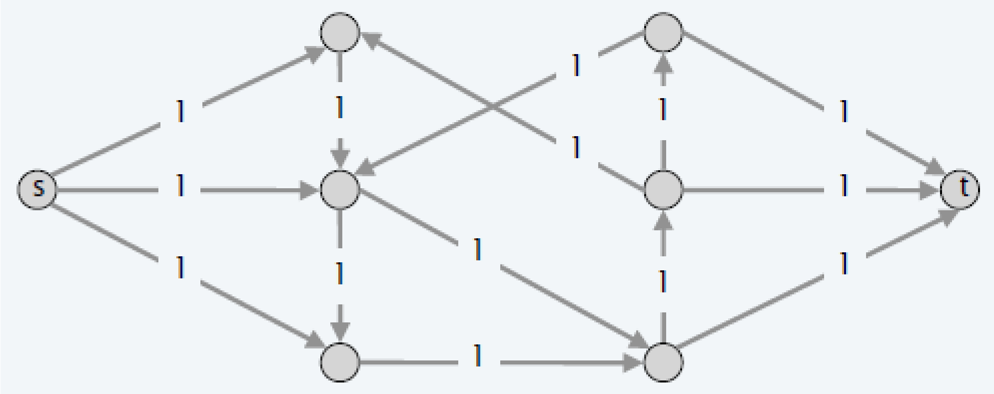
Edge-Disjoint Paths

Max-flow formulation. Assign unit capacity to every edge.

Theorem. Max number of edge-disjoint $s \to t$ paths equals value of max flow.

Pf. $\leq$

- Suppose there are k edge-disjoint $s \to t$ paths $P_1, \ldots, P_k$.
- Set $f(e) = 1$ if e participates in some path P_j, else set $f(e) = 0$.
- Since paths are edge-disjoint, f is a flow of value k.



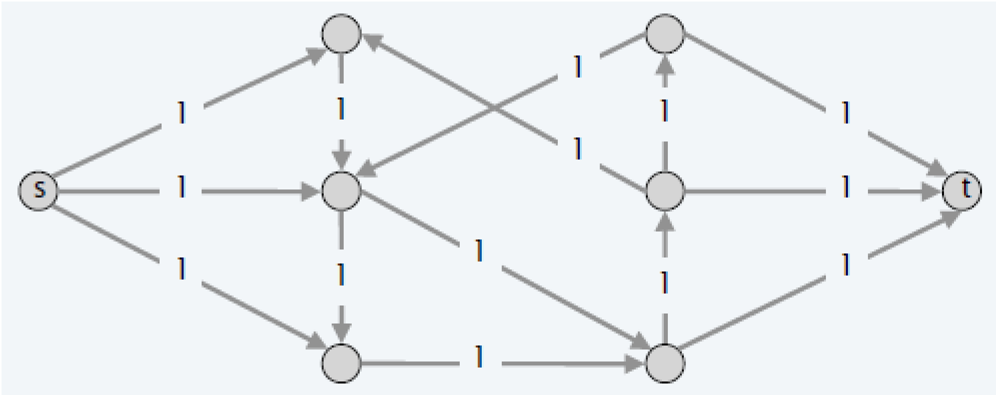
Edge-Disjoint Paths

Max-flow formulation. Assign unit capacity to every edge.

Theorem. Max number of edge-disjoint $s \to t$ paths equals value of max flow.

Pf. $\geq$

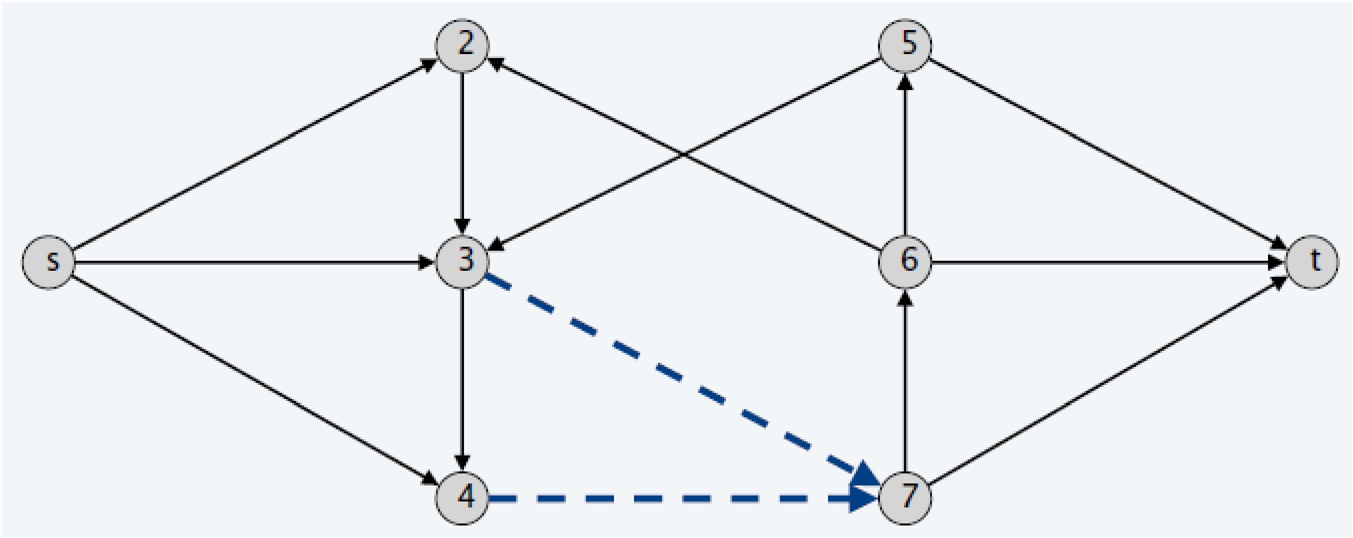
- Suppose max flow value is k.
- Integrality theorem $\Longrightarrow$ there exists 0-1 flow f of value k.
- Consider edge (s, u) with $f(s, u) = 1$.
  - By flow conservation, there exists an edge (u, v) with $f(u, v) = 1$
  - Continue until reach t, always choosing a new edge
- Produces k edge-disjoint paths.



Network Connectivity

Def. A set of edges $F \subseteq E$ disconnects t from s if every $s \rightarrow t$ path uses at least one edge in F.

Network connectivity. Given a digraph $G = (V, E)$ and two nodes s and t, find min number of edges whose removal disconnects t from s.

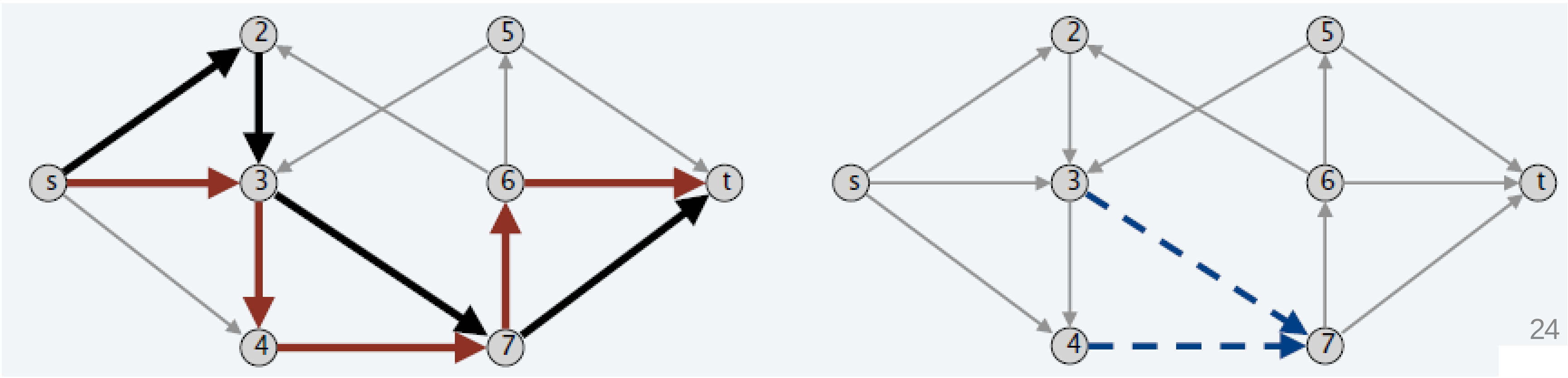


Menger's Theorem

Theorem. The max number of edge-disjoint $s \to t$ paths equals the min number of edges whose removal disconnects t from s.

Pr. $\leq$

- Suppose the removal of $F \subseteq E$ disconnects t from s, and $|F| = k$.
- Every $s \to t$ path uses at least one edge in F.
- Hence, the number of edge-disjoint paths is $\leq k$.

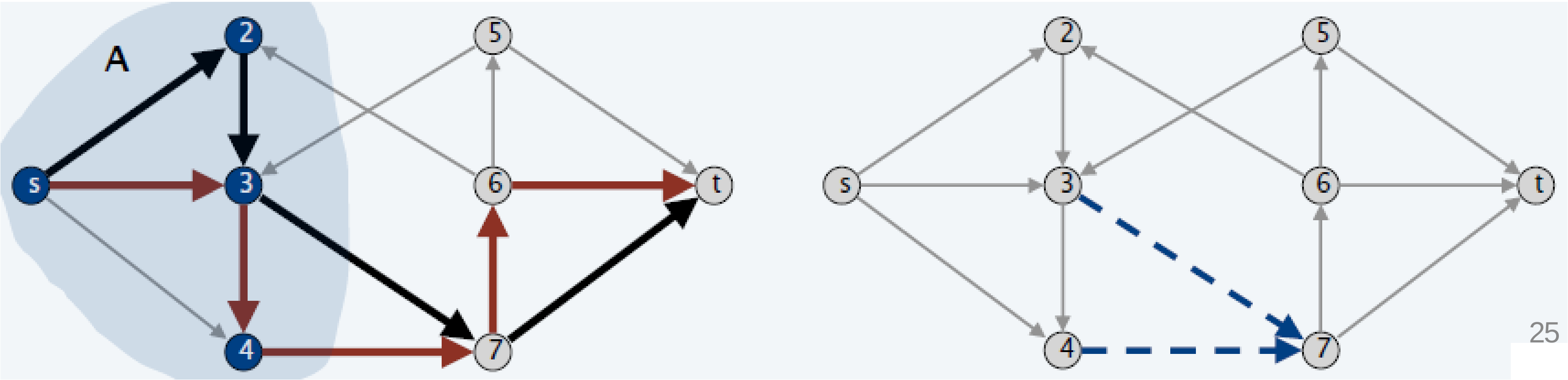


Menger's Theorem

Theorem. The max number of edge-disjoint $s \to t$ paths equals the min number of edges whose removal disconnects t from s.

Pr. $\geq$

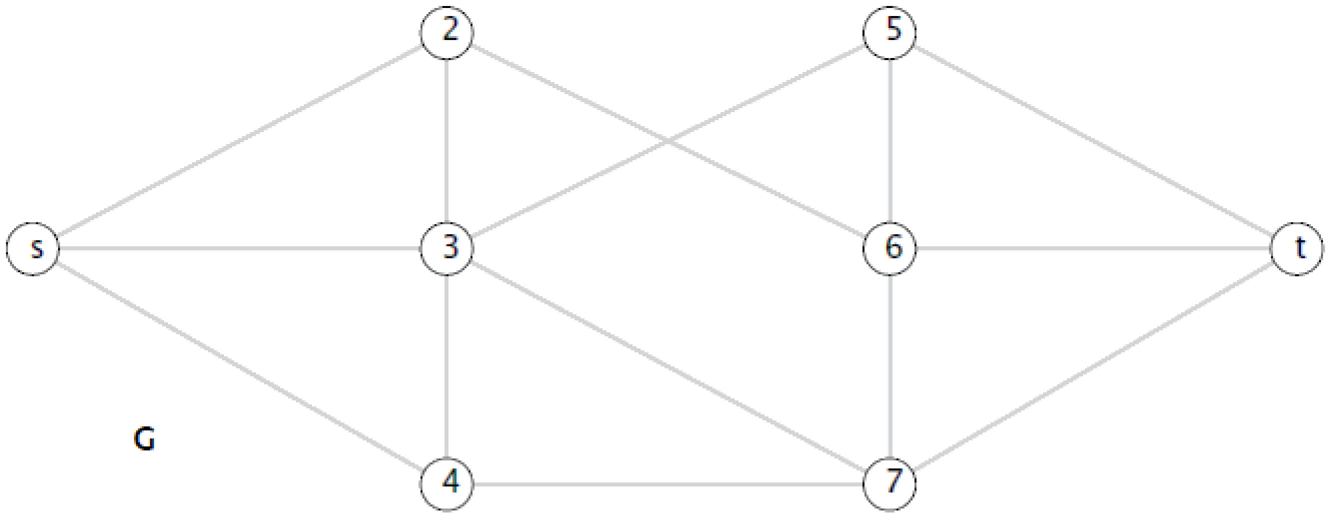
- Suppose max number of edge-disjoint paths is k.
- Then value of max flow $= k$.
- Max-flow min-cut theorem $\Longrightarrow$ there exists a cut (A, B) of capacity k.
- Let F be set of edges going from A to B.
- $|F| = k$ and disconnects t from s.



Edge-Disjoint Paths in Undirected Graphs

Def. Two paths are edge-disjoint if they have no edge in common.

Disjoint path problem in undirected graphs. Given a graph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint $s \to t$ paths.



G

Edge-Disjoint Paths in Undirected Graphs

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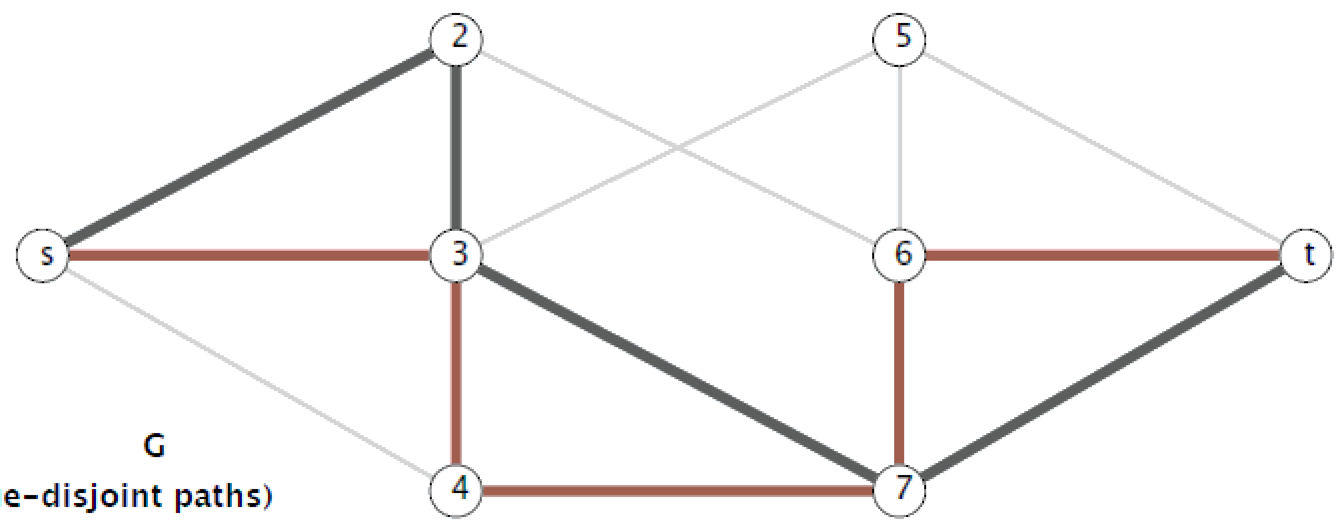
Disjoint path problem in undirected graphs. Given a graph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint $s \to t$ paths.



G
(2 edge-disjoint paths)

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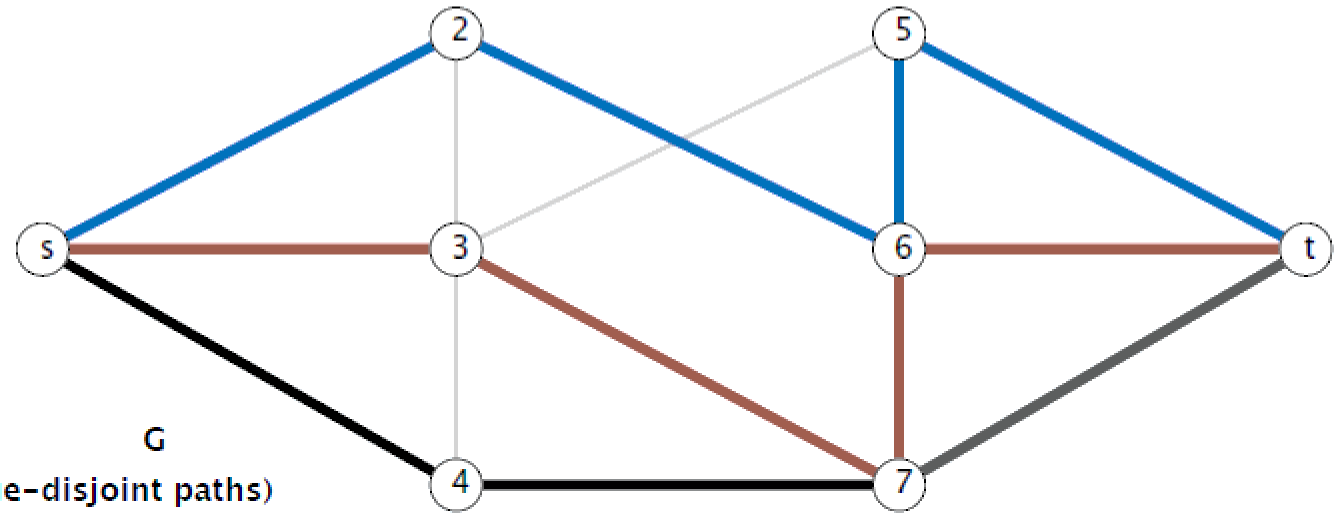


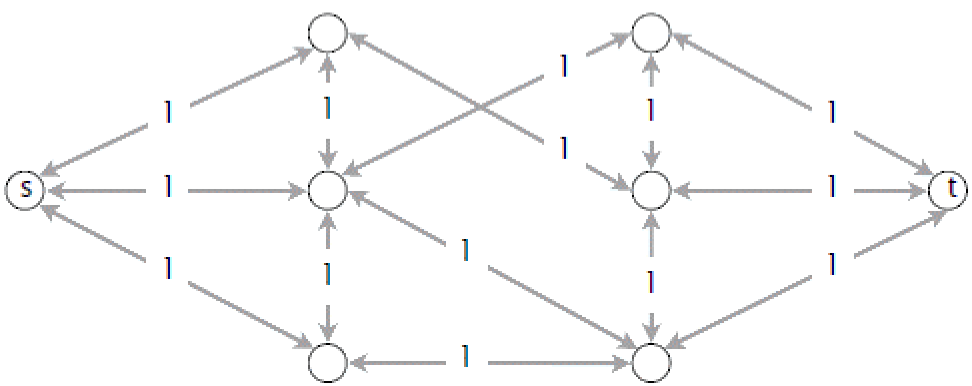
G
(3 edge-disjoint paths)

Edge-Disjoint Paths in Undirected Graphs

Max-flow formulation. Replace each edge with two antiparallel edges and assign unit capacity to every edge.

Observation. Two paths P_1 and P_2 may be edge-disjoint in the digraph but not edge-disjoint in the undirected graph.

If P_1 uses edge (u, v) and P_2 uses its antiparallel edge (v, u)



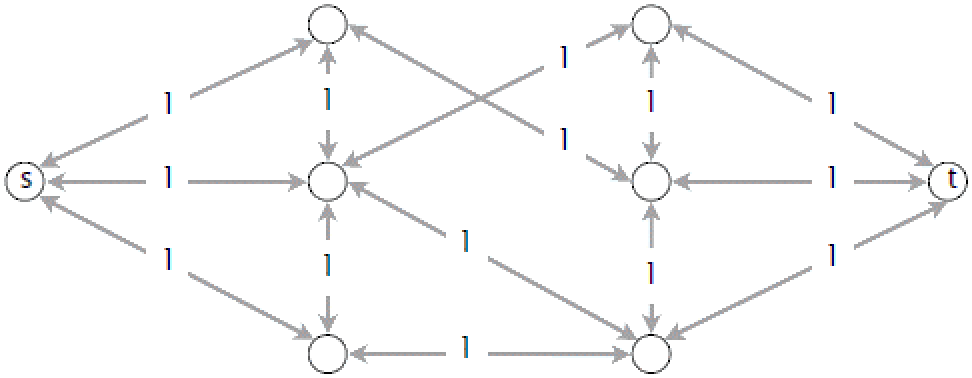
Edge-Disjoint Paths in Undirected Graphs

Max-flow formulation. Replace each edge with two antiparallel edges and assign unit capacity to every edge.

Lemma. In any flow network, there exists a maximum flow f in which for each pair of antiparallel edges e and e', either $f(e) = 0$ or $f(e') = 0$ or both. Moreover, integrality theorem still holds.

Pf. [by induction on number of such pairs of antiparallel edges]

- Suppose $f(e) > 0$ and $f(e') > 0$ for a pair of antiparallel edges e and e'.
- Set $f(e) = f(e) - \delta$ and $f(e') = f(e') - \delta$, where $\delta = \min\{f(e), f(e')\}$.
- f is still a flow of the same value but has one less such pair.



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Theorem. Max number edge-disjoint $s \to t$ paths equals value of max flow.