Modeling and Control of Elastic Joint Robots

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In this paper we study the modeling and control of robot manipulators with elastic joints. We first derive a simple model to represent the dynamics of elastic joint manipulators. The model is derived under two assumptions regarding dynamic coupling between the actuators and the links, and is useful for cases where the elasticity in the joints is of greater significance than gyroscopic interactions between the motors and links. In the limit as the joint stiffness tends to infinity, our model reduces to the usual rigid model found in the literature, showing the reasonableness of our modeling assumptions. We show that our model is significantly more tractable with regard to controller design than previous nonlinear models that have been used to model elastic joint manipulators. Specifically, the nonlinear equations of motion that we derive are shown to be globally linearizable by diffeomorphic coordinate transformation and nonlinear static state feedback, a result that does not hold for previously derived models of elastic joint manipulators. We also detail an alternate approach to nonlinear control based on a singular perturbation formulation of the equations of motion and the concept of integral manifold. We show that by a suitable nonlinear feedback, the manifold in state space which describes the dynamics of the rigid manipulator, that is, the manipulator without joint elasticity, can be made invariant under solutions of the elastic joint system. The implications of this result for the control of elastic joint robots are discussed.

1 Introduction

The proper choice of mathematical models for control system design is a crucial stage in the development of control strategies for any system. This is particularly true for robot manipulators due to their complicated dynamics. For simulation purposes one would like as detailed a model as possible, while for control design and implementation one would like to retain only the most significant dynamic effects in the model in order to simplify the analysis and minimize on-line computational requirements.

Because of the extreme complexity of the dynamic equations of motion for n-link manipulators with joint elasticity, most existing results on the control of such manipulators have relied either on computer programs to generate the equations [17] or have treated special configurations [26] and/or single link examples [27]. However, one generally obtains relatively little insight from symbolically generated equations, and an understanding of the physics underlying the model is of prime importance in understanding the control problem. For this reason we first investigate the problem of modeling the dynamics of elastic joint manipulators. For notational simplicity we treat the case of revolute joints driven by DC-motors whose rotors are elastically coupled to the links. It turns out that by making two rather simple approximating assumptions it is possible to derive a model of the system that is much more amenable to analysis and control than previous models.

Specifically, we assume

(A1) That the kinetic energy of the rotor is due mainly to its own rotation. Equivalently, the motion of the rotor is a pure rotation with respect to an inertial frame. We further assume.

(A2) The the rotor/gear inertia is symmetric about the rotor axis of rotation so that the gravitational potential of the system and also the velocity of the rotor center of mass are both independent of the rotor position.

Assumption (A2) hardly needs any justification and Assumption (A1) is easy to justify for a large class of robots, since roughly speaking it amounts to neglecting terms of order at most $1/m$ where m:1 is the gear ratio. In fact, most existing models of rigid manipulators are derived under precisely these same assumptions; see for example Paul [4], equation (6.49). The important point is to model the dynamic effects which are dominant, in this case the joint elasticity.

2 Modeling

We now consider an n-link manipulator with revolute joints actuated by DC-motors, and model the elasticity of the ith joint as a linear torsional spring with stiffness k_i. For notational simplicity we take $k_i \equiv k$ for all i. Because of the additional degrees of freedom introduced by the elastic coupling of the motor shaft to the links we model the rotor of each actuator as a "fictitious link," that is, as an additional rigid body in the chain with its own inertia. Thus the manipulator consists of n "actual" links and n "fictitious" or rotor links.

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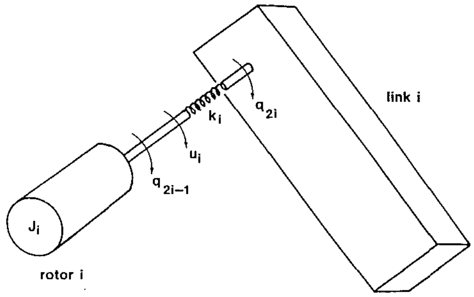

Fig. 1 Elastic Joint

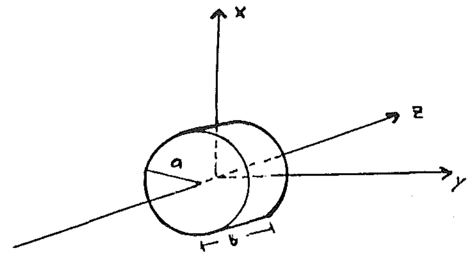

Fig. 2

The specific design of the manipulator will dictate the manner in which the actuators are coupled to the links. For simplicity we discuss the case in which the rotor is directly coupled to the link that it actuates as shown in Fig. 1. Other configurations, for example when the motors are located on link 1 and drive the distal links through cables, etc., can be handled by finding the corresponding transformation between "actuator space" and "joint space" as in [34]. The details are omitted.

Referring to Fig. 1, let $\mathbf{q}=(q_1, \ldots q_{2n})^T$ be a set of generalized of generalized coordinates for the system where

$$q_{2i} = \text{the angle of link } i, \; i=1, \ldots, n \tag{2.1}$$

$$q_{2i-1} = -\frac{1}{m_i}\theta_i, \; i=1, \ldots, n \tag{2.2}$$

where θ_i is the angular displacement of rotor i and m_i is the gear ratio. In this case then $q_{2i}-q_{2i-1}$ is the elastic displacement of link i.

Lagrangian Dynamics The rotor, as an intermediate link, now has its own coordinate frame and inertia tensor associated with it. We shall model the "rotor" link as a right circular cylinder of radius a and length b. From symmetry consideration we may establish the coordinate frame at the center of mass and assume that coordinate axes are principal axes of the cylinder, with the rotor angle θ_i measured about the z_{2i} axis. The inertia tensor of the rotor is then given by

$$I_i = \begin{bmatrix} I_{xx_i} & 0 & 0 \\ 0 & I_{yy_i} & 0 \\ 0 & 0 & I_{zz_i} \end{bmatrix} \tag{2.3}$$

where I_{xx}, I_{yy}, I_{zz} are the moments of inertia of the rotor about the principal axes. The kinetic energy of the rotor is

$$K_{r_i} = \frac{1}{2}M_i v_i^T v_i + \frac{1}{2}\,\omega_i^T I_i \omega_i \tag{2.4}$$

where v_i represents the velocity of the center of mass of the rotor, M_i is the rotor mass, and ω_i is the vector of angular velocities about the principal axes.

Now by the symmetry assumption (A2) the velocity v_i of the center of mass of the rotor can written as a function only of the link variables $q_2, \ldots, q_{2i-2}$. If we therefore include the rotor mass as part of link $2i-2$ for the purposes of calculating the inertia tensor of link $2i-2$ then the first term in (2.4) will be included with the kinetic energy of link $2i-2$.

We now invoke Assumption (A1) and model only the kinetic energy of the rotor about its principal axis of rotation, i.e., we assume that the second term in (2.4) above is given as

$$\frac{1}{2}\omega_i^T I_i \omega_i = \frac{1}{2} I_{zz_i}\dot{\theta}_i^2 \tag{2.5}$$

$$= \frac{1}{2} I_{zz_i} m_i^2 \dot{q}_{2i-1}^2$$

The following example gives a simple illustration of the effect of the above assumption.

Example Consider the cylinder shown in Fig. 2. The rotational kinetic energy is then

$$K = \frac{1}{2}(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2) \tag{2.6}$$

The principal moments of inertia of the cylinder with respect to the coordinate system shown are given by

$$I_{xx} = \frac{1}{4}Mb^2 = I_{yy} \tag{2.7}$$

$$I_{zz} = \frac{1}{2}Ma^2 \tag{2.8}$$

Due to the gear ratio m:1 the angular velocity ω_z will generally be a factor of m larger than the angular velocities about the other two axes. If we take therefore $\omega_z = m\omega_x = m\omega_y$ for the purposes of illustration, the kinetic energy becomes

$$K = \frac{1}{4}M\omega_z^2(a^2 + b^2/m^2) \tag{2.9}$$

We now approximate according to (A1) the kinetic energy K as

$$\bar{K} = \frac{1}{2}M\omega_z^2 a^2 \tag{2.10}$$

The percent error in the kinetic energy incurred by using the expression (2.10) instead of the true kinetic energy (2.9) is then

$$Error = \frac{K-\bar{K}}{K} \times 100 \tag{2.11}$$

$$= \frac{b^2}{b^2 + m^2 a^2} \times 100 \tag{2.12}$$

For example if $a=1$, $b=1/2$, and $m=100$, the percent error in kinetic energy is 0.01 percent.

Let us now partition the generalized coordinate vector $\mathbf{q}$ as $(\mathbf{q}_1, \mathbf{q}_2)^T$ where

$$\mathbf{q}_1 = (q_2, q_4, \ldots, q_{2n})^T \tag{2.13}$$

$$\mathbf{q}_2 = (q_1, q_3, \ldots, q_{2n-1})^T \tag{2.14}$$

In other words $\mathbf{q}_1$ is the vector of link variables and $\mathbf{q}_2$ is the vector of actuator variables (divided by the gear ratio).

We have shown by the previous discussion then that the kinetic energy of the system under our modeling assumption (A1) is

$$K = \frac{1}{2}\dot{\mathbf{q}}_1^T D(\mathbf{q}_1)\dot{\mathbf{q}}_1 + \frac{1}{2}\dot{\mathbf{q}}_2^T J \dot{\mathbf{q}}_2 \tag{2.15}$$

where $D(\mathbf{q}_1)$ is the inertia of the "rigid" robot

$$D(\mathbf{q}_1) = (d_{ij}(\mathbf{q}_1)) \tag{2.16}$$

which can be calculated using standard techniques, (e.g., formula 6.66 in [4]) once the rotor masses are included as part of the proximal links for the calculation of the latter's inertia tensor. The $n \times n$ matrix J is given by

$$J = diag\left[m_1^2 I_{zz_1}, \ldots, m_n^2 I_{zz_n}\right] \tag{2.17}$$

where the diagonal elements are the motor inertias about their principal axes of rotation multiplied by the square of the respective gear ratios.

We now invoke our second assumption (A2) again that the rotor inertia is symmetric about its axis of rotation. This implies that the gravitational potential is a function only of $\mathbf{q}_1$. Therefore the total potential energy of the system is

$$P = P_1(\mathbf{q}_1) + P_2(\mathbf{q}_1 - \mathbf{q}_2) \tag{2.18}$$

where, as in the case of the kinetic energy, the potential energy term P_1 is found from standard formulae for rigid robots (e.g., formula 6.54 in [4]). The second term above is due to the elastic potential of the spring and is given as

$$P_2 = \frac{1}{2}k(\mathbf{q}_1 - \mathbf{q}_2)^T(\mathbf{q}_1 - \mathbf{q}_2). \tag{2.19}$$

The Lagrangian $L = K - P$ of the system is now given by

$$L = \frac{1}{2}\dot{\mathbf{q}}_1^T D(\mathbf{q}_1)\dot{\mathbf{q}}_1 + \frac{1}{2}\dot{\mathbf{q}}_2 J\dot{\mathbf{q}}_2 - P_1(\mathbf{q}_1) - \frac{1}{2}k(\mathbf{q}_1 - \mathbf{q}_2)^T(\mathbf{q}_1 - \mathbf{q}_2) \tag{2.20}$$

and the equations of motion are found from the Euler-Lagrange equations [4] using (2.20) to be

$$D(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + c(\mathbf{q}_1, \dot{\mathbf{q}}_1) + k(\mathbf{q}_1 - \mathbf{q}_2) = 0 \tag{2.21}$$

$$J\ddot{\mathbf{q}}_2 - k(\mathbf{q}_1 - \mathbf{q}_2) = u. \tag{2.22}$$

The $n \times n$ matrix $D(\mathbf{q}_1)$ is symmetric, positive definite for each $\mathbf{q}_1$. The vector $C(\mathbf{q}_1, \dot{\mathbf{q}}_1)$ contains coriolis, centripetal, and gravitational forces and torques, and can be expressed as

$$c(\mathbf{q}_1, \dot{\mathbf{q}}_1) = \dot{D}\dot{\mathbf{q}}_1 - \frac{1}{2}\dot{\mathbf{q}}_1{}_1^T\frac{\partial D}{\partial \mathbf{q}_1}\dot{\mathbf{q}}_1 - \frac{\partial P_1}{\partial \mathbf{q}_1} \tag{2.23}$$

We note however that the gyroscopic forces between each rotor and the other links are not included in this expression as a result of Assumption (A1). It is interesting to compare the simplicity of our model (2.21)-(2.22) with other models of elastic joint manipulators that have been derived in the literature [13, 15, 18, 20, 23, 29]. It turns out that (2.21)-(2.22) is structurally similar to the models used in [26] and [27]. Thus our model can be viewed as the general n-degree-of-freedom extension to the models in the latter references.

Interestingly enough, our model is also the direct extension to the case of elastic joints of the familar rigid models that have become standard in the literature. In fact, we can easily see that the usual rigid model can be recovered from (2.21)-(2.22) as the joint stiffness parameters k_i tend to infinity. To see this we assume that in the limit as $\mathbf{k} \rightarrow \infty$ there is no elastic deformation, so that

$$\mathbf{q}_1 = \mathbf{q}_2, \dot{\mathbf{q}}_1 = \dot{\mathbf{q}}_2. \tag{2.24}$$

On the other hand the force $k(\mathbf{q}_1 - \mathbf{q}_2)$ transmitted through the coupling between the rotor and link remains finite in the rigid case, i.e., as $k \rightarrow \infty$, and it follows that the potential energy P_2 in (2.18) satifies

$$\frac{1}{2}k(\mathbf{q}_1 - \mathbf{q}_2)^T(\mathbf{q}_1 - \mathbf{q}_2) \rightarrow 0 \tag{2.25}$$

as $k \rightarrow \infty$ and $\mathbf{q}_2 - \mathbf{q}_1 \rightarrow 0$. Therefore the Lagrangian of the rigid system L_r is obtained from (2.20) as

$$L_r = \frac{1}{2}\dot{\mathbf{q}}_1^T(D(\mathbf{q}_1) + J)\dot{\mathbf{q}}_1 - P_1(\mathbf{q}_1) \tag{2.26}$$

which leads to the equations of motion for the rigid system by applying the Euler-Lagrange equations to (2.26)

$$D(\mathbf{q}_1 + J)\ddot{\mathbf{q}}_1 + c(\mathbf{q}_1, \dot{\mathbf{q}}_1) = u \tag{2.27}$$

The interesting implication of this is that the usual textbook model of rigid robots is subject to the same assumptions (A1) and (A2) that we use to derive the elastic joint model. Gyroscopic forces due to the rotation of the actuators are thus not considered in most existing rigid models. See [32] for an exception to this statement which does consider the modeling of these gyroscopic terms in the case of the rigid joints. It is of interest to note that [32] concluded that these gyroscopic terms can indeed by neglected in most cases.

3 Feedback Linearization

It is well known that the rigid robot equations (2.27) may be globally linearized and decoupled by nonlinear feedback. This is just the familiar inverse dynamics control scheme which transforms (2.27) into a set of double integrator equations which can then be controlled by adding an "outer loop" control [31].

The above technique of inverse dynamics control is now understood as a special case of a more general procedure for transforming a nonlinear system to a linear system, known as *external* or *feedback linearization*.

Definition 3.1: A nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{n} g_i(x)u_i \\ &= f(x) + G(x)u \end{aligned} \tag{3.1}$$

is said to be *feedback linearizable* in a neighborhood U_o of the origin if there is a diffeomorphism $T: U_o \rightarrow R^n$ and nonlinear feedback

$$u = \alpha(x) + \beta(x)v \tag{3.2}$$

such that the transformed state

$$y = T(x) \tag{3.3}$$

satisfies the linear system

$$\dot{y} = Ay + Bv \tag{3.4}$$

where (A, B) is a controllable linear system.

Necessary and sufficient conditions for a system of the form (3.1) to be feedback linearizable are given in [3]. In the case of elastic joint robots, the feedback linearization property was investigated in [13] using computer generated models of the manipulator dynamics. These models are sufficiently complex, even for two link examples that another computer program was used in [13] to check the conditions for feedback linearization. The answer was negative, i.e., the elastic joint model derived in [13] is not general linearizable in this fashion. In this section we show that the new model (2.21)-(2.22) is always globally feedback linearizable according to Definition 3.1. Moreover we do not need symbolic programs to check linearizability or to compute the required state space change of coordinates or the nonlinear feedback law. These can be found by inspection.

We first write the system (2.21)-(2.22) in state space by setting

$$\begin{array}{llll} x_1 = q_1 & & x_2 = \dot{q}_1 & \\ x_3 = q_2 & & x_4 = \dot{q}_2 & \end{array} \tag{3.5}$$

Then we have from (2.21)-(2.22)

$$\dot{x}_1 = x_2 \tag{3.6}$$

$$\dot{x}_2 = -D(x_1)^{-1}\{c(x_1, x_2)+k(x_1-x_3)\} \tag{3.7}$$

$$\dot{x}_3 = x_4 \tag{3.8}$$

$$\dot{x}_4 = J^{-1}k(x_1-x_3)+J^{-1}u \tag{3.9}$$

Since the nonlinearities enter into the second equation above, while the control appears only in the last equation, it is not obvious that the system is linearizable nor can u immediately be chosen to cancel the nonlinearities as in the case of the rigid equations (2.24).

In order to check feedback linearizability of the above system one needs, in principle, to check rank conditions and involutivity of certain sets of vector fields formed by taking Lie brackets of the vector fields defining the state equations (3.6)-(3.9). Our model is simple enough, however, that we can show global feedback linearizabilty by directly computing the required change of coordinates and nonlinear feedback law. Moreover the new coordinates themselves turn out to have physical significance for the control problem at hand.

Consider now the nonlinear state space change of coordinates.

$$y_1 = T_1(x)=x_1 \tag{3.10}$$

$$y_2 = T_2(x)=\dot{T}_1=x_2 \tag{3.11}$$

$$\begin{aligned} y_3 &= T_3(x)=\dot{T}_2 \\ &= -D(x_1)^{-1}\{c(x_1, x_2)+k(x_1-x_3)\} \end{aligned} \tag{3.12}$$

$$\begin{aligned} y_4 &= T_4(x)=\dot{T}_3 \\ &= -\frac{d}{dt}D(x_1)^{-1}\{c(x_1, x_2)+k(x_1-x_3)\} \\ &\quad -D(x_1)^{-1}\Big\{\frac{\partial c}{\partial x_1}x_2 \\ &\quad +\frac{\partial c}{\partial x_2}\,(-D(x_1)^{-1}(c(x_1,\ x_2)+k(x_1-x_3))) \\ &\quad +k(x_2-x_4)\Big\} \\ :&= f_4(x_1, x_2, x_3)+D(x_1)^{-1}kx_4 \end{aligned} \tag{3.13}$$

where for simplicity we define the function f_4 to be everything in the definition of y_4 above except the last term, which is $D^{-1}kx_4$. Note that x_4 appears only in this last term so that f_4 depends only on x_1, x_2, x_3.

The above mapping is actually a global diffeomorphism. Its inverse is likewise found by inspection to be

$$x_1 = y_1 \tag{3.14}$$

$$x_2 = y_2 \tag{3.15}$$

$$x_3 = y_1+k^{-1}(D(y_1)y_3+c(y_1, y_2)) \tag{3.16}$$

$$x_4 = k^{-1}D(y_1)(y_4-f_4(y_1, y_2, y_3)). \tag{3.17}$$

The linearizing control law can now be found from the condition

$$\dot{y}_4 = v \tag{3.18}$$

where v is a new control input. Computing $\dot{y}_4$ from (3.13) and suppressing function arguments for brevity yields

$$\begin{aligned} v &= \frac{\partial f_4}{\partial x_1}x_2-\frac{\partial f_4}{\partial x_2}D^{-1}(c+k(x_1-x_3)) \\ &\quad +\frac{\partial f_4}{\partial x_3}x_4+\left(\frac{d}{dt}D^{-1}\right)kx_4+D^{-1}k(J^{-1}k(x_1-x_3)+J^{-1}u) \\ :&= F(x_1, x_2, x_3, x_4)+D(x_1)^{-1}kJ^{-1}u \end{aligned} \tag{3.19}$$

where $F(x_1, x_2, x_3, x_4)=F(x)$ denotes all the terms in (3.19) but the last term, which involves the input u.

Solving the above expression for u yields

$$u=Jk^{-1}D(x_1)(v-F(x)) \tag{3.20}$$

With the nonlinear change of coordinates (3.10)-(3.13) and nonlinear feedback (3.20) the system (3.6)-(3.9) now has the linear block form

$$\dot{y} = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix} y+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} v \tag{3.21}$$

where $I=n\times n$ identity matrix, $0=n\times n$ zero matrix, $y^T=(y_1^T, y_2^T, y_3^T, y_4^T)\epsilon R^{4n}$, and $v\epsilon R^n$.

The nonlinear control law (3.20) is not completely determined until the function v is specified. We will detail next one design scheme for v which guarantees robust tracking for the above system.

Closed Loop Performance and Robustness It is easy to determine from the linear system (3.21) with linear feedback control (3.22) what the response of the system in the y_i coordinate system will be. The corresponding response of the original coordinates x_i is not necessarily easy to determine since the nonlinear coordinate transformation (3.10)-(3.13) must be inverted to find the x_i. However, in this case the transformed coordinates y_i are themselves physically meaningful. Inspecting (3.10)-(3.13) we see that the variables y_1, y_2, y_3, y_4 are n-vectors representing, respectively, the link positions, velocities, accelerations, and jerks (derivative of the acceleration). Since the motion trajectory of the manipulator is typically specified in terms of these quantities [4], they are natural variables to use for control.

The issue of robustness to parameter uncertainty is an important one at this point. In order to control the linear system (3.21) either the y_i coordinates must be physically measurable, or the y_i must be computed from the measured x_i variable according to (3.10)-(3.13), or a robust observer for these variables must be constructed. In the first case, the required measurements may be difficult to obtain, although solid state accelerometers are now available which could greatly simplify the problem. In the second case, that of computing the y_i via (3.10)-(3.130), one needs accurate estimates of the parameters in the manipulator model. In the third case there are results on the design of nonlinear observers which could be applied to this problem [35].

The computation of the overall nonlinear control law (3.20) also requires knowledge of the model parameters. In what follows we assume that both the original variables x_i and the transformed variables y_i may be used for feedback, and we consider the robust tracking problem.

Following [33] we consider the transformed system

$$\dot{y} = y_2 \tag{3.23}$$

$$\dot{y}_2 = y_3$$

$$\dot{y}_3 = y_4$$

$$\dot{y}_4 = F(x) + D^{-1}kJ^{-1}u$$

$$:= -\beta(x)^{-1}\alpha(x) + \beta(x)^{-1}u$$

that is, $\beta(x) = Jk^{-1}D(x)$ and $\alpha = \beta(x)F(x)$.

Now the control law

$$u = \alpha(x) + \beta(x)\nu \qquad (3.24)$$

that is, (3.20), which ideally linearizes the system is unachievable in practice due to parameter uncertainty, computational roundoff, unknown disturbances, etc. It is more reasonable to assume a control law of the form

$$u = \hat{\alpha}(x) + \hat{\beta}(x)\nu \qquad (3.25)$$

where $\hat{\alpha}(x)$ and $\hat{\beta}(x)$ are estimated or computed values of $\alpha(x)$ and $\beta(x)$, respectively. In addition, the functions α and β are extremely complicated so that $\hat{\alpha}$ and $\hat{\beta}$ may represent intentional model simplification to facilitate real-time computation. In what follows $\|x\|$ denotes the usual L_2-norm or Euclidean norm of a vector $x \epsilon R^n$ and, for any matrix M, $\|M\|$ is the corresponding induced matrix norm, i.e.,

$$\|M\| = \sqrt{\lambda_{\max}(M^TM)}$$

where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of a matrix. We make the following assumptions on the functions α, $\hat{\alpha}$, β, $\hat{\beta}$.

(A3) There exist positive constants $\bar{\beta}$ and β such that

$$\beta \le \|\beta^{-1}(x)\| \le \bar{\beta} \qquad (3.26)$$

(A4) There is a positive constant $a < 1$ such that

$$\|\beta^{-1}\hat{\beta} - I\| \le a \qquad (3.27)$$

(A5) There is a known function $\phi(x,, t)$ such that

$$\|\hat{a} - \alpha\| \le \phi < \infty \qquad (3.28)$$

We note that (A4) can always be satisfied by suitable choice of $\hat{\beta}$. For example, the choice $\hat{\beta} = 1/cI$, where I is the identity matrix and the constant c is $1/2\,(\bar{\beta} + \beta)$ results in [36]

$$\|\beta^{-1}\hat{\beta} - I\| \le \frac{\bar{\beta} - \beta}{\bar{\beta} + \beta} < 1.$$

Now we substitute the control law (3.25) into (3.23) which results in

$$\dot{y}_4 = \beta^{-1}\hat{\beta}\nu + \beta^{-1}\Delta\alpha \qquad (3.29)$$

$$= \nu + E\nu + \beta^{-1}\Delta\alpha$$

where $\Delta\alpha = \hat{\alpha} - \alpha$, and $E := \beta^{-1}\hat{\beta} - I$. Note that $\|E\| \le a < 1$ from assumption (A4).

To track a desired trajectory $y_i^d(t)$ we first find K such that $\bar{A} := A + BK$ is stable, where A and B are defined by (3.21), and we set $\nu = \dot{y}_4^d(t) + Ke + \Delta\nu$, where e is the vector tracking error.

$$e(t) = \begin{bmatrix} y_1 - y_1^d \\ y_2 - y_2^d \\ y_3 - y_3^d \\ y_4 - y_4^d \end{bmatrix}$$

The above system may now be written in ''error space'' as

$$\dot{e} = Ae + B\{\Delta\nu + \Psi\} \qquad (3.30)$$

where Ψ is the nonlinear function (hereafter referred to as the (''uncertainty'') defined by

$$\Psi = E(\dot{y}_4^d + Ke + \Delta\nu) + \beta^{-1}\Delta\alpha \qquad (3.31)$$

The problem of robust trajectory tracking now reduces to the problem of stabilizing the system (3.30) by suitable choice of the additional input $\Delta\nu$. The above formulation is valid for any system that is feedback linearizable as is shown in [33] and any number of techniques can now be used to design the input $\Delta\nu$. However, the problem of stabilizing (3.30) in nontrivial since Ψ is a function of both e and $\Delta\nu$ and hence Ψ cannot be treated merely as a disturbance to be rejected by $\Delta\nu$. A more sophisticated analysis and design is required to guarantee stability of (3.30).

Approaches that can be used to design $\Delta\nu$ in (3.30) to guarantee robust tracking include Lyapunov and sliding mode designs [6], [5], high gain [8]and other approaches. We shall outline one approach to robust stabilization of feedback linearizable systems based on Lyapunov's second method. See [33] for the details and proofs.

First we note that from our assumptions on the uncertainty we have

$$\|\Psi\| \le a(\|\dot{y}_4^d\| + \|Ke\| + \|\Delta\nu\|) + \bar{\beta}\phi \qquad (3.32)$$

$$\le \bar{\phi} + a\|\Delta\nu\|$$

where $\bar{\phi} := a(\|\dot{y}_4^d\| + \|Ke\|) + \bar{\beta}\phi$. Suppose that we can simultaneously satisfy the inequalities

$$\|\Psi\| \le \rho(e, t) \qquad (3.33)$$

$$\|\Delta\nu\| \le \rho(e, t) \qquad (3.34)$$

for a known function $\rho(e, t)$. The function ρ can be determined as follows. First suppose that $\Delta\nu$ satisfies (3.34). Then from (3.32) we have

$$\|\Psi\| \le \bar{\phi} + a\rho := \rho \qquad (3.35)$$

This definition of ρ is well-defined since $a < 1$ and we have

$$\rho = \frac{1}{1-a}\bar{\phi}. \qquad (3.36)$$

It now follows from [33] that the null solution of (3.30) is uniformly asymptotically stable (in a generalized sense) if $\Delta\nu$ is chosen as

$$\Delta\nu = \begin{cases} -\rho\dfrac{B^TPe}{\|B^TPe\|}; & \text{if } \|B^TPe\| \ne 0 \\ 0; & \text{if } \|B^TPe\| = 0 \end{cases} \qquad (3.37)$$

where P is the unique positive definite solution to the Lyapunov equation

$$\bar{A}^TP + P\bar{A} = -Q \qquad (3.32)$$

for a given symmetric, positive definite Q. The argument is completed by noting that indeed $\|\Delta\nu\| \le \rho$.

4 Integral Manifold Approach

The above feedback linearizing control scheme requires measurement of the link positions and velocities, the motor positions and velocities as well as the link accelerations and jerks for successful implementation. In this section we present a differnt approach based on a reformulation of the dynamic equations (2.21)-(2.22) as a singularly perturbed system and the concept of integral manifold. In the case of weakly elastic joints, such as arise in harmonic drive gear elasticity, this approach has the advantage that it may be applied even when only the link position and velocity are available for feedback, provided that the system has a degree of natural damping at the joints. We will make this precise later.

Returning to the original system, we set

$$z = k(\mathbf{q}_2 - \mathbf{q}_1);\ \mu = \frac{1}{k} \tag{4.1}$$

Then z is the elastic force at the joints. If we now choose coordinates z and $\mathbf{q}_1$ we have from (2.21) and (2.22)

$$\begin{aligned} \ddot{\mathbf{q}}_1 &= -D(\mathbf{q}_1)^{-1}c(\mathbf{q}_1, \dot{\mathbf{q}}_1) - D(\mathbf{q}_1)^{-1}z \qquad (4.2) \\ :&= a_1(\mathbf{q}, \dot{\mathbf{q}}) + A_1(\mathbf{q})z \end{aligned}$$

where we henceforth drop the subscript on $\mathbf{q}$ for convenience. Likewise,

$$\begin{aligned} \mu\ddot{z} &= \ddot{\mathbf{q}}_1 - \ddot{\mathbf{q}}_2 \qquad (4.3) \\ &= -D(\mathbf{q}_1)^{-1}c(\mathbf{q}_1, \dot{\mathbf{q}}) - (D(\mathbf{q}_1)^{-1} + J^{-1})z - J^{-1}u \\ &= a_2(\mathbf{q}, \dot{\mathbf{q}}) + A_2(\mathbf{q})z + B_2 u \end{aligned}$$

In this case not that $a_2 = a_1$ and B_2 is constant and invertible. The model (4.2)-(4.3) is singularly perturbed. In the limit as $\mu \to 0$(4.2)-(4.3) reduces to the rigid equations of motion. In other words, by formally setting $\mu = 0$ in (4.3) and eliminating z from the equations, one obtains the rigid equations (2.24), as we now show.

Setting $\mu = 0$ in (4.3) and solving for z yields

$$z = -(D^{-1} + J^{-1})^{-1}(D^{-1}c + J^{-1}u) \tag{4.4}$$

which, when substituted into (4.2) yields

$$\begin{aligned} \ddot{q} &= -D^{-1}c + D^{-1}(D^{-1} + J^{-1})^{-1}(D^{-1}c + J^{-1}u) \qquad (4.5) \\ &= -D^{-1}c + D^{-1}(D^{-1} + J^{-1})D^{-1}c + D^{-1}(D^{-1}J^{-1})^{-1}J^{-1}u \end{aligned}$$

Now a straightforward calculation shows that the first two term in (4.5) above may be combined to yield

$$\begin{aligned} &-D^{-1}c + D^{-1}(D^{-1} + J^{-1})^{-1}D^{-1}c \qquad (4.6) \\ &= -D^{-1}c + D^{-1}J(D+J)^{-1}c \\ &= (-D^{-1}(D+J) + D^{-1}J)(D+J)^{-1}c \\ &= -(D+J)^{-1}c \end{aligned}$$

Likewise the second term in (4.5) can be simplified as

$$\begin{aligned} &D^{-1}(D^{-1} + J^{-1})^{-1}J^{-1}u \qquad (4.7) \\ &= D^{-1}J(D+J)^{-1}DJ^{-1}u \\ &= [J\,D^{-1}(D+J)J^{-1}D]^{-1}u = (D+J)^{-1}u \end{aligned}$$

and so the reduced order system (4.5) simplifies to

$$\ddot{\mathbf{q}} = -(D+J)^{-1}c + (D+J)^{-1}u \tag{4.8}$$

which is just the rigid sytem (2.24).

Integral Manifold In the $4n$-dimensional state space of (4.2)-(4.3), a $2n$ dimensional manifold M_μ may be defined by the expressions,

$$z = h(\mathbf{q}, \dot{\mathbf{q}}, u, \mu) \tag{4.9}$$

$$\dot{z} = \dot{h}(\mathbf{q}, \dot{\mathbf{q}}, u, \mu) \tag{4.10}$$

The manifold M_μ is said to be an *integral manifold* (4.2)-(4.3) if it is invariant under solutions of the system. In other words, given an admissible input function $t \to u(t)$, if $\mathbf{q}(t)$, $z(t)$ are solutions of (4.2)-(4.3) for $t > t_0$ with initial conditions $\mathbf{q}(t_0) = \mathbf{q}^0$, $\dot{\mathbf{q}}(t_0) = \bar{\mathbf{q}}^0$, z^0, $\dot{z}(t_0) = \bar{z}^0$ then

$$z^0 = h(\mathbf{q}^0, \bar{\mathbf{q}}^0, u(t_0), \mu) \tag{4.11}$$

$$\bar{z}^0 = h(\mathbf{q}^0, \bar{\mathbf{q}}^0, u(t_0), \mu)$$

implies that for $t > t_0$

$$z(t) = h(\mathbf{q}(t), \dot{\mathbf{q}}(t), u(t), \mu) \tag{4.12}$$

$$\dot{z}(t) = \dot{h}(\mathbf{q}(t), \dot{\mathbf{q}}(t), u(t), \mu) \tag{4.13}$$

In otherwords, if the system lies initially on the manifold M_μ, then the solution trajectory remains on the manifold M_μ for $t > t_0$.

The integral manifold M_μ is characterized by the following partial differential equation, formed by substituting the expression (4.9) into the equation (4.3)

$$\mu\ddot{h} = a_2(\mathbf{q}, \dot{\mathbf{q}}) + A_2(\mathbf{q})h + B_2 u \tag{4.14}$$

In otherwords, if the system lies initially on the manifold M_μ, then the solution trajectory remains on the manifold M_μ for $t > t_0$.

$$\dot{h} = \frac{\partial h}{\partial \mathbf{q}}\dot{\mathbf{q}} + \frac{\partial h}{\partial \dot{\mathbf{q}}}(a_1 + A_1 z) + \frac{\partial h}{\partial u}\dot{u}$$

and $\ddot{h}$ is to be similarly expanded. Although the p.d.e. (4.14) is seemingly difficult we shall actually find an explicit solution.

Once h is determined from (4.14), the dynamics of the system (4.2)-(4.3) on the integral manifold are given by a reduced order system referred as the *reduced flexible system* formed by replacing z by h in (4.2)

$$\ddot{\mathbf{q}} = a_1(\mathbf{q}, \dot{\mathbf{q}}) + A_1(\mathbf{q})h(\mathbf{q}, \dot{\mathbf{q}}, u, \mu) \tag{4.15}$$

Equation (4.15) is of the same order as the rigid system, but as shown in [18] is a more accurate approximation of the flexible system than is the rigid model (2.24). We leave it to the reader to verify that the reduced flexible system reduces to the rigid system (2.24) as the perturbation parameter μ tends to zero.

We now utilize the concept of composite control [10] and choose the control input u of the form

$$u = u_s(\mathbf{q}, \dot{\mathbf{q}}, \nu, \mu) + u_f(\eta, \dot{\eta}) \tag{4.16}$$

where ν represents a new input to be specified. We also specify $u_\nu(0,0) = 0$ so that $u = u_s$ on the integral manifold. The variable η represents the deviation of the fast variables from the integral manifold, i.e.,

$$\eta = z - h(\mathbf{q}, \dot{\mathbf{q}}, u_s, \mu) \tag{4.17}$$

$$\dot{\eta} = \dot{z} - \dot{h}(\mathbf{q}, \dot{\mathbf{q}}, u_s, \mu).$$

Since $u = u_s$ on the integral manifold we may combine (4.3) and (4.14) to obtain

$$\begin{aligned} \mu\eta &= \mu\ddot{z} - \mu\ddot{h} \\ \mu\ddot{\eta} &= a_2 + A_2 z + B_2 u - (a_2 + A_2 h + B_2 u_s) \\ &= A_2\eta + B_2 u_f \end{aligned}$$

Therefore, in terms of the variables $\mathbf{q}$ and η, the system (4.2)-(4.3) is rewritten as

$$\ddot{\mathbf{q}} = a_1 + A_1 h(\mathbf{q}, \dot{\mathbf{q}}, u_s, \mu) + A_1\eta \tag{4.18}$$

$$\mu\ddot{d} = A_2(\mathbf{q})\eta + B_2 u_f \tag{4.19}$$

In order to solve the P.D.E. (4.14) defining the integral manifold, we expand the function h in terms of μ as

$$h(\mathbf{q}, \dot{\mathbf{q}}, u_s, \mu) = h_0(\mathbf{q}, \dot{\mathbf{q}}, u_0) + \mu h(\mathbf{q}, \dot{\mathbf{q}}, u_0 + \mu u_1) + \ \ldots \tag{4.20}$$

and we choose u_s as

$$u_s = u_0 + \mu u_1 \tag{4.21}$$

Substituting these expressions into the manifold condition (4.14) yields

$$\begin{aligned} \mu\{\ddot{h}_0 + \mu\ddot{h}_1 + \ \ldots\ \} &= a_2 + A_2(h_0 + \mu h_1 + \ \ldots\ ) \qquad (4.22) \\ &+ B_2(u_0 + \mu u_1). \end{aligned}$$

Equating coefficients of μ^k we obtain the sequence of equalities

$$0 = a_2 + A_2 h_0 + B_2 u_0 \tag{4.23}$$

$$\ddot{h} = A_2 h_1 + B_2 u_1 \tag{4.24}$$

$$\ddot{h}_{k-1} = A_2 h_k, \; k > 1. \tag{4.25}$$

Equation (4.23) may be solved for h_0 to yield

$$h_0 = -A_2^{-1}(a_2 + B_2 u_0) \tag{4.26}$$

The derivation now proceeds iteratively. The control u_0 is first computed at $\mu = 0$, that is, based on the rigid model, and can be any one of the many schemes that have been derived for control of rigid manipulators. Given u_0 then h_0 is computable from (4.26). From this, with the given u_0 we can compute $\ddot{h}_0$ and so we can write (4.24) as

$$A_2 h_1 = \ddot{h}_0 - B_2 u_1. \tag{4.27}$$

where the right-hand side contains only known quantities and the control. Since both A_2, and B_2 are invertible, we see that by setting

$$u_1 = B_2^{-1} \ddot{h}_0 \tag{4.28}$$

it follows that

$$h_1 \equiv 0 \text{ and therefore } \ddot{h}_1 \equiv 0 \tag{4.29}$$

From this it follows iteratively from (4.25) and the invertibility of A_2 that $h_k = 0$ for $k > 1$.

We have show therefore that the choice of control input

$$u_s = u_0 + \mu u_1 \tag{4.30}$$

with u_1 given by (4.28) results in $h = h_0$. Thus, on the integral manifold, i.e., when $\eta = 0$, the dynamics of the system are described by the reduced order system.

$$\ddot{\mathbf{q}} = a_1 - A_1 A_2^{-1} a_2 - A_1 A_2^{-1} B_2 u_0 \tag{4.31}$$

$$= -(D(\mathbf{q}) + J)^{-1}\{c(\mathbf{q}, \dot{\mathbf{q}}) + u_0\}$$

which is of course just the rigid system.

We see that we have produced a solution h_0 of the manifold condition (4.14). The fact that h_0, given by (4.26), satisfies (4.14) is significant. What this implies is that by adding the corrective control $\mu\mu_1$ the integral manifold h becomes the rigid manifold h_0. To put it another way, the rigid manifold h_0 is made an invariant manifold for the flexible system by the corrective control.

If the control u_0 is chosen to be the feedback linearizing control for the rigid system

$$u_0 = (D(\mathbf{q}) + J)\nu + c(\mathbf{q}, \dot{\mathbf{q}}) \tag{4.32}$$

we obtain the overall system

$$\ddot{\mathbf{q}} = \nu + A_1(\mathbf{q})\eta \tag{4.33}$$

$$\mu\ddot{\eta} = A_2(\mathbf{q})\eta + B_2 u_f \tag{4.34}$$

Since B_2 is nonsingular the fast subsystem (4.34), which is a linear system in η parameterized by $\mathbf{q}$, is controllable for each $\mathbf{q}$. Thus there exists a fast control $u_f(\eta, \dot{\eta})$ to place the poles of (4.34) arbitrarily. Note that we have not explicitly included damping in the model. Thus for each $\mathbf{q}$, since $-A_2$ is a positive definite matrix, the open loop poles of (4.34) are on the $j\omega$-axis. This shows clearly the resonance phenomonon whereby the elastic oscillations from (4.34) drive the slow variables through (4.33). Since A_2 is a function of $\mathbf{q}$ the resonant modes will be configuration dependent, a fact that was experimentally verified in [25]. In case the system (4.34) has some inherent natural damping one can show that the fast subsystem is of the form

Table 1 Parameters used for simulation

Mass	$m =$	1
Stiffness	$k =$	100
Length (2L)	$L =$	1
Gravity	$g =$	9.8
Inertias	$I =$	1
	$J =$	1

$$\mu\ddot{\eta} = A_2(\mathbf{q})\eta + \sqrt{\mu}A_3(\mathbf{q})\dot{\eta} + B_2 u_f \tag{4.35}$$

in which the fast variables, represented by η, decay to zero with $u_f = 0$. In other words the integral manifold, which in this case is the rigid manifold becomes an attracting set. Solutions off the manifold rapidily converge to the manifold after which the system equations are just the rigid equations. In this case the control u_s consisting of the rigid control plus the corrective control achieves the desired result. The point to note that this slow control is a function only of $\mathbf{q}$, $\dot{\mathbf{q}}$. Thus the corrective control compensates for the elasticity using a limited set of state measurements.

If there is no damping in the fast variables, or if the damping is insufficient then the fast control u_f must be added. Note that the choice

$$u_f = B_2^{-1}(\zeta - A_2(\mathbf{q})\eta) \tag{4.36}$$

where ζ is a new input, when applied to (4.34) results in

$$\mu\ddot{\eta} = \zeta \tag{4.37}$$

Inspecting (4.33), (4.37) we see that we have for all practical purposes produced an alternate feedback linearization of the original system, which exploits the two-time scale property of the elastic system. A linear control scheme can now be employed, for example

$$\nu = \alpha_1 \bullet \mathbf{q} + \alpha_2 \bullet \dot{\mathbf{q}} + r \tag{4.38}$$

$$\zeta = \beta_1 \bullet \eta + \sqrt{\mu}\beta_2 \bullet \dot{\eta} \tag{4.39}$$

to place the poles of the system arbitrarily. Note that implementation of the above control scheme requires either direct measurement of the fast variables, which in this case are the elastic forces at the joints and their time derivatives or else accurate knowledge of the system paramters in order that η and $\dot{\eta}$ can be computed from (4.17), which is an issue similar to that which arises in the feedback linearization approach of section 3.

5 An Example

For illustrative purposes consider the single link with the flexible joint of Fig. 1 with the parameters shown in Table 1. The equations of motion for this system in state space are easily computed to be

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -MgL/I \sin x_1 - k/I(x_1 - x_3) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= k/J(x_1 - x_3) + 1/Ju
\end{aligned} \tag{5.1}$$

where $x_1 = q_1$, $x_3 = q_2$, etc.

In the limit as $k \to \infty$ the resulting rigid system is

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -Mgl/(I+J)\sin x_1 + 1/(I+J)u
\end{aligned}$$

where we take here $x_1 = {}_1 = q_1 = q_2$.

The feedback linearizing control law for (5.2) may be chosen as

$$u = (I+J)(\nu + Mgl \sin x_1) \tag{5.3}$$

with ν given as a a simple linear control term

$$\nu = \dot{x}_2^d - a_1(x_1 - x_1^d) - a_2(x_2 - x_2^d) \tag{5.4}$$

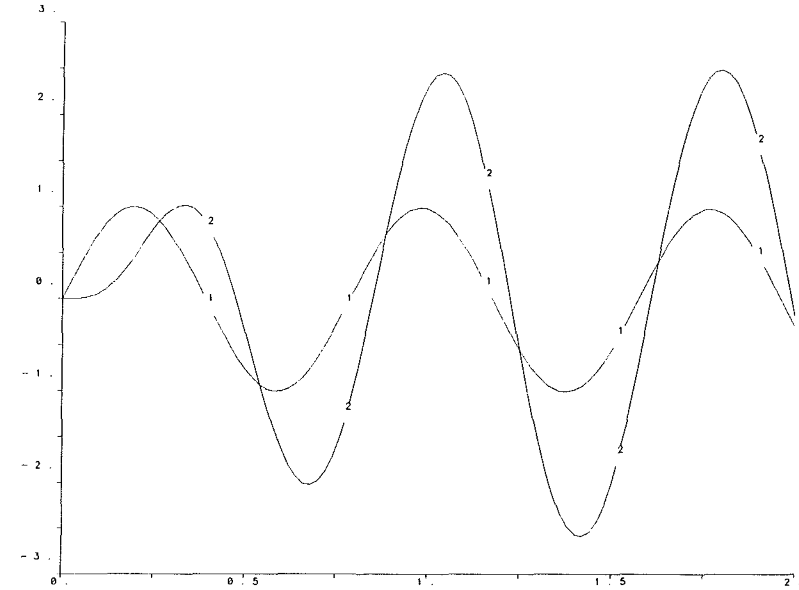

Fig. 3 Rigid control applied to flexible joint. 1 = reference trajectory; 2 = link angle.

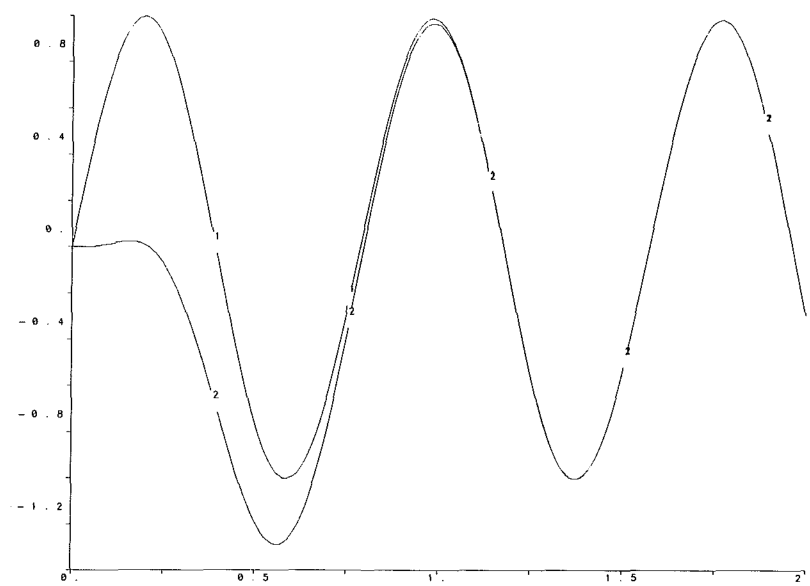

Fig. 4 Feedback linearization control. 1 = reference trajectory; 2 = link angle

designd to track a desired trajectory $t \rightarrow x_1^d(t)$.

It is interesting to see the response of the flexible joint system (5.1) to this "rigid control." At this point one must make a choice whether to use the motor variable q_2 or the link variable q_1 in this control law. Figure 3 shows the response of the link variable q_1 in the flexible joint system (5.1) using the motor variable $x_1 = q_2$ in (5.3)-(5.4), with a desired tracjectory $x_1^d = \sin 8t$. It is interesting to note that if one tries to feedback instead the link variable q_1 in (5.3)-(5.4) the system becomes unstable.

Feedback Linearization Control. The feedback linearizing transformation for this system is given by (3.10)-(3.13) as

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ y_3 &= -MgL/I \sin x_1 - k/I(x_1 - x_3) \\ y_4 &= -Mgl/I \cos x_1 \cdot x_2^2 - k/I(x_2 - x_4) \end{aligned} \tag{5.5}$$

The feedback linearizing control law computed from (3.19) and (3.20) turns out to be

$$u = \frac{IJ}{k}(\nu - F(x_1, x_2, x_3, x_4)) \tag{5.6}$$

where

$$\begin{aligned} F(x_1, x_2, x_3, x_4) &= MgL/I \sin x_1 \cdot x_2^2 \\ &+ (MgL/I \cos x_1 + k/I)(MgL/I \sin x_1 + k/I(x_1 - x_3)) \\ &+ k^2/IJ(x_1 - x_3) \end{aligned} \tag{5.7}$$

A simple linear control law for ν designed to track a desired trajectory $t \rightarrow y_i^d(t)$ can be expressed as

$$\nu = \dot{y}_4^d - \sum_{i=1}^{3} a_i (y_i - y_i^d) \tag{5.8}$$

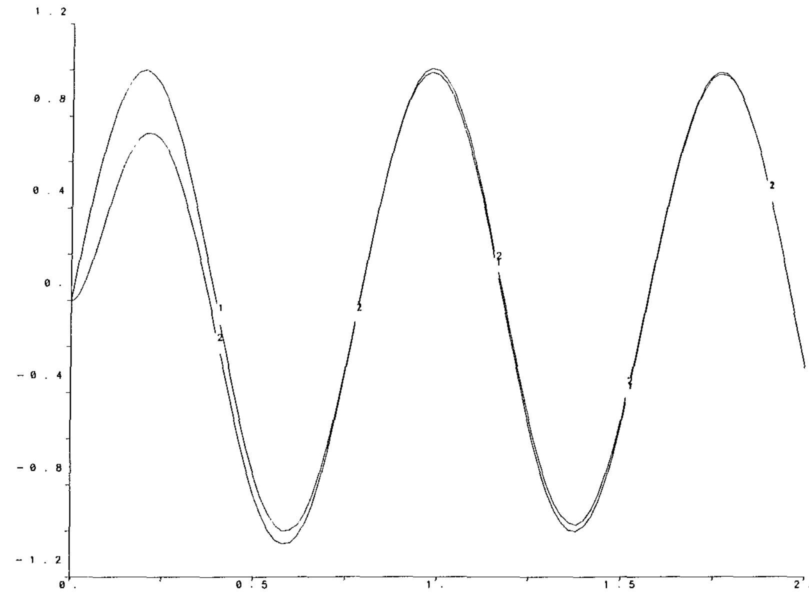

Fig. 5 Corrective control based on the integral manifold. 1 = reference trajectory; 2 = link angle

The zero-state response of the above system with a given desired trajectory $y_1 = \sin 8t$ is shown in Fig. 4. The gains in the control law (5.8) where chosen for simplicity to achieve a closed loop characteristic polynomial for the linearized system of $(s+10)^4$. This response illustrates the improved tracking resulting from basing the control design on the fourth-order flexible joint model rather than on the rigid model. The robust version of this control law, given by (3.37) is omitted.

Integral Manifold Control. In terms of the variables $q = q_1$ and $z = k(q_1 - q_2)$ the equations of motion for the system of Fig. 1 in singularly perturbed form with $\mu = 1/k$ are

$$\ddot{q} = -MgL\ \sin q - 1/I\ z \tag{5.5}$$

$$\mu \ddot{z} = -MgL\ \sin q - (1/I + 1/J)z - 1/Ju \tag{5.6}$$

In terms of the variables h and η the system (4.18)-(4.19) is

$$\ddot{q} = -MgL\ \sin q - 1/I\ h - 1/I\eta \tag{5.8}$$

$$\mu \ddot{\eta} = (1/I + 1/J)\eta - 1/Ju \tag{5.9}$$

where $\eta = z - h$ and h is determined from the manifold condition

$$\mu \ddot{\eta} = -MgL\ \sin q - (1/I + 1/J)h - 1/Ju \tag{5.10}$$

The detailed calculation of the asymptotic expansion of the function h and the corrective control are carried out in [23]. The interested reader is referred to that paper for the details and also more simulation results. In Fig. 5 the composite and corrective control law thus derived is applied to track the same desired trajectory $q^d = \sin 8t$.

6 Conclusions

In this paper we have rigorously derived a simple and rather intuitive model to represent the dynamics of elastic joint manipulators and presented two attractive control techniques for the resulting system. The first new result that we present is the global feedback linearization of the flexible joint system by nonlinear coordinate transformation and static state feedback. The importance of the property of feedback linearization is not necessarily that the nonlinearities in the system can by computed and exactly cancelled by feedback as this is never achievable in practice. Rather its significance is that once the proper coordinates are found in which to represent the system, the so-called matching conditions are satisfied, which is to say that the nonlinearities are all in the range space of the input. This property allows the design of control laws which are highly robust to parametric uncertainty. The second new result is based on the integral manifold formulation of the equations of motion. We have shown using the corrective control concept that the manifold in state space describing the dynamics of the rigid manipulator can be made invariant under solutions of the flexible joint system, independent of the joint stiffness. This result holds in general only for the model derived here. In previous models of elastic joint manipulators, as shown in [23], the results here hold up to $0(\mu^l)$ by applying a corrective control

$$u_s = u_0 + \mu u_1 + \ .\ .\ .\ + \mu^l u_l. \tag{6.1}$$

With the present model the result is exactly achieved to any order in μ and is done so only with a first order correction term μu_1.

There are several interesting research issues that arise at this point. Among are the design of robust state estimators to realize the feedback linearization control using only the joint positions and velocities and also the computational issues associated with computing in real-time what amounts to a very complicated nonlinear control algorithm. Also, the robustness of the integral manifold based corrective control strategy needs to be investigated.

References

1 Bejczy, A. C., "Robot Arm Dynamics and Control," JPL Technical Memo., 33-369, Feb. 1974.

2 Freund, "Fast Nonlinear Control with Arbitrary Pole-Placement for Industrial Robots and Manipulators," *Int. J. Robotics Res.*, Vol. 1, No. 1, 1982, pp. 65–78.

3 Hunt, L. R., Su, R., and Meyer, G., "Design for Multi-input Nonlinear Systems," *Differential Geometric Control Theory Conf.*, Birkhauser, Boston, 1983, pp. 268–298.

4 Paul, R. P., *Robot Manipilators: Mathematics, Programming and Control*, MIT Press, Cambridge, Mass., 1982.

5 Slotine, J. J., "Robust Control of Robot Manipulators," *Int. J. Robotics Res.*, Vol. 4, No. 2, 1985.

6 Spong, M. W., Thorp, J. S., and Kleinwaks, J. W., "The Control of Robot Manipulators with Bounded Control Part II: Robustness and Disturbance Rejection," *Proc. 23nd IEEE CDC*, Las Vegas, NV, Dec. 1984.

7 Tarn, T. J., Bejczy, A. K., Isidori, A., and Chen, Y., "Nonlinear Feedback in Robot Arm Control." *Proc. 23rd IEEE CDC*, Las Vegas, NV, Dec. 1984.

8 Samson, C., "Robust Non Linear Control of Robotic Manipulators," *Proc. 22nd IEEE CDC*, San Antonio, Dec. 1983.

9 Sweet, L. M., and Good, M. C., "Re'Definition of the Robot Motion Control Problem: Effects of Plant Dynamics, Drive System Constraints, and User Requirement," *Proc. 23rd IEEE Conf. on Decision and Control*, Las Vegas, NV, Dec. 1984, pp. 724-731.

10 Chow, J. H., and Kokotovic, P. V., "Two Time Scale Feedback Design of a Class of Nonlinear Systems," *IEEE Trans. on Automatic Control*, AC-23, 1978, pp. 438-443.

11 Kokotovic, P. V., "Applications of Singular Perturbation Techniques to Control Problems," *SIAM Review*, Vol. 26, No. 4, 1984, pp. 396-410.

12 Erzerber, H., "Analysis and Design of Model Following Systems by State Space Techniques," *Proc. JACC*, 1968, pp. 572-581.

13 Cesareo, G. and Marino, R., "On the Controllability Properties of Elastic Robots," *Sixth Int. Conf. on Analysis and Optimization of Systems*, INRIA, Nice, 1984.

14 Fenichel, "Geometric Singular Perturbation Theory for Ordinary Differential Equations," *J. Differential Equations*, Vol. 3, 1979, pp. 53-98.

15 Ficola, A., Marino, R., and Nicosia, S., "A Singular Peturbation Approach to the Control of Elastic Robots," *Proc. 21st. Annual Allerton Conf. on Communication, Control, and Computing*, Univ. of Illinois, 1983.

16 Hoppensteadt, F., "Properties of Solutions of Ordinary Differential Equations with Small Parameters," *Comm. Pure Appl. Math.*, Vol 34, 1971, pp. 807-840.

17 Hunt, L. R., Su. R., and Meyer, G., "Global Transformations of Nonlinear Systems," *IEEE Trans. on Automatic Control*, Vol. Ac-28, 1983, pp. 24-30.

18 Khorasani, K., and Spong, M. W., "Invariant Manifolds and their Applications to Robot Manipulators with Flexible Joints," *Proc. 1985 IEEE Int. Conf. on Robotics and Automation*, St. Louis, Mar. 1985.

19 Kokotovic, P. V., "Control Theory in the 80's: Trends in Feedback Design," *Proc. 9th World Congress of IFAC*, July 1984.

20 Marino, R., and Nicosia, S., "On the Feedback Control of Industrial Robots with Elastic Joints: A Singular Perturbation Approach," University of Rome, R-84.01, 1984.

21 Sobolev, V. A., "Integral Manifolds and Decompositions of Singularly Perturbed Systems," *Systems and Control Letters*, Vol. 5, 1984, pp. 1169-179.

22 Su, R., "On the Linear Equivalents of Nonlinear Systems," *Systems and Control Letters*, Vol 2, 1981, pp. 48-52.

23 Spong, M. W., Khorasani, K., and Kokotovic, P. V., "A Slow Manifold Approach to Feedback Control of Flexible Joint Robots," to appear in *IEEE Journal of Robotics and Automation*, 1986.

24 Rivin, E. I., "Effective Rigidity of Robot Structure: Analysis and Enhancement," *Proc. 1985, ACC*, Boston, June 1985.

25 Good, M., Strobel, K., and Sweet, L. M., "Dynamics and Control of Robot Drive Systems," General Electric Company, Corporate Research and Development, 1983.

26 Forrest-Barlach, M. G., and Babcock, S. M., "Inverse Dynamics Position Control of a Compliant Manipulator," *Proc. IEEE Conference of Robotics and Automation*, San Francisco, Apr. 1986, pp. 196-205.

27 Marino, R. W., and Spong, M. W., "Nonlinear Control Techniques for Flexible Joint Manipulators: A Single Link Case Study," *Proc. 1986 IEEE Conference on Robotics and Automation*, San Francisco, Apr. 1986, pp. 1030-1036.

28 Khorasani, K., and Kokotovic, P. V., "Feedback Linearization of a Flexible Manipulator Near its Rigid Body Manifold," *Systems and Control Letters*, Vol. 6, 1985, pp. 187-192.

29 De Luca, A., Isidori, A., and Nicolo, F., "An Application of Nonlinear Model Matching to the Dynamic Control of Robot Arm with Elastic Joints," preprint, 1985.

30 Bejczy, A. K., personal communication, San Francisco, Apr. 1986.

31 Spong, M. W., and Vidyasagar, M., "Robust Nonlinear Control of Robot Manipulators," *Proc. 24th IEEE Conf. on Decision and Control*, Fort Lauderdale, Dec. 1985, pp. 1767-1772.

32 Springer, H., Lugner, P., and Desoyer, "Equations of Motion for Manipulators Including Dynamic Effects of Active Elements," *Proc. IFAC Symposium and Robot Control*, SYROCO' 85, Barcelona, Nov. 1985.

33 Spong, M. W., "Robust Stabilization For a Class of Nonlinear Systems," *Proc. 7th Int. Symp. on the Math. Theory of Network and Systems*, (MTNS), Stockholm, June 1985, Springer-Verlag.

34 Craig, J., *Introduction to Robotics*, Addison-Wesley, 1985.

35 Krener, A. J., and Isidori, A., "Linearization by Output Injection and Nonlinear Observers," *Systems and Control Letters*, Vol. 3, 1983, pp. 47-52.

36 Shoureshi, R., Roesler, M. D., and Corlis, M. J., "Control of Industrial Manipulators with Bounded Uncertainty," *Japan-USA Symposium of Flexible Automation*, Osaka, Japan. 1986.