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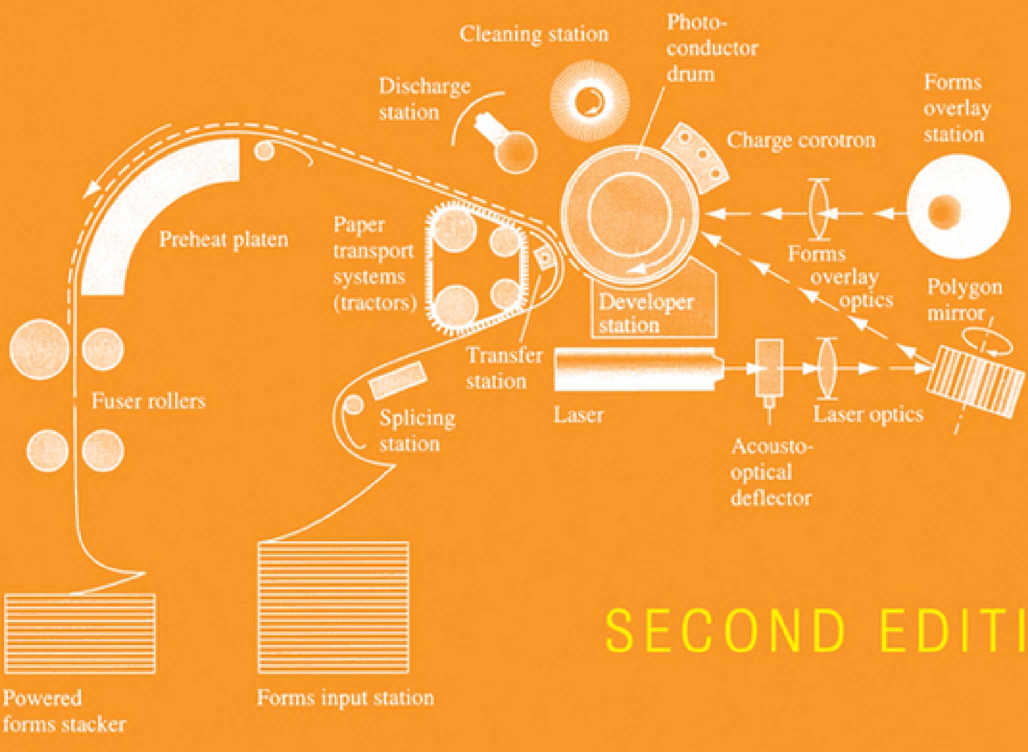
RELIABILITY ENGINEERING

ELSAYED A. ELSAYED



SECOND EDITION

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Second Edition

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CONTENTS

PREFACE

Reliability is one of the most important quality characteristics of components, products, and large and complex systems. Reliability is important to each one of us, every day, when we start a vehicle, attempt to place a phone call, or use a copier, a computer, or a fax machine. In all instances, the user expects the machine or the system to provide the designed functions when requested. As you probably have experienced, machines do not always function or deliver the desired quality of service when needed. Machines also experience failures and interruption, if not termination of service.

Engineers spend a significant amount of time and resources during the design, product (or service) development, and production phases of the product life cycle to ensure that the product or system will provide the desired service level. In doing so, engineers start with a concept design, select its components, test its functionality, and estimate its reliability. Modifications and design changes are usually made and these steps are repeated until the product (or service) satisfies its requirements. The prelude of this book presents these steps in the design of the "One-Hoss-Shay."

Designing the product may require redundancy of components (or subsystems), introduction of newly developed components or materials, or changes in design configuration. These will have a major impact on the product reliability. Once the product is launched and used in the field, data are collected so improvements can be made in the newer versions of the product. Moreover, these data become important in identifying potential safety issues or hazards for the users so recalls can be quickly made to resolve these issues. In other words, reliability is a major concern during the entire life of the product and is subject to continual improvements.

This book is an *engineering* reliability book. It is organized according to the same sequence followed when designing a product or service. The book consists of three parts. Part I focuses on system reliability estimation for time-independent and time-dependent models. Chapter 1 focuses on the basic definitions of reliability, its metrics, and methods for its calculations. Extensive coverage of different hazard functions is given. Chapter 2 describes, in greater detail, methods for estimating reliabilities of a variety of engineering systems configurations starting with series systems, parallel systems, series-parallel, parallel-series, consecutive k-out-of-$n : F$, k-out-of-n, and complex network systems. It also addresses systems with multistate devices and concludes by estimating reliabilities of redundant systems and the optimal allocation of components in a redundant system. The next step in product design is to study the effect of time on system reliability. Therefore, Chapter 3 discusses, in detail, time- and failure-dependent reliability and the calculation of mean time to failure of a variety of system configurations. It also introduces availability as a measure of system reliability.

Once the design is “firm,” the engineer assembles the components and configures them to achieve the desired reliability objectives. This may require conducting reliability tests on components or using field data from similar components. Therefore, Part II of the book, starting with Chapter 4, presents the concept of constructing the likelihood function and its use in estimating the parameters of a failure time distribution. Chapter 5 provides a comprehensive coverage of parametric and nonparametric reliability models for failure data. The extensive examples and methodologies presented in this chapter will aid the engineer in appropriately modeling the test data. Confidence intervals for the parameters of the models are also discussed. More important, the book devotes all of Chapter 6 to accelerated life testing and degradation testing. The main objective of this chapter is to provide varieties of statistical based models, physics-statistics based models, and physics-experimental based models to relate the failure time and data at accelerated conditions to the normal operating conditions at which the product is expected to operate.

Finally, once a product is produced and sold, the manufacturer must ensure its reliability objectives by providing preventive and scheduled maintenance and warranty policies. Part III of the book focuses on these topics. It begins with Chapter 7, which presents different methods (exact and approximate) for estimating the expected number of system failures during a specified time interval. These estimates are used in Chapter 8 in order to determine optimal preventive maintenance schedules and optimum inspection policies. Methods for estimating the inventory levels of spares required to ensure predetermined reliability and availability values are also presented. Finally, Chapter 9 presents different warranty policies and approaches for determining the product price, including warranty cost as well as the estimation of the warranty reserve fund.

Chapter 10 concludes the book. It presents actual case studies that demonstrate the use of the approaches and methodologies discussed throughout the book in solving real cases. The role of reliability during the design phase of a product or a system is particularly emphasized.

Every theoretical development in this book is followed by an engineering example to illustrate its application. Moreover, many problems are included at the end of each chapter. These two features increase the usefulness of this book as a comprehensive reference for practitioners and professionals in the quality and reliability engineering area. In addition, this book may be used for either a one- or two-semester course in reliability engineering geared toward senior undergraduates or graduate students in industrial and systems, mechanical, and electrical engineering programs. It can also be adapted for use in a life data analysis course in a graduate program in statistics. The book presumes a background in statistics and probability theory and differential calculus.

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This book represents the work of not just the author, but also many others whose work is referenced throughout the book. I have tried to give adequate credit to all whose work has influenced this book. Particular acknowledgment is made to the Institute of Electrical and Electronic Engineers, CRC Press, Institute of Mathematical Statistics, American Society of

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E.A. Elsayed
Piscataway, New Jersey

PRELUDE

DESIGN FOR RELIABILITY: A LOGICAL STORY

"The Deacon's Masterpiece, or The Wonderful One-Hoss-Shay" is a perfectly logical story that demonstrates the concept of designing a product for reliability. It starts by defining the objective of the product or service to be provided. The reliability structure of the system is then developed and its components and subsystem are selected. A prototype is constructed and tested. The failure data of the components are collected and analyzed. The system is then redesigned and retested until its reliability objectives are achieved. This is indeed what is considered today as "reliability growth." These logical steps are elegantly described below.

THE DEACON'S MASTERPIECE,

or The Wonderful One-Hoss-Shay[1]

I. System's Objective and Structural Design

Have you heard of the wonderful one-hoss-shay,
It ran a hundred years to a day,
And then, of a sudden, it—ah, but stay,
I'll tell you what happened without delay,
Scaring the parson into fits,
Frightening people out of their wits,—
Have you ever heard of that, I say?

Seventeen hundred and fifty-five.
Georgius Secundus was then alive,—
Snuffy old drone from the German hive.
That was the year when Lisbon-town
Saw the earth open and gulp her down,
And Braddock's army was done so brown,
Left without a scalp to its crown.
It was on the terrible Earthquake-day
That the Deacon finished the one-hoss-shay.

[1] Oliver Wendell Holmes, "The Deacon's Masterpiece," in *The Complete Poetical Works of Oliver Wendell Holmes*, Fourth Printing, Houghton Mifflin, 1908.

II. System Prototyping and Analysis of Failure Observations

Holmes' preface to the poem:

> Observation shows us in what point any particular mechanism is most likely to give way. In a wagon, for instance, the weak point is where the axle enters the hub or nave. When the wagon breaks down, three times out of four, I think, it is at this point that the accident occurs. The workman should see to it that this part should never give way, then find the next vulnerable place, and so on, until he arrives logically at the perfect result attained by the deacon.

This is a continuation of reliability growth methodology.

Now in building of chaises, I tell you what,
There is always <u>somewhere</u> a weakest spot,—
In hub, tire, felloe, in spring or thill,
In panel, or crossbar, or floor, or sill,
In screw, bolt, thoroughbrace,—lurking still,
Find it somewhere you must and will,—
Above or below, or within or without,—
And that's the reason, beyond a doubt,
That a chaise <u>breaks down</u>, but doesn't <u>wear out.</u>

But the Deacon swore (as Deacons do,
With an "I dew vum," or an "I tell <u>yeou</u>")
He would build one shay to beat the taown
'N' the keounty 'n' all the kentry raoun';
It should be so built that it <u>couldn'</u> break daown:
"Fur," said the Deacon, "'t's mighty plain
Thut the weakes' place mus' stan' the strain;
'N' the way t' fix it, uz I maintain, Is only jest
T' make that place uz strong uz the rest."

III. Design Changes and System Improvement

So the Deacon inquired of the village folk
Where he could find the strongest oak,
That couldn't be split nor bent nor broke,—
That was for spokes and floor and sills;
He sent for lancewood to make the thills;
The crossbars were ash, from the straightest trees,
The panels of white-wood, that cuts like cheese,
But last like iron for things like these;
The hubs of logs from the "Settler's ellum,"—

Last of its timber,—they couldn't sell 'em,
Never an axe had seen their chips,

And the wedges flew from between their lips,
Their blunt ends frizzled like celery-tips;
Step and prop-iron, bolt and screw,
Spring, tire, axle, and linchpin too,
Steel of the finest, bright and blue;
Thoroughbrace bison-skin, thick and wide;
Boot, top, dasher, from tough old hide
Found in the pit when the tanner died.
That was the way he "put her through."
"There!" said the Deacon, "naow she'll dew!"

Do! I tell you, I rather guess
She was a wonder, and nothing less!
Colts grew horses, beards turned gray,
Deacon and deaconess dropped away,
Children and grandchildren—where were they?
But there stood the stout old one-hoss-shay
As fresh as on Lisbon-earthquake-day!

IV. System Monitoring During Operation

EIGHTEEN HUNDRED;—it came and found
The Deacon's masterpiece strong and sound.
Eighteen hundred increased by ten;—
"Hahnsum kerridge" they called it then.
Eighteen hundred and twenty came;—
Running as usual; much the same.
Thirty and forty as last arrive,
And then come fifty, and FIFTY-FIVE.

Little of all we value here
Wakes on the morn of its hundredth year
Without both feeling and looking queer.
In fact, there's nothing that keeps its youth,
So far as I know, but a tree and truth.
(This is a moral that runs at large;
Take it. —You're welcome. —No extra charge.)

V. System Aging, Wear Out, and Replacement

FIRST OF NOVEMBER,—the Earthquake-day,—
There are traces of age in the one-hoss-shay,
A general flavor of mild decay,
But nothing local, as one may say.
There couldn't be,—for the Deacon's art
Had made it so like in every part
That there wasn't a chance for one to start.

For the wheels were just as strong as the thills,
And the floor was just as strong as the sills,
And the panels just as strong as the floor,
And the whipple-tree neither less nor more,
And the back crossbar as strong as the fore,
And spring and axle and hub <u>encore</u>.
And yet, <u>as a whole</u>, it is past a doubt
In another hour it will be <u>worn out</u>!

VI. System Reaches Its Expected Life

First of November, 'Fifty-five!
This morning the parson takes a drive.
Now, small boys, get out of the way!
Here comes the wonderful one-hoss-shay,
Drawn by a rat-tailed, ewe-necked bay.
"Huddup!" said the parson.—Off went they.
The parson was working his Sunday's text,—
Had got to <u>fifthly</u>, and stopped perplexed
At what the—Moses—was coming next.
All at once the horse stood still,
Close by the meet'n'-house on the hill.
First a shiver, and then a thrill,
Then something decidedly like a spill,—
And the parson was sitting upon a rock,
At half past nine by the meet'n'-house clock,—
Just the hour of the Earthquake shock!
What do you think the parson found,
When he got up and stared around?
The poor old chaise in a heap or mound,
As if it had been to the mill and ground!
You see, of course, if you're not a dunce,
How it went to pieces all at once,—
All at once, and nothing first,—
Just as bubbles do when they burst.

End of the wonderful one-hoss-shay.
Logic is logic. That's all I say.

CHAPTER 1

RELIABILITY AND HAZARD FUNCTIONS

1.1 INTRODUCTION

One of the quality characteristics that consumers require from the manufacturer of products is reliability. Unfortunately, when consumers are asked what reliability means, the response is usually unclear. Some consumers may respond by stating that the product should always work properly without failure or by stating that the product will always function properly when required for use, while others will completely fail to explain what reliability means to them.

What is reliability from your viewpoint? Take, for instance, the example of starting your car. Would you consider your car reliable if it starts immediately? Would you still consider your car reliable if it takes you two times to turn on the ignition key for the car to start? How about three times? As you can see, without quantification, it becomes more difficult to define or measure reliability. We define reliability later in this chapter, but for now, to further illustrate the importance of reliability as a field of study and research, we present the following cases.

On April 9, 1963, the USS *Thresher*, a nuclear submarine, slipped beneath the surface of the Atlantic and began a run for deep waters (1000 feet below surface). *Thresher* exceeded its maximum test depth and imploded. Its hull collapsed, causing the death of 129 crewmembers and civilians. It should be noted that the *Thresher* had been the most advanced submarine of its day, with a destructive power beyond that of the Navy's entire submarine force in World War II. Though it was designed to sustain stresses at this depth, it failed catastrophically.

In 1979, a DC-10 commercial aircraft crashed, killing all passengers aboard. The cause of failure was poor maintenance procedure. The engineers specified that the engine should have been taken off before the engine mounting assembly, because of the excessive weight of the engines. Apparently, those guidelines were not followed when maintenance was conducted, causing excessive stresses and forces that cracked the engine mounts.

On December 2, 1982, a team of doctors and engineers at Salt Lake City, Utah, performed an operation to replace a human heart by a mechanical one—the Jarvik heart. Two days later, the patient underwent further operations due to a malfunction of the valve of the mechanical heart. Here, a failure of the system may directly affect one human life at a time. In January 1990, the Food and Drug Administration stunned the medical community by recalling the world's first artificial heart because of deficiencies in manufacturing quality, training, and other areas. This heart affected the lives of 157 patients over an eight-year period. Now, consider the following case, where the failures of the systems have a much greater effect.

Reliability Engineering, Second Edition. Elsayed A. Elsayed.

On April 26, 1986, two explosions occurred at the newest of the four operating nuclear reactors at the Chernobyl site in the former USSR. It was the worst commercial disaster in the history of the nuclear industry. A total of 31 site workers and members of the emergency crew died as a result of the accident. About 200 people were treated for symptoms of acute radiation syndrome. Economic losses were estimated at $3 billion, and the full extent of the long-term damage has yet to be determined.

More recently, on July 25, 2000, a Concorde aircraft while taking off at a speed of 175 knots ran over a strip of metal from a DC-10 airplane, which had taken off a few minutes before. This strip cut the tire on wheel No. 2 of the left landing gear resulting in one or more pieces of the tire, which were thrown against the underside wing fuel tank. This led to the rupture of the tank causing fuel leakage and consequently resulting in a fire in the landing gear system. Fire spread to both engines of the aircraft causing loss of power and crash of the aircraft. Clearly, such field condition was not considered in the design process. This type of failure has ended the operation of the Concorde fleet indefinitely.

The explosions of the space shuttle *Challenger* in 1986 and the space shuttle *Columbia* in 2003, as well as the loss of the two external fuel tanks of the space shuttle *Columbia* in an earlier flight (at a cost of $25 million each), are other examples of the importance of reliability in the design, operation, and maintenance of critical and complex systems. Indeed, field conditions similar to those of the Concorde aircraft have lead to the failure of the *Columbia*. The physical cause of the loss of *Columbia* and its crew was a breach in the Thermal Protection System of the leading edge of the left wing. The breach was initiated by a piece of insulating foam that separated from the left bipod ramp of the External Tank and struck the wing in the vicinity of the lower half of Reinforced Carbon-Carbon panel 8 at 81.9 seconds after launch. During the reentry, reheated air penetrated the leading-edge insulation and progressively melted the aluminum structure until increasing aerodynamic forces caused loss of control, failure of the wing, and breakup of the Orbiter (Walker and Grosch, 2004).

Reliability plays an important role in the service industry. For example, to provide virtually uninterrupted communications for its customers, American Telephone and Telegraph Company (AT&T) installed the first transatlantic cable with a reliability goal of a maximum of one failure in 20 years of service. The cable surpassed the reliability goal and was replaced by new fiber optic cables for economic reasons. The reliability goal of the new cables is one failure in 80 years of service!

Another example of the reliability role in structural design is illustrated by the Point Pleasant Bridge (West Virginia/Ohio border), which collapsed on December 15, 1967, causing the death of 46 persons and the injuries of several dozen persons. The failure was attributed to the metal fatigue of a crucial eyebar, which started a chain reaction of one structural member falling after another. The bridge failed before its designed life.

The failure of a system can have a widespread effect and a far reaching impact on many users and on the society as a whole. On August 14, 2003, the largest power blackout in North American history affected eight U.S. states and the Province of Ontario, leaving up to 50 million people with no electricity. Controllers in Ohio, where the blackout started, were overextended, lacked vital data, and failed to act appropriately on outages that occurred more than an hour before the blackout. When energy shifted from one transmission line to another, overheating caused lines to sag into a tree. The snowballing cascade of shunted power that rippled across the Northeast in seconds would not have happened had the grid not been operating so near to its transmission capacity and assessment of the entire power network reliability when operating at its peak capacity were carefully estimated (The Industrial Physicist, 2003; U.S.-Canada Power System Outage Task Force, 2004).

Most of the above examples might imply that failures and their consequences are due to hardware. However, many systems' failures are due to human errors and software failures. For example, the Therac-25, a computerized radiation therapy machine, massively overdosed patients at least six times between June 1985 and January 1987. Each overdose was several times the normal therapeutic dose and resulted in the patient's severe injury or even death (Leveson and Turner, 1993). Overdoses, although they sometimes involved operator error, occurred primarily because of errors in the Therac-25's software and because the manufacturer did not follow proper software engineering practices. Other software errors might result from lack of validation of the input parameters. For example, in 1998, a crew member of the guided-missile cruiser USS Yorktown mistakenly entered a zero for a data value, which resulted in a division by zero. The error cascaded and eventually shut down the ship's propulsion system. The ship was dead in the water for several hours because a program did not check for valid input.

Another recent example of software reliability includes the Mars Polar Lander which was launched in January 1999 and was intended to land on Mars in December of that year. Legs were designed to deploy prior to landing. Sensors would detect touchdown and turn off the rocket motor. It was known and understood that the deployment of the landing legs generated spurious signals of the touchdown sensors. The software requirements, however, did not specifically describe this behavior and the software designers therefore did not account for it. The motor turned off at too high an altitude and the probe crashed into the planet at 50 mi/h and was destroyed. Mission costs exceeded $120 million (Gruhn, 2004). Reliability also has a great effect on the consumers' perception of a manufacturer. For example, consumers' experiences with car recalls, repairs, and warranties will determine the future sales and survivability of that manufacturer. Most manufactures have experienced car recalls and extensive warranties that range from as low as 1.2% to 6% of the revenue. Some car recalls are extensive and costly such as the recall of 8.6 million cars due to the ignition causing small engine fires. In 2010, an extensive recall of several car models due to sudden acceleration resulted in the shutdown of the entire production system and hundreds of lawsuits. One of the causes of the recall is lack of thoroughness in testing new cars and car parts under varying weather conditions; the gas-pedal mechanism tended to stick more as humidity increased. Clearly, the number and magnitude of the recalls are indicative of the reliability performance of the car and potential survivability of the manufacturer.

1.2 RELIABILITY DEFINITION AND ESTIMATION

A formal definition of reliability is given as follows:

1.2.1 Reliability

Reliability is the probability that a product will operate or a service will be provided properly for a specified period of time (design life) under the design operating conditions (such as temperature, load, volt …) without failure.

In other words, reliability may be used as a measure of the system's success in providing its function properly during its design life. Consider the following.

Suppose n_o identical components are subjected to a design operating conditions test. During the interval of time $(t - \Delta t, t)$, we observed $n_f(t)$ failed components, and $n_s(t)$ surviving components $[n_f(t) + n_s(t) = n_o]$. Since reliability is defined as the cumulative probability function of success, then at time t, the reliability $R(t)$ is

$$R(t) = \frac{n_s(t)}{n_s(t) + n_f(t)} = \frac{n_s(t)}{n_o}. \tag{1.1}$$

In other words, if T is a random variable denoting the time to failure, then the reliability function at time t can be expressed as

$$R(t) = P(T > t). \tag{1.2}$$

The cumulative distribution function (CDF) of failure $F(t)$ is the complement of $R(t)$, that is,

$$R(t) + F(t) = 1. \tag{1.3}$$

If the time to failure, T, has a probability density function (p.d.f.) $f(t)$, then Equation 1.3 can be rewritten as

$$R(t) = 1 - F(t) = 1 - \int_0^t f(\zeta) d\zeta. \tag{1.4}$$

Taking the derivative of Equation 1.4 with respect to t, we obtain

$$\frac{dR(t)}{dt} = -f(t). \tag{1.5}$$

For example, if the time to failure distribution is exponential with parameter λ, then

$$f(t) = \lambda e^{-\lambda t}, \tag{1.6}$$

and the reliability function is

$$R(t) = 1 - \int_0^t \lambda e^{-\lambda \zeta} d\zeta = e^{-\lambda t}. \tag{1.7}$$

From Equation 1.7, we express the probability of failure of a component in a given interval of time $[t_1, t_2]$ in terms of its reliability function as

$$\int_{t_1}^{t_2} f(t) dt = R(t_1) - R(t_2). \tag{1.8}$$

We define the failure rate in a time interval $[t_1, t_2]$ as the probability that a failure per unit time occurs in the interval given that no failure has occurred prior to t_1, the beginning of the interval. Thus, the failure rate is expressed as

$$\frac{R(t_1)-R(t_2)}{(t_2-t_1)R(t_1)}. \tag{1.9}$$

If we replace t_1 by t and t_2 by $t + \Delta t$, then we rewrite Equation 1.9 as

$$\frac{R(t)-R(t+\Delta t)}{\Delta t R(t)}. \tag{1.10}$$

The hazard function is defined as the limit of the failure rate as Δt approaches zero. In other words, the hazard function or the instantaneous failure rate is obtained from Equation 1.10 as

$$h(t)=\lim_{\Delta t\to 0}\frac{R(t)-R(t+\Delta t)}{\Delta t\, R(t)}=\frac{1}{R(t)}\left[-\frac{d}{dt}R(t)\right]$$

or

$$h(t)=\frac{f(t)}{R(t)}. \tag{1.11}$$

From Equations 1.5 and 1.11, we obtain

$$R(t)=e^{\left[-\int_0^t h(\zeta)d\zeta\right]}, \tag{1.12}$$

$$R(t)=1-\int_0^t f(\zeta)d\zeta, \tag{1.13}$$

and

$$h(t)=\frac{f(t)}{R(t)}. \tag{1.14}$$

Equations 1.5, 1.12–1.14 are the key equations that relate $f(t)$, $F(t)$, $R(t)$, and $h(t)$.

The following example illustrates how the hazard rate, $h(t)$, and reliability are estimated from failure data.

EXAMPLE 1.1

A manufacturer of light bulbs is interested in estimating the mean life of the bulbs. Two hundred bulbs are subjected to a reliability test. The bulbs are observed, and failures in 1000-h intervals are recorded as shown in Table 1.1.

Plot the failure density function estimated from data $f_e(t)$, the hazard-rate function estimated from data $h_e(t)$, the cumulative probability function estimated from data $F_e(t)$, and the reliability function estimated from data $R_e(t)$. The subscript e refers to *estimated*. Comment on the hazard-rate function.

TABLE 1.1 Number of Failures in the Time Intervals

Time interval (hours)	Failures in the interval
0–1000	100
1001–2000	40
2001–3000	20
3001–4000	15
4001–5000	10
5001–6000	8
6001–7000	7
Total	200

SOLUTION

We estimate $f_e(t)$, $h_e(t)$, $R_e(t)$, and $F_e(t)$ using the following equations:

$$f_e(t) = \frac{n_f(t)}{n_o \Delta t}, \tag{1.15}$$

$$h_e(t) = \frac{n_f(t)}{n_s(t)\Delta t}, \tag{1.16}$$

$$R_e(t) = \frac{f_e(t)}{h_e(t)} = \frac{n_s(t)}{n_o}, \tag{1.17}$$

and

$$F_e(t) = 1 - R_e(t). \tag{1.18}$$

Note that $n_s(t)$ is the number of surviving units at the beginning of the period Δt. Summaries of the calculations are shown in Tables 1.2 and 1.3. The plots are shown in Figures 1.1 and 1.2.

TABLE 1.2 Calculations of $f_e(t)$ and $h_e(t)$

Time interval (h)	Failure density $f_e(t) \times 10^{-4}$	Hazard rate $h_e(t) \times 10^{-4}$
0–1000	$\frac{100}{200\times10^3}=5.0$	$\frac{100}{200\times10^3}=5.0$
1001–2000	$\frac{40}{200\times10^3}=2.0$	$\frac{40}{100\times10^3}=4.0$
2001–3000	$\frac{20}{200\times10^3}=1.0$	$\frac{20}{60\times10^3}=3.33$
3001–4000	$\frac{15}{200\times10^3}=0.75$	$\frac{15}{40\times10^3}=3.75$
4001–5000	$\frac{10}{200\times10^3}=0.5$	$\frac{10}{25\times10^3}=4.0$
5001–6000	$\frac{8}{200\times10^3}=0.4$	$\frac{8}{15\times10^3}=5.3$
6001–7000	$\frac{7}{200\times10^3}=0.35$	$\frac{7}{7\times10^3}=10.0$

TABLE 1.3 Calculations of $R_e(t)$ and $F_e(t)$

Time interval	Reliability $R_e(t) = f_e(t)/h_e(t)$	Unreliability $F_e(t) = 1 - R_e(t)$
0–1000	$\frac{5.0}{5.0}=1.000$	0.000
1001–2000	$\frac{2.0}{4.0}=0.500$	0.500
2001–3000	$\frac{1.0}{3.33}=0.300$	0.700
3001–4000	$\frac{0.75}{3.75}=0.200$	0.800
4001–5000	$\frac{0.5}{4.0}=0.125$	0.875
5001–6000	$\frac{0.4}{5.3}=0.075$	0.925
6001–7000	$\frac{0.35}{10.0}=0.035$	0.965

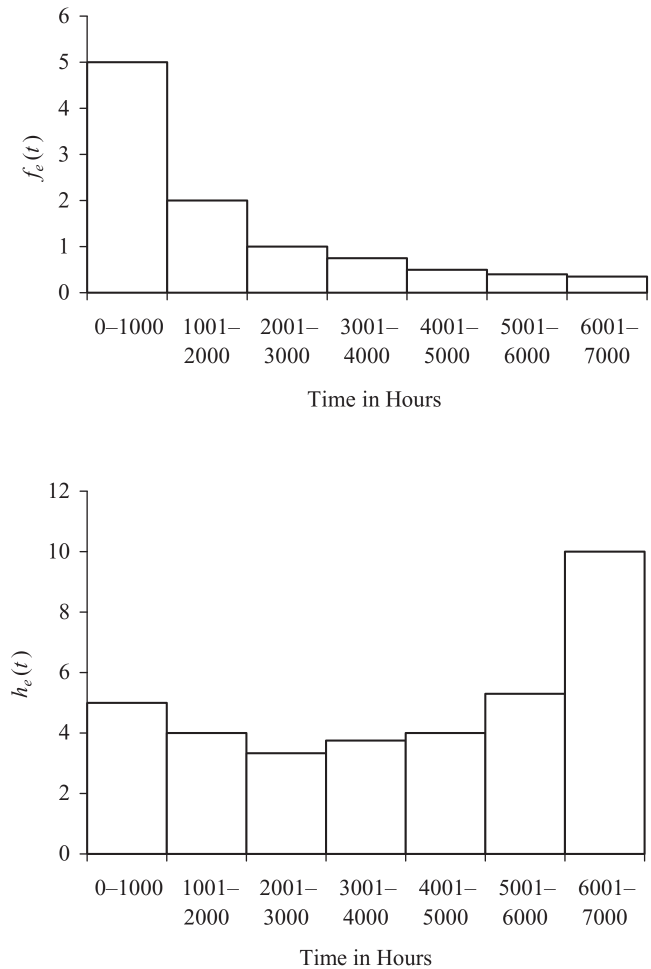
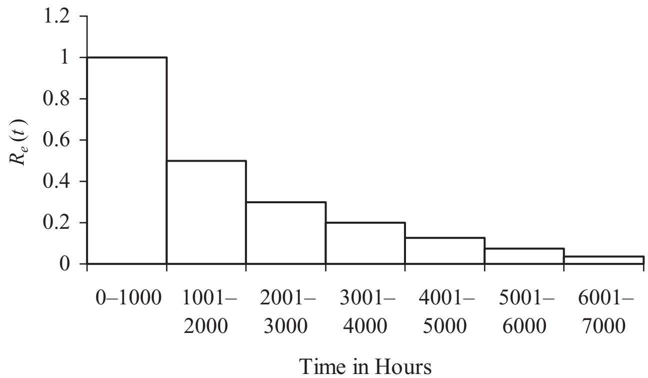


FIGURE 1.1 Plots of $f_e(t) \times 10^{-4}$ and $h_e(t) \times 10^{-4}$ versus time.



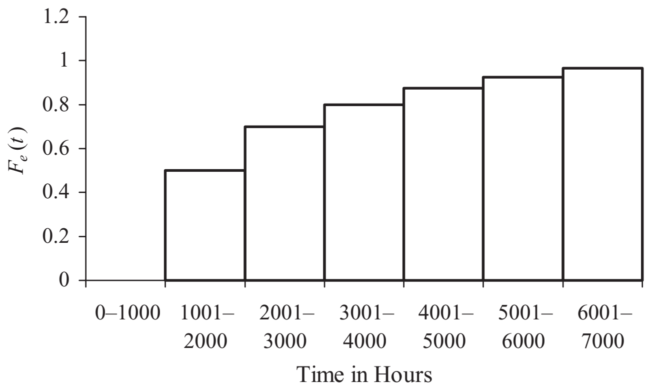


FIGURE 1.2 Plots of $R_e(t)$ and $F_e(t)$ versus time.

As shown in Figure 1.1, the hazard rate is constant until time of 5000 h and then increases linearly with t. Thus, $h_e(t)$ can be expressed as

$$h_e(t) = \begin{cases} \lambda_0 & 0 \le t \le 6{,}000 \\ \lambda_1 t & t > 6{,}000 \end{cases},$$

where λ_0 and λ_1 are constants. ■

The above example shows the hazard-rate function is constant for a period of time and then linearly increases with time. In other situations, the hazard-rate function may be decreasing, constant, or increasing, and the rate at which the function decreases or increases may be constant, linear, polynomial, or exponential with time. The following example is an illustration of an exponentially increasing hazard-rate function.

EXAMPLE 1.2

Facsimile (fax) machines are designed to transmit documents, figures, and drawings between locations via telephone lines. The principle of a fax machine is shown in Figure 1.3. The document on the sending unit drum is scanned in both the horizontal and rotating directions. The document is divided into graphic elements, which are converted into electrical signals by a photoelectric reading head. The signals are transmitted via telephone lines to the receiving end where they are demodulated and reproduced by a recording head.

The quality of the received document is affected by the reliability of the photoelectric reading head in converting the graphic elements of the document being sent into proper electrical signals. A manufacturer of fax machines performs a reliability test to estimate the mean life of the reading head by subjecting 180 heads to repeated cycles of readings. The threshold times, at which the quality of the received document is unacceptable, are recorded in Table 1.4.

Estimate the hazard rate and reliability function of the machines.

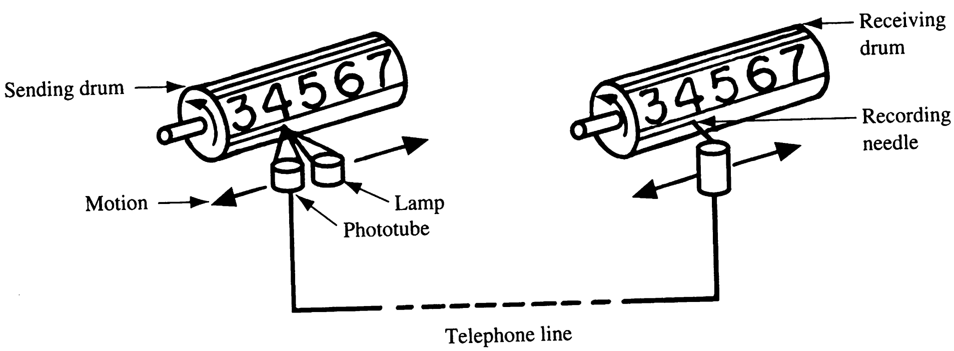


FIGURE 1.3 The principle of a fax machine.

TABLE 1.4 Failure Data of the Facsimile Machines

Time interval (hours)	0–150	151–300	301–450	451–600	601–750	751–900
Number of failures	20	28	27	32	33	40

SOLUTION

Using Equations 1.15–1.17, we calculate $f_e(t)$, $h_e(t)$, and $R_e(t)$ as shown in Table 1.5. Plots of the hazard rate and the reliability function are shown in Figures 1.4 and 1.5, respectively.

TABLE 1.5 Calculations for $f_e(t)$, $h_e(t)$, and $R_e(t)$

t	$f_e(t) \times 10^{-4}$	$h_e(t) \times 10^{-4}$	$R_e(t)$
0–150	7.407	7.407	1.000
151–300	10.370	11.666	0.889
301–450	10.000	13.636	0.733
451–600	11.852	20.317	0.583
601–750	12.222	30.137	0.406
751–900	14.815	66.667	0.222

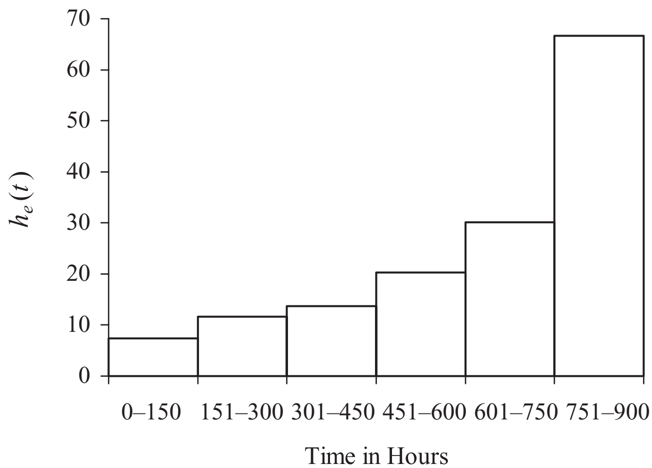


FIGURE 1.4 Plot of the hazard-rate function versus time.

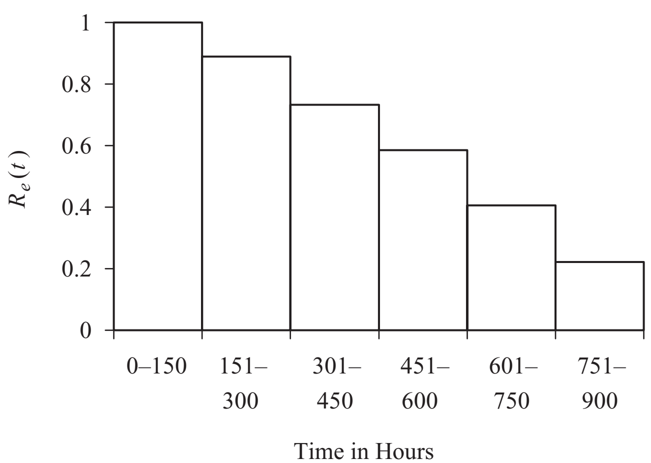


FIGURE 1.5 Plot of the reliability function versus time. ■

In some situations, it is possible to observe the exact failure time of every unit (component). In such situations, we utilize order statistics to obtain a "distribution free" reliability function and its associated characteristics. There are several approaches to do so starting with the "naïve" estimator followed by median rank estimators. Since all these estimators utilize the order of the observations (failure times) only, we refer to them as ordered statistics, and the empirical estimate of $F(t)$, denoted as $\hat{F}(t)$, is referred to as rank estimator which is then used to generate the density plot and reliability function plot. We present the commonly used rank estimators (mean and median) as follows.

We begin by ordering the failure times in an increasing order such that $t_1 \le t_2 \le \ldots \le t_{i-1} \le t_i \le t_{i+1} \le \ldots \le t_{n-1} \le t_n$ where t_i is the failure time of the i^{th} unit. Since we are interested in obtaining the naïve rank estimator $\hat{F}(t)$, we assign a probability mass of $1/n$ to each of the n failure times and set $\hat{F}(t_0) = 0$. The naïve mean rank estimator $\hat{F}(t)$ is expressed as

$$\hat{F}(t) = \frac{i}{n} \qquad t_i \le t \le t_{i-1}.$$

This estimator has a deficiency in that, for $t \ge t_n$, $\hat{F}(t) = 1.0$. Therefore, improvements are introduced that result in a more accurate estimate.

Among them is the most commonly used Herd–Johnson estimator (Herd, 1960; Johnson, 1964) which is expressed as

$$\hat{F}(t_i) = \frac{i}{n+1} \qquad i = 0,1,2,\ldots,n.$$

Others propose the use of the median rank instead. Several estimates of the median rank are commonly used; among them are Bernard's median rank estimator (Bernard and Bosi-Levenbach, 1953) and Blom's (1958) median rank estimator. They are expressed as

Bernard's estimator of $\hat{F}(t_i)$ is

$$\hat{F}(t_i) = \frac{i-0.3}{n+0.4} \qquad i = 0,1,2,\ldots,n.$$

Blom's estimator is

$$\hat{F}(t_i) = \frac{i-3/8}{n+1/4} \qquad i = 0,1,2,\ldots,n.$$

The corresponding p.d.f., reliability function, and the hazard-rate function are derived as follows.

We consider the mean rank estimator (the approach is also valid for median rank estimators). The mean rank estimator is

$$\hat{F}(t_i) = \frac{i}{n+1} \qquad i = 0,1,2,\ldots,n.$$

The reliability expression is

$$R(t_i) = 1 - F(t_i) = \frac{n+1-i}{n+1} \qquad t_i \le t \le t_{i+1} \qquad i = 0,1,2,...,n.$$

Since the p.d.f. is the derivative of the CDF, then

$$\hat{f}(t_i) = \frac{\hat{F}(t_{i+1}) - \hat{F}(t_i)}{\Delta t_i} \qquad \Delta t_i = t_{i+1} - t_i$$

or

$$\hat{f}(t_i) = \frac{1}{\Delta t_i.(n+1)}.$$

The hazard rate is $h(t_i) = \frac{f(t_i)}{R(t_i)} = \frac{1}{\Delta t_i\,(n+1-i)}$.

EXAMPLE 1.3

Nine light bulbs are observed, and the exact failure time of each is recorded as 70, 150, 250, 360, 485, 650, 855, 1130, and 1540. Estimate the CDF, reliability function, p.d.f., and hazard-rate function. Plot these functions with time.

SOLUTION

Figures 1.6–1.8 show $R(t)$, $f(t)$, and $h(t)$ graphs, and the corresponding calculations are given in Table 1.6.

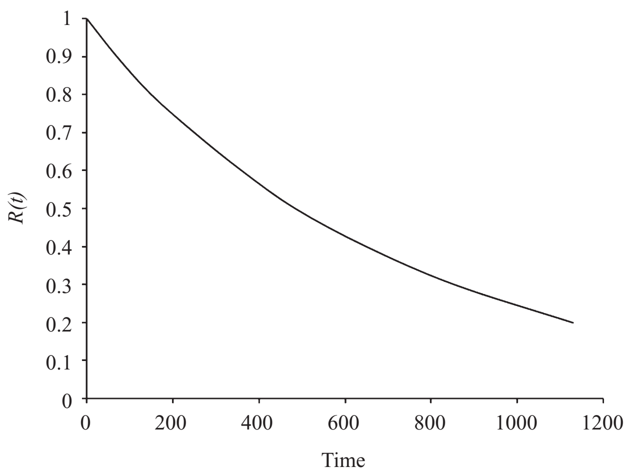


FIGURE 1.6 Plot of the reliability function versus time.

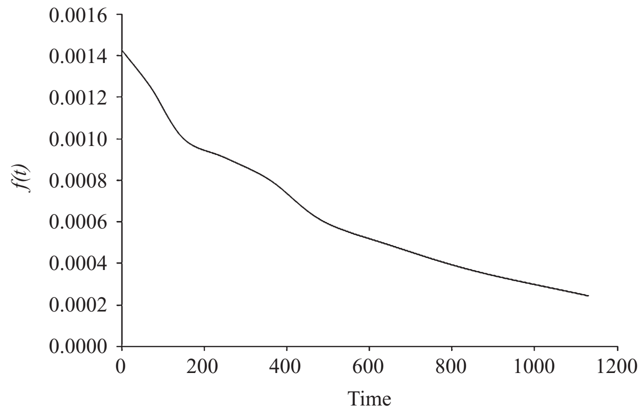


FIGURE 1.7 Plot of the probability density function versus time.

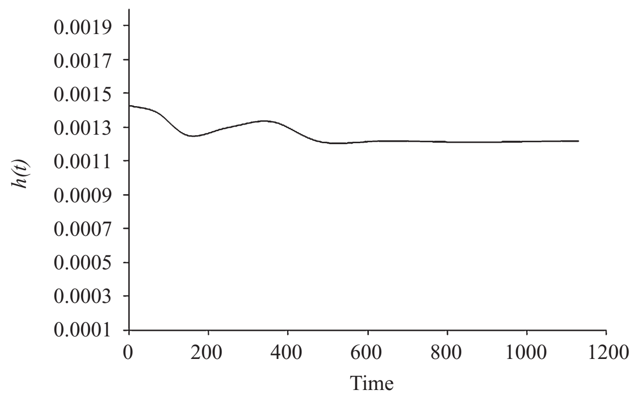


FIGURE 1.8 Plot of the hazard-rate function versus time.

TABLE 1.6 ***F*(*t*), *R*(*t*), *f*(*t*), and *h*(*t*) Calculations**

i	t_i	t_{i+1}	$\hat{F}(t_i)=\frac{i}{10}$	$R(t_i)=\frac{10-i}{10}$	$f(t_i)=\frac{1}{\Delta t_i(n+1)}$	$h(t_i)=\frac{1}{\Delta t_i(n+1-i)}$
0	0	70	0.0	1.0	0.001429	0.001429
1	70	150	0.1	0.9	0.001250	0.001389
2	150	250	0.2	0.8	0.001000	0.001250
3	250	360	0.3	0.7	0.000909	0.001299
4	360	485	0.4	0.6	0.000800	0.001333
5	485	650	0.5	0.5	0.000606	0.001212
6	650	855	0.6	0.4	0.000488	0.001220
7	855	1,130	0.7	0.3	0.000364	0.001212
8	1,130	1,540	0.8	0.2	0.000244	0.001220
9	1,540	–	0.9	0.1	–	–

■

The hazard-rate estimates using the mean rank and median rank are expressed respectively as

$$h_{mean-rank}(t_i) = \frac{1}{(n-i+1)(t_{i+1}-t_i)}$$

and

$$h_{median-rank}(t_i) = \frac{1}{(n-i+0.7)(t_{i+1}-t_i)}.$$

There are other estimates of the hazard rate such as Kaplan–Meier (to be discussed in Chapter 5) and Martz and Waller (1982) which is expressed as

$$\underset{\text{Martz–Waller}}{h(t_i)} = \frac{1}{(n-i+0.625)(t_{i+1}-t_i)}.$$

Martz and Waller's estimate is suitable when the sample size is small. It should be noted that hazard rates estimated by the above three estimators differ only slightly especially when the number of observed failure-time data is large.

Analysis of the historical data of failed products, components, devices, and systems resulted in widely used expressions for $h(t)$ and $R(t)$. We now consider the most commonly used expressions for $h(t)$.

1.3 HAZARD FUNCTIONS

The *hazard function* or *hazard rate* $h(t)$ is the conditional probability of failure in the interval t to $(t + dt)$, given that there was no failure at t divided by the length of the time interval dt. It is expressed as

$$h(t) = \frac{f(t)}{R(t)}. \tag{1.19}$$

The *cumulative hazard function* $H(t)$ is the conditional probability of failure in the interval 0 to t. It is also the total number of failures during the time interval 0 to t.

$$H(t) = \int_0^t h(\zeta)d\zeta. \tag{1.20}$$

The hazard rate is also referred to as the instantaneous failure rate. The hazard-rate expression is of the greatest importance for system designers, engineers, and repair and maintenance groups. The expression is useful in estimating the time to failure (or time between failures), repair crew size for a given repair policy, the availability of the system, and in estimating the warranty cost. It can also be used to study the behavior of the system's failure with time.

As shown in Equation 1.19, the hazard rate is a function of time. One may ask what type of function does the hazard rate exhibit with time? The general answer to this question is the bathtub-shaped function as shown in Figure 1.9. To illustrate how this function is obtained, consider a population of identical components from which we take a large sample N and place it in operation at time $T = 0$. The sample experiences a high failure rate at the beginning of the operation time due to weak or substandard components, manufacturing imperfections, design

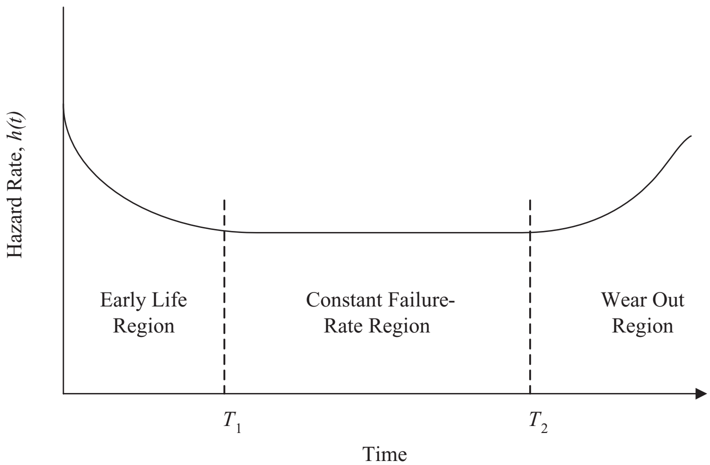


FIGURE 1.9 The general failure curve.

errors, and installation defects. As the failed components are removed, the time between failures increases which results in a reduction in the failure rate. This period of decreasing failure rate (DFR) is referred to as the “infant mortality region,” the “shakedown” region, the “debugging” region, or the “early failure” region. This is an undesirable region from both the manufacturer and consumer viewpoints as it causes an unnecessary repair cost for the manufacturer and an interruption of product usage for the consumer. The early failures can be minimized by employing burn-in of systems or components before shipments are made (burn-in is a common process where the unit is subjected to a slightly severer stress conditions than those at normal operating conditions for a short period), by improving the manufacturing process, and by improving the quality control of the products. Time T_1 represents the end of the early failure-rate region (normally this time is about 10^4 h for electronic systems).

At the end of the early failure-rate region, the failure rate will eventually reach a constant value. During the constant failure-rate region (between T_1 and T_2), the failures do not follow a predictable pattern, but they occur at random due to the changes in the applied load (the load may be higher or lower than the designed load). A higher load may cause overstressing of the component while a lower load may cause derating (application of a load in the reverse direction of what the component experiences under normal operating conditions) of the component and both will lead to failures. The randomness of the material flaws or manufacturing flaws will also lead to failures during the constant failure-rate region.

The third and final region of the failure-rate curve is the wear-out region, which starts at T_2. The beginning of the wear-out region is noticed when the failure rate starts to increase significantly more than the constant failure-rate value, and the failures are no longer attributed to randomness but are due to the age and wear of the components. Within this region, the failure rate increases rapidly as the product reaches its useful (designed) life. To minimize the effect of the wear-out region, one must use periodic preventive maintenance or consider replacement of the product.

Obviously, not all components exhibit the bathtub-shaped failure-rate curve. Most electronic and electrical components do not exhibit a wear-out region. Some mechanical components may not show a constant failure-rate region but may exhibit a gradual transition between the early failure-rate and wear-out regions. The length of each region may also vary from one component (or product) to another. The estimates of the times at which the bathtub curve changes from one region to another have been of interest to researchers. They are referred to as the change-point estimates. One of the approaches for estimating the change point is to equate the estimated hazard rate at the end of the region to the estimated hazard rate at the beginning of the following region.

We now describe the failure-time distributions that exhibit one or more of the regions as follows.

1.3.1 Constant Hazard

Many electronic components—such as transistors, resistors, integrated circuits (ICs), and capacitors—exhibit constant failure rate (CFR) during their lifetimes. Of course, this occurs at the end of the early failure region, which usually has a time period of 1 year (8760 h). The early failure region is usually reduced by performing burn-in of these components. Burn-in is performed by subjecting components to stresses slightly higher than the expected operating stresses for a short period in order to weed out failures due to manufacturing defects. The constant hazard-rate function, $h(t)$, is expressed as

$$h(t) = \lambda, \tag{1.21}$$

where λ is a constant. The p.d.f., $f(t)$, is obtained from Equation 1.19 as

$$f(t) = h(t)\exp\left[-\int_0^t h(\zeta)d\zeta\right] \tag{1.22}$$

or

$$f(t) = \lambda e^{-\lambda t} \tag{1.23}$$

and

$$F(t) = \int_0^t \lambda e^{-\lambda\zeta} d\zeta = 1 - e^{-\lambda t}. \tag{1.24}$$

The reliability function, $R(t)$, is

$$R(t) = 1 - F(t) = e^{-\lambda t}. \tag{1.25}$$

Plots of $h(t)$, $f(t)$, $F(t)$, and $R(t)$ are shown in Figures 1.10 and 1.11. At $t = 1/\lambda$, $f(1/\lambda) = \lambda/e$, $F(1/\lambda) = 1 - 1/e = 0.632$, and $R(1/\lambda) = 1/e = 0.368$. This is an important result since it states that the probability of failure of a product by its estimated mean time to failure (MTTF) $(1/\lambda)$ is 0.632. Also, note that the failure time for the constant hazard model is exponentially distributed.

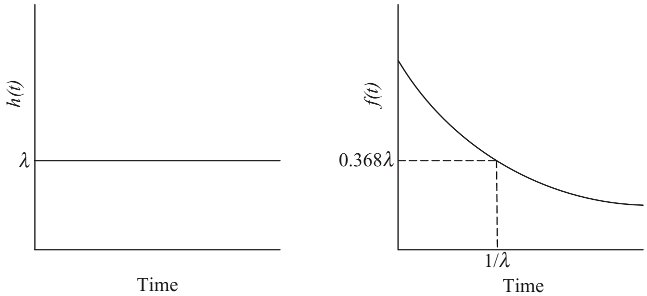


FIGURE 1.10 Plots of $h(t)$ and $f(t)$.

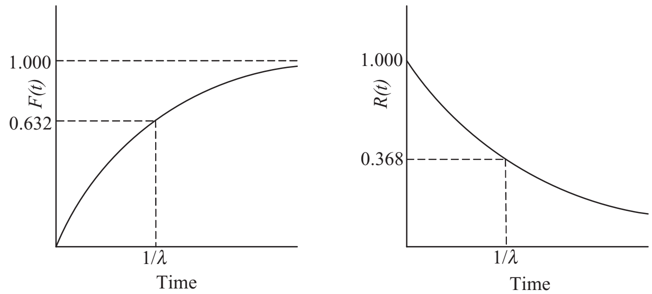


FIGURE 1.11 Plots of $F(t)$ and $R(t)$.

EXAMPLE 1.4

A manufacturer performs an Operational Life Test (OLT) on ceramic capacitors and finds that they exhibit CFR (used interchangeably with hazard rate) with a value of 3×10^{-8} failures per hour. What is the reliability of a capacitor after 10^4 h? In order to accept a large shipment of these capacitors, the user decides to run a test for 5000 h on a sample of 2000 capacitors. How many capacitors are expected to fail during the test?

SOLUTION

Using Equations 1.21 and 1.25, we obtain

$$h(t) = 3 \times 10^{-8} \text{ failures per hour}$$

and

$$R(10^4) = e^{-3\times10^{-4}} = 0.99970.$$

To determine the expected number of failed capacitors during the test, we define the following:

n_o number of capacitors under test,
n_s expected number of surviving capacitors at the end of test, and
n_f expected number of failed capacitors during the test.

Thus,

$$n_s = e^{-3\times 10^{-8}\times 5000} \times 2000 = 1999 \text{ capacitors}$$

and

$$n_f = 2000 - 1999 = 1 \text{ capacitor.}$$ ■

1.3.2 Linearly Increasing Hazard

A component exhibits an increasing hazard rate when it either experiences wearout or when it is subjected to deteriorating conditions. Most of mechanical components—such as rotating shafts, valves, and cams—exhibit linearly increasing hazard rate due to wearout whereas components such as springs and elastomeric mounts exhibit linearly increasing hazard rate due to deterioration. Few electrical components such as relays exhibit linearly increasing hazard rate. The hazard-rate function is expressed as

$$h(t) = \lambda t, \tag{1.26}$$

where λ is constant. The p.d.f., $f(t)$, is a Rayleigh distribution and is obtained as

$$f(t) = \lambda t e^{-\frac{\lambda t^2}{2}} \tag{1.27}$$

and

$$F(t) = 1 - e^{-\frac{\lambda t^2}{2}}. \tag{1.28}$$

The reliability function, $R(t)$, is

$$R(t) = e^{-\frac{\lambda t^2}{2}}. \tag{1.29}$$

Plots of $h(t)$, $f(t)$, $R(t)$, and $F(t)$ are shown in Figures 1.12 and 1.13. It should be noted that the failure-time distribution of the linearly increasing hazard is a Rayleigh distribution. The mean (expected value) and the variance of the distribution are $\sqrt{\pi/2\lambda}$ and $2/\lambda(1 - \pi/4)$, respectively.

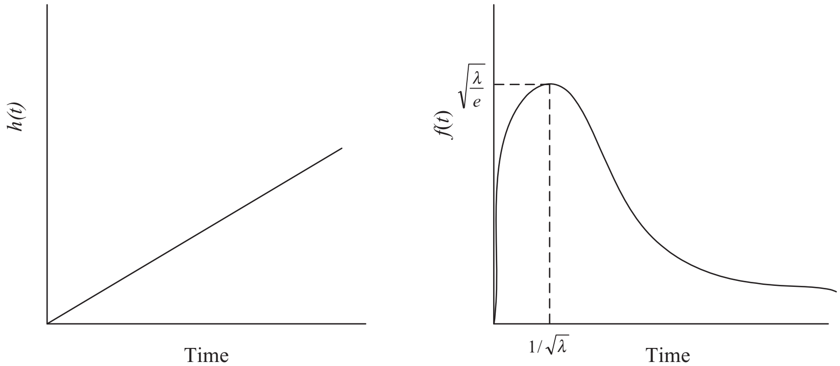


FIGURE 1.12 Plots of $h(t)$ and $f(t)$.

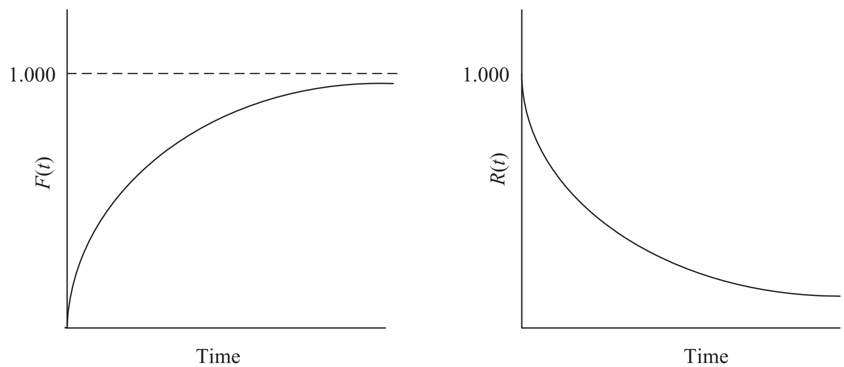


FIGURE 1.13 Plots of $R(t)$ and $F(t)$.

EXAMPLE 1.5

Rolling resistance is a measure of the energy lost by a tire under load when it resists the force opposing its direction of travel. In a typical car, traveling at 60 mi/h, about 20% of the engine power is used to overcome the rolling resistance of the tires. A tire manufacturer introduces a new material that, when added to the tire rubber compound significantly improves the tire rolling resistance but increases the wear rate of the tire tread. Analysis of a laboratory test of 150 tires shows that the failure rate of the new tire is linearly increasing with time (in hours). It is expressed as $h(t) = 0.50 \times 10^{-8}\, t$.

Determine the reliability of the tire after 1 year of use. What is the mean time to replace the tire?

SOLUTION

Using Equation 1.29 we obtain the reliability after 1 year as

$$R(8{,}760) = e^{-\frac{0.5}{2}\times 10^{-8}\times(8{,}760)^2} = 0.825.$$

The mean time to replace the tire is

$$\text{Mean time} = \sqrt{\frac{\pi}{2\lambda}} = \sqrt{\frac{\pi}{2 \times 0.5 \times 10^{-8}}} = 17{,}724 \text{ h},$$

and the standard deviation of the time to tire replacement is

$$\sigma = \sqrt{\frac{2}{\lambda}\left(1 - \frac{\pi}{4}\right)} = 9{,}265 \text{ h}.$$

■

1.3.3 Linearly Decreasing Hazard

Most components (both mechanical and electrical) show decreasing hazard rates during their early lives. The hazard rate decreases linearly or nonlinearly with time. In this section, we shall consider linear hazard functions while nonlinear functions will be considered in the next section. The linearly decreasing hazard-rate function is expressed as

$$h(t) = a - bt \tag{1.30}$$

and

$$a \geq bt,$$

where a and b are constants. Similar to the linearly increasing hazard-rate function, we can obtain expressions for $f(t)$, $R(t)$, and $F(t)$. The failure model and the reliability of a component exhibiting such hazard function depend on the values of a and b.

1.3.4 Weibull Model

A nonlinear expression for the hazard-rate function is used when it clearly cannot be represented linearly with time. A typical expression for the hazard function (decreasing or increasing) under this condition is

$$h(t) = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1}. \tag{1.31}$$

This model is referred to as the Weibull model, and its $f(t)$ is given as

$$f(t) = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1} e^{-\left(\frac{t}{\theta}\right)^{\gamma}} \qquad t > 0, \tag{1.32}$$

where θ and γ are positive and are referred to as the characteristic life and the shape parameter of the distribution, respectively. For $\gamma = 1$ this $f(t)$ becomes an exponential density. When $\gamma = 2$, the density function becomes a Rayleigh distribution. It is also well known that the Weibull

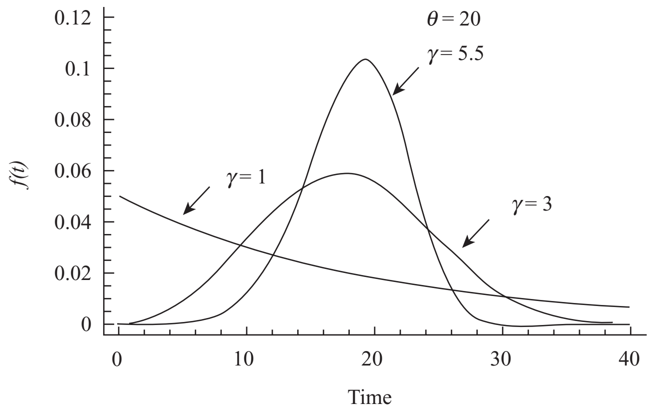


FIGURE 1.14 The Weibull p.d.f. for different γ.

p.d.f. approximates to a normal distribution if a suitable value for the shape parameter γ is chosen. Makino (1984) approximated the Weibull distribution to a normal using the mean hazard rate and found that the shape parameter that approximates the two distributions is $\gamma = 3.43927$. This value of γ is near to the value $\gamma = 3.43938$, which is the value of the shape parameter of the Weibull distribution at which the mean is equal to the median. The p.d.f.'s of the Weibull distribution for different γ's are shown in Figure 1.14. The distribution and reliability functions of the Weibull distribution $F(t)$ and $R(t)$ are given in Equations 1.34 and 1.35, respectively.

$$F(t) = \int_0^t \frac{\gamma}{\theta}\left(\frac{\zeta}{\theta}\right)^{\gamma-1} e^{-\left(\frac{\zeta}{\theta}\right)^{\gamma}} d\zeta \tag{1.33}$$

or

$$F(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^{\gamma}} \quad t > 0 \tag{1.34}$$

and

$$R(t) = e^{-\left(\frac{t}{\theta}\right)^{\gamma}} \quad t > 0 \tag{1.35}$$

The Weibull distribution is widely used in reliability modeling since other distributions such as exponential, Rayleigh, and normal are special cases of the Weibull distribution. Again, the hazard-rate function follows the Weibull model:

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1}. \tag{1.36}$$

When $\gamma > 1$, the hazard rate is a monotonically increasing function with no upper bound that describes the wear-out region of the bathtub curve. When $\gamma = 1$, the hazard rate becomes constant (constant failure-rate region), and when $\gamma < 1$, the hazard-rate function decreases with time (the early failure-rate region). This enables the Weibull model to describe the failure rate of many failure data in practice. The mean and variance of the Weibull distribution are

$$E[T(\text{time to failure})] = \theta\Gamma\left(1+\frac{1}{\gamma}\right), \tag{1.37}$$

$$\mathrm{Var}[T] = \theta^2\left\{\Gamma\left(1+\frac{2}{\gamma}\right) - \left[\Gamma\left(1+\frac{1}{\gamma}\right)\right]^2\right\}, \tag{1.38}$$

where $\Gamma(n)$ is the gamma function

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$$

and

$$\int_0^\infty x^{n-1}e^{-x/\theta}dx = \Gamma(n)\theta^n.$$

EXAMPLE 1.6

To determine the fatigue limit of specially treated steel bars, the Prot method (Collins, 1981) for performing fatigue test is utilized. The test involves the application of a steadily increasing stress level with applied cycles until the specimen under test fails. The number of cycles to failure is observed to follow a Weibull distribution with $\theta = 5$ (measurements are in 10^3 cycles) and $\gamma = 2$.

1. What is the reliability of a bar at 10^6 cycles? What is the corresponding hazard rate?
2. What is the expected life (in cycles) for a bar of this type?

SOLUTION

Since the shape parameter γ equals 2, the Weibull distribution becomes a Rayleigh distribution, and we have a linearly increasing hazard function. Its p.d.f. is given by Equation 1.32.

1. The reliability expression for the Weibull model is given by Equation 1.35:

$$R(10^6) = e^{-(10^6/5\times 10^3)^2}$$
$$= e^{-40,000} = 0.$$

The hazard rate at 10^6 cycles is

$$h(t) = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1} = \frac{2}{5000}\times\left(\frac{10^6}{5000}\right)$$

or $h(10^6) = 0.08$ failures/cycle.

2. The expected life of a bar is

$$E[T\,(\text{cycles to failure})] = \theta\,\Gamma\left(1+\frac{1}{\gamma}\right) = (5\times10^3)\,\Gamma\left(\frac{3}{2}\right)$$

$$=(5000)\left(\frac{1}{2}\right)\sqrt{\pi} = 4431.$$

The expected life of a bar from this steel is 4431 cycles. ■

In the above example, the Weibull model became a Rayleigh model since the failure rate is linearly increasing with time. In the following example, we consider the situation when the failure rate is nonlinearly increasing with time.

It is assumed that the failure time follows a known Weibull distribution and that the parameters of the distribution are known. In actual situations, the actual failure-time observations are the only known information. In this case, the failure-time data are used to obtain the failure-time distribution by fitting the data to the appropriate probability distribution. This can be achieved by plotting the frequency of failure times in a histogram and fitting a curve to them. The fitted curve is then used as a basis to select appropriate probability distribution that fits the data. The latter step is accomplished using standard software or probability papers. The following example illustrates these procedures.

EXAMPLE 1.7

A manufacturer of a tungsten-carbide cutting tool for highly abrasive rubber materials conducted a tool life experiments on 50 tools. The times to tool failure are given as

17	31	58	66	73	73	97	108	111	117
132	132	138	140	143	143	145	147	150	157
158	161	164	168	171	177	182	185	187	196
197	202	223	242	246	249	260	269	276	287
298	308	312	314	316	338	349	354	423	529

Use probability plot and fit the data with an appropriate probability distribution.

SOLUTION

Using a standard software such as STATGRAPHICS™ or SAS™, obtain a frequency distribution as shown in Figure 1.15. The fitted curve indicates an increasing hazard rate similar to the Weibull model discussed earlier in this chapter. Since Weibull is one of the most widely used distributions for analyzing reliability data, a Weibull probability plot is shown in Figure 1.16.

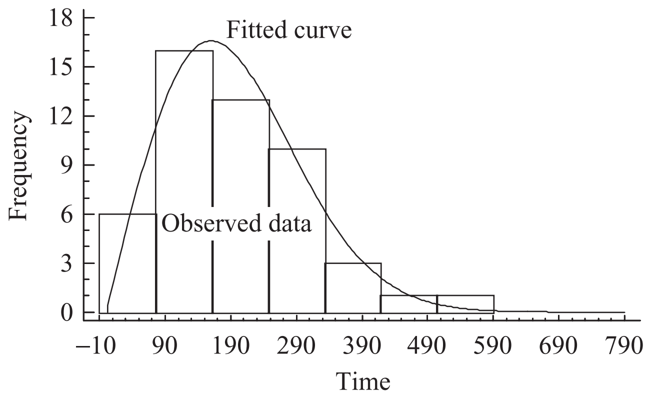


FIGURE 1.15 Frequency distribution of the failure times.

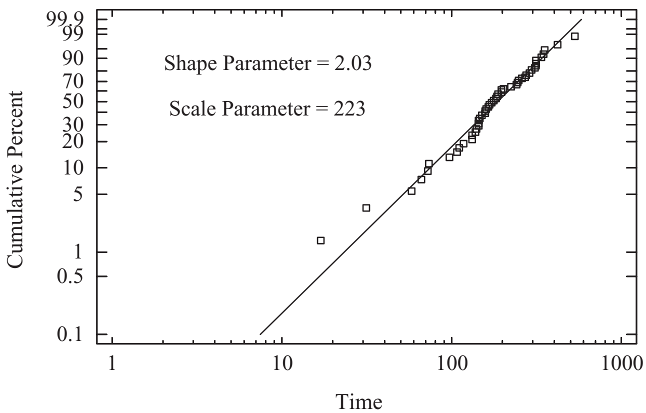


FIGURE 1.16 Probability plot of the failure times.

The straight line indicates that Weibull distribution is appropriate to describe the failure times. The parameters of the model are estimated as $\gamma = 2.03$ and $\theta = 223$ (using the software). Several methods for estimating these parameters are described in Chapter 5. ■

Alternatively, the parameters of the Weibull model can be obtained by using one the of the approaches discussed above for estimating the CDF $F(t)$ from failure data and fitting a linear regression model as described below.

The CDF is expressed as

$$F(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^{\gamma}} \qquad t > 0.$$

Taking the natural logarithm of $1 - F(t)$ results in

$$\ln(1-F(t)) = -\left(\frac{t}{\theta}\right)^{\gamma}.$$

Taking the logarithm one more time, we obtain

$$\ln\left[\ln\left(\frac{1}{1-F(t)}\right)\right] = \gamma \ln t - \gamma \ln \theta.$$

Fitting a linear regression model to the left hand side of the above expression and $\ln t$ we can then easily obtain the parameters of the Weibull model.

EXAMPLE 1.8

A manufacturing engineer observes the wear-out rate of a milling machine tool insert and fits a Weibull hazard model to the tool wear data. The parameters of the model are $\gamma = 2.25$ and $\theta = 30$. Determine the reliability of the tool insert after 10 h, the expected life of the insert, and the standard deviation of the mean life.

SOLUTION

The reliability after 10 h of operation is

$$R(10) = e^{-\left(\frac{10}{30}\right)^{2.25}} = 0.919.$$

The mean life of the insert is

$$\begin{aligned}\text{Mean life} &= \theta\Gamma\left(1+\frac{1}{\gamma}\right)\\ &= 30\Gamma\left(1+\frac{1}{2.25}\right),\end{aligned}$$

or Mean life = $30\Gamma(1.444) = 26.572$ h.

The value of $\Gamma(1.444)$ is obtained from the tables of the gamma function given in Appendix A.

Using Equation 1.38, we obtain the variance of the life as

$$\begin{aligned}\text{Variance} &= \theta^2\left\{\Gamma\left(1+\frac{2}{\gamma}\right) - \left[\Gamma\left(1+\frac{1}{\gamma}\right)\right]^2\right\}\\ &= 30^2\left\{\Gamma(1.888) - [\Gamma(1.444)]^2\right\}\end{aligned}$$

or

Variance = 156.140, and the standard deviation of the life is 12.50 h. ■

1.3.5 Mixed Weibull Model

This model is applicable when components or products experience two or more failure modes. For example, a mechanical component, such as a load-carrying bearing or a cutting tool, may fail due to wearout or when the applied stress exceeds the design strength of component material resulting in catastrophic failure (catastrophic failure is a failure that destroys the system, such as a missile failure). Each type of these failures may be modeled by a separate simple Weibull model. Since the component or the tool can fail in either of the failure modes, it is then appropriate to describe the hazard rate by a mixed Weibull model. It is expressed as

$$f(t) = p\frac{\gamma_1}{\theta_1}\left(\frac{t}{\theta_1}\right)^{\gamma_1 - 1} e^{-\left(\frac{t}{\theta_1}\right)^{\gamma_1}} + (1-p)\frac{\gamma_2}{\theta_2}\left(\frac{t}{\theta_2}\right)^{\gamma_2 - 1} e^{-\left(\frac{t}{\theta_2}\right)^{\gamma_2}} \tag{1.39}$$

for $\theta_1,\ \theta_2 > 0$.

The quantity $p(0 \le p \le 1)$ is the probability that the component or the tool fails in the first failure mode, and $1-p$ is the probability that it fails in the second failure mode. Clearly, if a product experiences more than two failure modes, the model given by Equation 1.39 can be expanded to include all failure modes and associated probabilities such that $\sum_{i=1}^{n} p_i = 1$ where p_i is the probability that the product fails in the ith failure mode, and n is the total number of failure modes.

Following Kao (1959), the time t_e at which the proportion of the catastrophic failure is equal to that of wear-out failure is obtained as

$$1 - e^{-\left(\frac{t_e}{\theta_1}\right)^{\gamma_1}} = 1 - e^{-\left(\frac{t_e}{\theta_2}\right)^{\gamma_2}}$$

or

$$t_e = \left(\frac{\theta_2^{\gamma_2}}{\theta_1^{\gamma_1}}\right)^{\frac{1}{\gamma_2 - \gamma_1}} = \exp\left(\frac{\gamma_2 \ln\theta_2 - \gamma_1 \ln\theta_1}{\gamma_2 - \gamma_1}\right). \tag{1.40}$$

The reliability expression of the mixed Weibull model is

$$R(t) = 1 - p\left[1 - e^{-\left(\frac{t}{\theta_1}\right)^{\gamma_1}}\right] - (1-p)\left[1 - e^{-\left(\frac{t}{\theta_2}\right)^{\gamma_2}}\right]. \tag{1.41}$$

Clearly, if the second failure mode occurs after a delay time δ, from the first failure mode, we rewrite Equations 1.39 and 1.41 as follows:

$$f_d(t) = p\frac{\gamma_1}{\theta_1}\left(\frac{t}{\theta_1}\right)^{\gamma_1 - 1} e^{-\left(\frac{t}{\theta_1}\right)^{\gamma_1}} + (1-p)\frac{\gamma_2}{\theta_2}\left(\frac{t-\delta}{\theta_2}\right)^{\gamma_2 - 1} e^{-\left(\frac{t-\delta}{\theta_2}\right)^{\gamma_2}} \tag{1.42}$$

and

$$R_d(t) = 1 - p\left[1 - e^{-\left(\frac{t}{\theta_1}\right)^{\gamma_1}}\right] - (1-p)\left[1 - e^{-\left(\frac{t-\delta}{\theta_2}\right)^{\gamma_2}}\right], \tag{1.43}$$

where the subscript d denotes delay.

1.3.6 Exponential Model (the Extreme Value Distribution)

The extreme value distribution is closely related to the Weibull distribution. It is useful in modeling cases when the hazard function is initially constant and then begins to increase rapidly with time.

The distribution is used to describe the failure time of products (or components) that will operate properly at normal operating conditions and will fail owing to a secondary cause of failure (such as overheating or fracture) when subjected to extreme conditions. In other words, the interest is in the tails of the failure distribution. Here, the hazard-rate function, the failure-time density function, and the reliability function are expressed as

$$h(t) = be^{\alpha t} \tag{1.44}$$

$$f(t) = be^{\alpha t} e^{-\int_0^t h(\zeta)d\zeta} \tag{1.45}$$

$$f(t) = be^{\alpha t} e^{-\frac{b}{\alpha}(e^{\alpha t}-1)} \tag{1.46}$$

$$R(t) = e^{-\frac{b}{\alpha}(e^{\alpha t}-1)}, \tag{1.47}$$

where b is a constant and e^{α} represents the increase in failure rate per unit time. For example, if it is found that the failure rate of a component increases about 10% each year, then $h(t) = b(1.1)^t$ where $\alpha = \ln(1.1) = 0.0953$. The function $f(t)$ as given by Equation 1.46 is also known as the *Gompertz distribution*.

Plots of the hazard rate and the reliability functions of the *extreme value distribution* for different values of α and b are shown in Figure 1.17. Some electronic components show such a hazard function. There are mechanical assemblies that exhibit extreme value hazard functions when subjected to high stresses. An example of such assemblies is a gearbox that operates properly at the recommended speeds. Excessive speeds may cause wearout of bearings that result in misalignments of shafts and an eventual failure of the assembly.

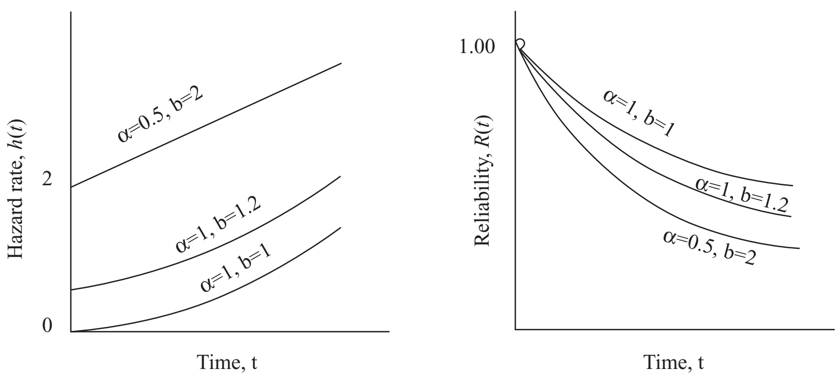


FIGURE 1.17 Plots of $h(t)$ and $R(t)$.

EXAMPLE 1.9

Excessive vibrations due to high speed cutting on a computer numerical control (CNC) machine may lead to the failure of the cutting tool. The failure time of the tool follows an extreme value distribution. The failure rate increases about 15% per hour. Assuming that $b = 0.01$, calculate the reliability of the tool at $t = 10$ h.

SOLUTION

Since the failure rate increases by 15% per hour, then $\alpha = \ln(1.15) = 0.1397$. Substituting the parameters α and b into Equation 1.47, we obtain

$$R(10) = e^{-\frac{0.01}{0.1397}\left(e^{0.1397\times 10}-1\right)}$$

$$R(10) = 0.8042. \qquad \blacksquare$$

1.3.7 Normal Model

There are many practical situations where the failure time of components (or parts) can be described by a normal distribution. For example, most of the mechanical components that are subjected to repeated cyclic loads, such as a fatigue test, exhibit normal hazard rates. Unlike other continuous probability distributions, there are no closed form expressions for the reliability or hazard-rate functions. The CDF of the life of a component is given by

$$F(t) = P[T \le t] = \int_{-\infty}^{t} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\tau-\mu}{\sigma}\right)^2\right] d\tau, \tag{1.48}$$

and

$$R(t) = 1 - F(t),$$

where μ and σ are the mean and the standard deviation of the distribution. Unlike other distributions, the integral of the cumulative distribution cannot be evaluated in a closed form. However, the standard normal distribution ($\sigma = 1$ and $\mu = 0$) can be utilized in evaluating the probabilities for any normal distribution. The p.d.f. for the standard normal distribution is

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \qquad -\infty < z < \infty, \tag{1.49}$$

where

$$z = \frac{\tau-\mu}{\sigma}.$$

The CDF is

$$\Phi(\tau) = \int_{-\infty}^{\tau} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz. \tag{1.50}$$

Therefore, when the failure time of a component is expressed as a normally distributed random variable T, with mean μ and standard deviation σ, one can easily determine the probability that the component will fail by time t (i.e., the unreliability of the component) by using the following equation:

$$P(T \le t) = P\left(\frac{T-\mu}{\sigma} \le \frac{t-\mu}{\sigma}\right) = \Phi\left(\frac{t-\mu}{\sigma}\right). \tag{1.51}$$

The right side of Equation 1.51 can be evaluated using the standard normal tables. The hazard function, $h(t)$, of the normal distribution is

$$h(t) = \frac{f(t)}{R(t)} = \frac{\phi\left(\frac{t-\mu}{\sigma}\right)\Big/\sigma}{R(t)}. \tag{1.52}$$

It can be shown that the hazard function for a normal distribution is a monotonically increasing function of t,

$$\begin{aligned} h(t) &= \frac{f(t)}{1-F(t)} \\ h'(t) &= \frac{(1-F)f' + f^2}{(1-F)^2}. \end{aligned} \tag{1.53}$$

The denominator is nonnegative for all t. Hence, it is sufficient to show that the numerator of Equation 1.53 is ≥0:

$$(1-F)f' + f^2 \ge 0. \tag{1.54}$$

The p.d.f. of the normal distribution is

$$f(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(t-\mu)^2/2\sigma^2}, \qquad -\infty < t < \infty,$$

and Equation 1.54 can be rewritten as

$$R(t)\frac{d}{dt}f(t) + f^2(t) \ge 0.$$

Now, the derivative term is

$$\begin{aligned} \frac{d}{dt}f(t) &= \frac{1}{\sqrt{2\pi\sigma^2}}\frac{d}{dt}e^{-(t-\mu)^2/2\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2}}\frac{-(t-\mu)}{\sigma^2}e^{-(t-\mu)^2/2\sigma^2} \\ &= \frac{-(t-\mu)}{\sigma^2}f(t), \end{aligned}$$

so now the condition that must be satisfied is

$$f(t)\left(\frac{-(t-\mu)}{\sigma^2}R(t) + f(t)\right) \ge 0.$$

Since $f(t) \geq 0$ by definition and $R(t) = \int_t^{\infty} f(x)dx$, we may use the condition

$$\frac{(t-\mu)}{\sigma^2}\int_t^{\infty} f(x)dx \leq \int_t^{\infty} \frac{(x-\mu)}{\sigma^2} f(x)dx = \int_t^{\infty} -df(x) = f(t)$$

to obtain

$$f(t) \geq \frac{t-\mu}{\sigma^2}\int_t^{\infty} f(x)dx$$

So

$$f(t)\left(f(x) - \frac{(t-\mu)}{\sigma^2}\int_t^{\infty} f(x)dx \right) \geq 0,$$

and therefore the Gaussian hazard function is a monotonically increasing function of time. The plots of $f(t)$, $F(t)$, $R(t)$, and $h(t)$ for $\mu = 20$ are shown in Figure 1.18.

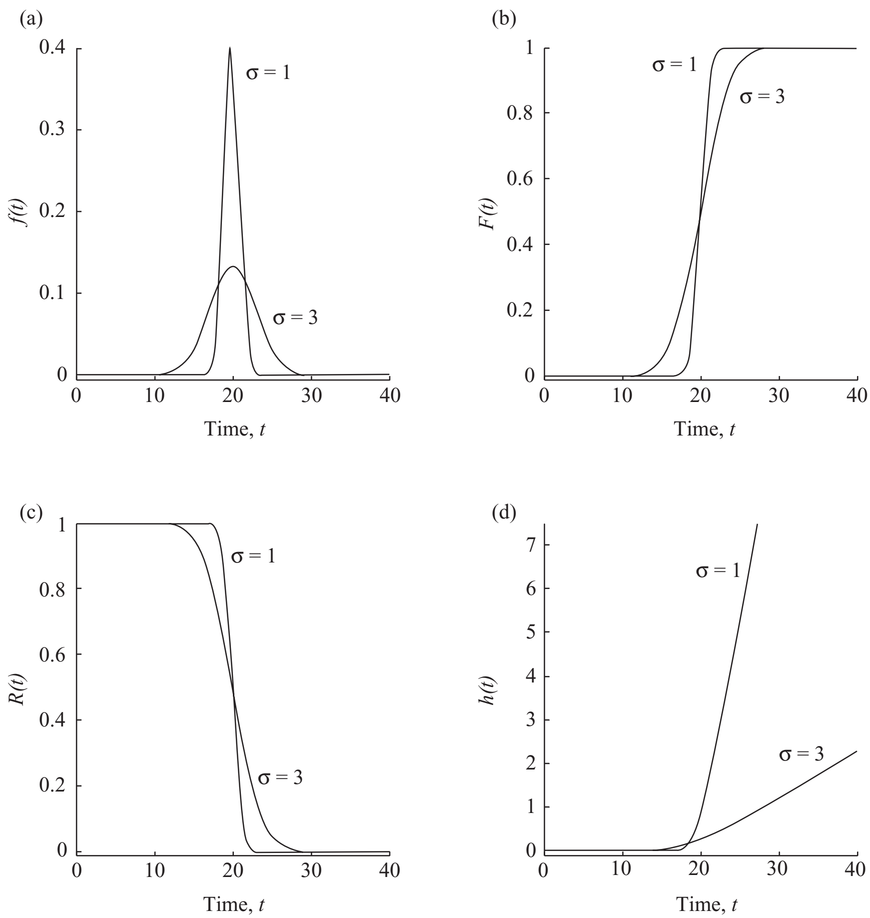


FIGURE 1.18 *f*(*t*), *F*(*t*), *R*(*t*), and *h*(*t*) for the normal model.

EXAMPLE 1.10

A component has a normal distribution of failure times with $\mu = 40{,}000$ cycles and $\sigma = 2000$ cycles. Find the reliability and hazard function at 38,000 cycles.

SOLUTION

The reliability function is

$$\begin{aligned} R(t) &= P\left(z > \frac{t-\mu}{\sigma}\right) \\ R(38{,}000) &= P\left(z > \frac{38{,}000 - 40{,}000}{2{,}000}\right) \\ &= P[z > -1.0] = \Phi(1.0) \\ &= 0.8413. \end{aligned}$$

The value of h (38,000) is

$$\begin{aligned} h(38{,}000) &= \frac{f(38{,}000)}{R(38{,}000)} = \frac{\phi\left(\frac{38{,}000 - 40{,}000}{2000}\right)\Big/2000}{R(38{,}000)} \\ &= \frac{\phi(-1.0)}{2000 \times 0.8413} = \frac{0.2420}{2000 \times 0.8413} \\ &= 0.0001438 \text{ failures per cycle.} \end{aligned}$$

■

1.3.8 Lognormal Model

One of the most widely used probability distributions in describing the life data resulting from a single semiconductor failure mechanism or a closely related group of failure mechanisms is the lognormal distribution. It is also used in predicting reliability from accelerated life test data. The p.d.f. of the lognormal distribution is

$$f(t) = \frac{1}{\sigma t \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln t - \mu}{\sigma}\right)^2\right] \qquad -\infty < \mu < \infty, \sigma > 0, t > 0.$$

Figure 1.19 shows the p.d.f. of the lognormal distribution for different μ and σ.

If a random variable X is defined as $X = \ln T$, where T is lognormal, then X is normally distributed with mean μ and standard deviation σ:

$$E[X] = E[\ln(T)] = \mu$$

$$\mathrm{Var}[X] = \mathrm{Var}[\ln(T)] = \sigma^2.$$

Since $T = e^X$, then the mean of the lognormal can be found by using the normal distribution:

$$E(T) = E\left(e^X\right) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[x - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$$

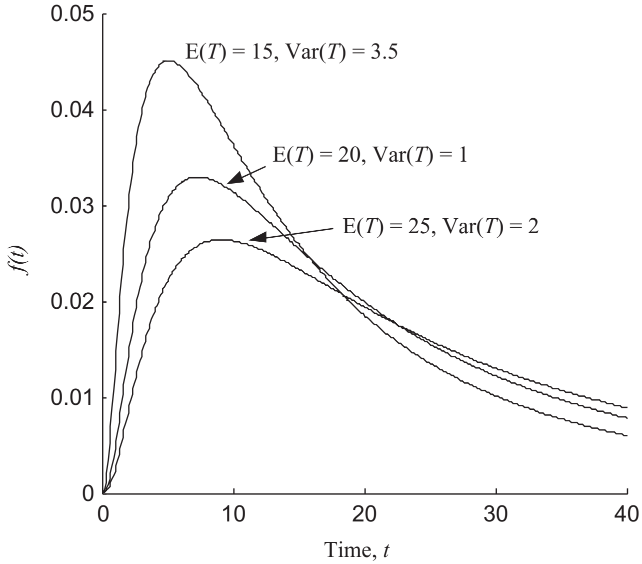


FIGURE 1.19 *f*(*t*) of the lognormal distribution for different μ and σ.

$$E(T) = \exp\left(\mu + \frac{\sigma^2}{2}\right)\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}\left(x - \left(\mu + \sigma^2\right)\right)^2\right] dx.$$

The mean of the lognormal is

$$E(T) = \exp\left[\mu + \frac{\sigma^2}{2}\right].$$

The second moment is obtained as

$$E\left(T^2\right) = E\left[e^{2X}\right] = \exp\left[2\left(\mu + \frac{\sigma^2}{2}\right)\right],$$

and the variance of the lognormal is

$$\text{Var}(T) = \left[e^{2\mu + \sigma^2}\right]\left[e^{\sigma^2} - 1\right].$$

The distribution function of the lognormal is

$$F(t) = \int_0^t \frac{1}{\tau\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln\tau - \mu}{\sigma}\right)^2\right] d\tau$$

or

$$F(t) = P(T \le t) = P\left[z \le \frac{\ln t - \mu}{\sigma}\right].$$

The reliability is

$$R(t) = P[T > t] = P\left[z > \frac{\ln t - \mu}{\sigma}\right]. \tag{1.55}$$

Thus, the hazard function is

$$h(t) = \frac{f(t)}{R(t)} = \frac{\phi\left(\frac{\ln t - \mu}{\sigma}\right)}{t\sigma\, R(t)}. \tag{1.56}$$

Figure 1.20 shows the reliability and the hazard-rate functions of the lognormal distribution for different values of μ and σ.

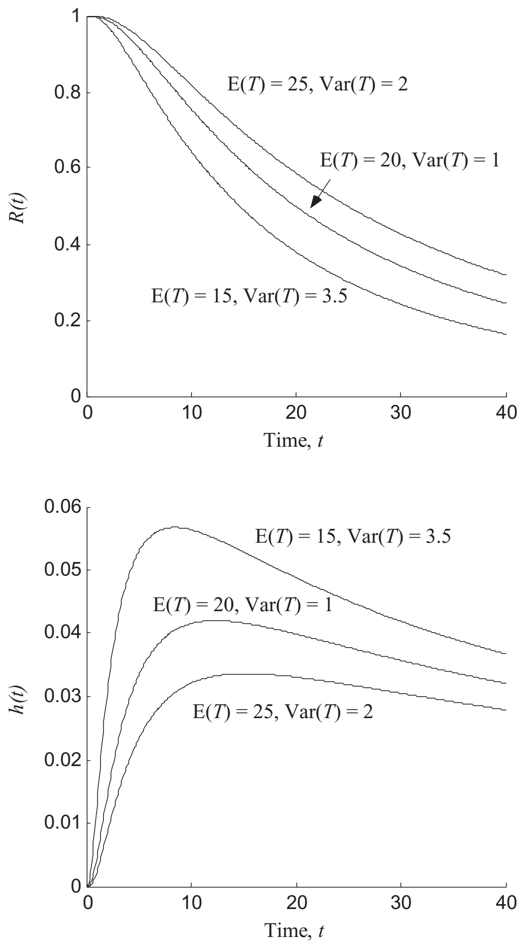


FIGURE 1.20 $R(t)$ and $h(t)$ for the lognormal model.

EXAMPLE 1.11

The failure time of a component is lognormally distributed with $\mu = 6$ and $\sigma = 2$. Find the reliability of the component and the hazard rate for a life of 200 time units.

SOLUTION

$$R(200) = P\left[z > \frac{\ln 200 - 6}{2}\right] = P[z > -0.350] = 0.6386.$$

The hazard function is

$$\begin{aligned} h(200) &= \frac{\phi\left(\frac{\ln 200 - 6}{2}\right)}{200 \times 2 \times 0.6386} \\ &= \frac{\phi(-0.350)}{200 \times 2 \times 0.6386} = \frac{0.3752}{200 \times 2 \times 0.6386} \\ &= 0.00147 \text{ failures per unit time.} \end{aligned}$$

■

1.3.9 Gamma Model

Like the Weibull model, the gamma model covers a wide range of the hazard-rate functions: decreasing, constant, or increasing hazard rates. The gamma distribution is suitable for describing the failure time of a component whose failure takes place in n stages or the failure time of a system that fails when n independent subfailures have occurred.

The gamma distribution is characterized by two parameters: shape parameter γ and scale parameter θ. When $0 < \gamma < 1$, the failure rate monotonically decreases from infinity to $1/\theta$ as time increases from 0 to infinity. When $\gamma > 1$, the failure rate monotonically increases from $1/\theta$ to infinity. When $\gamma = 1$, the failure rate is constant and equals $1/\theta$.

The p.d.f. of a gamma distribution is

$$f(t) = \frac{t^{\gamma-1}}{\theta^{\gamma}\Gamma(\gamma)} e^{-\frac{t}{\theta}}. \tag{1.57}$$

When $\gamma > 1$, there is a single peak of the density function at time $t = \theta(\gamma - 1)$. The CDF, $F(t)$, is

$$F(t) = \int_0^t \frac{\tau^{\gamma-1}}{\theta^{\gamma}\Gamma(\gamma)} e^{-\frac{\tau}{\theta}} d\tau.$$

Substituting $\tau/\theta = \mu$, we obtain

$$F(t) = \frac{1}{\Gamma(\gamma)} \int_0^{t/\theta} \mu^{\gamma-1} e^{-\mu} d\mu$$

or

$$F(t) = I\left(\frac{t}{\theta}, \gamma\right),$$

where $I(t/\theta, \gamma)$ is known as the incomplete gamma function and is tabulated in Pearson (1957). The reliability function $R(t)$ is

$$R(t) = \int_t^\infty \frac{1}{\theta\Gamma(\gamma)}\left(\frac{\tau}{\theta}\right)^{\gamma-1} e^{-\frac{\tau}{\theta}} d\tau. \tag{1.58}$$

When the shape parameter γ is an integer n, the gamma distribution becomes the well-known Erlang distribution. In this case, the CDF is written as

$$F(t) = 1 - e^{-\frac{t}{\theta}} \sum_{k=0}^{n-1} \frac{\left(\frac{t}{\theta}\right)^k}{k!} \tag{1.59}$$

and the reliability function is

$$R(t) = e^{-\frac{t}{\theta}} \sum_{k=0}^{n-1} \frac{\left(\frac{t}{\theta}\right)^k}{k!}. \tag{1.60}$$

The hazard rate of the gamma model, when γ is an integer n, is obtained by dividing Equation 1.57 by Equation 1.60:

$$h(t) = \frac{\frac{1}{\theta}\left(\frac{t}{\theta}\right)^{n-1}}{(n-1)! \sum_{k=0}^{n-1} \frac{\left(\frac{t}{\theta}\right)^k}{k!}}. \tag{1.61}$$

Figures 1.21–1.23 show the gamma density function, the reliability function, and the hazard rate for different γ values and a constant $\theta = 20$.

The mean and variance of the gamma distribution are obtained as

$$\begin{aligned}
\text{Mean life} &= \int_{-\infty}^{\infty} t\, f(t) dt \\
&= \int_0^\infty t\, t^{\gamma-1} \frac{1}{\Gamma(\gamma)\theta^\gamma} e^{-\frac{t}{\theta}} dt \\
&= \frac{1}{\Gamma(\gamma)\theta^\gamma} \int_0^\infty t^\gamma e^{-\frac{t}{\theta}} dt
\end{aligned}$$

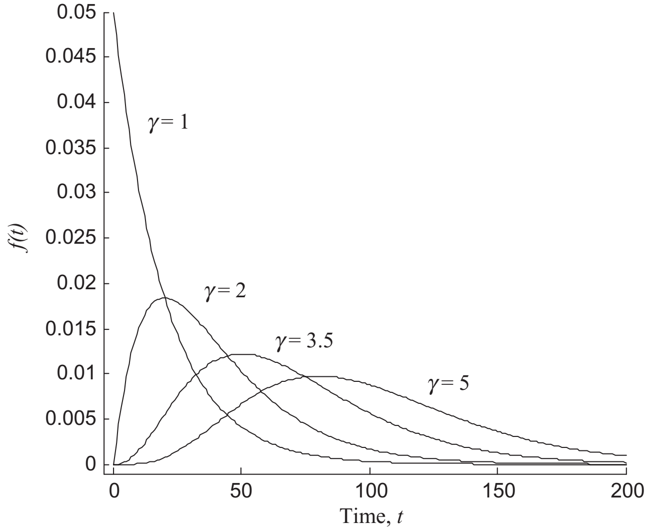


FIGURE 1.21 Gamma density function with different γ values, $\theta = 20$.

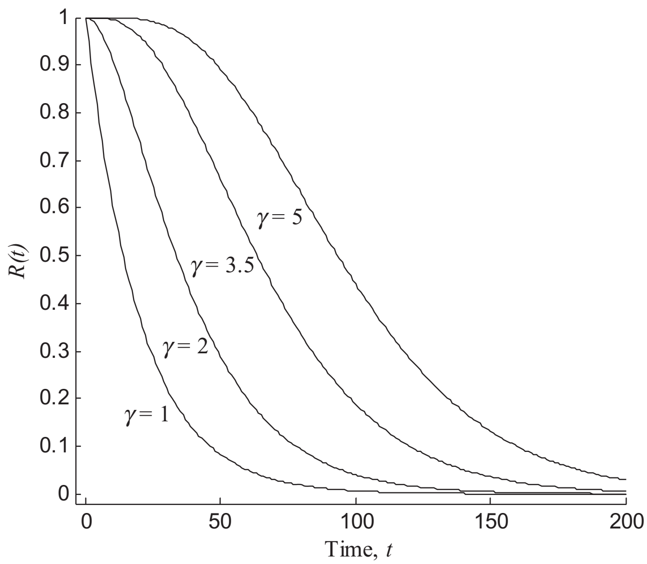


FIGURE 1.22 Gamma reliability function for different γ values, $\theta = 20$.

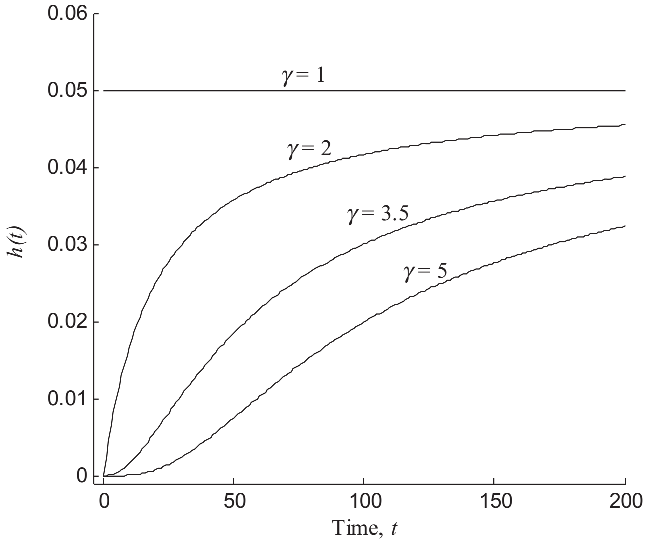


FIGURE 1.23 Gamma hazard rate for different γ values, $\theta = 20$.

or

$$\text{Mean life} = \frac{1}{\Gamma(\gamma)\theta^{\gamma}}\Gamma(\gamma+1)\theta^{\gamma+1} = \gamma\theta.$$

Similar manipulations yield $E[T^2] = \gamma\,(\gamma+1)\theta^2$, and the variance of the life is $\mathrm{Var}(T) = \gamma(\gamma+1)\theta^2 - \gamma^2\theta^2 = \gamma\theta^2$.

EXAMPLE 1.12

A mechanical system requires a constant supply of electric current, which is provided by a main battery having life length T_1 with an exponential distribution of mean 120 h. The main battery is supported by two identical backup batteries with mean lives of T_2 and T_3. When the main unit fails, the first backup battery provides the necessary current to the system. The second backup battery provides the current when the first backup unit fails. In other words, the batteries provide the current independently but sequentially.

Determine the reliability and the hazard rate of the mechanical system at $t = 280$ h. What is the mean life of the system?

SOLUTION

Since the life lengths of the batteries are independent exponential random variables each with mean 120, the distribution of the total life of the mechanical system, T_1, T_2, and T_3 has a gamma distribution with $\gamma = 3$ and $\theta = 120$. Using Equation 1.60 we obtain

$$R(280)=e^{\frac{-280}{120}}\sum_{k=0}^{2}\frac{\left(\frac{280}{120}\right)^k}{k!}=0.58723.$$

The hazard rate at 280 h is obtained by substituting into Equation 1.61

$$h(280)=\frac{\frac{1}{120}\left(\frac{280}{120}\right)^2}{2!(6.055)}=0.003746 \text{ failures per hour.}$$

The mean life of the mechanical system is given by

$$\text{mean life}=\gamma\theta=3(120)=360 \text{ h.}$$ ■

The above results can also be obtained using Special Erlang distribution as follows.

The Erlang distribution is the convolution of n identical units (times) each follows the exponential distribution with parameter λ. Since T_1, T_2, and T_3 are equal, then we can express the density function of the Erlang distribution for n units as

$$f_n(t)=\frac{\lambda^n e^{-t} t^{n-1}}{(n-1)!}$$

and

$$F_n(t)=1-\sum_{i=0}^{n-1}\frac{e^{-\lambda t}(\lambda t)^i}{i!}.$$

The reliability function of Erlang distribution is

$$R_n(t)=\sum_{i=0}^{n-1}\frac{e^{-\lambda t}(\lambda t)^i}{i!}.$$

The reliability and hazard-rate values obtained for this density function are identical to the values above.

1.3.10 Log-Logistic Model

If $T > 0$ is a random variable representing the failure time of a system and t represents a typical time instant in its range, we use $Y \equiv \log T$ to represent the log failure time (Kalbfleisch and Prentice, 2002). The log-logistic distribution for T is obtained if we express $Y = \alpha + \sigma W$ and W has the logistic density

$$f(w)=\frac{e^w}{\left(1+e^w\right)^2}. \tag{1.62}$$

The logistic density is symmetric with mean = 0 and variance = $\pi^2/3$ with slightly heavier tails than the normal density function (Kalbfleisch and Prentice, 2002). The p.d.f. of the failure time t is

$$f(t) = \lambda p(\lambda t)^{p-1}\left[1+(\lambda t)^p\right]^{-2}, \tag{1.63}$$

where $\lambda = e^{-\alpha}$ and $p = 1/\sigma$.

The reliability and hazard functions of the log-logistic model are

$$R(t) = \frac{1}{1+(\lambda t)^p} \tag{1.64}$$

and

$$h(t) = \frac{\lambda p(\lambda t)^{p-1}}{1+(\lambda t)^p}. \tag{1.65}$$

This model has the same advantage as both the Weibull and exponential models; it has simple expressions for $R(t)$ and $h(t)$.

Examination of Equation 1.65 reveals that the hazard function is monotonically decreasing when $p = 1$. If $p > 1$, the hazard rate increases from 0 to a peak at $t = (p-1)^{1/p}/\lambda$ and then decreases with time thereafter. The hazard rate is monotonically decreasing if $p < 1$. Figures 1.24 and 1.25 show the reliability function and the hazard rate for different values of p and a constant $\lambda = 20$.

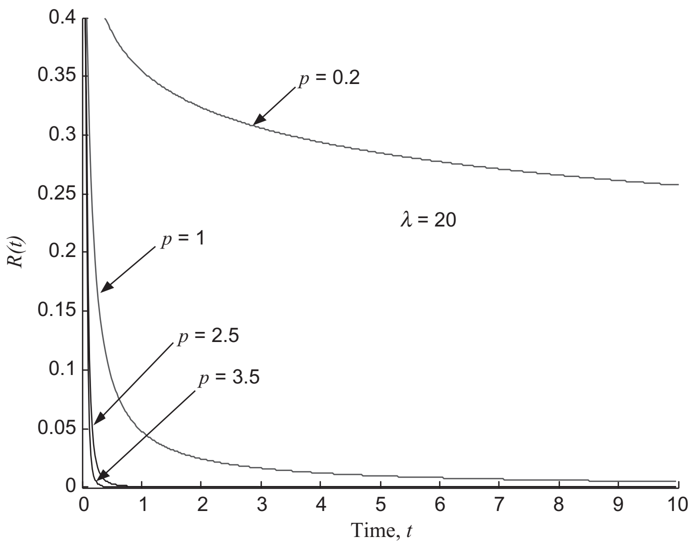


FIGURE 1.24 Reliability function for the log-logistic distribution.

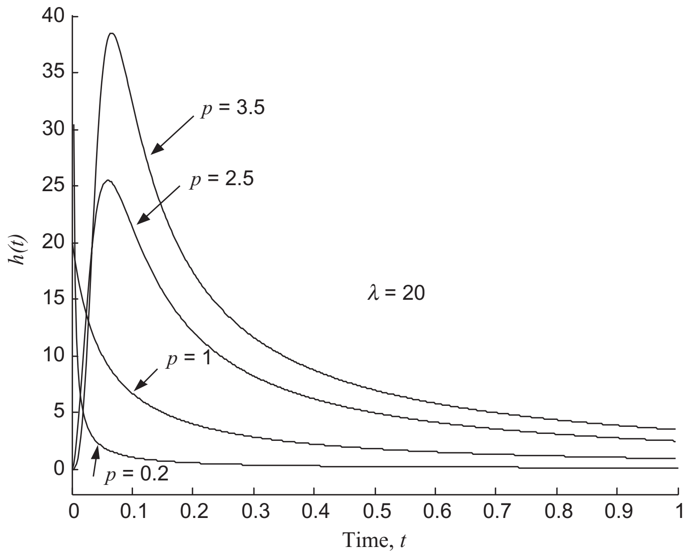


FIGURE 1.25 Hazard rate for the log-logistic distribution.

1.3.11 Beta Model

The hazard function models discussed thus far are defined as nonzero functions over the time range of zero to infinity. However, the life of some products or components may be constrained to a finite interval of time. In such cases, the beta model is the most appropriate model that can describe the reliability behavior of the product during the constrained interval (0, 1). Clearly, any finite interval can be transformed to a (0, 1) interval.

Like other distributions that describe three types of hazard functions—decreasing, constant, and increasing hazard rates—the two parameters of the beta model make it flexible to describe the above hazard rates. The standard form of the density function of the beta model is

$$f(t) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1} & 0 < t < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.66}$$

The parameters α and β are positive. Since

$$\int_0^1 f(t)dt = 1,$$

then,

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.67}$$

for positive α and β.

In general, there is no closed form expression for the cumulative distribution or the hazard-rate function. However, if α or β is a positive integer, a binomial expansion can be used to obtain $F(t)$ and consequently $h(t)$. $F(t)$ will be a polynomial in t, and the powers of t will be, in general, positive real numbers ranging from 0 through $\alpha + \beta - 1$.

The mean and variance of the beta distribution are

$$\text{Mean} = \frac{\alpha}{\alpha+\beta}$$

$$\text{Variance} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

1.3.12 The Inverse Gaussian Model

In most of the models presented so far, the reliability model is often selected based on how well the data appear to be fitted by the model. Clearly, incorporating the failure mechanism or the characteristics of the components (temperature effect, electric field effect, fatigue and cumulative damage effect, etc.) in the model will result in a more realistic model for the system. In other words, it is desirable to use the physical description of a failure to make a choice of distribution accordingly. This is demonstrated further in Chapter 6.

The Inverse Gaussian (IG) distribution is applicable when there is a high occurrence of early failures. Its failure rate is nonmonotonic; initially it increases and then decreases with a nonzero asymptotic value at the end. In effect the IG distribution is suitable for modeling the first two regions of the bathtub curve. Examples of its application are found in accelerated life testing and repair time situations whenever early failures dominate the lifetime distribution. The lognormal distribution could be used instead except when the asymptotic value of the failure rate is zero (Watson and Wells, 1961). However, there is difficulty in justifying the use of the lognormal distribution on physical basis (Chhikara and Folks, 1977). The physical aspect of Brownian motion or any Gaussian process gives rise to the IG as the first passage time distribution which implies its applicability in studying life testing or lifetime phenomenon (Cox and Miller, 1965). Like both the normal and lognormal distributions, the IG has two parameters μ and λ. The p.d.f. is

$$f(t;\mu,\lambda) = \sqrt{\lambda/2\pi t^3}\,\exp(-\lambda(t-\mu)^2/2\mu^2 t), \qquad t>0 \tag{1.68}$$

where μ and λ are assumed to be positive and are referred to as the mean and the shape parameters of the distribution. The variance is μ^3/λ and the p.d.f. is unimodal and skewed. The reliability function $R(t)$ and the hazard-rate function $h(t)$ are given by Equations 1.69 and 1.70 and are shown in Figures 1.26 and 1.27, respectively:

$$R(t) = \Phi\left(\sqrt{\frac{\lambda}{t}}\left(1-\frac{t}{\mu}\right)\right) - e^{2\lambda/\mu}\Phi\left(-\sqrt{\frac{\lambda}{t}}\left(1+\frac{t}{\mu}\right)\right) \tag{1.69}$$

$$h(t) = \frac{\sqrt{\lambda/2\pi t^3}\,\exp(-\lambda(t-\mu)^2/2\mu^2 t)}{\Phi\left(\sqrt{\lambda/t}(1-t/\mu)\right) - e^{2\lambda/\mu}\Phi\left(-\sqrt{\lambda/t}(1+t/\mu)\right)}, \tag{1.70}$$

where Φ denotes the CDF of the standard normal distribution.

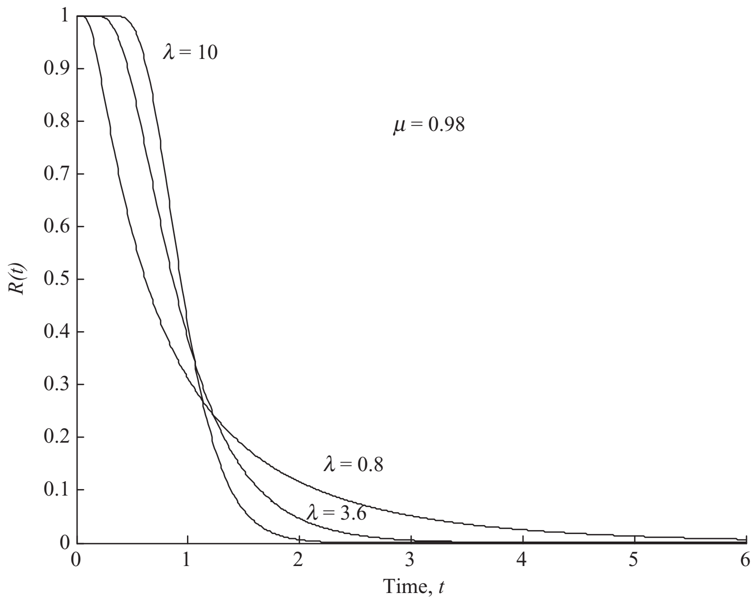


FIGURE 1.26 Reliability function for the Inverse Gaussian distribution.

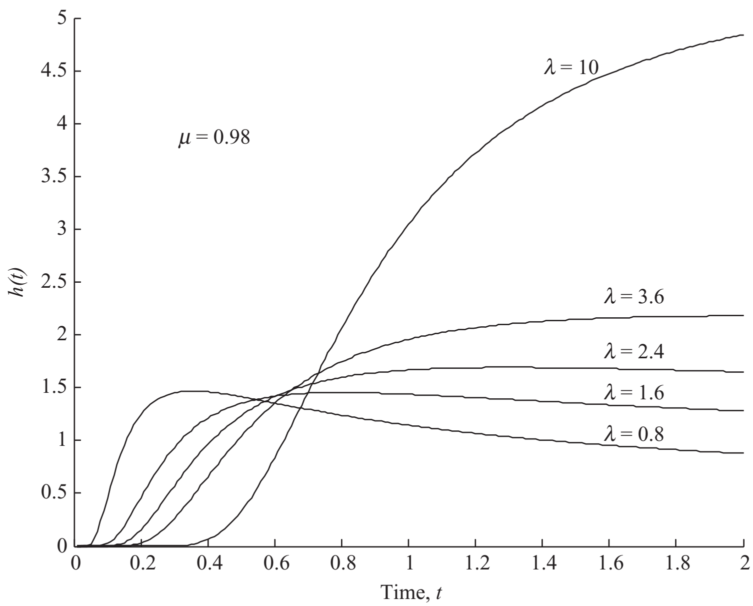


FIGURE 1.27 Hazard-rate function for the Inverse Gaussian distribution.

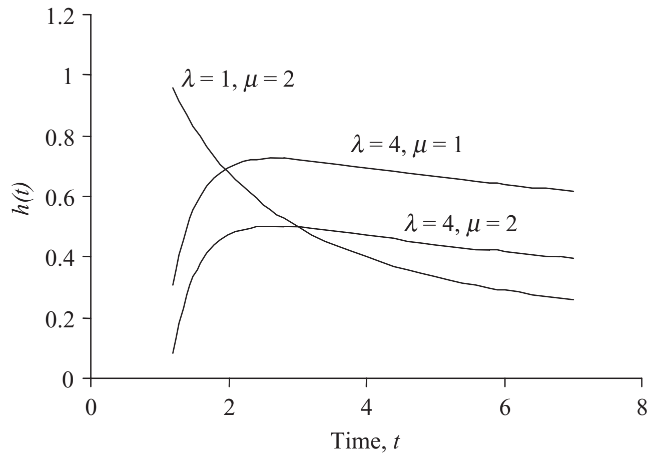


FIGURE 1.28 Maximum values of $h(t)$ for different μ and λ.

As shown in Figure 1.27, the failure rate is not monotonic for all μ and λ. However, the failure rate might be monotonic for some parameter values. It is also important to note that there exists a nonzero asymptotic value of $h(t)$ unlike the failure rate of the lognormal, which approaches zero asymptotically. Since the failure rate might increase then decrease with time, not common in practice, it becomes desirable to determine the time at which the failure rate is maximum in order to assess the system performance at the worst conditions and when it will occur. The maximum value of $h(t)$ can be found by differentiating log $h(t)$ with respect to t as given in Equation 1.71:

$$\begin{aligned}\frac{d}{dt}\log h(t) &= \frac{d}{dt}\log f(t) + \frac{f(t)}{R(t)} \\ &= -\frac{\lambda}{2\mu^2} - \frac{3}{2t} + \frac{\lambda}{2t^2} + h(t).\end{aligned} \tag{1.71}$$

The maximum value of $h(t)$ is obtained at t^* by setting Equation 1.71 to zero. Figure 1.28 shows the maximum values of $h(t)$ for different values of the distribution parameters.

EXAMPLE 1.13

The following failure data are reported on failure times (hours) of electronic capacitors under an accelerated test conditions. 1.0, 1.5, 2.5, 2.5, 2.5, 2.5, 3.0, 3.0, 3.5, 3.5, 3.5, 4.0, 4.0, 5.0, 5.0, 5.0, 5.0, 5.5, 6.5, 7.5, 7.5, 7.5, 7.5, 10.0, 10.0, 11.0, 12.5, 13.5, 15.0, 15.0, 16.5, 16.5, 20.0, 20.0, 22.5, 23.5, 25.0, 27.0, 27.0, 35.0, 37.5, 44.0, 45.0, 51.5, 110.0, 122.5. Estimate the parameters of the IG distribution.

SOLUTION

Let $T_i (i = 1, 2, \dots, n)$ be a random sample from an IG distribution. The maximum likelihood estimate (MLE's) of μ and λ are

$$\hat{\mu} = \bar{T} = \frac{1}{n}\sum_{i=1}^{n} T_i$$

$$\hat{\lambda}^{-1} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T_i} - \frac{1}{\bar{T}}\right).$$

The MLE of the variance is given by

$$\begin{aligned}\hat{\sigma}^2 &= \mu^3 / \hat{\lambda} \\ &= \frac{1}{n}\left[\sum_{i=1}^{n}\frac{\bar{T}^3}{T_i} - n\bar{T}^2\right]\end{aligned}$$

Using the above expressions, we obtain $\hat{\mu} = 18.032{,}61$, $\hat{\lambda} = 8.11398$, and $\hat{\sigma}^2 = 722.67301$. ■

1.3.13 The Frechet Model

The Frechet distribution is the only distribution defined on the nonnegative real numbers that is a well-defined limiting distribution for the maxima of random variables. Let $\{t_i{:}1 \le i \le n\}$ be a collection of independent and identically distributed random variables characteristic of a critical variable in an engineering or physical application. Often the essence of the application is dependent upon the statistical behavior of the maximum $M_n = \max\{t_i{:}1 \le i \le n\}$ or the $m_n = \min\{t_i{:}1 \le i \le n\}$, especially for large n. Classical extreme value theory is concerned with the distributions for M_n and m_n, when n is large. Of all possible nondegenerate limiting distributions, only Frechet distribution for M_n and the Weibull distribution for m_n are concentrated on the nonnegative real numbers (Harlow, 2001). This is useful in reliability applications when, for example, we are interested in estimating the time that a characteristic such as crack length will reach a maximum length that will cause failure (Lorén, 2003). The two-parameter Frechet p.d.f. (Kotz and Nadarajah, 2000) is

$$f(t) = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{-(\gamma+1)} e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}, \qquad t \ge 0, \gamma > 0, \theta > 0 \tag{1.72}$$

and its hazard rate $h(t)$ is given as

$$h(t) = \frac{\dfrac{\gamma}{\theta}\left(\dfrac{t}{\theta}\right)^{-(\gamma+1)} e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}}{1 - e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}}, \quad t \ge 0, \gamma > 0, \theta > 0, \tag{1.73}$$

where θ and γ are positive and are referred to as the characteristic scale and the shape parameters of the distribution, respectively. The p.d.f.'s and hazard function of the Frechet distribution with different γ's are shown in Figures 1.29 and 1.30, respectively.

The distribution and reliability functions of the Frechet distribution $F(t)$ and $R(t)$ are given by Equations 1.74 and 1.75, respectively.

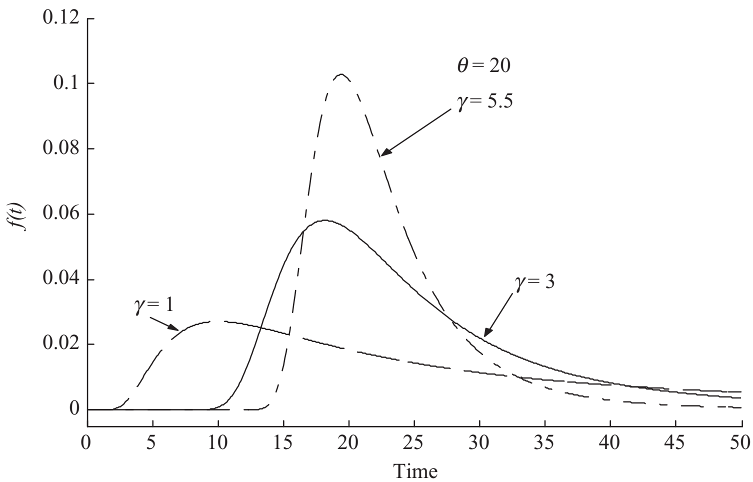


FIGURE 1.29 The Frechet p.d.f. for different γ.

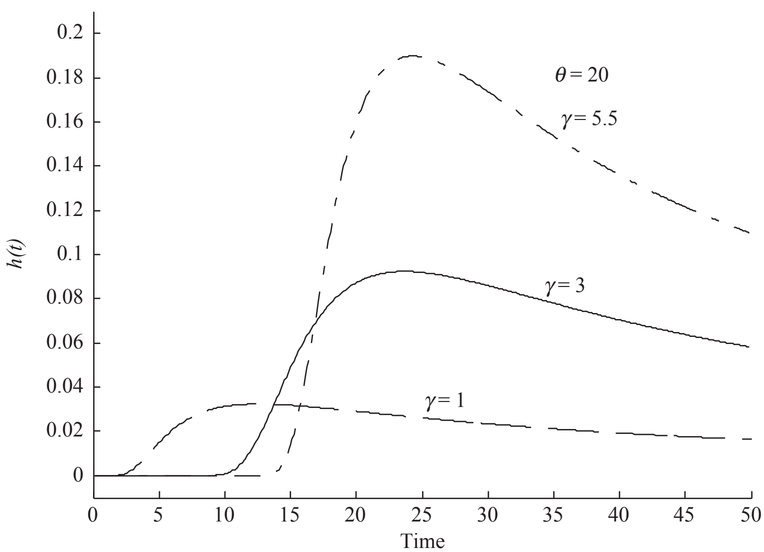


FIGURE 1.30 The hazard function of Frechet distribution for different γ.

$$F(t) = e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}, \quad t > 0 \tag{1.74}$$

$$R(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}, \quad t > 0. \tag{1.75}$$

Equation 1.74 is also referred to as CDF of the Inverse Weibull distribution. The reliability and distribution function of the Frechet distribution with different γ's are shown in Figures 1.31 and 1.32, respectively.

Again, the hazard rate of the Frechet distribution is given as

$$h(t) = \frac{\frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{-(\gamma+1)} e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}}{1 - e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}}, \quad t \geq 0, \gamma > 0, \theta > 0,$$

which is not monotonic. It initially increases to a maximum value and subsequently decreases. It can be shown that the maximum is unique, but its value must be determined numerically. Thus, like the IG distribution, the Frechet distribution may not be appropriate to describe the failure rate of many components or systems in the classical reliability modeling. However, it is commonly used in modeling the inclusion size distribution (inclusions are nonmetallic particles) to determine the mechanical properties of hard and clean metals. It is also used to model the extreme bursts (large file size, sudden increase in traffic) in network traffic.

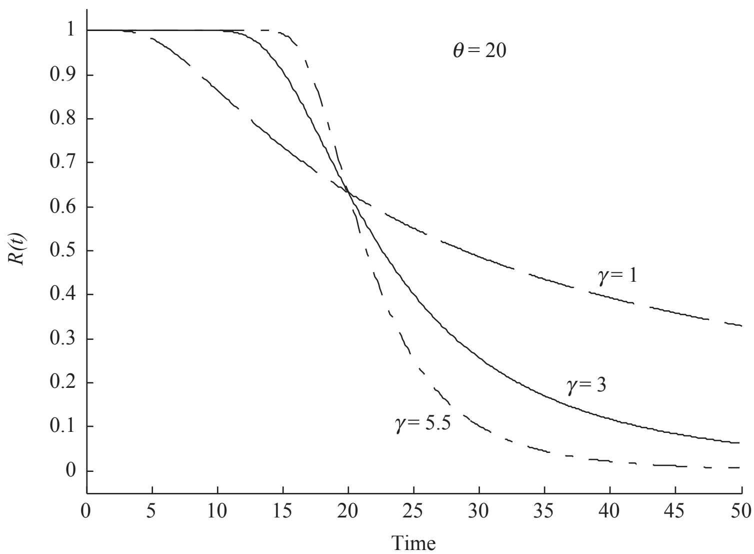


FIGURE 1.31 The reliability function of Frechet distribution for different γ.

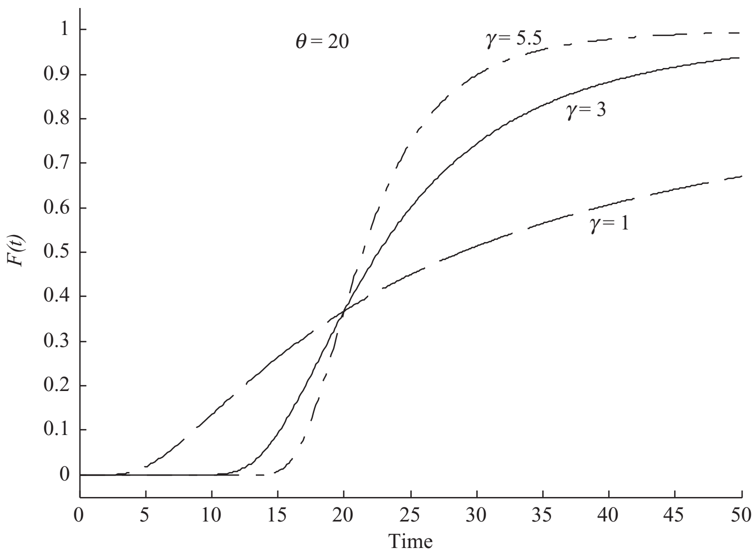


FIGURE 1.32 The CDF of Frechet distribution for different γ.

The kth moment of the Frechet distribution is given as

$$E[T^k] = \int_0^\infty t^k f(t)dt = \theta^k \Gamma\left(1 - \frac{k}{\gamma}\right),$$

where $\Gamma(x)$ is the gamma function. Notice that $E[T^k]$ only exists if $k < \gamma$. In particular, the mean and variance, and coefficient of variation CV could be derived as follows:

$$E[T(\text{time to failure})] = \theta\Gamma\left(1 - \frac{1}{\gamma}\right),$$

$$Var[T] = \theta^2\left[\Gamma\left(1 - \frac{2}{\gamma}\right) - \Gamma^2\left(1 - \frac{1}{\gamma}\right)\right],$$

$$CV = \sqrt{\frac{\left[\Gamma\left(1 - \frac{2}{\gamma}\right) - \Gamma^2\left(1 - \frac{1}{\gamma}\right)\right]}{\Gamma^2\left(1 - \frac{1}{\gamma}\right)}}.$$

Again, $E[T(\text{time to failure})]$ may be estimated from the above equation if $\gamma > 1$, and likewise for $Var[T]$ and CV if $\gamma > 2$. Using simple curve fitting, the CV is well approximated by

$$CV \approx 1/[1.55(\gamma - 2)^{0.7}], \qquad \gamma > 2.$$

Since CV depends on γ only, it is indicative of variability. As γ increases, the scatter decreases, and vice versa. If γ is sufficiently large, θ is approximately equal to the mean $E[T]$.

1.3.14 Birnbaum–Saunders (BS) Distribution

In some engineering applications, it is observed that the failure rate increases with time until it reaches a peak value then it begins to decrease; that is, it is unimodal. This type of behavior was observed by Birnbaum and Saunders (1969) who noted that the failure of units subject to fatigue stresses occurs when the crack length reaches a prespecified limit. It is assumed that the *jth* fatigue cycle increases the crack length by x_j. The cumulative growth in the crack length after n cycles is $\sum_{j=1}^{n} x_j$ which follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$. The probability that the crack does not exceed a critical length ω is expressed as

$$\Phi\left(\frac{\omega - n\mu}{\sigma\sqrt{n}}\right) = \Phi\left(\frac{\omega}{\sigma\sqrt{n}} - \frac{\mu\sqrt{n}}{\sigma}\right). \tag{1.76}$$

Assume that the unit fails when the crack length exceeds ω and that the lifetime is T (expressed either in time or number of fatigue cycles). The reliability at time t is then expressed as

$$R(t) = P(T < t) \approx 1 - \Phi\left(\frac{\omega}{\sigma\sqrt{t}} - \frac{\mu\sqrt{t}}{\sigma}\right) = \Phi\left(\frac{\mu\sqrt{t}}{\sigma} - \frac{\omega}{\sigma\sqrt{t}}\right). \tag{1.77}$$

Substituting $\beta = \omega/\mu$ and $\alpha = \sigma/\sqrt{\omega\mu}$, Equation 1.77 can be written as

$$R(t) = 1 - \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right] \quad 0 < t < \infty \qquad \alpha, \beta > 0. \tag{1.78}$$

Where $\Phi(.)$ is the CDF of the standard normal, α is the shape parameter and β is the scale parameter. Following Kundu et al. (2008), the p.d.f. of the two-parameter BS random variable T corresponding to the complementary CDF of Equation 1.78 is

$$f(t;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\sqrt{\frac{\beta}{t}} + \left(\frac{\beta}{t}\right)^{3/2}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t}\right) - 2\right], \tag{1.79}$$
$$0 < t < \infty, \quad \alpha, \beta > 0$$

This distribution is used to model situations when the maximum hazard rate occurs after several years of operations and then it decreases slowly over a fixed period. It is also applicable for modeling self-healing material or systems where its hazard rate increases up to a point of time then slowly decreases. The p.d.f.'s for different values of α and $\beta = 1$ are shown in Figure 1.33.

Kundu et al. (2008) consider the following transformation of a random variable T that follows BS (α, β)

$$X = \frac{1}{2}\left[\left(\frac{T}{\beta}\right)^{\frac{1}{2}} - \left(\frac{T}{\beta}\right)^{\frac{-1}{2}}\right],$$

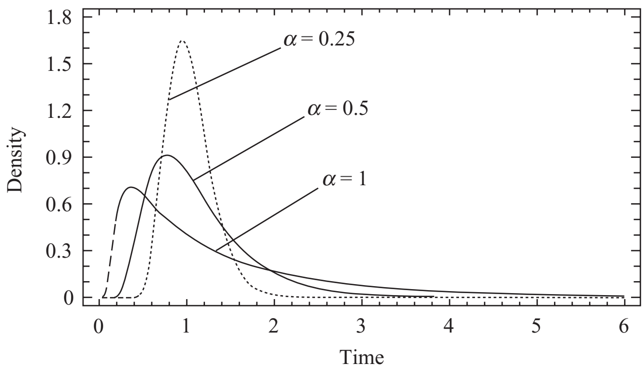


FIGURE 1.33 Probability density function of BS distribution.

which is equivalent to

$$T = \beta(1 + 2X^2 + 2X(1 + X^2)^{\frac{1}{2}}).$$

Then X is normally distributed with mean zero and variance $(\alpha^2/4)$. The above relationship is utilized to obtain several characteristics of the BS distribution (Johnson et al., 1995). They are

$$E(T) = \beta\left(1 + \frac{1}{2}\alpha^2\right). \tag{1.80}$$

$$V(T) = (\alpha\beta)^2\left(1 + \frac{5}{4}\alpha^2\right). \tag{1.81}$$

$$\text{Coefficient of Skewness} = \frac{16\alpha^2(11\alpha^2 + 6)}{(5\alpha^2 + 4)^3}. \tag{1.82}$$

$$\text{Coefficient of Kurtosis} = 3 + \frac{6\alpha^2(93\alpha^2 + 41)}{(5\alpha^2 + 4)^2}. \tag{1.83}$$

Note that Equation 1.80 is the mean life (or MTTF).

The hazard rate $h(t)$ is obtained by dividing Equation 1.79 by Equation 1.78. There is no closed form for $h(t)$, but it can be estimated numerically. Figure 1.34 shows the hazard-rate function for different values of α.

Kundu et al. (2008) show that the hazard rate is unimodal, and it increases to a peak value then slowly decreases with time. Assuming $\beta = 1$, they show that the change point of the hazard rate occurs approximately at

$$c(\alpha) = \frac{1}{(-0.4604 + 1.8417\alpha)^2}.$$

This approximation is for $\alpha > 0.25$ and works quite well for $\alpha > 0.60$. The change point moves closer to zero as the shape parameter increases which implies that the units exhibit a decreasing

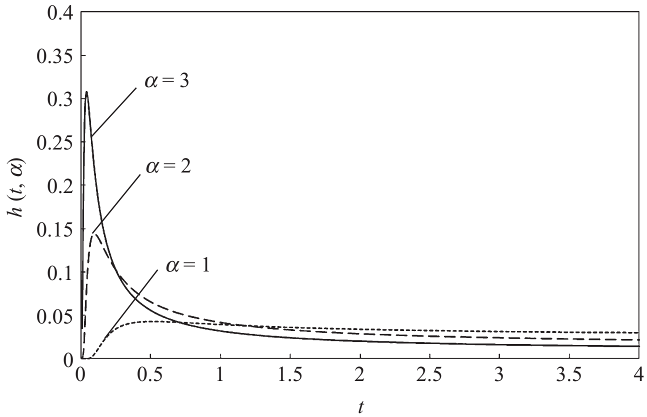


FIGURE 1.34 The hazard-rate function of the BS distribution.

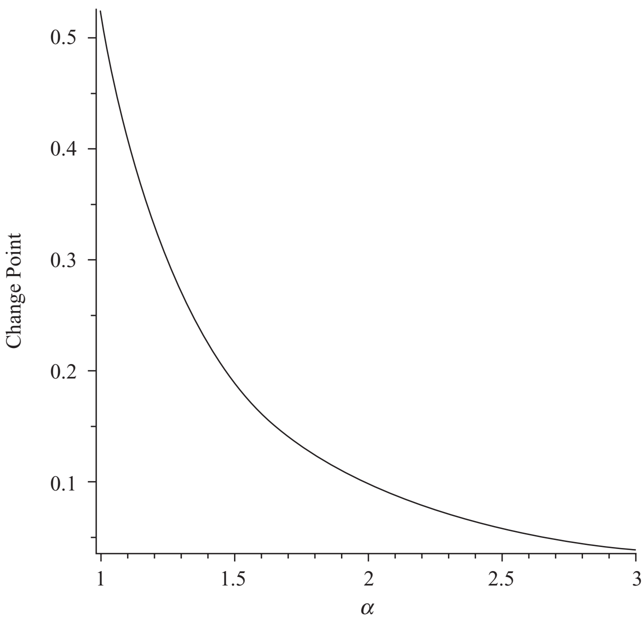


FIGURE 1.35 Effect of α on the change point of the hazard-rate function.

hazard rate as the shape parameter increases and the BS distribution might not be appropriate to model such hazard function. Indeed, a Weibull model with shape parameter less than one will result in a better fit. On the other hand, the distribution tends to normality as α tends to zero. The relationship between α and the change point is shown in Figure 1.35.

One of the interesting properties of the BS (α, β) is that T^{-1} also follows a BS distribution with parameters α and β^{-1}. The reliability function of BS (3,1) is shown in Figure 1.36.

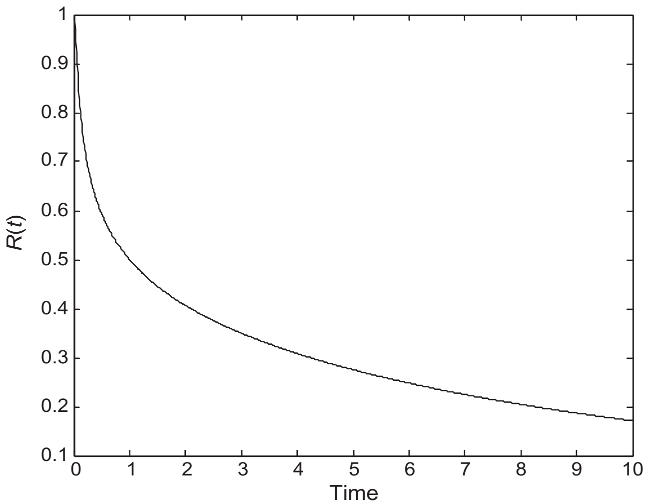


FIGURE 1.36 Reliability function of BS (3,1).

Assume that n time observations (t_1, t_2, . . . , t_n) corresponding to crack growth are recorded until the crack length reaches a critical threshold. These observations follow a BS distribution, and its parameters are estimated as follows (Kundu et al., 2008).

Let s and r denote the arithmetic mean and harmonic mean of the observations, respectively:

$$s = \frac{1}{n}\sum_{i=1}^{n} t_i$$

and

$$r = \left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{t_i}\right]^{-1}.$$

The modified moment estimator of the distribution parameters are

$$\hat{\alpha} = \left(2\left[\left(\frac{s}{r}\right)^2 - 1\right]\right)^{1/2} \tag{1.84}$$

and

$$\hat{\beta} = (sr)^{1/2} \tag{1.85}$$

Due to the bias of the sample size, Kundu et al. (2008) obtain the bias-corrected modified moment estimators as

$$\tilde{\alpha} = \left(\frac{n}{n-1}\right)\hat{\alpha}$$

$$\tilde{\beta} = \left(1 + \frac{\tilde{\alpha}^2}{4n}\right)^{-1} \hat{\beta}.$$

The reliability function and the hazard rate can be readily obtained.

EXAMPLE 1.14

An engineer conducts an axial fatigue test on a sample of alloy steel and measures the crack growth. The incremental increases in the length are set to equal values, and the corresponding times are recorded as follows:

200, 300, 390, 485, 560, 635, 695, 755, 810, 860, 905, 945, 985, 1020, 1053, 1100, 1150, 1200, 1280, 1370, 1400, 1600

Assume that a BS distribution fits these data. Determine the parameters of the distribution and plot the reliability function.

SOLUTION

The parameters of the distribution are obtained using Equation 1.84 and 1.85. The shape and scale parameters are $\hat{\alpha} = 1.1845$ and $\hat{\beta} = 783.94$. The unbiased estimates are $\tilde{\alpha} = 1.2409$ $\hat{\beta} = 89.93$.

Using the unbiased estimates, we obtain the reliability function shown in Figure 1.37.

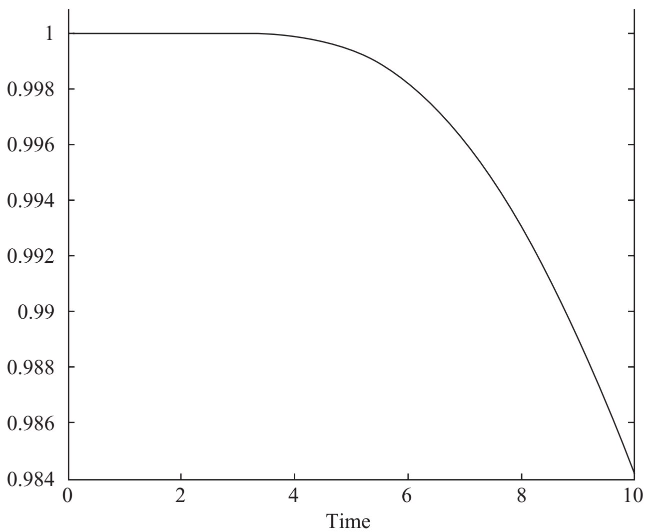


FIGURE 1.37 The reliability function of Example 1.14. ■

1.3.15 Other Forms

1.3.15.1 The Generalized Pareto Model When the hazard rate is either monotonically increasing or monotonically decreasing, it can be described by a three-parameter distribution with a hazard-rate function of the form

$$h(t) = \alpha + \frac{\beta}{t+\lambda}, \tag{1.86}$$

where α, β, and λ are the parameters of the model.

1.3.15.2 The Gompertz–Makeham Model This is a generalized model of the Gompertz hazard model with hazard rate

$$h(t) = \rho_0 + \rho_1 e^{\rho_2 t}, \tag{1.87}$$

where ρ_0, ρ_1, and ρ_2 are the parameters of the model.

1.3.15.3 The Power Series Model There are many practical situations where none of the above-mentioned models is suitable to accurately fit the hazard-rate values. In such a case, a general power series model can be used to fit the hazard-rate values. Clearly, the number of terms in the power series model relates to the desired level of fitness of the model to the empirical data. A good measure for the appropriateness of fitting the model to the data is the mean squared error between the hazard values obtained from the model and the actual data. The hazard-rate function of the power series model is

$$h(t) = a_o + a_1 t + a_2 t^2 + \ldots + a_n t^n. \tag{1.88}$$

The reliability function, $R(t)$, is

$$R(t) = \exp\left[-\left(a_o t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \ldots + \frac{a_n t^{n+1}}{n+1}\right)\right]. \tag{1.89}$$

EXAMPLE 1.15

Electromigration is a common failure mechanism in semiconductor devices. It is a phenomenon whereby a metal line in a device "grows" a link to another line or creates an open condition, due to movement (migration) of metal ions toward the anode at high temperatures or current densities (Comeford, 1989). Two hundred ICs are subjected to an elevated temperature of 250°C to accelerate their failures. The number of failures observed due to electromigration during the test intervals are given in Table 1.7.

Assume that the hazard-rate function is expressed as a power series function. Determine the hazard rate and the reliability after 10 h of operation at the same elevated temperature.

TABLE 1.7 Failure Data for the Integrated Circuits

Time interval (hours)	Failures in the interval
0–100	10
101–200	20
201–300	35
301–400	40
401–500	45
501–600	50
Total	200

SOLUTION

We calculate the hazard rate from the data as shown in Table 1.8.

We use the above hazard-rate data in Table 1.8 to fit the model given by Equation 1.88 using the least squares method to obtain

$$h(t) = 3.653\times10^{-3} - 0.171\times10^{-4}t + 4.86\times10^{-8}t^2$$
$$h(10\text{ h}) = 3.484\times10^{-3}.$$

The reliability is obtained using Equation 1.89 as

$$R(10) = \exp\left[-\left(3.653\times10^{-2} - \frac{0.171}{2}\times10^{-2} + \frac{4.86}{3}\times10^{-5}\right)\right]$$
$$= 0.9649.$$

TABLE 1.8 Hazard-Rate Calculation for Example 1.15

Time interval (hours)	Failures in the interval	Hazard rate × 10^{-3}
0–100	10	10/(200 × 100) = 0.50
101–200	20	20/(190 × 100) = 1.05
201–300	35	35/(170 × 100) = 2.05
301–400	40	40/(135 × 100) = 2.92
401–500	45	45/(95 × 100) = 4.73
501–600	50	50/(50 × 100) = 10.00

■

1.4 MULTIVARIATE HAZARD RATE

When a system is composed of two or more components, the joint life lengths are described by a multivariate distribution whose nature depends on the individual component life length. For example, consider a two-component system connected in parallel with each component having an exponentially distributed life length. The system fails when the two components fail.

When the effect of the operating conditions is accounted for, the joint life lengths of the components are shown to have a bivariate distribution whose marginals are univariate Paretos.

Assume that λ_i is the parameter of component $i(i = 1, 2)$. If the lives of the two components are assumed to be independent, then the reliability of the system is

$$R(t) = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}.$$

Suppose that the operating conditions affect the parameter λ_i by a common positive factor η. Then the system reliability is expressed as

$$R(t) = e^{-\eta\lambda_1 t} + e^{-\eta\lambda_2 t} - e^{-\eta(\lambda_1+\lambda_2)t}.$$

Following Lindley and Singpurwalla (1986), if η is an unknown quantity whose uncertainty is described by the distribution function $G(\eta)$, then the system reliability becomes

$$R(t) = G^*(\lambda_1 t) + G^*(\lambda_2 t) - G^*[(\lambda_1+\lambda_2)t],$$

where

$$G^*(y) = \int \exp(-\eta y)\, dG(\eta)$$

is the Laplace transform of G.

When $G(\eta)$ is a gamma distribution with density,

$$g(\eta) = \beta^{\alpha+1}\frac{\eta^{\alpha}}{\alpha!}e^{-\eta\beta}, \tag{1.90}$$

where $\alpha > -1$ and$\beta > 0$, then

$$R(t) = \left(\frac{\beta}{\lambda_1 t+\beta}\right)^{\alpha+1} + \left(\frac{\beta}{\lambda_2 t+\beta}\right)^{\alpha+1} - \left(\frac{\beta}{(\lambda_1+\lambda_2)t+\beta}\right)^{\alpha+1}. \tag{1.91}$$

The joint density of T_1 and T_2, the times to failure of the two components at t_1 and t_2, respectively, is

$$f(t_1, t_2, \lambda_1, \lambda_2, \alpha, \beta) = \frac{\lambda_1\lambda_2(\alpha+1)(\alpha+2)\beta^{\alpha+1}}{(\lambda_1 t_1+\lambda_2 t_2+\beta)^{\alpha+3}}. \tag{1.92}$$

Plots of Equation 1.92 for different values of λ_1, λ_2, α, and β are shown in Figures 1.38 and 1.39.

The bivariate hazard rate of the system is

$$h(t_1, t_2, \lambda_1, \lambda_2, \alpha, \beta) = \frac{(\alpha+1)(\alpha+2)\lambda_1\lambda_2}{(\beta+\lambda_1 t_1+\lambda_2 t_2)^2}. \tag{1.93}$$

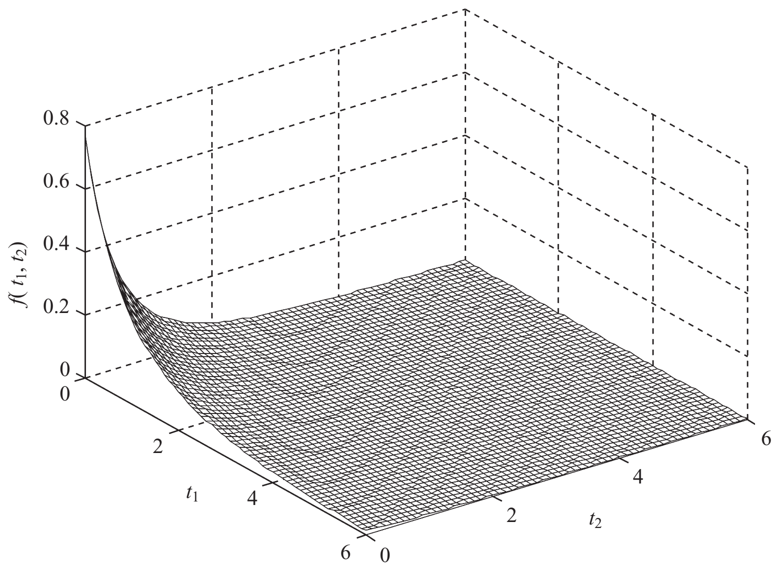


FIGURE 1.38 Plot of the bivariate gamma density ($\lambda_1 = 0.5$, $\lambda_2 = 0.3$, $\alpha_1 = 0.6$, $\beta = 0.9$).

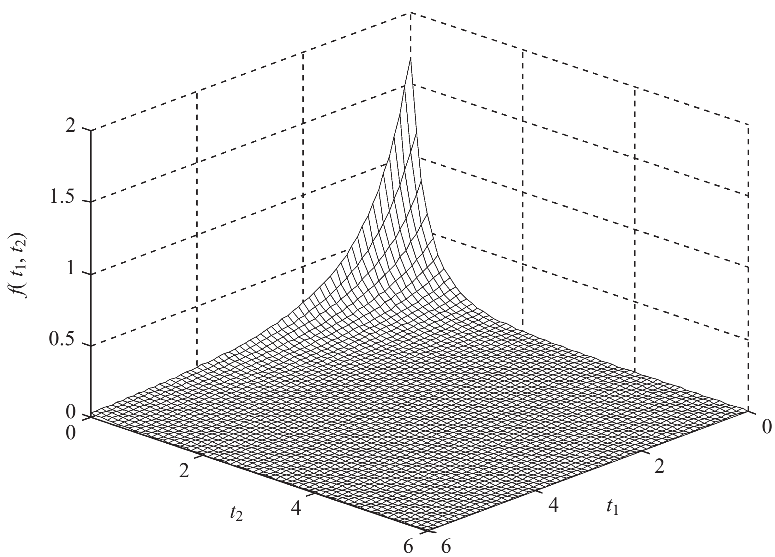


FIGURE 1.39 Plot of the bivariate gamma density ($\lambda_1 = 0.9$, $\lambda_2 = 0.3$, $\alpha = 0.3$, $\beta = 0.9$).

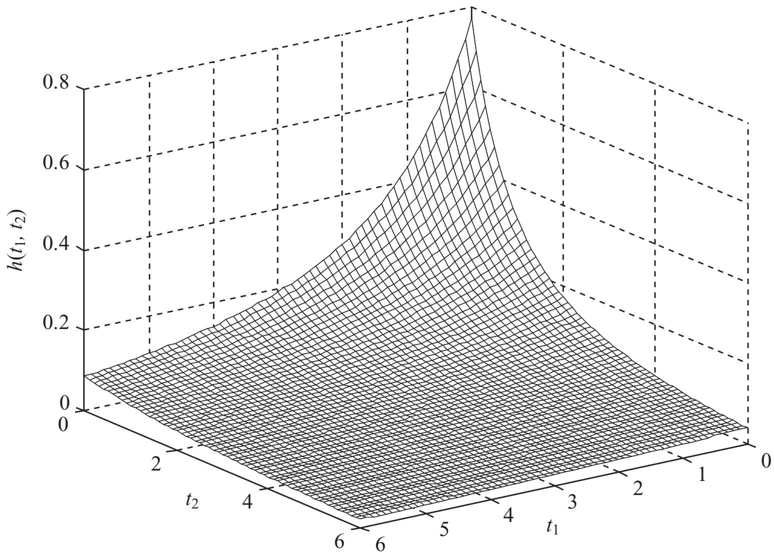


FIGURE 1.40 Plot of the bivariate hazard rate ($\lambda_1 = 0.5$, $\lambda_2 = 0.3$, $\alpha = 0.6$, $\beta = 0.9$).

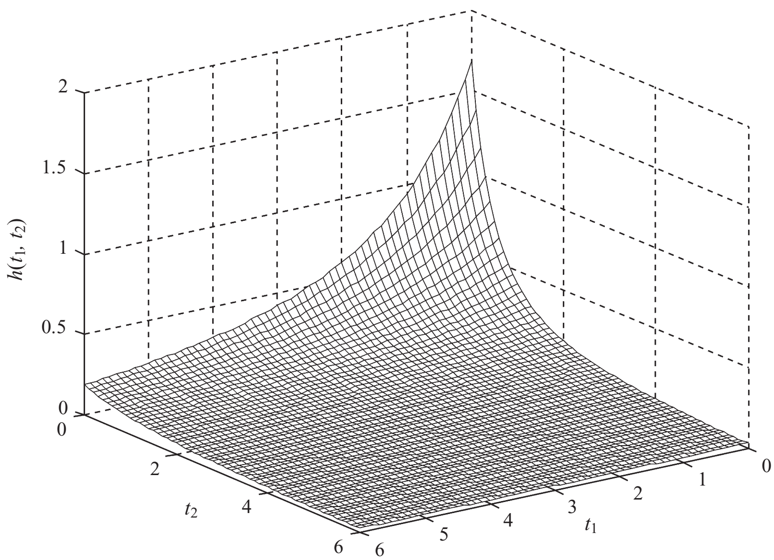


FIGURE 1.41 Plot of the bivariate hazard rate ($\lambda_1 = 0.9$, $\lambda_2 = 0.3$, $\alpha = 0.8$, $\beta = 0.9$).

The plots of the bivariate hazard rates for different λ_1, λ_2, α, and β are shown in Figures 1.40 and 1.41. Like univariate hazard rates, the bivariate hazard exhibits similar shapes—decreasing, constant, and increasing hazard rate.

The marginal density function of t_1 is obtained by integrating Equation 1.92 with respect to t_2 which yields

$$f(t_1, \lambda_1, \alpha, \beta) = \frac{\lambda_1(\alpha+1)\beta^{\alpha+1}}{(\lambda_1 t_1 + \beta)^{\alpha+2}}. \tag{1.94}$$

The density function given by Equation 1.94 is a Pearson Type VI whose mean and variance exist only for certain values of the shape parameter α. This distribution is also referred to as the "Pareto distribution of the second kind" (Lindley and Singpurwalla, 1986). Johnson and Kotz (1972) refer to Equation 1.94 as the *Lomax distribution*.

1.5 COMPETING RISK MODEL AND MIXTURE OF FAILURE RATES

Sometimes the failure data cannot be modeled by a single failure-time distribution. This is common in situations when a unit fails in different failure modes due to different failure mechanisms. For example, it has been shown that humidity has detrimental effects on semiconductor devices as it could induce failures due to large increases in threshold current in lasers (Osenbach et al., 1995; Chand et al., 1996; Osenbach and Evanosky, 1996; Osenbach et al., 1997) or could induce mechanical stresses due to polymeric layers' volume expansion in micromechanical devices (Buchhold et al., 1998). Humidity in silver-based metallization in microelectronic interconnects has caused metal corrosion and dendrites due to migration (Manepalli et al., 1999). In such situations, the failure data can be modeled using competing risk models or mixture of failure-rates models. We now discuss the necessary conditions for using either type of models.

1.5.1 Competing Risk Model

The competing failure model (also known as compound model, series system model, or multirisk model) plays an important role in reliability engineering as it can be used to model failure of units with several failure causes. There are three necessary conditions for this model: (1) failure modes are independent of each other, (2) the unit fails when the first of all failure mechanisms reaches the failure state, and (3) each failure mode has its own failure-time distribution. The model is constructed as follows,

Consider a unit that exhibits n failure modes and that the time to failure T_i due to failure mechanism i is distributed according to $F_i(t)$, $i = 1, 2, \ldots, n$. The failure time of the unit is the minimum of $\{T_1, T_2, \ldots, T_n\}$and the distribution function $F(t)$ is

$$F(t) = 1 - [1 - F_1(t)][1 - F_2(t)]\ldots[1 - F_n(t)]. \tag{1.95}$$

The reliability function is

$$R(t) = \prod_{i=1}^{n} R_i(t) \tag{1.96}$$

and the hazard function is

$$h(t) = \sum_{i=1}^{n} h_i(t). \tag{1.97}$$

To illustrate the application of the competing risk model we consider a product that experiences two different failure modes and each follows a Weibull distribution. The reliability of the product is

$$R(t) = R_1(t)R_2(t) = e^{-\left(\frac{t}{\theta_1}\right)^{\gamma_1}} e^{-\left(\frac{t}{\theta_2}\right)^{\gamma_2}}, \tag{1.98}$$

where θ_i and γ_i are the scale and shape parameters, respectively, of failure mode i. Upon differentiation we obtain the density function as (Jiang and Murthy, 1997)

$$\begin{aligned} f(t) &= R_1(t)f_2(t) + R_2(t)f_1(t) \\ &= R(t)\left[\frac{\gamma_1}{\theta_1}\left(\frac{t}{\theta_1}\right)^{\gamma_1 - 1} + \frac{\gamma_2}{\theta_2}\left(\frac{t}{\theta_2}\right)^{\gamma_2 - 1}\right] \end{aligned} \tag{1.99}$$

and the hazard-rate function is

$$h(t) = h_1(t) + h_2(t) = \gamma_1\theta_1^{-\gamma_1}t^{\gamma_1 - 1} + \gamma_2\theta_2^{-\gamma_2}t^{\gamma_2 - 1}. \tag{1.100}$$

The characteristics of the resultant $f(t)$ and $h(t)$ depend on the values of the parameters θ_1, θ_2, γ_1, and γ_2. Of course, the hazard rate $h(t)$ exhibits different characteristics: decreasing, constant, and increasing depending on the values and relationships among these parameters.

EXAMPLE 1.16

Consider a product that fails in two failure modes. Each failure is characterized independently by a Weibull model, and the parameters of failure mode 1 are θ_1 = 10,000 and γ_1 = 2.0 and the parameters of the failure mode 2 are θ_2 = 15,000 and γ_2 = 2.5. Plot the reliability function based on the competing risk model and compare it with the reliability function of each failure mode independently.

SOLUTION

The reliability function based on the competing risk model is (Fig. 1.42)

$$R(t) = R_1(t)R_2(t) = e^{-\left(\frac{t}{10{,}000}\right)^{2}} e^{-\left(\frac{t}{15{,}000}\right)^{2.5}}.$$

It is obvious that the competing risk model results in more accurate reliability estimates than modeling each failure mode separately.

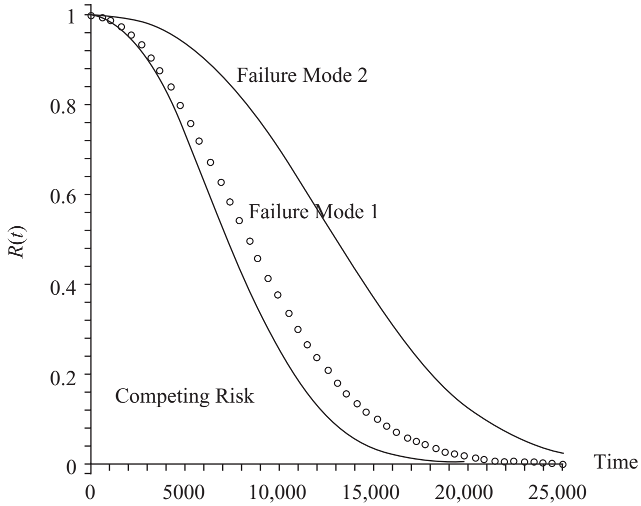


FIGURE 1.42 Reliability of the competing risk model. ■

1.5.2 Mixture of Failure-Rates Model

It is obvious that the mixtures of distributions with decreasing failure rates (DFRs) are always DFR. On the other hand, it may be intuitive to assume that the mixtures of distributions with increasing failure rates (IFRs) are also IFR. Unfortunately, some mixtures of distributions with IFR may exhibit DFR. In this section we discuss the conditions that guarantee that mixtures of IFR distributions will exhibit a DFR.

This is very important since, in practice, different IFR distributions are usually pooled in order to enlarge the sample size. In doing so, the analysis of data may actually reverse the IFR property of the individual samples to a DFR property for the mixture. Proschan (1963) shows that the mixture of two exponential distributions (each has a CFR) exhibits the DFR property.

Based on the work of Gurland and Sethuraman (1993), we consider mixtures of two arbitrary IFR distribution functions $F_i(t)$, $i = 1, 2$. The pooled distribution function of the mixture of the two distributions is $F_p(t) = p_1F_1(t) + p_2F_2(t)$ where $p = (p_1, p_2)$with $0 \le p_1, p_2 \le 1$, and $p_1 + p_2 = 1$ is a mixing vector.

We use the notation

$$h_i'(t) = H_i''(t) \quad \text{and} \quad \Re_i(t) = p_iR_i(t), \qquad i = 1, 2,$$

where $h_i(t)$, $H_i(t)$, and $R_i(t)$ are the hazard-rate function, the cumulative hazard function, and the reliability function of component i at time t. From Section 1.2, $R_i(t) = 1 - F_i(t)$, $H_i(t) = -\ln R_i(t)$ and $h_i(t) = H_i'(t)$.

The reliability function of the mixture of the two IFR distributions is

$$R_p(t) = p_1R_1(t) + p_2R_2(t).$$

But

$$H_p(t) = -\ln R_p(t)$$
$$H_p(t) = -\ln\left[p_1 R_1(t) + p_2 R_2(t)\right]$$

and

$$\begin{aligned} h_p(t) = H_p'(t) &= \frac{p_1 R_1(t) h_1(t) + p_2 R_2(t) h_2(t)}{p_1 R_1(t) + p_2 R_2(t)} \\ &= \frac{\Re_1(t) h_1(t) + \Re_2(t) h_2(t)}{\Re_1(t) + \Re_2(t)}. \end{aligned} \tag{1.101}$$

A hazard-rate function $h_p(t)$ is a DFR if $h_p'(t) \le 0$. Therefore, we take the derivative of Equation 1.101 with respect to t to obtain

$$\begin{aligned} (\Re_1(t) + \Re_2(t))^2 h_p'(t) &= [\Re_1(t) + \Re_2(t)]\{[\Re_1(t) h_1'(t) + \Re_2(t) h_2'(t)] \\ &\quad + [-\Re_1(t) h_1^2(t) - \Re_2(t) h_2^2(t)]\} + [\Re_1(t) h_1(t) + \Re_2(t) h_2(t)]^2 \\ &= (\Re_1(t) + \Re_2(t))(\Re_1(t) h_1'(t) + \Re_2(t) h_2'(t)) - \Re_1(t)\Re_2(t)(h_1(t) - h_2(t))^2. \end{aligned} \tag{1.102}$$

Using the fact that $\Re_i'(t) = -\Re_i(t) h_i(t)$ in the above equation, we show that the necessary and sufficient condition for $h_p'(t) \le 0$ and thus, for the mixture $F_p(t)$ to be DFR is

$$[\Re_1(t) + \Re_2(t)][\Re_1(t) h_1'(t) + \Re_2(t) h_2'(t)] \le \Re_1(t)\Re_2(t)[h_1(t) - h_2(t)]^2. \tag{1.103}$$

EXAMPLE 1.17

The failure-time distribution of a failure mode of a system is described by a truncated extreme distribution whose failure rate is $h_1(t) = \theta e^t$. Another mode of the system's failure exhibits a CFR $h_2(t) = \lambda$. Although one failure mode of the system exhibits IFR while the other is a CFR if treated separately, the analyst pools the data from both failure modes to obtain a pooled hazard-rate function. Prove that the pooled hazard rate is a DFR.

SOLUTION

The reliability functions of the failure modes of the system are

$$R_1(t) = e^{-\theta(e^t - 1)}$$

and

$$R_2(t) = e^{-\lambda t}.$$

The corresponding hazard rates are

$$h_1(t) = \theta e^t$$

and

$$h_2(t) = \lambda.$$

Let $F_p(t) = (1 - p)F_1(t) + pF_2(t)$. Then, the failure rate of the pooled data is

$$h_p(t) = \frac{(1-p)R_1(t)h_1(t) + pR_2(t)h_2(t)}{(1-p)R_1(t) + pR_2(t)}.$$

The necessary and sufficient condition that makes $h_p(t)$ a DFR function is given by Equation 1.102. Substituting the parameters of the individual distributions and $\Re_1(t) = (1-p)e^{-\theta(e^t-1)}$ and $\Re_2(t) = pe^{-\lambda t}$, it is easy to check that there is a $t_0(p)$ such that the derivative of the pooled hazard rate with respect to t is negative for $t \geq t_0(p)$ for each value of p. Thus, the mixture is DFR. ■

The class of IFR distributions that, when mixed with an exponential, becomes DFR is large; this is referred to as a mixture-reversible by exponential (MRE) distribution. It includes, for example, the Weibull, truncated extreme, gamma, truncated normal, and truncated logistic distributions. This phenomenon of the reversal of IFRs could be troublesome in practice when much of the data conform to an IFR distribution and the remainder (perhaps a small amount) of the data conform to an exponential distribution, and yet the overall pooled data would conform to a DFR distribution (Gurland and Sethuraman, 1994, 1995). For example, consider a mixture of an IFR gamma distribution with

$$f(t) = \frac{t^{\gamma-1}}{\theta^\gamma \Gamma(\gamma)} e^{-\frac{t}{\theta}}$$

where $\gamma > 1$ and $t > 0$ with an exponential distribution with parameter λ which satisfies the necessary conditions (Eq. 1.102) when $1/\theta > \lambda$. Thus, the IFR Gamma is MRE.

We note that the mixture failure rate for two populations is extensively studied. Gupta and Warren (2001) show that the mixture of two gamma distributions with IFRs (but have the same scale parameter) can result either in the increasing mixture failure rate or in the modified bathtub (MBT) mixture failure rate (the failure rate initially increases and then behaves like a bathtub failure rate). Jiang and Murthy (1998) show that the failure rate of the mixtures of two Weibull distributions with IFRs is similar to the failure rate of the mixture of two gamma distributions with IFRs. Likewise, Navarro and Hernandez (2004) state that the mixture failure rate of two truncated normal distributions depending on parameters involved can also be increasing, bathtub shaped, or MBT-shaped. Block et al. (2003) obtain explicit conditions for possible shapes of the mixture failure rate for two increasing linear failure rates.

Before concluding the presentation of the hazard functions, it is important to mention that some recent work argue that the bathtub curve is not a general failure-rate function that describes the failure rate of most, if not all, components. For example, Wong (1989) claims that the "roller-coaster" hazard-rate curve is more appropriate to describe the hazard rate of electronic systems than the bathtub curve. It is shown that semiconducting devices exhibit a generally decreasing hazard-rate curve with one or more humps on the curve. Data from a burn-in test of some electronic board assemblies demonstrate the trimodal (hump) characteristic on the cumulative failure rate. The wear-out (IFR) region starts immediately at the end of the decreasing failure-rate region without experiencing the constant failure-rate region, a main characteristic of the bathtub curve.

1.6 DISCRETE PROBABILITY DISTRIBUTIONS

Before we conclude the continuous probability distributions, we briefly present and discuss the use of discrete probability distributions in the reliability engineering area.

As presented so far, reliability is considered a continuous function of time. However, there are situations when systems, units, or equipment are only used on demand such as missiles that are normally stored and used when needed. Likewise, when systems operate in cycles, only the number of cycles before failure is observed. In such situations, the reliability and system performance are normally described by discrete reliability distributions. In this section, we briefly describe relevant distributions for reliability modeling.

1.6.1 Basic Reliability Definition

Assume that a discrete lifetime is the number K of system demands until the first failure. Then, K is a random variable defined over the set N of positive integers (Bracquemond and Gaudoin, 2003). The probability function and CDF are expressed, respectively, as $p(k) = P(K = k)$ and $F(k) = P(K \le k) = \sum_{i=1}^{k} p(i) \ \ \forall k \in N$. Consequently,

The reliability of a discrete lifetime distribution is

$$R(k) = P(K \ge k) = \sum_{i=k+1}^{N} p(i) \qquad \forall k \in N.$$

The MTTF is the expectation of the random variable K expressed as

$$\text{MTTF} = E(K) = \sum_{i=1}^{\infty} ip(i).$$

Similar to the continuous time case, we define the failure rate as the ratio of the probability function and the reliability function, thus

$$h(k) = \frac{P(K = k)}{P(K \geq k)} = \frac{p(k)}{R(k)}.$$

Likewise, we express other reliability characteristics such as the *mean residual life* (MRL) function, $L(k)$ as described in Section 1.8:

$$L(k) = E(K - k \mid K > k).$$

Of course, this can be generalized for the corresponding continuous time distributions. Since the practical use of such discrete lifetime distributions is limited, we show the above expressions for the geometric distribution case. Other distributions are found in (Bracquemond and Gaudoin, 2003).

1.6.2 Geometric Distribution

This distribution exhibits the memoryless property of the exponential distribution, and the system failure probabilities for each event (demand or request for use) are independent and all equal to p. In other words, the failure rate is constant or $P(K > i + k | k > i) = P(K > k)$. The probability of failure, reliability, and failure rate respectively, are

$$p(k) = p(1 - p)^{k-1},$$
$$R(k) = (1 - p)^k,$$
$$h(k) = \frac{p}{1 - p}.$$

Other use of discrete probability distributions arise when modeling system reliability, such as in the case of a four-engine aircraft, where its reliability is defined as the probability of at least two out of four engines function properly, and modeling the number of incidences (failures) of some characteristic in time as well as modeling warranty policies. We describe two commonly used distributions.

1.6.3 Binomial Distribution

In many situations, the reliability engineer might be interested in assessing system reliability by determining the probability that the system functions when k or more units out of n units function properly such as the case of the number of wires in a strand. This can be estimated using a binomial distribution. Let p be the probability that a unit is working properly; n is the total number of units; and k is the minimum number of units for the system to function properly. The probability of k units operating properly is

$$f(k) = \frac{n!}{k!(n-k)!} p^k q^{n-k} \quad k = 0, 1, \ldots, n \quad q = 1 - p.$$

The reliability of the system is then the sum of the probabilities that k, $k + 1$, . . . n units operate properly, that is,

$$\text{Reliability} = \sum_{i=k}^{n} \binom{n}{i} p^i q^{n-i},$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

The expectation of the distribution is

$$E(K) = \sum_{k=1}^{n} \left[\frac{kn!}{k(k-1)!(n-k)!} p^k q^{n-k} \right] = np.$$

The variance is

$$V(K) = E\left(K^2\right) - [E(K)]^2 = n^2 p^2 + np(1-p) - (np)^2 = np(1-p).$$

1.6.4 Poisson Distribution

Poisson distribution describes the probability that an event occurs in time t. The event may represent the number of defectives in a production process or the number of failures of a system or group of components. The Poisson distribution is derived based on the binomial distribution. This is achieved by taking the limit of the binomial distribution as $n \to \infty$ with $p = \lambda/n$. Substitution $p = \lambda/n$ in the binomial distribution results in

$$p(k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n} \right)^k \left(1 - \frac{\lambda}{n} \right)^{n-k}.$$

Taking limit as $n \to \infty$

$$\begin{aligned} \lim_{n\to\infty} p(k) &= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n} \right)^k \left(1 - \frac{\lambda}{n} \right)^{n-k} \\ &= \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n(n-1)\ldots(n-k+1)(n-k)!}{n^k (n-k)!} \left(1 - \frac{\lambda}{n} \right)^n \left(\frac{n-\lambda}{n} \right)^{-k}, \end{aligned}$$

which is reduced to

$$\lim_{n\to\infty} p(k) = \frac{\lambda^k}{k!} \lim_{n\to\infty} \left(1 - \frac{\lambda}{n} \right)^n = \frac{e^{-\lambda} \lambda^k}{k!} \quad k = 0,1,2,\ldots$$

Thus, the probability function of the Poisson distribution is

$$f(k)=\frac{e^{-\lambda}\lambda^{k}}{k!}\quad k=0,1,2,\ldots$$

Its expectation is

$$E(K)=\sum_{y=0}^{\infty}k\left[\frac{e^{-\lambda}\lambda^{k}}{k!}\right]=\lambda.$$

The variance is

$$V(K)=E\left(K^{2}\right)-[E(K)]^{2}=\lambda.$$

1.6.5 Hypergeometric Distribution

The hypergeometric distribution is used to model systems when successive events must occur before the failure of a system. Consider, for example, a system which is configured with implicit redundancy which requires the failure of two consecutive components for the system to fail. In this case, the reliability of the system is assessed using a hypergeometric distribution. Consider a population of size N with k working devices. A sample of size n is taken from the population; the number of working devices in the sample (y) is a random variable Y, and its probability function is

$$p(y)=\frac{\binom{k}{y}\binom{N-k}{n-y}}{\binom{N}{n}}\quad y=0,1,\ldots,\min(n,k).$$

The expectation and variance are

$$E(Y)=n\left(\frac{k}{N}\right)$$
$$V(Y)=n\left(\frac{k}{N}\right)\left(\frac{N-k}{N}\right)\left(\frac{N-n}{N-1}\right).$$

1.7 MEAN TIME TO FAILURE

One of the measures of the systems' reliability is the MTTF. It should not be confused with the mean time between failures (MTBF). We refer to the expected time between two successive failures as the MTTF when the system is nonrepairable. Meanwhile, when the system is repairable we refer to it as the MTBF.

Now, let us consider n identical nonrepairable systems and observe the time to failure for them. Assume that the observed times to failure are $t_1, t_2, \ldots, t_n$. The mean time to failure, $\widehat{MTTF}$, is

$$\widehat{MTTF} = \frac{1}{n}\sum_{i=1}^{n} t_i. \tag{1.104}$$

Since t_i is a random variable, then its expected value can be determined by

$$MTTF = \int_0^\infty t\, f(t)dt. \tag{1.105}$$

But $R(t) = 1 - F(t)$ and $f(t) = d\,F(t)/dt = -d\,R(t)/dt$. Substituting in Equation 1.105, we obtain

$$\begin{aligned} MTTF &= -\int_0^\infty t\frac{d\,R(t)}{dt}dt \\ &= -\int_0^\infty t\,d\,R(t) \\ &= -t\,R(t)\Big|_0^\infty + \int_0^\infty R(t)dt. \end{aligned}$$

Since $R(\infty) = 0$ and $R(0) = 1$, then the first part of the above equation is 0 and the MTTF is

$$MTTF = \int_0^\infty R(t)dt. \tag{1.106}$$

The MTTF for a constant hazard-rate model is

$$MTTF = \int_0^\infty e^{-\lambda t}dt = \frac{1}{\lambda}. \tag{1.107}$$

The MTTF of a linearly increased hazard-rate model is

$$MTTF = \int_0^\infty e^{-\frac{\lambda t^2}{2}}dt = \frac{\Gamma\left(\frac{1}{2}\right)}{2\sqrt{\frac{\lambda}{2}}} = \sqrt{\frac{\pi}{2\lambda}}. \tag{1.108}$$

Similarly, the MTTF for the Weibull model is

$$MTTF = \int_0^\infty e^{-\left(\frac{t}{\theta}\right)^\gamma}dt.$$

Substituting $x = \left(\frac{t}{\theta}\right)^\gamma$, the above equation becomes

$$\begin{aligned} MTTF &= \frac{\theta}{\gamma}\int_0^\infty e^{-x}x^{\frac{1}{\gamma}-1}dx \\ &= \frac{\theta}{\gamma}\Gamma\left(\frac{1}{\gamma}\right) \\ &= \theta\Gamma\left(1+\frac{1}{\gamma}\right). \end{aligned} \tag{1.109}$$

EXAMPLE 1.18

The MTTF for a robot controller that will be operating in different stress conditions is specified to be warranted for 20,000 h. The hazard-rate function of a typical controller is found to fit a Weibull model with $\theta = 3000$ and $\gamma = 1.5$. Does the controller meet the warranty requirement? If not, what should the value of θ be to meet the requirement (measurements are in hours)?

SOLUTION

Substituting $\theta = 3000$ and $\gamma = 1.5$ in Equation 1.109, we obtain the MTTF as

$$MTTF = (3000)\Gamma\left(1 + \frac{1}{1.5}\right) = 2700.8.$$

Thus, the MTTF is 2700.8 h. The MTTF does not meet the warranty requirement. The characteristic life that meets the requirement is calculated as 20,000 = $\theta\Gamma(1.666)$.

Thus, θ should equal 22,155. ■

EXAMPLE 1.19

The failure time of an electronic device is described by a Pearson type V distribution. The density function of the failure time is

$$f(t) = \begin{cases} \dfrac{t^{-(\alpha+1)} e^{-\beta/t}}{\beta^{-\alpha}\Gamma(\alpha)} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The shape parameter $\alpha = 3$ and the scale parameter $\beta = 4000$ h. Determine the MTTF of the device.

SOLUTION

Using Equation 1.105, we obtain

$$\begin{aligned} MTTF &= \int_0^\infty \frac{t^{-\alpha} e^{-\beta/t}}{\beta^{-\alpha}\Gamma(\alpha)} dt \\ &= \frac{1}{\beta^{-\alpha}\Gamma(\alpha)} \int_0^\infty t^{-\alpha} e^{-\beta/t} dt \end{aligned}$$

or

$$MTTF = \frac{\beta}{\alpha - 1} = \frac{4,000}{3-1} = 2,000 \text{ h.}$$

■

1.8 MEAN RESIDUAL LIFE (MRL)

A measure of the reliability characteristic of a product, component, or a system is the *MRL* function, $L(t)$. It is defined as

$$L(t) = E[T - t \mid T \geq t], \qquad t \geq 0. \tag{1.110}$$

In other words, the mean residual function is the expected remaining life, $T - t$, given that the product, component, or a system has survived to time t (Leemis, 1995).

The conditional p.d.f. for any time $\tau \geq t$ is

$$f_{T|T\geq t}(\tau) = \frac{f(\tau)}{R(t)} \qquad \tau \geq t. \tag{1.111}$$

The conditional expectation of the function given in Equation 1.111 is

$$E[T \mid T \geq t] = \int_t^\infty \tau f_{T|T\geq t}(\tau)\, d\tau = \int_t^\infty \tau \frac{f(\tau)}{R(t)}\, d\tau. \tag{1.112}$$

Since the component, product, or system has survived up to time t, the MRL is obtained by subtracting t from Equation 1.112, thus

$$\begin{aligned} L(t) &= E[T - t \mid T \geq t] \\ &= \int_t^\infty (\tau - t)\frac{f(\tau)}{R(t)}\, d\tau = \int_t^\infty \tau \frac{f(\tau)}{R(t)}\, d\tau - t \end{aligned}$$

or

$$L(t) = \frac{1}{R(t)} \int_t^\infty \tau f(\tau)\, d\tau - t. \tag{1.113}$$

EXAMPLE 1.20

A manufacturer uses rotary compressors to provide cooling liquid for a power-generating unit. Experimental data show that the failure times (between 0 and 1 year) of the compressors follow beta distribution with $\alpha = 4$ and $\beta = 2$. What is the MRL of a compressor given that the compressor has survived 5 months?

SOLUTION

The p.d.f. of the failure time is

$$f(t) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1} & 0 < t < 1 \\ 0 & \text{otherwise,} \end{cases}$$

or

$$f(t) = \frac{\Gamma(6)}{\Gamma(4)\Gamma(2)} t^3 (1-t)$$
$$= 20\left(t^3 - t^4\right).$$

But $R(t) = 1 - F(t) = 1 - \int_0^t 20\left(\tau^3 - \tau^4\right) d\tau.$

The value of t corresponding to 5 months is 5/12 = 0.416, thus

$$R(0.416) = 1 - 20\int_0^{0.416} \left(t^3 - t^4\right) dt = 0.900.$$

Using Equation 1.113, we obtain the MRL of a compressor that survived 5 months as

$$L(0.416) = \frac{20}{0.900}\int_{0.416}^{1} t\left(t^3 - t^4\right) dt - 0.416 = 0.288$$

or the MRL is 3.46 months. ■

1.9 TIME OF FIRST FAILURE

The advances in the design and production of medical devices, sensors, and nonmanufacturing have resulted in a wide range of medical devices and implants. Most of the implants are metallic due to their superior mechanical properties, such as hardness and fatigue strength, but one of their drawbacks is that electrochemical reactions take place on metallic surfaces in the human body which causes corrosion and degradation of the implants that might lead to extreme consequences. This has generated the interest in a different measure of reliability for such devices. One such measure is the time to first failure of N devices. In other words, we are interested in determining the time when the first failure occurs.

Consider a batch of N devices and assume that the failure time of a single device follows an exponential distribution. Let $f(t)$ be the *p.d.f.* for a single device, that is,

$$f(t) = \frac{dF(t)}{dt} = \frac{1}{T} e^{\frac{-t}{T}}, \tag{1.114}$$

where T is the design life (duration of interest). We are interested in determining $dF_1(t)/dt$ that the first failure in a batch of N devices occurs in $[t, t + dt]$. This can be expressed as

$$f_1(t) = \frac{dF_1(t)}{dt} = N\, f(t)\left(\int_t^{\infty} f(t')dt'\right)^{N-1}, \tag{1.115}$$

where $f(t)$ is the probability that a device fails in $[t, t + dt]$ and $\left(\int_t^{\infty} f(t')dt'\right)^{N-1}$ is the probability that $N - 1$ devices fail in $[t, \infty]$. Note that N is a combinatorial factor giving a number of choices to the devices which fail in $[t, t + dt]$ (Elsen and Schätzel, 2005). Normalization of Equation 1.115 yields the mean time of first failure as

$$\int_0^\infty t\left(\frac{dF_1(t)}{dt}\right)dt = \int_0^\infty tN\,f(t)\left(\int_t^\infty f(t')dt'\right)^{N-1} dt = \frac{T}{N} \tag{1.116}$$

The probability of the first failure $f_1(t)$ for given N and T can be obtained using Equation 1.115 and the mean time of the first failure is obtained from Equation 1.116.

EXAMPLE 1.21

Historical data show that most transistors exhibit CFR and are widely used in many applications. Consider the case where a manufacturer has the choice of releasing a batch of 100 or 200 devices that include one of the transistors and observe the time of the first failure of each batch for 5000 h. Show the failure-time distributions.

SOLUTION

Using Equation 1.115, we obtain the p.d.f of the first failure in a group of N devices over a period of time T as

$$f_1(t) = \frac{N}{T}e^{\frac{-tN}{T}}.$$

The failure-time distributions are shown in Figure 1.43.

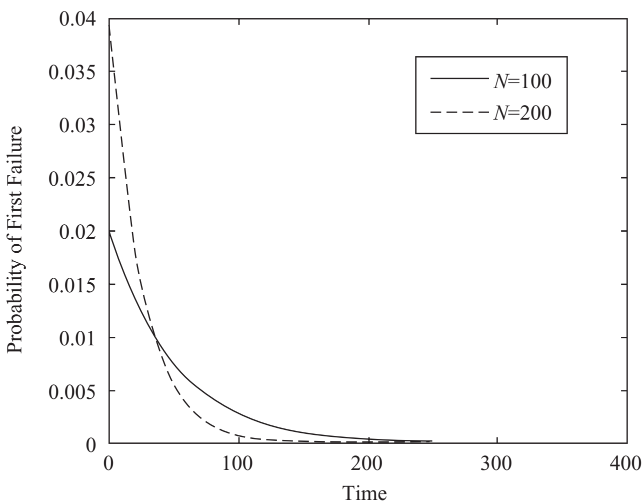


FIGURE 1.43 Time of first failure distributions. ■

Equation 1.115 can be generalized to obtain the time of the *jth* failure for N components. For example, we calculate the probability $dF_2(t)/dt$ that the second failure in a batch of N devices occurs in $[t, t + dt]$ as

$$f_2(t) = \frac{dF_2(t)}{dt} = N(N-1)f(t)\int_0^t f(x)dx\left(\int_t^\infty f(y)dy\right)^{N-2}. \tag{1.117}$$

The time to the second failure is the expectation of $f_2(t)$.

We conclude this chapter by providing a summary of the hazard-rate functions and their corresponding parameters, as shown in Table 1.9.

Table 1.9 summarizes the characteristics of the hazard functions discussed in this chapter.

TABLE 1.9 Characteristics of the Hazard Functions

Hazard function	$h(t)$	$f(t)$	$R(t)$	Parameters
Constant	λ	$\lambda e^{-\lambda t}$	$e^{-\lambda t}$	λ
Linearly increasing	λt	$\lambda t e^{-\frac{\lambda t^2}{2}}$	$e^{-\frac{\lambda t^2}{2}}$	λ
Weibull	$\frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1}$	$\frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1} e^{-\left(\frac{t}{\theta}\right)^{\gamma}}$	$e^{-\left(\frac{t}{\theta}\right)^{\gamma}}$	γ, θ
Exponential	$be^{\alpha t}$	$be^{\alpha t}e^{\frac{-b}{a}(e^{\alpha t}-1)}$	$e^{\frac{-b}{a}(e^{\alpha t}-1)}$	α, b
Normal	$\dfrac{\phi\left(\frac{t-\mu}{\sigma}\right)}{\sigma R(t)}$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$	$1-\int_{-\infty}^{t}\dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}d\tau$	μ, σ
Lognormal	$\dfrac{\phi\left(\frac{\ln t-\mu}{\sigma}\right)}{t\sigma R(t)}$	$\dfrac{1}{\sigma t\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln t-\mu}{\sigma}\right)^2}$	$1-\int_{0}^{t}\dfrac{1}{\tau\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln\tau-\mu}{\sigma}\right)^2}d\tau$	μ, σ
Gamma	$\dfrac{f(t)}{R(t)}$	$\dfrac{t^{\gamma-1}}{\theta^{\gamma}\Gamma(\gamma)}e^{\frac{-t}{\theta}}$	$\int_{t}^{\infty}\dfrac{1}{\theta\Gamma(\gamma)}\left(\frac{\tau}{\theta}\right)^{\gamma-1}e^{\frac{-\tau}{\theta}}d\tau$	θ, γ
Log-logistic	$\dfrac{\lambda p(\lambda t)^{p-1}}{1+(\lambda t)^p}$	$\dfrac{\lambda p(\lambda t)^{p-1}}{\left[1+(\lambda t)^p\right]^2}$	$\dfrac{1}{1+(\lambda t)^p}$	λ, p

PROBLEMS

1.1 Determine the mean and the variance of a uniform random variable X whose p.d.f. is

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

1.2 Determine the first and second moments for a normal distribution with parameters μ and σ^2.

1.3 The p.d.f. of the lognormal distribution is given by

$$f(t)=\frac{1}{\sigma t\sqrt{2\pi}}e^{-\frac{1}{2}\left\{\frac{\ln t-\mu}{\sigma}\right\}^2}.$$

Determine the variance and the median. (Hint: Median is defined as $\int_{med}^{\infty} f(x)\,dx = 1/2$).

1.4 A mechanical fatigue test is conducted on 100 specimens of a new polymer. The applied stress is identical for all specimens. The number of cycles observed and the corresponding numbers of failed specimens are given in Table 1.10.

a. Plot graphs for $f_e(t)$, $R_e(t)$, $h_e(t)$, and $F_e(t)$.

b. Comment on the above results.

c. Derive an analytical expression for $h_e(t)$ and estimate the MTTF of a bar made of the same material and is subjected to the same loading conditions.

TABLE 1.10 Fatigue Test Results

Number of cycles $\times 10^5$	Cumulative number of failed specimens
10	35
20	59
30	72
40	84
50	93
60	100

1.5 The reliability of disk drives can be predicted by increasing the operational machine hours accumulated in the field or in the laboratory as part of the initial design process. The failures have been accumulated and given in Table 1.11.

a. Plot graphs for $f_e(t)$, $R_e(t)$, $h_e(t)$, and$F_e(t)$.

b. Comment on the above results.

TABLE 1.11 Failure Data for Problem 1.5

Hour of operation $\times 10^3$	Number of failed disks
0–10.0	0
10.1–14.0	10
14.1–18.0	15
18.1–22.0	18
22.1–26.0	20
26.1–30.0	16
30.1–34.0	22
34.1–38.0	20

c. Derive an analytical expression for $h_e(t)$ and estimate the MTTF of a bar made of the same material and is subjected to the same loading conditions.

d. Would you buy a disk produced by the above manufacturer? Why?

1.6 One of the modern methods for stress screening is called highly accelerated stress screening (HASS), which use the highest possible stresses (well beyond the normal operating level) to attain time compression on the screens. The HASS exhibits an exponential acceleration of screen strength with stress level. A manufacturer employs a HASS test on newly designed leaf springs for light trucks. A cyclic load was applied on a number of springs and the failure times are recorded in Table 1.12.

a. Fit a nonlinear polynomial hazard function to describe the hazard rate of the springs.

b. What is the reliability at $t = 8$?

c. Assume that we obtained 500 springs that require testing under the same conditions. What is the expected time to failure? What is the least time needed to ensure that all units fail under test?

TABLE 1.12 Failure Data for Problem 1.6

Time interval (minutes)	Number of failed units
0–1.999	10
2–3.999	15
4–5.999	22
6–7.999	34
8–9.999	49
10–11.999	63
12–14	70

1.7 A reliability engineer subjected 10 steel specimens to High-Cycle Fatigue (HCF) that occurs at relatively large numbers of cycles and is caused by high frequency vibrations in both static and rotating hardware. The number of cycles to failure is recorded for each specimen and is reported as follows:

200,000, 250,000, 280,000, 300,000, 350,000, 370,000, 380,000, 400,000, 420,000, 460,000

a. Use the improved mean rank to obtain the p.d.f., $R(t)$ and $h(t)$.

b. Use two median rank approaches to obtain the p.d.f., $R(t)$ and $h(t)$.

c. Compare the results obtained from (a) with those obtained from (b).

1.8 Show that the variance of a component whose hazard rate can be described by $h(t) = \gamma/\theta(t/\theta)^{\gamma-1}$ is

$$\operatorname{Var}[T] = \theta^2\left\{\Gamma\left(1+\frac{2}{\gamma}\right)-\left[\Gamma\left(1+\frac{1}{\gamma}\right)\right]^2\right\},$$

where

$$\Gamma(n) = \int_0^\infty \tau^{n-1}e^{-\tau}d\tau$$

and

$$\int_0^\infty \tau^{n-1}e^{-\tau/\theta}d\tau = \Gamma(n)\theta^n.$$

1.9 Use the Weibull graph paper to estimate the parameters of a Weibull distribution that fits the data given in Problem 1.6.

1.10 Plot $h(t)$ and $R(t)$ for $t = 0$ to 1000, for different shape parameters of 0.5–3.5 with an increment of 0.5 and for different characteristic lives of 200–300 with an increment of 25. What is the effect of the characteristic life on the hazard-rate function? What is the best combination of shape parameter and characteristic life that results in the highest reliability at $t = 1000$? (Weibull distribution).

1.11 Dhillon (1979) proposes a hazard-rate model given by

$$h(t) = k\lambda c t^{c-1} + (1-k) b t^{b-1} \beta e^{\beta t^b}$$

for

$$b, c, \beta, \lambda > 0 \qquad 0 \le k \le 1 \qquad t \ge 0,$$

where

b, c = shape parameters,
β, λ = scale parameters, and
t = time.

Derive the reliability function and determine the conditions that make the hazard rate increasing, decreasing, or constant.

1.12 A rolling bearing rotating under load may ultimately suffer from material fatigue. Typically, fatigue damage is characterized by a small piece of material breaking away from the raceway leaving a cavity. This cavity may then propagate into a crack and the bearing will fail. If a large batch of identical bearings is run under the same conditions until 10% of the batch has failed from the material fatigue damage, then the batch is said to have attained its L_{10} life. In other words, the remaining 90% of the bearings in the batch will survive for periods longer than the L_{10} life. Consider a rolling bearing which has a hazard-rate function in the form

$$h(t) = \frac{\frac{1}{\theta}\left(\frac{t}{\theta}\right)^{n-1}}{(n-1)!\sum_{k=0}^{n-1}\frac{(t/\theta)^k}{k!}},$$

where $n = 3$ and $\theta = 290$ h. Determine the reliability of the bearing at $t = 100$ h. Assuming $L_{10} = 100$ h, determine the MRL of the bearing.

1.13 Find $f(t)$, $h(t)$, $R(t)$, and MTTF, assuming

$$F(t) = 1 - \frac{8}{7}e^{-t} + \frac{1}{7}e^{-8t}.$$

1.14 Find $f(t)$, $F(t)$, $R(t)$, and MTTF, assuming

$$h(t) = \frac{1}{25}t^{-1/4}.$$

If 200 units are placed in operation at the same time, how many failures are expected during 1 year of operation?

1.15 The failure rate of a brake system is found to be $h(t) = 0.006(1.5 + 2t + 3t^2)$ failures per year.

a. What is the reliability at $t = 10^4$ h?

b. If 20 systems are subjected to a test at the same time, how many would have survived at time $t = 10^3$ h? What is the expected number of failures in 1 year of operation?

1.16 The failure rate of a hydraulic system is found to be $h(t) = 0.003(1 + 2.5e^{-3t} + e^{-t/50})$ failures per year.

a. What is the reliability at $t = 10^5$ h?

b. What is the MTTF?

c. If 10 systems are subjected to a test at the same time, how many would have survived at time $t = 10^3$ h. What is the expected number of failures in 1 year of operation?

1.17 Consider the general hazard failure rate (Hjorth, 1980) that is given by $h(t) = \delta t + \theta/(1 + \beta t)$. Special cases are

$\theta = 0$ The Rayleigh distribution,

$\delta = \beta = 0$ The exponential distribution,

$\delta = 0$ DFR,

$\delta \geq \theta\beta$ IFR, and

$0 \leq \delta \leq \theta\beta$ The bathtub curve.

The reliability function corresponding to this general hazard rate is

$$R(t) = \frac{e^{-\delta t^2/2}}{(1+\beta t)^{\theta/\beta}}, \qquad t \geq 0.$$

Let T have the above reliability function, and define

$$I(a,b) = \int_0^\infty \frac{e^{-at^2/2}}{(1+t)^b}\, dt.$$

Find the mean and the variance of T. Plot the hazard rate for different values of the parameters.

1.18 The viscosity of a lubricant used in a heavy machinery (at 70°C) is measured in centipoise at equal intervals of times (days) as shown in Table 1.13. The lubricant needs to be replaced when the threshold value of the viscosity is 1400 centipoise. Assuming that the measurements follow a BS distribution, determine its parameters and plot the reliability function with time. Determine the change point of the hazard-rate function.

1.19 The p.d.f. of the early failure times of the circuit boards used in high-speed modems is found to follow a Pearson type V distribution given by

$$f(t) = \begin{cases} \dfrac{t^{-(\alpha+1)} e^{-\beta/t}}{\beta^{-\alpha}\Gamma(\alpha)} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

where α and β are the shape and scale parameters, respectively. Find the reliability function, the hazard rate, and the MTTF for the special case when $\beta = 1$ and $\alpha = 3$. Is the hazard rate increasing, decreasing, or constant?

TABLE 1.13 Viscosity Data for Problem 1.18

11	28	43	56	84	108	129	170	238	354
15	31	44	58	86	109	141	175	246	383
15	34	46	59	89	109	146	177	261	396
15	36	47	61	90	115	155	177	264	417
22	36	47	62	95	119	161	177	272	425
23	37	47	62	97	119	162	180	281	448
24	38	49	68	97	123	168	184	283	472
24	41	50	68	98	127	169	196	301	646
25	42	50	79	106	127	169	227	303	777
27	42	56	83	108	129	170	238	318	1181

1.20 Let t denote the time to failure of a component whose p.d.f. is given by

$$f(t) = \frac{1}{\ln 2}\frac{1}{t}, \qquad 25{,}000 < t < 50{,}000 \text{ h.}$$

a. Verify that f is a density for a continuous random variable.

b. What is the hazard function of this component?

c. What is the expected life of the component?

1.21 A manufacturer of medical equipment introduces three different prototype machines, Machine A, Machine B, and Machine C, all capable of sensing contrast or saline pooling under a patient's skin during a chemotherapy procedure. This task approximately equals one unit of time for every patient. The manufacturer records the incidents of each machine in terms of the number of patients served before the machine fails. Assume that when the machine fails it is repaired to be as good as new. The data are shown in Tables 1.14–1.16.

TABLE 1.14 Failure Data for Machine A

Incident #	Cumulative patients	Patients between failures
1	1	1
2	7	6
3	94	87
4	193	99
5	217	24
6	367	150
7	390	23
8	411	21
9	654	243
10	779	125
11	1,016	237
12	1,035	19
13	1,038	3
14	1,074	36

TABLE 1.15 Failure Data for Machine B

Incident #	Cumulative patients	Patients between failures
1	13	13
2	20	7
3	59	39
4	67	8
5	71	4
6	91	20
7	123	32
8	128	5
9	129	1
10	140	11
11	155	15
12	166	11
13	192	26
14	203	11
15	241	38
16	253	12
17	255	2
18	282	27
19	305	23
20	344	39
21	356	12
22	413	57
23	432	19
24	485	53
25	498	13
26	501	3
27	518	17
28	565	47
29	631	66
30	651	20
31	672	21
32	718	46
33	761	43
34	865	104
35	876	11
36	913	37
37	946	33
38	978	32
39	1,045	67

TABLE 1.16 Failure Data for Machine C

Incident #	Cumulative patients	Patients between failure
1	67	67
2	178	111
3	240	62
4	411	171
5	427	16
6	445	18
7	454	9
8	457	3
9	464	7
10	482	18
11	524	42
12	529	5
13	698	169
14	706	8
15	744	38
16	757	13
17	780	23
18	791	11
19	802	11
20	815	13
21	830	15
22	853	23
23	860	7
24	874	14
25	918	44
26	935	17
27	957	22
28	1,016	59
29	1,034	18
30	1,071	37
31	1,075	4
32	1,084	9

a. Analyze the failure data and compare the hazard-rate functions for the three machines.

b. Plot the reliability functions and estimate the MTTF for each machine.

c. What are your suggestions to the manufacturer?

1.22 In most electronic manufacturing operations, the role of process control has traditionally fallen to automated board-test systems. These systems are typically placed at the end of the manufacturing line in order to monitor fault trends and thus help control the process. The failure data collected at a board-test system show that the failure time follows a triangular distribution with the following p.d.f.

$$f(t)=\begin{cases}\dfrac{2(t-a)}{(b-a)(c-a)} & \text{if } a\le t\le c\\[2mm] \dfrac{2(b-t)}{(b-a)(c-a)} & \text{if } c<t\le b\\[4mm] 0 & \text{otherwise.}\end{cases}$$

where a, b, and c are real numbers with $a < c < b$. a is a location parameter, $b - a$ is a scale parameter, c is a shape parameter. Assume that $a = 2$, $b = 4$, and $c = 3$. What is the expected MTTF? What is its variance?

1.23 A manufacturer intends to introduce a new product. Five products are subjected to a reliability test. The mean of the failure times is 300 h and the variance is 90,000 h^2. Since the number of failure data is limited, it is difficult to determine with an acceptable confidence level the type of the failure-time distribution.

a. What is the expected number of failures at 500 h?

b. The similarity between this product and another product that has already been in the market for the last 10 years indicates that the failure-time distribution is likely to follow gamma distribution. What is the expected number of failures under these conditions at 500 h? Compare the results with (a) above. What do you conclude?

1.24 The failure time of a new brake drum design is observed to follow a gamma distribution with a p.d.f. of

$$f(t)=\frac{\lambda(\lambda t)^{\gamma-1}e^{-\lambda t}}{\Gamma(\gamma)}.$$

For $\gamma = 2$ and $\lambda = 0.0002$, determine

a. The expected number of failures in 1 year of operation,

b. The MTTF, and

c. The reliability at t = 1000 h.

1.25 Solve the above problem when $\gamma = 3$ and $\lambda = 0.0002$.Compare the results. Which brake system is better? Why?

1.26 Most fractional horsepower motor controllers use silicon-controlled rectifier (SCR) to vary the power applied to the motor and thereby control armature voltage and thus the motor's speed. The SCR is made of different layers of semiconductor materials. The heat dissipation from the motor increases the failure rate of the SCR. Failure data from the field show that the failure time follows a beta distribution with the following p.d.f.

$$f(t)=\begin{cases}\dfrac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}t^{\alpha}(1-t)^{\beta} & 0<t<1,\alpha>-1,\beta>-1\\ 0 & \text{otherwise.}\end{cases}$$

Assuming that $\alpha = 1.8$ and $\beta = 4.7$, what is the expected MTTF? What is its variance? What is the expected number of failures at $t = 2.5$?

1.27 Consider the case where the failure time of components follows a logistic distribution with p.d.f. of0

$$f(t) = \frac{(1/\beta)e^{-(t-\alpha)/\beta}}{\left(1+e^{-(t-\alpha)/\beta}\right)^2}, \qquad -\infty < \alpha < \infty, \beta > 0, -\infty < t < \infty.$$

Determine the expected number of failures in the interval $[t_1, t_2]$.

1.28 In order for a manufacturer to determine the length of the warranty period for newly developed ICs, 100 units are placed under test for 5000 h. The hazard-rate function of the units is $h(t) = 5 \times 10^{-9} t^{0.9}$.

What is the expected number of failures at the end of the test? Should the manufacturer make the warranty period longer or shorter if the ICs were redesigned and its new hazard-rate function became $h(t) = 6 \times 10^{-8} t^{0.75}$?

1.29 The manufacturer of diodes subjects 100 diodes to an elevated temperature testing for a 2-year period. The failed units are found to follow a Weibull distribution with parameters $\theta = 50$ and $\gamma = 2$ (in thousands of hours). What is the expected life of the diodes? What is the expected number of failures in a 2-year period?

1.30 In Problem 1.29, if a diode survives 1 year of operation, what is its MRL?

1.31 The hazard-rate function of a manufacturer's jet engines is a function of the amount of silver and iron deposits in the engine oil. If the metal deposit readings are "high," the engine is removed from the aircraft and overhauled. The hazard-rate function (Jardine and Buzacott, 1985) is

$$h(t; z(t)) = \frac{5.335}{3255.19}\left(\frac{t}{3255.19}\right)^{4.335} \exp[0.506\, z_1(t) + 1.25\, z_2(t)],$$

where t = flight hours,

$z_1(t)$ = iron deposits in parts per million at time t, and

$z_2(t)$ = silver deposits in parts per million at time t.

Analysis of the deposits over time shows that

$$z_1(t) = 0.0005 + 0.00006t$$
$$z_2(t) = 0.00008t + 8\times10^{-8} t^2.$$

Plot the reliability of the engine against flying hours. What is the MTTF?

1.32 A mixture model of the Inverse Gaussian (IG) and the Weibull (W) distributions, called the IG-W model, is capable of covering six different combinations of failure rates: one of the components has an upside-down bath tub failure rate (UBTFR) or IFR and the other component has a DFR, CFR, or IFR (Al-Hussaini and Abd-el-Hakim, 1989). The mixture density function of the IG-W model is

$$f(t) = p\, f_1(t) + q\, f_2(t),$$

where p is the mixing proportion, $0 \le p \le 1$ and $q = 1 - p$. The density functions $f_1(t)$ and $f_2(t)$ are those of the Inverse Gaussian $IG(\mu, \lambda)$and the Weibull $W(\theta, \beta)$ having the respective forms

$$f_1(t) = \sqrt{\lambda / 2\pi t^3}\, \exp(-\lambda(t-\mu)^2 / 2\mu^2 t), \qquad t > 0, \mu, \lambda > 0$$

$$f_2(t) = \left(\frac{\beta}{\theta}\right)\left(\frac{t}{\theta}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right], \qquad t > 0, \beta > 0, \theta > 0.$$

The reliability function $R(t)$ of the mixed model is

$$R(t) = p\,R_1(t) + q\,R_2(t).$$

The hazard-rate function, $h(t)$ of the mixed model is

$$h(t) = \frac{f(t)}{R(t)} = \frac{p\,f_1(t) + q\,f_2(t)}{p\,R_1(t) + q\,R_2(t)} = r(t)h_1(t) + (1 - r(t))h_2(t)$$

where

$$r(t) = \frac{1}{1 + g(t)}, \qquad g(t) = \frac{q\,R_2(t)}{p\,R_1(t)}.$$

Investigate the necessary conditions for an IFR, CFR, and DFR.

1.33 A beginner reliability engineer did not realize that the failures of the system should be grouped by type instead of having them in one group. The system was observed to fail because of two types of failures: electrical (E) and mechanical (M). The failure data for E are

316, 138, 87, 923, 921, 1113, 1152, 577, 480, 1401

The data for M are

746, 1281, 1304, 1576, 1386, 671, 2106, 660, 1149, 425

The true data for E comes from an exponential distribution with mean = 1000 h, and the data for M comes from Weibull with $\gamma = 2$ and $\theta = 1000$.

a. What is the reliability expression for the true distribution?

b. What is the reliability expression for the combined failures?

c. Is the analysis of the engineer correct? Why?

1.34 Determine the mean life and the variance of a component whose failure time is expressed by

$$f(t) = \sum_{i=1}^{n} p_i \frac{\gamma_i}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\gamma_i - 1} e^{-\left(\frac{t}{\theta_i}\right)^{\gamma_i}},$$

where $\sum_{i=1}^{n} p_i = 1$.

1.35 Assume that the mean hazard rate is given by

$$E[h(T)] = \int_0^{\infty} h(t) f(t)\, dt$$

and the MTTF $E[T]$ is

$$E[T] = \int_0^\infty R(t)dt.$$

Prove that $\{E[h(t)] \cdot E[T]\}$ is an increasing function of the shape parameter of the Weibull model.

1.36 Consider a Weibull distribution with a reliability function $R(t) = \exp(-\theta\, \lambda t^\gamma)$ for $t \geq 0$. For $\gamma > 1$, $\theta > 0$, and $\lambda > 0$, the Weibull density becomes an IFR distribution (the wear-out region of the bathtub curve). Suppose that the values of λ follow a gamma distribution with p.d.f. $f(\lambda)$ given by

$$f(\lambda) = \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha\lambda} \lambda^{\beta-1} \qquad \alpha > 0, \beta > 0, \lambda > 0.$$

The reliability function of the mixture is given by

$$R_{\text{mixture}}(t) = \int_0^\infty R(t) f(\lambda) d\lambda.$$

a. Show that the failure-rate function of the mixture is as given by (Gurland and Sethuraman, 1994),

$$h_{\text{mixture}}(t) = \beta \frac{\theta \gamma t^{\gamma-1}}{\alpha + \theta t^\gamma}.$$

b. Plot $h_{\text{mixture}}(t)$ for large values of t. What do you conclude?

c. Plot the hazard rate for different values of α, β, θ, and γ. What are the conditions at which $h_{\text{mixture}}(t)$ is an IFR function? A DFR function? A CFR function?

1.37 Data from a linearly increasing failure-rate distribution is mixed with some data from a constant failure-rate distribution. Assume that the linearly increasing failure rate is a Rayleigh distribution with $R_R(t) = e^{-\lambda t^2/2}$, where λ is a constant, and the reliability function of the CFR is $R_c(t) = e^{-\theta t}$. Investigate $h(t)$ of the mixture of the distributions.

1.38 The failure time of a component follows a Pareto distribution with a p.d.f. of

$$f(t) = \frac{\gamma \lambda^\gamma}{t^{\gamma+1}}, \qquad \lambda > 0, \gamma > 1, \lambda < t < \infty.$$

Determine the MTTF of the component and its MRL function.

1.39 Derive an expression for the probability that the first failure in a batch of N devices in $[t, t + dt]$ when every device has the same Weibull failure-time distribution. Estimate the mean time of the first failure. Plot the probability distribution for 200 devices with shape parameter of 2.5 and scale parameter of 4000.

1.40 The probability $dF_2(t)/dt$ that the second failure in a batch of N devices occurs in $[t, t + dt]$ is expressed as

$$f_2(t) = \frac{dF_2(t)}{dt} = N(N-1) f(t) \int_0^t f(x)dx \left(\int_t^\infty f(y)dy \right)^{N-2}.$$

Generalize the above expression for the jth failure in a batch of N devices.

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CHAPTER 2

SYSTEM RELIABILITY EVALUATION

2.1 INTRODUCTION

In Chapter 1, we presented definitions of reliability, hazard functions, and other measures of reliability, such as mean time to failure (MTTF) and the expected number of failures in a given time interval. These definitions and measures are applicable to both components and systems. A system (or a product) is a collection of components arranged according to a specific design in order to achieve desired functions with acceptable performance and reliability measures.

Clearly, the type of components used, their qualities, and the design configuration in which they are arranged have a direct effect on the system performance and its reliability. For example, a designer may use a fewer number of *higher* quality components (made of prime material) and configure them in such a way to result in a highly reliable system, or a designer may use a larger number of *lower* quality components and configure them differently in order to achieve the same level of reliability. A system configuration may be as simple as a series system where all components are connected in series; a parallel system where all components are connected in parallel; a series-parallel; or a parallel-series, where some components are connected in series and others in parallel and a complex configuration such as networks. Once the system is configured, its reliability must be evaluated and compared with an acceptable reliability level. If it does not meet the required level, the system should be redesigned and its reliability should be reevaluated. The design process continues until the system meets the desired performance measures and reliability level.

As seen above, system reliability needs to be evaluated as many times as the design changes. This chapter presents methods for evaluating reliability of systems with different configurations and methods for assessing the importance of a component in a complex structure. The presentation is limited to those systems that exhibit constant probability of failure. In the next chapter, time-dependent reliability systems are discussed.

2.2 RELIABILITY BLOCK DIAGRAMS

The first step in evaluating a system's reliability is to construct a reliability block diagram, which is a graphical representation of the components of the system and how they are connected. A block (represented by a rectangle) does not show any details of the component or the subsystem it represents. The second step is to create a reliability graph that corresponds to the block diagram. The reliability graph is a line representation of the blocks that indicates the

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path on the graph. The following examples illustrate the construction of both the reliability block diagram and the reliability graph.

EXAMPLE 2.1

A computer tomography system is used as a nondestructive method to inspect welds from outside when the inner surfaces are inaccessible. It consists of a source for illuminating the rotating welded part with a fan-shaped beam of X-rays or gamma rays as shown in Figure 2.1. Detectors in a circular array on the opposite side of the part intercept the beam and convert it into electrical signals. A computer processes the signals into an image of a cross section of the weld, which is displayed on a video monitor (Pascua and Jagatjit, 1990). Draw the reliability block diagram and the reliability graph of the system.

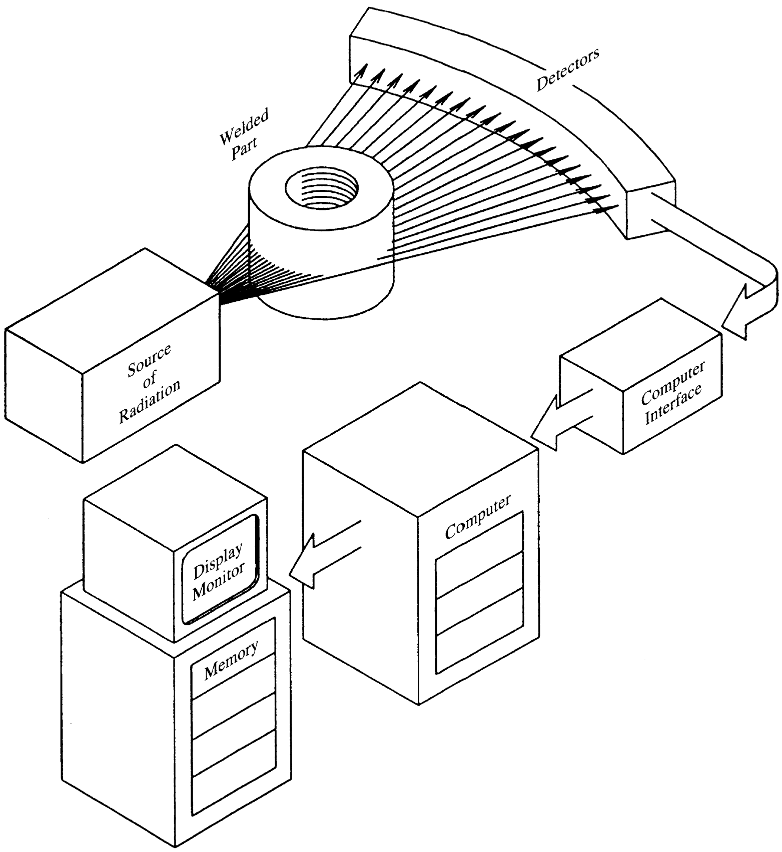

FIGURE 2.1 A computer tomography system.

SOLUTION

The reliability block diagram and the reliability graph are shown in Figure 2.2.

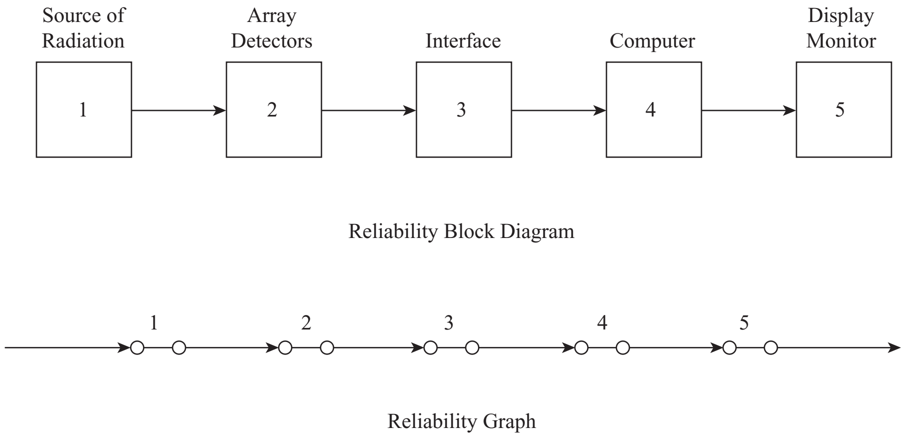

FIGURE 2.2 Reliability block diagram and reliability graph. ■

EXAMPLE 2.2

The operating principle of a typical laser printer is explained as follows (refer to Fig. 2.3). The main component of the laser printer is the photoconductor drum that rotates at a constant speed

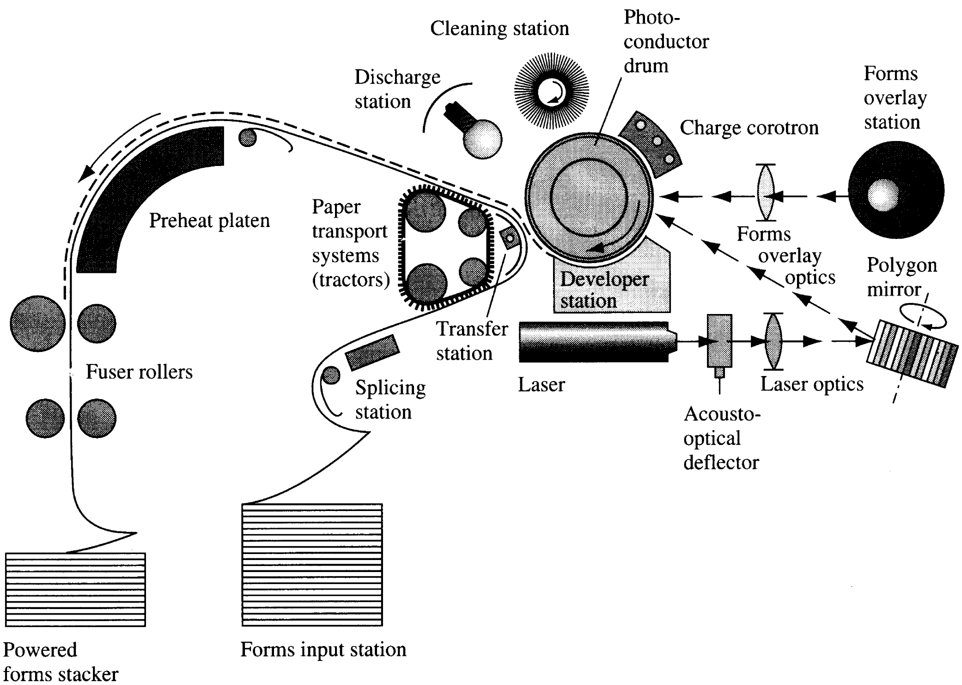

FIGURE 2.3 Operating principle of the laser printer.

with a semiconductor layer. During each revolution, this layer is electrostatically charged by a charge corotron. A laser beam, deflected vertically by an acousto-optical deflector and horizontally by a rotating polygon mirror, writes the print information onto the semiconductor layer by partially discharging this layer. Subsequently, as the drum passes through a "toner bath" in the developer station, the locations on the drum that have been discharged by the laser beam will then capture the toner. The print image thus produced on the photoconductor drum is transferred to paper in the transfer station and fused into the paper surface to prevent image smudging. After the printing operation is complete, a light source discharges the semiconductor layer on the drum and a brush removes any residual toner (Siemens Aktiengesellschaft, 1983).* Draw the reliability block diagram and the corresponding reliability graph.

SOLUTION

The reliability block diagram of the laser printer is a series system, and the failure of any component results in the failure of the system (Fig. 2.4).

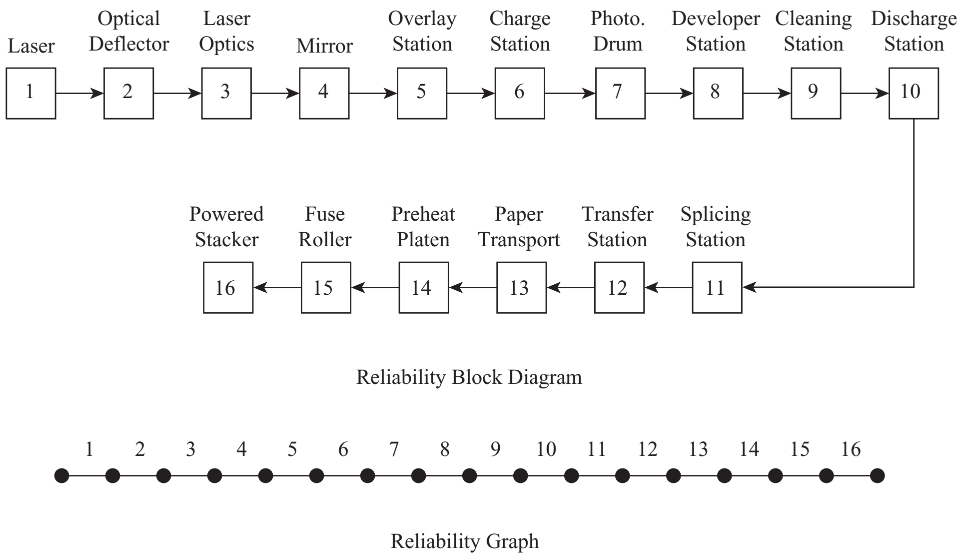

FIGURE 2.4 Reliability block diagram and the corresponding graph. ■

After constructing both the reliability block diagram and the reliability graph of the system, the next step is to determine the overall system reliability. The reliability graph can be as simple as pure series systems and parallel systems and as complex as a network with a wide range of many other systems in between such as series-parallel, parallel-series, and k-out-of-n and consecutive k-out-of-n systems. In the following sections, we present different approaches for

* Source: "The Laser Printer, bt020e," Siemens Aktiengesellschaft (1983).

determining the reliability of systems. We start with the simplest system and gradually increase the complexity of the system.

2.3 SERIES SYSTEMS

A series system is composed of n components (or subsystems) connected in series. A failure of any component results in the failure of the entire system. A car, for example, has several subsystems connected in series such as the ignition subsystem, the steering subsystem, and the braking subsystem. The failure of any of these subsystems causes the car not to perform its function, thus a failure of the system is considered to occur. In the car example, each subsystem may consist of more than one component connected in any of the configurations mentioned earlier. For example, the braking subsystem has three other subsystems including front brake, rear brake, and emergency brake. However, in estimating the reliability of the car, the subsystems are treated as components connected in series. In all situations, a system or a subsystem can be analyzed at different levels down to the component level. Another example of a simple series system is a flashlight system, which consists of four components connected in series—a bulb, housing, battery, and switch. The four must function properly for the flashlight to operate properly.

In order to determine the reliability of a series system, assume that the probability of success (operating properly) of every unit in the system is known at the time of system's evaluation. Assume also the following notations:

x_i	=	the ith unit is operational,
$\bar{x}_i$	=	failure of the ith unit,
$P(x_i)$	=	probability that unit i is operational,
$P(\bar{x}_i)$	=	probability that unit i is not operational (failed),
R	=	reliability of the system, and
P_f	=	unreliability of the system ($P_f = 1 - R$).

Since the successful operation of the system consisting of n components requires that all units must be operational, then the reliability of the system can be expressed as

$$R = P(x_1 x_2 \dots x_n)$$

or

$$R = P(x_1)P(x_2 \mid x_1)P(x_3 \mid x_2 x_1)\dots P(x_n \mid x_1 x_2 x_3 \dots x_{n-1}). \tag{2.1}$$

The conditional probabilities in Equation 2.1 reflect the case when the failure mechanism of a component affects other components' failure rates. A typical example of such a case is the heat dissipation from a potentially failing component may cause the failure rate of adjacent components to increase. When the components' failures are independent, then Equation 2.1 can be written as

$$R = P(x_1)P(x_2)\dots P(x_n)$$

or

$$R = \prod_{i=1}^{n} P(x_i). \tag{2.2}$$

Alternatively, the reliability of the system can be determined by computing the probability of system failure and subtracting it from unity. The system fails if any of the components fails. Thus,

$$P_f = P(\bar{x}_1 + \bar{x}_2 + \dots \bar{x}_n), \tag{2.3}$$

where "+" means the union of events.

From the basic laws of probability, the probability of either event A or B occurring is

$$P(A+B) = P(A) + P(B) - P(AB). \tag{2.4}$$

Following Equation 2.4, we rewrite Equation 2.3 as follows:

$$P_f = [P(\bar{x}_1) + P(\bar{x}_2) + \dots + P(\bar{x}_n)] - [P(\bar{x}_1\bar{x}_2) + P(\bar{x}_1\bar{x}_3) + \dots] + \dots + [-1]^{n-1} P(\bar{x}_1\bar{x}_2, \dots, \bar{x}_n). \tag{2.5}$$

The reliability of the system is

$$R = 1 - P_f.$$

It should be noted that the reliability of a series system is always less than that of the component with the lowest reliability.

EXAMPLE 2.3

Consider a series system that consists of three components and the probabilities that components 1, 2, and 3 being operational are 0.9, 0.8, and 0.75, respectively. Estimate the reliability of the system.

SOLUTION

Assuming independent failures, we use Equation 2.2 to obtain the reliability of the system

$$R = 0.9 \times 0.8 \times 0.75 = 0.54.$$

Alternatively, we use Equation 2.5 as

$$\begin{aligned}P_f &= [P(\bar{x}_1) + P(\bar{x}_2) + P(\bar{x}_3)] - [P(\bar{x}_1)P(\bar{x}_2) + P(\bar{x}_1)P(\bar{x}_3) + P(\bar{x}_2)P(\bar{x}_3)] + [P(\bar{x}_1)P(\bar{x}_2)P(\bar{x}_3)]\\ &= 0.55 - 0.095 + 0.005 = 0.46\end{aligned}$$

and

$$R = 1 - P_f = 1 - 0.46 = 0.54.$$

As shown above, the reliability of the system, 0.54, is less than the reliability of the worst component, 0.75. ■

2.4 PARALLEL SYSTEMS

In a parallel system, components or units are connected in parallel such that the failure of one or more paths still allows the remaining path(s) to perform properly. In other words, the reliability of a parallel system is the probability that any one path is operational. The block diagram and reliability graph of a parallel system consisting of n components (units) connected in parallel are shown in Figure 2.5.

Similar to the series systems, the reliability of parallel systems can be determined by estimating the probability that any one path is operational or by estimating the unreliability of the system then subtracting it from unity. In other words,

$$R = P(x_1 + x_2 + \ldots + x_n)$$

or

$$\begin{aligned}R &= [P(x_1) + P(x_2) + \ldots + P(x_n)] - [P(x_1x_2) + P(x_1x_3) + \ldots + P_{i\neq j}(x_ix_j)]\\ &\quad + \ldots + [-1]^{n-1}P(x_1x_2, \ldots, x_n).\end{aligned} \tag{2.6}$$

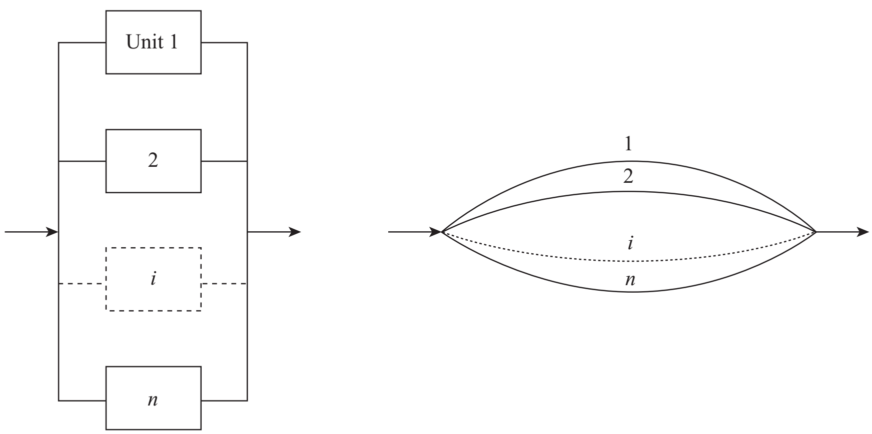

FIGURE 2.5 Block diagram and reliability graph of a parallel system.

Alternatively,

$$P_f = P(\bar{x}_1\bar{x}_2 \ldots \bar{x}_n)$$
$$R = 1 - P_f$$

or

$$R = 1 - P(\bar{x}_1)P(\bar{x}_2 \mid \bar{x}_1)P(\bar{x}_3 \mid \bar{x}_1\bar{x}_2)\ldots \tag{2.7}$$

Again, if the components are independent, then Equation 2.7 can be rewritten as $R = 1 - P(\bar{x}_1)P(\bar{x}_2)\ldots P(\bar{x}_n)$

or

$$R = 1 - \prod_{i=1}^{n} P(\bar{x}_i). \tag{2.8}$$

If the components are identical, then the reliability of the system is

$$R = 1 - (1 - p)^n,$$

where p is the probability that a component is operational.

EXAMPLE 2.4

Consider a system that consists of three components in parallel. The probabilities of the three components being operational are 0.9, 0.8, and 0.75. Determine the reliability of the system.

SOLUTION

The reliability of a parallel system is obtained by using Equation 2.6 as follows:

$$\begin{aligned} R &= P(x_1 + x_2 + x_3) \\ &= [P(x_1) + P(x_2) + P(x_3)] - [P(x_1)P(x_2) + P(x_1)P(x_3) + P(x_2)P(x_3)] + P(x_1)P(x_2)P(x_3) \end{aligned}$$

or

$$R = 2.450 - 1.995 + 0.540 = 0.995.$$

One can also obtain the reliability of the system by using Equation 2.8:

$$R = 1 - \prod_{i=1}^{n} P(\bar{x}_i)$$

$$R = 1 - (1 - 0.90)(1 - 0.80)(1 - 0.75) = 0.995.$$

■

The reliability of a parallel system is greater than the reliability of the most reliable unit (or component) in the system. This may imply that the more units we have in parallel, the more reliable the system. This statement is only valid for systems whose components exist only in two states, either an operational or a failure state. As we show later, there is an optimal number of multistate components (units) that can be connected in parallel, and adding more units in parallel results in lower values of reliability.

2.5 PARALLEL-SERIES, SERIES-PARALLEL, AND MIXED-PARALLEL SYSTEMS

The systems discussed in Sections 2.3 and 2.4 are referred to as pure series and pure parallel systems, respectively. There are many situations where a system is composed of a combination of series and parallel subsystems. This section considers three systems: parallel-series, series-parallel, and mixed-parallel.

2.5.1 Parallel-Series

A parallel-series system consists of m parallel paths. Each path has n units connected in series as shown in Figure 2.6. Let $P(x_{ij})$ be the reliability of component j ($j = 1, 2, \ldots, n$) in path i ($i = 1, 2, \ldots, m$) and x_{ij} be an indicator that component j in path i is operational. The reliability of path i is

$$P_i = \prod_{j=1}^{n} P(x_{ij}) \quad i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n.$$

The unreliability of path i is $\bar{P}_i$ and the reliability of the system is

$$R = 1 - \prod_{i=1}^{m} \bar{P}_i$$

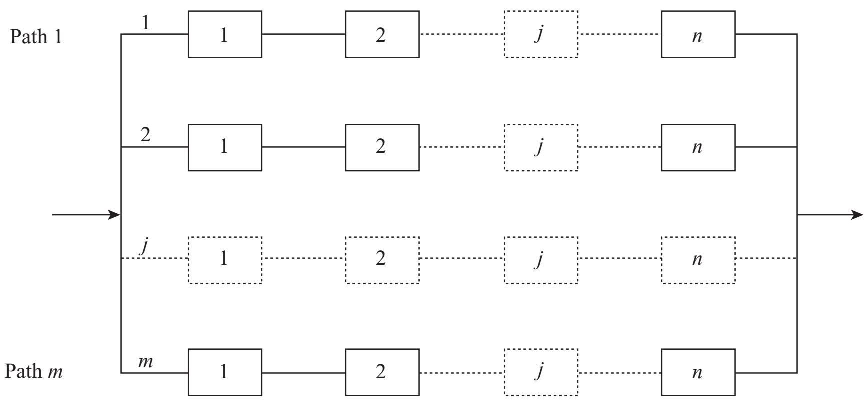

FIGURE 2.6 A parallel-series system.

or

$$R = 1 - \prod_{i=1}^{m}\left[1 - \prod_{j=1}^{n} P(x_{ij})\right].$$

If all units are identical and the reliability of a single unit is p, then the reliability of the system becomes

$$R = 1 - (1 - p^n)^m. \tag{2.9}$$

2.5.2 Series-Parallel

A general series-parallel system consists of n subsystems in series with m units in parallel in each subsystem as shown in Figure 2.7. Following the parallel-series systems, we derive the reliability expression of the system as:

$$R = \prod_{i=1}^{n}\left[1 - \prod_{j=1}^{m}(1 - P(x_{ij}))\right],$$

where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ and $P(x_{ij})$ is the probability that component j in subsystem i is operational. When all units are identical and the reliability of a single unit is p, then the reliability of the series-parallel system becomes

$$R = \left[1 - (1 - p)^m\right]^n. \tag{2.10}$$

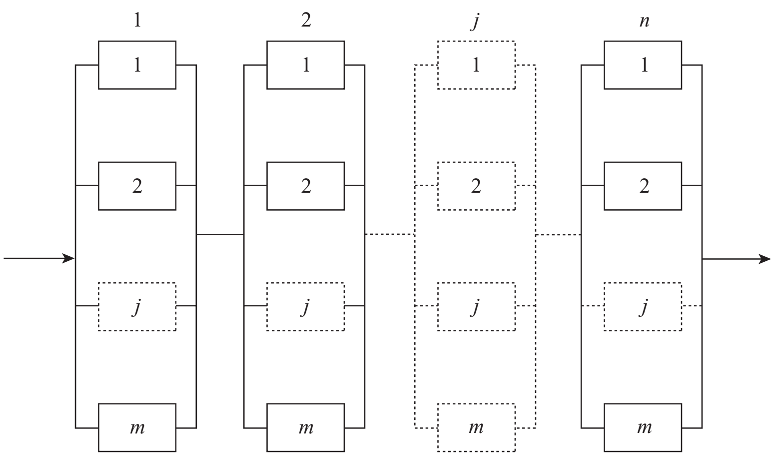

FIGURE 2.7 A series-parallel system.

In general, series-parallel systems have higher reliabilities than parallel-series systems when both have equal number of units and each unit has the same probability of operating properly.

2.5.3 Mixed-Parallel

A mixed-parallel system has no specific arrangement of units other than the fact that they are connected in parallel and series configurations. Figure 2.8 illustrates two possible mixed-parallel systems for eight units. The reliability of a mixed-parallel system can be estimated using Equations 2.2 and 2.8.

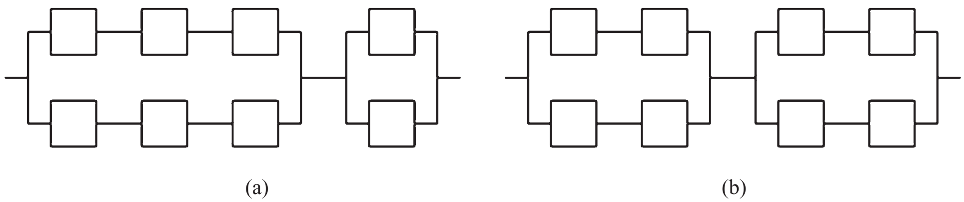

FIGURE 2.8 Mixed-parallel systems.

EXAMPLE 2.5

Given six identical units each having a reliability of 0.85, determine the reliability of three systems resulting from the arrangements of the units in parallel-series, series-parallel, and mixed-parallel configurations.

SOLUTION

Assume that the six units can be arranged in three series and parallel configurations, as shown in Figure 2.9. The reliabilities of the systems are as follows:

1. Parallel-series:

$$R = 1 - (1 - p^n)^m,$$

when $m = 2$, $n = 3$.

$$R = 1 - (1 - 0.85^3)^2 = 0.85110.$$

2. Series-parallel:

$$R = [1 - (1 - 0.85)^2]^3 = 0.934007.$$

3. Mixed-parallel:

$$R = \left[1 - (1 - 0.85^2)^2\right]\left[1 - (1 - 0.85)^2\right] = 0.902226.$$

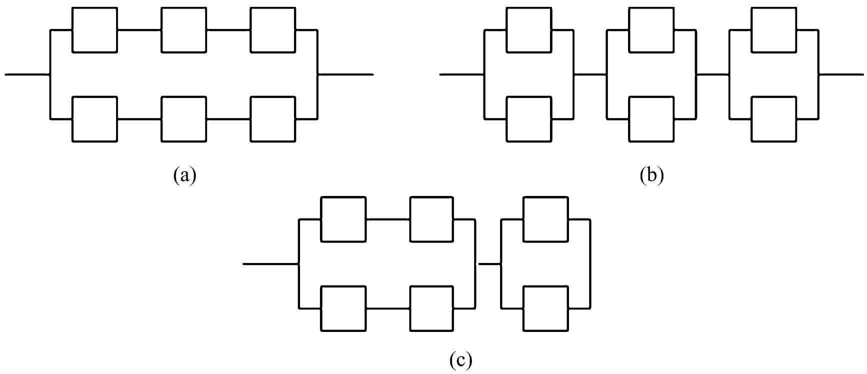

FIGURE 2.9 (a) Parallel-series, (b) series-parallel, (c) mixed-parallel. ■

2.5.4 Variance of System Reliability Estimate

The system reliability obtained in the previous sections is a point estimate of the reliability. However, in many situations, the point estimate is insufficient to make critical decisions such as warranty allocation. In such situations, it becomes important to provide an estimate of the variance of the system reliability. In this section, we describe the procedure for estimating the variance of the system reliability.

An exact expression for the estimated variance of the system reliability can be determined if the components' reliability estimates are independent and the system can be decomposed into an equivalent series-parallel system. An approximation of the variance can be computed when available data are used to estimate component reliability values. The variance estimation procedure can be divided into several steps. First, estimates of reliability are required for each component based on field-operation failure data or experimental data. Second, the variance of the component reliability estimate is required. The variability of the system's reliability is determined by aggregating the component information (Coit, 1997).

Estimation of component reliability and variance are often based on binomial data. For the ith component used in the system, consider that n_i components are placed under test for t hours and f_i failures are observed. The status of each component (survival/failure) is considered as an independent Bernoulli trial with parameter $r_i(t)$. An unbiased estimate of $r_i(t)$ and an approximation of the variance of the estimate are determined from the binomial distribution using the following well-known equations:

$$\hat{r}_i(t) = 1 - \frac{f_i}{n_i}$$

$$Var\left(\hat{r}_i(t)\right) = \frac{r_i(t)(1 - r_i(t))}{n_i}$$

$$V\hat{a}r\left(\hat{r}_i(t)\right) = \frac{\left(1 - \frac{f_i}{n_i}\right)\frac{f_i}{n_i}}{n_i},$$

where, n_i = testing sample size for the ith component.

f_i = number of failures out of the ith component.

$r_i(t)$ = reliability of ith component (an unknown constant).

$\hat{r}_i(t)$ = estimate of reliability of ith component.

$Var(\hat{r}_i(t))$ = variance of the reliability estimate for ith component.

$V\hat{a}r(\hat{r}_i(t))$ = estimate of variance $Var(\hat{r}_i(t))$.

Variance of the system reliability estimate can be determined based on component reliability and component estimate variance as demonstrated by Coit (1997). Exact expressions for the variance are given in Equations 2.11 and 2.12 for series and parallel systems, respectively.

$$Var\left(\hat{R}_{series}(t)\right) = \prod_i \left(r_i(t)^2 + Var(\hat{r}_i(t))\right) - \prod_i r_i(t)^2. \tag{2.11}$$

$$Var\left(\hat{R}_{parallel}(t)\right) = \prod_i \left((1-r_i(t))^2 + Var(\hat{r}_i(t))\right) - \prod_i (1-r_i(t))^2. \tag{2.12}$$

For any other system that can be decomposed into an equivalent series-parallel system, the variance of the system reliability estimate can be obtained using a decomposition methodology. The algorithm presented in this section is applied for series-parallel systems, but it can be readily adapted to other system configurations as well. The variance of the system reliability estimate for series-parallel (s-p) systems is given by Equation 2.13 as shown in Coit (1997). Equation 2.14 is a simplification of Equation 2.13.

$$Var\left(\hat{R}_{s-p}(t)\right) = \prod_{i=1}^{m}\left(\left(1-\prod_{j=1}^{s_i}(1-r_{ij}(t))\right)^2 + \prod_{j=1}^{s_i}\left((1-r_{ij}(t))^2 + Var(\hat{r}_{ij}(t))\right) - \prod_{j=1}^{s_i}(1-r_{ij}(t))^2\right) - \prod_{i=1}^{m}\left(1-\prod_{j=1}^{s_i}(1-r_{ij}(t))\right)^2 \tag{2.13}$$

$$Var\left(\hat{R}_{s-p}(t)\right) = \prod_{i=1}^{m}\left(1-2\prod_{j=1}^{s_i} q_{ij}(t) + \prod_{j=1}^{s_i}\left(q_{ij}(t)^2 + Var(\hat{r}_{ij}(t))\right)\right) - \prod_{i=1}^{m}\left(1-\prod_{j=1}^{s_i} q_{ij}(t)\right)^2, \tag{2.14}$$

where

$r_{ij}(t)$ = reliability of jth component in ith subsystem at time

$q_{ij}(t) = 1 - r_{ij}(t)$ and $\hat{q}_{ij}(t)$ is the estimate of $q_{ij}(t)$

m = total number of subsystems

s_i = total number of components in subsystem i, where $i = 1, 2, \ldots, m$.

If the reliability of each component is estimated from binomial data, then a variance estimate is given by Equation 2.15:

$$V\hat{a}r\left(\hat{R}_{s-p}(t)\right) = \prod_{i=1}^{m}\left(1-2\prod_{j=1}^{s_i} \hat{q}_{ij}(t) + \prod_{j=1}^{s_i}\left(\hat{q}_{ij}(t)^2 + \frac{\hat{r}_{ij}(t)\hat{q}_{ij}(t)}{n_{ij}}\right)\right) - \prod_{i=1}^{m}\left(1-\prod_{j=1}^{s_i} \hat{q}_{ij}(t)\right)^2, \tag{2.15}$$

where n_{ij} = units tested for jth component in subsystem i.

Jin and Coit (2001) extend this work to yield the system reliability estimate variance when there are arbitrarily repeated components used within the system, and thus, the subsystem reliability estimates may not be independent.

EXAMPLE 2.6

Assume that the variances of reliability estimate of the three components given in Examples 2.3 and 2.4 are 0.05, 0.03, and 0.01. Estimate the variance of the corresponding series and parallel systems.

SOLUTION

We use Equation 2.11 to estimate the variance of the series system reliability as

$$Var\left(\hat{R}_{series}(t)\right) = (0.9^2 + 0.05)(0.8^2 + 0.03)(0.75^2 + 0.01) - (0.9^2)(0.8^2)(0.75^2)$$

$$Var\left(\hat{R}_{series}(t)\right) = 0.03827$$

$$Var\left(\hat{R}_{parallel}(t)\right) = ((1-0.9)^2 + 0.05)((1-0.8)^2 + 0.03)((1-0.75)^2 + 0.01)$$
$$-(1-0.9)^2(1-0.8)^2(1-0.75)^2$$

$$Var\left(\hat{R}_{parallel}(t)\right) = 0.0002795$$ ■

It is important to note that not only the reliability of a parallel system is higher than that of a series composed of the same components but also its variance is an order of magnitude lower.

2.5.5 Optimal Assignments of Units

Clearly, the reliability of the system depends on how the units are placed in the system's configuration. When the components (units) are nonidentical, the problem of optimally assigning units to locations in the parallel-series, series-parallel, and mixed-parallel systems becomes highly combinatorial in nature. The problem of obtaining an optimal assignment of components in parallel-series or series-parallel systems with $p_{ij} = p_j$ where p_{ij} is the reliability of component j when it is assigned to position i has been analytically solved by El-Neweihi et al. (1986). More recently, Prasad et al. (1991) present three algorithms to determine the optimal assignments of components in such systems with the assumption that $p_{ij} = r_i p_j$ where r_i is a probability whose value is dependent on the position i.

In this section we present an approach, though not optimal, to illustrate how units can be assigned to a series-parallel configuration in order to maximize the overall system reliability.

As shown in Section 2.5.2 a series-parallel system consists of n subsystems connected in series with m units in parallel in each system. Let $K_1, K_2, \ldots, K_n$ represent the subsystems $1, 2, \ldots, n$. Consider the case when all subsystems have an equal number of units m. Thus, the total number of units of the entire system is $u = n \times m$. The problem of assigning units to the series-parallel system can simply be stated as: given a system of u units with reliabilities given by the vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_u)$ where p_i is the probability that unit C_i is operating, $(i = 1, 2, \ldots, u)$, we wish to allocate these units, which are assumed to be interchangeable, in such a manner that the reliability function $\mathfrak{R}$ of the system is maximized (Baxter and Harche, 1992).

The reliability function $\mathfrak{R}$ is

$$\mathfrak{R} = \prod_{i=1}^{n} R_i, \tag{2.16}$$

where

$$R_i = 1 - \prod_{j \in K_i} q_j \tag{2.17}$$

and

$$q_j = 1 - p_j, \quad j = 1, 2, \ldots u.$$

Revisiting the series-parallel configuration discussed in Section 2.5.2, we observe that the reliability of the system is maximum when the $R_i's$ (reliability of individual subsystems) are as equal as possible. Based on this, we utilize the *top-down heuristic* (TDH) proposed by Baxter and Harche (1992) to assign units to positions in the system configuration. The steps of the heuristic are

Step 1	Rank and label the units such that $p_1 \geq p_2 \geq \cdots \geq p_u$.
Step 2	Allocate units C_j to the subsystem K_j, $j = 1, 2, \ldots n$.
Step 3	Allocate units C_j to subsystem K_{2n+1-j}, $j = n + 1, \ldots, 2n$.
Step 4	Set $v := 2$.
Step 5	Evaulate $R_i^{(v)} = 1 - \prod_{j \in K_i} q_j$ for $i = 1, 2, \ldots, n$. Allocate unit C_{vn+j} to subsystem K_i for which $R_i^{(v)}$ is the jth smallest, $j = 1, 2, \ldots, n$.
Step 6	If $v < m$, set $v := v + 1$ and repeat Step 5. If $v = m$, stop.

It should be noted that there exists a bottom-up heuristic (BUH) corresponding to the TDH in which we start by allocating the n least reliable units, one to each subsystem, then allocating the n next least reliable units in reverse order and so on. Clearly, this heuristic is inferior to the TDH in practice, since (as we shall present later) the most important unit of a parallel system is the most reliable component. It should also be noted that the TDH results in optimal allocations when only two units are allocated to each subsystem.

EXAMPLE 2.7

An engineer wishes to design a redundant system that includes six resistors, all having the same resistance value, and their reliabilities are 0.95, 0.75, 0.85, 0.65, 0.40, and 0.55. The resistors are interchangeable within the system. Space within the enclosure where the resistors are connected limits the designer to allocate the resistor in a series-parallel arrangement of the form (3, 2),—that is, two subsystems connected in series and each subsystem consists of three resistors connected in parallel. Use the TDH to allocate units to the subsystems. Compare the reliability of the resultant system with that obtained from the application of the BUH.

SOLUTION

We follow the steps of the TDH to allocate units to the subsystems K_1 and K_2 as follows:

Step 1 Rank the resistors in a decreasing order of their reliabilities: 0.95, 0.85, 0.75, 0.65, 0.55, and 0.40.

Step 2 Allocate resistors C_1 to K_1 and C_2 and K_2.

Step 3 Allocate resistors C_3 to K_2 and C_4 to K_1.

Step 4 $\nu := 2$

Step 5 Calculate the reliabilities of the subsystems K_1 and K_2, respectively, as

$$R_1^{(2)} = 1-(1-0.95)(1-0.65) = 0.9825, \text{ and}$$
$$R_2^{(2)} = 1-(1-0.85)(1-0.75) = 0.9625$$

Since $R_2^{(2)} < R_1^{(2)}$, we allocate C_5 to K_2.

Allocate C_6 to K_1

Step 6 $\nu := 3$, stop.

The resultant allocation of the resistor is as follows:

Subsystem K_1	Subsystem K_2
C_1	C_2
C_4	C_3
C_6	C_5

The reliability of the system is

$$\Re_s = [1-(1-0.95)(1-0.65)(1-0.40)][1-(1-0.85)(1-0.75)(1-0.55)]$$

or

$$\Re_s = 0.972802.$$

Application of the BUH results in the following allocation:

Subsystem K_1	Subsystem K_2
C_6	C_5
C_3	C_4
C_2	C_1

The reliability of the system is

$$\Re_s = [1-(1-0.40)(1-0.75)(1-0.85)][1-(1-0.55)(1-0.65)(1-0.95)]$$

or

$$\Re_s = 0.9698.$$

In general, allocation of units to subsystems using the TDH results in a higher reliability of the system than the BUH. ■

System reliability optimization is an area of study for many years. The main objective is to maximize the reliability of a system considering some constraints such as cost, weight, space, and others. In general, reliability optimization is divided into two types, the first type deals with reliability redundancy allocation where determination of both optimal component reliability and the number of component redundancy allowing mixed components. This problem is considered of the NP-hard class (Nonpolynomial time is needed to solve large size problems). The second type seeks the determination of optimal component reliability to maximize the system reliability subject to constraints. This is also an NP-hard problem and resembles the well-known knapsack problems. We illustrate the formulation of the first type of problems under a subset of constraints.

Consider a series-parallel system with n subsystem each has m components in parallel. Let

i	=	index for the subsystem $i = 1, 2, \ldots, n$
j	=	index for the components $j = 1, 2, \ldots, m$
R_s	=	system reliability
R_i	=	reliability of subsystem i
r_{ij}	=	reliability of component j used in subsystem i
x_{ij}	=	number of components j used in subsystem i
u_i	=	maximum number of components used in subsystem i
w_{ij}	=	weight of component j available for subsystem i
C	=	maximum cost available for the entire system
W	=	maximum weight of the entire system

As shown above, the reliability of a series-parallel system is obtained by considering the subsystems as a series system while each subsystem is a separate parallel system. Therefore, the reliability of the system is

$$R_s = \prod_{i=1}^{n} R_i.$$

The reliability of a subsystem R_i is obtained using the reliability of its parallel components as

$$R_i = 1 - \prod_{j=1}^{m} [1 - r_{ij}]^{x_{ij}}.$$

The mathematical programming formulation of the optimization problem is

$$\text{Maximize } R_s = \prod_{i=1}^{n} R_i = \prod_{i=1}^{n} \left\{ 1 - \prod_{j=1}^{m} [1 - r_{ij}]^{x_{ij}} \right\}.$$

Subject to

$$\sum_{i=1}^{n} \sum_{j}^{m} c_{ij} x_{ij} \le C$$

$$\sum_{i=1}^{n} \sum_{j}^{m} w_{ij} x_{ij} \le W$$

$$1 \le \sum_{j=1}^{m} x_{ij} \le u_i, \quad i = 1, 2, \dots, n$$

$$x_{ij} \ge 0, i = 1, 2, \dots, n \quad j = 1, 2, \dots, m, \text{ integers.}$$

The first is the cost constraint, the second is the weight constraint, and the third is the maximum number of available components constraints. Finally, the decision variables are integers. As stated earlier, optimal solutions of this problem form small values of n and m. As the number of decision variables increases, the search space becomes larger and optimal solutions are difficult to obtain. Therefore, approaches such as heuristics, genetic algorithms, and others have been used to obtain "good" solutions (Coit and Smith, 1996; Kulturel-Konak et al., 2003; Chen and You, 2005; You and Chen, 2005).

2.6 CONSECUTIVE-*K-OUT-OF-N:F* SYSTEM

In Section 2.3 we presented a series system consisting of n components. In such a system, the failure of one component results in system failure. However, there exist series systems that are not considered failed until at least k components have failed. Moreover, those k components must be consecutively ordered within the system. Such systems are known as *consecutive-k-out-of-n:F systems*. An example of a consecutive-k-out-of-n:F system is presented in Chiang and Niu (1981), which considers a telecommunications system with n relay stations (either satellites or ground stations). The stations are named consecutively 1 to n. Suppose a signal emitted from Station 1 can be received by both Stations 2 and 3, and a signal relayed from Station 2 can be received by both Stations 3 and 4, and so on. Thus, when Station 2 fails, the telecommunications system is still able to transmit a signal from Station 1 to Station n. However, if both Stations 2 and 3 fail, a signal cannot be transmitted from Station 1 directly to Station 4, therefore the system fails. Similarly, if any two consecutive stations in the system fail, the system fails. This is considered a consecutive-2-out-of-*n:F* system.

Determining the reliability of the consecutive-2-out-of-*n:F* system is simple. First, we define the following notations:

n	=	the number of components in a system,
k	=	the minimum number of consecutive failed components that cause the system failure,
p	=	the probability that a component is functioning properly (all components have identical and independent life distributions),
$R(p, k, n)$	=	the reliability of a consecutive-k-out-of-*n:F* system whose components are identical and each component has a probability of p functioning properly,
x_i	=	the state of component i, $x_i = 0$ is failed state,
X	=	the vector component states,
Y	=	a random variable indicating the index of first 0 in X,
M	=	a random variable indicating the index of first 1 after the position Y in X,
$\lfloor a \rfloor$	=	the largest integer less than or equal to a, and
$\cup_k p$	=	$1 - (1 - p)^k$.

2.6.1 Consecutive-2-out-of-*n:F* System

We follow the work of Chiang and Niu (1981). The reliability of a consecutive-2-out-of-*n:F* system is

$$\begin{aligned} R(p,2,n) &= P[\text{the system is functioning}] \\ &= \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} P[\text{system is functioning and } j \text{ components failed}]. \end{aligned} \tag{2.18}$$

If the number of failed components is greater than $\lfloor (n + 1)/2 \rfloor$, then there exists two consecutive failed components in the system—that is, the system fails. Hence, the above expression of system reliability does not include the terms for $j > \lfloor (n + 1)/2 \rfloor$.

If j components have failed, $j \le \lfloor (n + 1)/2 \rfloor$, the system functions if there is at least one functioning component between every two failed components. The number of such combinations between functioning and failed components is

$$\binom{(j+1)+(n-2j+1)-1}{n-2j+1} = \binom{n-j+1}{j}, \tag{2.19}$$

which follows directly from Feller (1968) and Pease (1975). Substituting Equation 2.19 into Equation 2.18, we obtain

$$R(p,2,n)=\sum_{j=0}^{\lfloor (n+1)/2\rfloor}\binom{n-j+1}{j}(1-p)^j p^{n-j} \tag{2.20}$$

EXAMPLE 2.8

Consider four components that are connected in series. Each component has a reliability p. The system fails if two consecutive components fail. This system is referred to as consecutive-2-out-of-4:F system. Determine the reliability of the system when $p = 0.95$.

SOLUTION

Using Equation 2.20, we obtain

$$\begin{aligned}R(p,2,4)&=\sum_{j=0}^{2}\binom{4-j+1}{j}(1-p)^j p^{4-j}\\&=\binom{5}{0}(1-p)^0 p^4+\binom{4}{1}(1-p)p^3+\binom{3}{2}(1-p)^2p^2\\&=3p^2-2p^3.\end{aligned}$$

When $p = 0.95$, then

$$\begin{aligned}R(0.95,2,4)&=3(0.95)^2-2(0.95)^3\\&=0.992750.\end{aligned}$$

■

2.6.2 Generalization of the Consecutive-*k*-out-of-*n:F* Systems

We define X as a n-vector with element i having a value of 0 or 1 depending on whether component i is failing or not. The procedure for determining system reliability is based on observing the first sequence of consecutive 0's in the X vector. The system is considered to be failed if at least k consecutive 0's are observed in X. Since the reliability of a consecutive-k-out-of-n:F system for all $n < k$ is 1 by definition, we can recursively compute the reliability of consecutive-k-out-of-n:F system for $n \geq k$. The reliability (Chiang and Niu, 1981) is

$$\begin{aligned}R(p,k,n)&=P[\text{the system is functioning}]\\&=\sum_y\sum_m P[\text{system is functioning}|Y=y, M=m]P[Y=y, M=m]\\&=\sum_{y=1}^{n-k+1}\sum_{m=y+1}^{y+k-1}P[\text{system is functioning}|Y=y, M=m]p^y(1-p)^{m-y}+p^{n-k+1}.\end{aligned}$$

Since the system has less than k failed components for $Y > n - k + 1$ then P (system is functioning $|Y > n - k + 1) = 1$ and $P[Y > n - k + 1] = p^{n-k+1}$. When $m \geq y + k$, the system already has k failed components and is considered failed.

For $y + 1 \le m \le y + k - 1$, the first sequence of 0's does not constitute a cut-set. Furthermore, since $x_m = 1$, the event that the consecutive-k-out-of-$n{:}F$ system is functioning now is equivalent to the event that a consecutive-k-out-of-$(n - m){:}F$ system is functioning. Thus, the recursive formula for determining $R(p, k, n)$ is

$$\begin{aligned} R(p,k,n) &= \sum_{y=1}^{n-k+1} \sum_{m=y+1}^{y+k-1} R(p,k,n-m)p^y(1-p)^{m-y} + p^{n-k+1} \\ R(p,k,j) &= \begin{cases} 1, & 0 \le j < k \\ 0, & j < 0 \end{cases}. \end{aligned} \tag{2.21}$$

As discussed earlier, the failure of k consecutive components results in the system failure. Therefore, k consecutive components are the only minimum cut-sets and there is $n - k + 1$ such sets in the consecutive-k-out-of-$n{:}F$ system. If the system is functioning, then there is at least one functioning component in every cut-set. Hence, the lower bound for system reliability is

$$R_L(p,k,n) \ge (\cup_k p)^{n-k+1}. \tag{2.22}$$

Similarly, an upper bound for system reliability can be obtained as

$$R_U(p,k,n) \le (\cup_k p)^{\lfloor n/k \rfloor}. \tag{2.23}$$

EXAMPLE 2.9

Derive an expression for the reliability of a consecutive-2-out-of-7$:F$ system. Each component has a reliability p. Calculate the reliability and its lower and upper bounds when $p = 0.90$.

SOLUTION

Using Equation 2.20 we obtain

$$\begin{aligned} R(p,2,2) &= 2p - p^2 \\ R(p,2,3) &= p + p^2 - p^3 \\ R(p,2,4) &= 3p^2 - 2p^3 \text{ (see Example 2.8)} \\ R(p,2,5) &= p^2 + 3p^3 - 4p^4 + p^5 \\ R(p,2,6) &= 4p^3 - 2p^4 - 2p^5 + p^6. \end{aligned}$$

We use Equation 2.21 and the above expressions to obtain $R\ (p, 2, 7)$ as follows:

$$\begin{aligned} R(p,2,7) &= \sum_{y=1}^{6} R(p,2,6-y)p^y(1-p) + p^6 \\ R(p,2,7) &= p^3 + 6p^4 - 9p^5 + 3p^6 \\ R(0.9,2,7) &= 0.945513. \end{aligned}$$

The lower and upper bounds of the reliability are obtained using Equations 2.22 and 2.23, respectively:

$$R_L(0.9, 2, 7) \geq [1-(1-0.9)^2]^6$$

or

$$R_L(0.9, 2, 7) \geq 0.941480$$

and

$$R_U(0.9, 2, 7) \leq [1-(1-0.9)^2]^3$$

or

$$R_U(0.9, 2, 7) \leq 0.970299.$$

The effect of the component reliability on the system reliability is shown in Figure 2.10.

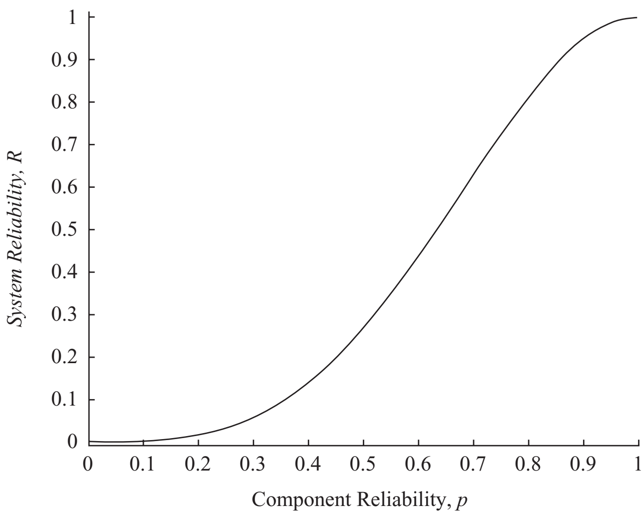

FIGURE 2.10 Effect of change of p on R. ■

2.6.3 Reliability Estimation of the Consecutive-*k*-out-of-*n:F* Systems

In the previous sections, we presented methods for estimating the reliability of 2-out-of-*n:F* systems for both cases when all units are identical and when the reliabilities of the units are equal. In this section, we present an algorithm for reliability estimation of the consecutive-*k*-out-of-*n:F* systems when p_i's are not equal. After an approach of reliability estimation of such systems is introduced, we present a general and more efficient algorithm.

The first approach is based on finding the reliability by determining the failure states of the system, that is, determining all the combinations of component failures which result in a system failure. For example, consider a system that consists of five components connected in series. The system fails when two consecutive components fail. The minimum possible states that cause the system failure are

$$\bar{x}_1\bar{x}_2,\ \bar{x}_2\bar{x}_3,\ \bar{x}_3\bar{x}_4, \text{ and } \bar{x}_4\bar{x}_5.$$

From these states, the reliability of the system can be estimated as

$$\begin{aligned}
R &= 1 - P\left[\bar{x}_1\bar{x}_2 + \bar{x}_2\bar{x}_3 + \bar{x}_3\bar{x}_4 + \bar{x}_4\bar{x}_5\right] \\
R &= 1 - \left[P(\bar{x}_1\bar{x}_2) + P(\bar{x}_2\bar{x}_3) + P(\bar{x}_3\bar{x}_4) + P(\bar{x}_4\bar{x}_5)\right. \\
&\quad - P(\bar{x}_1\bar{x}_2\bar{x}_3) - P(\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4) - P(\bar{x}_1\bar{x}_2\bar{x}_4\bar{x}_5) \\
&\quad - P(\bar{x}_2\bar{x}_3\bar{x}_4) - P(\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5) - P(\bar{x}_3\bar{x}_4\bar{x}_5) \\
&\quad + P(\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4) + P(\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5) \\
&\quad \left. + P(\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5) + P(\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5) - P(\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5)\right]
\end{aligned}$$

or

$$\begin{aligned}
R &= 1 - \left[P(\bar{x}_1\bar{x}_2) + P(\bar{x}_2\bar{x}_3) + P(\bar{x}_3\bar{x}_4) + P(\bar{x}_4\bar{x}_5)\right. \\
&\quad - P(\bar{x}_1\bar{x}_2\bar{x}_3) - P(\bar{x}_1\bar{x}_2\bar{x}_4\bar{x}_5) - P(\bar{x}_2\bar{x}_3\bar{x}_4) \\
&\quad \left. - P(\bar{x}_3\bar{x}_4\bar{x}_5) + P(\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5)\right]
\end{aligned} \tag{2.24}$$

EXAMPLE 2.10

Repeaters are devices that amplify signals and send them to other network segments. They play a vital role in building large networks. A repeater amplifies signals such that they can reach two repeaters away without loss or distortion. The repeaters are connected in series and the signals are considered lost when two consecutive repeaters fail.

Assume that five repeaters are connected in series and the probabilities of failure of repeaters 1 through 5 are: $q_1 = 0.62$, $q_2 = 0.079$, $q_3 = 0.25$, $q_4 = 0.22$, $q_5 = 0.42$. Determine the reliability of the system.

SOLUTION

The reliability of the system can be obtained by substituting the reliability values of the components in Equation 2.24:

$$\begin{aligned}
R = 1 - [&0.62 \times 0.079 + 0.079 \times 0.25 + 0.25 \times 0.22 + 0.22 \times 0.42 \\
&- 0.62 \times 0.079 \times 0.25 - 0.62 \times 0.079 \times 0.22 \times 0.42 \\
&- 0.079 \times 0.25 \times 0.22 - 0.25 \times 0.22 \times 0.42 \\
&+ 0.62 \times 0.079 \times 0.25 \times 0.22 \times 0.42]
\end{aligned}$$

or

$$R = 0.826.$$

■

Clearly, when n is large and $1 \le k \le n$, the above approach becomes quite complex and difficult to apply. This has prompted researchers to investigate different algorithms that can efficiently estimate system reliability. Among them are Bollinger (1982), Lambiris and Papastavridis (1985), Pham and Upadhyaya (1988), Shanthikumar (1982), and Zuo and Kuo (1990). We summarize Shanthikumar's algorithm as follows:

Step 1 Choose k, n. Test that $1 \le k \le n$.
$(i = 1, 2, \ldots, n)$. $q_i = 1 - p_i$.

Step 2 Set $F(r; k) = 0$ for $r = 0, 1, \cdots, k$.
Set $Q \leftarrow \prod_{i=1}^{k} q_i$.
Set $F(k; k) = Q$.

Step 3 Do for $r = k + 1$ to n.
$Q \leftarrow Q \times q_r / q_{r-k}$.
$F(r; k) = F(r-1; k) + [1 - F(r-k-1; k)] \times p_{r-k} \times Q$.

Step 4 $R(n; k) = 1 - F(n; k)$.
End.

The above algorithm is coded in a computer program and is listed in Appendix B.

EXAMPLE 2.11

Use the above algorithm to estimate the reliability of the system described in Example 2.10.

SOLUTION

We apply the steps of the algorithm as follows:

Step 1 $k = 2, n = 5$,
$p_1 = 0.38$, $p_2 = 0.921$, $p_3 = 0.75$, $p_4 = 0.78$, $p_5 = 0.58$.

Step 2 $F(0; 2) = 0$,
$F(1; 2) = 0$,
$Q = q_1 q_2 = 0.62 \times 0.079 = 0.0489$,
$F(2; 2) = 0.0489$.

Step 3 $r = 3$,
$Q = 0.0489 \times q_3 / q_1 = 0.0197$,
$F(3; 2) = F(2; 2) + [1 - F(0; 2)] \times 0.38 \times 0.0197 = 0.0563$.

Step 3 $r = 4$,

$Q = 0.0197 \times q_4 / q_2 = 0.0548.$

$F(4;2) = F(3;2) + [1 - F(1;2)] \times 0.921 \times 0.0548 = 0.1068.$

Step 3 $r = 5$,

$Q = 0.0548 \times q_5 / q_3 = 0.0920.$

$F(5;2) = F(4;2) + [1 - F(2;2)] \times 0.75 \times 0.0920 = 0.1724.$

Step 4 $R = 1 - F(5;2) = 0.827.$

The reliability of the system obtained by this algorithm is identical to that obtained by Equation 2.24. ■

2.6.4 Optimal Arrangement of Components in Consecutive-2-out-of-*n:F* Systems

As shown earlier, the reliability of a consecutive-2-out-of-n system, when the components have different failure probabilities, depends on the arrangement of the components in the system. The designer of such a system may wish to assign n components simultaneously to n positions within the system such that the reliability of the system is maximum. The components are ranked such that $p_1 < p_2 < \cdots < p_n$. (Derman et al., 1982) surmise that the optimal arrangement is (1, n, 3, $n - 2$, . . . , $n - 3$, 4, $n - 1$, 2), obtained by placing the least reliable pair of components outermost, followed by the most reliable pair, and so on in an alternating fashion. We shall now prove this conjecture. Define $\psi = (\psi(1), \ldots, \psi(n))$ as a policy where component $\psi(1)$ is assigned to position one in the system, $\psi(2)$ to the second position, . . . , $\psi(n)$ to position n. Let $r(\psi)$ be the reliability of the system when policy ψ is used.

Consider the case where $n = 2$. The optimum arrangement is either (1, 2) or (2, 1). When $n = 3$, the reliability $\psi(r)$ is

$$r(\psi) = 1 - q_{\psi(1)}q_{\psi(2)} - q_{\psi(2)}q_{\psi(3)} + \prod_{i=1}^{3} q_{\psi(i)},$$

where $q_{\psi(i)}$ is the unreliability of $\psi(i)$. For $r(\psi)$ to be maximum, the value of $[q_{\psi(1)}q_{\psi(2)} + q_{\psi(2)}q_{\psi(3)}]$ should be minimum. It can be verified that the arrangement $\psi = (1, 3, 2)$ yields maximum reliability. Similarly, when $n = 4$, the reliability of the system is

$$\begin{aligned} r(\psi) = 1 - [&q_{\psi(1)}q_{\psi(2)} + q_{\psi(2)}q_{\psi(3)} + q_{\psi(3)}q_{\psi(4)} - q_{\psi(1)}q_{\psi(2)}q_{\psi(3)} \\ &- q_{\psi(1)}q_{\psi(2)}q_{\psi(3)}q_{\psi(4)} - q_{\psi(2)}q_{\psi(3)}q_{\psi(4)} + q_{\psi(1)}q_{\psi(2)}q_{\psi(3)}q_{\psi(4)}] \\ r(\psi) = 1 - [&q_{\psi(1)}q_{\psi(2)} + q_{\psi(3)}q_{\psi(4)} + q_{\psi(2)}q_{\psi(3)}(1 - q_{\psi(1)} - q_{\psi(4)})]. \end{aligned}$$

The arrangement $\psi^* = (1, 4, 3, 2)$ maximizes $r(\psi)$. This follows from the observation that ψ^* simultaneously minimizes the sum of the first two terms and the last term within the bracket (Derman et al., 1982). Generalization of the above for any $n \geq 1$ yields

$$\psi^* = (1, n, 3, n-2, \ldots, n-3, 4, n, n-1, 2).$$

Appendix C is a listing of a computer program that obtains the optimal arrangement of components in a consecutive-2-out-of-*n:F* system and estimates its reliability.

EXAMPLE 2.12

A collision-avoidance system for articulated robot manipulators uses infrared proximity sensors grouped together in an array of sensor modules. The modules are distributed processing board-level products for acquiring data from proximity sensors mounted on robot manipulators. Each module consists of eight sensing elements, discrete electronics, a microcontroller, and communications components. The sensor system detects objects made of various materials at a distance of up to 50 cm. The module fails to detect the object if consecutive-2-out-of-8 sensing elements fail. The unreliabilities of the eight sensing elements are 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, and 0.08. Determine the optimal arrangement of these elements such that reliability of the module is maximized.

SOLUTION

We rank the sensing elements in decreasing order of the q_is:

Element	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
q_i	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01

Applying Derman et al. (1982) conjecture yields the following optimal arrangement of the sensing elements:

$$\psi^* = [1, 8, 3, 6, 5, 4, 7, 2].$$

The maximum reliability of the system is 0.9915. ■

Wei et al. (1983) partially support the conjecture developed by Derman et al. (1982). Malon (1985) characterizes all other values of k and n for which an optimal configuration can be determined without knowledge of the component failure probabilities. As we mentioned earlier, the reliability of the system depends upon the particular failure probabilities and the positions of the components in the system. However, for certain values of k and n, there is an arrangement that is optimal regardless of the failure probabilities. We refer to such an arrangement as an invariant optimal arrangement. We now characterize all values of k and n for such arrangements.

Rank the components such that $q_1 \geq q_2 \geq \cdots \geq q_n$. Malon (1985) states that the consecutive-*k*-out-of-*n:F* system admits an invariant arrangement if and only if $k \in \{1, 2, \ldots, n-2, n-1, n\}$. The optimal arrangements are given in Table 2.1.

EXAMPLE 2.13

Solve Example 2.12 for consecutive-6-out-of-8:*F* system when the sensing elements have the following unreliability values.

Element	1	2	3	4	5	6	7	8
q_i	0.4	0.35	0.32	0.28	0.25	0.21	0.18	0.15

SOLUTION

Using Table 2.1, any of the following arrangements results in a maximum reliability of 0.999651.

Arrangements (1, 4, any arrangement of elements [5, 6, 7, 8], 3, 2). ■

TABLE 2.1 Optimal Arrangements of Components

k	Invariant optimal arrangement
1	Any arrangement
2	[1, n, 3, n – 2, . . . , n – 3, 4, n – 1, 2]
n – 2	[1, 4, (any arrangement), 3, 2]
n – 1	[1, (any arrangement), 2]
n	Any arrangement

2.7 RELIABILITY OF *K*-OUT-OF-*N* SYSTEMS

In Section 2.6, we presented a consecutive-k-out-of-n:F system where a system fails if at least *k consecutive* components fail. In many cases, the k failures need not be consecutive and the system fails if any k or more components fail. For example, large airplanes usually have three or four engines but two engines may be the minimum number required to provide a safe journey. Similarly, in many power-generating systems that have two or three generators, one generator may be sufficient to provide the power requirements. Also, in a typical wire cable for cranes and bridges, the cable may contain thousands of wires and only a fraction of them is required to carry the desired load. Assuming that all units have identical and independent life distributions and the probability that a unit is functioning is p, then the probability of having exactly k functioning units out of n is

$$P(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k} \quad k = 0, 1, \ldots, n. \tag{2.25}$$

The system is considered to be functioning properly if k or $k + 1$ or . . . or $n - 1$ or n units are functioning. Therefore, the reliability of the system is

$$P(k; n, p) = \sum_{r=k}^{n} \binom{n}{r} p^r (1-p)^{n-r}. \tag{2.26}$$

If the units are all different, then in order to determine the reliability of the system, all possible operational combinations should be evaluated as shown in Example 2.14.

EXAMPLE 2.14

Consider a telecommunication system that consists of four different parallel channels. A system is considered operational if any three channels are operational. Determine the reliability of the system.

SOLUTION

This is a 3-out-of-4 system. Let x_1, x_2, x_3, and x_4 be indicators when channels 1 through 4 are functioning properly and $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, and $\bar{x}_4$ be the indicators when the channels fail. The reliability of the system is

$$R = P(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4). \tag{2.27}$$

Let

$$\begin{aligned} A_1 &= x_1x_2x_3 \\ A_2 &= x_1x_2x_4 \\ A_3 &= x_1x_3x_4 \\ A_4 &= x_2x_3x_4. \end{aligned}$$

In estimating the reliability one may include $A_5 = x_1x_2x_3x_4$ in Equation 2.27. However, the interaction terms will result in cancellations of some probabilities, which in turn causes Equation 2.27 to be valid without the inclusion of A_5.

We rewrite Equation 2.27 as

$$\begin{aligned} R &= P(A_1 + A_2 + A_3 + A_4) \\ &= P(A_1) + P(A_2) + P(A_3) + P(A_4) - P(A_1A_2) - P(A_1A_3) - P(A_1A_4) - P(A_2A_3) - P(A_2A_4) \\ &\quad - P(A_3A_4) + P(A_1A_2A_3) + P(A_1A_2A_4) + P(A_1A_3A_4) + P(A_2A_3A_4) - P(A_1A_2A_3A_4). \end{aligned} \tag{2.28}$$

But

$$\begin{aligned} A_1A_2 &= x_1x_2x_3x_4 \\ A_1A_3 &= x_1x_2x_3x_4 \\ A_1A_4 &= x_1x_2x_3x_4 \\ A_2A_3 &= x_1x_2x_3x_4 \\ A_2A_4 &= x_1x_2x_3x_4 \\ A_3A_4 &= x_1x_2x_3x_4 \\ A_1A_2A_3 = A_1A_2A_4 = A_1A_3A_4 &= A_1A_2A_3A_4 = x_1x_2x_3x_4. \end{aligned}$$

Substitution in Equation 2.28 yields

$$\begin{aligned} R &= P(x_1x_2x_3) + P(x_1x_2x_4) + P(x_1x_3x_4) + P(x_2x_3x_4) \\ &\quad - 6P(x_1x_2x_3x_4) + 4P(x_1x_2x_3x_4) - P(x_1x_2x_3x_4) \\ &= P(x_1x_2x_3) + P(x_1x_2x_4) + P(x_1x_3x_4) + P(x_2x_3x_4) - 3P(x_1x_2x_3x_4). \end{aligned}$$

If the units are independent and identical, then $R = 4p^3 - 3p^4$.

The above expression can also be obtained using Equation 2.26:

$$\begin{aligned} R &= \sum_{r=3}^{4} \binom{4}{r} p^r (1-p)^{4-r} \\ &= \binom{4}{3} p^3 (1-p) + \binom{4}{4} p^4 (1-p)^0 \end{aligned}$$

or

$$R = 4p^3 - 3p^4.$$ ■

2.8 RELIABILITY OF *K*-OUT-OF-*N* BALANCED SYSTEMS

In Section 2.7, we presented the general case of k-out-of-n system. There are cases when one unit fails another, which is arranged in some arrangement as explained later, is forced to shut down. We refer to this system as a k-out-of-n balanced system. Brown and Hirata (2001) and Brown et al. (2000) describe the descent systems for future crewed missions to Mars. Descent system dynamics require that if one of the engines in a pair fails while landing, the opposite engine in the pair must be shut off to maintain vehicle balance. Clearly this descent system must have even number of engines. Sarper (2005) considers two systems: four-engine descent system configuration and six-engine descent system configuration. Figure 2.11 shows a four-engine configuration. If an engine in the pair E1–E3 fails, the second one is forced down. Likewise, if an engine in the pair E2–E4 fails, the second one is forced down. The same engine balancing procedure is applicable for six-engine, eight-engine, or similarly configured descent systems.

We now estimate the reliability of such systems by using four-engine system as an example. We assume that all engines are identical and the probability that an engine functions properly during the mission time is p.

We define the status of the engine X_i as

$$X_i = \begin{cases} 1 & \text{if the } i\text{th engine functions properly} \\ 0 & \text{if it fails.} \end{cases}$$

Thus, $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$. We define the system reliability R, probability that the descent system descends successfully, as

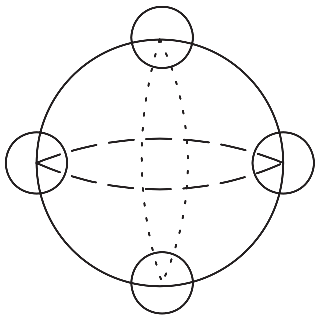

FIGURE 2.11 A four-engine descent system.

$$Y = \begin{cases} 1 & \text{if the system descends successfuly} \\ 0 & \text{if it fails.} \end{cases}$$

We also define the successful operation of engine pair i as $X^{(i)}$. Thus, the success of any pair is $P(X^{(i)} = 1) = p^2$ and $P(X^{(i)} = 0) = 1 - p^2$. The reliability of the four-engine system is the probability that at least one pair of engines is working during the mission. The probability of the failure of the two pairs (system unreliability) is

$$P(Y = 0) = P(X^{(1)} = 0) \times P(X^{(2)} = 0) = (1 - p^2)^2.$$

Thus, the reliability of the 2-out-of-4 balanced system is

$$R_b = 1 - P(Y = 0) = 1 - (1 - p^2)^2 = 2p^2 - p^4.$$

Clearly, the reliability of this system is lower than the reliability of the general k-out-of-n system where the reliability is defined as the probability that at least two out four engines are functioning properly. Consider the same four-engine descent system, the reliability of the general 2-out-of-4 is

$$R_g = 6p^2 - 8p^3 + 3p^4.$$

The difference $R_g - R_b$ is positive for any p value except when $p = 1$ where the two reliabilities are equal (Fig. 2.12).

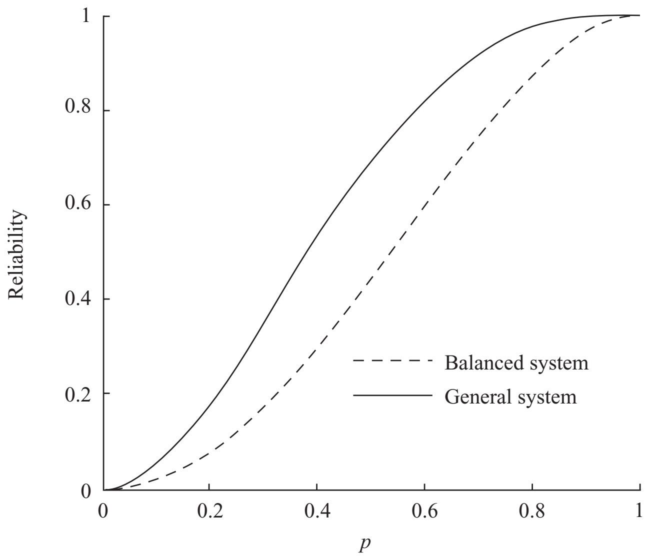

FIGURE 2.12 Reliability of balanced and general system.

2.9 COMPLEX RELIABILITY SYSTEMS

Telecommunication systems, computer networks, electric power utility systems, and water utility distribution systems are typical examples of complex networks. Some of the networks are referred to as *directed* networks when the flow from one node to another is unidirectional. When the flow is bidirectional we refer to the network as *undirected*. The examples, procedures, and problems discussed in this chapter are applicable to both the undirected and directed networks.

Consider the network shown in Figure 2.13. This network is a more complex system than those presented earlier in this chapter since it cannot be modeled (or is difficult to model)

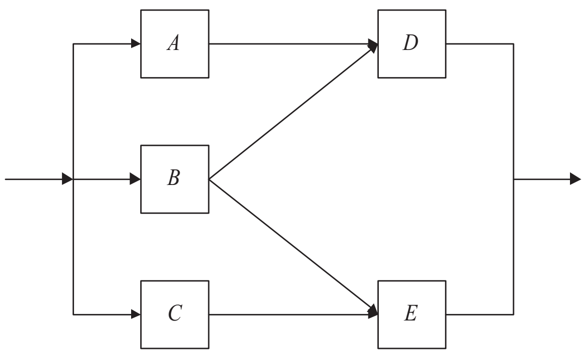

FIGURE 2.13 A complex reliability system.

as series, parallel, parallel-series, series-parallel, or k-out-of-n systems. The reliability of such systems can be determined using any of the following methods (Shooman, 1968).

2.9.1 Decomposition Method

This method begins by selecting a *keystone* component, x, which appears to link (bind) together the reliability structure of the system. The reliability may then be expressed in terms of the keystone component based on the theorem of total probability as follows:

$$R = P(\text{system good}|x)P(x) + P(\text{system good}|\bar{x})P(\bar{x}), \tag{2.29}$$

where $P(\text{system good}|x)$ is the probability that the system is functioning given that x is functioning, and $P(\text{system good}|\bar{x})$ is the probability that the system is functioning given that x is not functioning. Obviously, the choice of the keystone component has a direct effect on the necessary calculations for $P(\text{system good}|x)$ and $P(\text{system good}|\bar{x})$. An experienced engineer should be able to identify the keystone components. Nevertheless, if a component is selected as a keystone component, when in fact it is not, we still can determine the reliability of the system with little or no difficulty but using extra steps and calculations as shown later.

EXAMPLE 2.15

Determine the reliability of the network shown in Figure 2.13 when Component B is selected as the keystone component.

SOLUTION

In this case, B is the keystone component and the reliability of the network can be determined as

$$R = P(\text{system good}|B)P(B) + P(\text{system good}|\bar{B})P(\bar{B}). \tag{2.30}$$

Now, we estimate $P(\text{system good}|B)$ by determining the working paths in the network when B is functioning as shown in Figure 2.14. Similarly, the $P(\text{system good}|\bar{B})$ is obtained using the block diagram shown in Figure 2.15.

$$P(\text{system good}|B) = P(D) + P(E) - P(D)P(E) \tag{2.31}$$

$$P(\text{system good}|\bar{B}) = P(A)P(D) + P(C)P(E) - P(A)P(D)P(C)P(E) \tag{2.32}$$

Substituting Equations 2.31 and 2.32 into Equation 2.30, we obtain

$$\begin{aligned} R = {} & [P(D) + P(E) - P(D)P(E)]P(B) \\ & + [P(A)P(D) + P(C)P(E) - P(A)P(D)P(C)P(E)][1 - P(B)]. \end{aligned} \tag{2.33}$$

If all components have equal probabilities (p) of functioning properly, then

$$R = 4p^2 - 3p^3 - p^4 + p^5. \tag{2.34}$$

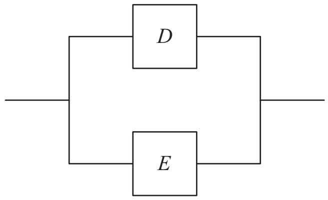

FIGURE 2.14 Block diagram when B is working.

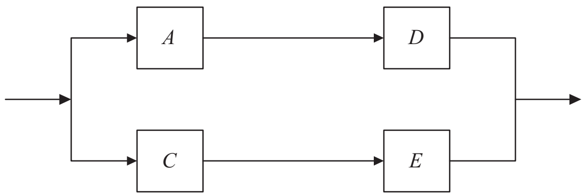

FIGURE 2.15 Block diagram when B fails. ■

EXAMPLE 2.16

Solve Example 2.15 using A as the keystone component.

SOLUTION

$$R = P(\text{system good}|A)P(A) + P(\text{system good}|\bar{A})P(\bar{A}). \tag{2.35}$$

Following Example 2.15, we estimate P(system good|A) using the diagrams shown in Figure 2.16. We start with diagram (a) which is reduced to diagram (b) and finally to graph (c).

Assume that all components are independent and identical. Then

$$\begin{aligned} P(\text{system good}|A) &= 1-(1-p)[1-p(1-(1-p)^2)] \\ &= p + 2p^2 - 3p^3 + p^4. \end{aligned} \tag{2.36}$$

Similarly, we consider the system reliability when Component A fails. The corresponding block diagram is shown in Figure 2.17. The diagram in Figure 2.17 is still complex and does not decompose into series/parallel arrangements. Therefore, we choose another keystone Component C, and the block diagram in Figure 2.17 can be redrawn as shown in Figure 2.18 to represent a subsystem of the main network.

$$P(\text{system good}|C) = 1-(1-p)(1-p^2). \tag{2.37}$$

$$P(\text{system good}|\bar{C}) = p(1-(1-p)^2). \tag{2.38}$$

The reliability of the subsystem is the same as P(system good|$\bar{A}$) and is obtained by using Equations 2.37 and 2.38 as follows:

$$\begin{aligned} P(\text{system good}|\bar{A}) &= [1-(1-p)(1-p^2)]p + p(1-(1-p)^2)(1-p) \\ &= 3p^2 - 2p^3. \end{aligned} \tag{2.39}$$

Substituting Equations 2.36 and 2.39 into Equation 2.35 we obtain

$$R = (p + 2p^2 - 3p^3 + p^4)p + (3p^2 - 2p^3)(1 - p),$$

which results in

$$R = 4p^2 - 3p^3 - p^4 + p^5. \tag{2.40}$$

$P(\text{system good}|\bar{C})$ is estimated using Figure 2.19.

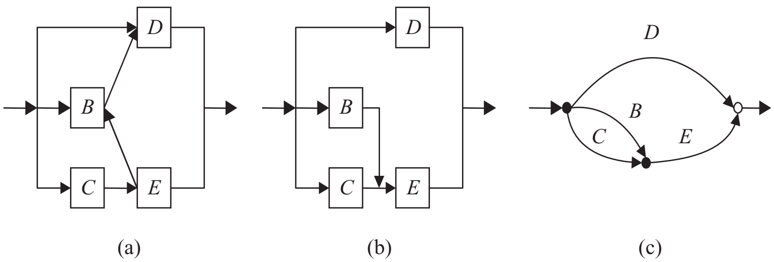

FIGURE 2.16 System diagram when *A* is working.

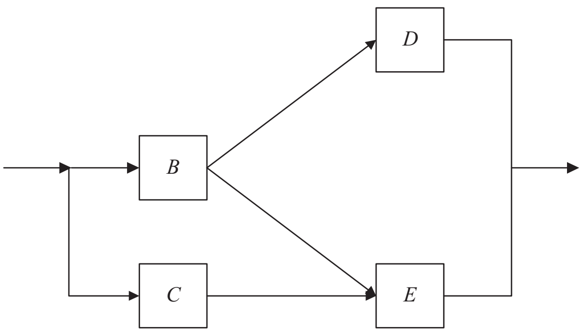

FIGURE 2.17 Block diagram when *A* fails.

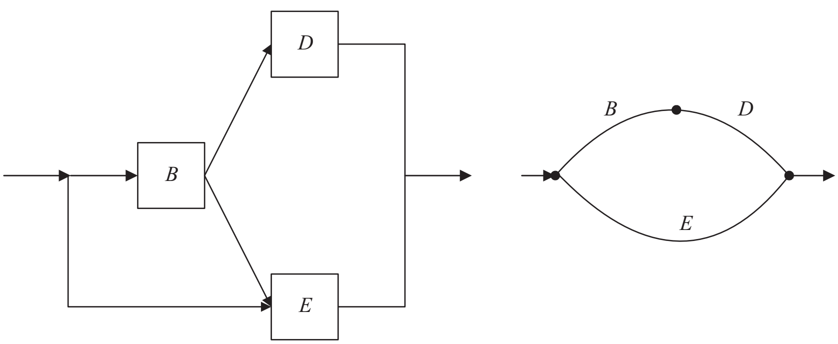

FIGURE 2.18 Block diagram and reliability graph when *C* is working.

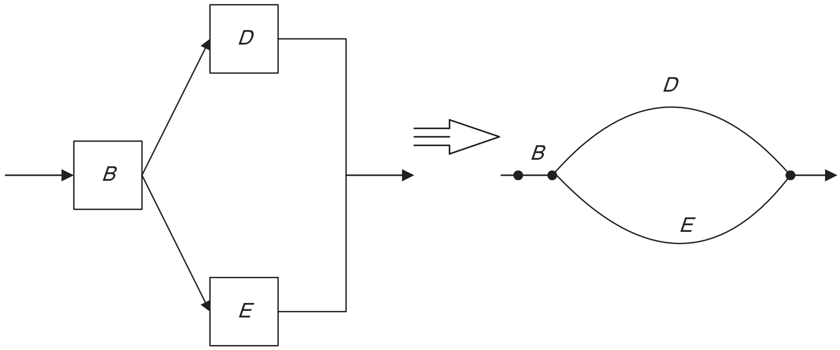

FIGURE 2.19 Subsystem when C fails.

Equations 2.34 and 2.40 are identical. However, much fewer steps are needed to determine system reliability when the keystone component is properly identified. ■

2.9.2 Tie-Set and Cut-Set Methods

The second approach for determining reliability of a complex system is based on the idea of a *tie-set* or a *cut-set*. A tie-set is a complete path through the reliability block diagram. It is not sufficient to determine all tie-sets since some of the tie-sets are contained within others. Therefore, it is important to define the *minimum* tie-set as the tie-set that contains no other tie-sets within it. The reliability of the system is given by the union of all minimum tie-sets.

A cut-set is a set of blocks (components) which interrupts all connections between the input and the output ends when removed from the reliability block diagram. A *minimum* cut-set is the one that contains no other cut-sets within it. The unreliability of the system is given by the probability that at least one minimal cut-set fails.

The following examples illustrate the use of tie-set and cut-set methods for estimating reliability of a complex system.

EXAMPLE 2.17

Consider the system shown in Figure 2.20. Use the tie-set and cut-set methods to estimate the system reliability.

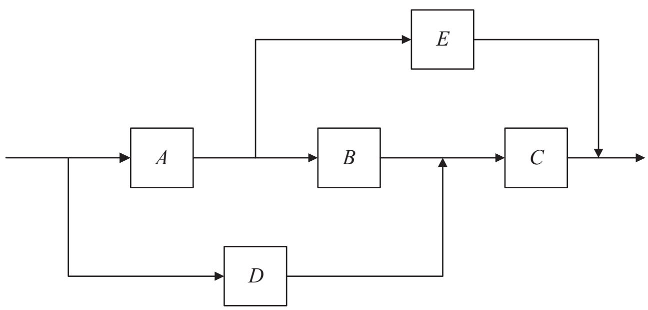

FIGURE 2.20 A complex system.

SOLUTION

The minimum tie-sets of the system are

$$T_1 = AE$$
$$T_2 = DC$$
$$T_3 = ABC.$$

The reliability of the system is the union of the tie-sets:

$$\begin{aligned} R &= P(AE \cup DC \cup ABC) \\ &= P(AE) + P(DC) + P(ABC) - P(AEDC) - P(AEBC) - P(DCAB) + P(AEDCB). \end{aligned} \tag{2.41}$$

Assuming independence of probabilities, then Equation 2.41 can be written as

$$\begin{aligned} R = {} & P(A)P(E) + P(D)P(C) + P(A)P(B)P(C) - P(A)P(E)P(D)P(C) - P(A)P(E)P(B)P(C). \\ & - P(D)P(C)P(A)P(B) + P(A)P(E)P(D)P(C)P(B). \end{aligned} \tag{2.42}$$

If all units are identical and each has a probability p of functioning properly, then Equation 2.42 becomes

$$R = 2p^2 + p^3 - 3p^4 + p^5. \tag{2.43}$$

We can also apply the cut-set method to determine R. The minimum cut-sets are

$$C_1 = \bar{A}\bar{D}$$
$$C_2 = \bar{E}\bar{C}$$
$$C_3 = \bar{A}\bar{C}$$
$$C_4 = \bar{B}\bar{E}\bar{D}.$$

The reliability of the system is

$$R = 1 - P(\bar{A}\bar{D} \cup \bar{E}\bar{C} \cup \bar{A}\bar{C} \cup \bar{B}\bar{E}\bar{D}). \tag{2.44}$$

Again, assuming independence of probabilities, Equation 2.44 becomes

$$\begin{aligned} R = 1 - [& P(\bar{A}\bar{D}) + P(\bar{E}\bar{C}) + P(\bar{A}\bar{C}) + P(\bar{B}\bar{E}\bar{D}) - P(\bar{A}\bar{D}\bar{E}\bar{C}) \\ & - P(\bar{A}\bar{D}\bar{C}) - P(\bar{A}\bar{D}\bar{B}\bar{E}) - P(\bar{E}\bar{C}\bar{A}) - P(\bar{E}\bar{C}\bar{B}\bar{D}) \\ & - P(\bar{A}\bar{C}\bar{B}\bar{E}\bar{D}) + P(\bar{A}\bar{D}\bar{E}\bar{C}) + P(\bar{A}\bar{D}\bar{C}\bar{B}\bar{E}) + P(\bar{A}\bar{D}\bar{C}\bar{B}\bar{E}) + P(\bar{E}\bar{C}\bar{A}\bar{B}\bar{D}) - P(\bar{A}\bar{B}\bar{C}\bar{D}\bar{E})]. \end{aligned}$$

Substituting

$$P(\bar{A}) = [1 - P(A)] \text{ and } P(A) = P(B) = P(C) = P(D) = P(E) = p$$

in the above equation, we obtain

$$R = 2p^2 + p^3 - 3p^4 + p^5 \tag{2.45}$$

Equations 2.43 and 2.45 are identical. ■

2.9.3 Event Space Method

The event-space method is based on listing all possible logical occurrences of the system. In other words, all components are considered functioning initially, and then they are allowed to fail individually, two at a time, three at a time, and so on. The reliability of the system is then determined by the union of all successful occurrences. Clearly, the number of occurrences depends on the number of components in the system. For example, a system with five components and each component can either be working or failing, will have $2^5 = 32$ occurrences. There is only one occurrence with no failure

$$\left\{\binom{5}{0} = 1\right\},$$

and five occurrences containing one failure

$$\left\{\binom{5}{1} = 5\right\},$$

and so on. The following example illustrates the use of the event-space method in estimating the reliability of complex systems.

EXAMPLE 2.18

Determine the reliability of the network given in Figure 2.20 using the event-space method.

SOLUTION

Since there are five blocks in the network, the number of system occurrences is $2^5 = 32$. These occurrences are shown in Table 2.2 (Shooman, 1968). The reliability of the system is the probability of the union of operational occurrences. Thus,

$$R = P(X_1 + X_2 + \cdots + X_7 + X_{10} + \cdots + X_{14} + X_{16} + X_{20} + X_{24}). \tag{2.46}$$

Assuming that all components are disjoint, then, Equation 2.46 can be written as

$$\begin{aligned} R = {} & P(X_1) + P(X_2) + \cdots + P(X_7) + P(X_{10}) + P(X_{11}) \\ & + P(X_{12}) + P(X_{13}) + P(X_{14}) + P(X_{16}) + P(X_{20}) + P(X_{24}). \end{aligned} \tag{2.47}$$

If all components are identical and independent and each has a probability of p of functioning properly, then

$$\begin{aligned} P(X_1) &= P(ABCDE) = p^5 \\ P(X_2) &= P(X_3) = \cdots P(X_6) = (1-p)p^4 \\ P(X_7) &= P(X_{10}) = \cdots = P(X_{14}) = P(X_{16}) = (1-p)^2 p^3 \\ P(X_{20}) &= P(X_{24}) = (1-p)^3 p^2. \end{aligned}$$

And

$$R = p^5 + 5(1-p)p^4 + 7(1-p)^2 p^3 + 2(1-p)^3 p^2$$

or

$$R = 2p^2 + p^3 - 3p^4 + p^5.$$

This is the same result obtained by both tie-set and cut-set methods.

TABLE 2.2 All Possible Logical Occurrences for Figure 2.20

Group 0 (no failures)	$X_1 = ABCDE$
Group 1 (one failure)	$X_2 = \bar{A}BCDE$, $X_3 = A\bar{B}CDE$,
	$X_4 = AB\bar{C}DE$, $X_5 = ABC\bar{D}E$, $X_6 = ABCD\bar{E}$
Group 2 (two failures)	$X_7 = \bar{A}\bar{B}CDE$, $X_8 = \underline{\bar{A}B\bar{C}DE}$,
	$X_9 = \underline{\bar{A}BC\bar{D}E}$, $X_{10} = \bar{A}BCD\bar{E}$,
	$X_{11} = A\bar{B}\bar{C}DE$, $X_{12} = A\bar{B}C\bar{D}E$,
	$X_{13} = A\bar{B}CD\bar{E}$, $X_{14} = AB\bar{C}\bar{D}E$,
	$X_{15} = \underline{AB\bar{C}D\bar{E}}$, $X_{16} = ABC\bar{D}\bar{E}$
Group 3 (three failures)	$X_{17} = \underline{AB\bar{C}\bar{D}\bar{E}}$, $X_{18} = \underline{A\bar{B}C\bar{D}\bar{E}}$,
	$X_{19} = \underline{A\bar{B}\bar{C}D\bar{E}}$, $X_{20} = A\bar{B}\bar{C}\bar{D}E$, $X_{21} = \underline{\bar{A}BC\bar{D}\bar{E}}$,
	$X_{22} = \underline{\bar{A}B\bar{C}D\bar{E}}$, $X_{23} = \underline{\bar{A}B\bar{C}\bar{D}E}$, $X_{24} = \bar{A}\bar{B}CD\bar{E}$,
	$X_{25} = \underline{\bar{A}\bar{B}C\bar{D}E}$, $X_{26} = \underline{\bar{A}\bar{B}\bar{C}DE}$
Group 4 (four failures)	$X_{27} = \underline{A\bar{B}\bar{C}\bar{D}\bar{E}}$, $X_{28} = \underline{\bar{A}B\bar{C}\bar{D}\bar{E}}$, $X_{29} = \underline{\bar{A}\bar{B}C\bar{D}\bar{E}}$,
	$X_{30} = \underline{\bar{A}\bar{B}\bar{C}D\bar{E}}$, $X_{31} = \underline{\bar{A}\bar{B}\bar{C}\bar{D}E}$
Group 5 (five failures)	$X_{32} = \underline{\bar{A}\bar{B}\bar{C}\bar{D}\bar{E}}$

Note: Underlined occurrence implies failure of the system. ■

2.9.4 Boolean Truth Table Method

This method is based on the construction of a Boolean truth table for the system. This method is tedious if done manually, but computer software can make it possible to construct large truth tables in a relatively small amount of time. A truth table is similar to the event-space method where every possible state of the system is listed. A state refers to the condition of a component as functioning or not. We create a column in the table for each component and a value of 1 or 0 is assigned to the column to indicate that the component is functioning or not, respectively. Each row in the table will then represent a state of the system. Each row is examined to determine the state of the system as functioning or not. This is indicated by assigning 1 or 0 to the system state column. The state probability for every functioning row is computed, and the reliability of the system is obtained by adding all functioning state probabilities.

EXAMPLE 2.19

Use the Boolean Truth Table Method to obtain the reliability of the system given in Figure 2.20.

SOLUTION

We construct the Boolean truth table for the system as shown in Table 2.3.

The reliability of the system is obtained by adding the probabilities of functioning states. Assume that the components are independent, identical, and have the same probability p of operating properly. The reliability is

$$R = p^5 + 5p^4(1-p) + 7p^3(1-p)^2 + 2p^2(1-p)^3$$

or

$$R = 2p^2 + p^3 - 3p^4 + p^5. \tag{2.48}$$

As shown above, the reliability obtained using the Boolean truth table is the same as that obtained using the tie-set and cut-set methods.

TABLE 2.3 Boolean Truth Table for Example 2.19

A	B	C	D	E	System state	State probability
1	1	1	1	1	1	$P(A)P(B)P(C)P(D)P(E)$
1	1	1	1	0	1	$P(A)P(B)P(C)P(D)P(\bar{E})$
1	1	1	0	1	1	$P(A)P(B)P(C)P(\bar{D})P(E)$
1	1	1	0	0	1	$P(A)P(B)P(C)P(\bar{D})P(\bar{E})$
1	1	0	1	1	1	$P(A)P(B)P(\bar{C})P(D)P(E)$
1	1	0	1	0	0	
1	1	0	0	1	1	$P(A)P(B)P(\bar{C})P(\bar{D})P(E)$
1	1	0	0	0	0	
1	0	1	1	1	1	$P(A)P(\bar{B})P(C)P(D)P(E)$
1	0	1	1	0	1	$P(A)P(\bar{B})P(C)P(D)P(\bar{E})$
1	0	1	0	1	1	$P(A)P(\bar{B})P(C)P(\bar{D})P(E)$
1	0	1	0	0	0	
1	0	0	1	1	1	$P(A)P(\bar{B})P(\bar{C})P(D)P(E)$
1	0	0	1	0	0	
1	0	0	0	1	1	$P(A)P(\bar{B})P(\bar{C})P(\bar{D})P(E)$
1	0	0	0	0	0	
0	1	1	1	1	1	$P(\bar{A})P(B)P(C)P(D)P(E)$
0	1	1	1	0	1	$P(\bar{A})P(B)P(C)P(D)P(\bar{E})$
0	1	1	0	1	0	

(*Continued*)

TABLE 2.3 (***Continued***)

A	B	C	D	E	System state	State probability
0	1	1	0	0	0	
0	1	0	1	1	0	
0	1	0	1	0	0	
0	1	0	0	1	0	
0	1	0	0	0	0	
0	0	1	1	1	1	$P(\bar{A})P(\bar{B})P(C)P(D)P(E)$
0	0	1	1	0	1	$P(\bar{A})P(\bar{B})P(C)P(D)P(\bar{E})$
0	0	1	0	1	0	
0	0	1	0	0	0	
0	0	0	1	1	0	
0	0	0	1	0	0	
0	0	0	0	1	0	
0	0	0	0	0	0	

■

2.9.5 Reduction Method

The reduction method is based on the standard Boolean truth table method and then applying the resulting mutually exclusive sum-of-products (s-o-p) terms (Case, 1977). The procedure starts by constructing a 2^n truth table (n is he number of components in the system). Each row in the table is then examined, and rows resulting in a system success (functioning properly) are indicated. A reduction table is then constructed by listing all success rows in Column 1. By a comparative process, product terms are formed for those terms in Column 1, which differ by a letter inverse. Once a term is used in a comparison, it is eliminated from all further comparisons ensuring that all remaining terms are still mutually exclusive. This procedure is repeated until no further comparisons are possible. The reliability of the system is the union of all terms that cannot be further compared.

It is important to note that the order of terms selected for the comparison process has no effect on the estimation of system reliability.

EXAMPLE 2.20

Use the reduction method to determine the reliability of the system given in Figure 2.20.

SOLUTION

We utilize the results obtained in Table 2.3. The functioning states of the system are listed under column 1 in Table 2.4.

The probability of the system is obtained by the union of all the states that cannot be further combined:

$$R = P\left(A\bar{B}C\bar{D}E + A\bar{B}CD + ABC + A\bar{C}E + \bar{A}CD\right)$$
$$= p^3(1-p)^2 + p^3(1-p) + p^3 + p^2(1-p) + p^2(1-p)$$

or

$$R = 2p^2 + p^3 - 3p^4 + p^5. \tag{2.49}$$

The reliability obtained by Equation 2.49 is the same as that obtained by other methods.

TABLE 2.4 Reduction Table for Figure 2.20

Column 1 functional states	Column 2	Column 3
$ABCDE$	$ABCD$	
$ABCD\overline{E}$		ABC
$ABC\overline{D}E$	$ABC\overline{D}$	
$ABC\overline{D}\overline{E}$		
$AB\overline{C}DE$	$AB\overline{C}E$	$A\overline{C}E$
$AB\overline{C}\overline{D}E$		
$A\overline{B}CDE$	$A\overline{B}CD$	
$A\overline{B}CD\overline{E}$		
$A\overline{B}C\overline{D}E$		
$A\overline{B}\overline{C}DE$	$A\overline{B}\overline{C}E$	
$A\overline{B}\overline{C}\overline{D}E$		
$\overline{A}BCDE$	$\overline{A}BCD$	
$\overline{A}BCD\overline{E}$		$\overline{A}CD$
$\overline{A}\overline{B}CDE$	$\overline{A}\overline{B}CD$	
$\overline{A}\overline{B}CD\overline{E}$		

■

2.9.6 Path-Tracing Method

The path-tracing method is simple and efficient in estimating the reliability of complex structures. The method starts by assuming that all blocks in the reliability diagram are missing initially, and the components are replaced singly, in pairs, in triplets, and so on. The successful paths found by using the least number of components are then used in calculating system reliability as shown below.

EXAMPLE 2.21

Use the path-tracing method to determine the reliability of the system given in Figure 2.20.

SOLUTION

As shown in the block diagram, no single component forms a successful path by itself, but the pairs *AE* and *DC* and the triplet components *ABC* form successful paths. The reliability of the system is then obtained by the probability of the union of these paths:

$$R = P(AE) + P(DC) + P(ABC) - P(AEDC) - P(AEBC) - P(DCAB) + P(ABCDE). \tag{2.50}$$

If all components are independent and identical and each has a probability p of functioning properly, then Equation 2.50 becomes

$$R = 2p^2 + p^3 - 3p^4 + p^5,$$

which is the same as the reliability estimated by other methods. ■

2.9.7 Factoring Algorithm

The complex structures presented in this chapter are simple and limited when compared with large-scale structures such as computer and telephone communication networks, electric power utility networks, and others. Reliability estimation of such networks is, in a sense, more difficult than many standard combinatorial optimization problems. However, researchers have developed algorithms, which can efficiently estimate the reliability of networks with specified characteristics. In this section, we present one of these algorithms, namely, the factoring algorithm.

Consider a complex structure that is represented by a reliability network or graph (a *graph* is a pictorial representation of the network). A typical graph consists of nodes and arcs where a node represents a location (or a point), which communicates with other nodes via arcs. An arc can represent a means of communication between the nodes, for example, components, cables, and pipes.

The factoring algorithm is based on the decomposition method discussed in Section 2.8.1. Following Equation 2.35, if $R(G \mid e)$ is the reliability of graph G under the condition that component e (arc or edge e of the graph) is working and $R(G \mid \overline{e})$ is the reliability of G under the condition that component e is not working, then the reliability of the graph G is

$$R(G) = p_e R(G \mid e) + (1 - p_e) R(G \mid \overline{e}), \tag{2.51}$$

where p_e is the reliability of component (edge) e.

The reliability of any graph G can be computed by repeated application of Equation 2.51. Undirected graphs have some special properties that can be used to simplify this method. If the vertices (nodes) are assumed to be working, then $R(G \mid e)$ coincides with $R(G_e)$ where G_e is the graph obtained from G by deleting edge e and merging its end points. Similarly, $R(G \mid \overline{e})$ equals $R(G - e)$, where $G - e$ is the graph with e deleted, and no vertex is deleted. It is important to note that this factoring algorithm can be employed using graph representation, but without knowing the minimal path sets. Moreover, unless some kind of probability reductions are performed (e.g., parallel and series reductions) after the deletion of an edge, the factoring algorithm will be equivalent to state space enumeration (Agrawal and Barlow, 1984). We now illustrate the use of the factoring algorithm in estimating network reliability.

EXAMPLE 2.22

In large cities, gas is produced by vaporizing liquefied natural gas (LNG). The quality of the gas is checked for calorific values, combustibility, and other characteristics. The gas is then sent out through transmission pipelines to distribution centers where gas pressure is decreased before delivery to customers. Figure 2.21 shows a simplified network of a gas distribution system. Node A represents the location at which gas is vaporized, nodes B, C, D, E, and F are major distribution centers where gas is received from A (directly or indirectly). Gas must reach the distribution center F since it provides gas to critical services of the city. Thus, the reliability of the network is the probability that gas sent from node A reaches the distribution center F. Assume that the reliability of every transmission pipe is p. Use the factoring algorithm to determine the reliability of the network.

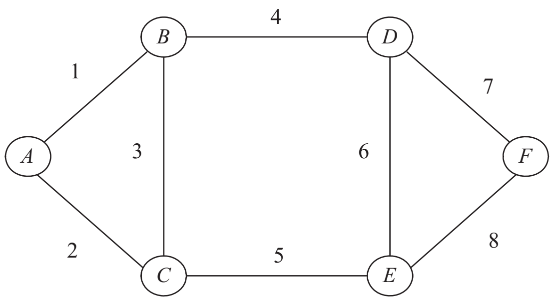

FIGURE 2.21 Simplified network of the gas distribution system.

SOLUTION

We use the operator $\oplus$ to correspond to calculating the reliability of parallel pipelines as

$$p_i \oplus p_j = p_i + p_j - p_i p_j.$$

The initial step in applying the algorithm is to select an arc, say arc 1, and form two subgraphs: G_1 corresponding to arc 1 working, and $G - 1$ corresponding to arc 1 failed. We now apply a parallel probability reduction by replacing the two arcs 2 and 3 in graph G_1 by a single arc with associated reliability $p_2 + p_3 - p_2p_3$. Likewise in G_1, this new arc and arc 5 form a series system; these two arcs can now be replaced by a single arc having reliability $(p_2 + p_3 - p_2p_3)p_5$. The factoring algorithm now proceeds by considering arc 4, which results in two additional subgraphs, each of which can be reduced to a single arc by series and parallel probability reduction. The algorithm continues until no further reductions can be made. Figure 2.22 shows the steps of the algorithm as described above (Agrawal and Barlow, 1984).

Using Equation 2.51 and the four subgraphs at the bottom of Figure 2.22 we obtain the reliability of the network as

$$
\begin{aligned}
R(G) &= p^2(((((p\oplus p)p)\oplus p)p)\oplus p) + p(1-p)(((p\oplus p)p)(p^2\oplus p)) \\
&\quad + p(1-p)((p(p\oplus p))\oplus p^2)p + (1-p)^2((p^3\oplus p)p^2) \\
&= (p^3+p^4+p^5-5p^6+4p^7-p^8)+(2p^4-p^5-4p^6+4p^7-p^8) \\
&\quad +(3p^4-4p^5-p^6+3p^7-p^8)+(p^3-2p^4+2p^5-3p^6+3p^7-p^8)
\end{aligned}
$$

or

$$R(G) = 2p^3+4p^4-2p^5-13p^6+14p^7-4p^8. \tag{2.52}$$

Derivations of $R(G)$ for a network whose pipelines have different reliability are straightforward and are similar to the above derivations.

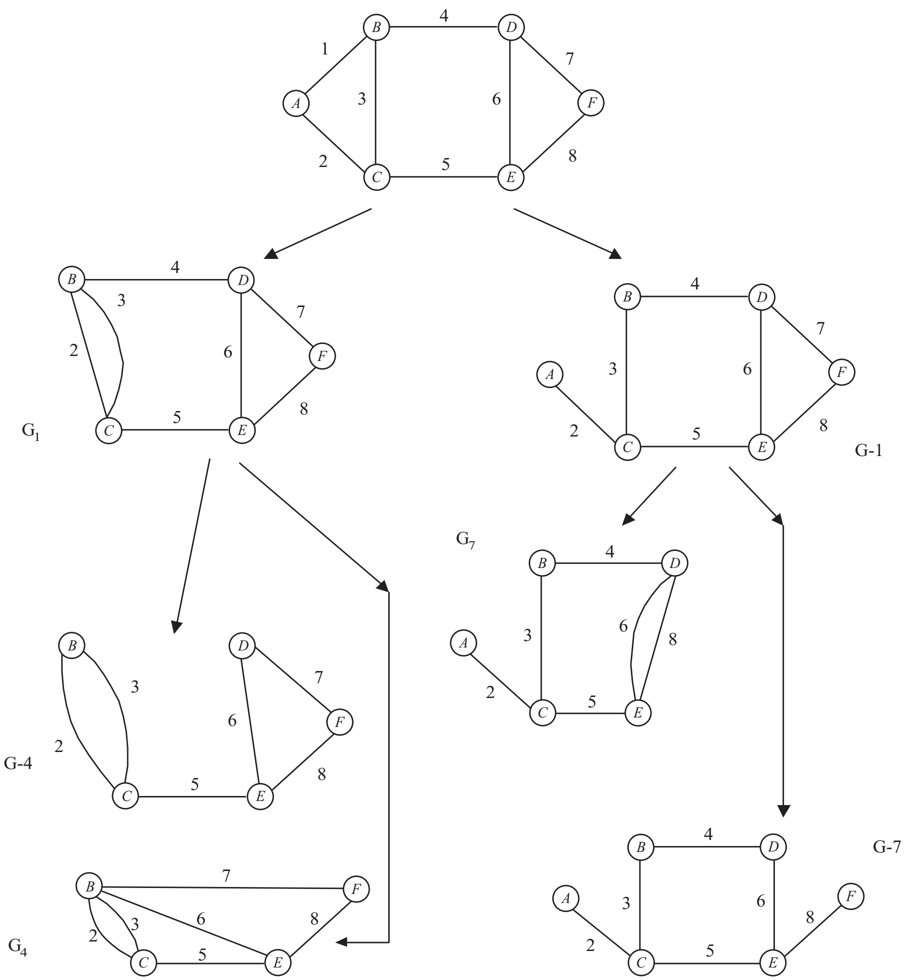

FIGURE 2.22 Formation of subgraphs. ■

2.10 SPECIAL NETWORKS

In many cases, the network may exhibit repeated patterns, but the number of arcs and nodes increases as the size of the network increases. Such networks exist in telecommunication systems. They provide fast and efficient interconnections and redundant paths in the system. These networks include Omega networks, binary n-cube, shuffle, Delta and Banyan networks which are considered full 2×2 crossbar networks. The network has n switches and several stages, and each switch (with the exception of those at the first and last stage) has two inputs and two outputs. Other networks such as Gamma networks are full 3×3 crossbar which means that in all but the first and last stage, each switching element has three inputs and three outputs. Switching elements at other stages have three possible interconnections determined by stage number.

Since such networks have repeated patterns, then the reliability of the network can be estimated using recursive equations. This is demonstrated by the ladder network (Parker and Raghavendra, 1984) in Example 2.23.

EXAMPLE 2.23

Consider a ladder network which has 10 switches (nodes). The network is a directed flow network as shown in Figure 2.23. Assume that every switch has a probability p of operating properly. Determine the reliability of the network.

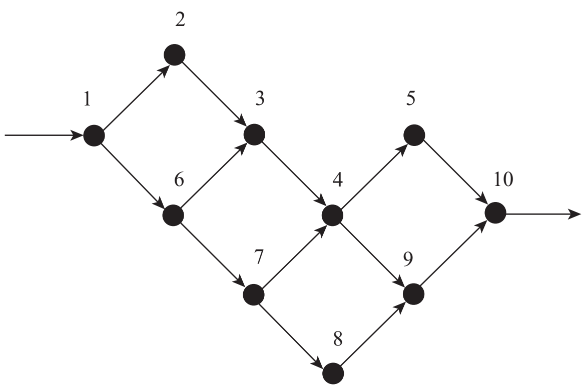

FIGURE 2.23 Ladder network with 10 switches.

SOLUTION

The network is shown in Figure 2.23.

Using a recurrence relationship for reliability, we obtain the reliability of the network as

$$R = R(1 \rightarrow 10)R(10) = R(1 \rightarrow 10)p,$$

where $R(1 \rightarrow 10)$ is the reliability of the network from switch 1 to switch 10. Following the above expression we obtain

$$
\begin{aligned}
R(1 \to 10) &= R(1 \to 9)R(9) + R(1 \to 4)R(4)R(5)(1 - R(9)) \\
R(1 \to 9) &= R(1 \to 4)R(4) + R(1 \to 7)R(7)R(8)(1 - R(4)) \\
R(1 \to 7) &= R(1)R(6) = p^2 \\
R(1 \to 4) &= R(1 \to 3)R(3) + R(1 \to 6)R(6)R(7)(1 - R(3)) \\
R(1 \to 3) &= 2p^2 - p^3 \quad \text{See reliability of parallel sytems} \\
R(1 \to 6) &= p \\
R(1 \to 4) &= (2p^2 - p^3)p + p^3(1 - p) = 3p^3 - 2p^4 \\
R(1 \to 9) &= 4p^4 - 3p^5 \\
R(1 \to 10) &= 7p^5 - 8p^6 + 2p^7
\end{aligned}
$$

Therefore, the reliability of the ladder network is $R = 7p^6 - 8p^7 + 2p^8$. ■

As shown through many of the examples presented in this chapter, all methods will produce the same reliability estimate, but in any individual case, one method may be considerably more convenient to apply. Of course, this depends on the structure of the block diagram. In fact, it may be more convenient to use different techniques for estimating the reliability of different parts of the same block diagram.

2.11 MULTISTATE MODELS

So far, we have assumed that a component can be in either one of two states, operational or failure. In many situations, a component may experience more than two states, for example, a three-state component may operate properly in its normal mode but may fail in either of two failure modes. Typical examples of three-state components are transistors and diodes. A transistor may operate properly or fail open or short. A *diode* is a device that passes current in the forward direction and blocks current in the reverse direction. When operating properly, the resistance in the forward direction is zero, whereas the resistance in the reverse direction is essentially infinite. The diode may operate properly or may fail in either state: (1) it may open circuit, that is, resistance in both directions is infinite, or (2) it may short-circuit, that is, resistance in both directions is zero. The same applies to mechanical systems when a three-way valve may operate properly or fails to close or open to allow flow in the proper direction.

There has been an increased interest in modeling and assessment of multistate reliability systems. They range from modeling multistate components to complex systems with multistate units (Lisnianski and Levitin, 2003; Lisnianski et al., 2010). The objective of this section is to explain the modeling of multistate devices and systems as well as highlight the fact that some of the rules such as the more redundancy the more reliable the system do not hold in the multistate case.

As mentioned earlier, redundancy is one of the means of increasing system reliability. Increasing the number of redundant components in a system whose components have only two states (operational or failure) increases the reliability of the system. Unlike the two-state components, adding multistate components may either increase or decrease the system reliability. This, of course, depends on the dominant mode of component failure, configuration of the system, and the number of redundant components (Dhillon and Singh, 1981).

In the following sections, we present reliability expressions for different system configurations composed entirely of multistate components. We also present methods for the determination of the optimum number of components in the system that achieves the highest levels of reliability.

2.11.1 Series Systems

This section considers components that have three states: x (good), $\bar{x}_s$ (fails short), $\bar{x}_o$ (fails open). In a series configuration of n three-state components, the system fails if any component fails in an open mode, whereas all components must fail in the short mode for the system to fail. Terms are defined as follows:

$\bar{x}_{si}$	=	the short-mode failure of component i,
$\bar{x}_{oi}$	=	the open-mode failure of component i,
x_i	=	the operating mode of component i,
n	=	the number of nonidentical but independent three-state components,
q_{si}	=	the probability of short-mode failure of component i, and
q_{oi}	=	the probability of open-mode failure of component i.

The reliability of a system composed of a one three-state component is

$$R = P(x_1) = 1 - P(\bar{x}_{o1}) - P(\bar{x}_{s1})$$

or

$$R = (1 - q_{o1}) - q_{s1}. \tag{2.53}$$

Consider now a system composed of two three-state components in series. Its reliability is obtained as

$$\begin{aligned} R &= 1 - P(\text{system failure}) \\ &= 1 - P(\bar{x}_{o1} + \bar{x}_{o2} + \bar{x}_{s1}\bar{x}_{s2}) \\ &= 1 - [P(\bar{x}_{o1}) + P(\bar{x}_{o2}) - P(\bar{x}_{o1}\bar{x}_{o2}) + P(\bar{x}_{s1}\bar{x}_{s2})] \end{aligned}$$

or

$$R = 1 - [(q_{o1} + q_{o2} - q_{o1}q_{o2}) + q_{s1}q_{s2}]. \tag{2.54}$$

Rewriting Equation 2.54, we obtain

$$R = \prod_{i=1}^{2}(1 - q_{oi}) - q_{s1}q_{s2}. \tag{2.55}$$

By induction from Equations 2.53 and 2.55, the reliability of n components system is

$$R = \prod_{i=1}^{n} (1 - q_{oi}) - \prod_{i=1}^{n} q_{si}. \tag{2.56}$$

If all components are independent and identical, then Equation 2.56 becomes

$$R = (1 - q_o)^n - q_s^n. \tag{2.57}$$

Unlike the standard series system with identical two-state components, the reliability of series systems with identical three-state components will reach its maximum by connecting an optimum number of components. Any number of components less or greater than the optimum will result in lower reliability values. To obtain the optimum number of three-state components in series that maximizes the reliability of the system, we take the derivative of Equation 2.57 with respect to n and equate it to zero. Then we solve the resultant equation to determine the optimum number (n^*) of components. Thus,

$$\frac{\partial R}{\partial n} = (1 - q_o)^n \ln(1 - q_o) - q_s^n \ln q_s = 0,$$

or

$$n^* = \frac{\ln[\ln q_s / \ln(1 - q_o)]}{\ln[(1 - q_o)/q_s]}. \tag{2.58}$$

If n^* is not an integer, then $\lfloor n^* \rfloor$ and $\lfloor n^* \rfloor + 1$ are also optimum solutions. Note that $\lfloor n^* \rfloor$ is the largest integer less than or equal to n^*.

2.11.2 Parallel Systems

We now consider a parallel system composed of two components connected in parallel. Using the same notations given in Section 2.11.1, we derive the reliability of the system as

$$\begin{aligned} R &= 1 - P(\bar{x}_{o1}\bar{x}_{o2} + \bar{x}_{s1} + \bar{x}_{s2}) \\ &= 1 - [P(\bar{x}_{o1})P(\bar{x}_{o2}) + P(\bar{x}_{s1}) + P(\bar{x}_{s2}) - P(\bar{x}_{s1})P(\bar{x}_{s2})] \\ &= 1 - [q_{o1}q_{o2} + q_{s1} + q_{s2} - q_{s1}q_{s2}], \end{aligned}$$

or

$$R = \prod_{i=1}^{2} (1 - q_{si}) - \prod_{i=1}^{2} q_{oi}. \tag{2.59}$$

Equation 2.59 can be generalized for systems with n components in parallel as

$$R = \prod_{i=1}^{n} (1 - q_{si}) - \prod_{i=1}^{n} q_{oi}. \tag{2.60}$$

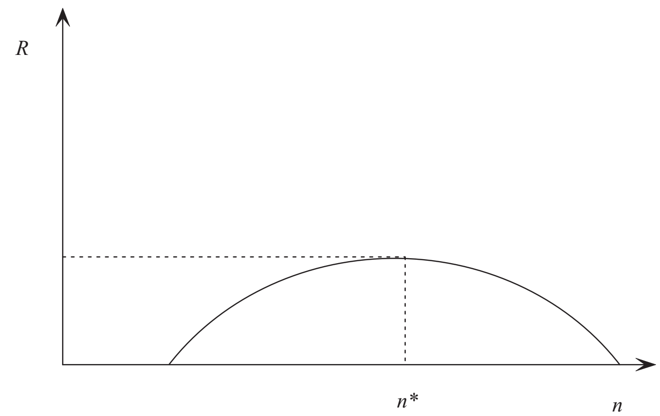

FIGURE 2.24 System reliability versus number of parallel components.

If all components are identical, the reliability of the system becomes

$$R = (1 - q_s)^n - q_o^n. \tag{2.61}$$

For any range of q_o and q_s the optimum number of parallel components that maximizes system reliability is one if $q_s > q_o$. For most practical values of q_o and q_s, the optimum number is two (Von Alven, 1964). In general, for a given q_o and q_s, the reliability function in terms of n would have the form shown in Figure 2.24. Therefore, we take the derivative of Equation 2.61 with respect to n and equate the resultant to zero to find the optimum number of components:

$$\frac{\partial R}{\partial n} = \frac{\partial[(1 - q_s)^n - q_o^n]}{\partial n}$$

$$0 = (1 - q_s)^n \ln(1 - q_s) - q_o^n \ln q_o$$

or

$$n^* = \frac{\ln\left[\dfrac{\ln q_o}{\ln[(1 - q_s)}\right]}{\ln[(1 - q_s)/q_o]}. \tag{2.62}$$

Again, if n^* is not an integer, then $\lfloor n^* \rfloor$ and $\lfloor n^* \rfloor + 1$ are also optimum solutions.

2.11.3 Parallel-Series and Series-Parallel

2.11.3.1 Parallel-Series Consider a parallel-series system that consists of four components as shown in Figure 2.25. The components in the same path are identical. The system is considered to be properly functioning if (1) at least one path has no open mode failures; and (2) each path has less than two shorts.

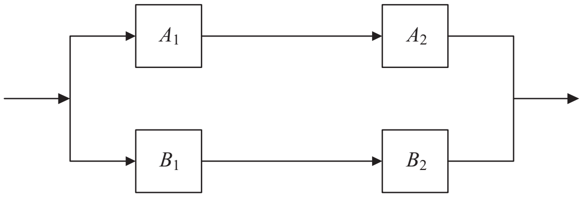

FIGURE 2.25 A parallel-series system.

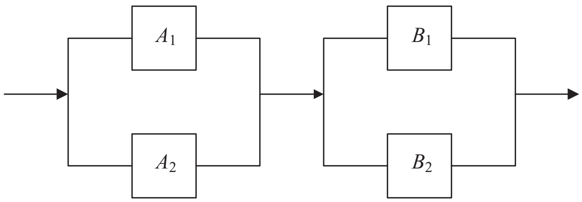

FIGURE 2.26 A series-parallel system.

The reliability of the system when $A_1 = B_1$ (denoted as 1) and $A_2 = B_2$ (denoted as 2) can then be obtained as

$$R = [1 - q_{s1}q_{s2}]^2 - [1 - (1 - q_{o1})(1 - q_{o2})]^2.$$

For m identical parallel paths each containing n elements in series,

$$R = \left[1 - \prod_{i=1}^{n} q_{si}\right]^m - \left[1 - \prod_{i=1}^{n} (1 - q_{oi})\right]^m. \tag{2.63}$$

If all elements are identical, the reliability of the system becomes

$$R = [1 - q_s^n]^m - [1 - (1 - q_o)^n]^m. \tag{2.64}$$

2.11.3.2 Series-Parallel Consider a system as shown in Figure 2.26. This series-parallel system is considered to be functioning properly if (1) both units have less than two open mode failures; and (2) at least one unit has no shorts.

The following reliability expression can easily be derived when $A_1 = B_1$ (denoted as 1) and $A_2 = B_2$ (denoted as 2) can then be obtained as

$$R = [1 - q_{o1}q_{o2}]^2 - [1 - (1 - q_{s1})(1 - q_{s2})]^2. \tag{2.65}$$

For n identical subsystems each containing m components in parallel, the reliability is

$$R = \left[1 - \prod_{i=1}^{m} q_{oi}\right]^n - \left[1 - \prod_{i=1}^{m} (1 - q_{si})\right]^n. \tag{2.66}$$

If all components are identical, the reliability of the system becomes

$$R = [1 - q_o^m]^n - [1 - (1 - q_s)^m]^n. \tag{2.67}$$

To find the optimum configurations for either the parallel-series or series-parallel, set the partial derivatives of R with respect to m and n equal to zero, and then iteratively solve the resulting equations simultaneously for n^* and m^*.

EXAMPLE 2.24

A series system consists of six identical three-state components. The probabilities that a component fails in an open-mode and a short-mode are 0.1 and 0.2, respectively. What is the reliability of the system? What is the optimum number of components that maximizes the system reliability?

$$q_o = 0.1$$
$$q_s = 0.2$$

SOLUTION

Using Equation 2.57, we obtain the reliability of the system as

$$R = (1 - 0.1)^6 - 0.2^6 = 0.53137.$$

The optimum number of components is obtained using Equation 2.58

$$n^* = \frac{\ln[\ln 0.2 / \ln 0.9]}{\ln[0.9/0.2]} = 1.8 \cong 2 \text{ units.}$$

The reliability corresponding to this system is 0.77. ■

EXAMPLE 2.25

Solve Example 2.24 when $q_o = 0.1$, $q_s = 0.2$ and the components are connected in parallel.

SOLUTION

$$q_s = 0.1$$
$$q_o = 0.2$$

From Equation 2.61,

$$R = (1 - 0.1)^6 - 0.2^6 = 0.53137.$$

The optimum number of components in parallel is obtained using Equation 2.62:

$$n^* = \frac{\ln[\ln 0.2 / \ln 0.9]}{\ln[0.9 / 0.2]} \cong 2.$$

The reliability corresponding to this system is 0.77. This is identical to the result obtained in Example 2.24.

In other words, a series system is equivalent to a parallel system if the same number of components is used in both systems and if the values of q_s and q_o are reversed from one system to the other. ■

2.12 REDUNDANCY

Redundancy is defined as the use of additional components or units beyond the number actually required for satisfactory operation of a system for the purpose of improving its reliability. A series system has no redundancy since a failure of any component causes failure of the entire system whereas a parallel system has redundancy since the failure of a component (or possibly more) does not result in a system failure. Similarly, consecutive-*k*-out-of *n:F* system, *k*-out-of-*n*, parallel-series, and series-parallel systems have explicit or implicit redundancy.

In a pure parallel system, redundancy is a function of the number and type of components connected in parallel. As stated earlier in this chapter, if only two-state components are used, then increasing the number of parallel components will increase the reliability of the system. However, if the components have more than two states, then there is an optimum number of components, which maximizes the system reliability. In other words, improving the system reliability through redundancy is not as simple as doubling, tripling, or adding more components in parallel.

There are two types of redundancy: active and inactive. In the *active redundancy*, all redundant components are in operation and are sharing the load with the main unit. Under the *inactive standby*, the redundant components do not share any load with the main components, and they only start operating when one or more operating components fail. When the failure rate of the standby component is the same as the main unit, we refer to this arrangement as *hot* standby. When the failure rate of the standby unit is less than that of the main unit, we then have a *warm* standby, and when the failure rate of the standby unit when it is not operating is zero, then we have a *cold* standby. Clearly, the application of the type of redundancy depends on the criticality of the system and the consequences of a major failure. For example, an airplane which requires two out of three engines for successful operation usually has all its engines in active redundancy whereas a computer system uses an uninterrupted power supply (UPS) in an inactive redundancy to provide the needed power when a failure occurs in the main power source. There is no difference between operating a system under active or inactive redundancy if the switching system (which connects the inactive components to the system) is perfect, that is, does not fail and if the redundant component has the same failure rate whether it is operating or not. The following example illustrates the difference between active and inactive redundancy.

EXAMPLE 2.26

A two-component system may be configured as active or inactive redundancy as shown in Figure 2.27. Assume that the switch S is perfect. What are the reliabilities of both systems?

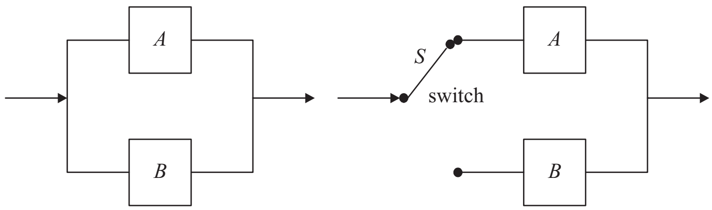

FIGURE 2.27 Active and inactive redundancy.

SOLUTION

The two-component active redundancy system fails only if both components A and B fail. Thus, the reliability of the system is

$$R_{\text{active}} = 1 - P(\bar{A}\bar{B}) = 1 - P(\bar{A})P(\bar{B} \mid \bar{A}). \tag{2.68}$$

If the components A and B are identical and independent, each having reliability p, then

$$R_{\text{active}} = 2p - p^2. \tag{2.69}$$

In the inactive redundancy, the system fails if Component A fails, the perfect switch switches to B, and then B fails. The reliability of the system is

$$R_{\text{inactive}} = 1 - P(\bar{A}\bar{B}) = 1 - P(\bar{A})P(\bar{B} \mid \bar{A}). \tag{2.70}$$

■

It appears that the active and inactive redundancies result in the same value of reliability. However, this is not true, since the interpretations of the conditional probabilities in Equations 2.68 and 2.70 are distinctly different. In Equation 2.68, $P(\bar{B} \mid \bar{A})$ may simply be $P(\bar{B})$ if events $\bar{A}$ and $\bar{B}$ are independent, or it may be slightly different if there is a small dependency. In the active redundancy case, Component B is assumed to have operated since time $t = 0$. In the inactive redundancy $P(\bar{B} \mid \bar{A})$ is always a dependent probability since Component B does not start to operate until A fails. Clearly, this conditional probability is a function of time. Further discussion of redundancy is presented in Chapter 3.

2.12.1 Redundancy Allocation for a Series System

As shown earlier, the reliability of a system composed entirely of two-state components increases by adding components in parallel with the main components of the system. An engineer may be interested in increasing the reliability of an n components series system. In order to do so, the engineer must allocate components in parallel with the main components of the system. We intend to determine the minimum number of redundant components that can be allocated to a series structure so that a given reliability level is achieved.

We utilize a sequential search method proposed by Barlow and Proschan (1965) and described as follows. Let S be the original series structure and S_i be the new structure obtained by doubling component x_i. *Doubling* is defined as placing an identical component in an active redundancy with the component to be doubled. First, use component x_i that maximizes the reliability of S_i. Then, denote the structure obtained by doubling component x_j (after doubling component x_i) as S_{ij}. The component x_j is chosen so that the reliability of S_{ij} is maximal. The process is continued until the desired reliability level is achieved. Choosing a component to be doubled depends on the reliability of the individual components as shown below.

Suppose that the original system S is composed of n components $x_1, x_2, \ldots, x_n$ connected in series and their respective reliabilities are $p_1, p_2, \ldots, p_n$. The reliability of the system is

$$R = p_1, p_2, \ldots, p_n \tag{2.71}$$

If component x_i is doubled, the reliability of the new system is

$$\begin{aligned} R_i &= p_1 p_2 \ldots [1-(1-p_i)^2] \ldots p_n \\ &= p_1 p_2 \ldots p_i (2-p_i) \ldots p_n \\ &= (2-p_i) p_1 p_2 \ldots p_n \end{aligned}$$

or

$$R_i = (2-p_i)R. \tag{2.72}$$

Thus, the reliability R_i is maximum when p_i is minimum. Therefore, doubling the least reliable component results in the largest gain in the reliability of the system. Repeating this reasoning, we either add another component in parallel with x_i or double the least reliable component other than x_i, and so on (Kaufmann et al., 1977).

EXAMPLE 2.27

A series system consists of three components x_1, x_2, and x_3, and their reliabilities are 0.70, 0.75, and 0.85, respectively. Determine the minimal number of components, which can be added in parallel (active redundancy) to the initial components such that the reliability becomes at least 0.82. Note: components used in active redundancy are identical to the components of the original system.

SOLUTION

The original system is shown in Figure 2.28. The reliability of the original system is

$$R = 0.70 \times 0.75 \times 0.85 = 0.4462.$$

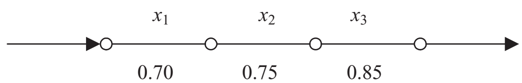

FIGURE 2.28 Original series system.

Now apply the procedure given above by doubling component x_1 using an identical component as shown in Figure 2.29. Then,

$$R_1 = [1-(1-0.70)^2]\times 0.75\times 0.85 = 0.5801.$$

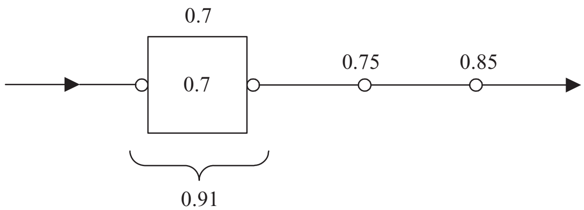

FIGURE 2.29 Doubling component x_1.

The least reliable component in the structure shown in Figure 2.29 is component x_2. Therefore, we choose to double x_2, and the resulting structure is shown in Figure 2.30. The reliability becomes

$$R_{12} = 0.91\times[1-(1-0.75)^2]\times 0.85 = 0.7251.$$

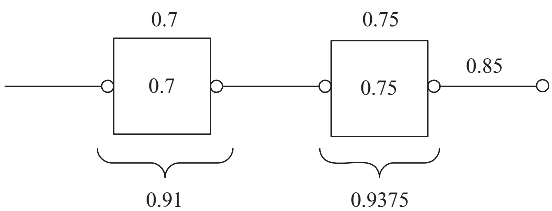

FIGURE 2.30 Doubling component x_2.

Now double the least reliable component, x_3, to get

$$R_{123} = 0.91\times 0.9375\times[1-(1-0.85)^2] = 0.8339.$$

The optimal solution that results in the desired reliability of 0.82 is obtained by using the following active redundancies: x_1 doubled, x_2 doubled, and x_3 doubled. ■

2.13 IMPORTANCE MEASURES OF COMPONENTS

After the reliability and design engineers configure a system which is usually composed of many components, they often face the problem of identifying design weaknesses and component failures that are crucial to the proper functioning of the system. By doing so, the designers may allocate additional resources or redundancy to these components in order to improve the overall system reliability. In this section, we present methods for measuring the importance of the components in terms of the system reliability. In assessing the importance of a component, most of the methods are based on observing the reliabilities (or unreliabilities) of the system when the component is functioning properly and when it is not. These reliabilities, in conjunction with the component reliability, are then algebraically manipulated to obtain different importance measures. To simplify the calculations necessary for these measures, we show how the reliabilities can be observed using the *structure function* of the system.

Consider a system consisting of n components represented by the set $N = (1, 2, \ldots, n)$. The system and the components can be in either of two states, working or not working as denoted by 1 or 0, respectively. The state of the system depends only on the state of its components. Let $X = (X_1, X_2, \ldots, X_n)$ be the random vector representing the state of the components at a given instant of time where X_i is the random variable denoting the state of component i at the given instant of time and $X_i = 1$ or 0 representing that component i is working or not for $i = 1, 2, \ldots, n$. Let $\phi(X)$ be the structure function of the system. Then the random variable $\phi(X)$ denotes the state of the system as $\phi(X) = 1$ when the system is working and $\phi(X) = 0$ when the system is not working (Seth and Ramamurthy, 1991). Obviously, $P[\phi(X) = 1] = E[\phi(X)]$. We shall assume that $(X_1, X_2, \ldots, X_n)$ are independently distributed binary random variables with $P[X_i = 0] = q_i$. In this case $E[\phi(X)]$, is a function of $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$. Let $G(\boldsymbol{q}) = 1 - E[\phi(X)]$, then $G(\boldsymbol{q})$ is called the unreliability (or unavailability) function of the system. This function will now be used in evaluating the structural importance measures discussed below.

2.13.1 Birnbaum's Importance Measure

The Birnbaum reliability importance $I_B^i(t)$ of component i is defined to be the probability that the ith component is critical to the functioning of the system at time t. It can be expressed as

$$I_B^i(t) = \frac{\partial G(\boldsymbol{q}(t))}{\partial q_i(t)} = G(1_i, \boldsymbol{q}(t)) - G(0_i, \boldsymbol{q}(t)) \tag{2.73}$$

or

$$I_B^i(t) \equiv \Delta G_i(t), \tag{2.74}$$

where $G(1_i, \boldsymbol{q}(t))$ is the unavailability of the system when component i is not working and $G(0_i, \boldsymbol{q}(t))$ is the unavailability when component i is working (Here 1_i means $q_i = 1$ and 0_i means $q_i = 0$). $I_B^i(t)$ can also be interpreted as (Henley and Kumamoto, 1981)

$$\begin{aligned} I_B^i(t) &= E[\phi(0_i, X(t)) - \phi(1_i, X(t))] \\ &= 1 \times P[\phi(1_i, X(t)) - \phi(0_i, X(t)) = 1] + 0 \times P[\phi(1_i, X(t)) - \phi(0_i, X(t)) = 0] \end{aligned}$$

or

$$I_B^i(t) = P[\phi(0_i, X(t)) - \phi(1_i, X(t)) = 1]. \tag{2.75}$$

EXAMPLE 2.28

The measurement of electrical resistance has many important applications such as the determination of continuity in an electrical circuit and the measurement of changes in resistance on the order of 10^{-6} ohms. One of the simplest methods of measuring resistance is accomplished by imposing a voltage across the unknown resistance and measuring the resulting current flow using a galvanometer. A manufacturer of such galvanometers requires standard cells (batteries) to provide the necessary voltage. The manufacturer has the following options for placing four batteries with constant failure rates of $\lambda_1 = 0.005$, $\lambda_2 = 0.009$, $\lambda_3 = 0.003$, and $\lambda_4 = 0.05$ failures per hour in any of the following configurations.

1. All batteries are connected in series;
2. Batteries 1 and 2 are connected in series with batteries 3 and 4 connected in parallel;
3. The four batteries are connected in parallel; or
4. Three-out-of-four batteries are needed for the galvanometer to function properly.

Find Birnbaum's importance measure of every battery in the above configurations at $t = 40$ h.

SOLUTION

We estimate the unreliability of the batteries at $t = 40$ h as

$$\begin{aligned} q_1 &= 1 - R_1(t) = 1 - e^{-\lambda_1 t} = 0.181 \\ q_2 &= 0.302 \\ q_3 &= 0.113 \\ q_4 &= 0.864. \end{aligned}$$

1. Batteries are connected in series. The structure function is obtained as

$$\phi(X) = X_1 X_2 X_3 X_4$$

and

$$G(\boldsymbol{q}) = 1 - (1 - q_1)(1 - q_2)(1 - q_3)(1 - q_4). \tag{2.76}$$

Birnbaum's importance measures for Batteries 1 through 4 are obtained using Equation 2.73 at $t = 40$ h:

$$\begin{aligned} I_B^1(40) &= (1 - q_2)(1 - q_3)(1 - q_4) = 0.084 \\ I_B^2(40) &= (1 - q_1)(1 - q_3)(1 - q_4) = 0.098 \\ I_B^3(40) &= (1 - q_1)(1 - q_2)(1 - q_4) = 0.077 \\ I_B^4(40) &= (1 - q_1)(1 - q_2)(1 - q_3) = 0.507. \end{aligned}$$

Battery 4 has the highest importance measure. Accordingly, it has the most impact on the overall system reliability. Therefore, in order to improve the system reliability, the designer may wish to replace this battery with one having a smaller failure rate or may add a redundant battery.

2. Batteries 1 and 2 are connected in series with Batteries 3 and 4 connected in parallel. The structure function of this system is

$$\phi(X) = (X_1 \wedge X_2) \wedge (X_3 \vee X_4),$$

where $\vee$ is the OR Boolean and $\wedge$ is the AND Boolean operators, respectively. Thus,

$$\phi(X) = X_1 X_2 [1 - (1 - X_3)(1 - X_4)]$$

or

$$\phi(X) = X_1 X_2 X_3 + X_1 X_2 X_4 - X_1 X_2 X_3 X_4,$$

and

$$G(\boldsymbol{q}) = q_1 + q_2 - q_1 q_2 + q_3 q_4 - q_1 q_3 q_4 - q_2 q_3 q_4 + q_1 q_2 q_3 q_4. \tag{2.77}$$

The Birnbaum's importance measures are

$$\begin{aligned} I_B^1(40) &= 1 - q_2 - q_3 q_4 + q_2 q_3 q_4 = 0.629 \\ I_B^2(40) &= 1 - q_1 - q_3 q_4 + q_1 q_3 q_4 = 0.739 \\ I_B^3(40) &= q_4 - q_1 q_4 - q_2 q_4 + q_1 q_2 q_4 = 0.493 \\ I_B^4(40) &= q_3 - q_1 q_3 - q_2 q_3 + q_1 q_2 q_3 = 0.065. \end{aligned}$$

In this case, the importance measure places more emphasis on Battery 2 since it is the most likely battery to fail.

3. Batteries are connected in parallel. The structure function is obtained as

$$\phi(X) = X_1 \vee X_2 \vee X_3 \vee X_4$$

and

$$G(\boldsymbol{q}) = q_1 q_2 q_3 q_4. \tag{2.78}$$

The importance measures are

$$\begin{aligned} I_B^1(40) &= q_2 q_3 q_4 = 0.029 \\ I_B^2(40) &= q_1 q_3 q_4 = 0.018 \\ I_B^3(40) &= q_1 q_2 q_4 = 0.047 \\ I_B^4(40) &= q_1 q_2 q_3 = 0.006. \end{aligned}$$

In the parallel configuration, Battery 3 is considered the most critical for the overall system reliability. This is rather unexpected since Battery 4 is the least reliable unit. This is because the Birnbaum's importance measure is related to the probability that the system is in a state at time t in which the functioning of a battery is critical. Since Battery 3 fails last among other batteries, then it is considered, according to Birnbaum's measure, to be most critical. This "anomalous" result in parallel systems exists in other importance measures as well. Note that the Birnbaum's importance measure for one-event cut-sets is always, and usually incorrectly, numerically equal to one (Henley and Kumamoto, 1981).

4. Batteries are connected in a 3-out-of-4 configuration. The structure function of the system is derived as

$$\begin{aligned}\phi(X) &= (X_1X_2X_3)\vee(X_1X_2X_4)\vee(X_1X_3X_4)\vee(X_2X_3X_4)\\ &= 1-[1-X_1X_2X_3][1-X_1X_2X_4][1-X_1X_3X_4][1-X_2X_3X_4].\end{aligned}$$

The unavailability function $G(\boldsymbol{q})$ is obtained as

$$\begin{aligned}G(\boldsymbol{q}) = q_1q_2 + q_1q_3 + q_1q_4 + q_2q_3 + q_2q_4 + q_3q_4 - 2q_1q_2q_3 \\ -2q_1q_2q_4 - 2q_1q_3q_4 - 2q_2q_3q_4 + 3q_1q_2q_3q_4.\end{aligned} \tag{2.79}$$

The Birnbaum's importance measures are

$$\begin{aligned}I_B^1(40) &= q_2 + q_3 + q_4 - 2q_2q_3 - 2q_2q_4 - 2q_3q_4 + 3q_2q_3q_4 = 0.522\\ I_B^2(40) &= q_1 + q_3 + q_4 - 2q_1q_3 - 2q_1q_4 - 2q_3q_4 + 3q_1q_3q_4 = 0.626\\ I_B^3(40) &= q_1 + q_2 + q_4 - 2q_1q_2 - 2q_1q_4 - 2q_2q_4 + 3q_1q_2q_4 = 0.499\\ I_B^4(40) &= q_1 + q_2 + q_3 - 2q_1q_2 - 2q_1q_3 - 2q_2q_3 + 3q_1q_2q_3 = 0.383.\end{aligned}$$

In this case, Battery 2 has the most critical effect on the system reliability. The result of this configuration can be explained in a similar way as that of the parallel system. This measure is not a useful importance criterion except for a simple series system whose results are obvious (Henley and Kumamoto, 1981). ■

2.13.2 Criticality Importance

Criticality importance corresponds to the conditional probability that the system is in a state at time t such that component i is critical and has failed, given that the system has failed by the same time (Gandini, 1990). This importance measure is based on the fact that it is more difficult to improve the more reliable components than to improve the less reliable components. The criticality importance measure is expressed as

$$I_{CR}^i(t) = \frac{\partial G(\boldsymbol{q}(t))}{\partial q_i(t)} \times \frac{q_i(t)}{G(\boldsymbol{q}(t))}.$$

The above equation can be rewritten as

$$I_{CR}^i(t) = \frac{[G(1_i, \boldsymbol{q}(t)) - G(0_i, \boldsymbol{q}(t))] \times q_i(t)}{G(\boldsymbol{q}(t))}. \tag{2.80}$$

We now illustrate the application of this measure.

EXAMPLE 2.29

Calculate the criticality importance measure for the four system configurations given in Example 2.28.

SOLUTION

1. Batteries are connected in series. We use the unavailability expression of the series system given by Equation 2.76:

$$I_{CR}^{1}(40) = \frac{(1-q_2)(1-q_3)(1-q_4)q_1}{1-(1-q_1)(1-q_2)(1-q_3)(1-q_4)} = \frac{0.084\times 0.181}{0.931} = 0.016$$

$$I_{CR}^{2}(40) = \frac{0.098\times 0.302}{0.931} = 0.032$$

$$I_{CR}^{3}(40) = \frac{0.077\times 0.113}{0.931} = 0.009$$

$$I_{CR}^{4}(40) = \frac{0.507\times 0.864}{0.931} = 0.471.$$

This importance measure results in the same ranking of the batteries' importance as the Birnbaum's measures when components are connected in a series configuration. It is important to note that the importance measure of a component in a series system changes with time and that the order of importance does not change with time (as long as the components exhibit the same failure time distribution), but the differences might change as shown in Figure 2.31. This does not hold for other conditions or system configurations.

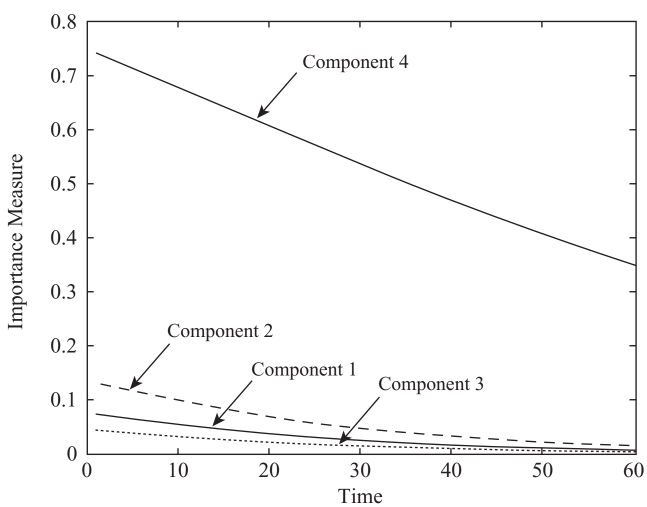

FIGURE 2.31 Criticality importance measure for a series system.

2. Batteries 1 and 2 are connected in series with batteries 3 and 4 connected in parallel. We use the unavailability of this configuration as given by Equation 2.77 to obtain the criticality importance measures of the batteries:

$$I_{CR}^{1}(40) = \frac{(1-q_2-q_3q_4+q_2q_3q_4)q_1}{q_1+q_2-q_1q_2+q_3q_4-q_1q_3q_4-q_2\,q_3\,q_4+q_1\,q_2\,q_3\,q_4} = \frac{0.629\times 0.181}{0.484} = 0.235$$

$$I_{CR}^{2}(40) = \frac{0.739\times 0.302}{0.484} = 0.461$$

$$I_{CR}^{3}(40) = \frac{0.493\times 0.113}{0.484} = 0.115$$

$$I_{CR}^{4}(40) = \frac{0.065\times 0.864}{0.484} = 0.116.$$

3. Batteries are connected in parallel. This measure places equal importance on all batteries as shown below:

$$I_{CR}^{1}(40) = \frac{q_2q_3q_4q_1}{q_1q_2q_3q_4} = 1$$

$$I_{CR}^{2}(40) = \frac{q_1q_3q_4q_2}{q_1q_2q_3q_4} = 1$$

$$I_{CR}^{3}(40) = \frac{q_1q_2q_4q_3}{q_1q_2q_3q_4} = 1$$

$$I_{CR}^{4}(40) = \frac{q_1q_2q_3q_4}{q_1q_2q_3q_4} = 1$$

4. Batteries are connected in a 3-out-of-4 configuration. We use Equation 2.79 to obtain the criticality importance measures as

$$I_{CR}^{1}(40) = \frac{[q_2+q_3+q_4-2(q_2q_3+q_2q_4+q_3q_4)+q_2q_3q_4]q_1}{G(\boldsymbol{q})}$$

$$= \frac{0.522\times 0.181}{0.492} = 0.220$$

$$I_{CR}^{2}(40) = \frac{0.626\times 0.302}{0.492} = 0.441$$

$$I_{CR}^{3}(40) = \frac{0.449\times 0.113}{0.429} = 0.118$$

$$I_{CR}^{4}(40) = \frac{0.383\times 0.864}{0.429} = 0.771.$$

The above measures show that Battery 4 has the most impact on the overall system unavailability. ■

2.13.3 Fussell–Vesely Importance

Fussell–Vesely importance measure of component i, I^i_{FV}, suggests consideration of the probability that the system's life coincides with the failure of a cut-set containing component i (Boland and El-Neweihi, 1995). The importance measure is given by

$$I^i_{FV}(t) = \frac{G_i(\boldsymbol{q}(t))}{G(\boldsymbol{q}(t))}, \tag{2.81}$$

where $G_i(\boldsymbol{q}(t))$ is the probability of component i contributing to cut-set failure.

EXAMPLE 2.30

Determine the Fussell–Vesely importance measures for the battery configurations given in Example 2.28.

SOLUTION

1. Batteries are connected in series. The probability of cut-sets containing Battery i in a series configuration is $G_i(\boldsymbol{q}(t)) = q_i(t) = q_i$, $i = 1, 2, 3$, and 4. The importance measures are

$$I^1_{FV}(40) = \frac{q_1}{G(\boldsymbol{q}(40))} = \frac{0.181}{0.931} = 0.194$$
$$I^2_{FV}(40) = \frac{q_2}{G(\boldsymbol{q}(40))} = \frac{0.302}{0.931} = 0.324$$
$$I^3_{FV}(40) = \frac{q_3}{G(\boldsymbol{q}(40))} = \frac{0.113}{0.931} = 0.121$$
$$I^4_{FV}(40) = \frac{q_4}{G(\boldsymbol{q}(40))} = \frac{0.864}{0.931} = 0.928.$$

   The importance rankings of the batteries are identical to those obtained by the Birnbaum's importance measures. Again, Battery 4 has the most impact on the overall system reliability.

2. Batteries 1 and 2 are connected in series with Batteries 3 and 4 connected in parallel. The probability of cut-sets containing battery i in this configuration is

$$G_1(\boldsymbol{q}(t)) = q_1$$
$$G_2(\boldsymbol{q}(t)) = q_2$$
$$G_3(\boldsymbol{q}(t)) = q_3q_4$$
$$G_4(\boldsymbol{q}(t)) = q_3q_4.$$

The importance measures of the batteries are

$$I_{FV}^{1}(40) = \frac{q_1}{G(\boldsymbol{q}(40))} = \frac{0.181}{0.484} = 0.373$$

$$I_{FV}^{2}(40) = \frac{q_2}{G(\boldsymbol{q}(40))} = 0.623$$

$$I_{FV}^{3}(40) = \frac{0.046}{0.484} = 0.201$$

$$I_{FV}^{4}(40) = \frac{0.046}{0.484} = 0.201.$$

3. Batteries are connected in parallel

$$G_i(\boldsymbol{q}(t)) = q_1 q_2 q_3 q_4 \quad i = 1, 2, 3, \text{and } 4$$
$$G(\boldsymbol{q}(t)) = q_1 q_2 q_3 q_4$$

Thus $I_{FV}^{i} = 1$ for i = 1, 2, 3, and 4. In other words, Fussell–Vesely importance measure ranks all the batteries equally in terms of their impact on the overall reliability of the system. This is a shortcoming of the measure since in a parallel system, the most reliable component has the most impact on the system reliability.

4. Batteries are connected in a 3-out-of-4 configuration

$$G_1(\boldsymbol{q}(t)) = q_1q_2 + q_1q_3 + q_1q_4 - q_1q_2q_3 - q_1q_2q_4 - q_1q_3q_4 + q_1q_2q_3q_4 = 0.165$$
$$G_2(\boldsymbol{q}(t)) = q_1q_2 + q_2q_3 + q_2q_4 - q_1q_2q_3 - q_1q_2q_4 - q_2q_3q_4 + q_1q_2q_3q_4 = 0.272$$
$$G_3(\boldsymbol{q}(t)) = q_1q_3 + q_2q_3 + q_3q_4 - q_1q_2q_3 - q_1q_3q_4 - q_2q_3q_4 + q_1q_2q_3q_4 = 0.104$$
$$G_4(\boldsymbol{q}(t)) = q_1q_4 + q_2q_4 + q_3q_4 - q_1q_2q_4 - q_1q_3q_4 - q_2q_3q_4 + q_1q_2q_3q_4 = 0.425.$$

The importance measures are

$$I_{FV}^{1}(40) = \frac{G_i(\boldsymbol{q}(t))}{G(\boldsymbol{q}(t))} = \frac{0.165}{0.429} = 0.384$$

$$I_{FV}^{2}(40) = \frac{0.272}{0.429} = 0.634$$

$$I_{FV}^{3}(40) = \frac{0.104}{0.429} = 0.242$$

$$I_{FV}^{4}(40) = \frac{0.425}{0.429} = 0.990.$$

In this configuration, component 4 is the most critical component since it has the highest failure rate, and if it is one of the three components needed for the 3-out-of-4 configuration, it will have a major impact on this specific configuration. ■

2.13.4 Barlow–Proschan Importance

This measure corresponds to the conditional probability that component i causes the system to fail in the time interval (t_0, t_F), given that the system has failed in the same period (Barlow and Proschan, 1974). It is expressed as

$$I_{BP}^{i} = \frac{\displaystyle\int_{t_o}^{t_F} \frac{\partial G(\boldsymbol{q}(t))}{\partial q_i} \frac{dq_i}{dt} dt}{\displaystyle\sum_{k=1}^{n} \int_{t_o}^{t_F} \frac{\partial G(\boldsymbol{q}(t))}{\partial q_k} \frac{dq_k}{dt} dt}, \tag{2.82}$$

where n is the total number of components in the system.

2.13.5 Upgrading Function

This function is developed by Lambert (1975). It is defined as the fractional reduction in the probability of the system failure when component failure rate λ_i is reduced fractionally. It is given by

$$I_{UF}^{i}(t) = \frac{\lambda_i}{G(\boldsymbol{q}(t))} \frac{\partial G(\boldsymbol{q}(t))}{\partial \lambda_i}. \tag{2.83}$$

Lambert and Yadigaroglu (1977) apply this measure of importance to the problem of determining the optimal choice of system upgrade. This function is limited to measuring the importance of components in nonrepairable systems.

EXAMPLE 2.31

An O-ring is a rubber doughnut squeezed into a groove between parts that are to be sealed. Pressure from the sealed gas pushes the O-ring ahead of it into the gap between the body parts so that the O-ring obstructs passage of the gas. This is called a self-energizing seal. The gas must exert pressure on the entire left side of the O-ring or, instead of pushing it forward and upward to block the escape route, the gas will push it down, out of the way of the escape route, and the gas will escape. Therefore, the O-ring groove must be wider than the compressed O-ring, otherwise, the O-ring will touch all four sides of its enclosure and will not seal as shown in Figure 2.32 (Kamm, 1991).

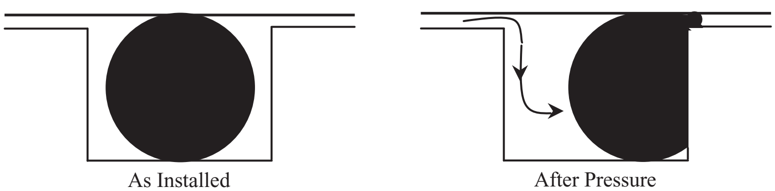

FIGURE 2.32 O-ring before and after gas pressure is applied.

A manufacturer of satellite booster rockets uses three O-rings located two inches away from each other to prevent leakage of gases. The manufacturer considers two designs, both of which meet the reliability requirements. They are

- All of the three O-rings must not leak under the maximum pressure and
- Two of the three O-rings must not fail under the maximum allowable pressure.

The O-rings exhibit constant failure rates of $\lambda_1 = 0.004$, $\lambda_2 = 0.009$, and $\lambda_3 = 0.025$ failures per hour. Determine the upgrading functions for each ring in both designs. Plot the functions against time. What are the most critical O-rings? Why?

SOLUTION

The unreliabilities of the O-rings are

$$q_1(t) = 1 - e^{-\lambda_1 t} = 1 - e^{-0.004t},$$
$$q_2(t) = 1 - e^{-0.009t},$$

and

$$q_3(t) = 1 - e^{-0.025t}.$$

We now consider the two designs.

1. All of the O-rings must operate. This is a series system and its $G(\boldsymbol{q}(t))$ is

$$G(\boldsymbol{q}(t)) = 1 - e^{-\lambda_1 t} e^{-\lambda_2 t} e^{-\lambda_3 t}$$
$$\frac{\partial G(\boldsymbol{q}(t))}{\partial \lambda_1} = te^{-(\lambda_1+\lambda_2+\lambda_3)t}$$
$$\frac{\partial G(\boldsymbol{q}(t))}{\partial \lambda_2} = te^{-(\lambda_1+\lambda_2+\lambda_3)t}$$
$$\frac{\partial G(\boldsymbol{q}(t))}{\partial \lambda_3} = te^{-(\lambda_1+\lambda_2+\lambda_3)t}$$

Thus,

$$I_{UF}^{i}(t) = \frac{\lambda_i te^{-(\lambda_1+\lambda_2+\lambda_3)t}}{1 - e^{-(\lambda_1+\lambda_2+\lambda_3)t}}$$

or

$$I_{UF}^{i}(t) = \frac{\lambda_i te^{-0.038t}}{1 - e^{-0.038t}} \quad i = 1, 2, \text{and } 3. \tag{2.84}$$

2. For the 2-out-of-3 O-ring system,

$$G(\boldsymbol{q}(t)) = q_1q_2 + q_2q_3 + q_3q_1 - 2q_1q_2q_3$$

or

$$G(\boldsymbol{q}(t)) = 1 - e^{-(\lambda_1+\lambda_2)t} - e^{-(\lambda_1+\lambda_3)t} - e^{-(\lambda_2+\lambda_3)t} + 2e^{-(\lambda_1+\lambda_2+\lambda_3)t}$$

$$\frac{\partial G(\boldsymbol{q}(t))}{\partial \lambda_1} = te^{-(\lambda_1+\lambda_2)t} + te^{-(\lambda_1+\lambda_3)t} - 2te^{-(\lambda_1+\lambda_2+\lambda_3)t}$$

$$\frac{\partial G(\boldsymbol{q}(t))}{\partial \lambda_2} = te^{-(\lambda_1+\lambda_2)t} + te^{-(\lambda_2+\lambda_3)t} - 2te^{-(\lambda_1+\lambda_2+\lambda_3)t}$$

$$\frac{\partial G(\boldsymbol{q}(t))}{\partial \lambda_3} = te^{-(\lambda_1+\lambda_3)t} + te^{-(\lambda_2+\lambda_3)t} - 2te^{-(\lambda_1+\lambda_2+\lambda_3)t}.$$

We now use Equation 2.83 to obtain

$$I_{UF}^{i}(t) = \frac{\lambda_i}{G(\boldsymbol{q}(t))}\frac{\partial G(\boldsymbol{q}(t))}{\partial \lambda_i} \quad i = 1, 2, \text{and } 3. \tag{2.85}$$

Graphs of Equations 2.84 and 2.85 are shown in Figures 2.33 and 2.34, respectively. As shown in the figures, the values of the importance measures for both systems decrease with time. Moreover, the differences between the importance measures within the same system decrease rapidly with time. This is a very important observation since allocation of resources for the improvement of the component reliability should ensure that the critical component's reliability is achieved at the desired time.

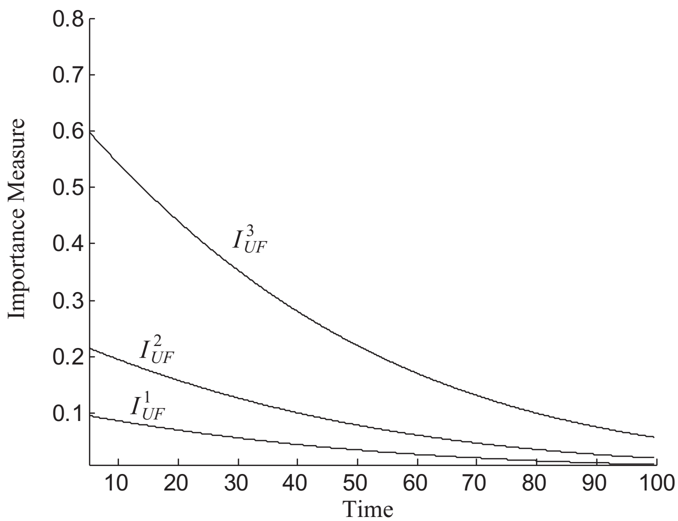

FIGURE 2.33 $I_{UF}^{i}(t)$ for the series system.

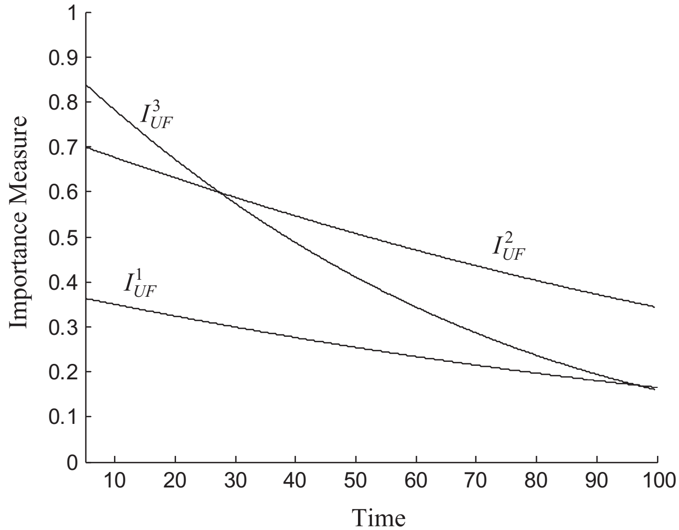

FIGURE 2.34 $I_{UF}^i(t)$ for the 2-out-of-3 system. ■

As shown in the previous examples, none of the importance measures is valid for all configurations, that is, the rankings of the system's components in terms of their importance are not consistently valid or intuitive. Therefore, it may be more appropriate to use an importance measure whose value is the weighted sum of several measures. Moreover, depending on the configuration and the objective of the system, the analyst may follow the derivations of the importance measures presented in this chapter and develop an appropriate measure accordingly.

Finally, recent research in importance measures has focused on determining component importance in systems with multistate components and component importance of consecutive-k-out-of-n:F systems. For example, Papastavridis (1987) gives a simple formula to determine the Birnbaum importance of a component in a consecutive-k-out-of-n:F system and proves that for independent and identical components, the most important ones are in the middle of the sequence. The formula for the Birnbaum's importance of component i is

$$I_B^i = \frac{R(i-1)R'(n-i) - R(n)}{q_i}, \tag{2.86}$$

where $R(j)$ is the reliability of consecutive-k-out-of-j:F subsystem consisting of components 1, 2, . . . , j and $R'(j)$ is the reliability of consecutive-k-out-of-j:F subsystem, consisting of components $(n-j+1)$, $(n-j+2)$, . . . , $(n-1)$, n.

The reliability of systems with multistate components is difficult to estimate for complex networks, and the importance measures of such systems have been under investigation by several researchers (see Ramirez-Marquez and Coit, 2007). The importance measures of a multistate component in such systems need to consider the reliability of the system when each component experiences all potential states. This generates an extensive list of the system states which makes it impractical for even a small system network. Meta-heuristics and algorithms might provide tools for modeling such a system.

The importance of a component does not only depend on its location in the reliability block diagram and its failure mode but also on its degradation with time. Therefore, it is important to consider degradation and aging of the components, since components that may be not as "critical" at some time may become "critical" as they age and deteriorate. This becomes important in the design and implementation of condition-based maintenance strategies as described in later chapters.

PROBLEMS

2.1 Figure 2.35 shows the block diagram of a closed-loop servo accelerometer. The accelerometer functions as follows: a pendulous mass reacts to an acceleration input and begins to move. A position sensor detects this minute motion and develops an output signal. This signal is demodulated, amplified, and applied as negative feedback to an electrical torque generator (torquer) coupled to the mass. The torquer develops a torque proportional to the current applied to it. The magnitude and direction of this torque just balance out the torque attempting to move the pendulous mass as a result of the acceleration input, preventing further movement of the mass.

Since both torques are equal and the torque generator output is proportional to its input current, the input current is, therefore, proportional to the torque attempting to move the pendulous mass. In fact, this torque is proportional to the product of moment of inertia and acceleration. Therefore, the torque generator current is proportional to applied acceleration. If this current is passed through a stable resistor, the voltage developed is proportional to applied acceleration.

a. Draw a reliability graph of the feedback system.

b. Assuming that the probability that a component functioning properly is 0.9, what is the total system reliability when all components have the same probability of success?

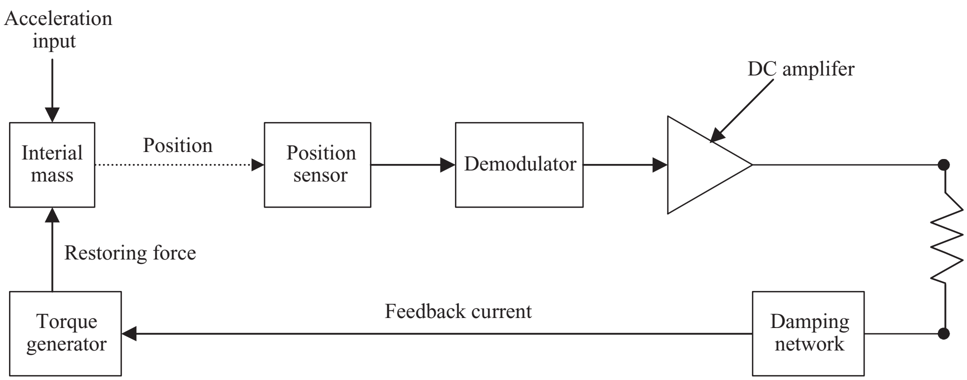

FIGURE 2.35 A block diagram for Problem 2.1.

2.2 Consider the head of a computer disk drive which must quickly transverse the radius of a rotating disk drive. The head moves from a known position to another known position on the disk. The head rests at the end of an arm. A large magnet surrounds the other end of the arm. The part of the arm next to the magnet is an electromagnet. Current is supplied to it as needed to produce a force to move the arm. A sensor detects the position of the head in relation to the target position and decreases the current (force) proportionally. Moreover, a damper is connected to the arm to ensure that the head will not oscillate around the desired position. Construct a block diagram and a reliability graph to represent the operation of the disk drive.

2.3 Cell phone is a commonly used communication device in modern life. It consists of seven components necessary for the proper function of the phone. These components are listed in Table 2.5. Construct a block diagram for the cell phone. Assuming that the cell towers are operating properly, and the components of a cell phone exhibit constant failure rates as given in Table 2.5, estimate the reliability of a successful communication from one phone to another.

TABLE 2.5 Components and Failure Rates of the Cell Phone

Component	Function	Failure rate
1	Circuit board which includes the analog-to-digital and digital-to-analog conversion chips that translate the outgoing audio signal from analog to digital and the incoming signal from digital back to analog. It also includes the microprocessor which controls all functions of the phone	0.0003
2	Antenna which sends and receives signals	0.0002
3	Liquid crystal display	0.0005
4	Keyboard	0.0008
5	Microphone	0.0001
6	Speaker	0.00025
7	Battery	0.0002

2.4 Determine the reliability and its variance of the system shown in Figure 2.36. The reliability and variance of all the components are given in Table 2.6.

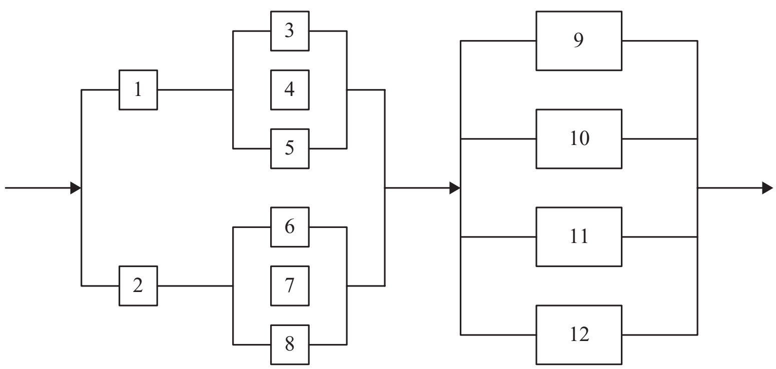

FIGURE 2.36 Reliability block diagram of the system.

TABLE 2.6 Reliability and Variance of the Components

Unit *i*	1	2	3	4	5	6	7	8	9	10	11	12
$R(i)$	0.96	0.87	0.98	0.88	0.77	0.99	0.95	0.70	0.90	0.96	0.98	0.75
$Var(R[i])$	0.03	0.02	0.01	0.06	0.04	0.02	0.01	0.03	0.05	0.08	0.05	0.04

2.5 Consider the system shown in Figure 2.37. The reliabilities of the components and their variances are shown in Table 2.7.

Using the following two expressions which are derived earlier in the chapter,

$$Var(R_{series}) = \prod_i (R_i^2 + Var(R_i)) - \prod_i R_i^2$$

$$Var(R_{parallel}) = \prod_i ((1-R_i)^2 + Var(R_i)) - \prod_i (1-R_i)^2$$

Determine the reliability and variance of the system. If you were to improve the variance of the system, what are the two components that have the most impact on its improvement?

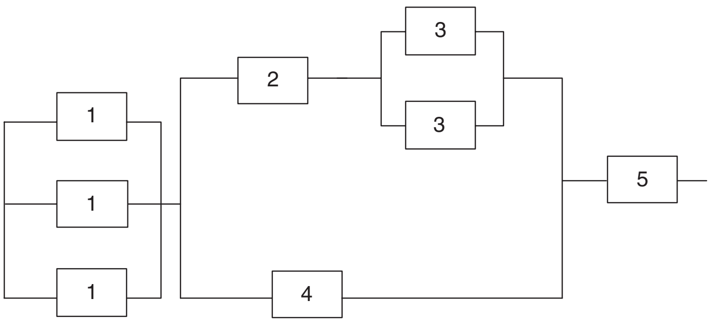

FIGURE 2.37 Reliability block diagram for Problem 2.5.

TABLE 2.7 Reliability and Variance of the Components for Problem 2.5

Component *i*	R_i	$Var(R_i)$
1	0.92	0.0052
1	0.92	0.0008
1	0.92	0.0002
2	0.98	0.0007
3	0.96	0.0006
3	0.96	0.0024
4	0.78	0.0095
5	0.90	0.0046

2.6 Investigate the effect of the components' reliabilities and their variances on the overall system reliability for both series and parallel systems. Begin by assuming that the system has six components with reliability values of 0.9, 0.8, 0.75, 0.70, 0.65, and 0.6 and the corresponding variances as 0.06, 0.05, 0.04, 0.03, 0.02, and 0.01, respectively. Then reverse the order of the variances and solve the problem. Conduct extensive analysis and provide conclusions about the system reliability and its variance.

2.7 The variance of reliability estimates is important in decision making in the warranty policy area and in the repair and maintenance area. In order to estimate the reliability of a system, we usually decompose the system into subsystems, and each is investigated by decomposing it to simple series or parallel arrangement. Therefore, it is helpful to discuss the variance properties of series and parallel systems. Let X and Y be two independent random variables with independent distributions. The variance of (XY) is

$$Var(XY) = E[(XY)^2] - E[XY]^2$$

Since X and Y are independent, then

$$Var(XY) = E[X^2]E[Y^2] - E[X]^2 E[Y]^2$$

and

$$\begin{aligned} Var(XY) &= (Var(x) + E[X]^2)(Var(Y) + E[Y]^2) - E[X]^2 E[Y]^2 \\ &= Var(X)Var(Y) + Var(X)E[Y]^2 + Var(Y)E[X]^2. \end{aligned}$$

Similarly, the variance of $X + Y$ is

$$\begin{aligned} Var(X+Y) &= E[(X+Y)^2] - E^2[X+Y] \\ &= E[X^2] + E[Y^2] + 2E[X]E[Y] - E^2[X] - E^2[Y] - 2E[X]E[Y] \\ &= Var(X) + Var(Y). \end{aligned}$$

Estimate the variance of the system reliability for the following configurations:

a. A series system consisting of n nonidentical components with component i having constant failure rate λ_i.

b. A parallel system consisting of n non identical components with component i having constant failure rate λ_i.

2.8 A reliability engineer has six components with reliability values of 0.9, 0.8, 0.75, 0.70, 0.65, and 0.6. The engineer wishes to configure them in several arrangements that yield the same (or close) values of reliability. The following configurations are considered:

a. The components are arranged in consecutive 2-out-of-6:F system.

b. The components are configured in series-parallel configuration.

c. The components are configured in a parallel-series configuration.

Arrange the components in item a configuration such that the system reliability is maximized. Then estimate the overall system reliability and arrange the components in items b and c to yield the same reliability value.

2.9 Walski and Pelliccia (1982) developed break-rate equations (hazard-rate equations) for the Binghamton, New York water system. These equations are

Pit cast iron (PCI)	$N(t) = 0.02577e^{0.027t}$
Sandspun cast iron (SCI)	$N(t) = 0.0627e^{0.0137t}$,

where

$$N(t) = \text{the break rate in breaks per mile per year and}$$
$$t = \text{the age of the pipe in years.}$$

An engineer wishes to design a new water distribution system as shown in Figure 2.38.

a. What is the reliability of the system after 2 years of service? (Reliability is measured as the probability of successful water delivery from node 1 to node 6).

b. What is the MTTF?

c. What do you suggest to ensure a reliability of 0.98 after 2 years of service?

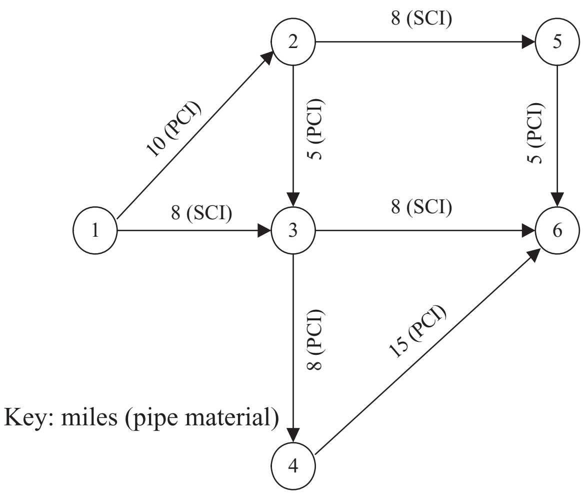

FIGURE 2.38 A new water distribution system.

2.10 Consider a diode that can function properly but can malfunction by either short circuiting or by open circuiting. Let the probabilities of these be

$$p = \text{probability of proper operation}$$
$$p_s = \text{probability of short circuit}$$
$$p_o = \text{probability of open circuit.}$$

If we consider four identical diodes for improving the total system reliability, these diodes can be arranged in two possible configurations as shown in Figure 2.39. What is the ratio of the reliability improvement for both configurations?

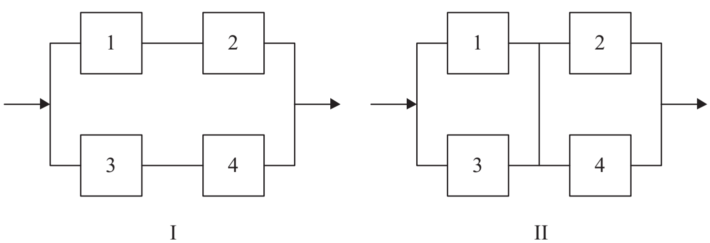

FIGURE 2.39 Configurations of diodes.

2.11 Consider the case of three elements, as shown in Figure 2.40. At least two of the three elements must function properly for the system to function properly. All three elements are different. Elements a and b have three states each. An element in State 1 implies that it is working properly, whereas State 2 represents a short failure mode, and State 3 represents an open failure mode of the unit. Let p_{ij} represent the probability that element $i(i = a, b)$ is in state (j = 1, 2, 3). Element c has two states only (working or not). Determine the reliability of the system.

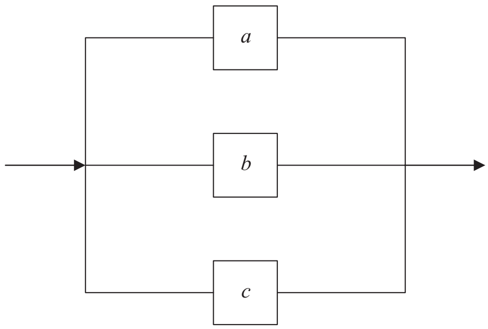

FIGURE 2.40 Reliability block diagram of the system.

2.12 When a fax machine receives a document, it converts electrical signals into a copy of the original document. There are two types of recording systems: (a) the thermal recording system which uses a set of fine wires positioned across the recording paper that produce hot spots as current passes through them, burning the image into the paper or (b) the electrostatic recording system which is shown in Figure 2.41. In this system, a charge is applied on the recording paper where a mark is needed. Black powder toner adheres to the charged areas, which are either fused to the paper with heat or pressed onto it by rollers. The paper

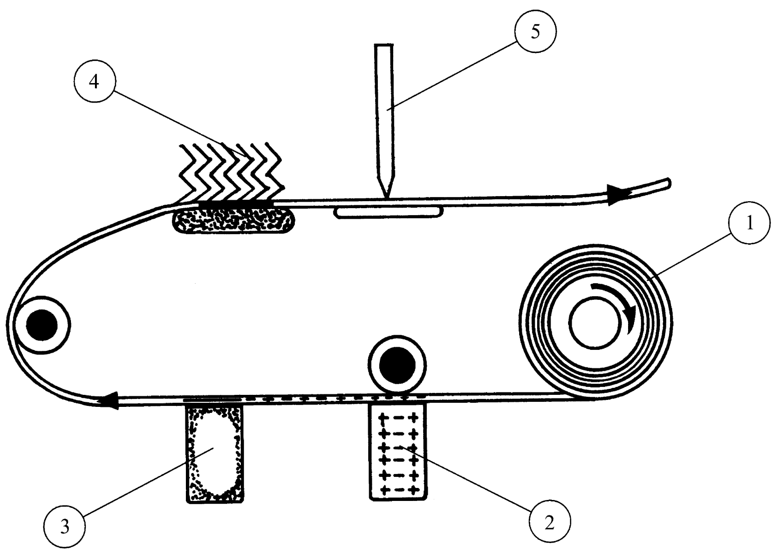

FIGURE 2.41 Schematic of an electrostatic fax machine.

is then cut to length. The main components of the fax machine and their constant failure rates are given in Table 2.8. All components exhibit constant failure rates per year. A fax machine uses the "group faxing" concept by sending the same document to a group of machines simultaneously. Assume that a fax machine uses the "group faxing" concept to send a document to six other machines. The reliability of the system is measured as the probability that all machines receive the document.

a. Draw a block diagram and reliability graph of the system.

b. Assuming that the communication links between machines do not fail, determine the reliability of the system at $t = 200$ h.

c. Calculate I_{FV}^{i} for all components of a fax machine at $t = 400$ h. What are the components that should be improved to increase the overall reliability of the fax machine?

TABLE 2.8 Components of the Fax Machine

Component	Description and function	Failure rate
1	Paper feeder	0.001
2	Printer head, applies charge	0.009
3	Toner, contains powder	0.0005
4	Heater, fuses powder onto paper	0.018
5	Cutter, cuts paper to length	0.0085

2.13 Consider the following reliability block diagram of a system (Fig. 2.42), which is composed of four subsystems: Sub 1, Sub 2, Sub 3, and Sub 4. These subsystems and the reliabilities of their components are

Sub 1 is a series subsystem with $p_1 = 0.95$, $p_2 = 0.98$, $p_3 = 0.999$;

Sub 2 is a redundant subsystem with $p_4 = 0.90$, $p_5 = 0.95$, $p_6 = 0.90$;

Sub 3 is a network subsystem with $p_7 = 0.98$, $p_8 = 0.85$, $p_9 = 0.93$, $p_{10} = 0.99$;

Sub 4 is a consecutive 2-out-of-4:F series system with

$$p_{11} = p_{12} = p_{13} = p_{14} = 0.96;$$

a. Determine the reliability of the system.

b. Use Birnbaum's importance measure to determine whether component 10 is more critical than component 13.

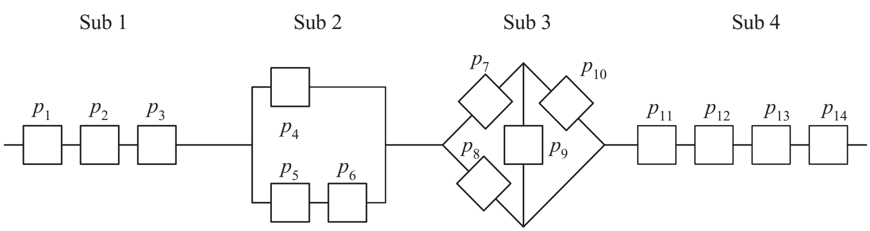

FIGURE 2.42 Reliability block diagram for Problem 2.13.

2.14 Four elements are configured as shown in Figure 2.43. At least two of the four elements must function properly for the system to operate properly. All four elements are different, and have reliabilities of p_a, p_b, p_c, and p_d, respectively. Find the reliability of the system.

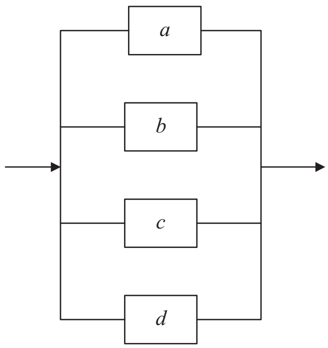

FIGURE 2.43 Reliability block diagram of the system.

2.15 Consider the following problem.

a. A system with three components in series is to attain a reliability of 0.95 at time $t = 120$ h. If the third component has twice the failure rate of the second, and the second twice the failure rate of the first, what must be their failure rates if $t = 120$ h?

b. What is the MTTF of the system?

c. What is the probability of having 0, 1, and 2 failures in 100 h?

d. What failure rates of components 1, 2, and 3 would you require if you demanded 0.95 reliability for the duration of the MTTF (use the same ratio of failure rates)?

e. If the redundancy is allowed for all three components and if the cost of each component is the same, how much and where would you impose redundancy if you require a reliability of 0.98 for a duration of 1000 h?

2.16 Consider the following.

a. In a 3-out-of-n system with components having a linearly increasing hazard rate $h(t) = 0.5 \times 10^{-3}t$ failures per hour, determine the number of components for the system such that a reliability of 0.98 is achieved at $t = 10^3$. What is the MTTF?

b. Solve (a) when the components are connected in parallel. Are the results identical? Explain why.

c. Plot the reliability of the system against time. When will the reliability reach 0.96?

2.17 Solve Problem 2.4, when (a) three out of four units (9 through 12) are needed for successful operation, (b) two out of units 3, 4, and 5 and two units out of 6, 7, and 8 must function in addition to at least three out of four of the remaining units (9 through 12).

2.18 A system consists of n components in series. Each component is subject to failure and its reliability is $p = 0.98$. The system fails if any two consecutive components fail. Determine the reliability of the system when $n = 6$.

2.19 Diodes are connected in a network as shown in Figure 2.44. Each diode can be in any of the following states:

- Fail open with probability p_o
- Fail short with probability p_s or

- Function properly with probability $1 - p_o - p_s$.
  a. What is the reliability of the network?
  b. Assuming all components are identical, what is the reliability of the network?

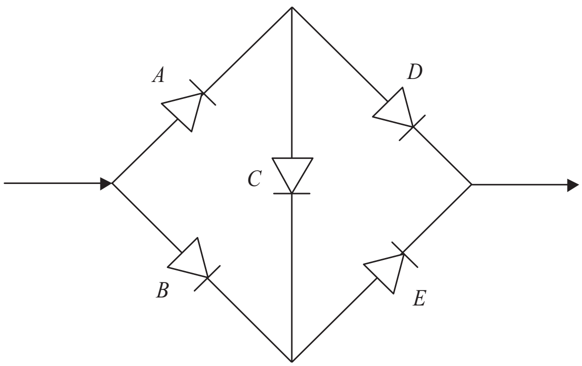

FIGURE 2.44 Network for Problem 2.19.

2.20 Consider a six-engine descent system of a large crewed vehicle missions to Mars. The system can land the vehicle safely as long as it experiences at most one engine pair failure to maintain balance of the vehicle. In Figure 2.45 the engine pairs are 1–4, 2–5, and 3–6. Assume that the engines are identical and each has a probability p of successfully operating during landing. Determine the reliability of the system.

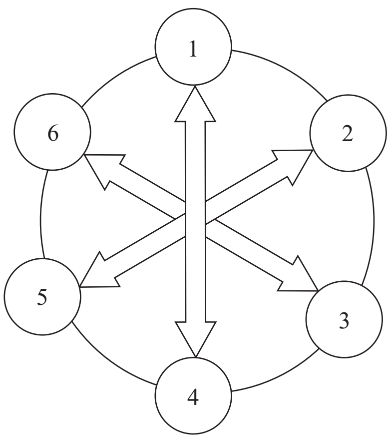

FIGURE 2.45 Six-engine vehicle system.

2.21 Consider a 2-out-of-4 system at time $t = 50$ h. The failure rates of the components are

$$h_1(t) = 0.005$$
$$h_2(t) = 0.005t$$
$$h_3(t) = 0.006t^{1.1}$$
$$h_4(t) = 0.009t^{1.05}.$$

Calculate the following importance measures

a. Birnbaum's importance $I_B^i(t)$.

b. Solve (a) if the system is a consecutive-2-out-of-4:F system.

2.22 What are the reliability expressions for the following systems whose components are identical, independent, and exhibit a constant failure rate λ?

a. Five components in series

b. A 2-out-of-5 system

c. A 3-out-of-5 system

d. A parallel system of five components.

2.23 Consider a *k-out-of-n* balanced system with the probability that a unit functions properly is p. Derive reliability expressions for the following:

a. $k = 2, n = 6$

b. $k = 4, n = 6$

c. $k = 2, n = 8$

d. $k = 4, n = 8$

e. Plot the reliability functions for different values of p.

f. Solve a through e for the general *k-out-of-n* system.

g. Determine the cases when reliability estimates for part f exceed that of a through e.

2.24 What is the MTTF of a system composed of two components having hazard rates $k_1 t^m$ and $k_2 t^m$? What is the reliability of the system at $t = k_1/k_2$?

2.25 In using the decomposition method to estimate the reliability of a complex system, one needs to identify a *keystone* component. If the identification is done properly, the reliability estimate can be made with the least amount of computation. A novice reliability engineer is not sure which of the components is a keystone component, and the engineer proceeds in estimating the reliability of the system by considering any one of the five components as a keystone component in Figure 2.46. Show that the reliability of the system is the same regardless of the choice of the keystone component.

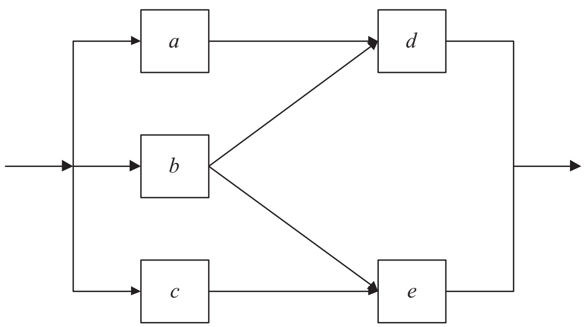

FIGURE 2.46 Reliability block diagram for Problem 2.25.

2.26 Repeat the above problem for the system shown in Figure 2.47.

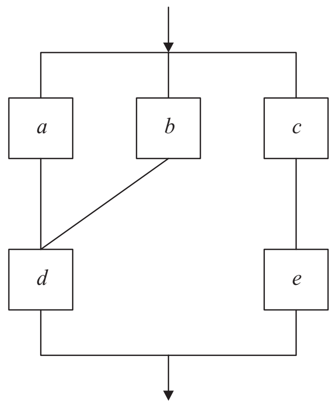

FIGURE 2.47 Reliability block diagram for Problem 2.26.

2.27 Figure 2.48 represents a four-node communications network. The four nodes, a, b, c, and d, represent the four stations. The six branches represent two-way communication links between each pair of stations.

a. Find the minimum cut-sets and tie-sets between a and b.

b. Approximate the system reliability when all links are independent and identical with probability of success p.

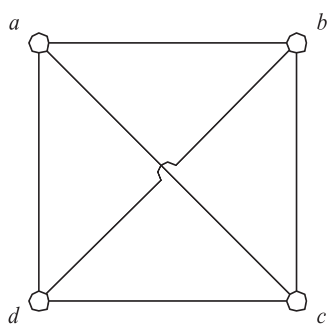

FIGURE 2.48 Reliability block diagram for Problem 2.27.

2.28 The reliability graph of a parallel-series system is shown in Figure 2.49. Assume that each component has a linearly increasing hazard function of the type $h(t) = a + bt$. The number of components connected in series is n whereas the number of parallel paths is m. What is the MTTF of this system? What is the effect of m and n on the MTTF?

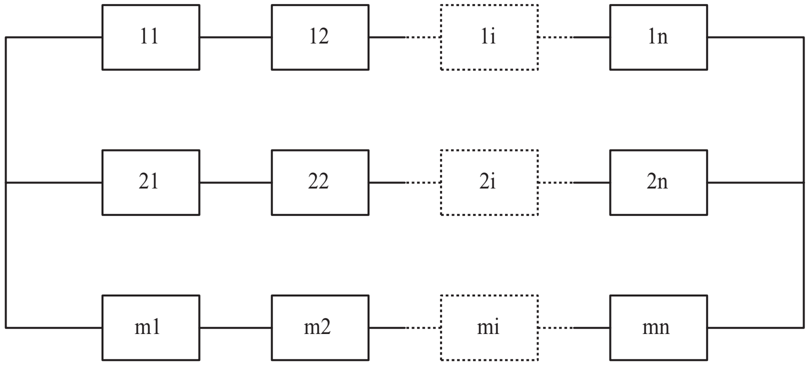

FIGURE 2.49 Figure for Problem 2.28.

2.29 A computer chip has 160,000 transistors connected in parallel, and k transistors are required to operate properly for the chip to perform its function. Assuming that each transistor has a constant hazard rate $h(t) = 5 \times 10^{-6}$ failures per hour, what is the value of k which ensures a chip reliability of 0.95 at $t = 10{,}000$ h?

2.30 A system consists of three components with hazard rates $h_1(t)$, $h_2(t)$, and $h_3(t)$. Assuming that the three components are connected in series, determine the reliability and MTTF when,

a. $h_1(t) = \lambda_1$, $h_2(t) = \lambda_2$, and $h_3(t) = \lambda_3$.

b. $h_1(t) = \lambda_1 t$, $h_2(t) = \lambda_2 t$, and $h_3(t) = \lambda_3 t$.

c. $h_1(t) = \lambda_1$, $h_2(t) = \lambda_2 t$, and $h_3(t) = \lambda_3 t^m$.

2.31 Solve Problem 2.30 when the three components are connected in parallel.

2.32 Solve Problem 2.31 when components 1 and 2 are connected in parallel while Component 3 is connected in series with them.

2.33 Using the numerical values given below, compare the MTTF for the three systems in Problems 2.30–2.32. Sketch the reliability functions for all conditions when: $\lambda_1 = 0.001$, $\lambda_2 = 0.003$, $\lambda_3 = 0.009$, and $m = 1.5$.

2.34 Consider Components 1, 2, 3, and 4. Their failure rates at $t = 40$ hours are $\lambda_1 = 0.006$, $\lambda_2 = 0.008$, $\lambda_3 = 0.002$, and $\lambda_4 = 0.07$. The following four configurations are to be made

a. Four components are connected in series.

b. Components 1 and 2 are connected in series with Components 3 and 4 connected in parallel.

c. The four components are connected in parallel.

d. Two-out-of-four components are needed for system functions.

e. Three-out-of-four components are needed for the system to operate properly.

Determine the reliability of each configuration.

2.35 Most color laser printers use a combination of four colors of cyan (C), magenta (M), yellow (Y), and black (K) to create colors. The printer has a heated print head to transfer pigment from a thin plastic ribbon onto paper or transparency film. The ribbon contains successive panels of pigment in C, M, Y, and K. After the printer has applied one color's dots, the drive mechanism pulls the media back for the next pass. After the application of the colors, the paper or transparency is transferred to the fuser that ensures the permanency of the colors. A diagram representing the elements and operation of the color laser printers is shown in Figure 2.50.

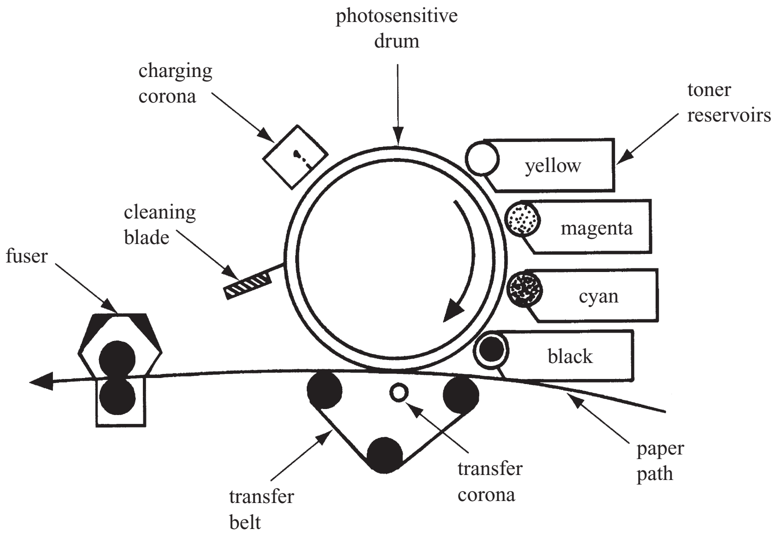

FIGURE 2.50 Figure for Problem 2.35.

a. Draw both the block reliability diagram and the reliability graph.

b. Assume that all toner reservoirs have the same constant failure rate with parameter λ_r and the drum's failure rate $\lambda_d = 2\lambda_r$. The failure rates of the fuser, transfer belt, transfer corona, cleaning blade, and the charging corona are $h_f(t) = at$, $h_{belt}(t) = ct$, $h_{tc} = e^{bt}$ $h_{blade}(t) = dt$, and $h_{cc}(t) = e^{-t^2}$, respectively.

c. Determine the reliability of the printer.

d. What is the most critical component of the printer?

e. Recommend an alternative design for the most critical component.

2.36 As discussed in this chapter, all component importance measures do not consider the variance of the component. Develop importance measures that take the variance of the components into account and determine the importance measures for the components in Problem 2.4.

2.37 Repeat Problem 2.36 for the system described in Problem 2.6.

2.38 The demand for energy and the increase in its cost prompted users to explore sources of renewable energy. Solar energy is considered an infinite source of such energy. However, the current solar systems are inefficient and unable to provide energy on a continuous basis. Therefore, the integration of energy generation from solar ponds (heating water using solar energy) and the power generation using gas turbine is a viable alternative that ensures continuous supply of power. Such systems are referred to as integrated power generation systems or cogeneration stations. A typical system (based on Grote and Antonsson, 2009) is shown in Figure 2.51. The solar field, which consists of 12 solar panels numbered 1 through 12, provides hot water to the heat recovery steam generator unit via pump A. The heat from hot water in conjunction with heat generated through the gas turbine provides sufficient heat to bring the water circulating in the heat recovery steam generator unit to steam (or supersaturated steam) which in turn operates the steam turbine to produce electric power. The steam is condensed and recirculated.

a. Draw both the block reliability diagram and the reliability graph.

b. Assume that the failure rates (failures/hour) of the components are constant as follows

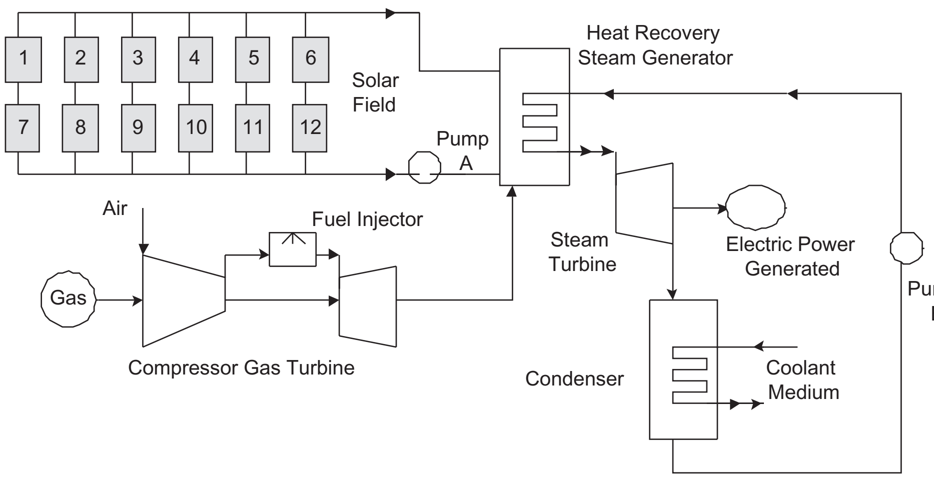

FIGURE 2.51 Figure for Problem 2.38.

Solar panel i ($i = 1, \ldots, 12$) = 0.000005

Pump A = 0.00006

Pump B = 0.00008

Condenser = 0.00004

Steam turbine = 0.0003

Gas turbine = 0.00007

Heat recovery steam generator = 0.00009

Compressor = 0.0006

Fuel injector = 0.000011

Also, assume that all units must function properly, what is the reliability of the co-generation station?

c. Determine the reliability of the cogeneration station if the solar field fails when two consecutive panels fail.

d. Assume that maximum energy generated is 100 MW and that the solar field contributes 25% to the total energy. The cogeneration station is considered "reliable" when it provides a minimum of 75 MW. What is the probability that the station meets the minimum requirements?

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CHAPTER 3

TIME- AND FAILURE-DEPENDENT RELIABILITY

3.1 INTRODUCTION

In Chapter 2, we present different system configurations and the appropriate methods for estimating their reliabilities. The reliability values are not time dependent since the reliabilities of the components are considered constant, and the failure-time distributions are not incorporated in estimating the reliability of the system. In other words, we only have a snapshot of the system at a specified instant and do not observe the reliability of the system over time (or over the life of the system). Moreover, we have not fully considered the dependence between component failures—that is, the effect of the failure of a component on the failure rates of other components in the system. Likewise, we have not considered the effect of repairs on the system performance in terms of its reliability, availability, mean time to failure (MTTF), and mean time between failures (MTBF).

In this chapter, we develop time-dependent reliability expressions for both nonrepairable and repairable systems. We also present different approaches for estimating the reliability of failure-dependent systems—for example, when the failure of a component affects the failure rate of other components in the system. Finally, we estimate different performance measures of the system such as MTTF, MTBF, and availability. We begin by presenting time-dependent reliability estimates of nonrepairable systems and progress gradually to the repairable systems.

3.2 NONREPAIRABLE SYSTEMS

The number of nonrepairable systems and products is on the rise due to the increasing cost of labor and the high rate of technological obsolescence of many products. For example, the rate of the technological advances in the development of computer chips renders the repair of a 2-year-old personal computer unnecessary since the advances in these 2 years may result in significantly less expensive and faster (clock speed) computers. Likewise, the advancements in hardware, software, and new features make cell phones nonrepairable products, in most cases. Other nonrepairable systems include, until recently, satellites, single-mission products such as rockets, and inexpensive radios and electronic devices.

Reliability Engineering, Second Edition. Elsayed A. Elsayed.

3.2.1 Series Systems

Assuming n independent components arranged in series with a reliability of one for each component at time $t = 0$—that is, $R_i(0) = 1$, $(i = 1, 2, \ldots, n)$. The reliability of the system at time t is the probability that all components survive to time t, thus

$$R_S(t) = R_1(t)R_2(t)\ldots R_n(t) = \prod_{i=1}^{n} R_i(t). \tag{3.1}$$

When each component has a constant hazard, the reliability of component i at time t is expressed as

$$R_i(t) = e^{-\lambda_i t}, \tag{3.2}$$

where $R_i(t)$ is the reliability of component i at time t, and λ_i is a constant failure rate of component i. Substituting Equation 3.2 into Equation 3.1, we obtain

$$R_s(t) = \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-\sum_{i=1}^{n} \lambda_i t}. \tag{3.3}$$

Thus, the effective failure rate of a series system composed of n components is the sum of the failure rates of the individual components.

Equation 3.3 is valid only under the assumptions that all components are independent and that each one of them exhibits a constant hazard. If the hazard rate of component i is $h_i(t)$ and the cumulative hazard is $H_i(t) = \int_0^t h_i(\zeta)d\zeta$, then we can generalize Equation 3.3 for a series system as

$$R_S(t) = \prod_{i=1}^{n} e^{-H_i(t)} = e^{-\sum_{i=1}^{n} H_i(t)}. \tag{3.4}$$

We now illustrate the use of Equation 3.4 to estimate the reliability of a series system when the components have different hazard rates.

- For components with linearly increasing hazard rates—$h_i(t) = k_i t$—the reliability of the system is obtained as

$$R_S(t) = \prod_{i=1}^{n} e^{-k_i t^2/2} = e^{-\sum_{i=1}^{n} \frac{k_i t^2}{2}}. \tag{3.5}$$

- For components with Weibull hazard—$h_i(t) = (\gamma_i/\theta_i)(t/\theta_i)^{\gamma_i - 1}$—the reliability of the system is obtained as

$$R_S(t) = \exp\left[-\sum_{i=1}^{n} \left(\frac{t}{\theta_i}\right)^{\gamma_i}\right]. \tag{3.6}$$

- When r components have constant hazard rates and $n - r$ components have Weibull hazard rates,

$$R_S(t) = \prod_{i=1}^{r} e^{-\lambda_i t} \prod_{i=r+1}^{n} e^{-\left(\frac{t}{\theta_i}\right)^{\gamma_i}}$$

or

$$R_S(t) = \exp\left[-\sum_{i=1}^{r} \lambda_i t - \sum_{i=r+1}^{n} \left(\frac{t}{\theta_i}\right)^{\gamma_i}\right]. \tag{3.7}$$

EXAMPLE 3.1

A series system consists of five components, three of which have constant failure rates $\lambda_1 = 5 \times 10^{-6}$, $\lambda_2 = 3 \times 10^{-6}$, and $\lambda_3 = 9 \times 10^{-6}$. The remaining two components exhibit Weibull hazards that have the following parameters: $\theta_1 = 1.5 \times 10^4$, $\gamma_1 = 2.2$, $\theta_2 = 2.5 \times 10^4$, and $\gamma_2 = 2.1$. Determine the reliability of the system at $t = 1000$ h.

SOLUTION

The exponent of Equation 3.7 at $t = 1000$ is

$$\begin{aligned}
&= -\sum_{i=1}^{3} \lambda_i t - \sum_{i=1}^{2} \left(\frac{t}{\theta_i}\right)^{\gamma_i} \\
&= -(17 \times 10^{-6})1000 - \left(\frac{1000}{1.5 \times 10^4}\right)^{2.2} - \left(\frac{1000}{2.5 \times 10^4}\right)^{2.1} \\
&= -0.021.
\end{aligned}$$

The reliability of the system is

$$R_S(1000) = e^{-0.021} = 0.9795.$$

Of course, the plot of $R_S(t)$ with time can be utilized in determining the time at which an unacceptable reliability value is attained. It can also be used to investigate the effect of a component on the overall system reliability. This will enable the system's designer to investigate the use of different components, redundancies, and other approaches for reliability improvements. ■

3.2.2 Parallel Systems

As shown in Chapter 2, a parallel system fails if and only if all components fail. The reliability of an n components parallel system is expressed as

$$R_S(t) = P(x_1 + x_2 + \ldots + x_n) = 1 - P(\bar{x}_1 \bar{x}_2 \ldots \bar{x}_n). \tag{3.8}$$

In the case of constant hazard independent components, the unreliability of component i is $1 - e^{-\lambda_i t}$, and the reliability of the system is obtained by using Equation 3.8 as follows:

$$R_S(t)=1-\prod_{i=1}^{n}(1-e^{-\lambda_i t}). \tag{3.9}$$

The effective hazard rate of a two-component parallel system is obtained as follows. Using Equation 3.9 and $n = 2$, the reliability of the system, $R_S(t)$, is

$$\begin{aligned} R_S(t) &= 1-(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t}) \\ &= e^{-\lambda_1 t}+e^{-\lambda_2 t}-e^{-(\lambda_1+\lambda_2)t}. \end{aligned} \tag{3.10}$$

Since $h(t) = f(t)/R(t)$ and $f(t) = -dR(t)/dt$, then using Equation 3.10,

$$f(t)=\lambda_1 e^{-\lambda_1 t}+\lambda_2 e^{-\lambda_2 t}-(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}, \tag{3.11}$$

and the effective hazard rate (failure rate) of the system is

$$h(t)=\frac{\lambda_1 e^{-\lambda_1 t}+\lambda_2 e^{-\lambda_2 t}-(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}}{e^{-\lambda_1 t}+e^{-\lambda_2 t}-e^{-(\lambda_1+\lambda_2)t}}. \tag{3.12}$$

It is important to note that the hazard rate of a series system whose components have constant hazard rates is also constant. On the other hand, the effective hazard rate of a pure parallel configuration is not constant. Indeed, it is a function of time and may stabilize at a constant value after extended time, depending on the values of the hazard rates of the components. Figure 3.1 shows the system hazard rate given by Equation 3.12 when $\lambda_1 = 0.009$ and $\lambda_2 = 0.008$ failures/unit time.

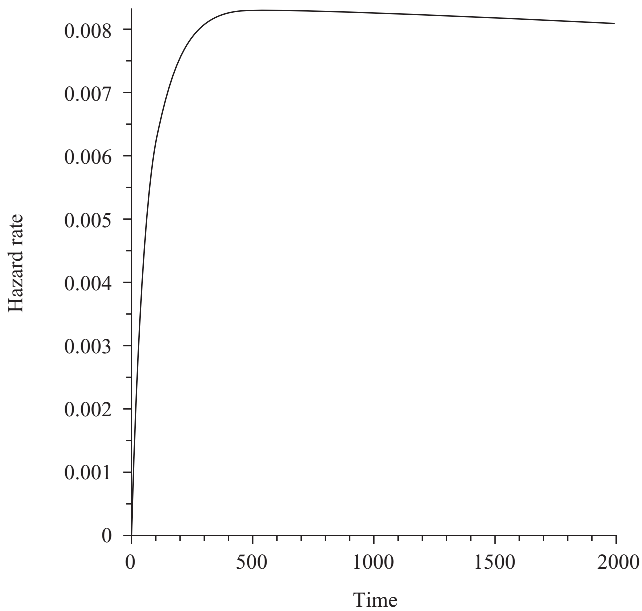

FIGURE 3.1 Effective hazard rate of a pure parallel system consisting of components with constant hazard rates.

EXAMPLE 3.2

Consider a parallel system with two components having constant hazard rates of $\lambda_1 = 0.5 \times 10^{-6}$ and $\lambda_2 = 0.3 \times 10^{-6}$ failures per hour. What is the reliability of the system at $t = 1000$ h? What is the effective hazard rate of the system? What is the effect of λ_1 and λ_2 on $h(t)$ at $t = 800$ h?

SOLUTION

From Equation 3.10, we obtain

$$R_S(1000) = e^{-0.5\times10^{-6}\times10^3} + e^{-0.3\times10^{-6}\times10^3} - e^{-0.8\times10^{-6}\times10^3}$$

or

$$R_S(1000) = 0.99999.$$

The effective hazard rate at 1000 h is obtained by substituting the hazard-rate parameters in Equation 3.12 as follows:

$$h(1000) = \left[0.5\times10^{-6} e^{-0.5\times10^{-6}\times1000} + 0.3\times10^{-6} e^{-0.3\times10^{-6}\times1000} - 0.8\times10^{-6} e^{-0.8\times10^{-6}\times1000}\right] / R_S(1000)$$

or

$$h(1000) = \frac{2.99820\times10^{-10}}{0.99999} = 2.99823\times10^{-10} \text{ failures per hour.}$$

The effect of λ_1 and λ_2 on the hazard rate $h(t)$ is shown in Figure 3.2. It is obvious that the effective hazard rate is much smaller than that of either of the components. In other words, the system reliability is higher than reliability of either component.

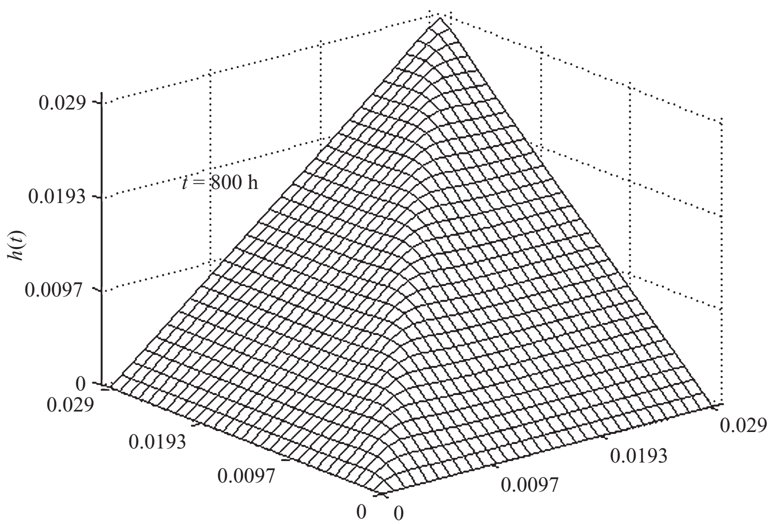

FIGURE 3.2 Effect of λ_1 and λ_2 on the effective hazard rate of a parallel system. ■

In general, the reliability of a parallel system with n components each having a hazard rate $h_i(t)$ is expressed as

$$R_S(t)=1-\prod_{i=1}^{n}(1-e^{-H_i(t)}), \tag{3.13}$$

where

$$H_i(t)=\int_0^t h_i(\zeta)d\zeta.$$

We now illustrate the use of Equation 3.13 in estimating the reliability of a parallel when the components have different hazard rates:

- For components with linearly increasing hazard rates,

$$R_S(t)=1-\prod_{i=1}^{n}(1-e^{-k_i t^2/2}), \tag{3.14}$$

and

- For components with Weibull hazard rates,

$$R_S(t)=1-\prod_{i=1}^{n}\left(1-e^{-\left(\frac{t}{\theta_i}\right)^{\gamma_i}}\right). \tag{3.15}$$

The expansion of Equation 3.13 can be easily obtained as follows (Shooman, 1968):

$$(1-y_1)(1-y_2)\ldots(1-y_n)=1-\sum_{i=1}^{n}y_i+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}y_iy_j-\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}y_iy_jy_k+\ldots+(-1)^n\prod_{i=1}^{n}y_i. \tag{3.16}$$

Using the above expansion, we now simplify Equation 3.13 as

$$R_S(t)=\left[\sum_{i=1}^{n}e^{-H_i(t)}\right]-\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}e^{-(H_i(t)+H_j(t))}\right]+\left[\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}e^{-(H_i(t)+H_j(t)+H_k(t))}\right]-\ldots \tag{3.17}$$

The rth parentheses in Equation 3.17 contains $n!/[r!(n-r)!]$ terms.

EXAMPLE 3.3

Determine the reliability of a three-component parallel system at time $t = 100$ h when the components exhibit linearly increasing hazard rates. The coefficients of the hazard rates are

$$k_1=2.5\times10^{-6}, k_2=4\times10^{-6}, \text{and } k_3=3.5\times10^{-6}.$$

SOLUTION

Using Equation 3.14, we obtain

$$R_S(100) = e^{\frac{-2.5}{2}\times10^{-6}\times10^4} + e^{\frac{-4}{2}\times10^{-6}\times10^4} + e^{\frac{-3.5}{2}\times10^{-6}\times10^4}$$
$$-\left[e^{\frac{-6.5}{2}\times10^{-6}\times10^4} + e^{\frac{-6}{2}\times10^{-6}\times10^4} + e^{\frac{-7.5}{2}\times10^{-6}\times10^4}\right] + \left[e^{\frac{-10}{2}\times10^{-6}\times10^4}\right]$$
$$R_S(100) = 0.9999957.$$

■

3.2.3 *k*-out-of-*n* Systems

In these types of systems, any combination of k operating components out of n independent components guarantees successful operation of the system. Wire ropes which consist of several wires to form a strand are typical k-out-of-n systems where a minimum of k wires must carry the load for the rope to function properly. Such ropes and cables exist in suspension bridges and cranes. If the components are not identical, we should investigate every possible successful path of the reliability structure in order to accurately estimate the reliability of the system. Fortunately, most k-out-of-n systems have independent and identical components, and the reliability of the system is much simpler to estimate by using the binomial distribution. In a typical k-out-of-n system with components having equal constant failure rates, the reliability of the system is

$$R_S(t) = \sum_{r=k}^{n} \binom{n}{r} \left(e^{-\lambda t}\right)^r \left(1 - e^{-\lambda t}\right)^{n-r}$$

or

$$R_S(t) = \sum_{r=k}^{n} \binom{n}{r} e^{-r\lambda t} \left(1 - e^{-\lambda t}\right)^{n-r} = 1 - \sum_{r=0}^{k-1} \binom{n}{r} e^{-r\lambda t} \left(1 - e^{-\lambda t}\right)^{n-r}. \tag{3.18}$$

Similarly, the reliabilities of a k-out-of-n system when the components exhibit linear or Weibull hazard rates are given by Equations 3.19 and 3.20, respectively. (In order to avoid confusion, for the moment, we replace the constant k of the linear hazard model by another constant λ).

$$R_S(t) = \sum_{r=k}^{n} \binom{n}{r} e^{-r\lambda t^2/2} \left(1 - e^{-\lambda t^2/2}\right)^{n-r} \tag{3.19}$$

and

$$R_S(t) = \sum_{r=k}^{n} \binom{n}{r} e^{-r\left(\frac{t}{\theta}\right)^{\gamma}} \left(1 - e^{-\left(\frac{t}{\theta}\right)^{\gamma}}\right)^{n-r}. \tag{3.20}$$

EXAMPLE 3.4

Consider a 2-out-of-3 system with components that exhibit constant failure rates with parameter λ. What is the reliability of the system? If $\lambda = 3.0 \times 10^{-5}$ failures per hour, determine the reliability at time $t = 1000$ h.

SOLUTION

Using Equation 3.18, we obtain

$$\begin{aligned} R_S(t) &= \sum_{r=2}^{3} \binom{3}{r} e^{-r\lambda t}\left[1-e^{-\lambda t}\right]^{3-r} \\ &= \binom{3}{2} e^{-2\lambda t}\left[1-e^{-\lambda t}\right] + \binom{3}{3} e^{-3\lambda t} \\ &= 3e^{-2\lambda t} - 3e^{-3\lambda t} + e^{-3\lambda t} = 3e^{-2\lambda t} - 2e^{-3\lambda t}. \end{aligned}$$

Substitute $\lambda = 3.0 \times 10^{-5}$ and $t = 1000$.

$$R_S(1000) = 3e^{-6\times 10^{-2}} - 2e^{-9\times 10^{-2}} = 0.9974.$$

■

EXAMPLE 3.5

In a 2-out-of-n system with components having a constant hazard rate 0.4×10^{-4} failures per hour, determine the number of components for the system such that a reliability of 0.966 is achieved at $t = 1000$ h.

SOLUTION

The reliability of any component in the system at $t = 1000$ h is $\exp(-0.4 \times 10^{-4} \times 1000) = 0.96078$. Thus,

$$\begin{aligned} 0.966 &= 1 - \sum_{r=0}^{1} \binom{n}{r} (0.96078)^r (0.03922)^{n-r} \\ &= 1 - \left[\binom{n}{0}(0.03922)^n + \binom{n}{1}(0.96078)(0.03922)^{n-1}\right] \\ &= 1 - 0.03922^n - n(0.96078)(0.03922)^{n-1} \end{aligned}$$

or

$$0.034 \le 0.03922^n + n(0.96078)(0.03922)^{n-1}.$$

Solving the above inequality results in $n = 2$ components. In other words, at most, two components should be used to achieve the desired reliability over a time period of 1000 h. ■

3.3 MEAN TIME TO FAILURE (MTTF)

MTTF is one of the most widely used measures of reliability. It is simply defined as the expected or mean value $E[T]$ of the failure time T. Thus,

$$MTTF = \int_0^\infty t f(t) dt. \tag{3.21}$$

The MTTF may be expressed directly in terms of the system reliability by substituting the following relationship into Equation 3.21:

$$f(t) = -\frac{dR(t)}{dt}$$

$$MTTF = -\int_0^\infty t \frac{dR(t)}{dt} dt$$

or

$$MTTF = -tR(t)\Big|_0^\infty + \int_0^\infty R(t) dt. \tag{3.22}$$

Since $tR(t) \to 0$ as $t \to 0$ and $tR(t) \to 0$ as $t \to \infty$, then Equation 3.22 can be written as

$$MTTF = \int_0^\infty R(t) dt. \tag{3.23}$$

The MTTF in itself is the mean of the failure time; it does not provide additional information about the distribution of the TTF (time to failure). In order to do so, we need to determine the standard deviation of the TTF. This serves as an indicator of the dispersion of the TTF which in turn has a direct impact on the warranty period and cost.

By definition, the standard deviation of the TTF is given as

$$\sigma_{TTF} = \sqrt{\int_0^\infty t^2 f(t) dt - MTTF^2}. \tag{3.24}$$

The following sections show how the MTTF is calculated for different systems.

3.3.1 MTTF for Series Systems

The MTTF for series systems with n components each having constant, linearly increasing, and Weibull hazard rates is given below.

3.3.1.1 Constant Hazard The reliability expression for a series system with constant hazard rates is given by Equation 3.3. The MTTF of such a system is

$$MTTF = \int_0^\infty e^{-\sum_{i=1}^{n} \lambda_i t} dt$$

or

$$MTTF = \frac{1}{\sum_{i=1}^{n} \lambda_i}.$$

3.3.1.2 Linearly Increasing Hazard The reliability of a component with linearly increasing hazard is

$$R(t) = e^{-kt^2/2},$$

and the MTTF is

$$MTTF = \int_0^\infty e^{-kt^2/2} dt = \frac{\Gamma(1/2)}{2\sqrt{k/2}} = \sqrt{\frac{\pi}{2k}}.$$

For a system with n components in series and each having a linearly increasing hazard, the MTTF is

$$MTTF = \sqrt{\frac{\pi}{2\sum_{i=1}^{n} k_i}}. \tag{3.25}$$

3.3.1.3 Weibull Hazard For a system composed of one component having a Weibull hazard rate, the MTTF is obtained as follows:

$$MTTF = \int_0^\infty R(t)dt$$

or

$$MTTF = \int_0^\infty e^{-\left(\frac{t}{\theta}\right)^\gamma} dt. \tag{3.26}$$

Let $x = \left(\frac{t}{\theta}\right)^\gamma$, then $dt = \frac{\theta}{\gamma} x^{\frac{1}{\gamma}-1} dx.$

Substituting in Equation 3.26, we obtain

$$MTTF = \frac{\theta}{\gamma} \int_0^\infty e^{-x} x^{\frac{1}{\gamma}-1} dx$$

or

$$MTTF = \theta \frac{1}{\gamma} \Gamma\left(\frac{1}{\gamma}\right) = \theta \Gamma\left(1 + \frac{1}{\gamma}\right). \tag{3.27}$$

The values of $\Gamma(x)$ for different x are given in Appendix A.

If n components form a series configuration and all components exhibit Weibull hazards with the same value of γ, then Equation 3.27 can be rewritten as

$$MTTF = \left(\frac{1}{\sum_{i=1}^{n} \left(\frac{1}{\theta_i} \right)^{\gamma}} \right)^{\frac{1}{\gamma}} \Gamma\left(1 + \frac{1}{\gamma}\right). \tag{3.28}$$

EXAMPLE 3.6

A series system consists of six components that exhibit the same shape parameter of a Weibull distribution. The shape parameter is 1.75, and the scale parameters of the components are 7.0×10^5, 8.2×10^5, 4.6×10^5, 6.5×10^5, 6.8×10^5, and 5×10^5. Determine the MTTF of the system.

SOLUTION

Using Equation 3.28, we obtain

$$MTTF = \left(2.162 \times 10^9\right)^{\frac{1}{1.75}} \Gamma\left(1 + \frac{1}{1.75}\right) = 1.9226 \times 10^5 \text{ h.}$$

■

3.3.2 MTTF for Parallel Systems

The calculations of the MTTF for parallel systems are similar to those of the series systems. Again, the MTTFs for different hazard functions are obtained as shown below.

3.3.2.1 Constant Hazard Consider a parallel system consisting of n independent components and that the failure rate, λ_i, of component i is constant. The MTTF of the system is

$$MTTF = \int_0^{\infty} R(t)dt = \int_0^{\infty} \left[\sum_{i=1}^{n} e^{-\lambda_i t} - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} e^{-(\lambda_i + \lambda_j)t} + \dots \right] dt$$

or

$$MTTF = \sum_{i=1}^{n} \frac{1}{\lambda_i} - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{\lambda_i + \lambda_j} + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \frac{1}{\lambda_i + \lambda_j + \lambda_k} - \dots + (-1)^{n+1} \frac{1}{\sum_{i=1}^{n} \lambda_i}. \tag{3.29}$$

If all components are identical and each component has a failure rate λ, then

$$R_S(t) = 1 - (1 - e^{-\lambda t})^n$$

and

$$MTTF = \frac{1}{\lambda}\left[1 + \frac{1}{2} + \ldots + \frac{1}{n}\right]. \tag{3.30}$$

Equation 3.30 implies that in active redundancy where each component exhibits one type of failure mode, the MTTF of the system exceeds the MTTF of the individual component, and the contribution of the second component and other additional components would have a diminishing return on the system's MTTF as n increases. In other words, there is an optimum n at which the cost of adding a component in parallel far exceeds the gained benefit in the MTTF.

3.3.2.2 Linearly Increasing Hazard The components are assumed to have linearly increasing hazard rates. In other words, each component i has a linearly increasing hazard, $k_i t$. The MTTF of such a system is

$$MTTF = \int_0^\infty \left[\sum_{i=1}^{n} e^{-1/2k_i t^2} - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} e^{-1/2(k_i+k_j)t^2} + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} e^{-1/2(k_i+k_j+k_k)t^2} - \ldots\right] dt$$

or

$$MTTF = \sum_{i=1}^{n}\sqrt{\frac{\pi}{2k_i}} - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sqrt{\frac{\pi}{2(k_i+k_j)}} + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\sqrt{\frac{\pi}{2(k_i+k_j+k_k)}} - \ldots \tag{3.31}$$

If all components are identical with a hazard rate kt, then

$$MTTF = \sqrt{\frac{\pi}{2k}}\left[n - \binom{n}{2}\sqrt{\frac{1}{2}} + \binom{n}{3}\sqrt{\frac{1}{3}} - \binom{n}{4}\sqrt{\frac{1}{4}} + \ldots\right] \tag{3.32}$$

where $\binom{n}{j} = \frac{n!}{j!(n-j)!}$.

EXAMPLE 3.7

An active redundant system consists of four identical parallel components each having a linearly increasing hazard rate, kt, with $k = 3.5 \times 10^{-6}$ failures per hour. Determine the MTTF of the system.

SOLUTION

Using Equation 3.32 with $n = 4$ and $k = 3.5 \times 10^{-6}$, we obtain the MTTF of the system as

$$MTTF = \sqrt{\frac{\pi}{7 \times 10^{-6}}}\left[4 - 6\sqrt{\frac{1}{2}} + 4\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{4}}\right]$$

or

$$MTTF = 970.184 \text{ h}.$$ ■

3.3.2.3 Weibull Hazard The MTTF of an active redundancy system that consists of n components in parallel and each component exhibits a Weibull hazard of the form $(\gamma/\theta_i)(t/\theta_i)^{\gamma-1}$, where θ_i is a constant for component i, and γ is the same shape parameter for all the components, is obtained as

$$MTTF = \Gamma\left(1+\frac{1}{\gamma}\right)\left\{\sum_{i=1}^{n}(\theta_i^{\gamma})^{\frac{1}{\gamma}} - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\frac{\theta_i^{\gamma}\theta_j^{\gamma}}{\theta_i^{\gamma}+\theta_j^{\gamma}}\right)^{\frac{1}{\gamma}} + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left(\frac{\theta_i^{\gamma}\theta_j^{\gamma}\theta_k^{\gamma}}{\theta_i^{\gamma}\theta_j^{\gamma}+\theta_j^{\gamma}\theta_k^{\gamma}+\theta_i^{\gamma}\theta_k^{\gamma}}\right)^{\frac{1}{\gamma}} - \ldots\right\}. \tag{3.33}$$

EXAMPLE 3.8

Solve Example 3.7 when the system consists of three components in parallel and their hazard rates are

$$h_1(t) = \frac{2.5}{10\times 10^6}(t)^{1.5}$$

$$h_2(t) = \frac{2.5}{12.5\times 10^6}(t)^{1.5}$$

$$h_3(t) = \frac{2.5}{10.25\times 10^6}(t)^{1.5}.$$

SOLUTION

From the above hazard-rate functions we obtain

$$\theta_1 = 631$$

$$\theta_2 = 690$$

$$\theta_3 = 637$$

$$MTTF = \Gamma\left(1+\frac{1}{2.5}\right)\{(631)+(690)+(637)-\left(\frac{631^{2.5}\times 690^{2.5}}{631^{2.5}+690^{2.5}}\right)^{\frac{1}{2.5}}$$

$$-\left(\frac{631^{2.5}\times 637^{2.5}}{631^{2.5}+637^{2.5}}\right)^{\frac{1}{2.5}} - \left(\frac{690^{2.5}\times 637^{2.5}}{690^{2.5}+637^{2.5}}\right)^{\frac{1}{2.5}}$$

$$+\left(\frac{631^{2.5}\times 690^{2.5}\times 637^{2.5}}{631^{2.5}\times 690^{2.5}+631^{2.5}\times 637^{2.5}+690^{2.5}\times 637^{2.5}}\right)^{\frac{1}{2.5}}$$

or

$$MTTF = 896.68\times 0.8873 = 795.62 \text{ h.}$$

Clearly, the MTTF is greater than the characteristic life (θ) of any of the components since the units are connected in parallel. However, the shape parameter is greater than 1, which implies that the failure rate is increasing with time and the system should be either redesigned or the components should be replaced by others with a much reduced failure rate. ■

3.3.3 *k*-out-of-*n* Systems

The reliability expression for a k-out-of-n system whose components are independent and identical is

$$R_S(t)=\sum_{r=k}^{n}\binom{n}{r}[p(t)]^r[1-p(t)]^{n-r}, \tag{3.34}$$

where $p(t)$ is the reliability of the component at time t. There is no general expression for the MTTF of a k-out-of-n system since it depends on the values of k and n. Therefore, we illustrate the procedure for obtaining the MTTF for a k-out-of-n system for different hazard rates through the following examples.

3.3.3.1 Constant Hazard Consider a k-out-of-n system whose components are independent and identical. Each component exhibits a constant hazard rate λ. The MTTF of this system is obtained by substituting $p(t) = e^{-\lambda t}$ into Equation 3.34:

$$MTTF=\int_0^{\infty}\sum_{r=k}^{n}\binom{n}{r}\left(e^{-\lambda t}\right)^r\left(1-e^{-\lambda t}\right)^{n-r}dt. \tag{3.35}$$

EXAMPLE 3.9

Determine the MTTF of a 2-out-of-4 system with independent components each having a constant hazard of 8.5×10^{-6} failures per hour.

SOLUTION

We first derive a reliability expression for the system, and then estimate its MTTF as $\int_0^{\infty} R_S(t)dt$.

$$\begin{aligned}R_S(t)&=\sum_{r=2}^{4}\binom{4}{r}(e^{-\lambda t})^r(1-e^{-\lambda t})^{4-r}\\&=\binom{4}{2}e^{-2\lambda t}\left(1-2e^{-\lambda t}+e^{-2\lambda t}\right)+\binom{4}{3}e^{-3\lambda t}\left(1-e^{-\lambda t}\right)+\binom{4}{4}e^{-4\lambda t}\\&=6e^{-2\lambda t}-12e^{-3\lambda t}+6e^{-4\lambda t}+4e^{-3\lambda t}-4e^{-4\lambda t}+e^{-4\lambda t}\end{aligned}$$

or

$$R_s(t)=6e^{-2\lambda t}-8e^{-3\lambda t}+3e^{-4\lambda t}$$

$$MTTF=\int_0^{\infty}R_s(t)dt=\frac{13}{12\lambda}=1.2745\times 10^5 \text{ h.}$$

■

3.3.3.2 Linearly Increasing Hazard The MTTF of a k-out-of-n system, when all components are independent, identical, and exhibit linearly increasing hazards, is determined by substituting $p(t) = e^{-kt^2/2}$ in Equation 3.34 to obtain a reliability expression of the system. Then the resulting expression is integrated with respect to t from 0 to ∞ as shown in the following example.

EXAMPLE 3.10

Determine the MTTF for the system given in Example 3.9 when the failure rates of the components are linearly increasing with parameter $k = 2.7 \times 10^{-4}$.

SOLUTION

The reliability of the system is

$$\begin{aligned} R_S(t) &= \binom{4}{2}\left(e^{-kt^2/2}\right)^2\left(1-e^{-kt^2/2}\right)^2 + \binom{4}{3}\left(e^{-kt^2/2}\right)^3\left(1-e^{-kt^2/2}\right) + \binom{4}{4}\left(e^{-kt^2/2}\right)^4 \\ &= 6e^{-kt^2}\left(1-2e^{-kt^2/2}+e^{-kt^2}\right) + 4e^{-3kt^2/2}\left(1-e^{-kt^2/2}\right) + e^{-2kt^2} \end{aligned}$$

or

$$R_S(t) = 6e^{-kt^2} - 8e^{-3kt^2/2} + 3e^{-2kt^2}.$$

The MTTF of the system is

$$MTTF = \int_0^\infty R_S(t)dt = 6\sqrt{\frac{\pi}{4k}} - 8\sqrt{\frac{\pi}{6k}} + 3\sqrt{\frac{\pi}{8k}} = 85.7 \times 10^3 \text{ h.}$$

■

3.3.3.3 Weibull Hazard Similar to the linearly increasing hazard, we calculate the MTTF of a k-out-of-n system composed of independent and identical components that exhibit Weibull hazard by first deriving an expression for the system reliability as shown in the following example.

EXAMPLE 3.11

Determine the MTTF of the system given in Example 3.9 if the components are independent, identical, and exhibit a Weibull hazard with parameters $\theta = 5 \times 10^2$ and $\gamma = 2.1$.

SOLUTION

The reliability of the system is

$$R_S(t) = \binom{4}{2}\left(e^{-\left(\frac{t}{\theta}\right)^\gamma}\right)^2\left(1-e^{-\left(\frac{t}{\theta}\right)^\gamma}\right)^2 + \binom{4}{3}\left(e^{-\left(\frac{t}{\theta}\right)^\gamma}\right)^3\left(1-e^{-\left(\frac{t}{\theta}\right)^\gamma}\right) + \binom{4}{4}\left(e^{-\left(\frac{t}{\theta}\right)^\gamma}\right)^4$$

or

$$R_S(t) = 6e^{-2\left(\frac{t}{\theta}\right)^\gamma} - 8e^{-3\left(\frac{t}{\theta}\right)^\gamma} + 3e^{-4\left(\frac{t}{\theta}\right)^\gamma}.$$

The MTTF is obtained as

$$MTTF = \int_0^\infty R_S(t)dt$$
$$= \frac{1}{\gamma}\left[6\frac{\theta}{2^{1/\gamma}}\Gamma\left(\frac{1}{\gamma}\right) - 8\frac{\theta}{3^{1/\gamma}}\Gamma\left(\frac{1}{\gamma}\right) + 3\frac{\theta}{4^{1/\gamma}}\Gamma\left(\frac{1}{\gamma}\right)\right]$$

or

$$MTTF = 0.88[2156.6 - 2370.6 + 775.2] = 493.83\ \text{h}.$$

Note that the characteristic life of a component is 500 h, but the system's MTTF is 493 even though it has an implicit redundancy. This is attributed to the shape parameter of the failure rate. ■

3.3.4 Other Systems

The estimation of the MTTF of any system requires the derivation of an expression for the reliability of the system. This expression is then integrated over time from 0 to ∞. When the system structure is not a standard structure such as series, parallel, k-out-of-n, we follow the same procedures described in Chapter 2 for the reliability estimation of complex structures to obtain an expression for $R_S(t)$ as shown in Example 3.12.

EXAMPLE 3.12

Determine the MTTF of the complex reliability structure system shown in Figure 3.3. Assume that the components are independent, identical, and exhibit a constant failure rate $\lambda = 3.5 \times 10^{-5}$ failures per hour.

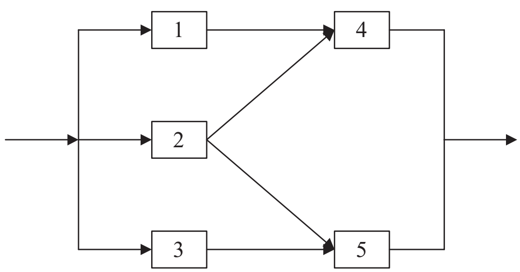

FIGURE 3.3 A complex reliability structure.

SOLUTION

The reliability of this structure can be estimated using any of the methods discussed in Chapter 2 such as cut-set or tie-set approaches. The reliability of the system is

$$R_S(t) = e^{-5\lambda t} - e^{-4\lambda t} - 3e^{-3\lambda t} + 4e^{-2\lambda t},$$

The MTTF is obtained as

$$MTTF = \int_0^\infty \left(e^{-5\lambda t} - e^{-4\lambda t} - 3e^{-3\lambda t} + 4e^{-2\lambda t}\right) dt$$
$$= \frac{1}{5\lambda} - \frac{1}{4\lambda} - \frac{3}{3\lambda} + \frac{4}{2\lambda} = 2.7142 \times 10^4 \text{ h}$$

or

$$MTTF \cong 3.098 \text{ years.}$$ ■

A summary of the MTTF expressions for different configurations and hazard rates is given in Table 3.1.

TABLE 3.1 MTTF for Different Configurations

Configuration	Hazard rate	MTTF
Series (n units in series)	λ_i	$\dfrac{1}{\sum_{i=1}^{n} \lambda_i}$
	$k_i t$	$\sqrt{\dfrac{\pi}{2\sum_{i=1}^{n} k_i}}$
	$\dfrac{\gamma}{\theta_i}\left(\dfrac{t}{\theta_i}\right)^{\gamma-1}$	$\left(\dfrac{1}{\sum_{i=1}^{n}\left(\dfrac{1}{\theta_i}\right)^{\gamma}}\right)^{\frac{1}{\gamma}} \Gamma\left(1+\dfrac{1}{\gamma}\right)$
Parallel (n units in parallel)	λ_i	$\sum_{i=1}^{n} \dfrac{1}{\lambda_i} - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \dfrac{1}{\lambda_i+\lambda_j} + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} \dfrac{1}{\lambda_i+\lambda_j+\lambda_k} - \cdots$
	$k_i t$	$\sum_{i=1}^{n} \sqrt{\dfrac{\pi}{2k_i}} - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \sqrt{\dfrac{\pi}{2(k_i+k_j)}} + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} \sqrt{\dfrac{\pi}{2(k_i+k_j+k_k)}} - \cdots$
	$\dfrac{\gamma}{\theta_i}\left(\dfrac{t}{\theta_i}\right)^{\gamma-1}$	$\Gamma\left(1+\dfrac{1}{\gamma}\right)\left\{\sum_{i=1}^{n} \theta_i - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \left(\dfrac{\theta_i^\gamma \theta_j^\gamma}{\theta_i^\gamma + \theta_j^\gamma}\right)^{\frac{1}{\gamma}} + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} \left(\dfrac{\theta_i^\gamma \theta_j^\gamma \theta_k^\gamma}{\theta_i^\gamma\theta_j^\gamma + \theta_j^\gamma\theta_k^\gamma + \theta_i^\gamma\theta_k^\gamma}\right)^{\frac{1}{\gamma}} \cdots\right\}$
k-out-of-n	λ	$\int_0^\infty \sum_{r=k}^{n} \binom{n}{r} \left(e^{-\lambda t}\right)^r \left(1-e^{-\lambda t}\right)^{n-r} dt$

3.4 REPAIRABLE SYSTEMS

Repairable systems are those systems that are repaired upon failure. Repairable systems include large and complex systems, automobiles, airplanes, HVAC (heating, ventilation, and air conditioning), mainframe computers, telephone networks, electric grid, water distribution networks, and many others. In Chapter 2, we illustrated the use of redundant components (or systems) to improve the overall system reliability. Other methods of improving system reliability include the use of "highly" reliable components (prime material free of manufacturing defects as an example) and the use of efficient repair and maintenance systems such as condition-based maintenance (to be discussed in a later chapter). Two of the most important performance criteria of repairable systems are availability (there are several measures of availability which will be discussed later in this chapter) and the MTBF. Conventionally, we use MTBF for repairable systems and MTTF for nonrepairable systems. For now, we define availability as the probability that the system is operating properly (or available for use) when it is requested for use. It is important to note that availability is the key measure of a repairable system's reliability.

Since repair and maintenance have a major impact on the system availability, we devote Chapter 8 to discuss in detail different repairs, replacements, and preventive maintenance policies.

In this section, we present two approaches for estimating the availability of repairable systems. The first is the alternating renewal process, and the second is the Markov process.

3.4.1 Alternating Renewal Process

Consider a repairable system that has a failure-time distribution with a probability density function (p.d.f.) $w(t)$, and a repair-time distribution with a p.d.f. $g(t)$. When the system fails, it is repaired and restored to its initial working condition. The process of failure and repair is repeated. We refer to this process as an alternating renewal process. We define $f(t)$ and $n(t)$ as the density function of the renewal process and the density function of the number of renewals, respectively. The underlying density function $f(t)$ of the renewal process is the convolution of w and g. In other words,

$$f^*(s) = w^*(s)g^*(s), \tag{3.36}$$

where $f^*(s)$, $w^*(s)$, and $g^*(s)$ are the Laplace transforms of the corresponding density functions.

As shown later in Chapter 7, the Laplace transform of the renewal density equation is

$$n^*(s) = \frac{f^*(s)}{1 - f^*(s)}, \tag{3.37}$$

where $f^*(s) = \int_0^\infty e^{-st} f(t)dt$.

Substituting Equation 3.37 into Equation 3.36 results in

$$n^*(s) = \frac{w^*(s)g^*(s)}{1 - w^*(s)g^*(s)}. \tag{3.38}$$

The component (or system) may be functioning at time t if either it had not failed during the time interval $(0, t]$ with probability $R(t)$ or the last repair occurred at time x, $0 < x < t$, and the component (or system) continued to function properly since that time with probability $\int_0^t R(t-x)n(x)dx$. Thus, the availability of the component (or system) at time t is the sum of the two probabilities, or

$$A(t) = R(t) + \int_0^t R(t-x)n(x)dx. \tag{3.39}$$

The Laplace transform of Equation 3.39 is

$$A^*(s) = R^*(s)\left[1 + n^*(s)\right]. \tag{3.40}$$

Substituting Equation 3.38 into Equation 3.40, we obtain

$$A^*(s) = R^*(s)\left[1 + \frac{w^*(s)g^*(s)}{1 - w^*(s)g^*(s)}\right]$$

or

$$A^*(s) = \frac{R^*(s)}{1 - w^*(s)g^*(s)}.$$

Since $R(t) = 1 - W(t) = 1 - \int_0^t w(\tau)d\tau$, then $R^*(s)$ is

$$R^*(s) = \frac{1 - w^*(s)}{s}$$

and

$$A^*(s) = \frac{1 - w^*(s)}{s\left[1 - w^*(s)g^*(s)\right]}. \tag{3.41}$$

The Laplace inverse of $A^*(s)$ results in obtaining the point availability $A(t)$. Often, a closed-form expression of the inverse of $A^*(s)$ is difficult to obtain, and numerical solutions or approximations become the only alternatives for obtaining $A(t)$. The steady state availability, A, is

$$A = \lim_{t\to\infty} A(t) = \lim_{s\to 0} sA^*(s).$$

When s is small, then $e^{-st} \cong 1 - st$ and

$$w^*(s) = \int_0^\infty e^{-st} w(t)\,dt \cong \int_0^\infty w(t)\,dt - s\int_0^\infty t w(t)\,dt$$
$$w^*(s) = 1 - \frac{s}{\alpha},$$

where $1/\alpha$ is the MTBF. Similarly, $g^*(s) = 1 - s/\beta$, where $1/\beta$ is the mean time to repair (MTTR). Therefore, the steady-state availability is obtained by taking the limit of Equation 3.41 multiplied by s as $s \to 0$:

$$A = \lim_{s\to 0} \frac{1-\left(1-\frac{s}{\alpha}\right)}{1-\left(1-\frac{s}{\alpha}\right)\left(1-\frac{s}{\beta}\right)} = \frac{\frac{1}{\alpha}}{\frac{1}{\alpha}+\frac{1}{\beta}}$$

or

$$A = \frac{MTBF}{MTBF + MTTR}. \tag{3.42}$$

EXAMPLE 3.13

The failure time of the system follows a Weibull distribution with a p.d.f. of the form

$$w(t) = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\gamma}\right],$$

and its repair time follows an exponential distribution with a p.d.f. given by

$$g(t) = \mu e^{-\mu t}.$$

Determine the point availability of the system $A(t)$ and its steady-state value.

SOLUTION

We first obtain the Laplace transforms of $w(t)$ and $g(t)$ as

$$w^*(s) = \int_0^\infty e^{-st} w(t)\,dt = \sum_{j=0}^{\infty} (-1)^j \frac{(\theta s)^j}{j!} \Gamma\left(\frac{j+\gamma}{\gamma}\right)$$

and

$$g^*(s) = \int_0^\infty e^{-st} g(t)\,dt = \frac{\mu}{s+\mu}.$$

Substituting $w^*(s)$ and $g^*(s)$ into Equation 3.41 results in

$$A^*(s) = \frac{1 - \sum_{j=0}^{\infty} (-1)^j \frac{(\theta s)^j}{j!} \Gamma\left(\frac{j+\gamma}{\gamma}\right)}{s\left[1 - \frac{\mu}{(s+\mu)} \sum_{j=0}^{\infty} (-1)^j \frac{(\theta s)^j}{j!} \Gamma\left(\frac{j+\gamma}{\gamma}\right)\right]}.$$

A closed form expression of $A(t)$ cannot be obtained from $A^*(s)$. Therefore, $A(t)$ can only be estimated numerically or by approximation. Though $A(t)$ is difficult to obtain, its steady-state value can be easily estimated by using Equation 3.42:

$$MTBF = \theta \Gamma\left(\frac{1+\gamma}{\gamma}\right)$$

and

$$MTTR = \frac{1}{\mu}.$$

If $\theta = 5 \times 10^6$, $\gamma = 2.15$, and $\mu = 10{,}000$, then

$$A = \frac{4.428 \times 10^6}{4.428 \times 10^6 + 10^{-4}} = 1.$$

■

An alternative to the use of Laplace transform in obtaining the availability of the system is now described. The pointwise availability of a system at time t is defined as the probability of the system being in a working state (operating properly) at t. As shown in Equation 3.42, the limiting availability of a system that has constant failure rate λ and repair rate μ is

$$A = \frac{\mu}{\lambda + \mu}.$$

The unavailability of the system $\bar{A}(t)$ is

$$\bar{A}(t) = 1 - A(t),$$

and the limiting unavailability is

$$\bar{A} = \frac{\lambda}{\lambda + \mu} = \frac{\beta\lambda}{1 + \beta\lambda}, \tag{3.43}$$

where $\beta = 1/\mu$ (the MTTR).

Equation 3.43 can be rewritten as the power series (Holcomb, 1981)

$$\bar{A} = \sum_{i=1}^{\infty} (-1)^{i+1} (\beta\lambda)^i = \beta\lambda - (\beta\lambda)^2 + (\beta\lambda)^3 - \ldots \tag{3.44}$$

Since $\lambda << 1/\beta$ (which implies that$\lambda << \mu$), the above series falls off very quickly. Thus, when $\beta\lambda$ is small, the first-order approximation of $\bar{A}$ is $\beta\lambda$. The same reasoning can be used to estimate the unavailability $\bar{A}(t)$ when the failure rate is time dependent. In this case, the expected number of failures, E, during the interval $(t - \beta, t)$ is

$$E = \int_{t-\beta}^{t} \lambda(x)dx \cong \bar{A}(t). \tag{3.45}$$

This is approximately equal to the probability of being nonworking at t as long as $E << 1$. Therefore, the integral of Equation 3.45 can be approximated as

$$\bar{A}(t) \approx \beta\lambda(t). \tag{3.46}$$

Equation 3.43 can now be rewritten as

$$\bar{A}(t) \cong \frac{\lambda(t)}{\lambda(t)+\mu}. \tag{3.47}$$

This approximation is reasonable if the failure rate changes relatively little near t—that is, over a time range on the order of β. For example, the failure rate of the Weibull model is

$$\lambda(t) = h(t) = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1}.$$

As t increases, the relative change of $\lambda(t)$ decreases. In other words, for a large enough time, the failure rate changes slowly enough to satisfy the conditions for Equations 3.46 and 3.47. Details of the derivation of the unavailability and how large t should be to ensure the validity of the approximations are given in Holcomb (1981).

EXAMPLE 3.14

Use the approximation given by Equation 3.47 to estimate the availability of the system described in Example 3.13 for different values of γ. Determine the availability obtained from the approximation at $t = 10^7$ h.

SOLUTION

The values of the availabilities for systems with $\theta = 2 \times 10^3$, $\mu = 10{,}000$, and different γ's are shown in Table 3.2. Figure 3.4 shows the effect of γ on $A(t)$. The availability value increases as γ decreases.

TABLE 3.2 Availability Values for Different γ

γ	**t (hours)**	$\lambda(t) = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1}$	$A(t)$
2.15	10^7	19.285	0.998075
2.0	10^7	5.000	0.999500
1.9	10^7	2.026	0.999797

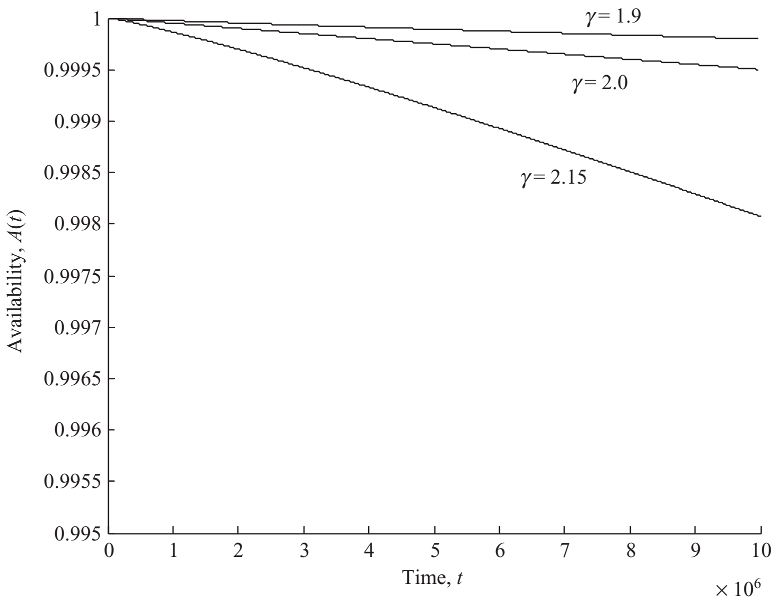

FIGURE 3.4 Effect of γ on $A(t)$. ■

In many cases, it is important to obtain the expected number of renewals during a time interval. For example, the manufacturers of home appliances are interested in the expected number of failures and repairs, $E(N[t])$, during a warranty period t in order to determine the optimum length of T that minimizes the total cost while meeting the users' expectations. This is achieved by obtaining the renewal density function $n(t)$ as described by Equation 3.37, repeated below

$$n^*(s) = \frac{f^*(s)}{1 - f^*(s)}.$$

The corresponding cumulative distribution function $N^*(s)$ is

$$N^*(s) = \frac{f^*(s)}{s(1 - f^*(s))}.$$

The challenge is to obtain $n(t)$ by inverting Laplace transform $n^*(s)$. Methods for obtaining Laplace transform inversion are investigated by Abate and Valkó (2004), Abate and Whitt (2006), and Rossberg (2008). One of the approaches for the inversion is Post's (1930) inversion formula which is given by

$$N(t) = \lim_{k \to \infty} \frac{(-1)^k}{k!} \left(\frac{k}{t}\right)^{k+1} N^{*(k)}\left(\frac{k}{t}\right)$$

for $t > 0$, where $N^{*(k)}$ denotes the kth derivative of N^*. The expected number is

$$E[N(t)] = \int_{t=0}^{\infty} t\,N(t)dt.$$

The inversion of N^* becomes difficult to obtain as the differentiation of N^* may yield complex expressions. We demonstrate the use of the above expression as follows.

Consider the reliability function of the constant failure-rate model given by

$$R(t) = e^{-\lambda t}.$$

Its Laplace transform is

$$R(s) = \frac{1}{s+\lambda}.$$

By induction, we obtain $R^{(k)}(s) = k!(-1)^k(s+\lambda)^{-k-1}$.

Substituting in the inversion formula results in

$$R(t) = \lim_{k\to\infty} \frac{k^{k+1}}{t^{k+1}}\left(\lambda + \frac{k}{t}\right)^{-k-1} = \lim_{k\to\infty}\left(1+\frac{\lambda t}{k}\right)^{-k-1}.$$

Taking the logarithm of both sides, we obtain $\ln R(t) = -\lambda t$ as $k \to \infty$ which results in the original expression of reliability.

It is of interest not only to obtain the expected number of renewals but also the variance of the number of renewals. The variance of $N(t)$ is expressed as

$$Var[N(t)] = E[N^2(t)] - \{E[N(t)]\}^2.$$

This requires the estimation of $E[N^2(t)]$. Following the estimation of $E(N[t])$, the Laplace transform of $E[N^2(t)]$ is

$$[N^2(s)]^* = \frac{f^*(s)(1+f^*(s))}{(1-f^*(s))^2}.$$

We follow Post's (1930) formula to obtain $E[N^2(t)]$ and consequently obtain the variance of the number of renewals. Note that Post's formula provides a numerical evaluation of the inverse of Laplace function as follows. Given a function $f(t)$ defined for $0 \le t < \infty$, its Laplace transform $f^*(s)$ is defined as

$$f^*(s) = \int_0^{\infty} e^{-st} f(t)dt.$$

Post's inversion formula is expressed as

$$f(t) = \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{t}\right)^{k+1} f^{*(k)}\left(\frac{k}{t}\right),$$

where $t > 0$ and $f^{*(k)}$ is the kth derivative of $f^*(s)$ with respect to s. To illustrate the use of Post's formula we consider the p.d.f. of the exponential distribution $f(t) = e^{-\lambda t}$, its Laplace transform is $f^*(s) = 1/(s+\lambda)$, and the kth derivative of $f^*(s)$ is $f^{*(k)}(s) = k!(-1)^k(s + \lambda)^{-k-1}$. Substituting in Post's inversion formula results in

$$\begin{aligned} f(t) &= \lim_{k\to\infty} \frac{k^{k+1}}{t^{k+1}} \left(\lambda + \frac{k}{t}\right)^{-k-1} \\ &= \lim_{k\to\infty} \left(1 + \frac{\lambda t}{k}\right). \end{aligned}$$

Taking the natural log of both sides and writing the resulting indeterminate form as $-\ln(1 + \lambda t/k)/(1/(k + 1))$ then applying L'Hopital's rule shows that the indeterminate approaches $-\lambda t$. Thus, $\ln f(t) = -\lambda t$ or $f(t) = e^{-\lambda t}$. Further explanation and examples are given in Bryan (2010) and Cain and Berman (2010).

3.4.2 Markov Models

This is the second approach that can be used to estimate the time-dependent availability of the system. This approach is valid when both the failure and repair rates are constant. When these rates are time dependent, the Markov process breaks down, except in some special cases. In this section we limit our presentation of the Markov models to constant failure and repair rates.

The first step in formulating a Markov model requires the definition of all the mutually exclusive states of the system. For example, a nonrepairable system may have two states: State $s_0 = x$, the system is working properly (i.e., good), and state $s_1 = \bar{x}$, the system is not working properly (i.e., failed), where x is the indicator that the system is good, and $\bar{x}$ is the indicator that the system is not working.

The second step is to define the initial and final conditions of the system. For example, it is reasonable to assume that initially the system is working properly at $t = 0$ with probability $R_S(0) = 1$. It is also reasonable to assume that the system will eventually fail as time approaches infinity—that is, $R_S(\infty) = 0$.

The third step involves the development of the Markov state equations, which describe the probabilistic transitions of the system. In doing so, the probability of transition from one state to another in a time interval Δt is $h(t)\Delta t$ where $h(t)$ is the rate associated with the two states ($h(t)$ can be a failure rate or a repair rate as shown later in this section). Moreover, the probability of more than one transition in Δt is neglected.

3.4.2.1 Nonrepairable Component We now consider a nonrepairable component that has a failure rate λ. We are interested in the development of a time-dependent reliability expression of the component. Define $P_0(t)$ and $P_1(t)$ as the probabilities of the component being in state s_0 (working properly) and in state s_1 (not working) at time t, respectively. Let us examine the states of the component at time $t + \Delta t$. The probability that the component is in state s_0 at $t + \Delta t$ is given by the probability of the component being in state s_0 at time t, $P_0(t)$, times the probability that the component does not fail in Δt, $1 - \lambda\Delta t$, plus the probability of the component being in state s_1 at time t, $P_1(t)$, times the probability that the component is repaired during Δt (this probability equals zero for nonrepairable components or systems). We write the state-transition equations as

$$P_0(t+\Delta t) = [1-\lambda\,\Delta t]P_0(t) + 0P_1(t). \tag{3.48}$$

Likewise, the probability that the component is in state s_1 at time $t + \Delta t$ is expressed as

$$P_1(t+\Delta t) = \lambda\,\Delta t\,P_0(t) + 1P_1(t). \tag{3.49}$$

Note that the transition probability $\lambda\Delta t$ is the probability of failure in Δt (change from state s_0 to state s_1) and the probability of remaining in state s_1 is unity (Grinstead and Snell, 1997).

Rearranging Equations 3.48 and 3.49 and dividing by Δt, we obtain

$$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda P_0(t)$$

and

$$\frac{P_1(t+\Delta t)-P_1(t)}{\Delta t} = \lambda P_0(t).$$

Taking the limits of the above equations as $\Delta t \to 0$, then

$$\frac{dP_0(t)}{dt} + \lambda P_0(t) = 0 \tag{3.50}$$

$$\frac{dP_1(t)}{dt} - \lambda P_0(t) = 0. \tag{3.51}$$

Using the initial conditions $P_0(t = 0) = 1$ and $P_1(t = 0) = 0$, we solve Equation 3.50 as

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t)$$

$$\ln P_0(t) = -\int_0^t \lambda d\xi + c$$

and

$$P_0(t) = c_1 \exp\left[-\int_0^t \lambda d\xi\right].$$

Since $P_0(t = 0) = 1$, then $c_1 = 1$ and

$$P_0(t) = e^{-\int_0^t \lambda d\xi}. \tag{3.52}$$

When λ is constant, Equation 3.52 becomes

$$P_0(t) = e^{-\lambda t}.$$

In other words, the reliability of the component at time t is

$$R(t) = P_0(t) = e^{-\lambda t}. \tag{3.53}$$

The solution of $P_1(t)$ is obtained from the condition $P_0(t) + P_1(t) = 1$,

$$P_1(t) = 1 - e^{-\lambda t}. \tag{3.54}$$

3.4.2.2 Repairable Component We now illustrate the development of a Markov model for a repairable component. Consider a component that exhibits a constant failure rate λ. When the component fails, it is repaired with a repair rate μ. Similar to the nonrepairable component, we define two mutually exclusive states for the repairable component: state s_0 represents a working state of the component, and state s_1 represents the nonworking state of the component. The state-transition equations of the component are

$$P_0(t + \Delta t) = [1 - \lambda \Delta t] P_0(t) + \mu \Delta t P_1(t) \tag{3.55}$$

$$P_1(t + \Delta t) = [1 - \mu \Delta t] P_1(t) + \lambda \Delta t P_0(t). \tag{3.56}$$

Rewriting Equations 3.55 and 3.56 as

$$\frac{dP_0(t)}{dt} = \dot{P}_0(t) = -\lambda P_0(t) + \mu P_1(t) \tag{3.57}$$

$$\frac{dP_1(t)}{dt} = \dot{P}_1(t) = -\mu P_1(t) + \lambda P_0(t). \tag{3.58}$$

Solutions of these equations can be obtained using Laplace transform and the initial conditions $P_0(0) = 1$ and $P_1(0) = 0$. Thus,

$$sP_0(s) - 1 = -\lambda P_0(s) + \mu P_1(s) \tag{3.59}$$

$$sP_1(s) = -\mu P_1(s) + \lambda P_0(s). \tag{3.60}$$

From Equation 3.60 we obtain

$$P_1(s) = \frac{\lambda}{s + \mu} P_0(s).$$

Substituting into Equation 3.59 to get $P_0(s)$,

$$P_0(s) = \frac{s + \mu}{s(s + \lambda + \mu)}. \tag{3.61}$$

Using the partial-fraction method, we write Equation 3.61 as

$$P_0(s) = \frac{\frac{\mu}{\lambda+\mu}}{s} + \frac{\frac{\lambda}{\lambda+\mu}}{(s+\lambda+\mu)}, \tag{3.62}$$

and the inverse of Equation 3.62 is

$$P_0(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}. \tag{3.63}$$

Of course, $P_0(t)$ is the availability of the component at time t. The unavailability $\bar{A}(t)$ is

$$\bar{A}(t) = 1 - P_0(t) = P_1(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}. \tag{3.64}$$

When the number of the mutually exclusive states is large, the solution of state-transition equations by using the Laplace transform becomes difficult, if not impossible, to obtain. In such a case, the equations may be solved numerically for different values of time. The following example illustrates the numerical solution of the transition equations.

EXAMPLE 3.15

An electric circuit which provides constant current for direct current (DC) motors includes one diode which may be in any of the following states: (1) s_0 represents the diode operating properly; (2) s_1 represents the short failure mode of the diode—that is, the diode allows the current to return in the reverse direction; (3) s_2 represents the open failure mode of the diode—that is, the diode prevents the passage of current in either direction; and (4) s_3 represents the assembly failure mode of the diode—that is, when the diode is not properly assembled on the circuit board, it generates hot spots that result in not providing the current for the motor to function properly. Let the failure rates from state s_0 to state s_i be constant with parameters λ_i (i = 1, 2, 3). The repair rate from any of the failure states to s_0 is constant with parameter μ. Transitions occur only between state s_0 and other states and vice versa. Graph the availability of the circuit against time for different values of failure and repair rates.

SOLUTION

Let $P_i(t)$ be the probability that the diode is in state i (i = 0, 1, 2, 3) at time t. The state-transition equations are

$$\dot{P}_0(t) = -[\lambda_1 + \lambda_2 + \lambda_3] P_0(t) + \mu P_1(t) + \mu P_2(t) + \mu P_3(t) \tag{3.65}$$

$$\dot{P}_1(t) = -\mu P_1(t) + \lambda_1 P_0(t) \tag{3.66}$$

$$\dot{P}_2(t) = -\mu P_2(t) + \lambda_2 P_0(t) \tag{3.67}$$

$$\dot{P}_3(t) = -\mu P_3(t) + \lambda_3 P_0(t). \tag{3.68}$$

The initial conditions of the diode are $P_0(0) = 1$, $P_i(0) = 0$ for $i = 1, 2, 3$. The solution of the above equations can be obtained by using the matrix-geometric approach and the analytical perturbations method (Schendel, 1989; Baruh and Altiok, 1991) or by using the Runge-Kutta method for solving differential equations (Lee and Schiesser, 2004). We utilize the computer program given in Appendix D and graph the availability, $P_0(t)$, of the circuit due the diode failure as shown in Figure 3.5. We choose $\lambda_1/\lambda_2 = \lambda_2/\lambda_3 = 0.5$ and $\mu/\lambda_1 = 10$, with $\lambda_1 = 0.0001$, 0.0005, 0.0008, and 0.0018. The availability decreases rapidly as t increases, then it reaches an asymptotic value for large values of t (steady-state availability).

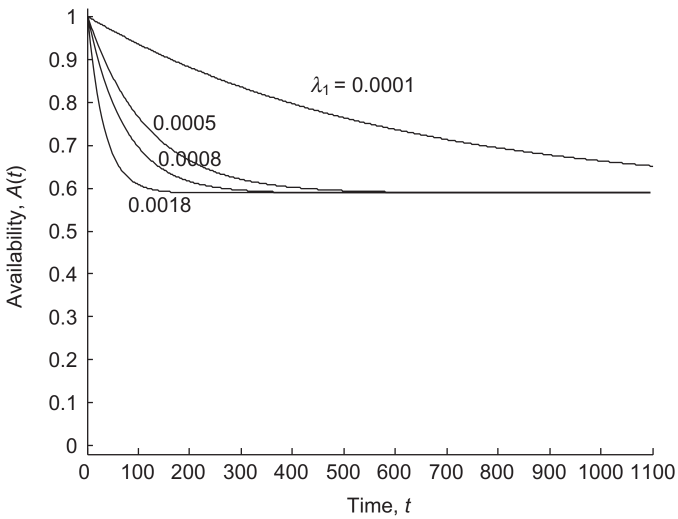

FIGURE 3.5 Effect of the failure rate on the circuit's availability. ■

3.5 AVAILABILITY

Availability is considered to be one of the most important reliability performance measures of maintained systems since it includes both the failure rates and repair rates of the systems. Indeed, the importance of availability has prompted manufacturers and users of critical systems to state the availability values in the systems' specifications. For example, manufacturers of mainframe computers that are used in large financial institutions and banks provide guaranteed availability values for their systems. In this section, we present different classifications of availability and methods for its estimation.

Availability can be classified either according to (1) the time interval considered or (2) the type of downtime (repair and maintenance). The time-interval availability includes instantaneous (or point availability), average up time, and steady-state availabilities. The availability classification according to downtime includes inherent, achieved, and operational availabilities (Lie et al., 1977). Other classifications include mission-oriented availabilities.

3.5.1 Instantaneous Point Availability, *A*(*t*)

Instantaneous point availability is the probability that the system is operational at any random time t. The instantaneous availability can be estimated for a system whose states are characterized by an alternating renewal process by using Equation 3.41

$$A^*(s) = \frac{1 - w^*(s)}{s\left[1 - w^*(s)g^*(s)\right]},$$

where $w^*(s)$ and $g^*(s)$ are the Laplace transforms of the failure-time and repair-time distributions, respectively.

When the failure and repair rates are constant, the availability can be obtained using the state-transition equations as described in Section 3.4. For the case when either the failure rate or the repair rate is time dependent, $A(t)$ can be estimated by using semi-Markov state-transition equations or by using an appropriate approximation method as given by Equation 3.47.

Sun and Han (2001) develop a stair-step approximation to time-varying failure rate. The approximation is based on the fact that the variation of failure rate during short time is hard to measure. The stair-step approximation therefore assumes that the failure rate is fixed during a very short time period, T_i. The failure-repair behavior of the system can be described by a nonhomogeneous Markov chain, and closed-forms for instantaneous availability and interval availability are readily obtained. This is explained as follows.

We express the failure rate during a short time interval as

$$h(t) = \begin{cases} \lambda_0 & t \le T_1 \\ \lambda_i & T_{i-1} < t < T_i \quad i > 1 \end{cases}.$$

The corresponding p.d.f. is

$$f(t) = \begin{cases} \lambda_0 \exp(-\lambda_0 t) & t \le T_1 \\ \alpha_{i-1}\lambda_{i-1} \exp(-\lambda_{i-1}(t - T_{i-1})) & T_{i-1} < t < T_i \quad i > 1 \end{cases}$$

where $\begin{cases} \alpha_1 = \exp(-\lambda_0 T_1) & t \le T_1 \\ \alpha_i = \alpha_{i-1} \exp(-\lambda_{i-1}(T_i - T_{i-1})) & T_{i-1} < t < T_i \quad i > 1 \end{cases}.$

Following the derivations of the instantaneous availability derived earlier, we obtain the instantaneous availability as

$$A(t) = \begin{cases} \mu/(\lambda_0 + \mu) + \lambda_0/(\lambda_0 + \mu)\exp(-(\lambda_0 + \mu)t) & t \le T_1 \\ \mu/(\lambda_i + \mu) + [A(T_i) - \mu/(\lambda_i + \mu)\exp(-(\lambda_i + \mu)(t - T_i)] & T_{i-1} < t < T_i \quad i > 1 \end{cases}.$$

The results of the approximation improve as the length of the time interval decreases.

3.5.2 Average Up-Time Availability, $A(T)$

In many applications, it is important to specify availability requirements in terms of the proportion of time in specified intervals $(0, T)$ that the system is available for use. We refer to this availability requirement as the average up-time availability or interval reliability. It is expressed as

$$A(T) = \frac{1}{T}\int_0^T A(t)dt. \tag{3.69}$$

$A(T)$ can be estimated by obtaining an expression for $A(t)$ as a function of time, if possible, and substituting in Equation 3.69, or by numerically solving the state-transition equations and summing the probabilities of the "up" states over the desired time interval T or by fitting a function to the probabilities of the up state and substituting this function in Equation 3.69.

EXAMPLE 3.16

Estimate the average up-time availability of the circuit described in Example 3.15 during the interval 0–1135 h under the following conditions: $P_i(0) = 0$ for $i = 1, 2, 3$, $\lambda_1/\lambda_2 = \lambda_2/\lambda_3 = 0.5$, $\mu/\lambda_1 = 10$, and $\lambda_1 = 0.0018$ failures per hour.

SOLUTION

We solve the state-transition Equations 3.65 through 3.68 using the Runge–Kutta method to obtain the probability of the up state, $P_0(t)$. A partial listing of $P_0(t)$ is shown in Table 3.3. The average up-time availability $A(t)$ is obtained by two methods.

- Adding $P_0(t)$ over the interval 0–1135 h and dividing by 1135

$$A(1135) = \frac{\sum_{i=1}^{1135} P_0(t_i)}{1135} = 0.6414196.$$

- Fitting a function of the form $P_0(t) = Ae^{Bt}$ to the data obtained from the solution of the state-transition equation, we obtain

$$A(t) = P_0(t) \cong 0.90661264\, e^{-0.0004t} \tag{3.70}$$

and

$$A(1135) \cong \frac{1}{1135}\int_0^{1135} A(t)dt$$

$$A(1135) \cong 0.7287.$$

TABLE 3.3 Point Availability $P_0(t)$

Time, t	$P_0(t)$
1.135	0.998867
2.270	0.997738
3.405	0.996611
4.540	0.995488
5.675	0.994368
6.810	0.993250
7.945	0.992136
. . .	. . .
. . .	. . .
993.111	0.625207
994.246	0.625105
995.381	0.625004
996.516	0.624903
997.651	0.624802
998.786	0.624701
999.921	0.624601
1001.056	0.624501

■

Obviously, the average up-time availability obtained by the second method is more accurate than that obtained by the first method since the availability is integrated over a continuous time interval.

The average up-time availability may be the most satisfactory measure for systems whose usage is defined by a duty cycle such as a tracking radar system which is called upon only when an object is detected and is expected to track the system continuously during a given time period (Lie et al., 1977).

3.5.3 Steady-State Availability, $A(\infty)$

The steady-state availability is the availability of the system when the time interval considered is very large. It is given by

$$A(\infty) = \lim_{T \to \infty} A(T).$$

The steady-state availability can be easily obtained from the state-transition equations of the system by setting $\dot{P}_i(t) = 0$, $i = 0, 1, \ldots$

EXAMPLE 3.17

Determine the steady-state availability of the system given in Example 3.16.

SOLUTION

Since we are seeking $A(\infty)$, we set $\dot{P}_i(t) = 0$ and let $P_i(t) = P_i$, $i = 0, 1, 2, 3$ in Equations 3.65 through 3.68. This results in

$$-(\lambda_1 + \lambda_2 + \lambda_3)P_0 + \mu P_1 + \mu P_2 + \mu P_3 = 0 \tag{3.71}$$

$$-\mu P_1 + \lambda_1 P_0 = 0 \tag{3.72}$$

$$-\mu P_2 + \lambda_2 P_0 = 0 \tag{3.73}$$

$$-\mu P_3 + \lambda_3 P_0 = 0 \tag{3.74}$$

Using the condition $P_0 + P_1 + P_2 + P_3 = 1$ and solving Equations 3.71 through 3.74, we obtain

$$A(\infty) = P_0 = \frac{\mu}{\lambda_1 + \lambda_2 + \lambda_3 + \mu} = 0.5882.$$

■

The steady-state availability may be a satisfactory measure for systems that operate continuously, such as a detection radar system, satellite communication system, and an undersea communication cable.

3.5.4 Inherent Availability, A_i

Inherent availability includes only the corrective maintenance of the system (the time to repair or replace the failed components) and excludes ready time, preventive maintenance downtime, logistics (supply) time, and waiting or administrative time. It is expressed as

$$A_i = \frac{MTBF}{MTBF + MTTR}. \tag{3.75}$$

The inherent availability is identical to the steady-state availability when the only repair time considered in the steady-state calculation is the corrective maintenance time.

3.5.5 Achieved Availability, A_a

Achieved availability, A_a, includes corrective and preventive maintenance downtime. It is expressed as a function of the frequency of maintenance, and the mean maintenance time as

$$A_a = \frac{MTBM}{MTBM + M}, \tag{3.76}$$

where $MTBM$ is the mean time between maintenance and M is the mean maintenance downtime resulting from both corrective and preventive maintenance actions (Lie, et al. 1977).

3.5.6 Operational Availability, A_o

Operational availability is a more appropriate measure of availability since the repair time includes many elements: the direct time of maintenance and repair and the indirect time which includes ready time, logistics time, and waiting or administrative downtime. The operational availability is then defined as

$$A_O = \frac{MTBM + \text{ready time}}{(MTBM + \text{ready time}) + MDT}, \tag{3.77}$$

where ready time = operational cycle − ($MTBM$ + MDT) and the mean delay time, $MDT = M$ + delay time.

3.5.7 Other Availabilities

Other availability definitions include *mission-availability*, $A_m(T_o, t_f)$, which is defined as

$$\begin{aligned} A_m(T_O, t_f) = {} & \text{Probability of each individual failure that occurs in a mission} \\ & \text{of a total operating time } T_o \text{ is repaired in a time} \le t_f. \end{aligned} \tag{3.78}$$

This definition is used for specifying the availabilities of military equipment and equipment or systems assigned to perform a specific mission with limited duration. Clearly, the repair time in Equation 3.78 includes all direct and indirect elements.

We follow Birolini (2010) and consider that the end of the mission falls within an operating period. The mission availability is obtained by summing over all the possibilities of having n failures (n = 1, 2, 3, . . .) during the total operating time T_o. Each failure can be repaired in a time shorter than (or equal to) t_f. In other words,

$$A_m(T_o, t_f) = 1 - F(T_o) + \sum_{n=1}^{\infty} [F_n(T_o) - F_{n+1}(T_o)](G(t_f))^n, \tag{3.79}$$

where

$F(t)$ = the distribution function of the failure time,

$F_n(T_o) - F_{n+1}(T_o)$ = the probability of n failures in T_o (see Chapter 7 for further details),

$(G(t_f))^n$ = the probability that the time of each of the n repairs is shorter than t_f, and

$G(t)$ = the cumulative distribution function of the repair time.

Assume that the system has a constant failure rate λ—that is, $f(t) = \lambda e^{-\lambda t}$. Then,

$$A_m(T_o, t_f) = e^{-\lambda T_o} + \sum_{n=1}^{\infty} \frac{(\lambda T_o)^n}{n!} e^{-\lambda T_o} (G(t_f))^n = e^{-\lambda T_o (1 - G(t_f))}. \tag{3.80}$$

EXAMPLE 3.18

In a musical play, the heroine is lowered to the stage on a 37,000-lb set inside of a mansion. The mansion set, along with the other scenery, is powered by an integrated hydraulic motor pump designed to orchestrate the operations of the stage reliably until the final act of the first part of the play.

A winch system consisting of steel cables controls the movements of the set. The cables are connected to a hydraulic brake, which is digitally regulated by proportional control valves. The hydraulic system is powered by an integrated motor pump that generates 30 hp and is capable of flow rates of up to 33 gal/min and 1000 lb per square inch (psi). To provide hydraulic power during the musical, the pump operates at flow rates of up to 28 gal/min and pressures of up to 1450 psi. The hydraulic system is regulated by a microprocessor-based controller (O'Connor, 1995). The heat generated by the electric motor and hydraulic pump raises the temperature inside the room where the equipment is installed and, in turn, affects the life of the controller.

The failure rate of the controller is constant with $\lambda = 0.006$ failures per hour. The repair follows a gamma distribution with a p.d.f. of

$$g(t) = \frac{t^{\beta-1}}{\alpha^{\beta}\Gamma(\beta)}\exp\left(\frac{-t}{\alpha}\right). \tag{3.81}$$

where $\beta = 3$, and $\alpha = 1000$.

The mansion and the other hydraulic equipment are used in the play for 60 min. Determine the mission availability of the system if $t_f = 2$ min.

SOLUTION

$$T_o = 1\ \text{h}$$
$$t_f = 0.0333\ \text{h}$$
$$1 - G(t_f) = \exp\left(\frac{-t_f}{\alpha}\right)\sum_{j=0}^{\beta-1}\left(\frac{t_f}{\alpha}\right)^j \frac{1}{\Gamma(j+1)}$$

or

$$1 - G(t_f) = 0.99996667[1 + 0.00000333 + 0.0]$$
$$1 - G(t_f) = 0.999970.$$

Substituting $[1 - G(t_f)]$ into Equation 3.80, we obtain

$$A_m(T_o, t_f) = e^{-0.006\times 0.999970} = 0.994.$$

■

The *work-mission availability*, $A_{wm}(T_o, t_d)$, is a variant of mission availability. It is defined as $A_{wm}(T_o, t_d)$ = probability of the sum of all repair times for failures occurring in a mission with total operating time

$$T_o \text{ is} \le t_d. \tag{3.82}$$

Using Equation 3.79 we rewrite Equation 3.82 as

$$A_{wm}(T_o, t_d) = 1 - F(T_o) + \sum_{n=1}^{\infty} [F_n(T_o) - F_{n+1}(T_o)] G_n(t_d), \tag{3.83}$$

where $G_n(t_d)$ is the probability that the sum of n repair times, which are distributed according to $G(t)$, is shorter than t_d (Birolini, 2010).

EXAMPLE 3.19

Membrane keyboards are widely used in the personal computer industry. A membrane keyswitch has a rubber dome-shaped actuator at the bottom of the keyswitch plunger as shown in Figure 3.6. When the key is depressed, the rubber dome compresses, and a small rubber nib or bump inside the dome is pushed down onto the membranes, bringing them together and closing the contacts (Johns, 1995).

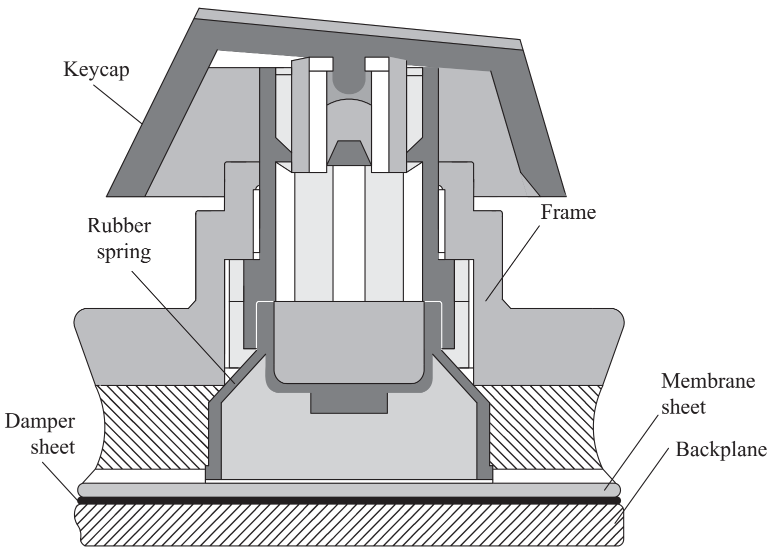

FIGURE 3.6 A sketch of a key in a membrane keyboard. Source: Redrawn with permission from "Membrane versus Mechanical Keyboards," Garden City: Electronic Product, June 1995, Don Johns.

The membrane consists of metallic pads that are screen-printed onto two membrane sheets. A third spacer membrane with holes is placed between the two printed membrane sheets. When the keyswitch is depressed, the top and bottom printed membranes are squeezed together, allowing the pad on one to touch its corresponding pad on the other through the hole on the third sheet, thus forming the contact. The expected life of a membrane keyboard is about 25 million keystrokes (approximately 3 years). The failure rate of the keyboard is constant with $\lambda = 0.000114$ failures per hour. The repair rate is also constant with $\mu = 0.002$. Assume that the keyboard is attached to a computer that is used for a 120-h task. What is the work-mission availability of the keyboard if the time required for repairing all failures does not exceed 1 h?

SOLUTION

$t_d = 1$ h

$T_o = 120$ h,

$\lambda = 0.000114$ failures per hour, and

$\mu = 0.002$ repairs per hour.

In addition,

$$1 - F(T_o) = e^{-\lambda T_o} = e^{-0.000114\times 120} = 0.986413.$$

Let x_1 and x_2 be the time to repair the first and second failures, respectively. Then,

$$G_1(t_d) = (1 - e^{-\mu t_d}) = 0.001998$$

$$G_2(x_1 + x_2 \le t_d) = \int_0^{t_d}\int_0^{t_d - x_1} \mu e^{-\mu x_1} \mu e^{-\mu x_2}\, dx_2 dx_1$$

$$G_2(t_d) = \int_0^{t_d} [\mu e^{-\mu x_1} - \mu e^{-\mu t_d}] dx_1$$

$$G_2(t_d) = 1 - \mu e^{-\mu t_d} - \mu t_d e^{-\mu t_d}$$

$$G_2(t_d) = 2\times 10^{-6}.$$

Substituting in Equation 3.83, we obtain

$$A_{wm}(120,1) = 0.986413 + 0.000114\times 120 e^{-0.000114\times 120}\times 0.001998$$
$$+\frac{1}{2}(0.000114\times 120)^2 e^{-0.000114\times 120}\times 2\times 10^{-6}$$
$$A_{wm}(120,1) = 0.98643996.$$

■

3.6 DEPENDENT FAILURES

In Chapters 2 and 3, we estimate the performance measures of the system reliability under the assumption that the failure-time distributions of the components are identical and independent. In other words, we consider only the situations where the failure of a component has no effect on the failure rate of other components in the system. This assumption, though valid in many situations, needs to be relaxed when the failure of a component or a group of components may change the failure rate of the remaining components. For example, consider a twin engine airplane. The engines have identical failure-time distributions, and they operate in parallel—that is, both engines share the load. When either one of the engines fails, the other engine provides the additional power requirement for safe operation of the airplane. This, in turn, causes the failure rate of the surviving engine to increase, and the reliability analysis of the system should reflect such change.

Similarly, the advances in computer technology have resulted in an increase in the number of components placed on a computer chip, which causes significant heat dissipation from the chip to the adjacent components. Insufficient cooling of the computer board results in an elevated operating temperature of the components, which, in turn, increases their failure rates.

Reliability analysis of systems whose components experience dependent failures can be performed using the Markov model. The model performs well when the number of state-transition equations is small and when the failure-time and repair-time distributions are exponential. When these conditions are not satisfied, alternative approaches, such as the *joint density function* (*j.d.f.*) and the *compound events*, can be used. Although both approaches are applicable for situations when the failure rates are time dependent, they rapidly break down, as the j.d.f. of the failure times is too complex to solve analytically. This section briefly presents approaches for reliability analysis of systems with dependent failures.

3.6.1 Markov Model for Dependent Failures

The Markov model for dependent failures is similar to the models discussed in Section 3.4.2 with the exception that the failure and repair rates are dependent on the state of the system (or component). The following example illustrates the development of such a Markov model.

EXAMPLE 3.20

Time-dependent dielectric breakdown (TDDB) of gate oxides of metal-oxide-semiconductor (MOS) transistors and of other thin oxide structures has been, and continues to be, one of the principal mechanisms of failure of metal-oxide-semiconductor integrated circuits (MOS-ICs) (Hawkins and Soden, 1986). Extensive studies of dielectric breakdown of MOS device structures show distribution of breakdown voltage and the effects of device processing, voltage, and temperature on the rate of failure of gate oxides, which are subject to an electric field (Crook, 1979; Edwards, 1982; Domangue et al., 1984; Dugan, 1986; Swartz, 1986). These studies show that the use of higher voltages is far more effective than the use of higher temperatures in screening to eliminate devices with defective oxide sites that would be susceptible to TDDB. This prompts the designers of ICs to improve the reliability of the circuits by using redundant devices.

Utilizing this information, a designer of an IC connects three oxide structures, such as transistors, in parallel in order to improve the reliability of the device. The device functions properly when no more than two transistors fail. Failure times of the transistors are exponentially distributed with the following parameters.

$\lambda_0 = 9 \times 10^{-5}$ failures per hour when all transistors are working properly,

$\lambda_1 = 16 \times 10^{-5}$ failures per hour when one unit fails, and

$\lambda_2 = 21 \times 10^{-5}$ failures per hour when two units fail.

Graph the reliability of the device over the period of 0–9500 h. Also, graph the reliability when $\lambda_2 = 2\lambda_1 = 4\lambda_0$.

SOLUTION

Let $P_{si}(t)$ be the probability that the three transistors are in state i ($i = 0, 1, 2, \ldots, 7$), where

$$\begin{aligned}
s0 &= x_1 x_2 x_3 \ \text{(no failures of the transistors)},\\
s1 &= \bar{x}_1 x_2 x_3 \ \text{(one unit fails)},\\
s2 &= x_1 \bar{x}_2 x_3 \ \text{(one unit fails)},\\
s3 &= x_1 x_2 \bar{x}_3 \ \text{(one unit fails)},\\
s4 &= \bar{x}_1 \bar{x}_2 x_3 \ \text{(two units fail)},\\
s5 &= \bar{x}_1 x_2 \bar{x}_3 \ \text{(two units fail)},\\
s6 &= x_1 \bar{x}_2 \bar{x}_3 \ \text{(two units fail), and}\\
s7 &= \bar{x}_1 \bar{x}_2 \bar{x}_3 \ \text{(all units fail)}.
\end{aligned}$$

The state-transition equations are

$$\begin{aligned}
\dot{P}_{s0}(t) &= (-3\lambda_0) P_{s0}(t)\\
\dot{P}_{s1}(t) &= (-2\lambda_1) P_{s1}(t) + \lambda_0 P_{s0}(t)\\
\dot{P}_{s2}(t) &= (-2\lambda_1) P_{s2}(t) + \lambda_0 P_{s0}(t)\\
\dot{P}_{s3}(t) &= (-2\lambda_1) P_{s3}(t) + \lambda_0 P_{s0}(t)\\
\dot{P}_{s4}(t) &= (-\lambda_2) P_{s4}(t) + \lambda_1 P_{s1}(t) + \lambda_1 P_{s2}(t)\\
\dot{P}_{s5}(t) &= (-\lambda_2) P_{s5}(t) + \lambda_1 P_{s1}(t) + \lambda_1 P_{s3}(t)\\
\dot{P}_{s6}(t) &= (-\lambda_2) P_{s6}(t) + \lambda_1 P_{s2}(t) + \lambda_1 P_{s3}(t)\\
\dot{P}_{s7}(t) &= \lambda_2 P_{s4}(t) + \lambda_2 P_{s5}(t) + \lambda_2 P_{s6}(t).
\end{aligned}$$

Solutions of the above equations under the conditions $P_{s0}(0) = 1$ and $P_{si}(0) = 0$ for $i = 1, 2, \ldots, 7$ can be obtained numerically using Appendix D. The reliability of the device over the time interval of 0–9500 h, $R(t) = \sum_{i=0}^{6} P_{si}(t)$, for dependent failures is shown in Figure 3.7. The reliability of the device decreases rapidly when λ_2 is significantly greater than λ_1, and λ_1 is significantly greater than λ_0.

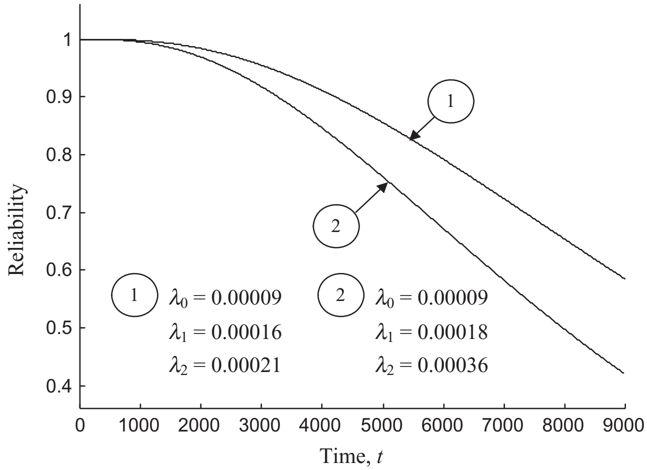

FIGURE 3.7 Reliability of the device for dependent failure rates. ■

3.6.2 The Joint Density Function Approach

This approach requires that the p.d.f. of the failure-time distribution of each component in the system as well as the j.d.f.'s of all components be known. Development of the j.d.f. is not an easy task since field or test failure data are usually collected under the assumption that the failure times of the components are independent. Careful testing of components under all possible configurations and failure conditions leads to the development of a j.d.f. that reflects the actual dependence of components. Once the configuration of the components in the system is known—that is, series, parallel, *k*-out-of-*n*, and so on—we proceed by developing the j.d.f. and then follow the reliability analysis using an appropriate method as described in Chapters 2 and 3.

We follow Shooman (1968) by considering two identical components having constant failure rates of λ_s when they operate singularly and λ_b when both operate simultaneously. Let τ be the time of the first failure and $g_1(t)$ be the density function for the first failure ($0 < \tau < t$). The time of the second failure is t and its dependent density function, $g_2(t/\tau)$, holds for $\tau < t$. In other words,

$$g_1(\tau) = 2\lambda_b e^{-2\lambda_b \tau} \qquad 0 < \tau < t$$
$$g_2(t \mid \tau) = \begin{cases} \lambda_s e^{-\lambda_s (t-\tau)} & 0 < \tau < t \\ 0 & \tau > t \end{cases}.$$

The density function, $g_1(\tau)$, is obtained as

$$g_1(\tau) = P\,[\text{if either of the two components fails first}]$$
$$g_1(\tau) = 2\lambda_b e^{-2\lambda_b \tau}.$$

The j.d.f. of the components can only be estimated once the configuration of the components in the system is known. For example, if the two components are connected in series, the system fails when either of the components fails and the j.d.f. is $\phi(\tau, t) = g_1(t)$. The system failure is governed by the marginal density function,

$$f(t) = \int_0^\infty \phi(\tau,t)dt = \int_0^\infty 2\lambda_b e^{-2\lambda_b t} dt = 2\lambda_b t e^{-2\lambda_b t}.$$

Thus,

$$R(t) = 1 - \int_0^t \phi(\tau,\zeta)d\zeta = e^{-2\lambda_b t}.$$

Similarly, if the two components are connected in parallel, then the j.d.f., $\phi(\tau, t)$, is defined as

$$\phi(\tau,t) = g_1(\tau)g_2(t \mid \tau) \qquad 0 < \tau < t. \tag{3.84}$$

The marginal density function, $f(t)$, is

$$\begin{aligned} f(t) &= \int_0^t \phi(\tau,t)d\tau \\ &= \int_0^t \left(2\lambda_b e^{-2\lambda_b \tau}\right)\lambda_s e^{-\lambda_s (t-\tau)} d\tau \\ &= 2\lambda_b \lambda_s e^{-\lambda_s t} \int_0^t e^{-(2\lambda_b - \lambda_s)\tau} d\tau \end{aligned}$$

or

$$f(t) = \frac{2\lambda_b \lambda_s}{2\lambda_b - \lambda_s}(e^{-\lambda_s t} - e^{-2\lambda_b t}),$$

and

$$R(t) = 1 - \int_0^t f(\zeta)d\zeta$$
$$R(t) = \frac{2\lambda_b}{2\lambda_b - \lambda_s} e^{-\lambda_s t} - \frac{\lambda_s}{2\lambda_b - \lambda_s} e^{-2\lambda_b t}.$$

The MTTF is obtained as

$$MTTF = \int_0^\infty R(t)dt = \frac{4\lambda_b^2 - \lambda_s^2}{2\lambda_b \lambda_s [2\lambda_b - \lambda_s]} = \frac{2\lambda_b + \lambda_s}{2\lambda_b \lambda_s}. \tag{3.85}$$

EXAMPLE 3.21

The main function of the generator regulator in a car is to set the battery voltage to a nominal value and to keep this value within a tight tolerance over the range of the operating conditions of the car (Fostner, 1994). The regulator monitors the generator (alternator) voltage and controls the alternator's current. Because of the critical role of the regulator, some manufacturers install two regulators in parallel. When the two regulators are functioning properly, their failure rates are identical and constant with parameter $\lambda_b = 6 \times 10^{-6}$ failures per hour. When either one of the regulators fails, the remaining unit operates at a lower temperature since there is no heat dissipation from the other unit. Consequently, the failure rate of the remaining unit decreases to $\lambda_s = 3 \times 10^{-6}$ failures per hour. Determine the reliability of the regulators at $t = 10{,}000$ h. What is the MTTF?

SOLUTION

Let τ be the time to the first failure and $g_1(\tau)$ be the p.d.f. for the first failure ($0 < \tau < t$). Thus,

$$g_1(\tau) = 2\lambda_b e^{-2\lambda_b \tau} = 12\times10^{-6} e^{-12\times10^{-6}\tau} \qquad 0 < \tau < t$$

and

$$g_2(t \mid \tau) = \begin{cases} 3\times10^{-6} e^{-3\times10^{-6}(t-\tau)} & 0 < \tau < t \\ 0 & \tau > t \end{cases}.$$

Since the two regulators are required for the battery protection, the p.d.f., $\phi(\tau, t)$, is

$$\phi(\tau,t) = g_1(\tau) g_2(t \mid \tau) \qquad 0 < \tau < t.$$

The reliability of the regulators is governed by the marginal density function, $f(t)$, which is obtained as

$$\begin{aligned} f(t) &= \int_0^\infty \phi(\tau,t)\,d\tau \\ &= \int_0^t \phi(\tau,\zeta)\,d\zeta \end{aligned}$$

and

$$\begin{aligned} R(t) &= 1 - \int_0^t f(\zeta)\,d\zeta \\ R(10{,}000) &= \frac{12\times10^{-6}}{12\times10^{-6} - 3\times10^{-6}} e^{-3\times10^{-6}\times10^4} - \frac{3\times10^{-6}}{12\times10^{-6} - 3\times10^{-6}} e^{-12\times10^{-6}\times10^4} \\ R(10{,}000) &= 0.9983 \end{aligned}$$

and

$$MTTF = 416666.67 \text{ h.}$$

■

Calculating the reliability for a system with a nonconstant failure rate becomes quite complex even when we deal with the simple linearly increasing failure-rate model. Let us consider a simple system with two identical components connected in parallel. The failure rates of the components, when both are operating, are constant with parameter λ. When either component fails, the failure rate of the surviving component becomes $h(t) = \lambda + kt$. The conditional densities are

$$g_1(\tau) = 2\lambda e^{-2\lambda\tau} \qquad 0 < \tau < t \tag{3.86}$$

$$g_2(t \mid \tau) = \begin{cases} [\lambda + k(t-\tau)]e^{-\left[\lambda(t-\tau)+k(t-\tau)^2/2\right]} & 0 < \tau < t \\ 0 & \tau > t \end{cases} \tag{3.87}$$

The j.d.f. is

$$\phi(\tau,t) = (2\lambda e^{-2\lambda\tau})[\lambda + k(t-\tau)]e^{-\left[\lambda(t-\tau)+k(t-\tau)^2/2\right]}. \tag{3.88}$$

The marginal density function that governs the reliability of the system is obtained as

$$\begin{aligned} f(t) &= \int_0^t \phi(\tau,t)\,d\tau \\ f(t) &= 2\lambda\int_0^t e^{-\lambda(t+\tau)}e^{-k(t-\tau)^2/2}[\lambda + k(t-\tau)]\,d\tau. \end{aligned} \tag{3.89}$$

The reliability $R(t)$ is

$$R(t) = 1 - \int_0^t f(\zeta)\,d\zeta. \tag{3.90}$$

Approximate results of Equation 3.89 can be obtained by expanding the exponentials in a truncated series or by using numerical integration. Although the formulation of the j.d.f. is straightforward, the solution of the marginal density function over the time period of interest is computationally difficult.

3.6.3 Compound-Events Approach

Before closing this section, we briefly mention the *compound-events* approach. This approach is based on computing the state probabilities in terms of the system failure rates. It is similar to the Markov model approach with the exception that the failure rates are nonconstant. Although it shares the straightforwardness in model formulation with both the Markov model and the j.d.f. approach, it also shares the difficulty of obtaining the results with the joint density approach.

3.7 REDUNDANCY AND STANDBY

In Section 3.6 we present reliability analysis approaches for systems with dependent failures. In such systems, when one of the components connected in parallel fails, the failure rates of the surviving components are affected. There is another type of failure dependency that arises when a component fails and a standby component replaces the failed one without affecting the failure rate of the standby component. In this section, we evaluate the reliability and availability of different standby and redundant systems.

Reliability of a system (or a component) *may* be improved by using redundant or standby systems (or components). Of course, as shown in Chapter 2, there is an optimum number of multistate devices that can be connected in parallel beyond which the overall system reliability begins to decrease. In addition to the explicit or physical number of units that can be added to the system to improve its reliability, there is an implicit type of redundancy, as in the case of consecutive-k-out-of-n system configuration and in the case of the factor of safety in engineering designs. The higher the factor of safety, the higher the level of redundancy, though not explicitly quantified. An example of this concept is the case of the supporting cables of suspension bridges. The cable contains thousands of wires arranged in a specific pattern. The number of wires required to carry the static load of the bridge, the wind effect, and the maximum applied dynamic load is significantly less than what the final design of the cable contains. If, for example, a factor of safety of two is used, the number of wires in the final design will increase significantly. This results in an implied redundancy in the system. Indeed, the reliability of a cable that contains n wires can be estimated by using the same procedure presented in Chapter 2 for k-out-of-n systems.

Other engineering designs include explicit component redundancies such as the number of components connected in parallel or the number of tires of a large transporter. In these cases, the failure of one or more component or a tire may not necessarily result in the system failure. Redundancy can also be achieved by requiring *system redundancy*—that is, having one or more systems capable of performing the same function such as the brake system of a car where two redundant brake systems are always in operation, the front and rear brake systems.

We classify redundancy as *active* or *inactive*. If the redundant systems are continuously energized and are sharing a portion of the load, there is *active redundancy*. If the redundant systems do not perform any function unless the primary system fails, there is a *standby redundancy* (*inactive redundancy*).

We further classify the standby redundancy according to the failure characteristics as follows:

- **Hot Standby:** Standby components have the same failure rates as the primary component. Since the failure rate of one component is not affected by the other components, the hot standby redundancy consists of statistically independent components (Henley and Kumamoto, 1981; Amari and Dill, 2010).
- **Cold Standby:** Standby components do not fail when they are in standby. The failure of the primary component results in the standby component being a primary component and in its failure rate becoming nonzero.
- **Warm Standby:** A standby component has a smaller failure rate than the primary component but is greater than zero.

If the primary component has a failure rate λ, a hot standby component experiences a failure rate $\lambda_{hot} = \lambda$; a cold standby component has a failure rate $\lambda_{cold} = 0$; and a warm standby experiences a failure rate $\lambda_{warm} < \lambda$.

We can also classify redundant and standby systems as repairable or nonrepairable. Examples of nonrepairable systems include satellites and devices of an IC. Repairable standby systems include electric power generators, automotive brake systems, and airplane jet engines.

In repairable standby systems, when the primary unit fails, it undergoes repair, and the standby unit assumes the functions of the primary unit. When the primary unit is repaired, it assumes the position of the standby unit. The units alternate positions as failures and repairs occur.

Far-reaching decisions on the use of standby redundancy to assure product reliability are typically made in early phases of a project, well before design details required for the usual reliability predictions are available. Three major considerations are: (1) number of standbys to be provided, (2) efficacy of the activation process, and (3) status of the standby when not used (Sears, 1990). In the following sections, we present methods for reliability and availability estimations of different redundant and standby systems.

3.7.1 Nonrepairable Simple Standby Systems

The simplest nonrepairable standby system is a two-unit system that functions successfully when the primary unit (Unit 1) does not fail, or if the primary unit fails during operating time t and the standby unit (Unit 2) assumes the function of the primary unit. The reliability of the system is the sum of the probability that Unit 1 does not fail until time t and the probability that Unit 1 fails at some time τ, $0 < \tau < t$, and the standby unit functions successfully from τ to time t. In other words,

$$R_{sb}(t) = R_1(t) + \int_{\tau=0}^{t} f_1(\tau) R_2(t-\tau)\, d\tau, \tag{3.91}$$

where

$R_{sb}(t)$ = the reliability of the standby system at t,

$R_1(t)$, $R_2(t)$ = the reliability of the primary Unit 1 and the standby Unit 2 at time t,

and

$f_1(t)$ = the p.d.f. of the failure-time distribution of the first unit.

Assume that the failure rates of the primary and the standby units are constant with parameters λ_1 and λ_2, respectively. Then

$$\begin{aligned} R_{sb}(t) &= e^{-\lambda_1 t} + \int_{\tau=0}^{t} \lambda_1 e^{-\lambda_1 \tau} e^{-\lambda_2 (t-\tau)}\, d\tau \\ &= e^{-\lambda_1 t} + \lambda_1 e^{-\lambda_2 t} \int_{\tau=0}^{t} e^{-(\lambda_1-\lambda_2)\tau}\, d\tau \end{aligned}$$

or

$$R_{sb}(t) = e^{-\lambda_1 t} + \frac{\lambda_1 e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}\left(1 - e^{-(\lambda_1-\lambda_2)t}\right). \tag{3.92}$$

We use L'Hopital's rule and differentiate the numerator and denominator of the second term of Equation 3.92 with respect to λ_2 and evaluate its limit as λ_2 approaches λ to obtain

$$R_{sb}(t) = (1+\lambda t)e^{-\lambda t}. \tag{3.93}$$

The MTTF of the two-unit standby unit is

$$MTTF = \int_0^\infty R_{sb}(t)dt = \frac{1}{\lambda} + \frac{\lambda}{\lambda^2} = \frac{2}{\lambda}. \tag{3.94}$$

3.7.2 Nonrepairable Multiunit Standby Systems

We extend the standby system presented in Section 3.7.1 by allowing $(n - 1)$ units in standby for the case where the failure rates are constant with parameter λ. When the primary unit fails, one of the $(n - 1)$ standby units assumes the functions of the primary unit. When the second unit fails, one of the remaining $(n - 2)$ units assumes the functions of the system. The replacements are repeated until the failure of the last unit. The reliability of the multiunit standby system is given by

$$R_{sb}(t) = e^{-\lambda t}\left[1+\lambda t + \frac{(\lambda t)^2}{2!} + \ldots + \frac{(\lambda t)^{n-1}}{(n-1)!}\right]. \tag{3.95}$$

The MTTF is

$$MTTF = \int_0^\infty R_{sb}(t)dt = \frac{n}{\lambda}. \tag{3.96}$$

From Equation 3.95 it is evident that the reliability of the system increases as the number of standby units increases. However, the rate of improvement of the system reliability decreases exponentially as the number of standby units increases. Hence, a decision regarding the number of standby units should consider the economics of adding standby units and the required reliability level of the system. Moreover, the failure modes of the units (such as multistate components) have a major effect on the number of standby or active units.

EXAMPLE 3.22

A thermocouple consists basically of two dissimilar metals, such as iron and constantan wires, joined to produce a thermal electromotive force when the junctions are at different temperatures. The measuring, or hot, junction is inserted into the medium where the temperature is to be measured. The reference, or cold, junction is the open end that is normally connected to the measuring instrument terminals. The electromagnetic force of a thermocouple increases as the difference in junction temperatures increases. Therefore, a sensitive instrument, capable of measuring electromagnetic force, can be calibrated and used to read temperature directly.

In order to measure the temperature around a retort (it is an oven-like equipment where canned food is immediately placed after canning to ensure a safe microbial level inside the can), any number of thermocouples may be used in parallel connections as shown in Figure 3.8. All thermocouples must be of the same type and must be connected by the proper wires.

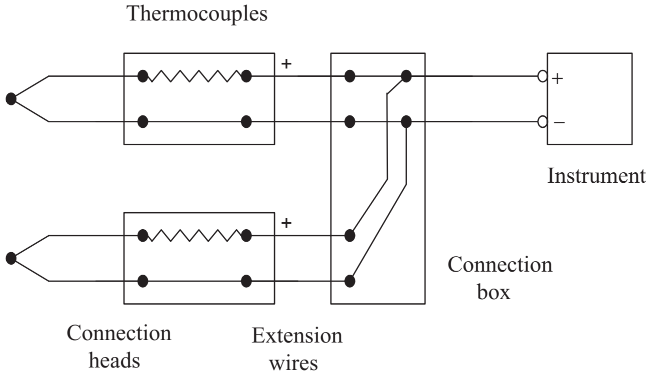

FIGURE 3.8 Thermocouples in parallel.

A producer of canned food uses thermocouples arranged in either a parallel or a standby configuration to ensure that the temperature of a retort is within an acceptable range (a lower temperature may result in a high microbial count while a higher temperature may result in a loss of food nutrition).

The thermocouples are identical and each has a constant failure rate $\lambda = 0.5 \times 10^{-6}$ failures per hour. Graph the reliability $R(t)$ for the parallel and the standby configuration. Determine the number of thermocouples needed in each configuration that ensures a system reliability of 0.999866 or higher at $t = 10^5$ h.

SOLUTION

The reliability of the system for the parallel configuration is given by Equation 3.97:

$$R_{\text{parallel}}(t) = 1 - (1 - e^{-\lambda t})^n. \tag{3.97}$$

The reliability of the multiunit standby system of n units is given by Equation 3.95 as

$$R_{sb}(t) = e^{-\lambda t}\left[1 + \lambda t + \frac{(\lambda t)^2}{2!} + \ldots + \frac{(\lambda t)^{n-1}}{(n-1)!}\right]. \tag{3.98}$$

Figures 3.9 and 3.10 show that the reliability of the parallel system is slightly lower than that of the standby system when $n \leq 3$ units. Indeed, the standby and the parallel systems require two and three units, respectively, to achieve a reliability of 0.999866 at $t = 10^5$ h. The standby system shows higher reliability values than the parallel systems when $n > 3$.

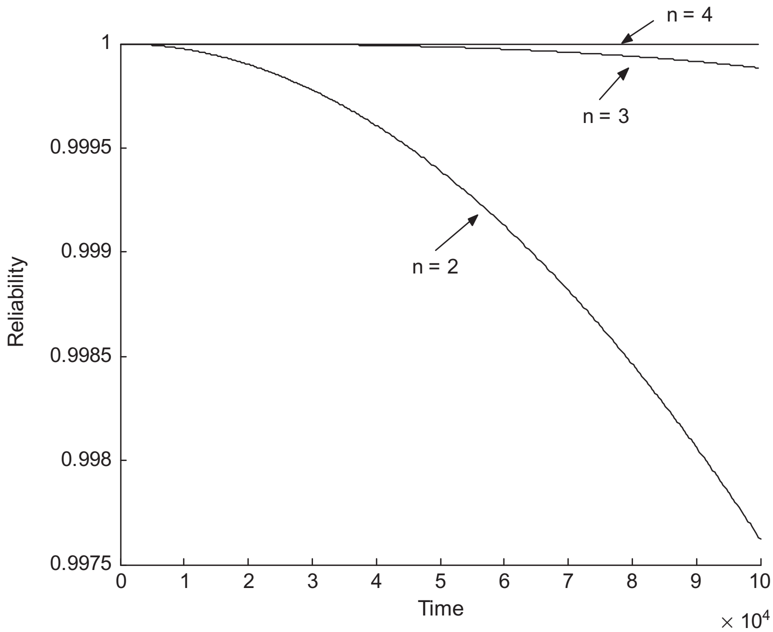

FIGURE 3.9 Reliability of the parallel system.

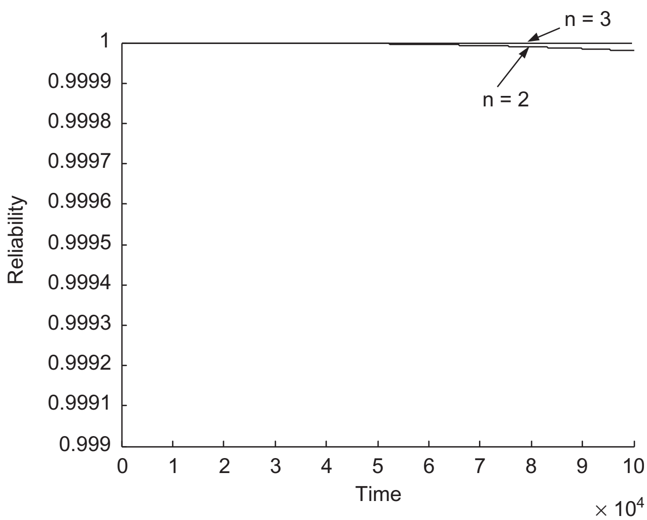

FIGURE 3.10 Reliability of the standby system. ■

3.7.3 Repairable Standby Systems

When the primary unit of a two-unit standby system fails, the standby unit assumes its functions while the primary unit undergoes repair. Upon completion of repairs, the primary unit returns to the system as a standby unit. The two units alternate positions as failures and repairs occur. Examples of such repairable standby systems include two mainframe computers that share the same software and are connected in parallel, electric power generators, pumps in a chemical plant, and scrubbers in coal mining. In all these examples, the repair rate of the failed unit has a major effect on the instantaneous availability of the system as illustrated below.

Electrical-discharge machining (EDM) is a method of removing metal by a series of rapidly recurring electrical discharges between an electrode (the cutting tool) and the workpiece in the presence of a liquid (usually hydrocarbon dielectric). Minute particles of metal or *chips* are removed by melting and vaporization, and are flushed from the gap between the tool and the workpiece (Dallas, 1976).

EDM usually requires a liquid to provide a path for the discharge of electric current to remove metal particles produced from the gap and to cool the tool and workpiece. The liquid is circulated through the system by two hydraulic pumps connected in parallel. When either of the pumps fails, it is repaired while the surviving pump provides the necessary functions. Pumps *A* and *B* are shown in Figure 3.11 as well as the basic components of the electrical discharge machine.

The pumps are identical and their failure rates are constant. The repair rate is also constant with parameter μ. We are interested in estimating the instantaneous availability of the two-pump system under three conditions of standby: hot, warm, and cold.

There are five possible states for the two pumps as shown in Figure 3.12. They are

- *s*1 Pump B is in operation, pump A is in standby, and neither is experiencing failure;
- *s*2 Pump A is in operation, pump B is in standby, and neither is experiencing failure;
- *s*3 Transition from state s_2 when *A* fails, *B* assumes *A*'s functions, and *A* undergoes repair;
- *s*4 Transition from state s_1 when *B* fails, *A* assumes *B*'s functions, and *B* undergoes repair; and
- *s*5 Transition from either state s_3 and s_4 when the two pumps are under repair.

We follow the same steps presented in Section 3.4.2 to develop the state-transition equations. Let $P_{si}(t)$ be the probability that the two-pump system is in state *si* ($i = 1, 2, \ldots$). Assume that the failure rate of the operating pump is λ and that of the standby pump is λ_s where

$\lambda_s = \lambda_h = \lambda$ for hot standby,

$\lambda_s = \lambda_w$ $(0 < \lambda_w < \lambda)$ for warm standby, and

$\lambda_s = \lambda_c = 0$ for cold standby.

The state-transition equations are

$$\dot{P}_{s1}(t) = -(\lambda + \lambda_s)P_{s1}(t) + \mu P_{s3}(t) \tag{3.99}$$

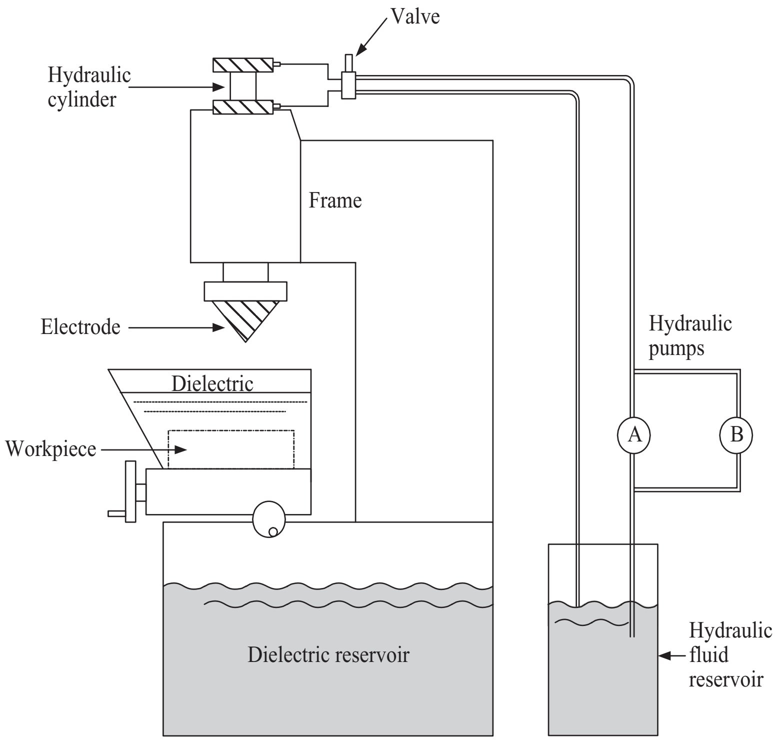

FIGURE 3.11 The basic components of an EDM.

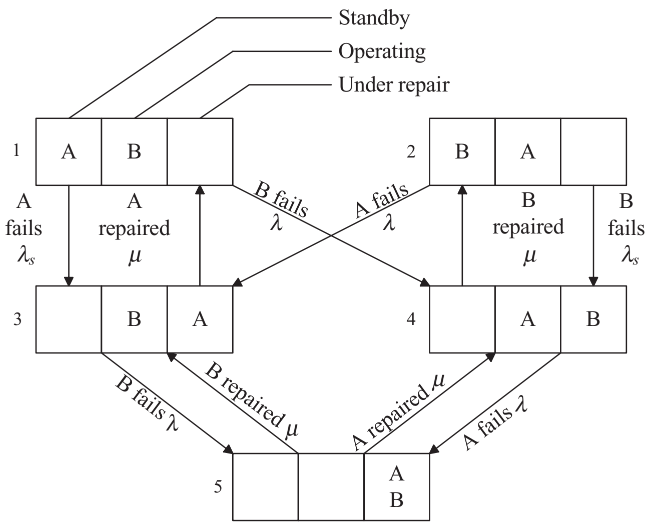

FIGURE 3.12 State-transition diagram of the two-pump system.

$$\dot{P}_{s2}(t) = -(\lambda+\lambda_s)P_{s2}(t)+\mu P_{s4}(t) \tag{3.100}$$

$$\dot{P}_{s3}(t) = -(\lambda+\mu)P_{s3}(t)+\lambda_s P_{s1}(t)+\lambda P_{s2}(t)+\mu P_{s5}(t) \tag{3.101}$$

$$\dot{P}_{s4}(t) = -(\lambda+\mu)P_{s4}(t)+\lambda P_{s1}(t)+\lambda_s P_{s2}+\mu P_{s5}(t) \tag{3.102}$$

$$\dot{P}_{s5}(t) = -2\mu P_{s5}(t)+\lambda P_{s3}(t)+\lambda P_{s4}(t) \tag{3.103}$$

The initial conditions of the two-pump system are $P_{s1}(0) = 1$, $P_{si}(0) = 0$ ($i = 2, 3, 4, 5$). Equations 3.99 through 3.103 can be solved numerically or simplified as follows.

Since the states $s1$ and $s2$ are similar (exchange A and B) and the states $s3$ and $s4$ are also similar in that A and B can be exchanged, we add Equation 3.99 to Equation 3.100, and Equation 3.101 to Equation 3.102 to obtain

$$\frac{d[P_{s1}(t)+P_{s2}(t)]}{dt} = -(\lambda+\lambda_s)[P_{s1}(t)+P_{s2}(t)]+\mu[P_{s3}(t)+P_{s4}(t)] \tag{3.104}$$

$$\frac{d[P_{s3}(t)+P_{s4}(t)]}{dt} = -(\lambda+\mu)[P_{s3}(t)+P_{s4}(t)]+(\lambda+\lambda_s)[P_{s1}(t)+P_{s2}(t)]+2\mu P_{s5}(t) \tag{3.105}$$

$$\frac{dP_{s5}(t)}{dt} = -2\mu P_{s5}(t)+\lambda[P_{s3}(t)+P_{s4}(t)]. \tag{3.106}$$

Define

$$P_1(t) = P_{s1}(t)+P_{s2}(t)$$
$$P_2(t) = P_{s3}(t)+P_{s4}(t)$$
$$P_3(t) = P_{s5}(t).$$

Substituting in Equations 3.104 through 3.106, we obtain

$$\dot{P}_1(t) = -(\lambda+\lambda_s)P_1(t)+\mu P_2(t) \tag{3.107}$$

$$\dot{P}_2(t) = -(\lambda+\mu)P_2(t)+(\lambda+\lambda_s)P_1(t)+2\mu P_3(t) \tag{3.108}$$

$$\dot{P}_3(t) = -2\mu P_3(t)+\lambda P_2(t) \tag{3.109}$$

The new initial conditions are $P_1(0) = 1$, and $P_2(0) = P_3(0) = 0$.

Equations 3.107 through 3.109 can be solved by substituting $P_3(t) = 1 - P_1(t) - P_2(t)$ into Equation 3.108, which results in

$$\dot{P}_2(t) = -(\lambda+3\mu)P_2(t)+(\lambda+\lambda_s-2\mu)P_1(t)+2\mu. \tag{3.110}$$

Equations 3.107 and 3.110 can be written as

$$\dot{P}_1(t) = -(\lambda+\lambda_s)P_1(t)+\mu P_2(t) \tag{3.111}$$

$$\dot{P}_2(t) = -(\lambda+3\mu)P_2(t)+(\lambda+\lambda_s-2\mu)P_1(t)+2\mu. \tag{3.112}$$

The instantaneous availability of the two-pump system, $A(t) = P_1(t) + P_2(t)$, is obtained by solving Equations 3.111 and 3.112 simultaneously.

EXAMPLE 3.23

Estimate the instantaneous availability for the two-pump system described above when the standby unit is considered hot, cold, or warm. Graph the availability for different λ and μ.

SOLUTION

Hot Standby: In hot standby configurations, the failure rate of the standby unit equals that of the operating unit—that is, $\lambda_s = \lambda_h = \lambda$. Assume $\lambda = 5 \times 10^{-5}$ failures per hour and $\mu = 0.008$ repairs per hour. Substituting in Equations 3.111 and 3.112, we obtain

$$\dot{P}_1(t) = -\left(10 \times 10^{-5}\right) P_1(t) + 0.008 P_2(t) \tag{3.113}$$

$$\dot{P}_2(t) = -(0.02405) P_2(t) - 0.0159 P_1(t) + 0.016. \tag{3.114}$$

Taking the Laplace transform of Equations 3.113 and 3.114 results in

$$\left(s + 10 \times 10^{-5}\right) P_1(s) = 1 + 0.008 P_2(s)$$

$$\left(s + 0.02405\right) P_2(s) = \frac{-0.0159}{\left(s + 10 \times 10^{-5}\right)} - \frac{0.0001272}{\left(s + 10 \times 10^{-5}\right)} P_2(s) + \frac{0.016}{s}$$

or

$$P_1(s) = \frac{(s + 0.02405)}{(s + 0.00805)(s + 0.0161)} + \frac{0.000128}{s(s + 0.00805)(s + 0.0161)} \tag{3.115}$$

and

$$P_2(s) = \frac{0.0001}{(s + 0.00805)(s + 0.0161)} + \frac{0.16 \times 10^{-5}}{s(s + 0.00805)(s + 0.0161)}. \tag{3.116}$$

We obtain the Laplace inverse of Equations 3.115 and 3.116 as

$$P_1(t) = 0.987616 + 0.0000385788 e^{-0.0161t} + 0.0123452 e^{-0.00805t} \tag{3.117}$$

and

$$P_2(t) = 0.0123452 - 0.0000771575 e^{-0.0161t} - 0.012268 e^{-0.00805t}. \tag{3.118}$$

The availability of the system is obtained by adding Equations 3.117 and 3.118 as

$$A(t) = 0.999961203 - 0.0000385787 e^{-0.0161t} + 0.0000772 e^{-0.00805t}. \tag{3.119}$$

In order to compare the availability of the system for different values of λ, we consider the hot standby pump by setting $\lambda_s = \lambda$, the warm standby pump by setting $0 < \lambda_s < \lambda$ ($\lambda = 0.0002$), and the cold standby system by setting $\lambda_s = 0$. As shown in Figure 3.13, the availability of the cold standby system is greater than the warm standby, which is greater than the hot standby system.

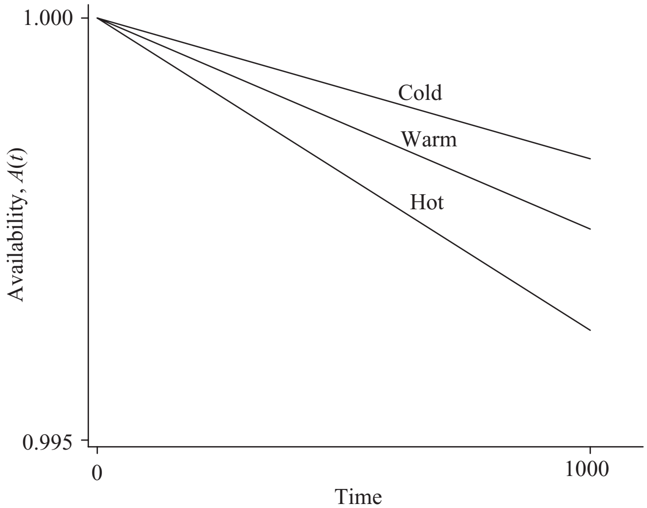

FIGURE 3.13 Availability of hot, warm, and cold standby systems. ■

PROBLEMS

3.1 Consider a consecutive-2-out-of-n:F system. What is the MTTF if the components are i.i.d. and each has a constant hazard rate λ?

3.2 Solve Problem 3.1 for both linearly increasing and Weibull hazard rates.

3.3 Solve Problem 3.1 for a consecutive-k-out-of-n:F system.

3.4 A system consists of three components that are connected in series with four components in parallel. The components are identical, and the failure rate of each component follows a Weibull model with parameters $\gamma = 1.2$ and $\theta = 1.5 \times 10^5$. Derive a reliability expression for the system. What is the MTTF and the effective hazard rate of the system?

3.5 The main components of an undersea lightwave communication system are found in part under water, called the *wet plant*, and in part on land, the *dry plant*. The wet plant components consist of a cabled fiber transmission medium and repeaters containing optical amplifiers. The cable also contains a copper conductor to carry electrical power to the repeaters. Moreover, the system contains a branching unit, which provides for greater flexibility in undersea network architecture by allowing traffic to be split or switched. The dry plant components consist of terminal transmitter equipment (TTE), line monitoring equipment (LME), and power feed equipment (PFE). The TTE provides communication between the "dry" land communication network and the "wet" undersea transmission link. The LME monitors the transmission system and locates failures and faults, and the PFE energizes the link, providing power to the repeaters (Mortenson et al., 1995).

Consider a point-to-point undersea repeater system as shown in Figure 3.14. The system has 150 repeaters and the failure of two consecutive repeaters interrupts the communication between the two points. The failure rate of each repeater is constant with $\lambda = 8.5 \times 10^{-7}$ failures per hour. The failure rates of the dry plant components are

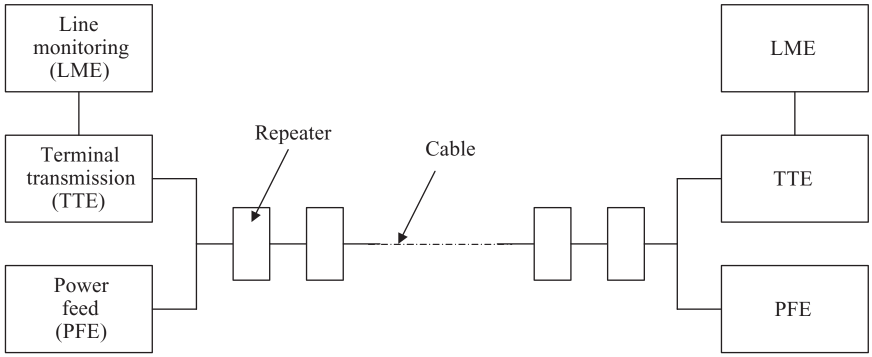

FIGURE 3.14 Point-to-point undersea cable.

$$h_{TTE}(t) = 5\times10^{-4}t^{1.25}$$
$$h_{LME}(t) = 12\times10^{-5}t$$
$$h_{PFE}(t) = 1.8\times10^{-6}.$$

Derive a reliability expression for the system. What is the MTTF? What do you recommend to improve the reliability of the system by 20%?

3.6 One of the important components of the wet plant described in Problem 3.5 is the repeater. It is a collection of optical and electrical components in a beryllium-copper (BeCu) housing. The housing can support up to four amplifier pairs that support transmission on two communication lines simultaneously. Assume that the failure rate of an amplifier is linearly increasing with time with a parameter 6×10^{-12} failures per hour. A minimum of one pair of the amplifiers in each repeater is required to operate properly in order to ensure the transmission between end points. Derive a reliability expression for each repeater. What is the MTTF of the total system? Graph $R(t)$ for different values of failure rates.

3.7 A nonredundant subsystem has 100 units, each having a constant failure rate with an MTTF of 5×10^3 h. What is the minimum number of units to be connected in parallel in order to achieve a system MTTF of 2 years? Provide alternative configurations that maintain a minimum reliability level of 0.999 after 3×10^3 h.

3.8 Assume a system with implicit redundancy such as consecutive k-out-of-$n{:}F$ that results in the same reliability level as that given in Problem 3.7. Determine the k units for such a system.

3.9 Repeat Problem 3.8 for k-out-of-n system.

3.10 One of the well-known approaches for improving system reliability is to add redundant components. This may be true for components with one type of failure mode. For components with multiple failure modes, there exists an optimum number of redundant components that maximizes the total system reliability. Consider a system that consists of m components in parallel. Each component has three modes: normal (operational), fail open, and fail short. Let

$q_{si}(t)$ be the probability that the component fails short at time t and

$q_{oi}(t)$ be the probability that the component fails open at time t.

a. Show that the reliability of the system is

$$R_S(t) = \prod_{i=1}^{m}(1 - q_{si}(t)) - \prod_{i=1}^{m} q_{oi}(t).$$

b. Assume $q_{oi}(t) = a_i t^{b_i}$ and $q_{si}(t) = \alpha_i t$, where a_i, b_i, and α_i are constants. Graph $R_S(t)$ for different values of the constants and investigate their effects on $R_S(t)$.

3.11 Consider a system with three components in series. The components have constant failure rates λ_1, λ_2, and λ_3. The failure rate of the third component is three times the failure rate of the second component, and the failure rate of the second component is twice that of the first component. It is desired to achieve a system reliability of 0.95 at time $t = 100$ h. Determine

a. The failure rates of the components.

b. The MTTF of the system.

c. The probability of having 0, 1, and 2 failures in 100 h of operation.

d. The failure rates of the components if a reliability of 0.95 is desired at the MTTF.

3.12 The Special Erlang distribution is useful in modeling the failure rate of many electronic components. The p.d.f. of the distribution is

$$f(t) = \frac{t}{\lambda^2}\exp\left(-\frac{t}{\lambda}\right) \quad t \geq 0.$$

a. Determine the MTTF of a component that exhibits such a failure-time distribution. What is the variance of the TTF?

b. Assume that at $t = 1000$ h, the hazard rate of a component is 5×10^{-4} failures per hour. What is the reliability of a system composed of three similar components connected in parallel at time $t = 10^4$? What is the MTTF of such a system?

3.13 A system is configured using four components as shown in Figure 3.15.

Assume that the components have the following hazard rates

$$h_1(t) = 0.5 \times 10^{-7} \text{ failures per hour,}$$
$$h_2(t) = 0.5 \times 10^{-7} \times t \text{ failures per hour,}$$
$$h_3(t) = 2.5 \times 10^{-6} \times t^{1.2} \text{ failures per hour, and}$$
$$h_4(t) = 0.5 \times 10^{-8} \times t^{1.3} \text{ failures per hour.}$$

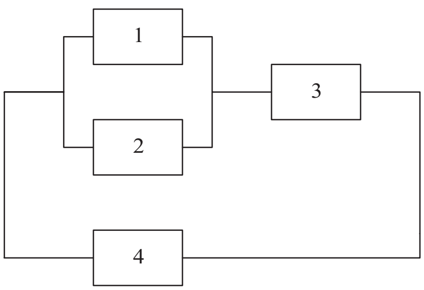

FIGURE 3.15 A four-component system for Problem 3.13.

Derive the reliability expression of the system. What is the reliability of the system at $t = 1000$ h? What is the MTTF?

3.14 Consider two nonrepairable systems, A and B, as shown in Figure 3.16. Each has the same number of components, n. The failure rate of component i is $h_i(t) = k_i t^{m_i}$.

a. Show that $R_A(t) \le R_B(t)$.

b. Replace System B components with identical and less reliable ones and obtain the conditions that result in $R_A(t) \le R_B(t)$.

c. Derive expressions for the MTTF for both systems in (a) and (b).

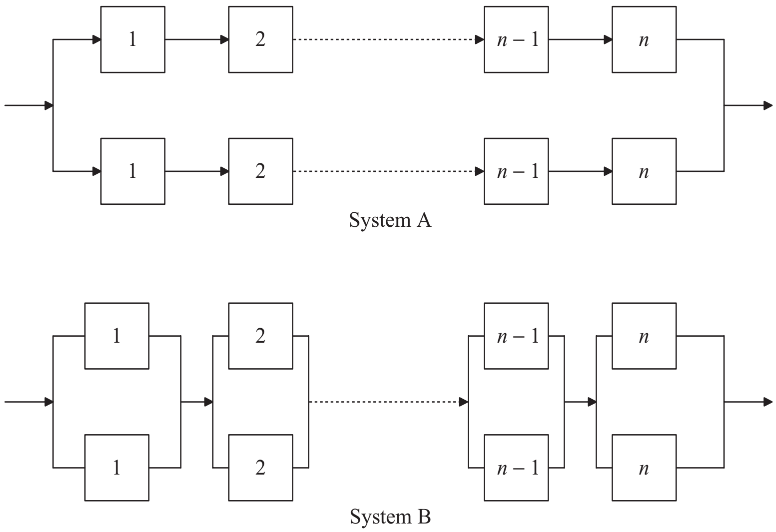

FIGURE 3.16 Two nonrepairable systems.

3.15 In typical large-scale ICs we observed beam bonding. Two types of open failures were observed on the failed ICs. The first type was a combination of silicon to beam interface separation and broken beam on the edge of the silicon chip. The second type was that of a broken beam at the heel or midspan. The failure rate of the first type is constant with parameter λ, and the failure rate of the second type is a Weibull model with parameters γ and θ. Graph the reliability of the ICs for different values of λ, γ, and θ.

3.16 Assume that the failure rates of the ICs described in Problem 3.15 are constant with parameters λ_1 and λ_2, and the failed ICs are repaired with constant repair rates μ_1 and μ_2 for the first and second type of failures, respectively.

a. Derive an expression for the ICs' availability at time t.

b. What is the steady-state availability of an IC?

c. If

$$\begin{aligned}
\lambda_1 &= 5\times10^{-6} \text{ failures per hour,}\\
\lambda_2 &= 6\times10^{-7} \text{ failures per hour,}\\
\mu_1 &= 0.5 \text{ repairs per hour, and}\\
\mu_2 &= 1 \text{ repair per hour,}
\end{aligned}$$

what is the availability at time $t = 10^6$ h?

d. What is the ratio between μ_1 and μ_2 that ensures a minimum availability of 0.9999 at time $t = 10^5$ h?

e. What is the MTBF?

3.17 Further analysis of the ICs given in Problem 3.15 shows that indeed the number of failure types is significantly more than two—for example, failure Type III consisted of darkened inclusions in the bond area, which acted as stress concentration centers, and resulted in degraded bond strength. In order to generalize the reliability and availability analysis of the ICs, the reliability engineers made the following assumptions.

- A typical IC may fail in any of the N failure modes. The failure rate of Type i is constant with parameter λ_i.
- The engineer also recommended that the failed IC be repaired and the repair rate of failure Type i is constant with parameter μ_i.

a. Derive expressions for $A(t)$.

b. For a wide range of λ_i and μ_i, determine the length of time which makes $A(t)$ equivalent to $A(\infty)$.

3.18 Pumping stations are considered major components of water supply systems. A typical pumping station consists of one or more pumping units supported by appropriate electrical, piping, control, and structural subsystems. The pumping unit is the primary subsystem that includes four components: pump, driver (motor), power transmission, and controls. A common design practice is to install sufficient pumps to handle peak flows and include a spare pump of equal size to accommodate any downtime of other pumps. Thus, the mechanical failure of the pumping station could be defined as the simultaneous failure of two or more pumping units while peak capacity is required (Mays, 1989).

The individual pumping units have two possible operating states: working and not working. The failure rates of the components of the individual units are

Component	Failure rate
Pump	Constant with parameter λ_p
Drive	Increasing with $h_d(t) = \lambda_d t$
Power transmission	Weibull with $h_s(t) = \dfrac{\gamma}{\theta}\left(\dfrac{t}{\theta}\right)^{\gamma-1}$
Controls	Exponential with $h_c(t) = be^{\alpha t}$

The configuration of the components of the individual pump unit can be considered as a series system. In order to meet the peak lead requirements of water supply, the planner of a city recommend the installation of four individual pump units in parallel in the major pumping station. Analysis of the data from an actual pumping station shows that

$\lambda_p = 0.00133$ failures per year,

$\lambda_d = 0.00288$,

$\gamma = 1.30$,

$\theta = 2.3 \times 10^3$ year,

$b = 1000$, and

$\alpha = 0.3$.

The time t is expressed in years. Derive an expression for the reliability of the system. What are the two most critical components in an individual pump unit? Recommend two methods that improve the overall reliability of the pump station. Explain the advantages and disadvantages of each method.

3.19 Consider a six-engine descent system of a large crewed vehicle missions to Mars. The system can land the vehicle safely as long as it experiences at most one engine pair failure to maintain balance of the vehicle. In Figure 3.17 the engine pairs are 1–4, 2–5, and 3–6. Assume that the engine pair 1 and 2 are identical and have constant failure rate of 0.000009 failures per hour, engine pair 3 and 4 are identical and the failure time follows Weibull distribution with $\theta = 5000$ and $\gamma = 1.5$, and the last pair, 5 and 6 are also identical with an Erlang failure-time distribution given by $f(t) = t/\lambda^2 \exp(-t/\lambda)$ $t \geq 0$ and $\lambda = 0.00003$. Determine the reliability of the system as a function of time.

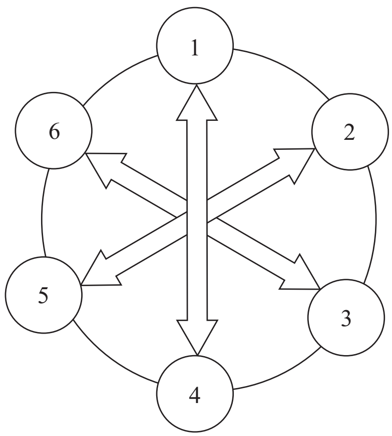

FIGURE 3.17 Six-engine vehicle system.

3.20 A high-voltage system consisting of a power supply and two transmitters, A and B, uses mechanically tuned magnetrons. When the two transmitters are used in parallel, each transmitter tunes one-half of the desired frequency range; however, if one transmitter fails, the other tunes the entire range with a resultant change in the expected TTF. Suppose that in order for this system to work, the power supply and at least one of the transmitters must operate properly (Pham, 1992). Let λ_A be the constant failure of transmitter A when transmitter B is operating in parallel with A. Let λ'_A be the failure rate when B fails. Similarly, suppose that transmitter B has a constant failure rate λ_B when A is operating in parallel with B and has a constant failure rate λ'_B when A fails.

a. Obtain an explicit reliability expression for the system.

b. Graph $R(t)$ for different ratios of λ_A/λ_B and λ'_A / λ'_B.

c. What is the MTTF of the system?

3.21 A repairable system consists of a primary unit and a standby unit. They alternate positions as failures and repairs occur. The units are identical and each has two failure modes: open and short. The failure and repair rates are constant with the following parameters.

λ_O = the failure rate of the open mode failure,

λ_S = the failure rate of the short mode failure,

μ_O = the repair rate of the open failure, and

μ_S = the repair rate of the short failure.

When the primary unit fails in either mode, it is immediately replaced with the standby unit at a constant rate α (Elsayed and Dhillon, 1979).

a. Derive the state-transition equations.

b. Solve (a) for $P_i(t)$ (probability that the system is in state i at time t).

c. What is the instantaneous availability of the system?

d. Investigate the effect of μ_O/μ_S and α on $A(t)$.

3.22 A maintained system with two components in parallel each has a failure rate of $\lambda = 0.001$ failures per day independent of the number of components in operation. At $t = 0$, the two components are in an operative state (state zero). If it is desired to have the system in this state at least 50% of the time and to be in an operative state with only one component working 25% of the time, what repair rates should be provided?

a. Consider that you could only work on one component at a time and the repair rate is therefore independent of the number of failed components. What is the required repair rate assuming the above availabilities? What are the repair rates if the repair rate is a function of the number of failed components?

b. If it costs \$1 for each percent decrease in failure rate and \$2 for each percent of increase in repair rate, determine the optimum policy that minimizes the total cost and maintains the availability at 0.95.

3.23 In a 3-out-of-n system with components having a linearly increasing hazard rate $h(t) = 0.5 \times 10^{-8}t$ failures per hour, determine the number of components for the system such that a reliability of 0.98 is achieved at $t = 10^3$ h. What is the MTTF?

3.24 Determine the number of components that can be connected in parallel and results in the same reliability values as that of Problem 3.23.

3.25 A computer chip has 200,000 transistors connected in parallel and k transistors are required to operate properly for the chip to perform its function. Assuming that each transistor has a constant hazard model, $h(t) = \lambda$, what is the value of k that ensures a chip reliability of 0.95 at $t = 10{,}000$ h?

3.26 *Partial Redundancy* is defined as the configuration where at least k out of m ($k < m$) possible paths must be successful. Consider a system of m diodes (a diode is an electronic device which allows current to pass in one direction and prevents it from passing in the other direction) connected in parallel and each diode is subject to either failures: (1) open, the current cannot pass in through the diode and (2) short, the diode allows current to pass either way in the circuit. Assume that the probability of a diode failing in the open failure mode is q_o, the probability of failing in the short mode is q_s, and the probability of working properly is p. Note that $q_o + q_s + p = 1$. Assuming that the system is successful if at least k of the m diodes are successful, develop an expression to describe the system's reliability. If $m = 6$, $k = 3$, $q_o = 0.02$, and $q_s = 0.05$, what is the reliability of the system?

3.27 In many pharmaceutical applications, the control system of the processes is crucial to ensure that the products meet quality requirements. Therefore, controllers of critical processes are designed such that some level of redundancy is provided. Figure 3.18 represents two simple configurations for a control system (System A and System B). The difference between the configurations is the addition of a redundant central processor for control Module I and a redundant communications network for System B. In these systems, the failure of the system occurs if there is a loss of communications network, loss of control Module I, and loss of both operators' consoles (Renner, 1988). The failure rate of the operators' consoles is constant with parameter λ_c, and their repair rate is also constant with parameter μ_c. The failure rate of

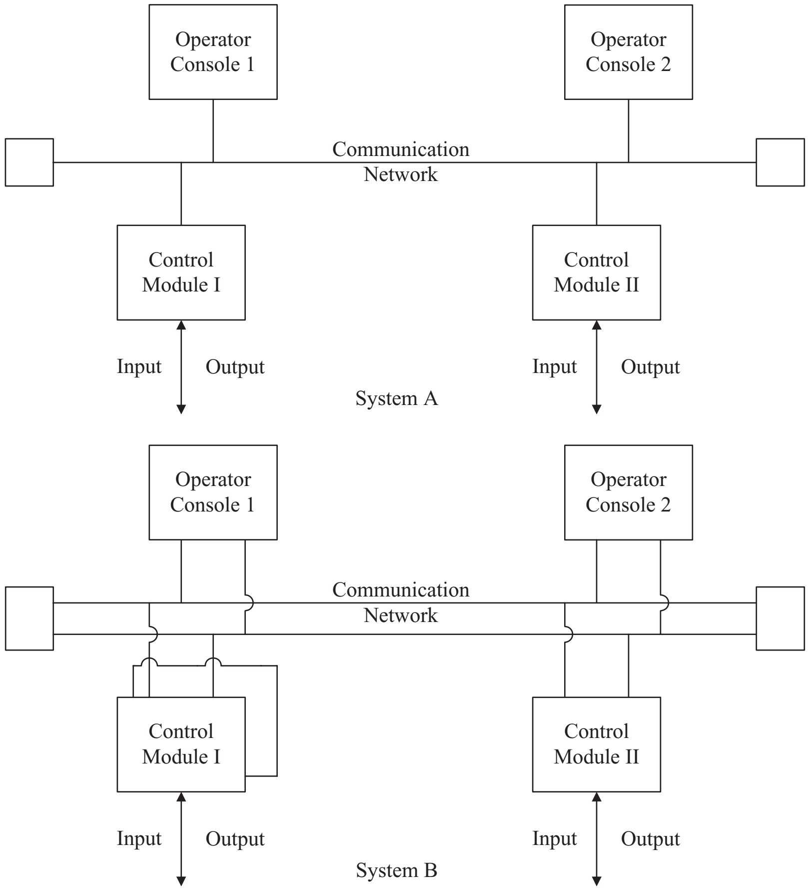

FIGURE 3.18 Two configurations for a control system.

the communication network is λ_n, and its repair rate is μ_n. Similarly, the failure and repair rates for Module I are λ_I, μ_I, and those for Module II are λ_{II} and μ_{II}.

a. Derive the state-transition equations of the system.

b. Derive expressions for the instantaneous availabilities of the two systems. What are the conditions that make the two systems equivalent?

c. Given the same number of components as that of System *B*, design a new control system that shows better performance than configuration *B*.

d. The failure rates of the components are

Operator consoles 1 and 2	24×10^{-6} failures per hour
Communication links	0.25×10^{-6} failures per hour
Control Module I	116.2×10^{-6} failures per hour
Control Module II	130.0×10^{-6} failures per hour

Use the above data to estimate the interval availability for the period 2000–7000 h for both systems. Make appropriate assumptions about repair times.

3.28 A redundant system consists of two components connected in parallel. Both components exhibit a constant failure rate λ when operating simultaneously. The failure rate of the surviving component increases linearly with time when one of the components fails. Use the joint probability distribution function approach to obtain a reliability expression for the system. What is the MTTF?

3.29 Consider the hot-standby system given in Example 3.23. Determine the expected number of renewals and the variance of the number of renewals for any given t.

3.30 A system consists of two units which are connected in parallel, and each has a failure rate that follows a Birnbaum–Saunders's distribution with parameters $\alpha = 1.5$ and $\beta = 200$. When one of the units fail, the failure rate of the second unit increases, and its failure-time distribution has two parameters $\alpha = 1.75$ and $\beta = 250$. Use the j.d.f. approach to obtain the reliability of the system.

3.31 Consider the system given in Problem 3.30 and assume that it only has one unit. When the unit fails, it is repaired with a constant repair rate μ that follows an exponential distribution. Obtain an approximation of the system's availability for different repair rates ranging from 10 to 20 with an increment of two repairs per unit time.

3.32 A typical car brake system has one master cylinder that pressurizes the brake lines which in turn activate the brake pads to apply pressure on the brake disks causing the wheels to slow down. As the brake pads wear out the brake pressure deceases, and it may not be sufficient to stop the car when needed. In other words, there is threshold value for the brake system to function properly. Assume that the failure time of the master cylinder follows a Weibull distribution with parameters $\gamma = 1.6$ and $\theta = 10{,}000$ h and the failure-time distribution for the brake pads (for a given threshold values) follows a Birnbaum–Saunders distribution with parameters $\alpha = 1.75$ and $\beta = 2500$. Safe braking of the car requires that a minimum of three wheels function properly with a probability of 0.85. Determine the minimum threshold level and the time for brake pad replacements.

3.33 Consider Example 3.12, its network is shown in Figure 3.19. Assume the components have the following constant failure rates:

$\lambda_1 = 0.00005$

$\lambda_2 = 0.00009$

$\lambda_3 = 0.00015$

$\lambda_4 = 0.00025$

$\lambda_5 = 0.00002$

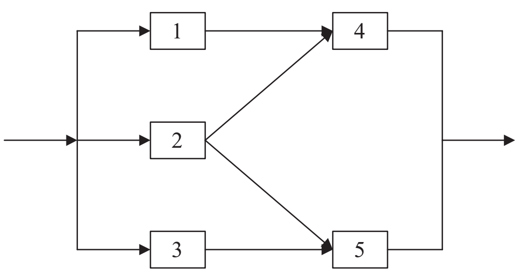

FIGURE 3.19 Network of Problem 3.33.

a. Derive an expression for the effective failure rate of the network.

b. Determine the conditions that ensure that the steady-state failure rate of the system is approximately constant.

3.34 Hong and Lie (1993) introduce the concept of joint reliability importance (JRI) of two components in binary system as

$$JRI(i,j) = \frac{\partial^2 h(\mathbf{p})}{\partial p_i \partial p_j},$$

where $h(\mathbf{p})$, reliability function of the system, is expressed as $h(\mathbf{p}) = E[\phi(x)]$, p_i and p_j are the reliabilities of components i and j, respectively. Obtain JRI for every pair of components 1 through 5 for a parallel system with 5 components.

3.35 The increasing generalized failure rate (IGFR) distributions are used in stochastic models of service systems (Lariviere and Porteus, 2001; Ziya et al., 2004; Paul, 2005). A failure-time distribution with density function $f(t)$ and distribution function F(t) is said to be IFR (increasing failure rate) if $f(t)/(1—F[t])$ is increasing in t. The failure rate is said to be IGFR if $t\, f(t)/(1—F[t])$ is increasing in t. Interestingly, some DFR (decreasing failure rate) distributions are IGFR. Show the conditions of a failure time that follows Weibull distribution when

a. The failure rate is IFR and indeed it is also an IGFR

b. The failure rate is DFR and indeed it is also an IGFR.

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CHAPTER 4

ESTIMATION METHODS OF THE PARAMETERS OF FAILURE-TIME DISTRIBUTIONS

4.1 INTRODUCTION

Reliability estimation requires the knowledge of the underlying failure-time distribution of the component or the system being modeled. Also, in order to predict the reliability or estimate the mean time to failure (MTTF) of components or systems subjected to an accelerated life test, we need to estimate the parameters of the probability distribution that describes the failure time of the population subjected to the test.

Clearly, the accuracy of the estimate of the parameters depends on the sample size and the method used for estimating the parameters. The statistics, calculated from the samples that are used to estimate population parameters, are called *estimators*. A good estimator should have the following properties.

- **Unbiased:** The estimator $\hat{\theta}$ is an unbiased estimator for a parameter θ if and only if $E\left[\hat{\theta}\right]=\theta$. In other words, an unbiased estimator should not consistently underestimate nor overestimate the true value of the parameter.
- **Consistent:** A consistent estimator is one that converges more closely to the true value of the population parameter as the sample size increases. In other words, the estimator $\hat{\theta}$ is said to be a consistent estimator of θ if the probability of making errors of any given size ε, tends to zero as n (sample size) tends to infinity—that is,

$$P\left[\left|\hat{\theta}-\theta\right|>\varepsilon\right]\to 0 \text{ as } n\to\infty, \text{ for any fixed positive } \varepsilon.$$

- **Efficient:** An efficient estimator is a consistent estimator whose standard deviation is smaller than the standard deviation of any other estimator for the same population parameter. We measure efficiency by

Reliability Engineering, Second Edition. Elsayed A. Elsayed.

Relative efficiency = $V(\hat{\theta}_2)/V(\hat{\theta}_1)$, where $V(\hat{\theta}_1)$ and $V(\hat{\theta}_2)$ are the variances of the two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ for the same population parameter: the better estimator has the smaller variance; and

The asymptotic relative efficiency (ARE): the relative efficiency is a function of n, and to avoid this we use the ARE as

$$\text{ARE} = \lim_{n\to\infty} \frac{V\left(\hat{\theta}_2\right)}{V\left(\hat{\theta}_1\right)}.$$

- **Sufficient:** A sufficient estimator is an estimator that utilizes all the *information* about the parameter that the sample possesses.

The statistic used to estimate the parameter of the population, θ, is called a *point estimator* for θ and is denoted by $\hat{\theta}$. The properties of the point estimator were discussed earlier.

Three of the most widely used methods for estimating the parameters of the population are *the method of moments*, *the maximum likelihood method*, and *the least squares method*. It should be made clear that the estimate of the parameters regardless of the method being used depends on the "quality" of the data. This means that the engineer should check for outliers, such as abnormally short or abnormally long failure times. There are many statistical tests that identify outliers in a data set such as the Natrella–Dixon test (Dixon and Massey, 1957; Natrella, 1963) and the Grubbs (1950) test. A comprehensive study of the outliers identification methods is presented in Hawkins (1980). In some situations, these methods may not provide estimates of the parameters due to the lack or the limited number of observations. They are also inappropriate when subjective values of the parameters are provided. The Bayesian approach for parameters estimation may be useful in providing initial estimates. This approach is presented later in this chapter.

We now present some of the most commonly used methods for parameter estimation.

4.2 METHOD OF MOMENTS

The main idea of the method of moments is to equate certain sample characteristics, such as mean and variance, to the corresponding population expected values and then solve the resulting equations to obtain the estimates of the unknown parameter values.

If $x_1, x_2, \ldots, x_n$ represent a set of data, then the kth sample moment is

$$M_k = \frac{1}{n}\sum_{i=1}^{n} x_i^k.$$

If $\theta_1, \theta_2, \ldots, \theta_m$ are the unknown parameters of the population, then the *moment estimators* $\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_m$ are obtained by equating the first m sample moments to the corresponding first m population moments and solving for $\theta_1, \theta_2, \ldots, \theta_m$.

EXAMPLE 4.1

Assume that $x_1, x_2, \ldots, x_n$ represent a random sample from an exponential distribution with parameter λ. What is the estimate of λ?

SOLUTION

The probability density function (p.d.f.) of the exponential distribution is

$$f(x) = \lambda e^{-\lambda x}$$

and

$$E[X] = \frac{1}{\lambda}.$$

Using the sample's first moment,

$$M_1 = \frac{\sum_{i=1}^{n} x_i}{n} = E[X] = \frac{1}{\lambda},$$

or the estimate of λ is

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}.$$

■

EXAMPLE 4.2

A manufacturer of a wireless data system uses infrared beams transmitted between devices mounted on the outside of buildings to provide a high-speed link data. The size of the infrared beam has a direct effect on the system reliability and its ability to reduce the effect of weather conditions such as snow and fog that obstruct the beam's path. Data are transmitted continuously using the infrared beams and the times to failure in hours (not receiving the transmitted data) are recorded as follows:

47, 81, 127, 183, 188, 221, 253, 311, 323, 360, 489, 496, 511, 725, 772, 880, 1509, 1675, 1806, 2008, 2026, 2040, 2869, 3104, 3205.

Assuming that the failure times follow an exponential distribution, determine the parameter of the distribution using the method of moments. Estimate the reliability of the system at time = 1000 h. (Note that the above data are generated from an exponential distribution with parameter $1/\lambda = 1000$.)

SOLUTION

The parameter of the exponential distribution is

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}$$

$$\hat{\lambda} = \frac{25}{26,209} = 0.00095387$$

$$\frac{1}{\hat{\lambda}} = 1048.36.$$

This is very close to the parameter value used in generating the data. Clearly, as the number of observations increases, the estimated parameter ($\hat{\lambda}$) quickly approaches the parameter of the actual distribution of the failure times.

The reliability of the system at 1000 h is

$$R(1000) = e^{-0.95387} = 0.385247.$$

■

We now illustrate the method of moments in estimating the parameters of a two-parameter distribution such as a gamma distribution.

EXAMPLE 4.3

Let $x_1, x_2, \ldots, x_n$ be a random sample from a gamma distribution whose p.d.f. is

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} \quad x > 0, \alpha \geq 0, \beta > 0.$$

Use the method of moments to obtain estimates of the parameters α and β.

SOLUTION

As shown in Chapter 1, the mean and variance of the gamma distribution, respectively, are

$$E[X] = \alpha\beta$$
$$V(X) = \alpha\beta^2 = E[X^2] - (E[X])^2.$$

We replace $E[X]$ and $E[X^2]$ by their estimators M_1 and M_2, respectively, to obtain

$$M_1 = \hat{\alpha}\hat{\beta}$$
$$M_2 - M_1^2 = \hat{\alpha}\hat{\beta}^2.$$

Solving the above two equations simultaneously yields

$$\hat{\beta} = \frac{(M_2 - M_1^2)}{M_1} \tag{4.1}$$

$$\hat{\alpha} = \frac{M_1^2}{(M_2 - M_1^2)}. \tag{4.2}$$

■

EXAMPLE 4.4

A manufacturer of personal computers performs a burn-in test on 20 computer monitors and obtains the following failure times (in hours):

130, 150, 180, 40, 90, 125, 44, 128, 55, 102, 126, 77, 95, 43, 170, 130, 112, 106, 93, 71.

Assume that the main population of the failure times follows a gamma distribution with parameters α and β. What are the estimates of these parameters?

SOLUTION

We first determine M_1 and M_2 as

$$M_1 = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{2067}{20} = 103.35$$

$$M_2 = \frac{1}{20}\sum x_i^2 = \frac{1}{20} \times 244{,}823 = 12{,}241.15.$$

Using Equations 4.1 and 4.2, we obtain

$$\hat{\beta} = \frac{12{,}241.15 - (103.35)^2}{103.35} = 15.09$$

$$\hat{\alpha} = \frac{(103.35)^2}{12{,}241.15 - (103.35)^2} = 6.847.$$

The expected mean life of a monitor is $\hat{\alpha}\hat{\beta} = 103.3$ h. ■

The following is another example that illustrates the use of the method of moments in estimating the parameters of a two-parameter distribution..

EXAMPLE 4.5

Use the method of moments to estimate the parameters μ and σ^2 of the normal distribution.

SOLUTION

The p.d.f. of the normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

The first moment, M_1, about the origin is

$$M_1 = \int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Let

$$z = \frac{x-\mu}{\sigma}.$$

Then $x = \mu + \sigma z$ and $dx = \sigma dz$, and the first moment becomes

$$M_1 = \int_{-\infty}^{\infty} \frac{(\mu+\sigma z)}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$M_1 = \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \sigma \int_{-\infty}^{\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

Since

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 1$$

and

$$\int_{-\infty}^{\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \int_{-\infty}^{\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} d\left(\frac{z^2}{2}\right) = \frac{-1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \Big|_{-\infty}^{\infty} = 0,$$

then

$$M_1 = \mu = \frac{1}{n}\sum_{i=1}^{n} x_i. \tag{4.3}$$

The second moment, M_2, about the origin is obtained as

$$M_2 = \int_{-\infty}^{\infty} \frac{x^2}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Again, let $z = (x - \mu)/\sigma$, then

$$M_2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} (\mu+\sigma z)^2 e^{-\frac{z^2}{2}} dz$$

$$M_2 = \mu^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + 2\sigma\mu \int_{-\infty}^{\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \sigma^2 \int_{-\infty}^{\infty} \frac{z^2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

The integral parts of the first two terms in the above equation have been obtained earlier. The integration of the third term is obtained by integration by parts.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{z^2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz &= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z d\left(e^{-\frac{z^2}{2}}\right) \\ &= \frac{-1}{\sqrt{2\pi}} z e^{-\frac{z^2}{2}}\Bigg|_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \\ &= 0 + 1 \\ &= 1.\end{aligned}$$

Therefore,

$$M_2 = \mu^2 + \sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2. \tag{4.4}$$

From Equations 4.3 and 4.4 the estimated parameters of the normal distribution are

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} x_i{}^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$

or

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \overline{x})^2.$$

■

The method of moments is a simple method for estimating the parameters of the failure-time distribution provided that the underlying distribution is known. The errors in estimating the parameters are minimum when the underlying distribution is symmetric with no skewness and when the failure times are not censored or truncated.

4.2.1 Confidence Intervals

After the determination of the point estimate of the parameters of the distribution, we may be interested in determining a confidence interval for which the estimated parameters are close to the true values of the population. This can be accomplished by defining two limits—the lower confidence limit (LCL) and the upper confidence limit (UCL)—that form an interval that has a probability $1 - \alpha$ of capturing the true value of parameters, where $1 - \alpha$ is called the confidence coefficient. In other words, the confidence interval for the parameter θ is

$$P[LCL \le \theta \le UCL] = 1 - \alpha \tag{4.5}$$

For brevity, we shall limit our presentation to the general distribution case. Other distributions can be easily treated in a similar fashion.

Suppose that a random sample $x_1, x_2, \ldots, x_n$ is taken from a population with mean μ and variance σ^2. Let $\bar{x}$ be the point estimator of μ. If n is large ($n \geq 30$), then $\bar{x}$ has approximately a normal distribution with mean μ and variance σ^2/n or

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

has a standard normal distribution. For any value of α we can find (using standard normal tables) a value of $Z_{\alpha/2}$ such that

$$P\left[-Z_{\alpha/2} \leq Z \leq +Z_{\alpha/2}\right] = 1 - \alpha.$$

Rewriting the above expression, we obtain

$$\begin{aligned} 1 - \alpha &= P\left[-Z_{\alpha/2} \leq \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leq +Z_{\alpha/2}\right] \\ &= P\left[\bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]. \end{aligned}$$

Thus, the interval

$$\left(\bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

forms the confidence interval of the estimated parameter $\bar{x}$ for μ with confidence coefficient of $(1 - \alpha)$. Note that when $n < 30$ the t-distribution is used instead of the normal distribution.

EXAMPLE 4.6

Consider the failure times of Example 4.4. Find a confidence interval for the mean failure time with confidence coefficient of 0.95.

SOLUTION

From the data, we obtain

$$\bar{x} = 103.35$$
$$s = 40.52.$$

s is the estimate of the standard deviation σ.

Since the sample size is small (<30), it is more appropriate to use the t distribution than the standard normal in determining the confidence interval. Thus, the confidence interval is

$$\bar{x} \pm t_{\alpha/2}\frac{\sigma}{\sqrt{n}} \text{ with } (1 - \alpha) = 0.95.$$

We obtain $t_{0.025} = 2.093$ and substitute s for σ to get

$$103.35 \pm 2.093 \times \frac{40.52}{\sqrt{20}}$$
$$103.35 \pm 18.96$$
$$(84.39, 122.31).$$

In other words, we have 95% confidence that the true mean failure time lies between 84.39 and 122.31 h. ■

4.3 THE LIKELIHOOD FUNCTION

The second most widely used method for estimating the parameters of a probability distribution is based on the likelihood function. This method plays a fundamental role in statistical inference and is applied in many practical problems. We first present the concept of the likelihood function and we then follow it by a description of the maximum likelihood method. Other likelihood methods such as the marginal and partial likelihood methods are variants of the maximum likelihood and will not be discussed in this chapter.

Consider a manufacturer who checks the quality of the products by taking a sample of 15 random products and inspects them for defective units. Assuming that θ is the proportion of defective units in the total production, then the probability of having x defective units in the sample is binomial.

$$P(x) = \binom{15}{x} \theta^x (1-\theta)^{15-x} \quad x = 0, 1, \ldots, 15. \tag{4.6}$$

The probability of having two defective units in the sample is

$$P(2) = \binom{15}{2} \theta^2 (1-\theta)^{13}.$$

This probability is a function of θ, and a plot of the $P(2)$ for different values of θ is shown in Figure 4.1. The numerical values of the probability, $P(2)$ and θ are shown in Table 4.1. The graph in Figure 4.1 is referred to as the likelihood function. One can deduce that the *likelihood function* is the joint probability of an observed sample as a function of the unknown parameters.

In situations where the sample size is very large, we may find that it is more convenient to calculate the logarithmic values of the likelihood functions than to calculate the function itself. Therefore, the plot of the likelihood function will be greatly simplified since the likelihood is usually obtained by multiplying the probabilities of independent events, and by considering the logarithm of the function we can eliminate (or use as a scale) the constant term of the logarithm. This is illustrated in the following example.

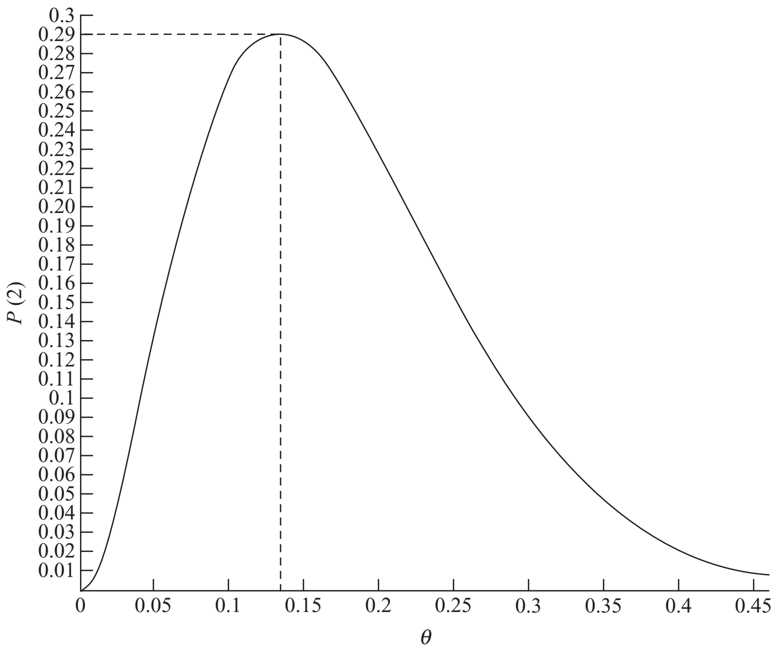

FIGURE 4.1 Plot of $P(2)$ versus θ.

TABLE 4.1 Values of θ and $P(2)$

θ	$P(2)$	θ	$P(2)$
0.02	0.0323	0.22	0.2010
0.04	0.0988	0.24	0.1707
0.06	0.1691	0.26	0.1416
0.08	0.2273	0.28	0.1150
0.10	0.2669	0.30	0.0916
0.12	0.2870	0.32	0.0715
0.14	0.2897	0.34	0.0547
0.16	0.2787	0.36	0.0411
0.18	0.2578	0.38	0.0303
0.20	0.2309	0.40	0.0219

EXAMPLE 4.7

The number of defectives in a production line is found to follow a Poisson distribution with an unknown mean μ. Two random samples are taken and the numbers of the defective units found are 10 and 12. What is the likelihood function of having 10 and 12 defective units?

SOLUTION

The probability of having x units from a Poisson distribution is

$$P(x) = \frac{e^{-\mu}\mu^x}{x!} \quad x = 0, 1, 2, \ldots$$

The probabilities of having 10 and 12 defectives, respectively, are

$$P(10) = \frac{e^{-\mu}\mu^{10}}{10!}$$

and

$$P(12) = \frac{e^{-\mu}\mu^{12}}{12!}.$$

The likelihood function $[L(x; \mu)]$ is the product of $P(10)$ and $P(12)$ —that is,

$$\begin{aligned} L(x;\mu) &= \frac{e^{-\mu}\mu^{10}}{10!} \times \frac{e^{-\mu}\mu^{12}}{12!} \quad (x = 10, 12) \\ &= \frac{e^{-2\mu}\mu^{22}}{(10!12!)}. \end{aligned} \tag{4.7}$$

Evaluation of Equation 4.7 for different values of μ can be simplified by taking the logarithm of $L(x; \mu)$. Let $l(x; \mu)$ be the logarithm of $L(x; \mu)$—that is,

$$l(x;\mu) = \log L(x;\mu),$$

and the logarithm of the likelihood function given in Equation 4.7 can now be written as

$$l(\mu) = 22 \log \mu - 2\mu - \log(10!12!). \tag{4.8}$$

Since the last term in Equation 4.8 is constant, we may drop it and plot the relative values of the log likelihood function as shown in Figure 4.2. The values of $l(\mu)$ corresponding to the Figure are shown in Table 4.2.

It is obvious from Figure 4.2 that the probability of having 10 and 12 defectives is maximum when the mean of the Poisson distribution is 11.

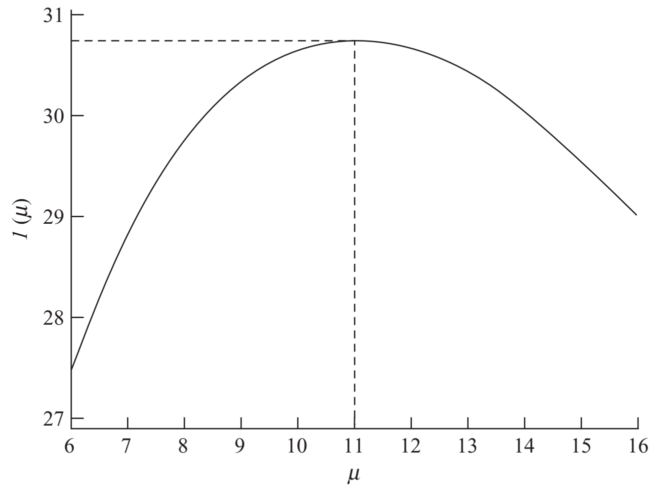

FIGURE 4.2 The log of the likelihood function versus μ.

TABLE 4.2 Values of $l(\mu)$ for Figure 4.2

μ	$l(\mu)$
6	27.4187
7	28.8100
8	29.7477
9	30.3389
10	30.6569
11	30.7537
12	30.6679
13	30.4289
14	30.0593
15	29.5771
16	28.9970

■

EXAMPLE 4.8

Suppose that the manufacturer of integrated circuits (ICs) takes three random samples from the same batch of sizes 10, 15, and 25 units. Upon inspection, it is found that these samples have 2, 3, and 5 defectives, respectively. What is the likelihood function for these probabilities?

SOLUTION

Since the three samples are taken from the same production batch, the underlying probability distribution has the same parameter θ. The probabilities of the three results are

$$\binom{10}{2}\theta^2(1-\theta)^8;\ \binom{15}{3}\theta^3(1-\theta)^{12};\ \binom{25}{5}\theta^5(1-\theta)^{20}.$$

The likelihood function is simply the product of the three probabilities

$$L(\theta)=\binom{10}{2}\theta^2(1-\theta)^8\binom{15}{3}\theta^3(1-\theta)^{12}\binom{25}{5}\theta^5(1-\theta)^{20}$$

$$L(\theta)=K\theta^{10}(1-\theta)^{40},$$

where K is a constant which includes all terms not involving θ. ■

So far, we have shown how a likelihood function can be developed for discrete probability distributions (binomial and Poisson). One can use the same procedure for developing the likelihood function for observations from continuous probability distributions.

Assume we have a distribution with a p.d.f. $f(x;\theta)$ with a single parameter θ, and let the observed results be $x_1, x_2, \ldots, x_n$. For any continuous random variable, the probability of obtaining exactly x is zero, for any x. However, the probability that an observation x occurs in an interval of length dx centered at x is $f(x;\theta)dx$. If $x_1, x_2, \ldots, x_n$ are independent, then the likelihood function is (Wetherill, 1981; Myung, 2003)

$$L(x_i;\theta)=\prod_{i=1}^{n} f(x_i;\theta)\,dx_i. \tag{4.9}$$

Since the product of the terms $dx_1, dx_2, \ldots, dx_n$ do not depend on θ, then we can rewrite Equation 4.9 as

$$L(x_i;\theta)=K\prod_{i=1}^{n} f(x_i;\theta).$$

The following example illustrates the procedure for developing a likelihood function for a continuous probability distribution.

EXAMPLE 4.9

Assume that the manufacturer in Example 4.8 randomly selects three samples having the same size and observes that they have 5, 7, and 9 defectives. It is also observed that the defectives in a production batch follow a normal distribution with an unknown mean, μ, but the variance equals 1. What is the likelihood function?

SOLUTION

The p.d.f. of observation x_i is

$$\frac{1}{\sqrt{2\pi}}\exp\left[-1/2(x_i-\mu)^2\right] \quad i=1,2,3.$$

The likelihood function is

$$L(\mu)=\frac{1}{2\pi\sqrt{2\pi}}\exp\left[-1/2(5-\mu)^2-1/2(7-\mu)^2-1/2(9-\mu)^2\right]. \tag{4.10}$$

Expanding the squared terms will result in terms that include μ^2, μx_i, x_i^2. The last term can be dropped since it does not involve μ.

We can simplify Equation 4.10 by rewriting the $x_i - \mu$ term as

$$x_i-\mu=x_i-\bar{x}+\bar{x}-\mu.$$

By squaring the above expression and adding, we have,

$$\sum_{i=1}^{n}(x_i-\mu)^2=\sum_{i=1}^{n}(x_i-\bar{x})^2+n(\bar{x}-\mu)^2.$$

Since the mean of the observations is 7 and the term $\exp([-1/2]\Sigma(x_i-\bar{x})^2)$ may be dropped since it does not include μ, we can rewrite Equation 4.10 as

$$l(\mu)=K-\frac{3}{2}(7-\mu)^2,$$

where K is a constant. ■

Clearly, if the probability distribution has more than one unknown parameter, we can develop a likelihood function in terms of these parameters as shown below.

EXAMPLE 4.10

Assume n observations $x_1, x_2, \ldots, x_n$ are randomly taken from a normal distribution with unknown mean μ and unknown variance σ^2. What is the likelihood function (Wetherill, 1981)?

SOLUTION

Following the same procedure of Example 4.9, we obtain the p.d.f. of observation x_i, as

$$\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-1/2\left(\frac{x_i-\mu}{\sigma}\right)^2\right] \quad i=1,2,\ldots,n.$$

The likelihood function is the product of these p.d.f.'s—that is,

$$L(x;\mu,\sigma^2)=\frac{(2\pi)^{\frac{-n}{2}}}{\sigma^n}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right].$$

The log of the likelihood function is

$$l(x;\mu,\sigma^2)=-n\log\sqrt{2\pi}-n\log\sigma-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2-\frac{n}{2\sigma^2}(\bar{x}-\mu)^2. \tag{4.11}$$

■

4.3.1 The Method of Maximum Likelihood

As discussed earlier, the likelihood function usually has a maximum at specific values of the distribution parameters. These values of the parameters are more likely to give rise to the observed data than other values. If we require a single value of the parameter to use as an estimate for the distribution, then it is clear that the value of the parameter that gives the maximum of the likelihood is the "best" estimate.

The objective is then to determine the best estimates of the parameters using the likelihood function. This can be accomplished by developing the likelihood function for the observations and obtaining its logarithmic expression. This expression is then differentiated with respect to the parameters, and the resulting equations are set to equal zero. These equations are then solved simultaneously to obtain the best estimates of the parameters that maximize the likelihood function.

It should be noted that it is not necessary in all cases to obtain the logarithmic expression of the likelihood function. When it is possible, the likelihood function itself can be maximized without resorting to its logarithmic expression.

EXAMPLE 4.11

What is the maximum likelihood estimate of μ for the Poisson distribution given in Example 4.7?

SOLUTION

Using the logarithm of the maximum likelihood function given by Equation 4.8,

$$l(\mu) = 22\log\mu - 2\mu - \log(10!12!).$$

The derivative of $l(\mu)$ with respect to μ is

$$\frac{dl(\mu)}{d\mu} = \frac{22}{\mu} - 2 = 0.$$

The "best" estimate of $\hat{\mu}$ is 22/2 = 11. ■

EXAMPLE 4.12

What is the maximum likelihood estimate of θ in Example 4.8?

SOLUTION

$$L(\theta) = K\theta^{10}(1-\theta)^{40}$$
$$l(\theta) = \log K + 10\log\theta + 40\log(1-\theta)$$
$$\frac{dl(\theta)}{d\theta} = 0 + \frac{10}{\theta} - \frac{40}{1-\theta} = 0,$$

and the "best" estimate of $\hat{\theta}$ is 1/5. ■

We now consider the maximum likelihood estimators (MLEs) for the parameters of the exponential, Rayleigh, and normal distributions.

4.3.2 Exponential Distribution

The p.d.f. of the exponential distribution with parameter λ is

$$f(x;\lambda)=\lambda e^{-\lambda x}.$$

The p.d.f. of n observations $x_1, x_2, \ldots, x_n$ is

$$f(x_i;\lambda)=\lambda e^{-\lambda x_i} \quad i=1,2,\ldots,n.$$

The likelihood function $L(x_1, x_2, \ldots, x_n; \lambda)$ is

$$\begin{aligned}L(x_1,x_2,\ldots x_n;\lambda) &= f(x_1;\lambda)f(x_2;\lambda)\ldots f(x_n;\lambda)\\ &=\prod_{i=1}^{n} f(x_i;\lambda)\\ &=\lambda^n\prod_{i=1}^{n} e^{-\lambda x_i}\\ &=\lambda^n e^{-\lambda\sum_{i=1}^{n}x_i}.\end{aligned}$$

The logarithm of the likelihood function is

$$l(x_1,x_2,\ldots,x_n;\lambda)=n\log\lambda-\lambda\sum_{i=1}^{n}x_i$$

and

$$\frac{\partial l(x_1,x_2,\ldots,x_n;\lambda)}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}x_i=0.$$

The "best" estimate of λ is $n/\sum_{i=1}^{n}x_i$.

This is the same estimate as that obtained by the method of moments.

EXAMPLE 4.13

A sample of six electronic components is subjected to a reliability test to estimate the mean time to failure. The following are the times to failure of the components: 25, 75, 150, 230, 430, and 700 h. What is the failure rate? Estimate the parameter(s) of the underlying failure-time distribution.

SOLUTION

The mean of the time to failure is 260 h, and the standard deviation is 232 h. Since the mean and the standard deviation are approximately equal, then it is reasonable to assume that the exponential distribution can be used to represent the failure-time distribution.

Therefore, the "best" estimate of $\hat{\lambda}$ (the parameter of the exponential distribution) as determined by the maximum likelihood method is

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i},$$

where x_i is the time of the ith failure

$$\hat{\lambda} = \frac{6}{1610} = 3.726 \times 10^{-3} \text{ failures per hour.}$$

■

4.3.3 The Rayleigh Distribution

As explained in Chapter 1, the Rayleigh distribution is used to represent the failure-time distribution of components that exhibit linearly increasing failure rates. The p.d.f. of the Rayleigh distribution is

$$f(x) = \lambda x e^{-\frac{\lambda x^2}{2}},$$

where λ is the parameter of the Rayleigh distribution.

The likelihood function for n observations is

$$L(x_1, x_2, \ldots x_n; \lambda) = f(x_1; \lambda) f(x_2; \lambda), \ldots, f(x_n; \lambda)$$

or

$$L(x_1, x_2, \ldots, x_n; \lambda) = \prod_{i=1}^{n} \lambda x_i e^{-\frac{\lambda x_i^2}{2}}.$$

Let

$$\prod_{i=1}^{n} x_i = X.$$

We can rewrite the likelihood function as

$$L(x_1, x_2, \ldots, x_n; \lambda) = \lambda^n X e^{-\frac{\lambda}{2}\sum_{i=1}^{n} x_i^2}.$$

The logarithm of the above function is

$$l(x_1, x_2, \ldots x_n; \lambda) = n \log \lambda + \log X - \frac{\lambda}{2}\sum_{i=1}^{n} x_i^2. \tag{4.12}$$

Taking the derivative of Equation 4.12 with respect to λ and equating the resultant to zero, we obtain

$$\frac{\partial l(x_1, x_2, \ldots, x_n; \lambda)}{\partial \lambda} = \frac{n}{\lambda} - \frac{1}{2}\sum_{i=1}^{n} x_i^2 = 0$$

or

$$\hat{\lambda} = \frac{2n}{\sum_{i=1}^{n} x_i^2}. \tag{4.13}$$

EXAMPLE 4.14

The following failure times are observed while conducting a reliability test: 15, 21, 30, 39, 52, and 68 h. Assume that a Rayleigh distribution is considered an appropriate distribution to represent these failure times. Determine the parameter of the distribution. What are the mean and standard deviation of the failure time?

SOLUTION

Using Equation 4.13, the parameter of the Rayleigh distribution is

$$\hat{\lambda} = \frac{2 \times 6}{10{,}415} = 0.00115 \text{ failures per hour.}$$

The mean and standard deviation of the failure times are

$$\hat{\mu} = \sqrt{\frac{\pi}{2\hat{\lambda}}} = 36.92 \text{ h}$$

$$\hat{\sigma} = \sqrt{\frac{2}{\hat{\lambda}}\left(1 - \frac{\pi}{4}\right)} = 19.3 \text{ h.}$$

■

4.3.4 The Normal Distribution

The p.d.f. of an observation x from a normal distribution with unknown mean μ and unknown variance σ^2 is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

The likelihood function for n observations is

$$L(x_1, x_2, \ldots, x_n; \mu, \sigma) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \prod_{i=1}^{n} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2},$$

and the logarithm of the above function is

$$l(x_1, x_2, \ldots, x_n; \mu, \sigma) = n \log \frac{1}{\sigma\sqrt{2\pi}} - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2. \tag{4.14}$$

Taking the derivative of Equation 4.14 with respect to μ results in

$$\frac{\partial l(x_1, x_2, \ldots x_n; \mu, \sigma)}{\partial \mu} = \frac{1}{\sigma^2}\left(\sum_{i=1}^{n} x_i - n\mu\right) = 0$$

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Similarly, taking the derivative of Equation 4.14 with respect to σ, we obtain

$$\begin{aligned}
\frac{\partial l(x_1, x_2, \ldots x_n; \mu, \sigma)}{\partial \sigma} &= \frac{\partial}{\partial \sigma}\left[n \log \frac{1}{\sqrt{2\pi}} - n \log \sigma - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2\right] \\
&= -\frac{n}{\sigma} - \sum \frac{(x_i-\mu)^2}{2\sigma^3}(-2) \\
&= \frac{1}{\sigma}\left[-n + \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}\right] = 0 \\
\sigma^2 &= \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2.
\end{aligned}$$

The estimate of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2.$$

These are the same results as those obtained by the method of moments.

EXAMPLE 4.15

Assume that applied stresses on components and the corresponding failure times form paired observations $(x_1, y_1), \ldots, (x_n, y_n)$ that follow the model

$$E(Y) = \alpha + \beta x$$
$$Var(Y) = \sigma^2,$$

where Y is an independently and normally distributed random variable. Use the maximum likelihood approach to estimate the parameters α and β.

SOLUTION

Since Y is independently and normally distributed, then the log likelihood is obtained using Equation 4.14 as

$$l[(x_1, y_1), \ldots, (x_n, y_n); \alpha, \beta] = \frac{-n}{2}\log 2\pi - n\log\sigma - \frac{1}{2\sigma^2}\sum (y_i - \alpha - \beta x_i)^2. \tag{4.15}$$

The first two terms of the right-hand side of Equation 4.15 are independent of α and β. Hence, to maximize the log likelihood, it is sufficient to minimize the term

$$K = \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2. \tag{4.16}$$

Taking the partial derivatives of K with respect to α and β and equating the derivatives to zero result in two linear equations in α and β. Their solutions yield

$$\hat{\beta} = \frac{\sum_{i=1}^{n} y_i (x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \tag{4.17}$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \tag{4.18}$$

where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

and

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

■

As we discussed earlier, in order to find the MLE, we set the derivatives of the log likelihood function with respect to the parameter being estimated to zero and solve the resulting equation for the value of the parameter. Unfortunately, sometimes there are no closed form expressions for the estimated parameter(s). In this case, we may employ other methods to obtain an estimate of the parameter. We now describe two of such methods (Wetherill, 1981).

4.3.4.1 The Gradient of the Likelihood Method This method is very effective when there is only one unknown parameter, θ. We simply calculate $dL/d\theta$ at various values of θ and plot $dL/d\theta$ versus θ to obtain a line which intersects with θ axis ($dL/d\theta = 0$) at the estimated value of θ. Drawing such a line rarely requires more than three or four calculations. The slope of the line is the second derivative, ($d^2L/d\theta^2$), of the likelihood function with respect to θ is an approximate estimate of the variance of the estimator.

4.3.4.2 Newton's Iterative Method Newton's iterative method for finding the roots of $f(x) = 0$ is well known. We apply Newton's method (see Appendix E) to solve the derivative of the log likelihood function with respect to θ:

$$f(\theta) = \left\{\frac{dl}{d\theta}\right\}_{\theta=\hat{\theta}} = 0. \tag{4.19}$$

If $\hat{\theta}_1$ is any rough estimate of θ, then using Taylor's expansion of Equation 4.19 about $\hat{\theta}_1$ we obtain

$$\left\{\frac{dl}{d\theta}\right\}_{\hat{\theta}} = \left\{\frac{dl}{d\theta}\right\}_{\hat{\theta}_1} + \left(\hat{\theta} - \hat{\theta}_1\right)\left\{\frac{d^2 l}{d\theta^2}\right\}_{\hat{\theta}_1} + \cdots = 0. \tag{4.20}$$

If $\hat{\theta}_1$ is sufficiently close to $\hat{\theta}$, we can simply ignore the higher terms of the expansion in Equation 4.20. Thus,

$$\left\{\frac{dl}{d\theta}\right\}_{\hat{\theta}_1} + \left(\hat{\theta} - \hat{\theta}_1\right)\left\{\frac{d^2 l}{d\theta^2}\right\}_{\hat{\theta}_1} \cong 0, \tag{4.21}$$

and a new estimate of $\hat{\theta}_2$ can be made as

$$\hat{\theta}_2 = \hat{\theta}_1 - \left\{\frac{dl}{d\theta}\right\}_{\hat{\theta}_1} \Big/ \left\{\frac{d^2 l}{d\theta^2}\right\}_{\hat{\theta}_1}. \tag{4.22}$$

We can use the above expression recursively until the estimate of the parameter has converged. Clearly, the rate of convergence depends on the selection of the initial value of $\hat{\theta}_1$. Unfortunately, there is no general method that enables us to provide a good initial estimate of $\hat{\theta}$, but a rough plot of the log likelihood function may provide an acceptable initial value for $\hat{\theta}_1$.

Under certain regularity, the asymptotic MLEs are consistent, efficient, and unbiased. The bias of the estimators decreases as the number of observations increases. The method requires simple calculations for single parameter distributions but may require extensive computations for two or more parameter distributions. Moreover, the method is applicable for both censored (or truncated) and noncensored data.

4.3.5 Information Matrix and the Variance-Covariance Matrix

One of the major benefits of the use of the MLE to obtain the parameters of distribution(s) is that the logarithm of the likelihood function can be utilized in constructing the so called *Fisher information matrix* (or *Hessian matrix*). The inverse of the matrix results in the well-known variance-covariance matrix.

Before we introduce the procedure for constructing the information matrix, we first need to define the variance-covariance matrix (or simply covariance matrix). If $X_1, X_2, \ldots, X_k$ are k mutually independent and identically distributed random variables within a p.d.f. $f(x, \theta_0)$, where θ_0 has components and is the true value of θ, then the covariance matrix is defined as

$$\begin{pmatrix} \text{Var}(\theta_1) & \text{Cov}(\theta_1, \theta_2) & \text{Cov}(\theta_1, \theta_3) & \ldots & \text{Cov}(\theta_1, \theta_k) \\ \text{Cov}(\theta_1, \theta_2) & \text{Var}(\theta_2) & \text{Cov}(\theta_2, \theta_3) & \ldots & \text{Cov}(\theta_2, \theta_k) \\ \vdots & & & & \\ \text{Cov}(\theta_1, \theta_k) & \text{Cov}(\theta_2, \theta_k) & \ldots & \ldots & \text{Var}(\theta_\text{k}) \end{pmatrix}$$

where $\text{Cov}(\theta_i, \theta_j)$ is the covariance of θ_i and θ_j, and $\text{Var}(\theta_i)$ is the variance of θ_i. This covariance matrix can be obtained from the information matrix as follows.

As we stated earlier in this chapter, when the sample size of data increases, the bias of the MLEs decreases, and they become asymptotically unbiased. In other words,

$$\lim_{n\to\infty} E\left[\hat{\theta}_i\right] = \theta_i, i = 1, 2, \ldots k.$$

To find the asymptotic variances and covariances of the estimators, we first construct the information matrix $\boldsymbol{I}$, regarding the likelihood as a function of random variables observed in a given sample.

The (ij)th element of the information matrix $\boldsymbol{I}$ is

$$\boldsymbol{I}_{ij} = E\left[-\frac{\partial^2 l(X; \theta)}{\partial\theta_i \partial\theta_j}\right]. \tag{4.23}$$

The inverse matrix, $\boldsymbol{I}^{-1}$, with the (ij)th element denoted by$\boldsymbol{I}^{ij}$, is the variance-covariance matrix of the $\hat{\theta}$'s, so that (Elandt-Johnson and Johnson, 1980)

$$\text{Var}\left(\hat{\theta}_i\right) = I^{ii} \quad \text{and} \quad \text{Cov}\left(\hat{\theta}_i, \hat{\theta}_j\right) = I^{ij}. \tag{4.24}$$

EXAMPLE 4.16

A random sample $x_1, x_2, \ldots, x_n$ follows a normal distribution with parameters μ and σ^2. Use the information matrix to obtain the variances of $\hat{\mu}$ and $\hat{\sigma}$.

SOLUTION

The logarithm of the likelihood function of the normal distribution is given by Equation 4.14 as

$$l(x_1, x_2, \ldots, x_n; \mu, \sigma) = n \log \frac{1}{\sigma\sqrt{2\pi}} - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2.$$

The partial derivatives of l with respect to μ and σ are

$$\frac{\partial l}{\partial \mu} = \frac{1}{\sigma^2}\left(\sum_{i=1}^{n} x_i - n\mu\right)$$

$$\frac{\partial l}{\partial \sigma} = \frac{1}{\sigma}\left[-n + \sum_{i=1}^{n}\frac{(x_i - \mu)^2}{\sigma^2}\right]$$

$$\frac{\partial^2 l}{\partial \mu^2} = \frac{-n}{\sigma^2} \tag{4.25}$$

$$\frac{\partial^2 l}{\partial \mu \partial \sigma} = \frac{-2}{\sigma^3}\left(\sum_{i=1}^{n} x_i - n\mu\right) \tag{4.26}$$

$$\frac{\partial^2 l}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{3}{\sigma^4}\sum_{i=1}^{n}(x_i - \mu)^2. \tag{4.27}$$

In order to construct the information matrix, we obtain the expectations of Equations 4.25–4.27. Then,

$$E\left[\frac{\partial^2 l}{\partial \mu^2}\right] = \frac{-n}{\sigma^2} = -I_{11}$$

$$E\left[\frac{\partial^2 l}{\partial \mu \partial \sigma}\right] = 0 = -I_{12} = -I_{21}$$

$$E\left[\frac{\partial^2 l}{\partial \sigma^2}\right] = \frac{-2n}{\sigma^2} = -I_{22}.$$

Thus, the information matrix $\boldsymbol{I}$ is constructed as

$$\boldsymbol{I} = \begin{pmatrix} n/\sigma^2 & 0 \\ 0 & 2n/\sigma^2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{I}^{-1} = \begin{pmatrix} \sigma^2/n & 0 \\ 0 & \sigma^2/2n \end{pmatrix}.$$

The variance-covariance matrix, $\boldsymbol{I}^{-1}$, is

$$\begin{pmatrix} \operatorname{Var}(\hat{\mu}) & \operatorname{Cov}(\hat{\mu}, \hat{\sigma}) \\ \operatorname{Cov}(\hat{\mu}, \hat{\sigma}) & \operatorname{Var}(\hat{\sigma}) \end{pmatrix} = \begin{pmatrix} \sigma^2/n & 0 \\ 0 & \sigma^2/2n \end{pmatrix}.$$

■

EXAMPLE 4.17

A check-weigher is a piece of equipment that has three major components: a scale, a controller, and a diverter. In typical high-speed production systems such as those found in the canned food industry or the pharmaceutical manufacturing industry, one or more check-weighers are usually installed in the system to ensure that the weights of the products are within acceptable specification limits. If a product fails to meet the specifications, it is diverted away from the acceptable products. The diverter, being a mechanical system, is the most susceptible component to failure. The following times to failure (in weeks) of a diverter are observed:

14, 18, 18, 20, 21, 22, 22, 20, 17, 17, 15, and 13.

Assume that the observations follow a normal distribution with mean μ and variance σ^2. Determine $\hat{\mu}$, $\hat{\sigma}$, and the variance-covariance matrix.

SOLUTION

From Section 4.3.4 we obtain $\hat{\mu}$ and $\hat{\sigma}$ as follows:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{12} \times 217 = 18.08$$

and

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \hat{\mu})^2 = 8.409$$

or

$$\hat{\sigma} = 2.9.$$

The variance-covariance matrix as shown in Example 4.16 is

$$\boldsymbol{I}^{-1} = \begin{pmatrix} \sigma^2/n & 0 \\ 0 & \sigma^2/2n \end{pmatrix} = \begin{pmatrix} 0.700 & 0 \\ 0 & 0.350 \end{pmatrix}.$$

Thus, the variance of $\hat{\mu}$ is 0.700 and that of $\hat{\sigma}$ is 0.350. ■

4.4 METHOD OF LEAST SQUARES

The method of least squares provides an efficient and unbiased estimator of the distribution parameters. The method defines the best fit as that which minimizes the sum of squared errors between the observed data and the fitted distribution (Elsayed and Boucher, 1994). Although the method is general and can be used for simple linear, multiple linear, and nonlinear regression models, we will limit our presentation to linear models.

Consider a set of data that may include extreme data points (or noise). We are interested in finding a function that reflects the pattern in the data and reduces the errors to a minimum. A simple plot of the data may reveal the underlying data-generating process as being linear or nonlinear. Assume that the data-generating process can be represented by the following linear model:

$$f(x_i) = \alpha + \beta x_i + \varepsilon_i, \tag{4.28}$$

where

$f(x_i)$ = observed value of the function at x_i,

α, β = intercept and slope, respectively,

x_i = independent variable such as time, and

ε_i = random noise in process at time x_i.

It is assumed that ε_i is normally an independently distributed random variable, with mean $\overline{\varepsilon}_i = 0$ and Var(ε_x) = σ^2. Based on the assumption of a linear process, we propose to fit a model of the form

$$\hat{f}(x) = \hat{\alpha} + \hat{\beta} x, \tag{4.29}$$

where

$\hat{f}(x) =$ the estimated value of the function at x, and

$\hat{\alpha}, \hat{\beta} =$ estimates of α and β.

Let $e(x_i) = \hat{f}(x_i) - f(x_i)$ be the value of the error between the proposed polynomial $\hat{f}(x_i)$ and the actual data $f(x_i)$. Then, we define the sum of squares of errors, SS_E, as

$$SS_E = \sum_{x=1}^{n} e^2(x_i), \tag{4.30}$$

where n is the total number of data points used for estimating $\hat{f}(x_i)$. Equation 4.30 can be rewritten as

$$SS_E = \sum_{x=1}^{n} \left[\hat{f}(x_i) - f(x_i)\right]^2. \tag{4.31}$$

The minimization of SS_E is accomplished by taking the partial derivatives of SS_E with respect to $\hat{\alpha}$ and $\hat{\beta}$ and setting the resulting equations to zero.

$$SS_E = \sum_{i=1}^{n} \left[f(x_i) - \hat{\alpha} - \hat{\beta} x_i\right]^2$$

$$\frac{\partial SS_E}{\partial \hat{\alpha}} = -2\sum_{i=1}^{n} \left[f(x_i) - \hat{\alpha} - \hat{\beta} x_i\right] = 0 \tag{4.32}$$

$$\frac{\partial SS_E}{\partial \hat{\beta}} = -2\sum_{i=1}^{n}\left[f(x_i) - \hat{\alpha} - \hat{\beta}x_i\right]x_i = 0. \tag{4.33}$$

Rewriting Equations 4.32 and 4.33 gives us

$$\sum_{i=1}^{n} f(x_i) = n\hat{\alpha} + \hat{\beta}\sum_{i=1}^{n} x_i \tag{4.34}$$

$$\sum_{i=1}^{n} x_i f(x_i) = \hat{\alpha}\sum_{i=1}^{n} x_i + \hat{\beta}\sum_{i=1}^{n} x_i^2, \tag{4.35}$$

which yields

$$\hat{\alpha} = \frac{\sum x_i^2 \sum f(x_i) - \sum x_i \sum x_i f(x_i)}{n\sum x_i^2 - \left(\sum x_i\right)^2} \tag{4.36}$$

$$\hat{\beta} = \frac{n\sum x_i f(x_i) - \sum x_i \sum f(x_i)}{n\sum x_i^2 - \left(\sum x_i\right)^2}. \tag{4.37}$$

After estimating the parameters of the model, one may wish to know how well the proposed model fits the data. The coefficient of determination and the coefficient of correlation are typical criteria that can be used for that purpose. They are, respectively, given by

$$r^2 = \frac{\sum\left(\hat{f}(x_i) - \overline{f}(x)\right)^2}{\sum\left(f(x_i) - \overline{f}(x)\right)^2} \tag{4.38}$$

and

$$\rho = \frac{\sigma_{x,f(x)}}{\sigma_x \sigma_{f(x)}} \tag{4.39}$$

where

r^2 = coefficient of determination $0 \le r^2 \le 1$,

ρ = coefficient of correlation $0 \le \rho \le 1$,

$\sigma_{x,f(x)}$ = covariance of x and $f(x)$,

$= \sum_{i=1}^{n}(x_i - \overline{x})\left(f(x_i) - \overline{f}(x)\right)$,

$\sigma_x = \sum(x_i - \overline{x})^2$, and

$\sigma_{f(x)} = \sum\left(f(x_i) - \overline{f}(x)\right)^2$.

Derivations of Equations 4.38 and 4.39 are given in Elsayed and Boucher (1994). A coefficient of determination of 0 indicates that the model does not fit the data, whereas when $r^2 = 1$, the model represents an ideal fit. Similarly when $\rho = 1$ or -1, the model indicates that there is a perfect positive correlation or a perfect negative correlation, respectively. When $r = 0$, then there is no correlation between the $f(x)$ and x. Therefore, when r^2 approaches 1 or r approaches ± 1, the model is a good fit of the data.

Equation 4.37 can be written as

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})f(x_i)}{\sum_{i=1}^{n}(x_i - \bar{x})^2}. \tag{4.40}$$

Since $\sum_{i=1}^{n}(x_i - \bar{x})^2 = 0$, then we obtain the expected value of $\hat{\beta}$ as

$$E[\hat{\beta}] = \frac{\sum_{i=1}^{n}(x_i - \bar{x})E[f(x_i)]}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(\alpha + \beta x_i)}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$= \beta \frac{\sum_{i=1}^{n}(x_i - \bar{x})x_i}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \beta.$$

Therefore, $\hat{\beta}$ is an unbiased estimator of β. Similarly, $\hat{\alpha}$ is an unbiased estimator of α. It is also important to obtain the variances of these two parameters:

$$Var\left(\hat{\beta}\right) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})Var(f(x_i))}{\left[\sum_{i=1}^{n}(x_i - \bar{x})^2\right]^2} = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}. \tag{4.41}$$

The variance of $\hat{\alpha}$ is

$$Var(\hat{\alpha}) = \frac{\sigma^2 \sum_{i=1}^{n} x_i^2}{n\sum_{i=1}^{n}(x_i - \bar{x})^2}. \tag{4.42}$$

The variance of $f(x)$ for a particular value x, that is, $\hat{f}(x) = \hat{\alpha} + \hat{\beta}x$, is obtained as (Weerahandi, 2003):

$$
\begin{aligned}
Var(\hat{f}(x)) &= Var(\hat{\alpha}) + x^2 Var(\hat{\beta}) + 2x Cov(\hat{\alpha}, \hat{\beta}) \\
&= \frac{\sigma^2 \sum_{i=1}^{n} x_i^2}{n \sum_{i=1}^{n} (x_i - \bar{x})^2} + \frac{x^2 \sigma^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} - \frac{2x\sigma^2 \bar{x}}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \\
&= \sigma^2 \left[\frac{1}{n} + \frac{(x - \bar{x})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right].
\end{aligned}
$$

EXAMPLE 4.18

The surface mount technology (SMT) enables the manufacturers of electronic components to create printed circuit assemblies with high component density. One problem with SMT, though, is that a surface mount device contact is attached to a printed circuit board (PCB) with only solder. As a result, the SMT attachment is only as reliable as the solder. Hence, manufacturers are required to perform accelerated reliability testing to determine the long-term reliability at normal operating conditions. The following failure-time data are obtained from such a test: 10, 20, 30, 40, 50, 60, 70, 80, 93, and 111 h. Assume that the failure rate is linearly increasing with t in the form $h(t) = \alpha + \beta t$. Determine the constants α and β and estimate the reliability at $t = 30$ h.

SOLUTION

Using the median rank approach presented in Chapter 1, we obtain the nonparametric estimate of the hazard rate as shown in Table 4.3. We then obtain the estimates of the constants α and β by fitting a linear regression to the hazard-rate values which is given by

$$h(t) = 0.007541 + 0.000434\, t.$$

TABLE 4.3 Failure-Rate Calculations

t	$h(t) \times 10^{-3}$ *Failures per hour*
10	10.00
20	11.11
30	12.50
40	14.28
50	16.66
60	20.00
70	25.00
80	33.33
93	38.40
111	55.45

The reliability at t = 30 h is

$$R(t) = e^{-\{0.0.0075541\,t + 0.0000217\,t^2\}}$$
$$R(30) = 0.78211.$$

The variances of α and β are obtained by substituting in Equations 4.41 and 4.42 which results in $Var(\hat{\alpha}) = 4.79923 \times 10^{-5}$ and $Var(\hat{\beta}) = 1.16008 \times 10^{-8}$, respectively. ■

The least squares method provides an efficient, consistent, and unbiased estimate of the distribution parameters. The least squares method is simple, computationally efficient, and can be used for simple linear, multiple linear, and nonlinear modeling. Moreover, many nonlinear forms can be transformed to linear expressions by using simple transformations. For example,

$$f(x) = ax^b \tag{4.43}$$

can be linearized by taking the logarithm of both sides of the equation which results in

$$\log f(x) = \log a + b \log x. \tag{4.44}$$

Let

$Y = \log f(x)$,
$X = \log x$, and
$A = \log a$.

Then Equation 4.43 can be written in the linear form of

$$Y = A + bX. \tag{4.45}$$

In summary, the least squares estimators of a linear model are (1) unbiased, (2) have minimum variance among linear unbiased estimators, and (3) obtained such that the residuals and estimators are uncorrelated (Wetherill, 1981).

4.5 BAYESIAN APPROACH

The previous methods for parameters estimation are based on determining the “best” fit distribution for failure data. They also assume that the obtained parameters are fixed. However, there are many situations where failure data are limited, perhaps nonexistent, which make it difficult to determine the “best” fit distribution. In such cases, the Bayesian approach might be a viable alternative to obtain estimates of the distribution parameters. The approach treats the parameters of the distribution as random variables. It utilizes prior knowledge of components’ failures, similarities with current ones, engineering experience, and subjective assessments to construct a prior distribution model.

The model uses initial assessment of the parameters with current data to obtain a posterior distribution using Bayes's formula. The process is repeated as new observations are obtained. The confidence intervals for the parameters can be easily obtained using standard procedures.

We now present a brief description of Bayes's theorem which is used later in estimating the parameters of distributions. Consider a sample space S of an experiment and $[B_1, B_2, \ldots, B_r]$ represent a partition of S. Let $\{P(A); A \subseteq S\}$denote a probability distribution defined on all events in S. For any events A and B in S and $P(A) > 0$, the conditional probability that B occurs given that A occurs is $P(B/A) = P(A \cap B)/P(A)$. Therefore,

$$P(B_i / A) = \frac{P(A \mid B_i)P(B_i)}{P(A)} \quad i = 1, 2, \ldots, r \tag{4.46}$$

whenever $P(A) > 0$, where it is calculated using the law of total probability (Leonard and Hsu, 1999) as

$$P(A) = \sum_{j=1}^{r} P(A / B_j) P(B_j).$$

Note that the events B_j are mutually exclusive and include the events A as well.

Let $g(\theta)$ be the prior distribution model for the parameter(s) θ and $g(\theta/t)$ be the posterior distribution model for θ given the observation t (failure time for example) and $f(t/\theta)$ be the probability model of the observed data t given the unknown parameter(s) θ. Then we rewrite Equation 4.46 as

$$g(\theta / t) = \frac{f(t / \theta) g(\theta)}{\int_0^\infty f(t / \theta) g(\theta) d\theta}. \tag{4.47}$$

The probability model $f(t/\theta)$ and the prior distribution $g(\theta)$ are called conjugate distributions and $g(\theta)$ is the conjugate prior for $f(t/\theta)$. Equation 4.47 is then used to obtain inferences and properties of the model parameters. We now demonstrate the use of the Bayesian approach in estimating the model parameters.

Consider a component that exhibits a constant failure rate $1/\theta$. The p.d.f. of the failure time is

$$f(t / \theta) = \frac{1}{\theta} e^{-\frac{t}{\theta}} \quad (0 < t < \infty, \theta > 0). \tag{4.48}$$

Assume that historical observations show that θ is a random variable whose prior density given by Bhattacharya (1967) is

$$g(\theta) = \frac{(a-1)(\alpha\beta)^{a-1}}{\beta^{a-1} - \alpha^{a-1}} \cdot \frac{1}{\theta^{a}} \quad (0 < \alpha < \theta \leq \beta) \\ = 0 \quad \text{otherwise.} \tag{4.49}$$

When $a = 0$, $g(\theta)$ is uniformly distributed on the range of $[\alpha, \beta]$. We also assume that n units of the same component type are subjected to a test at normal operating conditions. This is a complete test, that is, all units have failed, and their failure times are $t_1, t_2, \ldots, t_n$. The objective is to estimate the parameter $\hat{\theta}$ and its variance.

The sample likelihood conditional on θ is

$$L(t_1, t_2, \ldots, t_n / \theta) = \frac{n!}{\theta^n} e^{-\frac{T}{\theta}} \tag{4.50}$$

and

$$T = n\overline{t} = \sum_{i=1}^{n} t_i.$$

Using Bayes's formula in Equation 4.47 and the prior density given by Equation 4.49, we obtain a posterior distribution $g(\theta / t_1, t_2, \ldots, t_n)$ of θ as

$$g(\theta / t_1, t_2, \ldots, t_n) = \frac{\frac{1}{\theta^{n+a}} e^{-\frac{T}{\theta}}}{\int_{\alpha}^{\beta} \frac{1}{\theta^{n+a}} e^{-\frac{T}{\theta}} d\theta} \quad 0 \leq \alpha < \theta \leq \beta. \tag{4.51}$$

Equation 4.51 is a truncated "inverted gamma density," and the estimated value of $\hat{\theta}$ is expectation of $g(\theta / t_1, t_2, \ldots, t_n)$:

$$\hat{\theta} = \frac{\gamma\left(n+a-2, \frac{T}{\alpha}\right) - \gamma\left(n+a-2, \frac{T}{\beta}\right)}{\gamma\left(n+a-1, \frac{T}{\alpha}\right) - \gamma\left(n+a-1, \frac{T}{\beta}\right)} . T, \tag{4.52}$$

where

$$\gamma(n, x) = \int_0^x e^{-t} t^{n-1} dt$$

is the incomplete gamma function. The variance of the parameter $\hat{\theta}$ is obtained using the posterior density function given by Equation 4.51 and is obtained as (Bhattacharya, 1967)

$$Var\left(\hat{\theta}\right) = \frac{\{\gamma^*(n+a-3, T)\gamma^*(n-1, T) - [\gamma^*(n+a-2, T)]^2\}T}{[\gamma^*(n-1, T)]^2} \tag{4.53}$$

where $\gamma^*(n, y) = \gamma\left(n, \frac{y}{\alpha}\right) - \gamma\left(n, \frac{y}{\beta}\right)$.

EXAMPLE 4.19

The following failure times are obtained by subjecting a sample of nano-capacitors subjected to an electric field 0.237118, 2.48843, 20.9423, 30.5254, 62.339. Based on experience, the engineer believes that the parameter of the exponential distribution, θ, is uniformly distributed between 1100 and 1300. Obtain $\hat{\theta}$ and its variance.

SOLUTION

Using the data, we obtain $T = 116.5322$, $a = 0$ (for uniform distribution), $\alpha = 1100$, and $\beta = 1300$. Using Equation 4.52 we obtain

$$\hat{\theta} = \frac{\gamma\left(n-2, \frac{T}{\alpha}\right) - \gamma\left(n-2, \frac{T}{\beta}\right)}{\gamma\left(n-1, \frac{T}{\alpha}\right) - \gamma\left(n-1, \frac{T}{\beta}\right)} . T = \frac{0.000143}{0.000014} \times 116.5322 = 1190.29.$$

The variance of the parameter is estimated using Equation 4.53 as

$$Var(\hat{\theta}) = \frac{\{\gamma^*(n-3,T)\gamma^*(n-1,T) - [\gamma^*(n-2,T)]^2\}}{[\gamma^*(n-1,T)]^2} .T$$
$$= \frac{[(0.001445)(0.000014) - (0.000142)^2]}{(0.000014)^2} \times 116.5322 = 39.24.$$

■

It is interesting to note that the data in the example are generated from an exponential distribution with parameter $\theta = 1250$. Clearly, the assumption of the parameter model has a major effect on the parameter estimates. In most cases, the uniform distribution assumption is appropriate.

Consider Example 4.19 and assume that the parameter θ is a random variable Θ with prior density

$$g(\theta) = \frac{1}{\theta} e^{-\frac{1}{\theta}}.$$

We assume that failure data $t_1, t_2, \ldots, t_n$ exhibit an exponential distribution as

$$f(t_i / \theta) = \frac{1}{\theta} e^{-\frac{t_i}{\theta}}$$

and

$$f(t_1, t_2, \ldots, t_n / \theta) = \prod_{j=1}^{n} \frac{1}{\theta} e^{-\frac{t_i}{\theta}} = \left(\frac{1}{\theta}\right)^n e^{-\frac{1}{\theta}\sum_{j=1}^{n} t_j}.$$

Therefore,

$$\begin{aligned} g(\theta / t_1, t_2, \ldots, t_n) &\propto \left(\frac{1}{\theta}\right)^n e^{-\frac{1}{\theta}\sum_{j=1}^n t_j} \cdot \frac{1}{\theta} e^{-\frac{1}{\theta}} \\ &\propto \left(\frac{1}{\theta}\right)^{n+1} e^{-\frac{1}{\theta}\left(1+\sum_{j=1}^n t_j\right)}. \end{aligned} \tag{4.54}$$

For Equation 4.54 to be a proper p.d.f., its right-hand side must be multiplied by

$$\frac{\left(1+\sum_{j=1}^{n} t_j\right)^{n+2}}{\Gamma(n+2)}.$$

We then rewrite Equation 4.54 as

$$g(\theta / t_1, t_2, \ldots, t_n) = \frac{\left(1+\sum_{j=1}^{n} t_j\right)^{n+2}}{\Gamma(n+2)} \left(\frac{1}{\theta}\right)^{n+1} e^{-\frac{1}{\theta}\left(1+\sum_{j=1}^n t_j\right)}. \tag{4.55}$$

Equation 4.55 is a gamma distribution as that expressed in Equation 1.57 with parameters $(n + 2)$ and $1\big/(1+\sum_{j=1}^{n} t_j)$. Since the mean of the gamma distribution is the product of these two parameters, then Bayesian estimator of $\hat{\theta}$ is

$$\hat{\theta}(t_1, t_2, \ldots, t_n) = \frac{n+2}{\left(1+\sum_{j=1}^{n} t_j\right)}. \tag{4.56}$$

Its variance is

$$Var(\hat{\theta}) = \frac{n+2}{\left(1+\sum_{j=1}^{n} t_j\right)^2}.$$

4.6 GENERATION OF FAILURE-TIME DATA

Generation of failure times from distributions with known parameters might be useful in validating a methodology or simulating the failure times of test units. In this section, we briefly describe the methodology for generating random failure times from known distributions. There are several techniques for generating random variates, but each has its own *exactness*, that is, its ability to generate variates with exactly the desired distribution barring external limitations of the computer accuracy and the exactness of the uniform U(0,1) random number generator. We will only present the most widely used technique, namely, the inverse transform technique.

The inverse transformation is explained as follows: Let X be a random variate to be generated from a distribution function $F(x)$ which is continuous and strictly increasing in x. Also, let F^{-1} denote the inverse of F. The algorithm of this technique is

1. Generate $U \sim \mathrm{U}(0, 1)$, generate a random number U from a (0, 1) uniform distribution
2. Return a random variate X after substituting U in the inverse of F. Thus, $X = F^{-1}(U)$.

We illustrate this technique by generating random variates from several failure-time distributions.

4.6.1 Exponential Distribution

Generate random failure times that follow an exponential distribution given by Equation 4.57.

$$F(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.57}$$

We first obtain the inverse of the distribution function, F^{-1}. Set $u = F(t)$ and solve for t:

$$F^{-1}(u) = -\frac{1}{\lambda}\ln(1-u).$$

Thus, to generate the random variates, we generate a $U \sim \mathrm{U}(0, 1)$ and then let

$$t = -\frac{1}{\lambda}\ln(U).$$

Since U has the same distribution as $(1 - U)$, then

$$t = -\frac{1}{\lambda}\ln(1-U).$$

4.6.2 Weibull Distribution

Generate random failure times that follow a Weibull distribution given by Equation 4.58:

$$F(t) = 1 - e^{-(t/\theta)^{\gamma}}. \tag{4.58}$$

Similar to the exponential distribution, we obtain the inverse of the distribution function as

$$F^{-1}(u) = \theta[-\ln(1-u)]^{1/\gamma}.$$

Thus, generate a $U \sim \mathrm{U}(0, 1)$ and then let $t = \theta[-\ln(U)]^{1/\gamma}$.

4.6.3 Rayleigh Distribution

Generate random failure times that follow a Rayleigh distribution given by Equation 4.59.

$$F(t) = 1 - e^{\left[-\left(\frac{t-a}{b}\right)^2\right]} \quad t \geq a. \tag{4.59}$$

We obtain the inverse of the distribution function as

$$F^{-1}(u) = a + b[\ln(1-u)]^{1/2}.$$

Thus, generate a $U \sim \mathrm{U}(0, 1)$ and then let $t = a + b[-\ln(U)]^{1/2}$.

4.6.4 Birnbaum–Saunders Distribution

Generate random failure times that follow a Birnbaum–Saunders (BS) distribution whose reliability function is given by Equation 4.60.

$$R(t) = 1 - \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right] 0 < t < \infty \quad \alpha, \beta > 0, \tag{4.60}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal, α is the shape parameter, and β is the scale parameter. Kundu et al. (2008) consider the following transformation of a random variable T that follows BS (α, β):

$$X = \frac{1}{2}\left[\left(\frac{T}{\beta}\right)^{\frac{1}{2}} - \left(\frac{T}{\beta}\right)^{\frac{-1}{2}}\right],$$

which is equivalent to

$$T = \beta\left(1 + 2X^2 + 2X(1 + X^2)^{\frac{1}{2}}\right). \tag{4.61}$$

Then X is normally distributed with mean zero and variance ($\alpha^2/4$).

Equation 4.61 is then used to generate the random failure times from BS (α, β).

PROBLEMS

4.1 Given the following failure-time data,

40, 45, 55, 68, 78, 85, 94, 99, 120, 140, 160, and 175 h,

a. Assuming that the data follow an exponential distribution, derive an expression for the failure-rate function.

b. Use the methods of moments to estimate the parameter of the exponential distribution.

c. What is the reliability of a component which belongs to the same population of tested units at time $t = 49$ h?

d. Plot the reliability of the component against time.

4.2 The range of the parameters of the beta distribution enables it to model a wide variety of failure-time data. Hence, beta distribution is used widely in many reliability engineering applications. The p.d.f. of the beta distribution is

$$f(t)=\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}t^{\alpha}(1-t)^{\beta},$$

where $0 < t < 1$, $\alpha > -1$, $\beta > -1$, and α, β are shape parameters. The mean and the standard deviation of the distribution are

$$\mu=\frac{\alpha+1}{\beta+\alpha+2}$$

$$\sigma=\left[\frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^{2}(\alpha+\beta+3)}\right]^{\frac{1}{2}}.$$

The following failure-time data are obtained from a laboratory test:

50, 100, 130, 140, 142, 150, 160, 172, 179, 200, and 220 days.

(*Hint*: Since $0 < t < 1$, use 1 year to represent one unit of time.)

a. Use the method of moments to estimate the parameters of the beta distribution.

b. Derive expressions for $f(t)$, $F(t)$, $R(t)$, and $h(t)$.

c. Plot $R(t)$ and $h(t)$ against time.

4.3 The following failure times of the Weibull distribution are obtained from a reliability test:

320, 370, 410, 475, 562, 613, 662, 770, 865, and 1000 h.

a. Use the method of moments to determine the parameters that fit the above data.

b. Use the maximum likelihood method to obtain the parameters of the Weibull distribution. Use Newton's approach to solve for the values of the parameters. (Use γ and θ obtained from (a) as starting values for Newton's method).

c. Solve (b) using the least squares method.

d. Compare the results obtained from a, b, and c. Draw your conclusions.

4.4 Use the maximum likelihood method to obtain the parameters of a two-parameter exponential distribution having a p.d.f. of

$$f(t)=\lambda e^{-\lambda(t-\gamma)},\ f(t)\geq 0,\ \lambda>0,\ t\geq\gamma,$$

where γ is the location parameter of the distribution.

4.5 Solve Problem 4.4 using the least squares method.

4.6 Plot the contours of the likelihood function for the two-parameter exponential distribution for different values of λ and γ.

4.7 Plot the contours of the likelihood function for the normal distribution for different μ and σ.

4.8 Use the maximum likelihood approach to estimate the parameters of the lognormal distribution whose p.d.f. is

$$f(t)=\frac{1}{\sigma t\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\ln t-\mu}{\sigma}\right)^2\right].$$

a. Construct the information matrix and determine the covariance matrix.

b. Construct the confidence limits for μ.

4.9 Consider a system whose components fail if they enter either of two stages of failure mechanisms: the first mechanism is due to excessive voltage, and the second is due to excessive temperature. Suppose that the failure mechanism enters the first stage with probability θ, and the p.d.f. of the failure time is $\lambda_1 e^{-\lambda_1 t}$. It enters the second stage with probability $(1-\theta)$, and the p.d.f. of its failure time is $\lambda_2 e^{-\lambda_2 t}$. The failure of a component occurs at the end of either stage. Hence, the p.d.f. of the failure time is

$$f(t)=\theta\lambda_1 e^{-\lambda_1 t}+(1-\theta)\lambda_2 e^{-\lambda_2 t}.$$

a. Use the three methods—the method of moments, the maximum likelihood approach, and the least squares method—to obtain the parameters of the above distribution.

b. Which method do you prefer? Why?

4.10 Construct the Fisher information matrix for the p.d.f. of the failure time given in Problem 4.9. Determine the variance of λ_1 and λ_2.

4.11 A producer of light emitting diodes (LED) subjects 25 units to reliability test conditions similar to those of the normal operating conditions. They are subjected to a temperature of 70°F and an electric field of 5 V. The failure-time data are recorded and observed to follow a special Erlang distribution of the form

$$f(t)=\frac{t}{\lambda^2}\exp\left(\frac{-t}{\lambda}\right)\quad t\geq 0.$$

a. Use the method of moments to estimate λ.

b. Use the method of the maximum likelihood to estimate λ.

c. Compare the results obtained in (a) and (b).

4.12 A typical proportional, integral, and derivative (PID) controller consists of a stand-alone regulator (which adjusts the control variable of a process), a front end where the controller constants are manually entered, and a processor (or a computer) where the control algorithm is implemented. When a controller observes deviations in a process output from predefined reference values, the regulator automatically adjusts the process parameter to compensate for such deviations. The regulator is a mechanical, electrical, or an electromechanical system that implements the appropriate control action on the process parameter. Twenty controllers are placed in service, and the times to failure of the regulators are recorded as follows:

551, 571, 571, 575, 583, 588, 590, 592, 594, 598, 606, 610, 611, 611, 613, 615, 615, 626, 629, and 637.

a. Assuming that the failure data follow an exponential distribution, use the method of moments to obtain its parameter.

b. Assume that the failure data follow a Weibull distribution. Use the maximum likelihood approach to estimate the parameters of the distribution.

c. Compare the results of a and b. What do you conclude about the failure-time distribution? Which method is preferred?

4.13 A manufacturer of hydraulic turbomachinery produces turbines, impellers, pumps, and similar equipment. The manufacturer is interested in estimating the expected life of the components of power-generating turbines. The manufacturer subjects the turbines to high-speed flows through the components, resulting in pressure differences that can cause the flow to vaporize and form bubbles. When the bubbles collapse because of a change in pressure, liquid particles bombard the surface of the machinery at high velocities. Such high-velocity, high-pressure liquid particles can chip metal out of the structure and create local fatigue regions in the equipment that eventually results in the failure of the machinery.

The manufacturer subjects 15 turbines to a high-speed flow test and obtains the following failure times:

46, 70, 76, 78, 81, 86, 87, 92, 93, 95, 101, 105, 148, 154, and 158.

Assume that the failure-time data follow a log-logistic distribution of the form

$$f(t) = \lambda p(\lambda t)^{p-1}\left[1+(\lambda t)^{p}\right]^{-2},$$

where $\lambda = e^{-\alpha}$ and $p = 1/\sigma$. Use the method of moments to estimate the parameters α and σ. Estimate the reliability at $t = 200$ h.

4.14 The following are the p.d.f.'s for four failure-time distributions:

- Cauchy

$$f(x) = \left\{\pi\beta\left[1+\left(\frac{x-\alpha}{\beta}\right)^{2}\right]\right\}^{-1},$$

where $-\infty < \alpha < \infty$, $\beta > 0$, $-\infty < x < \infty$;

- Gumbel (or extreme value)

$$f(x) = \frac{1}{\beta}\exp\left[-e^{-(x-\alpha/\beta)} + (x-\alpha)/\beta\right],$$

where $-\infty < \alpha < \infty$, $\beta > 0$, $-\infty < x < \infty$;

- Logistic

$$f(x) = \frac{(1/\beta)e^{-(x-\alpha)/\beta}}{\left(1+e^{-(x-\alpha)/\beta}\right)^{2}},$$

where $-\infty < \alpha < \infty$, $\beta > 0$, $-\infty < x < \infty$; and

- Pareto

$$f(x) = \frac{\alpha_2 c^{\alpha_2}}{x^{\alpha_2+1}},$$

where $c > 0$, $\alpha_2 > 0$, $x > c$.

a. Use an appropriate method to estimate the parameters of the above distributions. You may or may not use the same methods for the four functions.

b. Explain the situations and conditions under which each one of the above distributions can be used in reliability modeling.

4.15 The p.d.f. of the gamma distribution is given by

$$f(x)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}e^{-x/\beta}\qquad x>0.$$

The mean and variance of the gamma distribution are $\alpha\beta$ and $\alpha\beta^2$, respectively.

a. Use the method of moments to obtain the parameters of the distribution.

b. Develop the likelihood function for the distribution and plot it against the parameters α and β.

4.16 The p.d.f. of the chi-squared distribution is

$$f(t)=\frac{t^{1/2v-1}}{\Gamma(1/2v)2^{1/2v}}e^{-1/2t}\qquad t>0,$$

where v is the number of degrees of freedom of the chi-squared distribution.

a. Use the method of moments to obtain the parameter of the chi-squared distribution if the mean and the variance are v and $2v$, respectively.

b. Use the least squares method to obtain the parameter of the chi-squared distribution.

4.17 A fatigue test is conducted, and the failure times shown in Table 4.4 are recorded at equal growth in crack length.

a. Fit a BS distribution to the data and obtain the mean life of the test units.

b. Assume that the only available observations are the first six observations. Use the Bayesian approach to estimate the parameters of the distribution.

c. Use the estimated parameters from (a) and (b) to generate random failure times. Compare the generated data with the data given in Table 4.4.

4.18 Use the failure data in Example 4.19 to obtain the parameter of the failure-time distribution when the prior model of the parameter θ follows an exponential distribution given by

$$g(\theta)=\frac{1}{\theta}e^{-\frac{1}{\theta}}.$$

TABLE 4.4 Failure-Time Data for Problem 4.17

32	70	116	133	171
36	71	118	133	175
52	75	120	138	178
53	76	122	141	178
56	77	123	143	184
59	90	126	152	188
60	97	128	158	199
61	111	130	165	200
62	111	132	165	200
65	116	132	166	204

Compare the estimated value with that of the example. Explain why they are different.

4.19 Develop an algorithm to obtain random failure times from a two-parameter exponential distribution using the inverse transformation approach. The p.d.f. of the distribution is

$$f(t) = \frac{1}{b} e^{-(t-a)/b}.$$

Verify the exactness of the variates.

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CHAPTER 5

PARAMETRIC RELIABILITY MODELS

5.1 INTRODUCTION

One of the most important factors which influences the design process of a product or a system is the reliability functions of its components. For example, a piece of equipment, such as a personal computer, often uses a large number of integrated circuits in a single system. Clearly, a small percentage of the integrated circuits that exhibit early life failures can have a significant impact on the reliability of the system. It is commonly observed that the failure rate of semiconductor devices (components of the integrated circuits) is initially high, and then decreases to a steady state value with time.

There are many other examples that illustrate that the reliability of the system is highly dependent on the reliability of the individual components that comprise the system. Submarine optical fiber transmission systems, weather satellites, telecommunication networks, air traffic control, and supercomputers are typical systems that require "highly reliable" components in order to achieve the reliability goals of the total system. Designers of such systems usually set the long-term reliability of the systems to unusually high values, for example, 0.999999 or higher (note that reliability is a monotonically decreasing function with time, and availability is indeed the long-term measure of reliability for repairable systems). Such values may appear unrealistic, but they are achievable when appropriate subsystems are designed and highly reliable components are incorporated in the systems.

In order to estimate the reliability of the individual components or the entire system, we may follow one or more of the following approaches.

5.2 APPROACH 1: HISTORICAL DATA

The failure data for components can be found in data banks such as GIDEP (Government-Industry Data Exchange Program), MIL-HDBK-217D (which includes failure data for components as well as procedures for reliability prediction), *AT&T Reliability Manual* (Klinger et al., 1990), and *Bell Communications Research Reliability Manual* (Bell Communications Research, 1986, 1995). In such data banks and manuals, the failure data are collected from different manufacturers and are presented with a set of multiplying factors that relate the failure rates to different manufacturer's quality levels and environmental conditions. For example, the general expression used to determine steady-state failure rate λ_{SS} of an electronic or electrical

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component (most of, if not all, electronic components exhibit constant failure rates) that includes different devices is

$$\lambda_{SS} = \pi_E \left[\sum_{i=1}^{n} N_i (\lambda_G \pi_Q \pi_S \pi_T)_i \right], \tag{5.1}$$

where

π_E = environmental factor for a component,

N_i = quantity of the ith device,

n = number of different N_i devices in the system,

λ_G = generic failure rate for the ith device,

π_Q = quality factor for the ith device,

π_S = stress factor for the ith device, and

π_T = temperature factor for the ith device.

The factors π_E, π_Q, π_S, and π_T are estimated empirically and are found in Klinger et al. (1990) and Bell Communications Research (1986, 1995). Some of these factors are difficult to determine for highly reliable devices.

It should be mentioned that the collection (and analysis) of field data poses challenging and important problems, yet it has not been discussed much in the statistical literature (Lawless, 1983).

5.3 APPROACH 2: OPERATIONAL LIFE TESTING

An *operational life test* (OLT) is one in which prototypes of a product—whether it is a single component product such as a telephone pole, or multicomponent products such as cars and computers—are subjected to stresses and environmental conditions at typical normal operating conditions.

The duration of the test is determined by the number of products under test (sample size) and the expected number of failures. In all cases, the test should be terminated when its duration reaches the expected life of the product. Clearly, this test requires extensive durations especially when the product's life is rather long, the case of many electronic devices.

An example of the operational life testing is the testing of utility poles by taking a sample and placing it under the same environmental and weather conditions and observing the failure times over an extended period that ranges from 1 year to several years. Similar operational life testing is performed on electric switching systems and mechanical testing machines.

Usually, the OLT equipment is designed to be capable both of operating components and of testing them on a scanning basis. As mentioned earlier, the test conditions are not accelerated but rather designed to simulate the field operating conditions (such as temperature fluctuations and power on/off).

Analyses of test results are used to monitor and estimate the reliabilities and failure rates of products in order to achieve the desired specifications.

Although the results obtained from OLT are the most useful among other tests, the duration of the test is relatively long and the costs associated with the tests may make them prohibitive to run. Indeed, this test is not classified as an accelerated life testing (ALT) since no real acceleration of time or stress is performed.

5.4 APPROACH 3: BURN-IN TESTING

It is often found that in a large population of components (or products), some individual components have quality defects which considerably affect the component's life. In order to "weed out" these individual components, a *burn-in test* is performed at stress conditions—that is, the time or applied stresses are accelerated. It is important to note that the test conditions must be determined such that the majority of failures are detected without significantly overstressing the remaining components. Additionally, an optimal burn-in period should be estimated such that the total cost to the producer and the user of the product is minimized. There are two cost elements that should be considered in estimating the optimal burn-in period. They are: (1) cost per unit time of the test (long test periods are costly to the producer), and (2) cost of premature failures since short test periods may not completely "weed out" the defective components which in turn results in significant costs for both producers and consumers. Mathematical models for estimating the optimal burn-in period are given in the literature of Jensen and Petersen (1982), Bergman (1985), Kuo et al. (2001), and Wu and Su (2002).

5.5 APPROACH 4: ACCELERATED LIFE TESTING

Accelerated life testing (ALT) is used to obtain information quickly on life distributions, failure rates, and reliabilities. ALT is achieved by subjecting units and components to test conditions such that failures occur sooner. Thus, prediction of the long-term reliability can be made within a short period of time. Results from the ALT are used to extrapolate the unit characteristic at any future time t and at given normal operating conditions. There are two methods used for conducting an accelerated life test. In the first method, it is possible to accelerate the test by using the product more intensively than in normal use. For example, in evaluating the life distribution of a light bulb of a telephone set which is used on the average 1 h a day, a usage of the bulb during its expected life of 40 years can be compressed into 18 months by cycling the power on/off continuously during the test period. Another example, the endurance limit of a crankshaft of a car with an expected life of 15 years (3 h of driving per day), can be obtained by compressing the test into 2 years. However, such time compression (accelerating time) may not be possible for a product that is in constant use, such as a mainframe computer. Moreover, in such cases, the prediction of reliability must consider the aging effect on the component's life.

When time cannot be compressed, the test is usually conducted at higher stress levels than those at normal use. For example, assuming the normal operating temperature of a computer is 25°C, we may accelerate the test by subjecting the critical components of the computer to a temperature of 100°C or higher. This causes the failure of the components to occur in a shorter time. Obviously, the higher the stress, the shorter the time needed for the failures to occur. Such accelerated testing should be carefully designed in order not to induce different failure modes than those that occur at normal operating conditions. The types of stresses, stress levels, test durations, and others are discussed in details in Chapter 6.

A variant of ALT is to perform the test at very high stress levels in order to induce failures in very short times. We refer to this approach as *highly accelerated life testing* (HALT). This test attempts to greatly reduce the test time for both burn-in and life test. The ceramic capacitor is a good example for using HALT to evaluate both life test and production burn-in. Generally, the duration of the burn-in of the ceramic capacitor is about 100 h. However, the use of the HALT approach can reduce the burn-in time significantly and in turn increase the throughput of the production facility. To apply HALT for ceramic capacitors, one or both of the two factors—temperature and voltage—may be used. Obviously, there are maximum stress levels beyond which the tested product will be damaged. Moreover, there are other dangers associated with accelerated tests. For example, voltage increases can create dangerous situations for personnel and equipment since the fuses, used to protect the bias supply, often explode. HALT is not intended to be a "true" reliability test for estimating the reliability of the units. Rather, its purpose is to determine the extreme stresses that the unit (component) experiences before failure in order to improve its design.

One of the main objectives of ALT is to use the test results at the accelerated stress conditions to predict reliability at design stress levels (i.e., normal operating conditions) using an appropriate physics-based or statistics-based models (Meeker and Hahn, 1985; Shyur et al., 1999a, 1999b; Elsayed and Liao, 2004).

Two important statistical problems in ALT are *model identification* and *parameter estimation*. While model identification is the more difficult of the two, they are interrelated. Lack of fit of a model can be due in part to the use of an inefficient method of parameter estimation.

The model usually portrays a valid relationship between the results at accelerated conditions and the normal conditions when the failure mechanisms are the same at both conditions. Once an appropriate model has been identified, it is reasonable to ask which method of parameter estimation is better in terms of such criteria as root mean square (RMS) error and bias. The methods commonly used for parameter estimation are maximum likelihood estimator (MLE), method of moments (MMs), and best linear unbiased estimator (BLUE). Methods for parameter estimation are discussed in Chapter 4. Development of ALT methods and test plans are discussed in detail in the following chapters.

A valid statistical analysis does not require that all test units fail. This is especially true in situations where the accelerated stress conditions are close to the normal operating conditions and failures may not occur during the predetermined test time. The information about nonfailed units at such stress levels is more important than the information about failed units, which are tested at much higher stress levels than the operating conditions. Therefore, the information about the nonfailed units (censored) must be incorporated into the analysis of the data. Recognizing this fact, it is important to present the different types of censoring used in the ALT.

5.6 TYPES OF CENSORING

One common aspect of reliability data which cause difficulties in the analysis is the censoring that takes place since not all units under test will fail before the test-stopping criterion is met. There are many types of censoring, but we limit our presentation to the widely used types of censoring.

5.6.1 Type 1 Censoring

Suppose we place n units under test for a period of time T. We record the failure times of r failed units as $t_1, t_2, \ldots, t_r \leq T$. The test is terminated at time T with $n - r$ surviving (nonfailed) units. The number of failures, r, is a random variable that depends on the duration of the test and the applied stress level and stress type.

Analysis cannot be performed about the reliability and failure rate of the units if no failures occur during T. Therefore, it is important to determine T such that at least some units fail during the test. The time T at which the test is terminated is referred to as the test censoring time, and this type of censoring is referred to as Type 1 censoring.

5.6.2 Type 2 Censoring

Suppose we place n units under test and the exact failure times of failed units are recorded. The test continues to run until exactly r failures occur.

The test is terminated at t_r. Since we specify r failures in advance, we know exactly how much data will be obtained from the test. It is obvious that this type of testing guarantees that failure times will occur and reliability analysis of the data is assured. Of course, the accuracy of reliability analysis is dependent on the number of failure times recorded. The test duration, T, is a random variable which depends on the value of r and the applied stress level.

In this type of test, the censoring parameter is the number of failures, r, during the test. It is usually preferred to Type 1 censoring.

5.6.3 Random Censoring

Random censoring arises when, for example, n units (devices) are divided among two or more independent test equipment. Suppose after time t_f has elapsed, we observe a failure of one of the test equipment. The units placed on this test equipment are removed from the test while the remaining units on the other test equipment continue until the test is completed.

The time at which we observe the failure of the test equipment is called the censoring time of units. Since the failure time of the test equipment is a random variable, we refer to this type of censoring as *random censoring*.

There are other types of censoring which are used for specific purposes. Suppose for example that n units are placed on a test at the same time and at predetermined time τ_1, r_1 surviving units are randomly removed from the test and $(n - r_1)$ units continue on test. At a second predetermined time τ_2, r_2 surviving units are randomly removed from the test and the remaining $(n - r_1 - r_2)$ continue on test. This process continues until a predetermined time point

τ_s is reached (test termination time) or when all the units fail. When $r_1 = r_2 = \ldots = r_{s-1} = 0$, the progressive censoring becomes Type 1 censoring. If the $r_1 \ldots r_{s-1}$ surviving units are removed from the test following the first to the $(s-1)^{th}$ failure, the test is generalized to progressive Type 2 censoring.

In the following sections, we analyze the failure data obtained from reliability testing assuming that the testing is conducted at normal conditions. We start by using parametric fittings for the data when failure times of all units under test are known and when censoring exists.

5.6.4 Hazard-Rate Calculations under Censoring

As discussed in Chapter 1, the hazard rate for a time interval is the ratio between the number of units failed during the time interval and the number of surviving units at the beginning of the interval divided by the length of the interval. Censored units during an interval should not be counted as part of the failed units during that interval. Otherwise, the hazard rate will be inflated. The following example illustrates the necessary calculations for both the hazard rate and the cumulative hazard under censoring.

EXAMPLE 5.1

Two hundred ceramic capacitors are subjected to a highly accelerated life test. The failure times of some of the capacitors are censored since the equipment used during testing these capacitors failed during the test. The number of surviving and censored units is shown in Table 5.1. Compute both the hazard rate and the cumulative hazard.

SOLUTION

The hazard rate at time t_i is estimated as

$$h(t_i) = \frac{N_f(\Delta t_i)}{N_S(t_{i-1})\Delta t_i},$$

where

$h(t_i)$ = the hazard at time t_i,

$N_f(\Delta t_i)$ = the number of failed units during the interval Δt_i,

$N_S(t_{i-1})$ = the number of surviving units at the beginning of the interval Δt_i, and

Δt_i = the length of the time interval (t_{i-1}, t_i).

It should be noted that $N_f(\Delta t_i)$ does not include censored units. The calculations for the hazard rate and the cumulative hazard are given in Table 5.1. It is apparent that the hazard rate is constant with a mean value of 0.0319. Therefore, the mean time to failure (MTTF) is 31.35 h.

TABLE 5.1 Hazard Rate and Cumulative Hazard

Time interval	Number of failed units	Number of censored units	Survivors at end of interval	Hazard rate × 10	Cumulative × 10
0–10	0	3	197	0.0000	0.0000
10–20	6	8	183	0.0304	0.0304
20–30	7	9	167	0.0382	0.0686
30–40	6	8	153	0.0359	0.1045
40–50	6	15	132	0.0392	0.1437
50–60	5	20	107	0.0373	0.1810
60–70	4	18	85	0.0373	0.2183
70–80	3	20	62	0.0352	0.2535
80–90	2	30	30	0.0322	0.2857
90–100	1	29	0	0.0333	0.3190

■

In the following sections, we present parametric models to fit failure data from reliability testing or field data to known failure-time distributions such as exponential, Weibull, lognormal, and gamma.

5.7 THE EXPONENTIAL DISTRIBUTION

Since the exponential distribution has a constant failure rate and is commonly used in practice, we shall illustrate how to assess the validity of using the exponential distribution as a failure-time model. Let

t_1 = the time of the first failure,

t_i = the time between $i-1$th and ith ($i = 2, 3, \ldots$) failures or the time to failure i (depending on the time being observed),

r = the total number of failures during the test (assurming no censoring),

T = the sum of the times between failures, $T = \sum_{i=1}^{r} t_i$, and

X = a random variable to represent time to failure.

In order to check whether or not the failure times follow an exponential distribution, we use the Bartlett's test whose statistic is

$$B_r = \frac{2r\left[\ln\left(\frac{T}{r}\right) - \frac{1}{r}\left(\sum_{i=1}^{r} \ln t_i\right)\right]}{1+(r+1)/6r}, \tag{5.2}$$

where B_r is chi-square distributed statistics with $r - 1$ degrees of freedom.

The Bartlett's test does not contradict the hypothesis that the exponential distribution can be used to model a given time to failure data if the value of B_r lies between the two critical values of a two-tailed chi-square test with a $100(1 - \alpha)$ significance level. The lower critical value is $\chi^2_{(1-\alpha/2),r-1}$, and the upper critical value is $\chi^2_{\alpha/2,r-1}$.

EXAMPLE 5.2

Twenty transistors are tested at 5 V and 100°C. When a transistor fails, its failure time is recorded, and the failed unit is replaced by a new one. The times between failures (in hours) are recorded in an increasing order as shown in Table 5.2. Test the validity of using a constant hazard rate for these transistors.

TABLE 5.2 Times between Failures of the Transistors

Times between failures in hours(t_i)	
200	32,000
400	34,000
2,000	36,000
6,000	39,000
9,000	42,000
13,000	43,000
20,000	48,000
24,000	50,000
26,000	54,000
29,000	60,000

SOLUTION

Since all the transistors failed during the test, the failure times can be assumed to be from a sample of 20 transistors, and each t_i is a value of the random variable, X, time between failures.

$$\sum_{i=1}^{20} \ln t_i = 193.28$$

$$T = \sum_{i=1}^{20} t_i = 567{,}600$$

$$B_{20} = \frac{2 \times 20 \left[\ln\left(\frac{567{,}600}{3}\right) - \frac{1}{20} \times 193.28 \right]}{1 + (21)/(6 \times 20)}$$

$$B_{20} = 20.065.$$

The critical values for a two-tailed test $\alpha = 0.10$ are

$$\chi^2_{0.95,19} = 10.117 \quad \text{and} \quad \chi^2_{0.05,19} = 30.144.$$

Therefore, B_{20} does not contradict the hypothesis that the failure times can be modeled by an exponential distribution. ■

The following example includes Type 2 censoring.

EXAMPLE 5.3

In a test similar to the previous example, 20 transistors are subjected to an accelerated life test (temperature 200°C and 2.0 V). The test is discontinued when the tenth failure occurs. Determine whether the failure data in Table 5.3 follow an exponential distribution.

TABLE 5.3 Times between Failures in Hours

600	2,800
700	3,000
1,000	3,100
2,000	3,300
2,500	3,600

SOLUTION

$$\sum_{i=1}^{10} \ln t_i = 75.554$$

$$T = \sum_{i=1}^{10} t_i = 22{,}600$$

$$B_{10} = \frac{2\times 10\left[\ln\left(\frac{22{,}600}{10}\right) - \frac{1}{10}(75.554)\right]}{1+(10+1)/(6\times 10)}$$

$$B_{10} = 2.834.$$

The critical values for a two-sided test with $\alpha = 0.10$ are

$$\chi^2_{0.95,9} = 3.325 \quad \text{and} \quad \chi^2_{0.05,9} = 16.919.$$

Therefore, B_{10} contradicts the hypothesis that the failure data can be modeled by an exponential distribution. However, at a significance level of 98%, the critical values of the two-sided test become

$$\chi^2_{0.99,9} = 2.088 \quad \text{and} \quad \chi^2_{0.01,9} = 21.666,$$

and the test does not contradict the hypothesis that the failure times can be modeled by an exponential distribution. ■

Of course, the exponential distribution assumption can also be validated graphically as follows.

The exponential cumulative distribution function $F(t) = 1 - e^{-\lambda t}$, $t > 0$ can be rewritten as $t = (1/\lambda)\ln[1/(1 - F(t))]$. Using the mean or median rank approaches described in Chapter 1 to estimate $F(t)$, the exponential distribution assumption holds when a linear plot of t versus $\ln[1/(1-F(t))]$ is demonstrated.

5.7.1 Testing for Abnormally Short Failure Times

Short failure times may occur due to manufacturing defects such as the case of *freak* failures. These failure times do not actually represent the true failure times of the population. Therefore, it is important to determine whether the failure times are abnormally short before fitting the data to an exponential distribution. If the failure times are abnormally short, they should be discarded and not considered in determining the parameters of the failure-time distribution.

Let $(t_1, t_2, \ldots, t_r)$ be a sequence of r independent and identically distributed exponential random variables that represent the time between failures for the first r failures. Then the quantity $2t_i/\theta$ is chi-square distributed with two degrees of freedom, where θ is the mean of the exponential distribution.

Therefore, if t is the time to first failure which follows an exponential distribution with mean = θ—that is,

$$f(t) = \frac{1}{\theta} e^{-t/\theta}$$

—then the random variable $y = 2t/\theta$ is a χ^2 with two degrees of freedom. This is explained further as follows.

The density function of the random variable $\boldsymbol{t}$ is known and is given above. Our objective is to find the density function $g(y)$ for the random variable $\boldsymbol{y}$. We have

$$dy = \frac{2}{\theta} dt \quad \text{and} \quad t = \frac{\theta}{2} y.$$

Using the random variable transformation $g(y)dy = f(t)dt$, we write $g(y)$ as

$$g(y) = \frac{\theta}{2} f\left(\frac{\theta}{2} y\right),$$

which yields

$$g(y) = \frac{1}{2} e^{-\frac{y}{2}} \; y \geq 0.$$

This expression is indeed the probability density function (p.d.f.) of a χ^2 distribution with two degrees of freedom. Note that the general expression for the p.d.f. of χ^2 distribution with ν degrees of freedom is

$$g(y) = \frac{e^{\frac{-y}{2}} y^{\frac{\nu}{2}-1}}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)}.$$

The sum of two or more independent χ^2 distribution random variables is a new variable that follows χ^2 with a degree of freedom equal to the sum of the degrees of freedom of the individual random variables (Kapur and Lamberson, 1977). So,

$$\frac{2}{\theta}\sum_{i=2}^{r} t_i \text{ is } \chi^2 \text{ with } 2r-2 \text{ degrees of freedom.}$$

Thus,

$$F = \frac{\frac{2}{\theta} t_1 / 2}{\frac{2}{\theta}\sum_{i=2}^{r} t_i / (2r-2)}.$$

This means that the F-distribution can be formed as

$$F_{2,2r-2} = \frac{(r-1)t_1}{\sum_{i=2}^{r} t_i},$$

where t_1, the short failure time, follows F distribution with degrees of freedom 2 and $2r-2$.

If t_1 is small, then the F ratio becomes very small—that is,

$$F_{1-\alpha,2,2r-2} > \frac{(r-1)t_1}{\sum_{i=2}^{r} t_i}.$$

This inequality is equivalent to

$$F_{\alpha,2r-2,2} < \frac{\sum_{i=2}^{r} t_i}{(r-1)t_1}.$$

It is important to note that failure data should be ordered according to an increasing failure-time arrangement. In other words, the shortest failure time is listed first, followed by the second shortest time, and so forth.

EXAMPLE 5.4

Consider the failure data shown in Table 5.4 which represent cycles to failure for 20 turbine blades. The test is performed by subjecting a turbine to an accelerated load, replacing it by a new turbine upon failure, and recording the time to failure. Is the first failure time abnormally short?

TABLE 5.4 Failure Data of Turbine Blades

120	2,112	2,689	4,256
1,300	2,192	2,892	4,368
1,680	2,215	2,999	4,657
1,990	2,290	3,565	4,933
2,010	2,581	3,873	5,832

SOLUTION

The total failure time (except the first one) is

$$\sum_{i=2}^{20} t_i = 5832 - 120 = 5712$$

$$t_i = 120$$

$$F_{calculated} = \frac{5712}{(19)120} = 2.51.$$

The critical value of F at 95% confidence is

$$F_{0.05,38,2} = 19.47.$$

Thus, the first failure is a representative of the rest of the data. In other words, the hypothesis that the first failure time is abnormally short should be rejected. ■

5.7.2 Testing for Abnormally Long Failure Times

Following the above procedure, the failure time t_{ab} is considered to be an abnormally long failure time if

$$F_{\alpha,2,2r-2} < \frac{(r-1)t_{ab}}{\sum_{i \neq ab}^{r} t_i}.$$

If t_r is the largest failure time, then the above equation can be rewritten as

$$F_{\alpha,2,2r-2} < \frac{(r-1)t_r}{\sum_{i=1}^{r-1} t_i}.$$

EXAMPLE 5.5

Consider the following failure-time data. Test whether the last failure time is abnormally long at 5% level of significance (Table 5.5).

TABLE 5.5 Failure Times

30,000	46,585	63,200	77,990
34,500	49,970	66,600	80,330
37,450	54,430	70,000	84,450
39,950	57,600	73,120	88,960
43,760	59,990	75,690	99,550

SOLUTION

To check if the failure time 99,550 is abnormally long, we obtain

$$\sum_{i=1}^{19} t_i = 1{,}134{,}575$$

$$F_{\text{calculated}} = \frac{19 \times 99{,}550}{1{,}134{,}575} = 1.667.$$

Since $F_{calculated} < F_{0.05,2,38} = 3.25$, then the last failure time (99,550) is not abnormally long. ■

Suppose that n units are placed under test and the exact failure times of the units are recorded. They are $t_1, t_2, \ldots, t_n$. Since all units have failed and no censoring exists, the maximum likelihood estimate of λ as described in Chapter 4 is

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} t_i}, \tag{5.3}$$

where $\hat{\lambda}$ is the MLE of the failure rate.

The mean μ of the exponential distribution is $1/\lambda$, and the MLE of μ is

$$\hat{\mu} = \frac{1}{\hat{\lambda}} = \frac{\sum_{i=1}^{n} t_i}{n} = \bar{t}. \tag{5.4}$$

$\hat{\mu}$ is referred to as the MLE of the mean life.

EXAMPLE 5.6

Assume that the data given in Example 5.2 represent the failure times of the units under test. Determine the mean life of a transistor from this population.

SOLUTION

Using Equation 5.3, we obtain

$$\bar{t} = \frac{567{,}600}{20} = 28{,}380 \text{ h.}$$

■

It can be shown that $2n\hat{\mu}/\mu$ has an exact chi-square distribution with $2n$ degrees of freedom. Since $\lambda = 1/\mu$ and $\hat{\lambda} = 1/\hat{\mu}$, then a $100(1 - \alpha)$ percent confidence interval for $\hat{\lambda}$ (assuming zero minimum life) is

$$\frac{\hat{\lambda}\chi^2_{1-\alpha/2,2n}}{2n} < \lambda < \frac{\hat{\lambda}\chi^2_{\alpha/2,2n}}{2n}, \tag{5.5}$$

where is the $\chi^2_{\alpha,2n}$ is the 100α percentage point of the chi-square distribution with $2n$ degrees of freedom—that is,

$$P[\chi^2_{2n} > \chi^2_{\alpha,2n}] = \alpha.$$

The confidence interval of the mean life corresponding to Equation 5.5 is

$$\frac{2n\hat{\mu}}{\chi^2_{\alpha/2,2n}} < \mu < \frac{2n\hat{\mu}}{\chi^2_{1-\alpha/2,2n}}.$$

When n is large ($n \geq 25$), we can obtain an approximate interval for λ by approximating $\hat{\lambda}$ by a normal distribution with mean λ and variance λ^2/n. Thus,

$$\hat{\lambda} - \frac{\hat{\lambda}Z_{\alpha/2}}{\sqrt{n}} < \lambda < \hat{\lambda} + \frac{\hat{\lambda}Z_{\alpha/2}}{\sqrt{n}}, \tag{5.6}$$

where $Z_{\alpha/2}$ is $100(\alpha/2)$ percent point $[P(Z > Z_{\alpha/2}) = \alpha/2]$ of the standard normal distribution.

EXAMPLE 5.7

Determine the 95% two-sided confidence interval for the mean life of the transistors given in Example 5.6.

SOLUTION

Since $\hat{\mu} = 28{,}380$ h, then at 95% confidence level, $\chi^2_{0.975,40} = 24.423$ and $\chi^2_{0.025,40} = 59.345$, the confidence interval for μ is

$$\frac{2 \times 567{,}600}{59.345} < \mu < \frac{2 \times 567{,}600}{24.423}$$

or

$$19{,}218 \leqslant \mu \leqslant 46{,}480 \text{ h.}$$

■

EXAMPLE 5.8

A mechanical engineer conducts a fatigue test to determine the expected life of rods made of a specific type of steel by subjecting 25 specimens to an axial load that causes a stress of 9000 lb per square inch (psi). The number of cycles is recorded at the time of failure of every specimen. Assuming that the test is run at 10 cycles per minute, determine the reliability of a rod made of this steel at 10 h. Results of the test are shown in Table 5.6.

TABLE 5.6 Number of Cycles to Failures

Cycles to failure				
200	720	1,950	5,570	10,660
280	850	2,460	6,590	11,670
340	990	2,590	7,600	12,680
460	1,200	3,520	8,630	13,685
590	1,420	4,560	9,650	14,690

SOLUTION

Using B_{25}, we first check if the failure data can be represented by an exponential distribution. This requires the determination of the cycles between failures from the data given above as shown in Table 5.7.

$$T = \sum_{i=1}^{25} CBF_i = 14{,}690$$

$$\sum_{i=1}^{25} \ln CBF_i = 149.211$$

$$B_{25} = \frac{2 \times 25\left[\ln \dfrac{14{,}690}{25} - \dfrac{1}{25} \times 149.211\right]}{1 + \dfrac{26}{6 \times 25}}$$

$$B_{25} = 17.370.$$

The critical values for a two-tailed test with $\alpha = 0.10$ are

$$\chi^2_{0.95,24} = 13.848$$

TABLE 5.7 Number of Cycles to Failures and between Failures

Rod No.	Cycles to failure (CTF)	Cycles between failures (CBF)
1	200	200
2	280	80
3	340	60
4	460	120
5	590	130
6	720	130
7	850	130
8	990	140
9	1,200	210
10	1,420	220
11	1,950	530
12	2,460	510
13	2,590	130
14	3,520	930
15	4,560	1,040
16	5,570	1,010
17	6,590	1,020
18	7,600	1,010
19	8,630	1,030
20	9,650	1,020
21	10,660	1,010
22	11,670	1,010
23	12,680	1,010
24	13,685	1,005
25	14,690	1,005

and

$$\chi^2_{0.05,24} = 36.415.$$

Hence, the B_{25} statistic does not contradict the hypothesis that the failure times can be modeled by an exponential distribution.

The reliability of a rod at 10 h is obtained as

$$R(t = 10\text{ h}) = e^{-\hat{\lambda} t},$$

where

$$\hat{\lambda} \text{ is } \frac{25}{14{,}690} \text{ failures per cycle.}$$

$$R(t = 10\text{ h}) = e^{\frac{-25}{14{,}690}\times 60\times 10\times 10}$$

$$R(t = 10\text{ h}) = 0.3676\times 10^{-4}.$$

■

We now consider the effect of censoring on the estimation of λ. First, we present Type 1 censoring to be followed by Type 2 censoring.

5.7.3 Data with Type 1 Censoring

Assume that n units are placed under test and that the failure times t_i's of the failed units are recorded and reordered in an increasing order. Let T be the censoring time of the test. Thus, $t_1 \le t_2 \le t_3 \le \ldots \le t_r \le t_1^+ = \ldots = t_{n-r}^+ = T$ where t_i^+ is the censoring time of censored unit i. We use the MLE approach to estimate the distribution parameters, and the likelihood function is

$$L = \prod_{i=1}^{r} f(t_i) \prod_{i=1}^{n-r} R(t_i^+) = \prod_{i=1}^{r} \lambda e^{-\lambda t_i} \prod_{i=1}^{n-r} e^{-\lambda t_i^+}. \tag{5.7}$$

Taking the logarithm of Equation 5.7 yields (we use l to refer to $\ln L$)

$$l = r\ln\lambda - \sum_{i=1}^{r} \lambda t_i - \sum_{i=1}^{n-r} \lambda t_i^+.$$

The derivative of l with respect to λ is

$$\frac{dl}{d\lambda} = \frac{r}{\lambda} - \sum_{i=1}^{r} t_i - \sum_{i=1}^{n-r} t_i^+.$$

Equating the derivative to zero results in the MLE of λ as

$$\hat{\lambda} = \frac{r}{\sum_{i=1}^{r} t_i + \sum_{i=1}^{n-r} t_i^+}, \tag{5.8}$$

and the mean life of units can be estimated as

$$\hat{\mu} = \frac{1}{\hat{\lambda}} = \frac{1}{r}\left[\sum_{i=1}^{r} t_i + \sum_{i=1}^{n-r} t_i^+\right]. \tag{5.9}$$

The statistic $2r\lambda/\hat{\lambda}$ has a chi-square distribution with $2r$ degrees of freedom, and the mean and variance of $\hat{\lambda}$ are $r\lambda/(r-1)$ and $\lambda^2/(r-1)$, respectively (Lee, 1992). The $100(1-\alpha)$ confidence interval for λ is

$$\frac{\hat{\lambda}\chi^2_{1-\alpha/2,2r}}{2r} < \lambda < \frac{\hat{\lambda}\chi^2_{\alpha/2,2r}}{2r}. \tag{5.10}$$

The confidence interval for the mean life, μ, is

$$\frac{2r\hat{\mu}}{\chi^2_{\alpha/2,2r}} < \mu < \frac{2r\hat{\mu}}{\chi^2_{1-\alpha/2,2r}}.$$

When n is large ($n \geq 25$), the distribution of $\hat{\lambda}$ can be approximated by a normal distribution with mean λ and variance $\lambda^2/(r-1)$. The $100(1-\alpha)$ percent confidence interval is

$$\hat{\lambda} - \frac{\hat{\lambda}Z_{\alpha/2}}{\sqrt{r-1}} < \lambda < \hat{\lambda} + \frac{\hat{\lambda}Z_{\alpha/2}}{\sqrt{r-1}}. \tag{5.11}$$

EXAMPLE 5.9

A manufacturer of end mill cutters introduces a new ceramic cutter material. In order to estimate the expected life of a cutter, the manufacturer places 10 units under continuous test and monitors the tool wear. A failure of the cutter occurs when the wearout exceeds a predetermined value. Because of budgeting constraints, the manufacturer decides to run the test for 50,000 min. The times to failure are recorded as shown in Table 5.8. Determine the mean life of a cutter made from this material. What is the 90% confidence interval for the expected life? What is the reliability at 60,000 min?

TABLE 5.8 Times to Failure of Cutters

Cutter's life in minutes
3,000
7,000
12,000
18,000
20,000
30,000

SOLUTION

We check if the failure data follow an exponential distribution by estimating B_6 using the time between failures as shown in Table 5.9. We calculate

$$T = \sum_{i=1}^{6} TBF_i = 30{,}000$$

$$\sum_{i=1}^{6} \ln TBF_i = 50.33$$

$$B_6 = 1.2973.$$

TABLE 5.9 Failure Data of the Cutters

Cutter number	Time to failure (TTF)	Time between failures (TBF)
1	3,000	3,000
2	7,000	4,000
3	12,000	5,000
4	18,000	6,000
5	20,000	2,000
6	30,000	10,000

The chi-squared statistics are

$$\chi^2_{0.95,5} = 1.145 \quad \text{and} \quad \chi^2_{0.05,5} = 11.070.$$

Thus, the data follow an exponential distribution.

Using Equation 5.9, we estimate $\hat{\mu}$ as

$$\hat{\mu} = \frac{1}{6}[90{,}000 + 4 \times 50{,}000] = 48{,}333 \text{ min},$$

and the 90% confidence interval for μ is

$$\frac{2 \times 6 \times 48{,}333}{21.026} < \hat{\mu} < \frac{2 \times 6 \times 48{,}333}{5.226}$$

or

$$21{,}584 < \hat{\mu} < 110{,}982.$$

The probability that a cutter will survive for 60,000 min is

$$\hat{R}(60{,}000) = e^{-\hat{\lambda} t} = e^{\frac{-1}{48{,}333} \times 60{,}000}$$

$$\hat{R}(60{,}000) = 0.289,$$

where $\hat{R}(t)$ is the estimated reliability at time t. ■

It is important to note that $\hat{\mu}$ in Equation 5.9 is an unbiased estimator of μ since it is estimated using MLE. This can also be verified as follows.

The total time on test (T_{total}) is

$$\left[\sum_{i=1}^{r} t_i + \sum_{i=1}^{n-r} t_i^+\right] \quad \text{or} \quad \left[\sum_{i=1}^{r} t_i + (n-r)t^*\right],$$

where $t^* = t_r$ is the common censoring time of the $(n - r)$ units. It can also be written as

$$T_{total} = nt_1 + (n-1)(t_2 - t_1) + (n-2)(t_3 - t_2) + \ldots + (n-r+1)(t_r - t^*).$$

However t_1 is the minimum of n exponential variables and its expectation $E(t_1) = 1/(n\lambda)$. Similarly, $(t_2 - t_1)$ is the smallest of the remaining $n - 1$ variables, and its expectation (based on the memoryless property of the exponential distribution) is $E(t_2 - t_1) = 1/[(n - 1)\lambda]$. Following the same for $t_{i+1} - t_i$, we obtain the expectation of the total time on test as

$$\begin{aligned} E(T_{total}) &= nE(t_1) + (n-1)E(t_2 - t_1) + (n-2)E(t_3 - t_2) + \ldots + (n-r+1)E(t^* - t_{r-1}) \\ &= \frac{n}{n\lambda} + \frac{n-1}{(n-1)\lambda} + \frac{n-2}{(n-2)\lambda} + \ldots + \frac{n-r+1}{(n-r+1)\lambda} = \frac{r}{\lambda}. \end{aligned}$$

Therefore, the expected total test time given r failures is $E(T_{total}/r) = (1/r)E(T_{total}) = (1/r)(r/\lambda) = \mu$.

5.7.4 Data with Type 2 Censoring

Suppose that n units are placed under test at time zero and their failure times are recorded in an increasing order. Suppose that the test is terminated when r of the n units fail. The failure times of the n units are $t_1 \le t_2 \le t_3 \ldots t_r, t_1^+ \ldots = t_{n-r}^+$, where t_i is the failure time of unit i and t_i^+ is the censoring time of censored unit i which is also the censoring time of the test.

Following the same procedure of data analysis with Type 1 censoring, we obtain the same equations for the MLE for both λ and μ. Thus, there is no difference in results when either Type 1 or Type 2 censoring is applied.

We now examine the random censoring situation when n units undergo a reliability test at time zero. The test is terminated at time T. Let r be the number of units that fail before T, and $n - r$ be the number of units that either survive the test time T or their test equipment fails during T while the units are operating. The data collected may be observed as follows: $t_1, t_2, \ldots, t_r, t_1^+, t_2^+, \ldots, t_2^+, \ldots, t_{n-r}^+$. The + sign indicates censoring. We now order the failure times and the censoring times in ascending order: $t_1 \le t_2 \le \ldots \le t_r, t_1^+, t_2^+, \ldots, t_{n-r}^+$. Using the MLE method we obtain

$$\hat{\lambda} = \frac{r}{\sum_{i=1}^{r} t_i + \sum_{i=1}^{n-r} t_i^+}.$$

This is the same result as Type 1 censoring.

What happens when all observations are censored? In this case, one estimates $\hat{\mu}$ as the sum of the censored time of all units

$$\hat{\mu} = \sum_{i=1}^{n} t_i^+.$$

Clearly, this estimate of μ has little practical value, and this reliability test is considered to be poorly designed. Methods for handling all censored data will be discussed in the next chapter.

When the number of units under test, n, is large($\geq$25), the distribution of $\hat{\lambda}$ is approximately normal with mean λ and variance (Lee, 1980):

$$\mathrm{Var}(\hat{\lambda}) = \frac{\lambda^2}{\sum_{i=1}^{n}\left(1 - e^{-\lambda T_i}\right)},$$

where T_i is the time that the ith component is under observation (time until failure or end of test). If T_i is unknown due to an abnormal termination of the test, then the variance can be approximated as

$$\mathrm{V\hat{a}r}(\hat{\lambda}) \cong \frac{\hat{\lambda}^2}{r},$$

where r is the number of failed components before the termination of the test.

The 100(1 − α) percent confidence interval is

$$\hat{\lambda} - Z_{\alpha/2}\sqrt{\mathrm{V\hat{a}r}(\hat{\lambda})} < \lambda < \hat{\lambda} + Z_{\alpha/2}\sqrt{\mathrm{V\hat{a}r}(\hat{\lambda})},$$

and distribution of $\hat{\mu}$ is approximated by a normal distribution with mean μ and an estimated variance of

$$\mathrm{V\hat{a}r}(\hat{\mu}) = \frac{\hat{\mu}^2}{\sum_{i=1}^{n}\left(1 - e^{-\lambda T_i}\right)}.$$

If T_i is unknown, then

$$\mathrm{V\hat{a}r}(\hat{\mu}) = \frac{\hat{\mu}^2}{r}.$$

The upper and lower bounds for μ are

$$\hat{\mu} - Z_{\alpha/2}\sqrt{\mathrm{V\hat{a}r}(\hat{\mu})} < \mu < \hat{\mu} + Z_{\alpha/2}\sqrt{\mathrm{V\hat{a}r}(\hat{\mu})}.$$

5.8 THE RAYLEIGH DISTRIBUTION

The Rayleigh distribution exhibits a linearly increasing hazard function with time. This implies that when the time to failure follows the Rayleigh distribution, an intense aging of the equipment takes place, and the failures do not satisfy the conditions of a stationary random process. More importantly, during the early life of a product where the hazard rate is small, the probability of failure-free operation of the product (or system) decreases with time more slowly than in the case of the exponential distribution. However, as the time increases, the probability of failure-free operation decreases with time at a faster rate than the exponential distribution. Rayleigh distribution is useful in modeling rapidly fading communication channels where the amplitude of the signal can be described by such a distribution. The p.d.f. of the Rayleigh distribution is

$$f(t) = \lambda t e^{\frac{-\lambda t^2}{2}}. \tag{5.12}$$

Following the exponential distribution, we estimate the parameter of the Rayleigh distribution for both noncensored and censored failure data as shown below.

5.8.1 Estimation of Rayleigh's Parameter for Data without Censored Observations

Suppose that n devices are subjected to an accelerated life test and that the exact failure time of every device is recorded. The failure times are $t_1, t_2, \ldots, t_n$. Since all devices have failed, then the maximum likelihood estimate of Rayleigh's parameter is obtained as follows:

$$\begin{aligned} L(\lambda, t) &= \prod_{i=1}^{n} f(t_i) \\ &= \prod_{i=1}^{n} \lambda t_i e^{\frac{-\lambda t_i^2}{2}} \end{aligned}$$

or

$$L(\lambda, t) = \lambda^n \prod_{i=1}^{n} t_i e^{\frac{-\lambda t_i^2}{2}}. \tag{5.13}$$

The logarithm of Equation 5.13 is

$$l(\lambda, t) = n \ln \lambda + \sum_{i=1}^{n} \ln t_i - \frac{\lambda}{2} \sum_{i=1}^{n} t_i^2. \tag{5.14}$$

In order to estimate λ, we take the derivative of Equation 5.14 with respect to λ and equate the resultant equation to zero. Thus,

$$\frac{dl(\lambda, t)}{d\lambda} = \frac{n}{\lambda} - \frac{1}{2} \sum_{i=1}^{n} t_i^2$$

or

$$\hat{\lambda} = \frac{2n}{\sum_{i=1}^{n} t_i^2}. \tag{5.15}$$

The variance of Rayleigh distribution is

$$\text{Var}\,(t) = \frac{2}{\hat{\lambda}}\left(1 - \frac{\pi}{4}\right).$$

EXAMPLE 5.10

A manufacturer of an automotive speed sensor subjects 10 sensors to a reliability test that simulates the environmental conditions (temperature and speed) at which the sensors normally operate. A sensor is classified failed when its output falls outside 5% tolerance. The miles accumulated before the failures of the sensors are

110,000, 130,000, 150,000, 155,000, 159,000, 163,000, 166,000, 168,000, 169,000, 170,000

Assume that the miles to failure follow a Rayleigh distribution. Determine the parameter of the distribution, the mean life of a sensor, and the variance of its life.

SOLUTION

The parameter of the Rayleigh distribution is obtained using Equation 5.15 as

$$\hat{\lambda} = \frac{2n}{\sum_{i=1}^{n} t_i^2}$$

$$\hat{\lambda} = \frac{2 \times 10}{2.40616 \times 10^{11}} = 8.31199 \times 10^{-11}.$$

The mean life is

$$\text{Mean life} = \sqrt{\frac{\pi}{2\hat{\lambda}}} = \sqrt{\frac{\pi}{2 \times 8.31199 \times 10^{-11}}} = 137{,}470 \text{ mi},$$

and the variance of the life is

$$\text{Variance} = \frac{2}{\hat{\lambda}}\left(1 - \frac{\pi}{4}\right) = 5.1636 \times 10^{9}.$$

The standard deviation is 71,859. ■

5.8.2 Estimation of Rayleigh's Parameter for Data with Censored Observations

Suppose that n devices are subjected to a test and that the failure times of the r failed units are recorded and listed in an ascending order as $t_1 \le t_2 \le \ldots \le t_r$. The remaining $n - r$ units are censored, that is, these units have not failed before the test is terminated. We assume that the censoring is either of Type 1 or Type 2 only and the censored times are $t_1^+ = t_2^+ = \ldots t_{n-r}^+$.

The likelihood function is

$$L(\lambda, t) = \prod_{i=1}^{r} \lambda t_i e^{\frac{-\lambda t_i^2}{2}} \prod_{i=1}^{n-r} e^{\frac{-\lambda t_i^{+^2}}{2}},$$

and the logarithm of the likelihood function is

$$l(\lambda, t) = r \ln \lambda + \sum_{i=1}^{r} \ln t_i - \frac{\lambda}{2}\left(\sum_{i=1}^{r} t_i^2 + \sum_{i=1}^{n-r} t_i^{+^2} \right).$$

The estimated value of the parameter $\hat{\lambda}$ is obtained as

$$\hat{\lambda} = \frac{2r}{\sum_{i=1}^{r} t_i^2 + \sum_{i=1}^{n-r} t_i^{+^2}}. \tag{5.16}$$

EXAMPLE 5.11

As an alternative to an automobile air-bag crash testing, a test engineer develops a sensor test system which uses a mechanical vibration shaker to replay measured actual crashes. The sensors are subjected to the same conditions measured during a crash test. Ten sensors are placed under test for 50 h and the following failure times are recorded:

10, 20, 30, 35, 39, 42, 44, 50^+, 50^+, 50^+.

Determine Rayleigh's parameter, the mean life of the sensors, and the standard deviation of the life.

SOLUTION

Using Equation 5.16 we obtain

$$\hat{\lambda} = \frac{2 \times 7}{7846 + 7500} = 9.12289 \times 10^{-4}.$$

The mean life is

$$\text{Mean life} = \sqrt{\frac{\pi}{2 \times 9.12289 \times 10^{-4}}} = 41.49 \text{ h.}$$

The standard deviation of the life is

$$\text{Standard deviation} = \sqrt{\frac{2}{\hat{\lambda}}\left(1 - \frac{\pi}{4}\right)} = 21.70 \text{ h.}$$

■

5.8.3 Best Linear Unbiased Estimate for Rayleigh's Parameter for Data with and without Censored Observations

The MLE of the Rayleigh's parameter is biased when the number of observations is small. The bias increases as the number of observations decreases. If the p.d.f. of the failure time can be linearized, the bias in estimating the parameter(s) of the distribution can be decreased when the least square method is used in estimating the parameters. We refer to such an estimate as the best linear unbiased estimate (BLUE). In this section, we obtain the best linear unbiased estimate of Rayleigh's parameter for both censored and noncensored observations.

5.8.3.1 BLUE for Rayleigh's Parameter Suppose that the failure times for n devices subjected to a reliability test are $t_1 \leq t_2 \leq \ldots t_n$ where t_i is the ith order statistics. Assume that the n ordered failure times follow the Rayleigh distribution with

$$f(t) = \frac{1}{\theta_2^2}(t - \theta_1)e^{-(t-\theta_1)^2/2\theta_2^2} \quad (t > \theta_1 \geq 0, \theta_2 > 0) \tag{5.17}$$

and

$$f(t) = 0 \text{ elsewhere,}$$

where θ_1 is the location (threshold) parameter and θ_2 is the scale parameter. Note that Equation 5.17 is identical to Equation 5.12 when $\theta_1 = 0$ and $\lambda = 1/\theta_2^2$.

The BLUE θ_2^* of θ_2, when θ_1 is known, can be estimated by

$$\theta_2^* = \sum_{i=1}^{n} b_i t_i - \theta_1 \frac{K_3}{K_2}. \tag{5.18}$$

If the location parameter $\theta_1 = 0$, the density function, given by Equation 5.17, becomes

$$f(t) = \frac{1}{\theta_2^2} t e^{-t^2/2\theta_2^2}, \tag{5.19}$$

and the estimate θ_2^* becomes

$$\theta_2^* = \sum_{i=1}^{n} b_i t_i. \tag{5.20}$$

The coefficients b_i's are given in Appendix F for $i = 1, \ldots, n$ for noncensored samples, and for censored samples with r largest observations censored, where $r = 0, 1, 2, \ldots, (n - 2)$ and n is the sample size for $n = 5(1)25(5)45$. The variance of θ_2^* in terms of θ_2^2/n and K_3/K_2 is given in Appendix G.

EXAMPLE 5.12

A manufacturer of biosensors produces an electrochemical sensor array that is small enough to fit inside a blood vessel. The device is inserted into an artery within a catheter that has an inside diameter of 650 μ. It measures the levels of oxygen, carbon dioxide, and pH in the blood. The producer subjects 20 sensors to a functional test and observes the following failure times in hours:

0.9737	8.0327	13.1911	19.4369
1.0590	8.0833	13.4695	22.5168
3.3152	8.1957	14.0578	24.4470
3.3161	9.3706	14.8812	24.9225
5.2076	11.1886	17.9624	30.0000

(Note: These data are actually generated randomly from a Rayleigh distribution with $\theta_1 = 0$ and $\theta_2 = 10.5$). Estimate the Rayleigh parameter and its variance.

SOLUTION

To estimate the scale parameter θ_2, we use the coefficients b_i's in Appendix F and K_3/K_2 in Appendix G for $n = 20$.

$$\begin{aligned}\theta_2^* &= (0.00767)(0.9737) + \ldots + (0.072142)(30.0000) - (0)(0.63995)\\ &= 10.7303\\ &\cong 10.73.\end{aligned}$$

Using Appendix G, for $n = 20$ and $r = 0$, where the variances are given in terms of θ_2^2, we obtain

$$\begin{aligned}\mathrm{Var}(\theta_2^*) &= 0.01260 \times \theta_2^{*2} = 0.01260 \times (10.73)^2\\ &= 1.4507\\ &\cong 1.45.\end{aligned}$$

Standard deviation of θ_2^* is $= \sqrt{1.45} = 1.20$. ■

EXAMPLE 5.13

A manufacturer of an automotive speed sensor subjects 10 sensors to a reliability test that simulates the environmental conditions (temperature and speed) at which the sensors will normally operate. A sensor is classified failed when its output falls outside 5% tolerance. The miles accumulated before the failures of the sensors are

110,000, 130,000, 150,000, 155,000, 159,000, 163,000, 166,000, 168,000, 169,000, 170,000.

Assume that the miles to failure follow a Rayleigh distribution with a location parameter = 0. Find an estimate for the parameter θ_2, the mean life of sensor, and the standard deviation of the estimate of θ_2.

SOLUTION

Using Appendix F for n = 10 and r = 0, we get

$$\begin{aligned}\theta_2^* &= (110{,}000)(0.02149) + (130{,}000)(0.03171) + \ldots \\ &\quad + (170{,}000)(0.13149) - (0)(0.65170) \\ &= 105{,}061.18 \\ &\cong 105{,}061 \text{ h.}\end{aligned}$$

The mean life is

$$\text{Mean life} = \theta_2^* \sqrt{\frac{\pi}{2}} = 131{,}674 \text{ mi.}$$

Using Appendix G, for n = 10 and r = 0, where the variances are given in terms of θ_2^2, we find

$$\text{Var}(\theta_2^*) = (0.02537)\theta_2^2.$$

Substituting the estimate θ_2^* for θ_2, we obtain

$$\begin{aligned}\text{Var}(\theta_2^*) &= (0.02537)(105{,}061)^2 \\ &= 2.8 \times 10^8.\end{aligned}$$

The standard deviation of the estimate θ_2^* is 16,733 h. ■

EXAMPLE 5.14

Considering the data in Example 5.13, assume that the six largest values of failure times are censored. The miles accumulated before the failures of the sensors are 110,000, 130,000, 150,000, and 155,000. Estimate the parameter and its standard deviation.

SOLUTION

Using Appendix F for $n = 10$ and $r = 6$ (where r in this Appendix refers to the number of censored observations from the right), we find

$$\begin{aligned}\theta_2^* &= (110{,}000)(0.05301) + (130{,}000)(0.07777) + \\ &\quad (150{,}000)(0.09821) + (155{,}000)(0.90085) \\ &\quad - (0)(1.12984) \\ &= 170{,}304.5.\end{aligned}$$

Using Appendix G for $n = 10$ and $r = 6$, we find

$$\begin{aligned}\mathrm{Var}(\theta_2^*) &= 0.064634\theta^{*2} \\ &= 0.06434 \times (170{,}304)^2 \\ &= 1.866 \times 10^9.\end{aligned}$$

Standard deviation of the estimate θ_2^* is 43,198 mi. ■

5.8.3.2 Confidence Interval Estimate for θ_2^2 for Noncensored Observations

If $t_1 \leq \ldots \leq t_n$ are order statistics from Rayleigh distribution

$$f(t) = \frac{1}{\theta_2^2} t e^{-t^2/2\theta_2^2},$$

then $y_1 = t_1^2/2, y_2 = t_2^2/2, \ldots, y_n = t_n^2/2$ are order statistics from exponential distribution of the form

$$f(y) = \frac{1}{\theta_2^2} e^{-y/\theta_2^2}.$$

Also, $2/\theta_2^2 \sum_{i=1}^n y_i = 1/\theta_2^2 \sum_{i=1}^n t_i^2$ follows a χ^2 distribution with ($2n$) degrees of freedom.

Thus, $100(1-\alpha)$ confidence interval for θ_2^2

$$\frac{1}{\chi^2_{\alpha/2}} \sum_{i=1}^n t_i^2 \leq \theta_2^2 \leq \frac{1}{\chi^2_{1-\alpha/2}} \sum_{i=1}^n t_i^2. \tag{5.21}$$

Also, $100(1-\alpha)$ confidence interval for θ_2 is

$$\sqrt{\frac{1}{\chi^2_{\alpha/2}} \cdot \sum_{i=1}^n t_i^2} \leq \theta_2 \leq \sqrt{\frac{1}{\chi^2_{1-\alpha/2}} \cdot \sum_{i=1}^n t_i^2}, \tag{5.22}$$

where $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$ are respectively the lower and upper $\alpha/2$ points of χ^2 with $2n$ degrees of freedom.

5.8.3.3 *Confidence Interval Estimate for θ_2^2 for Censored Observations*

Suppose that the sample size is n with r largest censored observations. The $100(1-\alpha)$ percent confidence interval for θ_2^2, using $(n-r)$ noncensored observations is

$$\frac{1}{\chi^2_{\alpha/2}}\sum_{i=1}^{(n-r)} t_i^2 \le \theta_2^2 \le \frac{1}{\chi^2_{1-\alpha/2}}\sum_{i=1}^{(n-r)} t_i^2. \tag{5.23}$$

Also $100(1-\alpha)$ confidence interval for θ_2 is

$$\sqrt{\frac{1}{\chi^2_{\alpha/2}}\cdot\sum_{i=1}^{(n-r)} t_i^2} \le \theta_2 \le \sqrt{\frac{1}{\chi^2_{1-\alpha/2}}\cdot\sum_{i=1}^{(n-r)} t_i^2}, \tag{5.24}$$

where $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$ are respectively the lower and upper $\alpha/2$ points of χ^2 distribution with $2(n-r)$ degrees of freedom.

EXAMPLE 5.15

Determine the 95% confidence interval for the BLUE of θ_2^* for the data given in Example 5.13.

SOLUTION

We are using noncensored data in this example, therefore we utilize Equation 5.22:

$$\sum_{i=1}^{10} t_i^2 = 2406.16\times 10^8.$$

From the tables of percentiles of the χ^2 distribution for 20 degrees of freedom, we find

$$\chi^2_{0.025} = 34.17, \quad \chi^2_{0.975} = 9.59.$$

Therefore, a 95% confidence interval for θ_2^* is

$$\sqrt{\frac{1}{34.17}\times 2406.16\times 10^8} \le \theta_2^* \le \sqrt{\frac{1}{9.59}\times 2406.16\times 10^8}$$
$$88{,}181.17 \le \theta_2^* \le 158{,}399.18.$$

■

EXAMPLE 5.16

For the data given in Example 5.14 where the six largest values of failure times are censored, find a 95% confidence interval for the estimate θ_2^*.

SOLUTION

We are using censored data in this example. We utilize Equation 5.24:

$$\sum_{i=1}^{4} t_i^2 = 755.25 \times 10^8.$$

From the tables of percentiles of the χ^2 distribution for eight degrees of freedom, we find

$$\chi^2_{0.025} = 17.53, \quad \chi^2_{0.975} = 2.18.$$

Therefore a 95% confidence interval for θ_2^* is

$$\sqrt{\frac{1}{17.53} \times 755.25 \times 10^8} \le \theta_2^* \le \sqrt{\frac{1}{2.18} \times 755.25 \times 10^8}$$
$$65{,}637.86 \le \theta_2^* \le 186{,}130.31.$$ ■

5.9 THE WEIBULL DISTRIBUTION

In Chapter 1, we presented the p.d.f. of the Weibull distribution as

$$f(t) = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1} e^{-\left(\frac{t}{\theta}\right)^{\gamma}}, \quad t \ge 0, \gamma > 0, \theta > 0, \tag{5.25}$$

where γ and θ are the shape and scale parameters, respectively.

When the failure data are assumed to follow Weibull distribution, the estimated parameters of the distribution, $\hat{\theta}$ and $\hat{\gamma}$, can be obtained by using the MLE procedures proposed by Cohen (1965), Harter and Moore (1965), and discussed in Lee (1980, 1992). This, as with the exponential distribution, is presented for two cases.

5.9.1 Failure Data without Censoring

The exact failure times of n units under test are recorded as $t_1, t_2, \ldots, t_n$. Assume that the failure data follow a Weibull distribution. The likelihood function is

$$L(\gamma, \theta, t) = \left(\frac{\gamma}{\theta^{\gamma}}\right)^n \prod_{i=1}^{n} t_i^{\gamma-1} e^{-\left(\frac{t_i}{\theta}\right)^{\gamma}}. \tag{5.26}$$

Following the same procedure as the exponential and Rayleigh cases, we take the logarithm of Equation 5.26. We then take the derivatives of the logarithmic function with respect to γ and θ. This results in the following two equations:

$$\frac{n}{\hat{\gamma}} - n\ln\hat{\theta} + \sum_{i=1}^{n}\ln t_i - \frac{1}{\hat{\theta}^{\hat{\gamma}}}\sum_{i=1}^{n} t_i^{\hat{\gamma}}(\ln t_i - \ln\hat{\theta}) = 0 \tag{5.27}$$

$$-n + \frac{1}{\hat{\theta}^{\hat{\gamma}}}\sum_{i=1}^{n} t_i^{\hat{\gamma}} = 0. \tag{5.28}$$

The MLE of γ and θ can be obtained by solving Equations 5.27 and 5.28 simultaneously. Substituting $\hat{\theta}$ obtained from Equation 5.28 into Equation 5.27, we obtain a difference $D(\hat{\gamma})$:

$$D(\hat{\gamma}) = \frac{\sum_{i=1}^{n} t_i^{\hat{\gamma}}\ln t_i}{\sum_{i=1}^{n} t_i^{\hat{\gamma}}} - \frac{1}{\hat{\gamma}} - \frac{1}{n}\sum_{i=1}^{n}\ln t_i = 0. \tag{5.29}$$

The above equation in $\hat{\gamma}$ can be solved numerically by using the Newton–Raphson method (described in Appendix E) or by trial and error. Once $\hat{\gamma}$ is estimated, we obtain $\hat{\theta}$ as

$$\hat{\theta} = \left[\sum_{i=1}^{n}\frac{t_i^{\hat{\gamma}}}{n}\right]^{\frac{1}{\hat{\gamma}}}.$$

Similar to the exponential distribution and assuming a large number of failure data, the $100(1-\alpha)$ percent confidence intervals for γ and θ are

$$\hat{\gamma} - Z_{\alpha/2}\sqrt{\mathrm{Var}(\hat{\gamma})} < \gamma < \hat{\gamma} + Z_{\alpha/2}\sqrt{\mathrm{Var}(\hat{\gamma})} \tag{5.30}$$

$$\left[\hat{\theta}_1 - Z_{\alpha/2}\sqrt{\mathrm{Var}\left(\hat{\theta}_1\right)}\right]^{\frac{1}{\hat{\gamma}}} < \theta < \left[\hat{\theta}_1 + Z_{\alpha/2}\sqrt{\mathrm{Var}\left(\hat{\theta}_1\right)}\right]^{\frac{1}{\hat{\gamma}}}, \tag{5.31}$$

where $\hat{\theta}_1 = \hat{\theta}^{\hat{\gamma}}$, $\mathrm{Var}(\hat{\gamma})$, and $\mathrm{Var}(\hat{\theta}_1)$ for large n are obtained as follows.

Define

$$S_0 = \sum_{i=1}^{n} t_i^{\hat{\gamma}}$$

$$S_1 = \sum_{i=1}^{n} t_i^{\hat{\gamma}}(\ln t_i)$$

$$S_2 = \sum_{i=1}^{n} t_i^{\hat{\gamma}}(\ln t_i)^2.$$

Then,

$$\mathrm{Var}(\hat{\gamma}) \cong \frac{\hat{\gamma}^2 S_0^2}{n\left(S_0^2 + \hat{\gamma}^2 S_0 S_2 - \hat{\gamma}^2 S_1^2\right)}$$

$$\mathrm{Var}(\hat{\theta}_1) \cong \frac{S_0}{n^2}\left(\frac{S_0}{\hat{\gamma}^2} + S_2\right)\mathrm{Var}(\hat{\gamma})$$

$$\mathrm{Cov}(\hat{\gamma}, \hat{\theta}_1) \cong \frac{S_1}{n}\mathrm{Var}(\hat{\gamma}).$$

Unbiased estimates of the parameters $\hat{\theta}$, $\hat{\gamma}$, $\mathrm{Var}(\hat{\theta})$ and $\mathrm{Var}(\hat{\gamma})$ are discussed in Section 5.9.3.

EXAMPLE 5.17

Ten diodes are tested to failure at accelerated conditions. The failure times (in minutes) are recorded in Table 5.10. Suppose that the data follow Weibull distribution. Find the parameters $\hat{\gamma}$ and $\hat{\theta}$.

TABLE 5.10 Failure Data of the Diodes

31,000	51,000
36,000	54,500
40,000	54,000
44,000	57,000
50,000	63,000

SOLUTION

The first step is to obtain a good initial value for $\hat{\gamma}$ to be substituted in Equation 5.29. This can be achieved by using the relationship developed by Cohen (1965) between $\hat{\gamma}$ and CV (coefficient of variation), which is the ratio of sample standard deviation and mean. We may also use the following approximation to obtain $\hat{\gamma}$:

$$\hat{\gamma} = \frac{1.05}{\mathrm{CV}}. \tag{5.32}$$

From the data,

$$\bar{t} = \frac{477{,}500}{10} = 47{,}750 \text{ min}.$$

The sample standard deviation, s, is

$$s = 9886$$

and

$$\mathrm{CV} = 0.207.$$

Using Equation 5.32 we obtain an initial value for $\hat{\gamma}$ as 5. Substituting in Equation 5.29,

$$\sum_{i=1}^{10} t_i^{\hat{\gamma}} \ln t_i = 3.74 \times 10^{25}$$
$$\sum_{i=1}^{n} \ln t_i = 107.53$$
$$\sum_{i=1}^{10} t_i^{\hat{\gamma}} = 3.43 \times 10^{24}$$
$$D(5) = -0.04921.$$

We now try $\hat{\gamma} = 2.1$:

$$\sum_{i=1}^{10} t_i^{\hat{\gamma}} \ln t_i = 75.758 \times 10^{10}$$
$$\sum_{i=1}^{10} \ln t_i = 107.53$$
$$\sum_{i=1}^{10} t_i^{\hat{\gamma}} = 6.99 \times 10^{10},$$

and

$$D(2.1) = -0.397345.$$

As $\hat{\gamma}$ decreases, the value of $D(\hat{\gamma})$ decreases and moves further away from zero. Therefore, we try a higher value of $\hat{\gamma}$. We now try $\hat{\gamma} = 3.64$:

$$\sum_{i=1}^{10} t_i^{\hat{\gamma}} \ln t_i = 13.676 \times 10^{18}$$
$$\sum_{i=1}^{10} \ln t_i = 107.53$$
$$\sum_{i=1}^{10} t_i^{\hat{\gamma}} = 1.2575 \times 10^{18}$$
$$D(3.64) = -0.15381.$$

We now try $\hat{\gamma} = 5.907$:

$$\sum_{i=1}^{10} t_i^{\hat{\gamma}} \ln t_i = 74.636 \times 10^{28}$$

$$\sum_{i=1}^{n} \ln t_i = 107.53$$

$$\sum_{i=1}^{10} t_i^{\hat{\gamma}} = 6.836 \times 10^{28}$$

$$D(5.907) = 0.00396145.$$

We now try $\hat{\gamma} = 6.0$:

$$\sum_{i=1}^{10} t_i^{\hat{\gamma}} \ln t_i = 20.519 \times 10^{29}$$

$$\sum_{i=1}^{n} \ln t_i = 107.53$$

$$\sum_{i=1}^{10} t_i^{\hat{\gamma}} = 1.8791 \times 10^{29}$$

$$D(6.0) = 0.0001322.$$

The exact value as obtained by the Newton–Raphson method (see computer listing in Appendix H) is $\hat{\gamma} = 5.99697278$. Thus, using this value of $\hat{\gamma}$, we obtain

$$\sum_{i=1}^{10} t_i^{\hat{\gamma}} \ln t_i = 19.852 \times 10^{29}$$

$$\sum_{i=1}^{10} \ln t_i = 107.53$$

$$\sum_{i=1}^{10} t_i^{\hat{\gamma}} = 1.817998 \times 10^{29}$$

$$D(5.99697278) = 2.69800180 \times 10^{-16}.$$

Thus, an approximate value of $\hat{\gamma}$ is 6 and $\hat{\theta}$ is obtained as

$$\hat{\theta} = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{\hat{\gamma}}\right]^{\frac{1}{\hat{\gamma}}}$$

$$\hat{\theta} = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{\hat{\gamma}}\right]^{\frac{1}{\hat{\gamma}}} = \left[\frac{1}{10} \times 1.8791 \times 10^{29}\right]^{\frac{1}{6}} = (1.8791 \times 10^{28})^{\frac{1}{6}} = 5.1561 \times 10^{4}.$$

Thus,

$$R(t) = e^{-\left(\frac{t}{\theta}\right)^{\gamma}}.$$

The reliability at t = 40,000 h is

$$R(40{,}000) = 0.8041.$$

The MTTF is

$$MTTF = \theta\Gamma\left(1+\frac{1}{\gamma}\right) = 5.1561\times10^{4}\times0.9277 = 4.7835\times10^{4}.$$

■

5.9.2 Failure Data with Censoring

Assume that the units under test are subjected to censoring of Type 1 or Type 2. The failure data can be represented by

$$t_1 \le t_2 \le t_3 \ldots \le t_r = t_{r+1}^{+} = \ldots = t_n^{+}.$$

Suppose that the failure data follow a Weibull distribution. Following Equations 5.27 and 5.28, we obtain

$$\frac{r}{\hat{\gamma}} - r\ln\hat{\theta} + \sum_{i=1}^{r}\ln t_i - \frac{1}{\hat{\theta}^{\hat{\gamma}}}\left[\sum_{i=1}^{r} t_i^{\hat{\gamma}}(\ln t_i - \ln\hat{\theta}) + (n-r)t_r^{\hat{\gamma}}(\ln t_r - \ln\hat{\theta})\right] = 0 \tag{5.33}$$

$$-r + \frac{1}{\hat{\theta}^{\hat{\gamma}}}\left[\sum_{i=1}^{r} t_i^{\hat{\gamma}} + (n-r)t_r^{\hat{\gamma}}\right] = 0. \tag{5.34}$$

Again, substituting $\hat{\theta}$ from Equation 5.34 into Equation 5.33, we obtain $D(\hat{\gamma})$ as

$$D(\hat{\gamma}) = \frac{\sum_{i=1}^{r} t_i^{\hat{\gamma}}\ln t_i + (n-r)t_r^{\hat{\gamma}}\ln t_r}{\sum_{i=1}^{r} t_i^{\hat{\gamma}} + (n-r)t_r^{\hat{\gamma}}} - \frac{1}{r}\sum_{i=1}^{r}\ln t_i - \frac{1}{\hat{\gamma}} = 0. \tag{5.35}$$

Using Equation 5.35, the value of $\hat{\gamma}$ can be estimated by trial and error or by using the Newton–Raphson method. The estimate of $\hat{\theta}$ is

$$\hat{\theta} = \left\{\frac{1}{r}\left[\sum_{i=1}^{r} t_i^{\hat{\gamma}} + (n-r)t_r^{\hat{\gamma}}\right]\right\}^{\frac{1}{\hat{\gamma}}}. \tag{5.36}$$

We now present a procedure for obtaining unbiased estimates of $\hat{\theta}$ and $\hat{\gamma}$.

5.9.3 Variance of the MLE Estimates

Since the MLE cannot be presented in a closed form expression, determining properties of the estimators such as their bias, distribution, and so on is not straightforward. However, Bain and

TABLE 5.11 Asymptotic Value of the Coefficients to Be Used for the Calculations of Variances and Covariances of the MLE for Complete and Censored Sampling

p	c_{11}	c_{22}	c_{12}
1.0	1.108665	0.607927	0.257022
0.9	1.151684	0.767044	0.176413
0.8	1.252617	0.928191	0.049288
0.7	1.447258	1.122447	–0.144825
0.6	1.811959	1.372781	–0.446603
0.5	2.510236	1.716182	–0.937566
0.4	3.933022	2.224740	–1.785525
0.3	7.190427	3.065515	–3.438610
0.2	16.478771	4.738764	–7.375310
0.1	60.517110	9.744662	–22.187207

Engelhardt (1991) address these properties through Monte Carlo simulation. Following the procedure for the construction of the information matrix as presented in Section 4.3.5, the asymptotic variances and covariances of the MLE for complete or censored sampling are obtained as

$$\begin{bmatrix} \mathrm{Var}(\hat{\theta}) & \mathrm{Cov}(\hat{\theta},\hat{\gamma}) \\ \mathrm{Cov}(\hat{\theta},\hat{\gamma}) & \mathrm{Var}(\hat{\gamma}) \end{bmatrix} = \begin{bmatrix} c_{11}\hat{\theta}^2 / n\hat{\gamma}^2 & c_{12}\hat{\theta} / n \\ c_{12}\hat{\theta} / n & c_{22}\hat{\gamma}^2 / n \end{bmatrix}, \tag{5.37}$$

where c_{11}, c_{22}, and c_{12} depend on $p = r/n$ and are shown in Table 5.11.

5.9.4 Unbiased Estimate of $\hat{\gamma}$

The MLE may be used to provide a point estimate of $\hat{\gamma}$, but it is quite biased for a small n, particularly when heavy censoring occurs. Bain and Engelhardt (1991) suggest the use of an *unbiasing factor* G_n. Using this factor, the unbiased estimation of $\hat{\gamma}$ is

$$\hat{\gamma} = G_n \hat{\gamma}_{MLE}. \tag{5.38}$$

Tables for determining G_n are available in the literature. Alternatively, G_n can be computed using the following approximation:

$$G_n = 1.0 - 1.346 / n - 0.8334 / n^2. \tag{5.39}$$

For complete sampling, the asymptotic results for $\mathrm{Var}(\hat{\gamma})$ when $n \to \infty$ is $\mathrm{Var}(\hat{\gamma}) = c_{22}\hat{\gamma}^2 = 0.6079\hat{\gamma}^2$. However, if $n < 100$, instead of using $c_{22} = 0.6079$, a more accurate estimate of $\mathrm{Var}(\hat{\gamma})$ is obtained using C_n. Again, tables for C_n can be found in the literature or, alternatively, C_n can be computed using

$$C_n = 0.617 + \frac{1.8}{n} + \frac{78.25}{n^3} \tag{5.40}$$

and

$$\mathrm{Var}(\hat{\gamma}) = \frac{C_n \hat{\gamma}^2}{n}. \tag{5.41}$$

EXAMPLE 5.18 (Complete Sample)

Ten units are tested until failure. The data (time to failure) are

20, 22, 24, 25, 26, 27, 30, 35, 42, 52.

Fit a Weibull distribution to the data.

SOLUTION

Using Equations 5.27 and 5.28, we obtain the MLE estimates

$$\hat{\gamma} = 3.275$$

and

$$\hat{\theta} = 33.75.$$

The corresponding variances can be calculated using Equation 5.37 and Table 5.11 for $p = r/n = 1$ (no censoring):

$$\mathrm{Var}(\hat{\gamma}) = c_{22}\hat{\gamma}^2 / n = 0.608(3.275)^2 / 10 = 0.6521$$
$$\mathrm{Var}(\hat{\theta}) = c_{11}\hat{\theta}^2 / \hat{\gamma}^2 n = 1.109(33.75)^2 / (3.275^2 \times 10) = 11.78$$
$$\mathrm{Cov}(\hat{\theta}, \gamma) = c_{12}\hat{\theta} \ / n = 0.257(33.75) / 10 = 0.8674.$$

Now, using Equations 5.39 and 5.38 we obtain the unbiased estimate of $\hat{\gamma}$ as

$$G_n = 1.0 - (1.346 / 10) - (0.8334 / 10^2) = 0.857$$
$$\hat{\gamma} = G_n \hat{\gamma}_{MLE} = 0.857 \times 3.275 = 2.81$$

and the corresponding value of $\hat{\theta}$ is

$$\hat{\theta} = \left(\sum_{i=1}^{r} \frac{t_i^{\hat{\gamma}}}{r} \right)^{1/\hat{\gamma}} = 33.01.$$

Also, we can calculate the variance of the unbiased estimate of $\hat{\gamma}$ using Equations 5.40 and 5.41:

$$C_n = 0.617 + (1.8 / 10) + (78.25 / 10^3) = 0.875$$
$$\mathrm{Var}(\hat{\gamma}) = \frac{0.875 \times 2.81^2}{10} = 0.691.$$

■

EXAMPLE 5.19 (Censored Sample)

Thirty units are under test that is terminated after 22 failures occur. The times to failure are

18.5, 20, 20.5, 21.5, 22, 22.5, 23.5, 24, 24.3, 24.6, 25, 25.3, 25.6, 26, 26.3, 26.7, 27, 28, 29, 30, 32, 33.

Fit a Weibull distribution to the data.

SOLUTION

Using Equations 5.27 and 5.28, the MLE estimates are

$$\hat{\gamma} = 5.106$$

and

$$\hat{\theta} = 30.58.$$

The corresponding variances can be calculated using Equation 5.37 and interpolating from Table 5.11 for $p = 22/30 = 0.73$:

$$\begin{aligned}\text{Var}(\hat{\gamma}) &= c_{22}\hat{\gamma}^2 / n = 1.06(5.106)^2 / 30 = 0.921\\ \text{Var}(\hat{\theta}) &= c_{11}\hat{\theta}^2 / (\hat{\gamma}^2 n) = 1.38(30.58)^2 / (5.106^2 \times 30) = 1.65,\end{aligned}$$

and the unbiased estimate of $\hat{\gamma}$ is

$$\begin{aligned}G_n &= 1.0 - (1.346/30) - (0.8334/30^2) = 0.954\\ \hat{\gamma} &= G_n \hat{\gamma}_{MLE} = 4.87.\end{aligned}$$

The corresponding value of $\hat{\theta}$ is

$$\hat{\theta} = \left[\frac{1}{r}\left\{\sum_{i=1}^{r} t_i^{\hat{\gamma}} + (n-r)t_r^{\hat{\gamma}}\right\}\right]^{\frac{1}{\hat{\gamma}}} = 30.59.$$

■

5.9.5 Confidence Interval for $\hat{\gamma}$

Asymptotic results derived by Bain and Engelhardt (1991) indicate that for heavy censoring, $\hat{\gamma}$ approximately follows a chi-squared distribution with $2(r-1)df$ (degrees of freedom), and it follows a chi-squared distribution with $(n-1)df$ when the sample is complete.

In order to take into account this transition, Bain and Engelhardt (1991) suggest the following approximation:

$$df = c(r-1) \tag{5.42}$$

where

$$c = 2/[(1+p^2)^2 p c_{22}].$$

Once the *df* has been calculated using these expressions, the 100(1 − α) percent confidence interval for $\hat{\gamma}$ can be computed using

$$\hat{\gamma}^L = \hat{\gamma}\left[\frac{\chi^2_{(1-\alpha/2),df}}{cr}\right]^{\frac{1}{1+p^2}} \tag{5.43}$$

$$\hat{\gamma}^U = \hat{\gamma}\left[\frac{\chi^2_{\alpha/2,df}}{cr}\right]^{\frac{1}{1+p^2}}. \tag{5.44}$$

The superscripts L and U denote lower and upper limits, respectively.

EXAMPLE 5.20 (Complete Sample)

Find the 90% confidence interval for $\hat{\gamma}$ estimated in Example 5.18.

SOLUTION

Since this is a complete sample, then $\hat{\gamma}$ approximately follows a chi-squared distribution with (n − 1) degrees of freedom

$$\chi^2_{0.95,9} = 3.33 \quad \text{and} \quad \chi^2_{0.05,9} = 16.92$$

$$c = 2/[(1+1^2)^2(1)\times 0.608] = 0.822$$

$$\hat{\gamma}^L = \hat{\gamma}\left[\frac{\chi^2_{0.95,9}}{cr}\right]^{1/(1+p^2)}$$

$$\hat{\gamma}^L = 2.81\left[\frac{3.33}{0.822\times 10}\right]^{1/2} = 1.788$$

$$\hat{\gamma}^U = \hat{\gamma}\left[\frac{\chi^2_{0.05,9}}{cr}\right]^{1/(1+p^2)}$$

$$\hat{\gamma}^U = 2.81\left[\frac{16.92}{0.822\times 10}\right]^{1/2} = 4.032.$$

■

The following example illustrates the procedure for calculating the confidence interval for $\hat{\gamma}$ when some of the failure times are censored.

EXAMPLE 5.21 (Censored Sample)

Find the 90% confidence interval for $\hat{\gamma}$ estimated in Example 5.19.

SOLUTION

$\hat{\gamma}$ approximately follows a chi-squared distribution with *df* given by Equation 5.42:

$$c = 1/[(1+p^2)^2 pc_{22} = 2/[1+0.73^2)^2 \times (0.73) \times 1.06] = 1.10$$
$$df = c(r-1) = 1.10(21) = 23$$
$$\chi^2_{0.95,23} = 13.09 \quad \text{and} \quad \chi^2_{0.05,23} = 35.17.$$

Using Equations 5.43 and 5.44 we obtain the upper and lower limits of the confidence interval for $\hat{\gamma}$ as

$$\hat{\gamma}^L = 4.87\left[\frac{13.09}{(1.10\times 22)}\right]^{1/(1+p^2)} = 3.28$$
$$\hat{\gamma}^U = 4.87\left[\frac{35.17}{(1.10\times 22)}\right]^{1/(1+p^2)} = 6.24.$$

■

5.9.6 Inferences on $\hat{\theta}$

The bias of $\hat{\theta}$ is a function of both θ and γ, and it is not easily assessed. Fortunately, in general, θ is not very biased, and the use of an unbiased $\hat{\gamma}$ in Equation 5.39 provides a reasonable estimate of $\hat{\theta}$.

Confidence intervals for $\hat{\theta}$ can be constructed using the distribution of $U = \sqrt{n}\hat{\gamma}\ln(\hat{\theta}/\theta)$. It can be shown that the $100(1-\alpha)$ percent confidence intervals for $\hat{\theta}$ are

$$\theta^L = \hat{\theta}\exp(-U_{1-\alpha/2}/(\sqrt{n}\cdot\hat{\gamma})) \tag{5.45}$$
$$\theta^U = \hat{\theta}\exp(-U_{\alpha/2}/(\sqrt{n}\cdot\hat{\gamma})). \tag{5.46}$$

Bain and Engelhardt (1991) provide tables with the percentage points U_α such that $p(U \le U_\alpha) = \alpha$. Alternatively, $U_{0.05}$ and $U_{0.95}$ can be computed using the following approximation:

$$U_{0.05} = -1.715 - (3.868/n) - (44.23/\exp(n)) \tag{5.47}$$
$$U_{0.95} = 1.72 + (3.163/n) + (18.25/\exp(n)). \tag{5.48}$$

These expressions hold for complete samples only.

EXAMPLE 5.22 (Complete Sample)

Find the 90% confidence interval for $\hat{\theta}$ estimated in Example 5.18.

SOLUTION

Using Equations 5.47 and 5.48, we calculate the values of the U distribution,

$$U_{0.05} = -1.715 - (3.868/10) - (44.23/\exp(10)) = -2.09$$

and

$$U_{0.95} = 1.72 + (3.163/10) + (18.25/\exp(10)) = 2.04.$$

Now, using Equations 5.45 and 5.46, the lower and upper limits for $\hat{\theta}$ are estimated as

$$\hat{\theta}^L = \hat{\theta}\exp(-U_{0.95}/(\sqrt{n}\times\hat{\gamma}))$$
$$\hat{\theta}^L = 33.01\exp(-2.04/(\sqrt{10}\times 2.81)) = 26.2$$

and

$$\hat{\theta}^U = \hat{\theta}\exp(-U_{0.05}/(\sqrt{n}\times\hat{\gamma}))$$
$$\hat{\theta}^U = 33.01\exp(2.09/(\sqrt{10}\times 2.81)) = 41.8.$$

For censored samples, U_α is a function of $p = r/n$ and n. Some tabulated results for U_α are provided in Bain and Engelhardt (1991). Alternatively, $U_{0.05}$ and $U_{0.95}$ for censored samples can be computed using the following approximations ($p = r/n$):

$$U_{0.05} = -7.72 + 12.99p - 7.02p^2 + \frac{24.83}{n} + \frac{47.72}{n^2} - \frac{26.57}{np} - \frac{66.46}{(np)^2} \tag{5.49}$$

$$U_{0.95} = 4.08 - 4.76p + 2.43p^2 + \frac{11.41}{n} - \frac{9.85}{np} + \frac{10.46}{(np)^2}. \tag{5.50}$$

These expressions are valid for $5 \le n < 120$ and $0.5 \le p \le 1.0$. ■

EXAMPLE 5.23 (Censored Sample)

Find the 90% confidence interval for $\hat{\theta}$ estimated in Example 5.19.

SOLUTION

In Example 5.19, we have $n = 30$ and $p = 22/30 = 0.733$. Using Equations 5.49 and 5.50, we calculate the values of the U distribution as

$$U_{0.05} = -2.44$$
$$U_{0.95} = 1.85.$$

Then, using Equation 5.45 and 5.46, the lower and upper limits for $\hat{\theta}$ are

$$\hat{\theta}^L = \hat{\theta}\exp(-U_{0.95}/(\sqrt{n}\hat{\gamma}))$$
$$\hat{\theta}^L = 26.24\exp(-1.85/(\sqrt{30}\times 4.87)) = 24.5$$
$$\hat{\theta}^U = \hat{\theta}\exp(-U_{0.05}/(\sqrt{n}\hat{\gamma}))$$
$$\hat{\theta}^U = 26.24\exp(+2.44/(\sqrt{30}\times 4.87)) = 28.7.$$

■

Murthy et al. (2004) provide a comprehensive treatment of the Weibull models with variants of the two parameters model given in Chapter 1. They analyze six types (variants of the Weibull models) and describe different procedures for estimating their parameters.

As indicated in Chapter 4, when the sample size is rather small and the failure-time observations are limited, it might be appropriate to utilize the Bayesian approach to obtain estimates of the distribution parameters. Kundu (2008) uses Lindley's (1980) approximation to construct the Bayes's estimates of the Weibull distribution parameters and Markov chain Monte Carlo (MCMC) technique to compute the credible (confidence) intervals of the parameters.

5.10 LOGNORMAL DISTRIBUTION

When the failure times are assumed to follow a lognormal distribution, the p.d.f., $f(t)$, is given by

$$f(t) = \frac{1}{\sigma t\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\ln t-\mu}{\sigma}\right)^2\right] \quad t \geq 0.$$

Let $x = \ln t$, where x is normally distributed with a mean μ and standard deviation σ—that is,

$$E[x] = E[\ln t] = \mu$$
$$V[x] = \mathrm{Var}[\ln t] = \sigma^2.$$

Since $t = e^x$,

$$E[t] = E[e^x] = \int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}\exp\left[x-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]dx$$
$$E[t] = \exp\left[\mu+\frac{\sigma^2}{2}\right]\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}[x-(\mu+\sigma)]^2\right]dx$$
$$E[t] = \exp\left[\mu+\frac{\sigma^2}{2}\right]$$
$$E[t^2] = E[e^{2x}] = \exp\left[2\left(\mu+\sigma^2\right)\right]$$
$$\mathrm{Var}[t] = \left[e^{2\mu+\sigma^2}\right]\left[e^{\sigma^2}-1\right].$$

But

$$F(t)=\int_0^t \frac{1}{\tau\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\ln\tau-\mu}{\sigma}\right)^2\right]d\tau$$

$$F(t)=P(\mathbf{t}\le t)=P\left[z\le\frac{\ln t-\mu}{\sigma}\right]$$

$$R(t)=P[\mathbf{t}>t]=P\left[z>\frac{\ln t-\mu}{\sigma}\right]$$

$$h(t)=\frac{f(t)}{R(t)}=\frac{\phi\left(\frac{\ln t-\mu}{\sigma}\right)}{t\sigma R(t)}.$$

Estimations of the parameters of the lognormal distribution when the failure data are not censored and when the failure data are censored are discussed next.

5.10.1 Failure Data without Censoring

When the failure time, T, follows a lognormal distribution with p.d.f. $f(t)$,

$$f(t)=\frac{1}{t\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}(\ln t-\mu)^2\right] \tag{5.51}$$

with a mean of

$$\exp\left(\mu+\frac{\sigma^2}{2}\right)$$

and a variance of

$$\left[e^{\sigma^2}-1\right]\left[e^{2\mu+\sigma^2}\right],$$

the estimation of the parameters $\hat{\mu}$ and $\hat{\sigma}$ can be obtained directly from Equation 5.51. However, one of the simplest ways to obtain μ and σ^2 with optimum properties is by considering the distribution of $Y=\ln T$.

Assume that $t_1, t_2, \ldots, t_n$ are the exact failure times of n units that are subjected to a test. The MLE of μ and σ^2 of Y are

$$\hat{\mu}=\frac{1}{n}\sum_{i=1}^{n}\ln t_i \tag{5.52}$$

$$\hat{\sigma}^2=\frac{1}{n}\left[\sum_{i=1}^{n}(\ln t_i)^2-\frac{\left(\sum_{i=1}^{n}\ln t_i\right)^2}{n}\right]. \tag{5.53}$$

The estimate of $\hat{\mu}$ is unbiased. However, the estimate of $\hat{\sigma}^2$ is not unbiased. Therefore, to ensure that $\hat{\sigma}^2$ is unbiased, we use

$$s^2 = \hat{\sigma}^2[n/(n-1)],$$

where s^2 is the sample variance.

Obviously, $s^2 \approx \hat{\sigma}^2$ when n is large. The estimates of the mean and variance of T are

$$\exp\left(\hat{\mu}+\hat{\sigma}^2/2\right) \quad \text{and} \quad \left[e^{\hat{\sigma}^2}-1\right]\left[e^{2\hat{\mu}+\hat{\sigma}^2}\right],$$

respectively, and the 100(1 − α) percent confidence interval for μ is obtained as

$$\hat{\mu} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \hat{\mu} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}. \tag{5.54}$$

If σ is unknown or when the sample size is relatively small (n < 25), then we replace it by s and use the Student t-distribution instead. Thus,

$$\hat{\mu} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} < \mu < \hat{\mu} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}. \tag{5.55}$$

Similarly, the 100(1 − α) percent confidence interval for $\hat{\sigma}^2$ [$(n\hat{\sigma}^2/\sigma^2)$ has a chi-square distribution with (n − 1) degrees of freedom] is

$$\frac{n\hat{\sigma}^2}{\chi^2_{\alpha/2,(n-1)}} < \sigma^2 < \frac{n\hat{\sigma}^2}{\chi^2_{1-\alpha/2,(n-1)}}. \tag{5.56}$$

Engineers are normally interested in estimating the MTTF and confidence intervals for components whose failure-time distribution is lognormal. Indeed, this is a main concern in many applications. To obtain a 100(1 − α) percent confidence interval for the MTTF, $\hat{\tau}$, of the lognormal, we let

$$\tau = \mu + \frac{\sigma^2}{2} \tag{5.57}$$

and

$$\hat{\tau} = \hat{\mu} + \frac{n}{(n-1)}\frac{\hat{\sigma}^2}{2}, \tag{5.58}$$

where $\hat{\mu}$ and $\hat{\sigma}$ are obtained from Equations 5.52 and 5.53, respectively. For large samples, $\hat{\tau}$ is approximated by a normal distribution with variance $\sigma^2_{\hat{\tau}}$ as given in (Shapiro and Gross, 1981)

$$\hat{\sigma}^2_{\hat{\tau}} = \mathrm{Var}(\hat{\mu}) + \mathrm{Var}[n\hat{\sigma}^2/(n-1)]/4 \tag{5.59}$$

However, $\mathrm{Var}[n\hat{\sigma}^2/(n-1)] = n^2\sigma^4/(n-1)^3$ and $\mathrm{Var}(\hat{\mu}) = \hat{\sigma}^2/(n-1)$.

Therefore, we rewrite Equation 5.59 as

$$\sigma_{\hat{\tau}}^2 = \frac{\hat{\sigma}^2}{n-1} + \frac{n^2\hat{\sigma}^4}{4(n-1)^3}. \tag{5.60}$$

Once $\hat{\tau}$ and $\hat{\sigma}_{\hat{\tau}}^2$ are obtained using Equations 5.58 and 5.60, respectively, the 100(1 − α) percent confidence interval for MTTF of the population can be determined as

$$\exp\left(\hat{\tau} - Z_{1-\alpha/2}\hat{\sigma}_{\hat{\tau}}\right) < MTTF < \exp\left(\hat{\tau} + Z_{1-\alpha/2}\hat{\sigma}_{\hat{\tau}}\right). \tag{5.61}$$

EXAMPLE 5.24 (Complete Sample)

A production engineer performs a burn-in test on eight Video Display Terminals (VDT). The following failure times (in hours) are recorded:

20, 28, 35, 39, 42, 44, 46, 47.

Suppose that the failure times follow a lognormal distribution. Determine the mean failure time and its standard deviation. What are the 95% confidence intervals for μ and σ^2?

SOLUTION

In order to calculate the parameters of the distribution we construct Table 5.12.

TABLE 5.12 Failure Times of the VDT

t_i	$\ln t_i$	$(\ln t_i)^2$
20	2.995	8.974
28	3.332	11.103
35	3.555	12.640
39	3.663	13.421
42	3.737	13.970
44	3.784	14.320
46	3.828	14.658
47	3.850	14.823
Sum	28.747	103.912

Using Equations 5.52 and 5.53, we obtain

$$\hat{\mu} = \frac{\sum \ln t_i}{n} = \frac{28.747}{8} = 3.593$$

$$\hat{\sigma}^2 = \frac{1}{8}\left[103.912 - \frac{28.747^2}{8}\right] = 0.0766.$$

The mean failure time is

$$\exp\left[\hat{\mu}+\frac{\hat{\sigma}^2}{2}\right]=\exp[3.6313]=37.76 \text{ h.}$$

The standard deviation of the failure time is

$$\sigma=\left\{\left[\exp\left(\hat{\sigma}^2\right)-1\right]\exp\left[2\hat{\mu}+\hat{\sigma}^2\right]\right\}^{1/2}$$
$$\sigma=\{[\exp(0.0766)-1]\exp[2\times 3.593+0.0766]\}^{1/2}$$
$$\sigma=10.654 \text{ h.}$$

We now determine the 95% confidence interval for μ. Since n is less than 25, we estimate the variance σ^2 as

$$s^2=\frac{8\hat{\sigma}^2}{8-1}=0.0876$$
$$\hat{\mu}-t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}<\mu<\hat{\mu}+t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$$
$$3.593-2.365\times\sqrt{0.0876}/\sqrt{8}<\mu<3.593+2.365\times\sqrt{0.0876}/\sqrt{8}$$
$$3.3456<\mu<3.8404.$$

The 95% confidence interval for σ^2 is

$$\frac{n\hat{\sigma}^2}{\chi^2_{\alpha/2,n-1}}<\sigma^2<\frac{n\hat{\sigma}^2}{\chi^2_{1-\alpha/2,n-1}}$$
$$\frac{8\times 0.0766}{16.013}<\sigma^2<\frac{8\times 0.0766}{1.689}$$

or

$$0.0382<\sigma^2<0.3628.$$

The 95% confidence interval for the mean life can be estimated using Equation 5.61. We first estimate $\hat{\tau}$ using Equation 5.58, then we estimate $\sigma^2_{\hat{\tau}}$ using Equation 5.60. Thus,

$$\hat{\tau}=3.593+\frac{7}{8}\times\frac{0.0766}{2}=3.62651$$
$$\hat{\sigma}^2_{\hat{\tau}}=\frac{0.0766}{7}+\frac{8^2\times 0.0766^2}{4(8-1)^3}=0.0112165.$$

The 95% confidence interval for the MTTF is

$$e^{(3.62651-1.96\times 0.0112165)}<MTTF<e^{(3.62651+1.96\times 0.0112165)}$$
$$36.7612<MTTF<38.416.$$

■

5.10.2 Failure Data with Censoring

Consider the placement of n units under test and the exact failure times of r units are

$$t_1 \leqslant t_2 \leqslant \ldots \leqslant t_r.$$

Since the test is censored after the occurrence of the rth failure or at time T_c, we can assume r failures occurred within T_c. Thus, we have either Type 2 or Type 1 censoring. As we discussed earlier, we use the fact that $Y = \ln T$ has a normal distribution with mean μ and variance σ^2. We can estimate μ and σ^2 from the transformed data $y_i = \ln t_i$. We use the method of Sarhan and Greenberg (1956, 1957, 1958, 1962) to estimate μ and σ^2. They propose that the best estimates are linear combinations of the logarithms of the r exact failure times

$$\hat{\mu} = \sum_{i=1}^{r} a_i \ln t_i \tag{5.62}$$

and

$$\hat{\sigma} = \sum_{i=1}^{r} b_i \ln t_i, \tag{5.63}$$

where a_i and b_i are given by Sarhan and Greenberg for $n \leq 20$ as well as the variance and covariance of $\hat{\mu}$ and $\hat{\sigma}$.

If the sample size is greater than 20, the MLEs for normal distribution can be utilized in estimating the parameters of the lognormal (with censoring) as shown below:

$$\bar{y} = \frac{1}{r}\sum_{i=1}^{r} \ln t_i \tag{5.64}$$

$$s^2 = \frac{1}{r}\left[\sum_{i=1}^{r} (\ln t_i)^2 - \frac{1}{r}\left(\sum_{i=1}^{r} \ln t_i\right)^2\right], \tag{5.65}$$

and the MLEs of $\hat{\mu}$ and $\hat{\sigma}^2$ are

$$\hat{\mu} = \bar{y} - \lambda(\bar{y} - \ln t_r) \tag{5.66}$$

$$\hat{\sigma}^2 = s^2 + \lambda(\bar{y} - \ln t_r)^2. \tag{5.67}$$

The coefficient λ is a function of α and β (Cohen, 1961), where

$$\alpha = s^2/(\bar{y} - \ln t_r)^2 \tag{5.68}$$

$$\beta = (n - r)/n. \tag{5.69}$$

As shown in Equation 5.69, 100β is the percentage of censored units. Cohen (1961) provides tabulated results for λ as a function of α and β. Alternatively, λ can be calculated using the following approximation:

$$\lambda = [1.13\alpha^3 - \ln(1-\alpha)][1 + 0.437\beta - 0.25\alpha\beta^{1.3}] + 0.08\alpha(1-\alpha). \tag{5.70}$$

Equation 5.70 provides a good approximation of λ with a maximum error of 5%.

The asymptotic variances of $\hat{\mu}$ and $\hat{\sigma}$ can be estimated as

$$\text{Var}(\hat{\mu}) = m_1\hat{\sigma}^2 / n \tag{5.71}$$

$$\text{Var}(\hat{\sigma}) = m_2\hat{\sigma}^2 / n. \tag{5.72}$$

Cohen also provides tabulated values of m_1 and m_2 as a function of $\hat{c}$, where $\hat{c} = (\ln t_r - \hat{\mu})/\hat{\sigma}$. Alternatively, m_1 and m_2 can be calculated using the following approximation:

For $y < 0$,

$$m_1 = 1 + 0.51e^{2.5y} \tag{5.73}$$

$$m_2 = 0.5 + 0.74e^{1.6y}. \tag{5.74}$$

For $y > 0$,

$$m_1 = 0.52 + e^{(1.838y + 0.354y^2)} - 0.391y - 0.676y^2 \tag{5.75}$$

$$m_2 = 0.24 + e^{(y + 0.384y^2)} + 0.0507y + 0.2735y^2, \tag{5.76}$$

where $y = -\hat{c}$.

EXAMPLE 5.25 (Censored Sample)

Ten units are subjected to a fatigue test. The test is terminated when seven units fail and their failure times (in weeks) are

30, 37, 42, 45, 47, 48, 50.

Suppose that the failure times follow a lognormal distribution. Determine $\hat{\mu}$ and $\hat{\sigma}$.

SOLUTION

In this case n = 10, r = 7, and $n - r$ = 3. Using Equations 5.62 and 5.63, and Appendix I, we obtain

$$\begin{aligned}\hat{\mu} &= 0.0244\ln 30 + 0.0636\ln 37 + 0.0818\ln 42 + 0.0962\ln 45\\ &\quad + 0.1089\ln 47 + 0.1207\ln 48 + 0.5045\ln 50 = 3.8447\\ \hat{\sigma} &= -0.3252\ln 30 - 0.1758\ln 37 - 0.1058\ln 42 - 0.0502\ln 45\\ &\quad - 0.0006\ln 47 + 0.0469\ln 48 + 0.6107\ln 50 = 0.2409.\end{aligned}$$

The estimated mean failure time is

$$\mu = \exp[\hat{\mu} + \hat{\sigma}^2/2] = \exp\left[3.8447 + \left(\frac{0.2409^2}{2}\right)\right]$$
$$\mu = 48.12 \text{ weeks.}$$

And its estimated standard deviation is

$$\sigma = \left[\left(e^{\hat{\sigma}^2} - 1\right)\left(e^{2\hat{\mu} + \hat{\sigma}^2}\right)\right]^{1/2}$$
$$\sigma = \left[\left(e^{0.2409^2} - 1\right)\left(e^{2\times 3.8447 + 0.2409^2}\right)\right]^{1/2}$$
$$\sigma = [0.05975 \times 2315.6201]^{1/2} = 11.76.$$

■

5.11 THE GAMMA DISTRIBUTION

The Gamma hazard model is useful in estimating the reliability of many practical situations such as in the case of repeated shocks on equipment subject to impact testing. The gamma hazard model can also be used to analyze test results of cell phones during its design and development stage as they are subject to repeated drops from different heights. The two-parameter Gamma density is given by

$$f(t;\theta,\gamma) = \frac{t^{\gamma-1}}{\theta^{\gamma}\Gamma(\gamma)} e^{-\frac{t}{\theta}}, \tag{5.77}$$

where $\Gamma(x)$ denotes the Gamma function, γ is the shape parameter, and $\theta = 1/\lambda$ (λ is sometimes referred to as the scale parameter). It is worth noting that the chi-square distribution is a special case of the Gamma distribution when $\theta = 2$ and $\gamma = v/2$ (where v is an integer and is also the number of degrees of freedom of the chi-square distribution). Also the exponential distribution is a special case of Gamma distribution, when $\gamma = 1$. The value of $\Gamma(x)$ can be found in Appendix A.

5.11.1 Failure Data without Censoring

We consider the case where n units are subjected to a reliability test and the failure times of all units are recorded. We assume that the failure times are expressed by a two-parameter Gamma distribution, and its p.d.f. is given by Equation 5.77. To estimate γ and θ we use either the MMs or the maximum likelihood estimation. However, the maximum likelihood estimation provides a more accurate result and will now be considered.

The likelihood function for a complete sample is

$$L(\theta, \gamma) = \frac{1}{\theta^{n\gamma}\Gamma^{n}(\gamma)}\left[\prod_{i=1}^{n} t_i\right]^{(\gamma-1)} e^{-\sum_{i=1}^{n} t_i/\theta^{n\gamma}}. \tag{5.78}$$

Taking partial derivatives of the logarithm of Equation 5.78 with respect to θ and γ and equating the resultant equations to zero we obtain

$$\hat{\theta} = \bar{t}/\hat{\gamma} \tag{5.79}$$

$$\ln\hat{\gamma} - \psi(\hat{\gamma}) - \ln\bar{t} + \ln\tilde{t} = 0, \tag{5.80}$$

where

$$\bar{t} = \text{the arithmetic mean, } \bar{t} = \frac{\sum_{i=1}^{n} t_i}{n}$$

$$\tilde{t} = \text{the geometic mean, } \tilde{t} = \left(\prod_{i=1}^{n} t_i\right)^{1/n},$$

and the ψ function is defined as $\psi(x) = \Gamma'(x)/\Gamma(x)$, where $\Gamma'(\hat{\gamma})$ is the derivative of $\Gamma(\hat{\gamma})$ or

$$\Gamma'(\hat{\gamma}) = \int_0^{\infty} x^{\hat{\gamma}-1} \ln x\, e^{-x} dx.$$

Using ψ function tables, Equation 5.80 can be solved iteratively. However, an easier approach is to use the approximation suggested by Greenwood and Durand (1960)

$$\hat{\gamma} = \left(0.50009 + 0.16488M - 0.054427M^2\right)/M; \quad 0 < M \le 0.5772 \tag{5.81a}$$

$$\hat{\gamma} = \frac{8.8989 + 9.0599M + 0.97754M^2}{M\left(17.797 + 11.968M + M^2\right)}; \quad 0.5772 < M \le 17 \tag{5.81b}$$

$$\hat{\gamma} = 1/M; \quad M > 17, \tag{5.81c}$$

where $M = \ln(\bar{t}/\tilde{t})$. This approximation provides good estimates of the parameters obtained by the MLE method. Once we estimate $\hat{\gamma}$ we can easily estimate the value $\hat{\theta}$ using Equation 5.79.

As usual, for small n, the estimates of the parameters obtained using the MLE method are noticeably biased. The following expressions are suggested to provide unbiased estimates of γ and θ

$$\hat{\gamma}_{\text{unbiased}} = \hat{\gamma}\frac{(n-3)}{n} + \frac{2}{3n} \tag{5.82}$$

$$\hat{\theta}_{\text{unbiased}} = \frac{\bar{t}}{\hat{\gamma}_{\text{unbiased}}\left[1 - 1/\left(\hat{\gamma}_{\text{unbiased}} n\right)\right]}. \tag{5.83}$$

Equation 5.82 is based on Bain and Engelhardt (1991), while Equation 5.83 is based on Lee (1992).

EXAMPLE 5.26 (Complete Sample)

A mechanical engineer conducts a fatigue test by subjecting 10 identical steel rods to a stress level significantly higher than the endurance limit of the rod material. The numbers of cycles to failure are recorded as

20,000, 35,000, 47,000, 58,000, 68,000, 77,000, 85,000, 92,000, 97,000, 102,000.

Assume that the cycles to failure follow gamma distribution. What are the parameters of the distribution?

SOLUTION

We calculate $\bar{t}$, $\tilde{t}$, and M,

$$\bar{t} = \frac{\sum_{i=1}^{10} t_i}{10} = 68{,}100$$

$$\tilde{t} = \left[\prod_{i=1}^{10} t_i\right]^{\frac{1}{10}} = 61{,}492.22$$

$$M = \ln\left(\frac{\bar{t}}{\tilde{t}}\right) = 0.102.$$

Using Equation 5.81a, we obtain $\hat{\gamma}$ as

$$\hat{\gamma} = \frac{1}{0.102}\left(0.50009 + 0.16488 \times 0.102 - 0.054427 \times 0.102^2\right)$$

$$\hat{\gamma} = 5.0588.$$

The unbiased estimates of $\hat{\gamma}$ and $\hat{\theta}$ are

$$\hat{\gamma}_{\text{unbiased}} = 5.0588 \times \frac{7}{10} + \frac{2}{30}$$

$$\hat{\gamma}_{\text{unbiased}} = 3.60$$

$$\hat{\theta}_{\text{unbiased}} = \frac{68{,}100}{3.60\left(1 - \frac{1}{3.6 \times 10}\right)} = 19{,}457.$$

The mean life = $\hat{\theta}_{\text{unbiased}}\hat{\gamma}_{\text{unbiased}} = 70{,}045$ cycles. ■

5.11.2 Failure Data with Censoring

When there are censored observations in the failure data, the estimation of the parameters becomes considerably more difficult. Wilk et al. (1962a, 1992b) and Bain and Engelhardt (1991) provide tables to aid in computing $\hat{\gamma}$ and $\hat{\theta}$. Alternatively, an approximation can be obtained using the following algorithm.

1. Compute the arithmetic and geometric mean of the available observations in the censored sample (n is the total sample size and r is the number of failed units):

$$\bar{t}_c = \frac{\sum_{i=1}^{r} t_i}{r} \tag{5.84}$$

$$\tilde{t}_c = \left(\prod_{i=1}^{r} t_i\right)^{\frac{1}{r}}. \tag{5.85}$$

2. Compute NR, S, and Q as

$$NR = \frac{n}{r}$$

$$S = \frac{\bar{t}_c}{t_r}$$

$$Q = \frac{1}{\left(1 - \frac{\tilde{t}_c}{\bar{t}_c}\right)}.$$

3. Finally, compute $\hat{\gamma}_{\text{unbiased}}$ as a function of NR, S, and Q:
   - If $S < 0.42$, use

$$\begin{aligned}\hat{\gamma}_{\text{unbiased}} &= 1.061\left(1 - \sqrt{Q}\right) + 0.2522Q(1 + (\sqrt{S} / NR^4)) \\ &\quad + 1.953\left(\sqrt{S} - 1/Q\right) - 0.220 / NR^4 + 0.1308Q / NR^4 + 0.4292 / \left(Q\sqrt{S}\right).\end{aligned} \tag{5.86}$$

   - When $0.42 < S < 0.80$, use

$$\begin{aligned}\hat{\gamma}_{\text{unbiased}} &= 0.5311Q\left((1 / NR^2) - 1\right) + 1.436\log Q + 0.7536(QS - S) \\ &\quad - 2.040 / NR - 0.260QS / NR^2 + 2.489 / (Q/NR)^{1/2}.\end{aligned} \tag{5.87}$$

- When $S > 0.80$, use

$$\hat{\gamma}_{\text{unbiased}} = 1.151 + 1.448(Q(1-S)/NR) - 1.024(Q+S) + 0.5311\log Q + 1.541QS - 0.515(Q/NR)^{1/2}. \tag{5.88}$$

Once $\hat{\gamma}$ is estimated, then $\hat{\theta}$ can be estimated using the following expression:

$$\hat{\theta} = \frac{\left(\sum_{i=1}^{r} t_i + (n-r)t_r\right)/n}{\hat{\gamma}_{\text{unbiased}}\left[1 - 1/(\hat{\gamma}_{\text{unbiased}} r)\right]}.$$

The expressions for estimating $\hat{\gamma}$ in step 3 provide good approximations when $S \geq 0.12$, $1.2 \leq Q \leq 12$ and $NR \leq 3.0$. The accuracy of the estimation was not verified outside these limits. The standard error is approximately 0.04. For small values of $\hat{\gamma}(\hat{\gamma} < 1)$, the maximum percentile error can be large, about 10%. For large values of $\hat{\gamma}(\hat{\gamma} < 2)$, the maximum percentile error is less than 5%.

EXAMPLE 5.27 (Censored Sample)

Ten components are subjected to a test. Seven of the components fail during the test, and three components survive the predetermined test period of 4900 h. The failure times are

1000, 3000, 4000, 4400, 4700, 4800, 4900, 4900^+, 4900^+, 4900^+.

Fit a gamma distribution to these data points and estimate its parameters. What is the mean life of a component from this population?

SOLUTION

We calculate $\bar{t}_c$, $\tilde{t}_c$, NR, S, and Q

$$\bar{t}_c = \frac{\sum_{i=1}^{7} t_i}{7} = 3829$$

$$\tilde{t}_c = \left(\prod_{i=1}^{7} t_i\right)^{\frac{1}{7}} = 3452$$

$$NR = \frac{10}{7}$$

$$S = \frac{\bar{t}_c}{t_r} = \frac{3829}{4900} = 0.7814$$

$$Q = \frac{1}{1 - 0.9015} = 10.15$$

Since $0.42 < S < 0.8$, then we use Equation 5.87 to obtain the unbiased estimate of $\hat{\gamma}$,

$$\hat{\gamma}_{\text{unbiased}} = 0.5311 \times 10.15\left(\frac{1}{1.428^2} - 1\right) + 1.436 \log 10.15 + 0.7536(10.15 \times 0.7814 - 0.7814) - \frac{2.040}{1.428} - \frac{0.260 \times 10.15 \times 0.7814}{1.428^2} + \frac{2.489}{\sqrt{10.15/1.428}} = 2.58$$

and

$$\hat{\theta}_{\text{unbiased}} = \frac{\left(\sum_{i=1}^{7} t_i + 3 \times 4900\right)/10}{2.85 \times 0.91445} = 1759$$

The mean life of a component from this population is $\hat{\gamma}_{\text{unbiased}}\hat{\theta}_{\text{unbiased}} = 4{,}538$ h. ■

5.11.3 Variance of $\hat{\gamma}$ and $\hat{\theta}$

For the case of a complete sample, the variances of $\hat{\gamma}$ and $\hat{\theta}$ are functions of γ itself and n (sample size). Bain and Engelhardt (1991) provide a table with coefficients (C_γ and C_θ) that permit estimates of the corresponding variances as

$$\text{Var}(\hat{\gamma}) = C_\gamma \hat{\gamma}^2 / n \tag{5.89}$$

$$\text{Var}(\hat{\theta}) = C_\theta \hat{\theta}^2 / n. \tag{5.90}$$

Alternatively, these coefficients can be calculated using the following approximate expressions

$$C_\gamma = 2.076A - 0.3697A^2 + 0.01654A^3 + 5.463B - 0.3917B^2 - 7.274\sqrt{B} + 0.0006823BA^4 \tag{5.91}$$

$$C_\theta = 1.976 + \frac{0.608}{(\gamma_{\text{unbiased}})^{1.2}} - \frac{1.8942}{n}, \tag{5.92}$$

where

$$A = 8 - (1/\hat{\gamma}_{\text{unbiased}})$$
$$B = n/(n-6).$$

For the case of censored samples, the variances of $\hat{\gamma}$ and $\hat{\theta}$ also depend on $p = r/n$. Asymptotic results ($n \to \infty$, $r/n \to p$) provided by Harter (1969) indicate that C_γ and C_θ, and thus $\text{Var}(\hat{\gamma})$ and $\text{Var}(\hat{\theta})$, are always larger when there is censoring (as would be expected).

TABLE 5.13 Coefficients C_1 and C_2

	C_1			C_2		
p	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$
1.00	1	1	1	1	1	1
0.75	1.293	1.343	1.370	1.691	1.600	1.573
0.50	1.806	1.944	2.027	3.337	2.921	2.799
0.25	3.237	3.592	3.843	10.069	7.696	7.040

Thus, for censored sample, $\mathrm{Var}(\hat{\gamma})$ and $\mathrm{Var}(\hat{\theta})$ can be calculated using

$$\mathrm{Var}(\hat{\gamma}) = C_1 C_\gamma \hat{\gamma}^2 / n \tag{5.93}$$

$$\mathrm{Var}(\hat{\theta}) = C_2 C_\theta \hat{\theta}^2 / n, \tag{5.94}$$

where C_1 and C_2 are coefficients greater than one. Based on limited results provided by Harter (1969), Table 5.13 presents results for the asymptotic case—that is, the case where $n \to \infty$. The results can also be used to obtain approximations for $\mathrm{Var}(\hat{\gamma})$ and $\mathrm{Var}(\hat{\theta})$ when n is small.

Alternatively, C_1 and C_2 can be calculated using the following (approximate) expressions:

$$C_1 = 1 + 0.2942(1-p) + 0.5744\frac{(1-p)}{p} + 0.1021\frac{(1-p)}{p}\hat{\gamma}_{\text{unbiased}} \tag{5.95}$$

$$C_2 = 1 + 2.848(1-p) - 6.736(1-p)^2 + 14.49(1-p)^3 + 0.3832\frac{(1-p)}{\hat{\gamma}_{\text{unbiased}}p^2}. \tag{5.96}$$

Equations 5.91 and 5.92 are valid for $\hat{\gamma} > 0.2$ and $n > 8$, while Equations 5.95 and 5.96 are valid for $0.5 < \hat{\gamma} < 5$ and $p > 0.25$. For all these expressions, the maximum error of estimate is approximately 2%.

5.11.4 Confidence Intervals for γ

For large γ and a complete sample, a $100(1 - \alpha)$ percent confidence interval is given by (Bain and Engelhardt, 1991)

$$\gamma^L = \frac{\chi^2_{\alpha/2}(n-1)}{2nS} \quad \text{and} \quad \gamma^U = \frac{\chi^2_{1-\alpha/2}(n-1)}{2nS}, \tag{5.97}$$

where $S = \ln(\bar{t}/\tilde{t})$. The number of degrees of freedom approaches $2(n - 1)$ as γ approaches 0, and it approaches $n - 1$ as γ approaches ∞. In other words, for small values of γ, one may use $df = 2(n - 1)$. Otherwise, $n - 1$ degrees of freedom should be used instead.

For small values of $\hat{\gamma}$, the *df* and also the denominator of the expressions given by Equation 5.97 must be corrected. The correction is done by following an iterative procedure. For example, in order to find γ^L, begin with $\gamma^L = \hat{\gamma}_{\text{unbiased}}$ and then

1. Calculate

$$c = c(\gamma^L, n) = \frac{n\phi_1(\gamma^L) - \phi_1(n\gamma^L)}{n\phi_2(\gamma^L) - \phi_2(n\gamma^L)} \tag{5.98}$$

$$v = v(\gamma^L, n) = [n\phi_1(\gamma^L) - \phi_1(n\gamma^L)]c, \tag{5.99}$$

where

$$\phi_1(x) = 1 + 1/(1+6x) \tag{5.100}$$

$$\phi_2(x) = \begin{cases} 1 + 1/(1+2.5x) & 0 < x < 2 \\ 1 + 1/(3x) & 2 \le x < \infty. \end{cases} \tag{5.101}$$

2. Update γ^L using

$$\gamma^L = \frac{\chi^2_{1-\alpha/2}(v)}{2ncS}. \tag{5.102}$$

3. Return to Step 1.

These calculations are continued until convergence is obtained. Likewise, γ^U can be estimated by replacing $\chi^2_{1-\alpha/2}$ by $\chi^2_{\alpha/2}$ and following the same procedure.

In the case of censored sample, the same procedure can be applied, but $\bar{t}$ and $\tilde{t}$ must be replaced by A_r and G_r, respectively, where

$$A_r = \frac{1}{n}\left[\sum_{i=1}^{r} t_i + (n-r)t_r\right] \tag{5.103}$$

$$G_r = \left[\left(\prod_{i=1}^{r} t_i\right)\left(t_r^{n-r}\right)\right]^{1/n}. \tag{5.104}$$

The correction provided by Equations 5.98 through 5.102 is necessary because it can be shown that while for large γ,

$$2n\hat{\gamma}S \simeq \chi^2(n-1),$$

for small γ (i.e., where $\gamma \to 0$),

$$2n\hat{\gamma}S \simeq \chi^2(2(n-1)).$$

5.12 THE EXTREME VALUE DISTRIBUTION

The extreme value distribution is useful in modeling the reliability of components that experience significant wearout—that is, highly increasing failure rate. The extreme value distribution is derived from the Weibull distribution as follows.

The p.d.f. of the Weibull distribution is

$$f(t)=\frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1}\exp\left[-\left(\frac{t}{\theta}\right)^{\gamma}\right] \quad t\geq 0, \gamma\geq 1, \theta>0,$$

where γ and θ are the shape and scale parameters of the distribution, respectively. The reliability function of the Weibull distribution is

$$R(t)=e^{-\left(\frac{t}{\theta}\right)^{\gamma}} \quad t\geq 0.$$

Assume n units are subjected to a reliability test that is terminated after $r(r \leq n)$ failures. This is Type 2 censoring. The failure times of the r units follow a Weibull distribution and their values are

$$t_1 \leq t_2 \leq t_3 \ldots \leq t_r \leq t_{r+1}^{+} = t_{r+2}^{+} = \ldots = t_n^{+}.$$

Let $x_1 \leq x_2 \leq x_3 \ldots \leq x_r \leq x_{r+1}^{+} \leq x_{r+2}^{+} = \ldots = x_n^{+}$ be the corresponding extreme value lifetime, where $x_i = \ln t_i$ ($i = 1, 2, \ldots, r$) and $x_i^{+} = \ln t_i^{+}\ (i = r, r+1, \ldots, n)$.

The p.d.f. and the reliability function of the extreme value distribution are

$$f(x,a,b)=\frac{1}{b}e^{\left(\frac{x-a}{b}\right)}e^{-e^{\left(\frac{x-a}{b}\right)}} \quad -\infty<x<\infty,$$

and

$$R(x,a,b)=e^{-e^{\left(\frac{x-a}{b}\right)}} \quad -\infty<x<\infty,$$

where $a = 1/\gamma \log\theta$ and $b = 1/\gamma$ are the location and scale parameters of the extreme value distribution, respectively.

Following the derivations in Section 5.8.2, the likelihood function for complete or Type 2 censored lifetimes is obtained as

$$L(a,b)=\prod_{i=1}^{r} f(x_i,a,b)\prod_{i=1}^{n-r} R(x_i^{+},a,b)$$

or

$$L(a,b)=\frac{1}{b^r}\exp\left[\sum_{i=1}^{r}\left(\frac{x_i-a}{b}\right)-\sum_{i=1}^{r}e^{\left(\frac{x_i-a}{b}\right)}-(n-r)e^{\left(\frac{x_{r+1}^{+}-a}{b}\right)}\right].$$

The logarithm of the likelihood function is

$$l(a,b) = -r\ln b + \sum_{i=1}^{r}\left(\frac{x_i - a}{b}\right) - \sum_{i=1}^{r} e^{\left(\frac{x_i - a}{b}\right)} - (n-r)e^{\left(\frac{x_{r+1}^{+} - a}{b}\right)}. \tag{5.105}$$

The maximum likelihood estimates of a and b are obtained by taking the derivatives of Equation 5.105 with respect to a and b and equating the resulting equations to zero, and then solving them simultaneously. These two equations are

$$\frac{\partial l(a,b)}{\partial a} = \frac{-r}{b} + \frac{1}{b}\sum_{i=1}^{r} e^{\left(\frac{x_i - a}{b}\right)} + \frac{n-r}{b} e^{\left(\frac{x_{r+1}^{+} - a}{b}\right)} = 0 \tag{5.106a}$$

$$\begin{aligned}\frac{\partial l(a,b)}{\partial b} &= \frac{-r}{b} - \frac{1}{b}\sum_{i=1}^{r}\left(\frac{x_i - a}{b}\right) + \frac{1}{b}\sum_{i=1}^{r}\left(\frac{x_i - a}{b}\right) e^{\left(\frac{x_i - a}{b}\right)} \\ &\quad + \left(\frac{n-r}{b}\right)\left(\frac{x_{r+1}^{+} - a}{b}\right) e^{\left(\frac{x_{r+1}^{+} - a}{b}\right)}.\end{aligned} \tag{5.106b}$$

Solving Equation 5.106a for $\hat{a}$ in terms of $\hat{b}$ (Leemis, 1995), we obtain

$$\hat{a} = \hat{b}\ln\left[\frac{1}{r}\sum_{i=1}^{r} e^{x_i/\hat{b}} + \left(\frac{n-r}{r}\right) e^{x_{r+1}^{+}/\hat{b}}\right]. \tag{5.107}$$

Substituting Equation 5.107 into Equation 5.106b, we obtain

$$-\hat{b} - \frac{1}{r}\sum_{i=1}^{r} x_i + \frac{\sum_{i=1}^{r} x_i e^{x_i/\hat{b}} + (n-r)x_{r+1}^{+} e^{x_{r+1}^{+}/\hat{b}}}{\sum_{i=1}^{r} e^{x_i/\hat{b}} + (n-r)e^{x_{r+1}^{+}/\hat{b}}} = 0. \tag{5.108}$$

Solving Equation 5.108 results in the maximum likelihood estimate of $\hat{b}$. Approximate estimates of variances of $\hat{a}$ and $\hat{b}$ and their covariance ($\mathrm{Cov}(\hat{a}, \hat{b})$) are given in Balakrishnan and Varadan (1991).

EXAMPLE 5.28

Manufacturers of flight data recorders conduct reliability testing by subjecting the recorders to extremely high impacts, pressures, and temperatures, and to corrosive fluids. The last test is performed by completely immersing the recorder for 48 h in each of the several different fluids found on an airplane, including hydraulic oil, jet fuel, and de-icing fluid. The recorder is also dipped in fire fighting fluid, such as Halon foam, for 8 h (O'Connor, 1995).

Thirteen recorders are subjected to the corrosive fluid test, which is terminated at the time of the tenth failure. The times to failure are

2.25, 5.6, 8.9, 10.6, 13.8, 13.9, 15.7, 17.4, 25.3, 30.5 h.

Assuming that the data follow a Weibull distribution, obtain the parameters of the corresponding extreme value distribution. What is the reliability of a recorder immersed in such fluids for 20 h?

SOLUTION

In order to obtain estimates of the parameters of the extreme value distribution, we transform the failure-time data by taking the logarithms of the observations

0.812, 1.723, 2.186, 2.361, 2.625, 2.632, 2.754, 2.856, 3.231, 3.418.

We also have

$$n = 13, r = 10, n - r = 3.$$

Substituting in Equation 5.108, we obtain

$$-\hat{b} - \frac{1}{10} \times 24.598 + \frac{\sum_{i=1}^{10} x_i e^{x_i/\hat{b}} + 3 \times 3.418 e^{3.418/\hat{b}}}{\sum_{i=1}^{10} e^{x_i/\hat{b}} + 3e^{3.418/\hat{b}}} = 0.$$

Using the Newton–Raphson method, we solve the above equation and obtain $\hat{b} = 0.692$. Substituting $\hat{b} = 0.692$ into Equation 5.107, we obtain $\hat{a} = 3.143$. The reliability of a recorder immersed in the fluids for 20 h is

$$R(20) = e^{-e^{\left(\frac{2.996-3.143}{0.692}\right)}} = 0.446. \quad \blacksquare$$

5.13 THE HALF-LOGISTIC DISTRIBUTION

The half-logistic distribution is commonly used in modeling the failure times of components that exhibit increasing failure rates. A unique characteristic of the standard half-logistic distribution is that its hazard rate is a monotonically increasing function of $x(x = (t - \mu)/\sigma)$ and tends to 1 as $x \to \infty$, where t is the failure time and μ and σ are the parameters of the distribution. The p.d.f. and the cumulative distribution function of the half-logistic distribution, respectively, are given by

$$g(y;\mu,\sigma)=\frac{2\exp\left(-\frac{y-\mu}{\sigma}\right)}{\sigma\left[1+\exp\left(-\frac{y-\mu}{\sigma}\right)\right]^2} \tag{5.109}$$

$$G(y;\mu,\sigma)=\frac{1-\exp\left(-\frac{y-\mu}{\sigma}\right)}{1+\exp\left(-\frac{y-\mu}{\sigma}\right)} \quad y\geq\mu,\sigma>0. \tag{5.110}$$

The above half-logistic distribution can be transformed into a standard half-logistic distribution by letting the random variable $X = (Y - \mu)/\sigma$. Thus, the p.d.f. and cumulative distribution function of the standard half-logistic distribution are

$$f(x)=\frac{2e^{-x}}{\left(1+e^{-x}\right)^2} \tag{5.111}$$

$$F(x)=\frac{1-e^{-x}}{1+e^{-x}} \quad 0\leq x<\infty. \tag{5.112}$$

The reliability and the hazard functions are

$$R(x)=\frac{2e^{-x}}{1+e^{-x}} \tag{5.113}$$

and

$$h(t)=\frac{f(x)}{R(x)}=\frac{1}{1+e^{-x}}. \tag{5.114}$$

Figures 5.1 and 5.2 show $f(x)$ and $h(x)$, respectively, for different values of μ and σ.

The *rth* moment of X can be found by direct integration

$$E[X^r]=2\int_0^\infty \frac{x^r e^{-x}}{\left(1+e^{-x}\right)^2}dx$$

$$E(X^r)=2(r!)\sum_{i=1}^{\infty}(-1)^{i-1}i^{-r}.$$

The mean of the distribution is

$$E[X]=\ln 4.$$

The variance is obtained as

$$\mathrm{Var}[X]=E[X^2]-(\ln 4)^2=\frac{\pi^2}{3}-(\ln 4)^2.$$

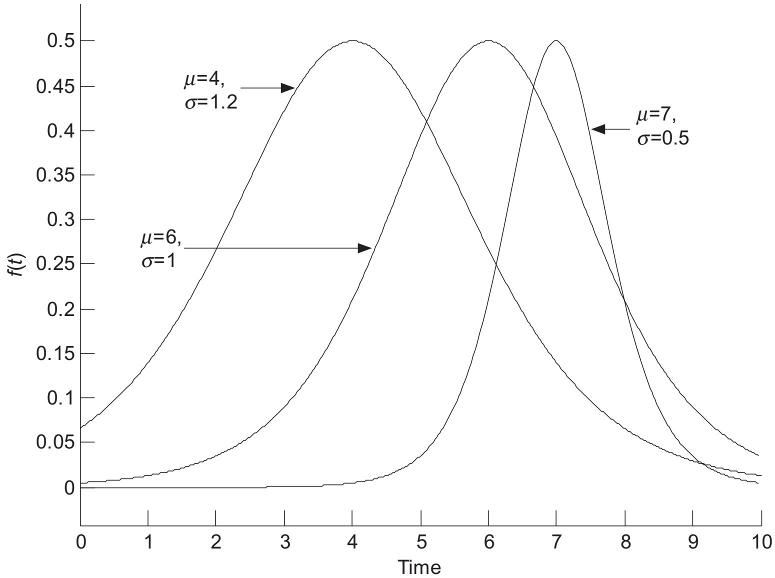

FIGURE 5.1 The p.d.f. of the standard half-logistic distribution.

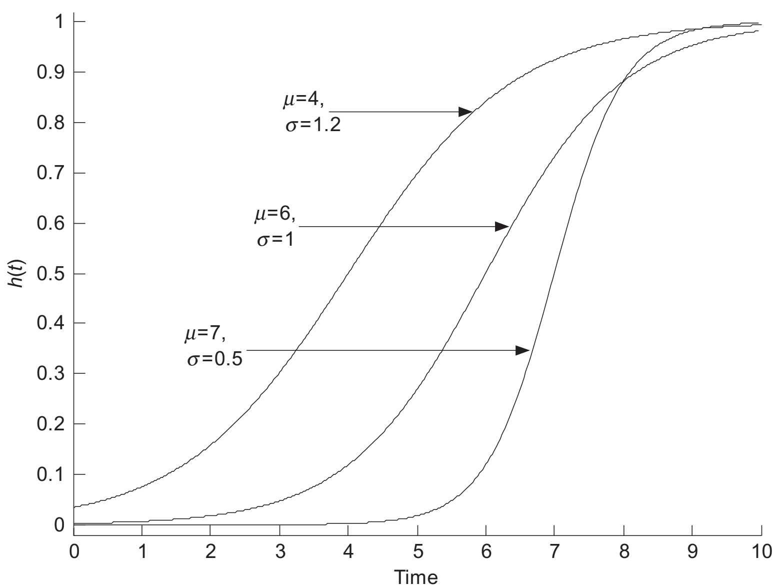

FIGURE 5.2 The hazard function of the standard half-logistic distribution.

Assume n components are subjected to a reliability test and their failure times are recorded. The test is terminated after r failures and the remaining $n - r$ components are Type 2 censored. The failure times are

$$y_1 \leqslant y_2 \leqslant \ldots \leqslant y_r.$$

The likelihood function based on Type 2 censoring is

$$L = \frac{n!}{(n-r)!}\left[1-\bar{G}(y_r)\right]^{n-r}\prod_{i=1}^{r} g(y_i). \tag{5.115}$$

Since L is a monotonically increasing function of μ, then the MLE of μ is

$$\hat{\mu} = y_1. \tag{5.116}$$

Substituting $x = (y - \mu)/\sigma$ in Equation 5.115, we obtain

$$L = \frac{n!}{(n-r)!}\frac{1}{\sigma^r}\left[1-F(x_r)\right]^{n-r}\prod_{i=1}^{r} f(x_i). \tag{5.117}$$

Substituting

$$f(x) = \frac{1}{2}\{[1-F(x)][1+F(x)]\}$$

into Equation 5.117 and taking the logarithm, we obtain the derivative of the logarithm l with respect to σ as

$$\frac{\partial l}{\partial \sigma} = \frac{-1}{2\sigma}\left[2r-(n-r)x_r-(n-r)x_rF(x_r)-2\sum_{i=1}^{r} x_iF(x_i)\right] = 0. \tag{5.118}$$

Equation 5.118 does not provide a closed-form expression for σ. We expand the function $F(x_i)$ in a Taylor series around point $F^{-1}(p_i) = \ln((1 + p_i)/q_i)$ as given in Arnold and Balakrishnan (1989), Balakrishnan and Wong (1991), David and Johnson (1954), and David (1981) and then approximate it by

$$F(x_i) \cong \alpha_i + B_i x_i,$$

where

$$\alpha_i = p_i - \frac{1}{2}q_i(1+p_i)\ln\left(\frac{1+p_i}{q_i}\right)$$
$$\beta_i = \frac{q_i(1+p_i)}{2}$$
$$p_i = \frac{i}{n+1}$$
$$q_i = 1 - p_i.$$

Following Balakrishnan and Wong (1991), we approximate Equation 5.118 by

$$\frac{\partial l}{\partial \sigma} = \frac{-1}{2\sigma}\left[2r-(n-r)(1+\alpha_r)x_r - 2\sum_{i=1}^{r}\alpha_i x_i - (n-r)\beta_r x_r^2 - 2\sum_{i=1}^{r}\beta_i x_i^2\right] = 0$$

or

$$\hat{\sigma} = \frac{1}{4r}\left[B + \left(B^2 + 8rC\right)^{\frac{1}{2}}\right], \tag{5.119}$$

where

$$B = (n-r)(1+\alpha_r)(y_r - y_1) + 2\sum_{i=2}^{r}\alpha_i(y_i - y_1)$$

$$C = (n-r)\beta_r(y_r - y_1)^2 + 2\sum_{i=2}^{r}\beta_i(y_i - y_1)^2.$$

The estimator of $\hat{\sigma}$ given in Equation 5.119 remains the same when y_i is replaced by $y_i + \alpha(1 \le i \le r)$, and it becomes $\beta\hat{\sigma}$ when y_i is replaced by $\beta y_i(1 \le i \le r)$. Therefore, realizing that the estimator of $\hat{\sigma}$ is statistically biased for small sample sizes, Balakrishnan and Wong (1991) simulated censored samples from the half-logistic population and estimated values of the unbiased factor (b) of $\hat{\sigma}$ as shown in Table 5.14. Note that $s = n - r$.

The unbiased estimate of $\hat{\sigma}$ is referred to as σ and is expressed as

$$\sigma = b\hat{\sigma} \tag{5.120}$$

We now need to determine the unbiased estimator of μ. From Equation 5.116,

$$E[\hat{\mu}] = E[y_1] = \mu + \sigma E[x_1].$$

The unbiased estimator of μ is

$$\mu = \hat{\mu} - \sigma E[x_1]. \tag{5.121}$$

The value of $E[x_1]$ required for Equation 5.121 can be obtained from Table 5.15, which is prepared by Balakrishnan (1985) for sample sizes up to 15, or can be found for larger sample sizes as discussed in Balakrishnan (1985).

TABLE 5.14 Unbiasing Factor *(b)* for $\hat{\sigma}$

N	s = 0	s = 1	s = 2	s = 3	s = 4	s = 5	s = 6	s = 7	s = 8	s = 9	s = 10	s = 11	s = 12	s = 13	s = 14	s = 15	s = 16	s = 17	s = 18
2	1.882																		
3	1.458	2.054																	
4	1.296	1.523	2.085																
5	1.209	1.333	1.536	2.082															
6	1.179	1.258	1.369	1.566	2.119														
7	1.147	1.203	1.267	1.376	1.563	2.117													
8	1.129	1.171	1.217	1.279	1.389	1.579	2.141												
9	1.115	1.144	1.182	1.225	1.290	1.387	1.583	2.127											
10	1.101	1.125	1.153	1.186	1.231	1.291	1.386	1.567	2.135										
11	1.090	1.112	1.134	1.157	1.185	1.226	1.284	1.376	1.565	2.117									
12	1.085	1.101	1.117	1.135	1.158	1.189	1.230	1.291	1.393	1.580	2.133								
13	1.076	1.090	1.103	1.121	1.139	1.160	1.189	1.228	1.288	1.378	1.556	2.121							
14	1.075	1.087	1.100	1.113	1.127	1.145	1.167	1.198	1.237	1.295	1.383	1.559	2.079						
15	1.067	1.079	1.089	1.101	1.114	1.128	1.145	1.167	1.193	1.232	1.290	1.382	1.560	2.081					
16	1.059	1.069	1.076	1.087	1.098	1.111	1.123	1.140	1.162	1.189	1.224	1.279	1.370	1.536	2.075				
17	1.061	1.069	1.076	1.085	1.095	1.104	1.116	1.130	1.150	1.172	1.198	1.233	1.288	1.373	1.540	2.067			
18	1.057	1.064	1.071	1.080	1.087	1.097	1.106	1.118	1.131	1.146	1.165	1.192	1.230	1.282	1.371	1.543	2.065		
19	1.056	1.064	1.069	1.075	1.081	1.089	1.097	1.107	1.118	1.135	1.149	1.173	1.201	1.238	1.290	1.384	1.559	2.070	
20	1.048	1.053	1.059	1.063	1.070	1.076	1.083	1.090	1.101	1.114	1.129	1.146	1.165	1.189	1.228	1.284	1.374	1.556	2.084

Source: Reprinted from *IEEE Transactions on Reliability*, Vol. 40, No. 2, June 1991 by N. Balakrishnan and K.H.T. Wong. © 1991 IEEE.

TABLE 5.15 Means of Order Statistics

n	$E[x_1]$	n	$E[x_1]$
1	1.38629	9	0.20326
2	0.77259	10	0.18430
3	0.54518	11	0.16860
4	0.42369	12	0.15538
5	0.34738	13	0.14410
6	0.29475	14	0.13435
7	0.25617	15	0.12584
8	0.22663		

Source: Abridged from Balakrishnan (1985).

EXAMPLE 5.29

Flight recorders are insulated by a ceramic fiber impregnated with a phase-change material designed to control the memory module's temperature. By changing from a liquid to a gas, the material absorbs energy and delays a rise in temperature. Twelve flight recorders are subjected to 250ºC to determine the effectiveness of its insulation in protecting the contents of the memory module. The failure times of the recorders' insulators, in minutes, are

18, 33, 37, 42, 64, 70, 105, 112, 144, 147, 208, 208^+.

Determine the parameters of the half-logistic distribution that fits the failure data. What is the reliability of a recorder after being subjected to this temperature for 2.5 h?

SOLUTION

The censoring time is 208 since the test is terminated at the eleventh failure. We estimate the biased $\hat{\sigma}$ by using the calculations shown in Table 5.16.

$$B = 1\times[1+0.4932]\times 190 + 2\times 202.2549 = 688.2178$$

$$C = 1\times\left[(0.1420)(190)^2 + 2\times 20596.230\right] = 46319.66$$

$$\hat{\sigma} = \frac{1}{44}\left[688.2178 + \left(688.2178^2 + 8\times 11\times 46319.66\right)^{\frac{1}{2}}\right] = 64.119.$$

Using Table 5.14, we obtain the unbiasing factor of 1.101, thus

$$\sigma = 1.101\times 64.119 = 70.59.$$

The approximate MLE of μ is

$$\hat{\mu} = E[y_1] = 18.$$

TABLE 5.16 Calculations for *B* and *C*

i	p_i	q_i	$y_i - y_1$	α_i	$\alpha_i(y_i - y_1)$	β_i	$\beta_i(y_i - y_1)^2$
1	0.0769	0.9230	0.0	0.0003	0.0000	0.4970	0.000
2	0.1538	0.8461	15.0	0.0024	0.0365	0.4881	109.837
3	0.2307	0.7692	19.0	0.0082	0.1573	0.1733	170.887
4	0.3076	0.6923	24.0	0.0198	0.4752	0.4526	260.733
5	0.3846	0.6153	46.0	0.0391	1.7999	0.1260	901.491
6	0.4615	0.5384	52.0	0.0686	3.5685	0.3934	1064.000
7	0.5384	0.4615	87.0	0.1110	9.6583	0.3550	2687.219
8	0.6153	0.3846	94.0	0.1695	15.9399	0.3106	2744.911
9	0.6923	0.3076	126.0	0.2484	31.3069	0.2603	4133.396
10	0.7692	0.2307	129.0	0.3534	45.5908	0.2041	3397.127
11	0.8461	0.1538	190.0	0.4932	93.7208	0.1420	5126.627
Total					202.2549		20596.23

The unbiased approximate MLEs of μ and σ are obtained using Equation 5.121 as

$$\mu = 18 - 70.59 \times E[x_1],$$

where $E[x_1]$ is obtained from Table 5.15. Therefore,

$$\mu = 18 - 70.59 \times 0.1553 = 7.037.$$

Using $\mu = 7.037$ and $\sigma = 70.59$, we obtain an unbiased estimate of the mean failure time as

$$\text{Mean failure time} = \mu + \sigma \ln 4 = 7.037 + 70 \ln 4 = 104 \text{ min}.$$

The reliability at $t = 2.5 \times 60 = 150$ min is

$$x = \frac{150 - 104}{70.59} = 0.6516$$

$$R(0.6516) = \frac{2e^{-0.6516}}{1 + e^{-0.6516}} = 0.685. \qquad ■$$

Further details about the half-logistic and other continuous univariate distributions are given in Johnson et al. (1994, 1995).

5.14 FRECHET DISTRIBUTION

In Chapter 1, we presented the p.d.f. of two-parameter Frechet distribution as

$$f(t) = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{-(\gamma+1)} e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}, t \geqslant 0, \gamma > 0, \theta > 0, \tag{5.122}$$

where γ and θ are the shape and scale parameters, respectively.

When the failure data are assumed to follow Frechet distribution, the estimated parameters of the distribution, $\hat{\gamma}$ and $\hat{\theta}$, can be obtained by using either graphical estimation or maximum likelihood estimation procedures proposed by Harlow (2001). The MLE procedures for both data without censoring and data with censoring are presented.

5.14.1 Failure Data without Censoring

The exact failure times of n units under test are recorded as $t_1, t_2, \ldots, t_n$. Assume that the failure data follow a Frechet distribution. The parameter estimates are found by maximizing the likelihood function

$$L(\gamma, \theta; t_1, t_2, \ldots, t_n) = \prod_{i=1}^{n} f(\gamma, \theta, t_i) = \left(\frac{\gamma}{\theta}\right)^n \prod_{i=1}^{n} \left(\frac{t_i}{\theta}\right)^{-(\gamma+1)} e^{-\left(\frac{t_i}{\theta}\right)^{-\gamma}}. \tag{5.123}$$

The logarithm of Equation 5.123 is

$$l(\gamma, \theta; \boldsymbol{t}) = n(\ln\gamma - \ln\theta) - (\gamma+1)\left(\sum_{i=1}^{n} \ln t_i - n\ln\theta\right) - \sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^{-\gamma}. \tag{5.124}$$

Then we take the derivatives of the logarithmic function with respect to γ and θ, and equate the resulting equations to zero. This results in the following two equations

$$\frac{\partial l}{\partial \gamma} = \frac{n}{\gamma} - \left(\sum_{i=1}^{n} \ln t_i - n\ln\theta\right) + \sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^{-\gamma} \ln\left(\frac{t_i}{\theta}\right) = 0 \tag{5.125}$$

$$\frac{\partial l}{\partial \theta} = -\frac{n}{\theta} + n(\gamma+1)\frac{1}{\theta} - \gamma\theta^{(\gamma-1)}\sum_{i=1}^{n} t_i^{-\gamma} = 0. \tag{5.126}$$

After straightforward algebraic manipulation of Equations 5.125 and 5.126, the MLE estimate for γ, denoted by $\hat{\gamma}$, is found as the solution of the nonlinear equation

$$\left\{\frac{1}{n}\sum_{i=1}^{n} t_i^{-\hat{\gamma}}\right\}\left\{\frac{n}{\hat{\gamma}} - \sum_{i=1}^{n} \ln t_i\right\} + \sum_{i=1}^{n} t_i^{-\hat{\gamma}} \ln t_i = 0, \tag{5.127}$$

and subsequently, $\hat{\theta}$ is given by

$$\hat{\theta} = \left\{\frac{1}{n}\sum_{i=1}^{n} t_i^{-\hat{\gamma}}\right\}^{-\frac{1}{\hat{\gamma}}}. \tag{5.128}$$

It is advantageous to change variables in Equation 5.127 by setting $y_i = \ln t_i$. Then $\hat{\gamma}$ is the root of

$$[1 - \hat{\gamma}\bar{y}]\sum_{i=1}^{n} e^{(-\hat{\gamma}y_i)} + \hat{\gamma}\sum_{i=1}^{n} y_i e^{(-\hat{\gamma}y_i)} = 0, \tag{5.129}$$

where

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

is the sample average of the transformed data. It can be shown that the solution to Equation 5.129 is unique, that is, $\hat{\gamma}$ is unique, if it exists. To make the computation more numerically stable, set $z_i = y_i - \bar{y}$. Multiplying Equation 5.129 by $e^{(\gamma\bar{y})}$ and simplifying yields the following equation:

$$\sum_{i=1}^{n}(1+\hat{\gamma}z_i)e^{-\hat{\gamma}z_i} = 0, \tag{5.130}$$

which is considerably more concise than Equation 5.127.

Standard search methods are very efficient since the solution of Equation 5.130 is unique. Harlow (2001) provides an excellent initial value for γ as

$$\hat{\gamma} = 2 + [1.55CV(n)]^{-\frac{1}{0.7}}, \tag{5.131}$$

where $CV(n)$ is the sample coefficient of variation. Convergence is typically obtained in less than 10 iterations for errors less than 10^{-10}.

5.14.2 Failure Data with Censoring

Assume that the units under test are subjected to censoring of Type 1 or Type 2. The failure data can be represented by $t_1 \le t_2 \le t_3 \ldots \le t_r = t_{r+1}^{+} = \ldots = t_n^{+}$. Suppose that the failure data follow a Frechet distribution. The parameter estimates are found by maximizing the likelihood function

$$\begin{aligned} L(\gamma,\theta;t_1,t_2,\ldots,t_n) &= \left\{\prod_{i=r+1}^{n} R(\gamma,\theta;t_i^{+})\right\}\left\{\prod_{i=1}^{r} f(\gamma,\theta;t_i)\right\} \\ &= \left[1-e^{-\left(\frac{t_r}{\theta}\right)^{-\gamma}}\right]^{(n-r)}\left(\frac{\gamma}{\theta}\right)^{r}\prod_{i=1}^{r}\left(\frac{t_i}{\theta}\right)^{-(\gamma+1)} e^{-\left(\frac{t_i}{\theta}\right)^{-\gamma}}. \end{aligned} \tag{5.132}$$

The logarithm of Equation 5.132 is

$$\begin{aligned} l(\gamma,\theta;\boldsymbol{t}) &= (n-r)\ln\left[1-e^{-\left(\frac{t_r}{\theta}\right)^{-\gamma}}\right] + r(\ln\gamma - \ln\theta) \\ &\quad -(\gamma+1)\left(\sum_{i=1}^{r}\ln t_i - r\ln\theta\right) - \sum_{i=1}^{r}\left(\frac{t_i}{\theta}\right)^{-\gamma}. \end{aligned} \tag{5.133}$$

We then take the derivatives of the logarithmic function with respect to γ and θ, and equate the resulting equations to zero. This results in the following two equations:

$$\frac{\partial l}{\partial \gamma} = -\frac{(n-r)(t_r/\theta)^{-\gamma}(\ln t_r - \ln\theta)}{e^{(t_r/\theta)^{-\gamma}} - 1} + \frac{r}{\gamma} - \left(\sum_{i=1}^{r} \ln t_i - r\ln\theta\right) + \sum_{i=1}^{r}\left(\frac{t_i}{\theta}\right)^{-\gamma} \ln\left(\frac{t_i}{\theta}\right) = 0 \tag{5.134}$$

$$\frac{\partial l}{\partial \theta} = \frac{(n-r)\gamma\theta^{(\gamma-1)}t_r^{-\gamma}}{e^{(t_r/\theta)^{-\gamma}} - 1} - \frac{r}{\theta} + r(\gamma+1)\frac{1}{\theta} - \gamma\theta^{(\gamma-1)}\sum_{i=1}^{r} t_i^{-\gamma} = 0. \tag{5.135}$$

To obtain the MLEs of γ and θ, we solve Equations 5.134 and 5.135 simultaneously with respect to γ and θ. These equations have no closed-form solution. Therefore, we use a numerical method such as the Newton–Raphson method to obtain the solution.

5.15 BIRNBAUM–SAUNDERS DISTRIBUTION

In Chapter 1, we presented the characteristics of the two-parameter Birnbaum–Saunders distribution, especially the unimodal characteristic of its hazard-rate function. As indicated, this distribution is commonly used in the analysis of fatigue data and in situations where the hazard rate increases to a peak (after repeated loads) then decreases gradually. This is analogous to the yield point of a ductile material. In essence, both the peak point of the hazard-rate function and the yield point define limits for the design of components and parts that exhibit such behavior. The p.d.f. of the two-parameter Birnbaum–Saunders distribution is

$$f(t;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\sqrt{\frac{\beta}{t}} + \left(\frac{\beta}{t}\right)^{3/2}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t}\right) - 2\right]. \tag{5.136}$$
$$0 < t < \infty, \alpha, \beta > 0$$

When the failure data are assumed to follow Birnbaum–Saunders distribution, the estimated parameters of the distribution, $\hat{\alpha}$ and $\hat{\beta}$, can be obtained by using the maximum likelihood estimation procedures (Ng et al., 2003, 2006). The MLE procedures for both data without censoring and data with censoring are presented.

5.15.1 Failure Data without Censoring

The exact failure times of n units under test are recorded as $t_1, t_2, \ldots, t_n$. Assume that the failure data follow a Birnbaum–Saunders distribution. The parameter estimates are found by maximizing the likelihood function

$$\begin{aligned} L(\alpha,\beta;t_1,t_2,\ldots,t_n) &= \prod_{i=1}^{n} f(\alpha,\beta,t_i) \\ &= \frac{1}{2\sqrt{2\pi}\alpha\beta}\prod_{i=1}^{n}\left[\sqrt{\frac{\beta}{t_i}} + \left(\frac{\beta}{t_i}\right)^{3/2}\right] \\ &\quad \exp\left[-\frac{1}{2\alpha^2}\left(\frac{t_i}{\beta} + \frac{\beta}{t_i}\right) - 2\right], \\ &\quad 0 < t_i < \infty, \alpha, \beta > 0. \end{aligned} \tag{5.137}$$

The logarithm of Equation 5.137 is

$$l(\alpha, \beta; t_1, t_2, \ldots, t_n) = -1.6144 - (\ln\alpha + \ln\beta) + \ln\left(\prod_{i=1}^{n}\left[\sqrt{\frac{\beta}{t_i}} + \left(\frac{\beta}{t_i}\right)^{3/2}\right]\right) - \frac{1}{2\alpha^2}\sum_{i=1}^{n}\left(\frac{t_i}{\beta} + \frac{\beta}{t_i}\right) - 2n. \tag{5.138}$$

Taking the partial derivatives of Equation 5.138 with respect to α and β and equating the resulting equations to zero then solving them simultaneously, we obtain the estimated values of the Birnbaum–Saunders distribution parameters. Ng et al. (2003) simplify the procedure as follows:

The sample's arithmetic and harmonic means are calculated as

$$s = \frac{1}{n}\sum_{i=1}^{n} t_i, \quad r = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{-1}\right]^{-1}.$$

Furthermore, they define a harmonic mean function K as

$$K(x) = \left[\frac{1}{n}\sum_{i=1}^{n}(x + t_i)^{-1}\right]^{-1} \qquad x \geq 0.$$

The MLE of β is obtained by solving the nonlinear function given by Equation 5.139:

$$\beta^2 - \beta[2r + K(\beta)] + r[s + K(\beta)] = 0. \tag{5.139}$$

The estimated α is explicitly obtained as

$$\hat{\alpha} = \sqrt{\frac{s}{\hat{\beta}} + \frac{\hat{\beta}}{r} - 2}. \tag{5.140}$$

EXAMPLE 5.30

One of the main failures in oil refineries is its piping. For example, a typical heat exhanger is configured so that one process stream flows through the inside of a tube and a different stream flows over the outside of the tube, exchanging heat through the tube wall. The integrity of the tube wall is affected by corrosion, and when it becomes too thin (threshold thickness of the tube wall), the tube must be replaced. The thickness of the tube is measured using an ultrasonic system. The following meaurements represent the times (in days) taken to reach equal amounts of change in the wall thickness before failure. This is analogous to the crack length growth in fatigue testing:

38, 39, 40, 40, 42, 44, 44, 46, 48 and 49.

Determine the parameters of Birnbaum–Saunders distribution that fits the failure data. What is the MTTF?

SOLUTION

The sample's arithmetic and harmonic means are 42.9788 and 42.6733, respectively.

Solving the nonlinear Equation 5.139 using Newton–Raphson method results in $\hat{\beta} = 42.825{,}83$. Substituting in Equation 5.140 results in $\hat{\alpha} = 0.084$. The MTTF is

$$E(T) = \beta\left(1 + \frac{1}{2}\alpha^2\right) = 43.131.$$

■

5.15.2 Failure Data with Censoring

Assume that the units under test are subjected to censoring of Type 2. The ordered failure data are represented by $t_1 \le t_2 \le t_3 \ldots \le t_r = t_{r+1}^+ = \ldots = t_n^+$. The largest $(n - r)$ lifetimes are censored. Suppose that the failure data follow a Birnbaum–Saunders distribution. The parameter estimates are found by maximizing the likelihood function (Balakrishnan and Cohen, 1991 and Ng et al., 2006) as

$$L(\alpha, \beta; t_1, t_2, \ldots, t_r) = \frac{n!}{(n-r)!}\left\{1 - \Phi\left[\frac{1}{\alpha}\xi\left(\frac{t_r}{\beta}\right)\right]\right\}^{n-r} \times \left\{\frac{1}{\sqrt{2\pi}\alpha\beta}\left[\prod_{i=1}^{r}\xi'\left(\frac{t_i}{\beta}\right)\right]\exp\left[-\frac{1}{2\alpha^2}\xi^2\left(\frac{t_i}{\beta}\right)\right]\right\}. \tag{5.141}$$

The log-likelihood function is

$$\begin{aligned} l(\alpha, \beta; t_1, t_2, \ldots, t_r) = K + (n-r)\ln\left\{1 - \Phi\left[\frac{1}{\alpha}\xi\left(\frac{t_r}{\beta}\right)\right]\right\} - r\ln\alpha - r\ln\beta \\ + \sum_{i=1}^{r}\xi'\left(\frac{t_i}{\beta}\right) - \frac{1}{2\alpha^2}\sum_{i=1}^{r}\xi^2\left(\frac{t_i}{\beta}\right). \end{aligned} \tag{5.142}$$

Where K is constant

$$\xi(t) = t^{1/2} - t^{-1/2}$$
$$\xi^2(t) = t + t^{-1} - 2.$$

Substituting $t^* = \dfrac{t_i}{\beta}$, $H(y) = \dfrac{\phi(y)}{1 - \Phi(y)}$ in Equation 5.142 and taking the partial derivative of the log-likelihood function with respect to the two parameters of the distribution, we obtain

$$\frac{\partial l}{\partial \alpha} = \frac{n-r}{\alpha^2}H\left[\frac{1}{\alpha}\xi(t_r^*)\right]\xi(t_r^*) - \frac{r}{\alpha} + \frac{1}{\alpha^3}\sum_{i=1}^{r}\xi^2(t_i^*) = 0 \tag{5.143}$$

$$\frac{\partial l}{\partial \beta} = \frac{n-r}{\alpha\beta}H\left[\frac{1}{\alpha}\xi(t_r^*)\right]t_r^*\xi'(t_r^*) - \frac{r}{\beta} - \frac{1}{\beta}\sum_{i=1}^{r}\frac{t_i^*\xi''(t_i^*)}{\xi'(t_i^*)} + \frac{1}{\alpha^2\beta}\sum_{i=1}^{r}t_i^*\xi(t_i^*)\xi'(t_i^*) = 0. \tag{5.144}$$

Solving Equations 5.431 and 5.144 simultaneously results in the estimated values of the distribution parameters.

5.16 LINEAR MODELS

When several hazard-rate functions can fit the same data, it becomes necessary to discriminate among the functions to determine the "best fit" function. This can be achieved by substituting the reliability estimators obtained by the different hazard functions into a likelihood function. The "best fit" function is the one that maximizes the likelihood function. This can be easily accomplished when the hazard functions are linear. As we have seen in Chapter 1, most of the hazard functions are nonlinear. However, simple transformations can change some of the nonlinear functions to linear ones. In this section, we illustrate the use of linear hazard functions in conjunction with a likelihood function in determining the "best" hazard function that fits a given set of failure data.

Consider the following hazard-rate functions:

Constant hazard $h(t) = \lambda$

Weibull hazard $h(t) = \dfrac{\gamma}{\theta}\left(\dfrac{t}{\theta}\right)^{\gamma-1}$

Rayleigh hazard $h(t) = \lambda t$

Gompertz hazard $h(t) = \exp(a + bt)$

Linear Exponential hazard $h(t) = a + bt$.

All the above hazard-rate functions are either already linear functions of time or can be linearized by taking the logarithm of both sides of the hazard-rate function. In all cases, we can write the linear hazard-rate function as

$$y_i = a + bx_i,$$

where y_i is the estimated hazard function or its logarithm at the ith interval, x_i is the midpoint of the time interval t_i or its logarithm, and a and b are constants. Using the weighted least-squares method, we express the weighted sum of squares (WSS) of the differences between the actual y_i and the estimated $\hat{y}_i = \hat{a} + \hat{b}x_i$ for N intervals as

$$WSS = \sum_{i=1}^{N} w_i\left(y_i - \hat{a} - \hat{b}x_i\right)^2, \tag{5.145}$$

where w_i is the weight for interval t_i. Researchers considered the case where $w_i = 1$ or $w_i = n_i b_i$ where b_i and n_i are the width and number of components under test in the ith interval. Other weights can also be assigned. In order to minimize WSS, we take the derivatives of Equation 5.145 with respect to $\hat{a}$ and $\hat{b}$ to obtain two equations. These resultant equations are set equal to zero and solved simultaneously to obtain $\hat{a}$ and $\hat{b}$ follows.

$$\hat{b} = \frac{\sum_{i=1}^{N} w_i (x_i - \overline{x}_w)(y_i - \overline{y}_w)}{\sum_{i=1}^{N} w_i (x_i - \overline{x}_w)}$$

and

$$\hat{a} = \overline{y}_w - \hat{b}\overline{x}_w,$$

where $\overline{x}_w$ and $\overline{y}_w$ are the weighted averages of x_i's and y_i's, respectively. They are expressed as

$$\overline{x}_w = \frac{\sum_{i=1}^{N} w_i x_i}{\sum_{i=1}^{N} w_i}$$

$$\overline{y}_w = \frac{\sum_{i=1}^{N} w_i y_i}{\sum_{i=1}^{N} w_i}.$$

Having estimated $\hat{a}$ and $\hat{b}$ for each of the above hazard-rate functions, we estimate the corresponding reliabilities using

Constant	$R(t) = e^{-\lambda t}$
Weibull	$R(t) = e^{-\left(\frac{t}{\theta}\right)^{\gamma}}$
Rayleigh	$R(t) = e^{-\lambda t^2/2}$
Gompertz	$R(t) = \exp\left[\frac{-e^{a}}{b}(\exp(bt) - 1)\right]$
Linear exponential	$R(t) = \exp\left[-\left(at + \frac{1}{2}bt^2\right)\right].$

We then substitute in the logarithm of the likelihood function, L

$$L = \prod_{i=1}^{N-1}\left[1 - \frac{\hat{R}(t_{i+1})}{\hat{R}(t_i)}\right]^{r_i}\left[\frac{\hat{R}(t_{i+1})}{\hat{R}(t_i)}\right]^{n_i - r_i}.$$

The logarithm of the likelihood function is

$$l = \sum_{i=1}^{N-1} r_i \ln\left[1 - \frac{\hat{R}(t_{i+1})}{\hat{R}(t_i)}\right] + \sum_{i=1}^{N-1}(n_i - r_i)\ln\left[\frac{\hat{R}(t_{i+1})}{\hat{R}(t_i)}\right], \tag{5.146}$$

where n_i and r_i are the number of units under test and the number of failed units in the interval i, respectively. We finally compare the log-likelihood values of the observed data under the various hazard-rate functions. We chose the hazard-rate function that results in the maximum value of l as the "best" function.

5.17 MULTICENSORED DATA

So far we have presented parametric fitting of failure-time data for Type 1, Type 2, and random censoring. There are situations when a combination of censoring may occur during the same test. For example, consider a reliability test where some of the units under test are removed due to the malfunction of the test equipment and the remaining units continue their testing until the test is terminated (when a specified number of failures occurs or when a specified test time is reached). The test results contain multicensored data. There is no known parametric method that can accommodate such multicensored data. However, there are two well-known simple approaches that can easily deal with such data. They are referred to as the *product-limit estimator* (PLE), which is developed by Kaplan and Meier (1958), and the *cumulative-hazard estimator* (CHE), which is developed by Nelson (1979, 1982). In the following two sections, we present these estimators and compare their reliability estimates.

5.17.1 Product-Limit Estimator (PLE) or Kaplan–Meier (K-M) Estimator

As stated earlier, the main advantage of this estimator lies in its ability of handling multicensored data and the simplicity of calculations. The estimator relies on the fact that the probability of a component surviving during an interval of time (t_i, t_{i+1}) is estimated as the ratio between the number of units that did not fail during the interval and the units that were under the reliability test at the beginning of the interval (time t_i). The reliability estimate at that interval is then obtained as the product of all ratios from time zero until time t_{i+1}. In other words, the reliability function using the PLE at the distinct lifetime t_j is

$$\hat{R}_{pl}(t_j) = \prod_{l=1}^{j} (n_l - d_l)/n_l = \prod_{l=1}^{j} (1 - x_l), \tag{5.147}$$

where t_j is jth ordered distinct lifetime $j = 1, 2, \ldots k$, and t_0 is the start time of the reliability test, n_j and d_j are the number of units under test and the number of failed units, respectively. Some of the units may be right-censored (Type 2 censored) during the test interval (t_j, t_{j+1}); we refer to the censored observations during this interval as e_j. Thus,

$$n_j = \sum_{l=j}^{k} (d_l + e_l), \quad n_0 = n,$$

where

n = the total number of units under test,

k = the distinct failure times, and

$x_j = d_j/n_j$.

Equation 5.147 is a nonparametric MLE of the reliability function. When there is a multiplicity m_l of the failure time t_l, then we rewrite Equation 5.147 as

$$\hat{R}_{pl}(t) = \prod_{\substack{l=1 \\ l \leq t}} \left(1 - \frac{m_l}{n_l}\right). \tag{5.148}$$

The standard deviation of the K-M point estimator $\hat{R}_{pl}(t)$ is estimated by

$$\sigma_{R(t)} = \hat{R}_{pl}(t)\sqrt{Var\left(\log\left(\hat{R}_{pl}(t)\right)\right)}, \tag{5.149}$$

where

$$Var\left(\log\left(\hat{R}_{pl}(t)\right)\right) = \sum_{\substack{l \\ t_l < t}} \frac{m_l}{n_l(n_l - m_l)}$$

is the well-known Greenwood's formula. The $(1 - \alpha)$ level left-sided confidence interval for $\hat{R}_{pl}(t)$ is given by $\left(\hat{R}_{pl}(t) - u_{(1-\alpha)}\sigma\left[\hat{R}_{pl}(t)\right], 1\right)$, where u_α is the quantile of order α for the standard normal distribution $N(0, 1)$.

5.17.2 Cumulative-Hazard Estimator (CHE)

This is an alternative procedure for dealing with multicensored data. The estimates of the hazard rate and cumulative hazard functions are

$$\hat{h}(t_j) = \frac{d_j}{n_j} = x_j \tag{5.150}$$

$$\hat{H}(t_j) = \sum_{l=1}^{j} \hat{h}(t_l) = \sum_{l=1}^{j} x_l. \tag{5.151}$$

From the relationships in Chapter 1, we calculate the reliability using the CHE as

$$\hat{R}_{CH}(t_j) = \exp\left(-\sum_{l=1}^{j} x_i\right)$$

or

$$\hat{R}_{CH}(t_j) = \prod_{l=1}^{j} \exp(-x_l) \text{ for all } j \in \{1, 2, \ldots, k\}. \tag{5.152}$$

Bohoris (1994) shows that the reliability estimates obtained using CHE are greater than those obtained by the product limit estimator. The following example confirms the observation.

EXAMPLE 5.31

Nonmetallic bearings (dry bearings) are made from polymers and polymer composites. They are preferred in operating environments where there is no adequate lubrication present or where a combination of high load, low speed, or intermittent motion makes lubrication difficult. The main disadvantages of such bearings are their poor creep strength, low softening temperature, high thermal expansion coefficients, and the ability to absorb liquids. Therefore, producers of the nonmetallic bearings perform extensive reliability experiments to determine the failure rates at different operating conditions.

A manufacturer of dry bearings subjects 25 units to a creep test and observes the following failure times (in hours):

$70, 180, 190^{+}, 200, 210, 230, 275, 295, 310, 370^{+}, 395, 420, 480, 495, 560, 600^{+}, 620^{+}, 680, 750, 780, 800, 900, 980^{+}, 1010^{+}, 1020^{+}.$

The "+" sign indicates censoring. Estimate the reliability functions using the product limit and the CHEs.

SOLUTION

We use Equations 5.147, 5.150–5.152 to obtain the estimates shown in Table 5.17. As shown in the table, the reliability estimates obtained by the product limit method are always less than those obtained by the cumulative hazard method. These two methods, though simple, are quite useful in many applications that have multicensored data. The reliability can be estimated at any time t by fitting a parametric exponential function to the reliability values given in the last two columns of Table 5.17.

TABLE 5.17 Estimates of the Reliability Functions

i	t_i	j	t_j	n_j	d_j	e_j	$\hat{h}(t_j)$	$\hat{H}(t_j)$	$\hat{R}_{CH}(t_j)$	$\hat{R}_{pl}(t_j)$
1	70	1	70	25	1	0	0.040	0.040	0.961	0.960
2	180	2	180	24	1	1	0.042	0.082	0.921	0.920
3	190^{+}	—	—	—	—	—	—	—	—	—
4	200	3	200	22	1	0	0.045	0.127	0.881	0.879
5	210	4	210	21	1	0	0.048	0.175	0.840	0.840
6	230	5	230	20	1	0	0.050	0.225	0.799	0.798
7	275	6	275	19	1	0	0.053	0.278	0.757	0.756
8	295	7	295	18	1	0	0.056	0.334	0.716	0.714
9	310	8	310	17	1	1	0.059	0.393	0.675	0.672
10	370^{+}	—	—	—	—	—	—	—	—	—
11	395	9	395	15	1	0	0.067	0.460	0.631	0.627
12	420	10	420	14	1	0	0.071	0.531	0.588	0.582
13	480	11	480	13	1	0	0.077	0.608	0.544	0.537

TABLE 5.17 (*Continued*)

i	t_i	j	t_j	n_j	d_j	e_j	$\hat{h}(t_j)$	$\hat{H}(t_j)$	$\hat{R}_{CH}(t_j)$	$\hat{R}_{pl}(t_j)$
14	495	12	495	12	1	0	0.083	0.691	0.501	0.492
15	560	13	560	11	1	2	0.091	0.782	0.457	0.447
16	600^{+}	—	—	—	—	—	—	—	—	—
17	620^{+}	—	—	—	—	—	—	—	—	—
18	680	14	680	8	1	0	0.125	0.907	0.404	0.391
19	750	15	750	7	1	0	0.143	1.050	0.350	0.335
20	780	16	780	6	1	0	0.167	1.217	0.296	0.279
21	800	17	800	5	1	0	0.200	1.417	0.242	0.223
22	900	18	900	4	1	3	0.250	1.667	0.189	0.167
23	980^{+}	—	—	—	—	—	—	—	—	—
24	$1{,}010^{+}$	—	—	—	—	—	—	—	—	—
25	$1{,}020^{+}$	—	—	—	—	—	—	—	—	—

■

EXAMPLE 5.32

Table 5.18 contains data by Koucky et al. (2011) obtained from a military system field operation. The indicator $d_l = 1$ implies failure and 0 implies censoring. Obtain the reliability, standard deviation at the indicated failure times, and plot reliability $\hat{R}_{pl}(t)$ against time.

TABLE 5.18 Field Data from System Operation

l	t_l	d_l
1	177	1
2	185	1
3	185	1
4	190	1
5	196	1
6	198	1
7	199	1
8	200	0
9	200	0
10	200	0
11	200	0
12	200	0
13	200	0
14	200	0
15	200	0
16	200	0

(*Continued*)

TABLE 5.18 (*Continued*)

I	t_I	d_I
17	200	0
18	200	0
19	200	0
20	200	0
21	200	0
22	200	0
23	200	0
24	200	0
25	200	0

SOLUTION

We use Equations 5.148 and 5.149 to construct Table 5.19 and plot the reliability $\hat{R}_{pl}(t)$ values against time as shown in Figure 5.3.

TABLE 5.19 Reliability and Standard Deviations

I	t_I	d_I	m_I	n_I	$\hat{R}_{pl}(t)$	Greenwood	$\sigma_{R(t)}$
1	177	1	1	25	0.96	0.001,7	0.039,2
2,3	185	1	2	24	0.88	0.005,5	0.065,0
4	190	1	1	22	0.84	0.007,6	0.073,3
5	196	1	1	21	0.8	0.010,0	0.080,0
6	198	1	1	20	0.76	0.012,6	0.085,4
7	199	1	1	19	0.72	0.015,6	0.089,8
8–25	200	0	—	—	—	—	—

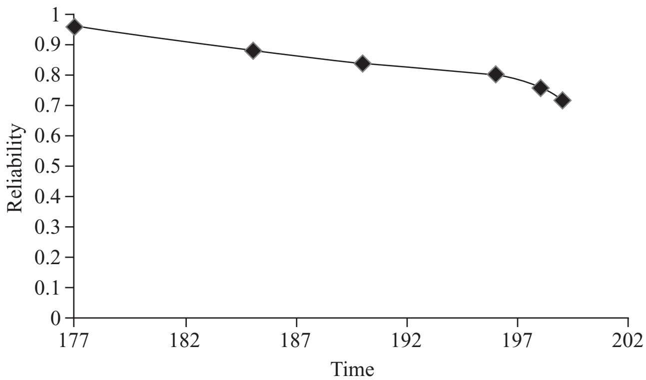

FIGURE 5.3 Reliability versus time. ■

PROBLEMS

5.1 The following failure data are obtained from a fatigue test of helical gears. (The times between failures are in hours.)

Times between failures in hours	
220	25,000
400	28,000
590	31,000
790	35,000
1,200	40,500
1,900	45,000
3,000	49,000
4,900	53,000
6,500	58,000
9,000	64,000
14,000	68,000
19,000	75,000
22,000	100,000

a. Does the exponential distribution fit this data?

b. Is the first failure abnormally short?

c. Is the last failure abnormally long?

d. If the data fit an exponential distribution, what is the estimated mean time between failures (MTBF)?

5.2 Consider the following failure times of a wear test of composite tires

4000
4560
5800
7900
10,000
12,000
15,000
17,000
23,000
26,000
30,000
36,000
40,000
48,000
52,000
70,000

a. Check if an exponential distribution can be used to fit the failure data (failure data are measured in miles).

b. What are the parameters of the exponential distribution?

c. Set a 95% confidence limit on the parameters of the distribution.

d. Set a 90% confidence limit on the reliability at 30,000 mi.

e. Would you buy a tire from this population? If yes, state why. If not, state why not.

5.3 Power supplies are major units for most electronic products. The manufacturers usually use a reliability demonstration test to establish a measure of reliability. For example, to demonstrate a 20,000 h MTTF, 13 power supplies must be operated at full load for 60 days without observing any failure. To extend the demonstrated MTTF to 200,000 h, 127 units must be operated over the same duration (Eimar, 1990). A manufacturer subjects 10 power supplies to a reliability test and observes the following times to failure

10,000, 18,000, 21,000, 22,000, 22,500, 23,000, 25,000, 30,000, 40,000, 70,000 h.

a. Fit an exponential distribution to the above failure-time data (check for abnormal failure times).

b. Estimate the parameters of the distribution using three different methods.

c. If you were to own a power supply produced by this manufacturer, what would the reliability of your unit be at $t = 20{,}000$ h?

d. Do the units meet the conditions for the reliability demonstration test?

5.4 Assume that the manufacturer in the above problem conducts a reliability test using 15 power supplies instead. The times to failure of 10 units are identical to those obtained in Problem 5.3. However, the manufacturer terminates the test for the remaining units at $t = 60{,}000$ h. Answer questions (a) through (d) of Problem 5.3 using the results of the new test.

5.5 Twenty-one units are subjected to a fatigue test. The times to failure in hours are

8, 8, 8, 9, 13, 15, 18, 25, 26, 8^+, 10^+, 13^+, 19^+, 22^+, 33^+, 36^+, 40^+, 45^+, 47^+, 49^+.

a. Plot the hazard function of these data.

b. Assuming an exponential failure-time distribution, estimate the failure rate and the MTTF.

c. What is the reliability of a unit at time $t = 52$ h?

d. Assuming that the observations fit a Rayleigh distribution, estimate its parameter using both the maximum-likelihood method and the best linear unbiased estimate (BLUE). Compare the results and explain the causes of differences, if any.

5.6 Fit a Weibull distribution for the data given in Problem 5.5 and estimate the reliability of a unit at time $t = 52$ h. Compare the reliability estimates at $t = 52$ h with that obtained in the above problem. Explain the difference in the results.

5.7 Suppose that a manufacturer of tires makes a new prototype and provides 20 customers with a pair of these tires. The failure times (time when tread reaches a predetermined threshold level) measured in miles of driving are

3000, 4000, 6000, 9000, 9000, 11,000, 12,000, 14,000, 16,000, 18,000, 30,000, 35,000, 38,000, $8{,}000^+$, $13{,}000^+$, $22{,}000^+$, $28{,}000^+$, $36{,}000^+$, $45{,}000^+$, and $46{,}000^+$.

The "+" sign indicates that the customers left the study at the indicated miles.

a. Fit a Weibull distribution to these data. Determine the parameters of the distribution.

b. What is the reliability of a tire at $t = 50{,}000$ mi?

c. What is the MTTF?

d. Determine the MTTF when the censored observations are ignored? Compare the results with (c) and explain which of these MTTFs should be recommended to the manufacturer. Why?

5.8 A manufacturer of long life-cycle toggle switches observes 15 switches under test and records the number of switch activations to failure as follows

Failure number	Number of activations
1	50,000
2	51,000
3	60,000
4	72,000
5	80,000
6	85,000
7	89,000
8	94,000
9	97,000
10	99,000
11	110,000
12	115,000
13	116,000^{+}
14	117,000^{+}
15	118,000^{+}

The "+" sign indicates censoring.

a. Assuming that the activations to failure follow a lognormal distribution, determine the parameters of the distribution.

b. Determine the 95% confidence interval for the parameters of the distribution.

c. What are the variances of the parameters?

d. Suppose that the manufacturer ignores the censored activations and limits the analysis to the noncensored data only. Compare the mean lives when the noncensored and censored data are included in the analysis.

5.9 Being unsure of the failure distribution, the manufacturer in Problem 5.8 uses a gamma distribution instead.

a. Solve items (a) through (d) of the same problem using gamma distribution.

b. Compare the mean lives obtained from the gamma and the lognormal distribution.

c. Obtain the variances of the parameters of the gamma distribution.

5.10 A manufacturer of utility power tubing analyzes the failures of two super-heater tubes that are operating under conditions of 540°C. The analyses reveal creep voids near the rupture of the tubes. Therefore, the manufacturer designs an accelerated stress test where fifteen tubes are tested at 750°C. The following failure times are observed.

Failure number	Failure time
1	173.90
2	188.91
3	124.10
4	177.71
5	105.31
6	45.44
7	101.24
8	243.57
9	34.54
10	269.87
11	85.67
12	134.73
13	42.70
14	258.39
15	29.75

The manufacturer also accumulates the following failure times from units operating under normal conditions.

Failure number	Failure time
1	867.20
2	1,681.22
3	1,785.56
4	1,088.08
5	347.90
6	819.30
7	1,035.16
8	816.99
9	1,214.37
10	1,094.08
11	1,453.07
12	715.79
13	294.70
14	867.42
15	434.52

Assume that the failure mechanism at the accelerated stresses (failure due to creep voids) is the same as that occurring at normal conditions. Determine the parameters of the failure-time distributions at the accelerated and normal operating conditions (assume Weibull distribution). What is the ratio between the mean lives at the accelerated conditions and the normal conditions?

5.11 In order to reduce the test time, 12 ceramic capacitors are subjected to a HALT. The test is terminated after nine capacitors fail. The survival times in hours are

6, 9, 10, 11, 13, 16, 22, 23, 27, 27^+, 27^+, 27^+.

a. Assume that the failure times follow the lognormal distribution. Determine the parameters of the distribution and their 95% confidence intervals.

b. A Weibull distribution can also fit the failure times of the capacitors. What are the parameters of the Weibull distribution and their 95% confidence intervals?

5.12 Suppose that 20 products are placed under a vibration test and the time to failure (in months) is recorded as follows.

1, 2, 3, 3, 4, 4, 4, 5, 5, 6, 7, 7, 8, 9, 9, 10, 15, 16, 20, 25.

The time to failure can be described by a Weibull distribution. Determine the parameters of the distribution and the MTTF. What are the variances of the parameters?

5.13 An automated testing laboratory conducts an experiment using a sample of 10 devices. The failure rate of the units is observed to follow a linear model.

$$h(t) = a + bt.$$

The failure-time data are

20, 50, 80, 110, 130, 150^+, 150^+, 150^+, 150^+, 150^+.

The "+" implies censoring.

a. Use the maximum likelihood estimation procedure to estimate the parameters a and b.

b. Use the failure-time data to estimate the reliability of a device at time = 100 h.

5.14 The most frequently employed environmental test is the 85/85 temperature and humidity stress test. The purpose of the test is to determine the ability of the device to withstand long-term exposure to warm and humid environments. The test involves subjecting a sample of devices to 85°C and unsaturated moisture of 85% RH (relative humidity) under static electrical bias for 1000–2000 h. The devices are then analyzed to determine if the metallic wire bonds have corroded. The test usually lasts for about 12 weeks. A manufacturer of high-capacity hard disk drives uses the test to demonstrate that the MTBF of the drives is greater than 100,000 h at the 85/85 test.
A sample of 25 drives is subjected to this test and the following failure-time data are obtained:

1000, 1100, 1300, 1450, 1520, 1600, 1720, 1750, 1800, 1910, 2000, 2000^+ h.

The "+" implies censoring time of the remaining devices. The manufacturer is not sure which failure-time distribution should be used to model the failure times. Since the Weibull and the Gamma distributions are widely used, the manufacturer decides to use both distributions.

a. The p.d.f. of the Weibull model is

$$f(t) = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\gamma}\right] \quad t \geq 0, \gamma > 0, \theta > 0.$$

Estimate the parameters of the Weibull distribution for the failure-time data. What is the MTTF?

b. The p.d.f. of the Gamma model is

$$f(t) = \frac{\lambda}{\Gamma(\gamma)}(\lambda t)^{\gamma-1} \exp(-\lambda t) \quad t \geq 0, \gamma, \lambda > 0.$$

Estimate the parameters of the Gamma model. What is the MTTF?

c. What is the probability that the MTTF is greater than 100,000 h for the following cases?

- Weibull is used.
- Gamma is used.

5.15 A mining company owns a 1400 car fleet of 80-ton high side, rotary dump gondolas. A car will accumulate about 100,000 mi/year. In their travels from the mines to a power plant, the cars are subjected to vibrations due to track input in addition to the dynamic effects of the longitudinal shocks coming through the couplers. Consequently, the couplers encounter high dynamic impacts and experience fatigue failures and wear. Twenty-eight cars are observed, and the miles driven until the coupler is broken are recorded as follows.

Car	Number of miles	Car	Number of miles
1	131,375	12	199,284
2	153,802	13	202,996
3	167,934	14	203,754
4	171,842	15	204,356
5	178,770	16	209,866
6	184,104	17	213,354
7	189,838	18	218,898
8	193,242	19	226,196
9	196,150	20	234,634
10	198,949	21	233,567
11	199,986	22	235,987

The remaining six cars left the service after 151,345, 154,456, 161,245, 167,876, 175,547, and 177,689 mi. None of them experienced a broken coupler.

a. Fit a Weibull distribution to the failure miles and determine the parameters of the distribution.

b. Obtain unbiased estimates of the parameters and their variances.

c. Construct 90% confidence intervals for the parameters.

d. What is the probability that a car's coupler will break after 150,000 mi have been accumulated?

5.16 A manufacturer of resistors conducts an accelerated test on 10 resistors and records the following failure times (in days):

2, 3.8, 6, 9, 12, 15, 20, 33, 45, 60

a. Assume that the failure times follow the lognormal distribution. Determine the parameters of the distribution and their 95% confidence intervals.

b. A Weibull distribution can also fit the failure times of the capacitors. What are the parameters of the Weibull distribution and their 95% confidence intervals? Determine the variances of the parameters.

5.17 Assume n units are subjected to a test and r different failure times are recorded as $t_1 \le t_2 \le \ldots \le t_r$. The remaining $n - r$ units are censored and their censoring times are $t_r = t_1^+ = t_2^+ = \ldots = t_{n-r}^+$. Assuming that the failure times follow the Special Erlang distribution, whose p.d.f. is

$$f(t) = \frac{t}{\lambda^2} e^{-\frac{t}{\lambda}} \quad t \ge 0,$$

a. Estimate the distribution's parameter.

b. Obtain an expression for its reliability function and determine the MTTF.

5.18 The following failure times are recorded and follow the Special Erlang distribution on Problem 5.17:

200, 300, 390, 485, 570, 640, 720, 720^+, 720^+, 720^+

The "+" indicates censoring. What is the Erlang's parameter? What is the MTTF? What is the reliability of a component from this population at $t = 500$ h? Derive the hazard-rate function expression. What is your assessment of the hazard rate?

5.19 One of the techniques for performing stress screening is referred to as highly accelerated stress screening (HASS) which uses the highest possible stresses beyond "qualification" level to attain time compression on the tested units. Assume that 15 units are subjected to a HASS and the failure times of the first 11 units are recorded in minutes. The remaining four units are still operating properly when the test is terminated. The failure times are

1.5, 4.0, 7.0, 11.0, 14.0, 16.5, 19.0, 22.0, 24.0, 26.4, 28.5.

The test is terminated after 30 min.

a. Assume that the engineer in charge suspects that the data follow an exponential distribution. In order not to limit the analysis, the engineer suggests that the Weibull distribution would be a better fit for the data. Fit both the exponential and the Weibull distributions to the data and estimate their parameters.

b. Determine the 90% confidence intervals for all the parameters obtained above.

c. Obtain the reliability of a unit from the above population using both distributions at time = 16 min. What do you conclude?

5.20 Prove that the reliability estimates obtained by the CHE are larger than those obtained by the PLE for all cases when sample is complete or when it has censored observations.

5.21 Repeaters are used to connect two or more ethernet segments of any media type. As segments exceed their maximum number of nodes or maximum length, signal quality begins to deteriorate. Repeaters provide the signal amplification and retiming required to connect segments. It is, therefore, necessary that repeaters used in high traffic networks have low failure rates. A manufacturer subjects 20 repeaters to a reliability vibration test and obtains the following failure times (in hours):

25, 50, 89, 102, 135, 136, 159, 179, 254, 300, 360, 395, 460, 510, 590, 670, 699, 780^+, 780^+, 780^+.

The "+" indicates censoring. The manufacturer believes that an extreme value distribution of the form

$$f(t;\mu;\sigma)=\frac{1}{\sigma}e^{(y-\mu)/\sigma}\exp\left(-e^{(y-\mu)/\sigma}\right)$$

is appropriate to fit the failure times where μ and σ are the parameters of the distribution. The cumulative distribution function is

$$F(t;\mu,\sigma)=1-\exp\left(-e^{(y-\mu)/\sigma}\right)\quad -\infty<y<\infty, -\infty<\mu<\infty, \sigma>0.$$

a. Determine the parameters of the distribution.

b. Estimate the reliability of a repeater obtained from the same population as that of the test units at time $t = 500$ h.

c. Assume that your estimate of μ and σ has ±20 percent error from the actual values. What is the range of the reliability estimate at $t = 500$ h?

d. Which parameter has more effect on reliability?

5.22 The most common choice of metallic spring material is carbon steel, either wire or flat form. The majority of spring applications require significant deflections, but however good the spring material, there are limits over which it can be expected to work consistently and exhibit a reasonable fatigue life. Most spring failures result from too high forces creating too high material stresses for too many deflections. Manufacturers use different methods such as shot peening or vapor blasting to increase the working stresses under fatigue conditions.

A reliability test is conducted on 20 springs to determine the effect of shot peening on their expected lives. The following failure times (in hours) are obtained.

610, 1090, 1220, 1430, 2160, 2345, 3535, 3765, 4775, 4905, 6500, 7250, 7900, 8348, 9000,
9650, 9980, $11{,}000^{+}$, $11{,}000^{+}$, $11{,}000^{+}$.

The "+" indicates censoring. Assume that the failure data follow a half-logistic distribution. Determine the parameters of the distribution, the reliability of a spring at time $t = 9700$ h, and the unbiased estimate of the mean failure time.

5.23 Assume that the failure data of Problem 5.22 can also fit a logistic distribution of the form

$$f(t)=\frac{\pi e^{-\pi(t-\mu)/\sqrt{3}\sigma}}{\sqrt{3}\sigma\left(1+e^{-\pi(t-\mu)/\sqrt{3}\sigma}\right)^2}\quad t<\infty, 0<\mu<\infty, 0<\sigma<\infty.$$

The cumulative distribution function and the hazard function are

$$F(t)=\frac{1}{1+e^{-\pi(t-\mu)/\sqrt{3}\sigma}}$$

$$h(t)=\frac{f(t)}{1-F(t)}=\frac{\pi}{\sqrt{3}\sigma}F(t).$$

The density function is similar to the gamma density in that the hazard function approaches a constant, and thus it may be a useful alternative to the Weibull model.

a. Determine the parameters (μ, σ) of the logistic distribution that fit the data of Problem 5.22.

b. What is the mean life of a spring from the same population as that of the test units?

5.24 A manufacturer produces micromotors that rotate at hundreds of thousands of revolutions per minute. The medical devices that utilize such micromotors may eventually be used to perform neurosurgery, unclog arteries, and study abnormal cells. The reliability of the motors is of special concern for the users of such devices. Therefore, the manufacturer conducts a reliability test by subjecting the motors to 1.16 micronewton torque and observes the number of cycles to failure. The average rotation of a motor is 150,000 rpm (revolutions per minute). Twenty-five motors are subjected to the test, and the following numbers of cycles multiplied by 10^7 are observed:

150, 170, 180, 190^+, 195^+, 199, 210, 230, 260, 270^+, 295, 330, 380, 390^+, 420, 460, 500, 560^+, 590, 675, 725, 794, 830, 850, 950^+.

The sign "+" indicates censoring.

a. Use both the CHE and PLE methods to develop reliability functions for the motors.

b. Determine the reliability of a motor after 8.5×10^9 cycles of operations using both methods.

c. What is the estimated MTTF?

5.25 Magnetic abrasive machining is used to achieve 2 μm surface roughness of round steel bars. This process reduces the number of surface notches or scratches, which contribute to the initiation of cracks when the bars are subjected to a fatigue test. Twenty bars are tested, and the time to failure can be expressed by a Special Erlang distribution having the following p.d.f.:

$$f(t) = \frac{t}{\lambda^2} e^{-t/\lambda},$$

where λ is the parameter of the distribution. Derive expressions to estimate λ for both complete samples and right-censored samples. The failure times of the units are

1000, 1500, 1700, 1900, 2200, 2350, 2880, 3309, 3490, 3600, 3695, 3825, 4050, 5000, 6000, 6750, 8000^+, 8000^+, 8000^+, 8000^+.

The sign "+" indicates censoring.

a. What is the estimated value of λ?

b. What is the variance of the estimator of λ?

c. Determine the 95% confidence interval for λ.

d. What is the MTTF?

5.26 Use the product limit estimator to develop a reliability expression for the failure data in Problem 5.25.

a. Compare the MTTF obtained by using the developed reliability expression with that obtained from the Special Erlang distribution.

b. Explain the source of differences.

5.27 Assume that a reliability engineer fits a half-logistic distribution to the failure data given in Problem 5.25. An additional reliability test is conducted and the following data are obtained:

1500, 2000, 2200, 2800, 3500, 3900, 4500, 4900, 5200, 5750, 6125, 6680, 7125, 7795, 8235, 8699, 9000^+, 9000^+, 9000^+.

The sign "+" indicates censoring. The engineer fits an exponential distribution to the data. Realizing that some of the data fitted by the half-logistic distribution exhibits an increasing hazard rate and that the data fitted by the exponential distribution exhibits a constant hazard rate, the engineer decides to combine the two data sets and fit them using one distribution. In doing so, the hazard rate of the mixture may exhibit a decreasing failure rate.

a. Determine the conditions that result in a decreasing hazard rate (if it exists in this case).

b. What is the estimated MTTF based on the mixed distribution?

5.28 The inverse Weibull distribution is sometimes used to describe the failure time of components since it is the limiting distribution of the largest order statistics. Its distribution function is expressed as

$$F(t) = \exp^{-(t/\theta)^{-\gamma}}$$

Assume n units are subjected to a test and r different failure times are recorded as $t_1 \le t_2 \le \ldots \le t_r$. The remaining $n - r$ units are censored and their censoring times are $t_r = t_1^+ = t_2^+ = \ldots = t_{n-r}^+$. Assuming that the failure times follow the inverse Weibull distribution,

a. Use the MLE to obtain the parameters of the distribution

b. Derive expressions for the reliability function and the MTTF.

5.29 Fit the following failure-time observations to an inverse Weibull distribution. Obtain the parameters of the distribution and expressions for its reliability function and MTTF.

0.84	1.17	2.55
0.88	1.29	2.91
0.91	1.29	3.24
0.93	1.38	3.73
0.95	1.43	4.16
1.01	1.86	4.79
1.10	1.87	6.67
1.10	1.89	8.39
1.14	2.01	8.60
1.15	2.40	10.77

5.30 Fit the following failure-time observations to a Birnbaum–Saunders distribution. Obtain the parameters of the distribution and expressions for its reliability function and MTTF.

3	12	24	33
3	12	24	34
5	13	26	40
5	13	26	44
7	16	26	49
7	19	28	51
8	21	28	51
8	21	30	51
9	23	30	77
10	24	32	81

5.31 In addition to the failure-time observations in Problem 5.30, there are 10 censored observations at time = 90. Obtain the parameters of the distribution and expressions for its reliability function and MTTF.

5.32 One of the main causes of failure of water pumps in cars is the bearings. When subjected to high stresses and high speeds, the bearings tend to fatigue and fail after about 60,000 mi. The failure times of 25 bearings that were placed in service at the same time are recorded below (note that the last three observations are censored).

3,632	29,367	41,679	53,730	82,787
11,241	29,538	41,679	56,780	90,004
17,508	29,786	44,790	58,992	110,000
26,020	31,813	51,639	58,992	110,000
28,602	38,983	53,060	82,099	110,000

a. Use the CHE and plot the reliability of the pumps versus time.

b. Use Kaplan–Meier estimator and plot the reliability of the pumps versus time.

c. Estimate the MTTF using both methods (note that the data indeed follow a Weibull model with $\gamma = 1.5$ and $\theta = 60{,}000$). Which approach results in a more accurate estimate of the MTTF?

d. Determine the standard deviation for every point estimate of the observations.

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CHAPTER 6

MODELS FOR ACCELERATED LIFE TESTING

6.1 INTRODUCTION

In Chapter 5, various methods for estimating the parameters of the underlying failure-time distributions were presented. Confidence intervals for the values of the parameters were also presented and discussed. Once the failure-time distribution is identified and its parameters are estimated, we can then estimate reliability metrics of interest, such as the expected number of failures during a specified time interval [T_1, T_2] and the mean time to failure (MTTF). Obviously, such metrics are useful when they are estimated from failure data obtained at the normal operating conditions of the components, products, or systems of interest. However, the high rate of technological advances and innovations are spurring the continuous introduction of new products and services which are expected to perform the intended functions satisfactorily for extended periods of time. Testing under normal operating conditions requires a very long time especially for components and products with long expected lives, and it requires extensive number of test units, so it is usually costly and impractical to perform reliability testing under normal conditions. Therefore, reliability engineers seek alternative methods to "predict" the reliability metrics using data and test conditions other than normal operating conditions. We refer to these methods as accelerated testing methods. The main objective of these methods is to induce failures or degradation of the components, units, and systems in a much shorter time and to use the failure data and degradation observations at the accelerated conditions to estimate the reliability at normal operating conditions.

Careful reliability testing of systems, products, and components at the design stage is crucial in order to achieve the desired reliability at the field operating conditions. During the design stage of many products, especially those used in military, the elimination of design weaknesses inherent to intermediate prototypes of complex systems is conducted via the test, analyze, fix, and test (TAFT) process. This process is generally referred to as "reliability growth." Specifically, reliability growth is the improvement in the true but unknown initial reliability of a developmental item as a result of failure mode discovery, analysis, and effective correction. Corrective actions generally assume the form of fixes, adjustments, or modifications to problems found in the hardware, software, or human error aspects of a system (Hall et al., 2010). Likewise, field test results are used in improving product design and consequently its reliability.

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The above examples and requirements have magnified the need for providing more accurate estimates of reliability by performing testing of materials, components, and systems at different stages of product development.

There is a wide variety of reliability testing methodologies and objectives. They include testing to determine the potential failure mechanisms, reliability demonstration testing, reliability acceptance testing, reliability prediction testing using accelerated life testing (ALT), and others. In this chapter, we provide a brief description of these different types of testing with emphasis on ALT, design of test plans and modeling, and analysis of degradation testing.

6.2 TYPES OF RELIABILITY TESTING

As stated above, there is a wide variety of testing each has its objective and method of conducting the test. They include the following.

6.2.1 Highly Accelerated Life Testing (HALT)

The objective of this test is to determine the operational limits of components, subsystems, and systems. These are the limits beyond which different failure mechanisms occur other than those that occur at normal operating conditions. Moreover, extreme stress conditions are applied to determine all potential failures (and failure modes) of the unit under test. HALT is primarily used during the design phase of a product. In a typical HALT test, the product (or component) is subject to increasing stress levels of temperature and vibration (independently and in combination) as well as rapid thermal transitions (cycles) and other specific stresses related to the actual use of the product. In electronics, for example, HALT is used to locate the causes of the malfunctions of an electronic board. These tests often consist of testing the product under temperature, vibration, and moisture. However, the effect of humidity on the product's failure mechanism requires a long time. Consequently, HALT is conducted only under two main stresses—temperature and vibration. The defects which may appear are reversible or irreversible in nature. The reversible defects make it possible to define the functional limits. However, the irreversible defects make it possible to estimate the limits of destruction. The test also reveals the potential types of failures and the limitations of the design. The results are used to improve the product's quality and its reliability. It is difficult to use the results from the HALT test for reliability prediction due to the short test periods and the extreme stress levels used in the test. Indeed, HALT is not an ALT as its focus is on testing the product to induce failures that are unlikely to occur under normal operating conditions in order to determine the root cause of potential failures. The stress range and method of its application (cyclic, constant, step) are dependent on the type of component to be tested.

6.2.2 Reliability Growth Test (RGT)

The objective of RGT is to provide continuous improvement of the unit during its design phase. This test is conducted at normal operating conditions, and test results are analyzed to verify whether a specific reliability goal has been reached. In general, the first prototypes of a product are likely to contain design deficiencies most of which can be discovered through a rigorous testing program. It is also unlikely that the initial design will meet the target reliability metrics. It is rather common that the initial design will experience iterative changes (design changes) that will lead to improvements in the reliability of a product. Once the design is released to production, the product is tested and monitored in the field for potential design changes that will further improve its reliability. In summary, a reliability growth program is a well-structured process of finding reliability problems and failures by testing, incorporating corrective actions, and design changes that will improve the product's reliability throughout the design and test phases of the product. It is important that the reliability metrics are carefully and realistically defined in the early design stages by incorporating the users' expectations and experience with proven similar products. This will avoid expectations of unachievable reliability metrics under time and cost constraints.

6.2.3 Highly Accelerated Stress Screening (HASS)

The main objective of HASS is to conduct screening tests on regular production units in order to verify that actual production units continue to operate properly when subjected to the cycling of environmental variables used during the test (Lagattolla, 2005). It is also used to detect shift (and changes) in the "quality" of a production process. It uses the same stresses as those used during the HALT test except the stresses are derated since HASS is primarily used to detect process shift and not design marginality issues. Therefore, temperature stress is about ±15–20% of the operating limits, and vibration acceleration (in G) is about 50% of destruction limit. Other stresses would be reduced to within the component's specifications.

6.2.4 Reliability Demonstration Test (RDT)

Reliability demonstration test is conducted to demonstrate whether the units (components, subsystems, systems) meet one or more measures (metrics) of reliability. These include a quantile of population that does not fail before a defined number of cycles or a test duration. It may also include that the failure rate after test should not exceed a predetermined value. Usually, RDT is conducted at the normal operating conditions of the units.

There are several methods for conducting RDT such as the "success-runs" test. This test involves the use of a sample with a predetermined size, and if no more than a specific number of failures occurs by the end of the test, then the units are deemed acceptable. Otherwise, depending on the number of failures in the test, another sample is drawn, and the test is conducted once more or the units are deemed unacceptable. Formally, in a success-runs test, the units are acceptable when N units survive at least L cycles under test conditions. Let $S_L = N$ be the number of surviving units at the end of the test and p_L be the probability of failure at the time corresponding to the L cycles. Then the binomial distribution of the surviving units gives the significance equation (Feth, 2009)

$$(1 - p_L)^N = \alpha,$$

where α is the significance of the statistical test.

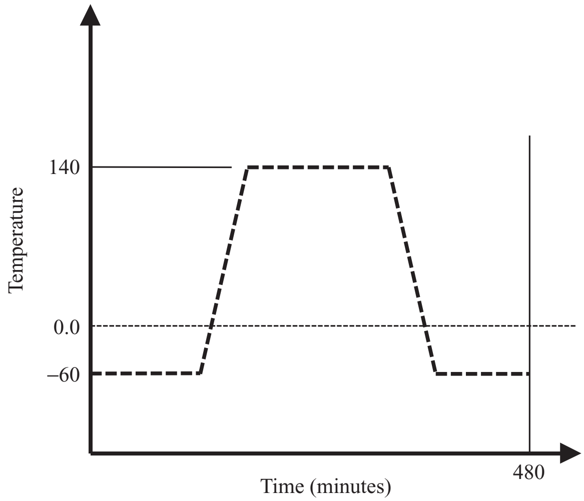

FIGURE 6.1 Temperature profile for an acceptance test.

6.2.5 Reliability Acceptance Test

The objective of the reliability acceptance test is similar to that of the RDT where units are subjected to a well-designed test plan, and decisions regarding the acceptance of the unit (units, prototypes, etc.) are made accordingly. An example of a reliability test plan of a major automotive company is shown in Figure 6.1.

In this figure, the test units are subjected to a temperature from –60°C to 140°C over a cycle of 480 min. The units are deemed acceptable when they survive 200 cycles without failure. The accepted units are expected to have a mean life equivalent to 100,000 mi of driving (based on the engineers' experience). Other acceptance test in automotive industry requires cycling the cable for the car hood 6000 times without failure. Such acceptance test is based on experience. However, optimum test plans can be designed for acceptance test based on some criterion and constraints.

6.2.6 Burn-In Test

The objective of the burn-in test is to screen out the substandard (less reliable) components from a population. This is usually performed at conditions slightly severer than normal operating conditions for a relatively short period of time. A typical burn-in test is performed at 80/80 (80°F and 80% relative humidity) for 24–48 h in order to remove defects contributing to infant mortality (early failure-time region). Almost all integrated circuit (IC) makers conduct burn-in by combining electrical stresses and temperature over time in order to activate temperature-dependent and voltage-dependent failure mechanisms in a short time. Since the "weak" components have a separate distribution than the remaining components, then the failure-time distribution of a unit from this population is modeled as a mixture of two distributions (see

Chapter 1). Some of the issues to be considered when designing a burn-in test include the duration of the test, the types and levels of the stresses, and the residual life after the burn-in test. Tsai and Tseng (2011) developed a cost model to determine the optimal termination time of a burn-in test considering the degradation of the units under test.

6.2.7 Accelerated Life Testing (ALT) and Accelerated Degradation Testing (ADT)

In many cases, ALT might be the only viable approach to assess whether the product meets the expected long-term reliability requirements. ALT experiments can be conducted using three different approaches. The first is conducted by accelerating the "use" of the unit at normal operating conditions such as in the cases of products that are used only a fraction of time in a typical day which includes home appliances and auto tires. The second is conducted by subjecting a sample of units to stresses severer-than-normal operating conditions in order to accelerate the failure. The third is conducted by subjecting units that exhibit some type of degradation such as stiffness of springs, corrosions of metals, and wearout of mechanical components to accelerated stresses. The last approach is referred to as ADT.

The reliability data obtained from the experiments are then utilized to construct a reliability model for predicting the reliability of the product under normal operating conditions through a statistical and/or physics-based inference procedure. The accuracy of the inference procedure has a profound effect on the reliability estimates and the subsequent decisions regarding system configuration, warranties, and preventive maintenance schedules. Specifically, the reliability estimate depends on two factors, the ALT model and the experimental design of the ALT test plans. A "good" model can provide an appropriate fit to the testing data and results in achieving accurate estimates at normal operating conditions. Likewise, an optimal design of the test plans, which determines the stress loadings (constant-stress, ramp-stress, cyclic-stress, etc.), allocation of test units to stress levels, number of stress levels, optimum test duration, and other experimental variables, can improve the accuracy of the reliability estimates. Indeed, without an optimum test plan, it is likely that a sequence of expensive and time-consuming tests result in inaccurate reliability estimates. This might also cause delays in product release, or the termination of the entire product as has been observed by the author.

We describe briefly the methods of stress application, types of stresses, and focus on the reliability prediction models that utilize the failure data at stress conditions to obtain reliability information at normal conditions.

6.3 ACCELERATED LIFE TESTING

Accelerated life testing is usually conducted by subjecting the product (or component) to severer conditions than those which the product will be experiencing at normal conditions or by using the product more intensively than in normal use without changing the normal operating conditions. We refer to these approaches as *accelerated stress* and *accelerated failure time* (AFT), respectively. It is clear that the accelerated failure-time approach is suitable for products or components that are used on a continuous time basis such as tires, toasters, heaters, and light bulbs. For example, in evaluating the failure-time distribution of light bulbs that are used on the average about 6 h/day, 1 year of operating experience can be compressed into 3 months by using the light bulb 24 h every day. Similarly, the failure-time distribution of automobile tires which are used on the average about 2 h/day (equivalent to 60 mi/day) can be obtained by observing the failure times during 70 days of continuous use (50,000 mi) at normal operating conditions. The AFT testing is preferred to the accelerated stress testing since no assumptions need to be made about the relationship of the failure-time distributions at both accelerated and normal conditions. Of course, this is possible only if the time can be compressed as discussed earlier. When it is not possible to compress the product life due to the constant use of the product—such as the case of components of a power-generating unit, communication satellites, and monitors of the air traffic controllers—then reliability estimates of such products or components can be obtained by conducting an accelerated test at stress (temperature, humidity, volt, vibration) levels higher than those of the normal operating conditions. The results at the accelerated stress testing are then related to the normal conditions by using appropriate models as illustrated later in this chapter.

Conducting ALT requires understanding of the types of stresses applied to the units at normal operating conditions as well as the physics of failure mechanisms. Stress loading procedures at accelerated conditions as well as the stress levels have major effects on the accuracy of the reliability prediction at normal operating conditions. Finally, the test results are used by proper reliability prediction (extrapolation) models with realistic assumptions to predict reliability at normal conditions (or other operating conditions). We begin by presenting the types of stresses and their loading procedures. We follow this by presenting reliability prediction models for both ALT and ADT.

6.3.1 Stress Loading

Traditionally, ALT is conducted under constant stresses during the entire test duration. The test results are used to extrapolate the product life at normal conditions. In practice, constant-stress tests are easier to carry out but need more test units and a long time at low stress levels to yield sufficient degradation or failure data. However, in many cases, the available number of test units and test duration are extremely limited. This has led to the consideration of different stress loading. Figure 6.2 shows examples of various stress loadings as well as their adjustable parameters. Some of these stress loadings have been widely utilized in ALT experiments. For instance, static-fatigue tests and cyclic-fatigue tests (Matthewson and Yuce, 1994) have been frequently performed on optical fibers to study their reliability; dielectric-breakdown of thermal oxides (Elsayed et al., 2006) have been studied under elevated constant electrical fields and temperatures; the lifetime of ceramic components subject to slow crack growth due to stress corrosion have been investigated under cyclic stress by the National Aeronautics and Space Administration (NASA) as stated in Choi and Salem (1997). These stress loadings are selected because of the ease and convenience of statistical analyses and familiarity of the existing analytical tools and industrial routines without following a systematic refinement procedure. Due to tight budgets and time constraints, there is an increasing need to determine the best stress loading in order to shorten the test duration and reduce the total cost while achieving an accurate reliability estimate. Figure 6.2a shows a constant stress loading which is usually conducted at different stress levels (high, medium, and low) where the test units are subjected to constant

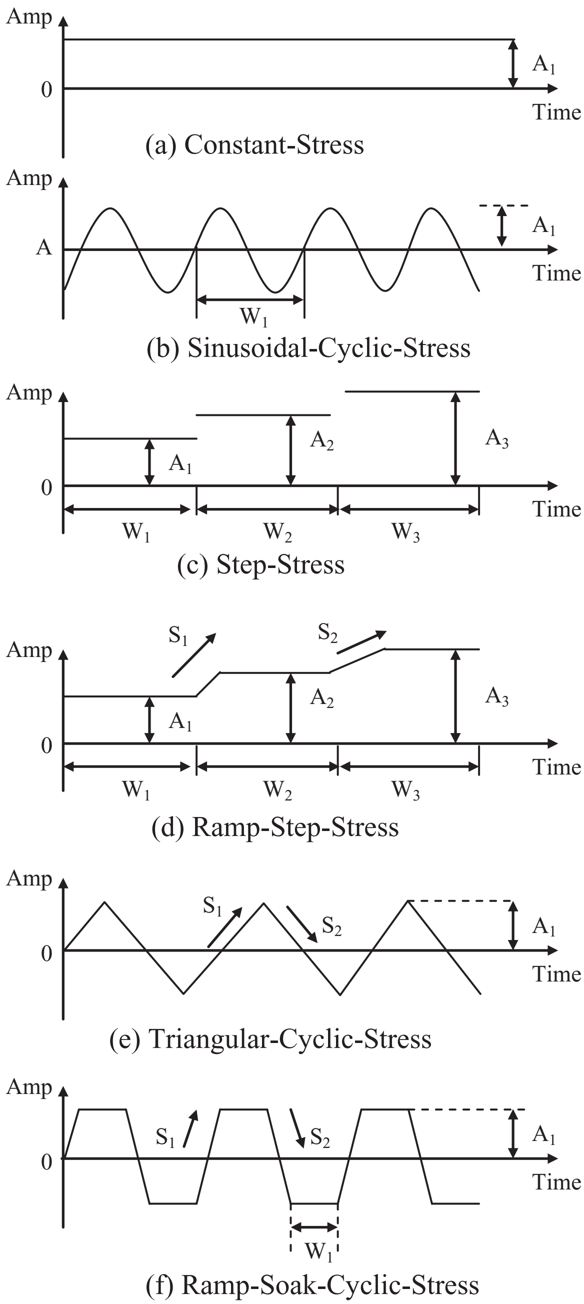

FIGURE 6.2 Various loadings of a single type of stress.

stress for the entire test duration. The main decision variables are the duration of the test and the stress level (provided that the stress type is properly selected).

Figure 6.2b shows cyclic stress loading which is commonly used in fatigue testing and power cycling. The frequency and amplitude are key factors in determining the severity of the stress. Figure 6.2c is a step-stress loading where stress is applied for a period of time then is increased and kept constant at a higher stress level for another period of time. The process is repeated until the maximum feasible stress level is applied. We refer to this step-stress as simple step-stress when it involves only two stresses. The decision variables of this stress loading are normally the stress level and the duration of the test at selected stress levels. Figure 6.2d is a variant of the step-stress in Figure 6.2c where shifting from one stress level to another is not instantaneous (case of step in temperature). Figure 6.2e,f are variants of the above stress loading. Of course, the loadings shown in Figure 6.2 are only examples of possible stress loadings. The choice of stress loading in the presence of multiple stresses is a challenging problem which is currently under investigation.

6.3.2 Stress Type

In order to determine the type of stresses to be applied in ALT, it is important to understand the potential failures of the components and the causes of such failures. As discussed in Section 6.2, this can be accomplished via HALT, physics of failures, and engineering experience. In general, the type of applied stresses depends on the intended operating conditions of the unit and the potential cause of failures. Of course, this is dependent on the materials of the unit, assembly process, and other factors. We group the type of stresses as follows.

6.3.2.1 Mechanical Stresses *Fatigue* stress is the most commonly used accelerated test for mechanical components. Fatigue is the cause of failures of all rotating mechanical components. When the components are subject to elevated temperature, then *creep* testing (which combines both temperature and static or dynamic loads) should be applied. The application of the load is similar to the cyclic load shown in Figure 6.2b. *Shock* and vibration testing is suitable for components or products subject to such conditions as in the case of bearings, shock absorbers, cell phones, tires, and circuit boards in airplanes and automobiles. Other mechanical stresses include combinations of the above.

Wearout is another cause of moving mechanical parts. Depending on the actual use of the unit at normal operating conditions, an accelerated test that mimics these conditions needs to be designed but with increased loads to cause significant wearout of the unit.

6.3.2.2 Electrical Stresses These include power cycling, electric field, current density, and electromigration. Electric field is one of the most common electrical stresses as it induces failures in relatively short times, and its effect is significantly higher than other types of stresses. Thermal fatigue which is induced by change in temperature of the solder joints due to power cycling is another major cause of failure of electronic components.

6.3.2.3 Environmental Stresses Temperature and thermal cycling are commonly used for most of products. As stated earlier, it is important to use appropriate stress levels that do not induce different failure mechanisms than those at normal conditions. Humidity is as critical as temperature, but its application usually requires a very long time before its effect is noticed. Other environmental stresses include ultraviolet light which affects the strength of elastomers, sulfur dioxide which causes corrosion in circuit boards, salt and fine particles, and alpha rays which cause the failure of the read access memory (RAM) and similar components. Likewise, high levels of ionizing can free electrons in outer orbits which results in electronic noise and signal spikes in digital circuits. Therefore, radiation is an environmental stress that should be applied to the units subject to deployment in space and other similar environments. Corrosion is yet another cause of failure of most ferrous material and is induced due to humidity and corrosive environment. Units that are subject to corrosion should then be tested using humidity and other corrosive environment as a stress.

Furthermore, as is often the case, products in actual use are usually exposed to multiple stresses such as temperature, humidity, electric current, electric field, and various types of shocks and vibration. Such units should be subjected to multiple types of stresses simultaneously in order to "mimic" the operating environments which normally result in different failure modes than those found when testing the units under these stresses separately.

6.4 ALT MODELS

Many ALT models have been developed and successfully implemented in a variety of engineering applications. The important assumption for relating the accelerated failures to those at normal operating conditions is that the components/products operating at the normal conditions experience the same failure mechanisms as those at accelerated conditions.

Using failure data at ALT condition to predict reliability at normal conditions requires accurate modeling of such relationship. For example, automotive electronics located under the hood are subject to multiple stresses where significant temperature fluctuation, vibration, corrosive gases, and dust contribute to various types of degradation leading to failures, such as cracks of solder joints, loss of connection of connectors, and sensor degradation. It is of interest to know with high confidence what the mileage of normal driving conditions is equivalent to, for each hour on test under accelerated conditions.

We classify the existing ALT models into three categories: *statistics-based models*, *physics-statistics-based models*, and *physics-experimental-based models*, as shown in Figure 6.3. In particular, the statistics-based models are generally used when the relationship between the applied stresses and the failure time of the product is difficult to determine based on physics or chemistry principles. In this case, AFTs are used to determine the model parameters statistically after assuming either a linear or nonlinear life–stress relationship.

The statistics-based models can be further classified into parametric models and semiparametric/nonparametric models. The most commonly used failure-time distributions in the parametric models are the exponential, Weibull, normal, lognormal, gamma, and extreme value distributions. The underlying assumption of these models is that the failure times of the products follow the same distributions at different stress levels. In reality, however, when the failure process involves complex and/or inconsistent failure-time distributions, the parametric models may not interpret the data satisfactorily and the reliability prediction will be far from

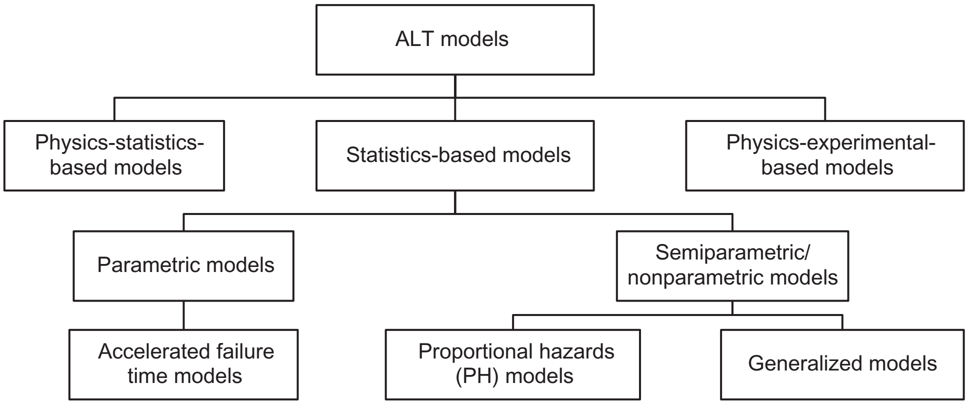

FIGURE 6.3 Classification of ALT models.

accurate. Consequently, semiparametric or nonparametric models appear to be attractive and more suitable for reliability estimation due to their "distribution-free" property. ALT models can also be classified based on their underlying assumptions that relate the reliability metrics at stress conditions to that at normal conditions. These include AFT models, proportional hazards models (PHMs), proportional odds models (POMs), and others. We describe these models as follows.

6.4.1 Accelerated Failure-Time Models

The most widely used class of ALT models is AFT models. For many products, there are well-established acceleration models that perform satisfactorily over the desired range of stresses. For instance, for temperature accelerated testing, the Arrhenius model has gained acceptance because of its many successful applications and general agreement of laboratory and test results with long-term field performance. In an AFT model, it is assumed that for a unit under the applied stress vector z, the log-lifetime $Y = \log T$ has a distribution with a location parameter $\mu(z)$ depending on the stress vector z, and a constant scale parameter $\sigma > 0$ in the form

$$Y = logT = \mu(z) + \sigma\varepsilon,$$

where ε is a random variable whose distribution does not depend on z. The location parameter $\mu(z)$ follows some assumed life–stress relationship, for example, $\mu(z_1, z_2) = \theta_0 + \theta_1 z_1 + \theta_2 z_2$, where z_1 and z_2 are some known functions of stresses. The popular Inverse Power Law and Arrhenius model are special cases of this simple life–stress relationship. The AFT models assume that the covariates (applied stresses) act multiplicatively on the failure time, or linearly on the log failure time, rather than multiplicatively on the hazard rate. The hazard function in the AFT model can then be written in terms of the baseline hazard function $\lambda_0(\cdot)$ as

$$\lambda(t; z) = \lambda_0\left(e^{\beta z} t\right) e^{\beta z}.$$

The main assumption of the AFT models is that the times to failure are inversely proportional to the applied stresses; for example, the time to failure at high stress is shorter than the time to failure at low stress. It also assumes that the failure-time distributions are of the same type. In other words, if the failure-time distribution at the higher stress is exponential, then the distribution at the low stress is also exponential.

In AFT models, we assume that the stress levels applied at the accelerated conditions are within a range of true acceleration—that is, if the failure-time distribution at a high stress level is known and time-scale transformation to the normal conditions is also known, we can mathematically derive the failure-time distributions at normal operating conditions or any other stress condition. For practical purposes, we assume that the time-scale transformation (also referred to as acceleration factor, $A_F > 1$) is constant, which implies that we have a true linear acceleration. Thus, the relationships between the accelerated and normal conditions are summarized as follows (Tobias and Trindade, 1986). Let the subscripts o and s refer to the operating conditions and stress conditions, respectively. Thus,

- The relationship between the time to failure at operating conditions and stress conditions is

$$t_O = A_F \times t_S. \tag{6.1}$$

- The cumulative distribution functions (CDFs) are related as

$$F_o(t) = F_s\left(\frac{t}{A_F}\right). \tag{6.2}$$

- The probability density functions (p.d.f.'s) are related as

$$f_o(t) = \left(\frac{1}{A_F}\right) f_s\left(\frac{t}{A_F}\right). \tag{6.3}$$

- The failure rates are given by

$$\begin{aligned} h_o(t) &= \frac{f_o(t)}{1 - F_o(t)} \\ &= \frac{\left(\frac{1}{A_F}\right) f_s\left(\frac{t}{A_F}\right)}{1 - F_s\left(\frac{t}{A_F}\right)} \\ h_o(t) &= \left(\frac{1}{A_F}\right) h_s\left(\frac{t}{A_F}\right). \end{aligned} \tag{6.4}$$

We now explain the AFT models.

6.4.2 Statistics-Based Models: Parametric

Statistics-based models are generally used when the exact relationship between the applied stresses (temperature, humidity, voltage) and the failure time of the component (or product) is difficult to determine based on physics or chemistry principles. In this case, components are tested at different accelerated stress levels $s_1, s_2, \ldots$. The failure times at each stress level are then used to determine the most appropriate failure-time probability distribution along with its parameters. As stated earlier, the failure times at different stress levels are linearly related to each other. Moreover, the failure-time distribution at stress level s_1 is expected to be the same at different stress levels $s_2, s_3, \ldots$ as well as at the normal operating conditions. The shape parameters of the distributions are the same for all stress levels (including normal conditions), but the scale parameters may be different.

When the failure-time probability distribution is unknown, we use the nonparametric models discussed later in this chapter. We now present the parametric models.

6.4.2.1 Exponential Distribution Acceleration Model This is the case where the time to failure at an accelerated stress s is exponentially distributed with parameter λ_S. The hazard rate at the stress is constant. The CDF at stress s is

$$F_s(t) = 1 - e^{-\lambda_s t}. \tag{6.5}$$

Following Equation 6.2, the CDF at the normal operating condition is

$$F_o(t) = F_s\left(\frac{t}{A_F}\right) = 1 - e^{\frac{-\lambda_s t}{A_F}}. \tag{6.6}$$

Similarly,

$$\lambda_o = \frac{\lambda_s}{A_F}. \tag{6.7}$$

The failure rate at stress level s can be estimated for both noncensored and censored failure data as follows:

$$\lambda_s = \frac{n}{\sum_{i=1}^{n} t_i} \text{ for noncensored data}$$

and

$$\lambda_s = \frac{r}{\sum_{i=1}^{r} t_i + \sum_{i=1}^{n-r} t_i^+} \text{ for censored data,}$$

where t_i is the time of the ith failure, t_i^+ is the ith censoring time, n is the total number of units under test at stress s, and r is the number of failed units at the accelerated stress s.

EXAMPLE 6.1

An accelerated life test is conducted on 20 ICs by subjecting them to 150°C and recording the failure times. Assume that the failure-time data exhibit an exponential distribution with an MTTF at stress condition, $MTTF_S = 6000$ h. The normal operating temperature of the ICs is 30°C, and the acceleration factor is 40. What are the failure rate, the MTTF, and the reliability of an IC operating at the normal conditions at time = 10,000 h (1 year)?

SOLUTION

The failure rate at the accelerated temperature is

$$\lambda_s = \frac{1}{MTTF_s} = \frac{1}{6000} = 1.666 \times 10^{-4} \text{ failures per hour.}$$

Using Equation 6.7, we obtain the failure rate at the normal operating condition as

$$\lambda_o = \frac{\lambda_s}{A_F} = \frac{1.666 \times 10^{-4}}{40} = 4.166 \times 10^{-6} \text{ failures per hour.}$$

The MTTF at normal operation condition, $MTTF_o$, is

$$MTTF_o = \frac{1}{\lambda_o} = 240{,}000 \text{ h.}$$

The reliability at 10,000 h is

$$R(10{,}000) = e^{-\lambda_o t} = e^{-4.166 \times 10^{-6} \times 10^4} = 0.9591. \qquad \blacksquare$$

Typical ALT plans allocate equal units to the test stresses. However, units tested at stress levels close to the design or operating conditions may not experience enough failures that can be effectively used in the acceleration models. Therefore, it is preferred to allocate more test units to the low stress conditions than to the high stress conditions (Meeker and Hahn, 1985) so as to obtain an equal expected number of failures at each condition. They recommend the use of 1:2:4 ratios for allocating units to high, medium, and low stresses, respectively. In other words, the proportions 1/7, 2/7, and 4/7 of the units are allocated to high, medium, and low stresses, respectively. When censoring occurs, we can use the methods discussed in Chapter 5 to estimate the parameters of the failure-time distribution. In the following example, we illustrate the use of failure data at accelerated conditions to predict reliability at normal conditions when the accelerated test is censored. Details of the design of ALT plans are given later in this chapter.

EXAMPLE 6.2

Accelerated life testing is an accelerated test where components are subjected to a high-temperature, high-humidity atmosphere under pressure to further accelerate the testing process. In order to observe the latch-up failure mode associated with complementary metal-oxide-silicon (CMOS) devices where the device becomes nonfunctional and draws excessive power supply current causing overheating and permanent device damage, a manufacturer subjects 20 devices to highly accelerated stress-testing conditions and observes the following failure times in minutes;

> 91, 145, 257, 318, 366, 385, 449, 576, 1021, 1141, 1384, 1517, 1530, 1984, 3656, 4000^+, 4000^+, 4000^+, 4000^+, and 4000^+.

The "+" sign indicates censoring time. Assuming an acceleration factor of 100 is used, what is the MTTF at normal operating conditions? What is the reliability of a device from this population at t = 10,000 min?

SOLUTION

The Bartlett test does not reject the hypothesis that the above failure-time data follow an exponential distribution. Therefore, we estimate the parameter of the exponential distribution at the accelerated conditions as follows:

$$n = 20$$

$$r = 15$$

$$\hat{\lambda}_s = \frac{r}{\sum_{i=1}^{r} t_i + \sum_{i=1}^{n-r} t_i^+}$$

$$\hat{\lambda}_s = \frac{15}{14{,}820 + 20{,}000} = 4.3078 \times 10^{-4} \text{ failures per minute.}$$

The failure rate at normal operating conditions is

$$\lambda_o = \frac{\hat{\lambda}_s}{A_F} = 4.3078 \times 10^{-6} \text{ failures per minute,}$$

and the MTTF is 2.321×10^5 min or 3868 h (about 5 months).

The reliability at 10,000 min is

$$R(10{,}000) = e^{-4.3078 \times 10^{-6} \times 10^4} = 0.9578.$$

■

6.4.2.2 Weibull Distribution Acceleration Model

Again, we consider the true linear acceleration case. Therefore, the relationships between the failure-time distributions at the accelerated and normal conditions can be derived using Equations 6.2 and 6.3. Thus,

$$F_s(t) = 1 - e^{-\left(\frac{t}{\theta_s}\right)^{\gamma_s}} \qquad t \geq 0, \gamma_s > 0, \theta_s > 0$$

and

$$F_o(t) = F_s\left(\frac{t}{A_F}\right) = 1 - e^{-\left(\frac{t}{A_F\theta_s}\right)^{\gamma_s}} = 1 - e^{-\left(\frac{t}{\theta_o}\right)^{\gamma_o}}. \tag{6.8}$$

The underlying failure-time distributions at both the accelerated stress and operating conditions have the same shape parameters—that is, $\gamma_s = \gamma_o$, and $\theta_o = A_F\theta_s$. If the shape parameters at different stress levels are significantly different, then either the assumption of true linear acceleration is invalid or the Weibull distribution is inappropriate to use for analysis of such data.

Let $\gamma_s = \gamma_o = \gamma > 0$. Then the p.d.f. at normal operating conditions is

$$f_o(t) = \frac{\gamma}{A_F\theta_s}\left(\frac{t}{A_F\theta_s}\right)^{\gamma-1} \exp\left[-\left(\frac{t}{A_F\theta_s}\right)^{\gamma}\right] \qquad t \geq 0, \theta_s \geq 0. \tag{6.9}$$

The MTTF at normal operating conditions is

$$MTTF_o = \theta_o\Gamma\left(1 + \frac{1}{\gamma}\right). \tag{6.10}$$

The hazard rate at the normal conditions is

$$h_o(t) = \frac{\gamma}{A_F\theta_s}\left(\frac{t}{A_F\theta_s}\right)^{\gamma-1} = \frac{h_s(t)}{A_F^{\gamma}}. \tag{6.11}$$

EXAMPLE 6.3

Gold bonding failure mechanisms are usually related to gold-aluminum bonds on the IC chip. When gold-aluminum beams are present in an IC, carbon impurities may lead to cracked beams. This is common with power transistors and with analog circuits due to the elevated temperature environment (Christou, 1994). An accelerated life test is designed to cause gold bonding failures in a newly developed transistor. Three stress levels s_1, s_2, and s_3 (mainly temperature) are determined where $s_1 > s_2 > s_3$. The sample sizes for s_1, s_2, and s_3 are 22, 18, and 22, respectively. The failure times at these stresses follow.

Stress level	Failure times in minutes
s_1	438, 641, 705, 964, 1136, 1233, 1380, 1409, 1424, 1517, 1614, 1751, 1918, 2044, 2102, 2440, 2600, 3352, 3563, 3598, 3604, 4473
s_2	427, 728, 1380, 2316, 3241, 3244, 3356, 3365, 3429, 3844, 3955, 4081, 4462, 4991, 5322, 6244, 6884, 8053
s_3	1287, 2528, 2563, 3395, 3827, 4111, 4188, 4331, 5175, 5800, 5868, 6221, 7014, 7356, 7596, 7691, 8245, 8832, 9759, 10,259, 10,416, 15,560

Assume an acceleration factor of 30 between the lowest stress level and the normal operating conditions. What is the MTTF at normal conditions? What is the reliability of a transistor at t = 1000 min?

SOLUTION

Using the maximum likelihood estimation procedure, the parameters of the Weibull distributions corresponding to the stress levels s_1, s_2, and s_3 are

$$\text{For } s_1 : \gamma_1 = 1.953, \theta_1 = 2260.$$
$$\text{For } s_2 : \gamma_2 = 2.030, \theta_2 = 4325.$$
$$\text{For } s_3 : \gamma_3 = 2.120, \theta_3 = 7302.$$

Since $\gamma_1 = \gamma_2 = \gamma_3 \cong 2$, then the Weibull distribution model is appropriate to describe the relationship between failure times at accelerated stress conditions and normal operating conditions. Moreover, we have a true linear acceleration. Thus,

The A_F from s_3 to s_2 = 1.68.

The A_F from s_2 to s_1 = 1.91.

The A_F from s_3 to s_1 = 3.24.

The relationship between the scale parameter θ_s at s_3 and the normal operating conditions is

$$\theta_o = A_F \theta_3 = 30 \times 7302 = 219{,}060.$$

The MTTF at normal conditions is

$$MTTF_o = 219{,}060\,\Gamma\left(\frac{3}{2}\right) = 194{,}130 \text{ min}.$$

The corresponding reliability of a component at 1000 min is

$$R(1000) = e^{-\left(\frac{1000}{219{,}060}\right)^2} = 0.999979161. \qquad \blacksquare$$

6.4.2.3 Rayleigh Distribution Acceleration Model The Rayleigh distribution appropriately describes linearly increasing failure-rate models. When the failure rates at two different stress levels are linearly increasing with time, we may express the hazard rate at the accelerated stress s as

$$h_s(t) = \lambda_s t. \tag{6.12}$$

The p.d.f. for the normal operating conditions is

$$f_o(t) = \frac{\lambda_s t}{(A_F)^2} e^{\frac{-\lambda_s t^2}{2(A_F)^2}} = \lambda_o t e^{\frac{-\lambda_o t^2}{2}},$$

where

$$\lambda_o = \frac{\lambda_s}{A_F^2}. \tag{6.13}$$

The reliability function at time t is

$$R(t) = e^{\frac{-\lambda_s t^2}{2(A_F)^2}}. \tag{6.14}$$

The MTTF at normal conditions is

$$MTTF = \sqrt{\frac{\pi}{2\lambda_o}}. \tag{6.15}$$

EXAMPLE 6.4

The failure of silicon and gallium arsenide substrate is the main cause for the reduction of yield and for the introduction of microcracks and dislocations during processing of ICs. Thermal fatigue crack propagation in the substrate reduces the reliability levels of many electronic products that contain such ICs as components. A manufacturer of ICs subjects a sample of 15 units to a temperature of 200°C and records their failure times in minutes as

> 2000, 3000, 4100, 5000, 5200, 7100, 8400, 9200, 10,000, 11,500, 12,600, 13,400, 14,000$^+$, 14,000$^+$, and 14,000$^+$.

The "+" sign indicates censored time. Assume that the acceleration factor between the accelerated stress and the operating condition is 20 and that the failure times follow a Rayleigh distribution. What is the MTTF of components at normal operating conditions? What is the reliability at t = 20,000 min?

SOLUTION

The parameter of the Rayleigh distribution at the accelerated stress is obtained as

$$\lambda_s = \frac{2r}{\sum_{i=1}^{r} t_i^2 + \sum_{i=1}^{n-r} t_i^{+2}}, \tag{6.16}$$

where r is the number of failed observations, n is the sample size, t_i is the failure time of the ith component, and t_i^+ is the censoring time of the ith component. Thus,

$$\lambda_s = \frac{2\times 12}{8.5803\times 10^8 + 5.88\times 10^8} = \frac{24}{14.4603\times 10^8}$$

$$\lambda_s = 1.6597\times 10^{-8} \text{ failures per minute.}$$

From Equation 6.13, we obtain

$$\lambda_o = \frac{1}{(A_F)^2}\lambda_s = \frac{1.6597\times 10^{-8}}{400} = 4.149\times 10^{-11}.$$

The MTTF is

$$MTTF = \sqrt{\frac{\pi}{2\lambda_o}} = 194{,}575 \text{ min (about 4.5 months).}$$

The reliability at t = 20,000 min is

$$R(20{,}000) = e^{\frac{-\lambda_o t^2}{2}} = 0.8471.$$ ■

6.4.2.4 Lognormal Distribution Acceleration Model Lognormal distribution is widely used in modeling failure times of the accelerated testing of electronic components when they are subjected to high temperatures, high electric fields, or a combination of both temperature and electric field. Indeed, the lognormal distribution is used for calculating the failure rates due to electromigration in discrete and integrated devices.

Another failure mechanism that results in failures of ICs that can be modeled by a lognormal distribution is the fracture of the substrate. For example, in field usage of IC packages, power on and off of the device makes the junction temperature fluctuate (due to the differences in the coefficients of thermal expansion of material in the package). This temperature cycle develops stresses in the substrate, which in turn may develop microcracks and cause failure.

The p.d.f. of the lognormal distribution (see Chapter 1) is

$$f(t)=\frac{1}{\sigma t\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln t-\mu}{\sigma}\right)^2} \qquad t\geq 0. \tag{6.17}$$

The mean and variance of the lognormal distribution are

$$\begin{aligned}\text{Mean} &= e^{\left(\mu+\frac{\sigma^2}{2}\right)}\\ \text{Variance} &= \left(e^{\sigma^2}-1\right)\left(e^{2\mu+\sigma^2}\right).\end{aligned} \tag{6.18}$$

The p.d.f. at accelerated stress s is

$$f_s(t)=\frac{1}{\sigma_s t\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln t-\mu_s}{\sigma_s}\right)^2},$$

and the p.d.f. at normal conditions is

$$f_o(t)=\frac{1}{\sigma_o t\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln t-\mu_o}{\sigma_o}\right)^2}=\frac{1}{\sigma_s t\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{t}{A_F}\right)-\mu_s}{\sigma_s}\right)^2}, \tag{6.19}$$

which implies that $\sigma_o = \sigma_s = \sigma$. This is similar to the Weibull model where the shape parameter is the same for all stress levels. The parameter σ for the lognormal distribution is equivalent to the shape parameter of the Weibull distribution. Therefore, when σ is the same for all stress levels, then we have a true linear acceleration. Equation 6.19 also implies that the parameters μ_s and μ_o are related as $\mu_o = \mu_s + \ln A_F$. The relationship between failure rates at different stress levels is time dependent, and it should be calculated at specified times.

EXAMPLE 6.5

Electric radiant element heaters are used in furnaces that hold molten potline aluminum before casting into ingots. Stainless steel sheet metal tubes are used to protect the elements from the furnace atmosphere and from splashes of molten aluminum, and they are expected to survive for 2.5 years (Esaklul, 1992). The furnace temperatures range from 700°C to 1200°C with metal being cast at 710°C. In order to predict the reliability of the tubes, an accelerated test is performed at 1000°C using 16 tubes, and their failure times (in hours) are

2617, 2701, 2757, 2761, 2846, 2870, 2916, 2962, 2973, 3069, 3073, 3080, 3144, 3162, 3180, and 3325.

Assume that the acceleration factor between the normal operating conditions and the acceleration conditions is 10. What is the mean life at normal conditions assuming a lognormal distribution?

SOLUTION

Using the maximum likelihood estimation procedure, we obtain the parameters of the lognormal distribution at the accelerated conditions as follows:

$$\mu_s = \frac{1}{n}\sum_{i=1}^{16} \ln t_i = \frac{1}{16} \times 127.879 = 7.9925$$

$$\sigma_s^2 = \frac{1}{n}\left[\sum_{i=1}^{16} (\ln t_i)^2 - \frac{1}{n}\left(\sum_{i=1}^{16} \ln t_i\right)^2\right] = 0.0042.$$

The mean life and the standard deviation at the accelerated conditions are

$$\text{Mean life} = e^{\mu_s + \frac{\sigma_s^2}{2}} = 2{,}964.75 \text{ h.}$$

$$\text{Standard deviation} = \sqrt{\left(e^{\sigma_s^2} - 1\right)\left(e^{2\mu_s + \sigma_s^2}\right)} = 192.39.$$

The parameters of the lognormal distribution at normal operating conditions are

$$\mu_o = \mu_s + \ln A_F = 7.9925 + \ln 10 = 10.295,$$

and $\sigma_o = \sigma_s = \sigma$. The mean life is then obtained as

$$e^{\mu_o + \frac{\sigma^2}{2}} = e^{10.2971} = 29{,}649 \text{ h.}$$

This implies that the stress acts multiplicable on the mean lifeor

$$\text{Mean life} = 3.39 \text{ years.}$$

■

It is recognized that the lognormal distribution can be effectively used to model the failure times of metal-oxide-semiconductor (MOS) ICs when they are subjected simultaneously to two types of stress accelerations—thermal acceleration and electric field acceleration. In this case, the p.d.f. at the normal operating conditions can be expressed as

$$f_o(t) = \frac{1}{\sigma_o t\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[\ln\left(\frac{t}{e^{\mu_s} A_T A_{EF}}\right)^{\frac{1}{\sigma_o}}\right]^2\right\}, \tag{6.20}$$

where $\sigma_o = \sigma_s = \sigma$—that is, the shape parameter of the lognormal is the same for all stress levels. Moreover, σ and $\mu_o = \mu_s + \ln A_T + \ln A_{EF}$ are the standard deviation and mean of the logarithmic failure time, respectively. A_T and A_{EF} are the thermal and electric field acceleration factors, respectively. The quantity $e^{\mu_s}/A_T A_{EF}$ can be viewed as a scale parameter. In spite of the popularity of the lognormal distribution as given in Equation 6.20, its hazard-rate function is not a representative model of the device behavior. McPherson and Baglee (1985) advocate the use of the lognormal distribution as given in Equation 6.20 to model the time-dependent dielectric breakdown (TDDB) of MOS ICs. The thermal acceleration factor A_T is the ratio of the reaction rate at the stress temperature T_s to that at the normal operating temperature T_o. It is given by

$$A_T = \exp\left[\frac{E_a}{k}\left(\frac{1}{T_o} - \frac{1}{T_s}\right)\right], \tag{6.21}$$

where E_a is the activation energy and k is Boltzmann's constant ($k = 8.623 \times 10^{-5}$ eV/Kelvin). A more detailed explanation of the activation energy is based on the transition-state theory. The theory predicts, approximately, that the rate of reaction (or breakdown rate) is given by E_aT/k where T is the temperature or the applied stress; E_a, and k are defined above. In effect, the activation energy represents the difference between the energy of reacting molecules at the final stress level and the energy of the molecules at the initial stress level.

The electric field acceleration factor, A_{EF}, between the accelerated electric field and the normal electric field is expressed as

$$A_{EF} = \exp\left(\frac{E_s - E_o}{E_{EF}}\right), \tag{6.22}$$

where E_s is the stress field in MV/cm, E_o is the normal operating field in MV/cm, and E_{EF} is the electric field acceleration parameter. Even though Equation 6.22 is commonly used, other functional forms for the electric field acceleration factor may be used instead. For instance, Chen and Hu (1987) propose that the logarithm of the electric field acceleration is inversely proportional to the stress field E_S—that is,

$$A_{EF} = \exp\left[C_{EF}\left(\frac{1}{E_o} - \frac{1}{E_s}\right)\right], \tag{6.23}$$

where C_{EF} is the proportionality constant. Interaction terms between temperature and electric field may be included in Equation 6.20. For example, the product A_TA_{EF} of the acceleration factor given in Equation 6.20 can be replaced by (McPherson and Baglee, 1985)

$$A_{\text{combined}} = A\exp\left[-\frac{Q}{kT_s}\right]\exp[-\gamma(T_s)E_s], \tag{6.24}$$

where A is a constant which normalizes the acceleration factor to 1 at the operating conditions; Q is an energy term associated with material breakdown; $\gamma(T_s)$ is a temperature-dependent parameter, given by

$$\gamma(T) = B + \frac{C}{T}, \tag{6.25}$$

where B and C are constants. Substitution of Equation 6.25 into Equation 6.24 shows that the combined acceleration factor is proportional to the exponential of

$$\left[\frac{C_1}{T_S}+C_2E_S+C_3\frac{E_S}{T_S}\right],$$

where C_1, C_2, C_3 are constants. Thus, an interaction term $C_3(E_s/T_s)$ is introduced.

EXAMPLE 6.6

Twenty-five long-life bipolar transistors for submarine cable repeaters are subjected to both temperature and electric field accelerated stresses in order to predict the expected mean life at normal operating conditions of 10°C and 5 eV. The accelerated stress conditions are 50°C and 15 eV. The following failure times are obtained from the accelerated test:

830, 843, 870, 882, 900, 932, 946, 953, 967, 992, 1005, 1010, 1019, 1023, 1028, 1035, 1036, 1044, 1054, 1064, 1078, 1099, 1106, 1115, and 1135.

Assume that the electric field acceleration parameter is 3.333 and the activation energy of the bipolar transistors is 0.07 eV. What are the mean life and standard deviation at the normal operating conditions?

SOLUTION

Similar to Example 6.5, we obtain the parameters of the lognormal distribution at the accelerated conditions as

$$\mu_s=\frac{1}{n}\sum_{i=1}^{25}\ln t_i=\frac{1}{25}\times 172.568=6.903$$

$$\sigma_s^2=\frac{1}{n}\left[\sum_{i=1}^{25}(\ln t_i)^2-\frac{1}{n}\left(\sum_{i=1}^{25}(\ln t_i)\right)^2\right]=0.00765.$$

Therefore, the mean life and standard deviation at the accelerated conditions are

$$\text{Mean life}=e^{\mu_s+\frac{\sigma_s^2}{2}}=999\text{ h}$$

$$\text{Standard deviation}=\sqrt{\left(e^{\sigma_s^2}-1\right)\left(e^{2\mu_s+\sigma_s^2}\right)}=87.55\text{ h}.$$

The mean life at normal conditions is obtained as

$$\mu_o = \mu_s + \ln A_T + \ln A_{EF}.$$

With

$$A_T = \exp\left[\frac{E_a}{k}\left(\frac{1}{T_o} - \frac{1}{T_s}\right)\right]$$
$$A_T = \exp\left[\frac{0.07}{8.623\times10^{-5}}\left(\frac{1}{283} - \frac{1}{323}\right)\right] = 1.426$$

and

$$A_{EF} = \exp\left[\frac{15-5}{3.333}\right] = 20.0915.$$

Therefore,

$$\mu_o = 6.903 + \ln 1.426 + \ln 20.0915$$
$$\mu_o = 10.258.$$

The mean life at normal operating conditions is

$$\text{Mean life} = e^{\mu_o + \frac{\sigma^2}{2}} = e^{10.258+0.0038} = 28{,}633 \text{ h}.$$

The mean life is approximately 2.9 years. ■

6.5 STATISTICS-BASED MODELS: NONPARAMETRIC

When the failure-time data involve complex distributional shapes which are largely unknown or when the number of observations is small, making it difficult to accurately fit a failure-time distribution and to avoid making assumptions that would be difficult to test, semiparametric or nonparametric statistics-based models appear to be a very attractive alternative to the parametric ones. There are several nonparametric models that can be used in modeling failure-time data.

In this section, we present two nonparametric models. The first is a widely used multiple regression model and is referred to as the linear model. The second is gaining acceptance in reliability modeling and is referred to as the PHM.

6.5.1 The Linear Model

The standard linear model is

$$T_i = \alpha + \beta x_i + e_i \tag{6.26}$$

or

$$T_i = \alpha + \boldsymbol{\beta}^T \mathbf{x}_i + e_i \quad i = 1, 2, \ldots, n, \tag{6.27}$$

where

T_i = the time to failure of the ith unit;

x_i = the vector of the covariates (stresses) associated with time to failure T_i;

$\boldsymbol{\beta}^T$ = the vector of regression coefficients; and

$e_1, e_2, \ldots, e_n$ = identical and independent error coefficients with a common distribution.

Linear models are connected to hazard models through an accelerated time model (Miller, 1981). Suppose t_o is a survival time with hazard rate

$$\lambda_o(t) = \frac{f_o(t)}{1 - F_o(t)}. \tag{6.28}$$

Also, assume that the survival time of a component with stress vector **x** has the same distribution as

$$t_x = e^{\boldsymbol{\beta}^T \mathbf{x}} t_o. \tag{6.29}$$

If $\boldsymbol{\beta}^T \mathbf{x} < 0$, then t_x is shorter than t_o and that the stress accelerates the time to failure and the acceleration factor is

$$A_F = e^{-\boldsymbol{\beta}^T \mathbf{x}}. \tag{6.30}$$

The hazard rate of t_x is

$$\lambda_x(t) = \frac{f_x(t)}{1 - F_x(t)} \tag{6.31}$$

or

$$\lambda_x(t) = \lambda_o\left(e^{-\boldsymbol{\beta}^T \mathbf{x}} t\right) e^{-\boldsymbol{\beta}^T \mathbf{x}}. \tag{6.32}$$

EXAMPLE 6.7

The reliability modeling of computer memory devices such as dynamic random access memory (DRAM) device is of particular interest to manufacturers and consumers. To predict the reliability of a newly developed DRAM, the manufacturer subjects 22 devices to combined accelerated stress testing of temperature and electric field. The test conditions and the time to failure are recorded in Table 6.1.

TABLE 6.1 Failure Times and Test Conditions

Failure time (hours)	Temperature (°C)	Electric field eV
19.00	200	15
19.00	200	15
19.10	200	15
19.20	200	15
19.30	200	15
19.32	200	15
19.38	200	15
19.40	200	15
19.44	200	15
19.49	200	15
110.00	150	10
110.50	150	10
110.70	150	10
111.00	150	10
111.40	150	10
111.80	150	10
1000.00	100	10
1002.00	100	10
1003.00	100	10
1004.00	100	10
1005.00	100	10
1006.00	100	10

Determine the time to failure at normal operating conditions of 25°C and 5 eV. What is the acceleration factor between the normal conditions and the most severe stress conditions?

SOLUTION

It is important to first convert the temperature from °C to K (Kelvin). Then we develop a multiple regression model in the form

$$t(\text{time to failure}) = \alpha + \beta_1 T + \beta_2 E, \tag{6.33}$$

where

α = constant;

β_1, β_2 = coefficients of the applied stresses;

T = temperature in Kelvin; and

E = electric field in eV.

Using the standard multiple linear regression method, we obtain

$$t = 6059.29 - 17.848T + 160.16E.$$

The time to failure at normal conditions is obtained as

$$t_o = 6059.29 - 17.848 \times 298 + 160.16 \times 5$$
$$t_o = 1541.38 \text{ h}.$$

The acceleration factor is

$$\frac{t_o}{t_s(\text{at } 200°\text{C}, 15\,\text{eV})} = \frac{1541.38}{19.26} = 80.02.$$

■

Of course the effect of the interactions between the applied stresses can be included in the regression model as additional terms with corresponding coefficients. The inclusion of such terms should be based on physics of failure models and the understanding of how such interactions occur. Other regression models such as nonlinear or exponential models can be considered based on the effect of the stresses on the failure time separately or in combination. The following model illustrates this point.

6.5.2 Proportional Hazards Model

The second set of the nonparametric models is the PHM, introduced by Cox (1972). The model is essentially "distribution-free" since no assumptions need to be made about the failure-time distribution. The only assumption that needs to be made about the failure times at the accelerated test is that the hazard-rate functions for different devices when tested at different stress levels must be proportional to one another. In other words, the hazard rate at a high stress level is proportional and higher than the hazard rate at a low stress level. However, the need for proportionality can be relaxed by using time-dependent explanatory variables (time-dependent stress levels) or stratified baseline hazards. One more advantage of the PHM is that it can easily accommodate the coupling effects (interactions) among applied stresses.

Unlike standard regression models, the PHM assumes that the applied stresses have a multiplicative (rather than additive) effect on the hazard rate—a much more realistic assumption in many cases (Dale, 1985). Moreover, the model takes into consideration censored failure times, tied values, and failure times equal to zero. Each of these commonly occurring phenomena causes difficulty when using standard analyses. The basic PHM has been widely used in the medical field to model the survival times of patients and other applications in biology and health care (O' Quigley, 2008). Only recently has the model been used in the reliability field (Dale, 1985; Elsayed and Chan, 1990; Yuan et al., 2011).

The basic PHM is given by

$$\lambda(t; z_1, z_2, \ldots, z_k) = \lambda_0(t)\exp(\beta_1 z_1 + \beta_2 z_2 \ldots + \beta_k z_k), \tag{6.34}$$

where

$\lambda(t; z_1, z_2, \ldots, z_k)$ = the hazard rate at time t for a device (unit) under test with regressor variable (covariates) $z_1, z_2, \ldots, z_k$;

$z_1, z_2, \ldots, z_k$ = regressor variables (these are also called explanatory variables or the applied stresses);

$\beta_1, \beta_2, \ldots, \beta_k$ = regression coefficients; and

$\lambda_0(t)$ = unspecified baseline hazard-rate function.

The explanatory variables account for the effects of environmental stresses (such as temperature, voltage, and humidity) on the hazard rate. We should note that the number of regressor variables may not correspond to the number of environmental stresses used in the ALT. For example, when a device (unit) is subjected to an accelerated temperature T and an electric field E, the hazard function may be explained by three regressors—$1/T$, E, and E/T. The first two terms refer to temperature and electric field effects, whereas the last term refers to the interaction between the two terms. To simplify our presentation, we assume that the number of regressors corresponds to the number of stresses used in the accelerated test.

It is important to reemphasize the fact that in the PHM, it is assumed that the ratio of the hazard rates for two devices tested at two different stresses (such as two temperatures, T_o and T_s); $\lambda(t; T_o)/\lambda(t; T_s)$ does not vary with time. In other words, $\lambda(t; T_o)$ is directly proportional to $\lambda(t; T_s)$—hence the term PHM.

The unknowns of the PHM are $\lambda_0(t)$ and β_i's. In order to determine these unknowns, we utilize Cox's partial likelihood function to estimate β_i's as follows. Suppose that a random sample of n devices under test gives d distinct observed failure times and $n - d$ censoring times. The censoring times are the times at which the functional devices are removed from test or when the test is terminated and $n - d$ devices are still properly functioning. The observed failure times are $t_{(1)} < t_{(2)} < \ldots < t_{(d)}$. To estimate β, we use the partial likelihood function $L(\beta)$ without specifying the failure-time distribution:

$$L(\beta) = \prod_{i=1}^{d} \frac{e^{(\beta z_{(i)})}}{\sum_r e^{(\beta z_{(r)})}}, \tag{6.35}$$

where $z_{(i)}$ is the regressor variable associated with the device that failed at $t_{(i)}$. The index, r, refers to the units under test at time $t_{(i)}$. We illustrate the construction of the likelihood function $L(\beta)$ with the following example.

EXAMPLE 6.8

In an accelerated life experiment, 100 devices are subjected to a temperature acceleration test at $T_1 = 130°\text{C} = 403$ K. One device fails at $t = 900$ h, and the test is discontinued at 1000 h. In other words, the remaining 99 devices survive the test. Another test is performed on five devices but at an elevated temperature of $T_2 = 250°\text{C} = 523$ K. Three devices fail at times 500, 700, and 950 h. Two devices are removed from the test at 800 h. The results of the experiments are summarized in Table 6.2.

TABLE 6.2 Failure Time Data of the Experiments

Time (hours)	Temperature (°C)	Observation
500	250	1 failed
700	250	1 failed
800	250	2 removed
900	130	1 failed
950	250	1 failed
1000	130	99 removed

SOLUTION

In order to simplify the analysis, we determine a scale factor s for the regressor variables (Temperature T_1 and T_2) z_1 and z_2 such that $z_1 = 0$ and $z_2 = 1$. To do so, we use

$$z = s\left(\frac{1}{T_1} - \frac{1}{T}\right)$$

$$z_1 = 0$$

$$z_2 = s\left(\frac{1}{T_1} - \frac{1}{T_2}\right) = 1$$

$$z_2 = s\left(\frac{1}{403} - \frac{1}{523}\right) = 1$$

$$s = 1756 \text{ K}.$$

Now, let us consider the total population of devices under test. There are 105 devices. The probability of failure occurring in a particular device of those tested at 130°C is

$$\frac{e^0}{100 + 5e^{\beta}}.$$

Similarly, the probability of the failure occurring in a particular device of those tested at 250°C is

$$\frac{e^{\beta}}{100 + 5e^{\beta}}.$$

When $\beta = 0$—that is, there is no activation energy, the two probabilities are equal to 1/105. This means that temperature has no effect on the hazard rate. When $\beta > 0$, we have $e^{\beta} > e^{0}$ and the probability of the failure occurring in a particular device of those tested at 250°C is higher than its 130°C counterpart.

We should note that the denominator ($100 + 5e^{\beta}$) used in computing the probability is a weighted sum in which the number of devices is weighted by a hazard coefficient—that is, 100 is weighted by 1, and 5 is weighted by e^{β}. If β is low (the activation energy is low), then it is more likely to observe the first failure from the 130°C group, because there are more devices under test at 130°C. On the other hand, if β is high, then it is more likely that the first failure will be from the 250°C group because the hazard coefficient $e^{\beta} > 1$ is dominant. Given the first failure did occur at 250°C, the probability of the failure occurring in the 250°C group is $e^{\beta}/(100 + 5e^{\beta})$; this is the first term in the partial likelihood function,

$$L(\beta) = \left(\frac{e^{\beta}}{100+5e^{\beta}}\right)\left(\frac{e^{\beta}}{100+4e^{\beta}}\right)\left(\frac{1}{100+e^{\beta}}\right)\left(\frac{e^{\beta}}{99+e^{\beta}}\right).$$

As shown above, $L(\beta)$ is simply the product of the probabilities. To obtain an estimate of β, we take the natural logarithm of the partial likelihood $L(\beta)$ and equate it to zero. ■

Thus,

$$\ln L(\beta) = \sum_{i=1}^{d} z_{(i)}\beta - \sum_{i=1}^{d} \ln\left(\sum_{r} e^{\beta z_{(i)}}\right),$$

and

$$\frac{\partial \ln L(\beta)}{\partial \beta} = 0.$$

Since $\lambda(t; \mathbf{Z}) = \lambda_0(t)e^{\boldsymbol{\beta}\mathbf{Z}}$, the reliability function $R(t; \mathbf{Z})$ is obtained as

$$\begin{aligned} R(t;\mathbf{Z}) &= e^{-\int_0^t \lambda_0(t)e^{\boldsymbol{\beta}\mathbf{Z}}dt} \\ &= R_0(t)^{\exp(\mathbf{Z}\boldsymbol{\beta})}, \end{aligned} \tag{6.36}$$

where $\mathbf{Z}$ is the vector of the applied stresses, $\boldsymbol{\beta}$ is the vector of the regression coefficients, and $R_0(t)$ is the underlying reliability function when $\mathbf{Z} = \mathbf{0}$. To obtain $R_0(t)$, we utilize the life table method proposed by Kalbfleisch and Prentice (1980, 2002). We first group data into intervals $I_1, I_2, \ldots, I_k$ such that $I_j = (b_0 + \ldots + b_{j-1}, b_0 + \ldots + b_j)$, $j = 1, \ldots, k$ is of width b_j with $b_0 = 0$ and $b_k = \infty$. The method then considers the hazard function to be a step function in the form

$$\lambda_0(t) = \lambda_j, \qquad t \in I_j, \qquad j = 1, \ldots, k.$$

Take $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}$ as estimated from the partial likelihood to obtain the maximum likelihood estimate of λ_j as

$$\hat{\lambda}_j = \frac{d_j}{S_j}, \tag{6.37}$$

where d_j is the number of failures in I_j and

$$S_j = b_j \sum_{l \in R_j} e^{Z_l \beta} + \sum_{l \in D_j} (t_l - b_l - \ldots - b_{j-1}) e^{Z_l \beta},$$

where R_j is the number of units under test at $b_0 + \ldots + b_j - 0$ and D_j is the set of units failing in I_j. The corresponding estimator of the baseline reliability function for $t \in I_j$ is

$$\hat{R}_0(t) = \exp\left[-\hat{\lambda}_j \left(t - \sum_{0}^{j-1} b_l\right) - \sum_{1}^{j-1} \hat{\lambda}_i b_i\right]. \tag{6.38}$$

The above estimator is a continuous function of time. The following example illustrates the use of the PHM in using failure data from accelerated condition to estimate the hazard rate at normal operating conditions. This example is based on the data presented in Nelson and Hahn (1978) and the results obtained by Dale (1985).

EXAMPLE 6.9

An accelerated life test is conducted by subjecting motorettes to accelerated temperatures. Four temperature stress levels are chosen and ten motorettes are tested at each level. The number of hours to failure at each stress level is shown in Table 6.3. Estimate the hazard-rate function at normal operating conditions of 130°C.

TABLE 6.3 Hours to Failure of Motorettes[a]

Temperature (°C)	Hours to failure
150	10 motorettes without failure at 8064 h
170	1764, 2772, 3444, 3542, 3780, 4860, 5196
	3 motorettes without failure at 5448 h
190	408, 408, 1344, 1344, 1440
	5 motorettes without failure at 1680 h
220	408, 408, 504, 504, 504
	5 motorettes without failure at 528 h

[a] Reprinted from *Reliability Engineering* 10, C. J. Dale, "Application of proportional hazard model in the reliability field," pp. 1–14, 1985, with permission from Elsevier Science Ltd, The Boulevard, Langford Lane, Kidlington OX5 1GB, UK.

SOLUTION

The failure data exhibit severe censoring since only 17 of the 40 motorettes failed before the end of the test, and none of those tested at 150° experienced failure before the end of the test. There is also a number of tied values: 17 failures occurred at only 11 distinct time values. Moreover, the number of failure-time observations at any stress level is too small to use parametric models to fit the data. Therefore, we use a PHM of the form

$$\lambda(t;\mathrm{Z}) = \lambda_0(t)\exp(\beta\mathrm{Z}),$$

where Z is the reciprocal of the absolute temperature. Fitting the data in Table 6.3 as described earlier (using the SAS software or equivalent), the estimated value of β is −19,725. Thus, with $\lambda_0(t)$ representing the hazard-rate function applying to operating conditions at 130°C, the fitted model is

$$\lambda(t;150°\mathrm{C}) = 10\lambda_0(t)$$
$$\lambda(t;170°\mathrm{C}) = 83\lambda_0(t)$$
$$\lambda(t;190°\mathrm{C}) = 568\lambda_0(t)$$
$$\lambda(t;220°\mathrm{C}) = 7594\lambda_0(t).$$

The baseline hazard-rate function $\lambda_0(t)$ can be estimated using a parametric model such as Weibull or a nonparametric method as discussed above. The nonparametric maximum likelihood estimates of the hazard-rate function at 130°C and 150°C are shown in Table 6.4. Fitting a nonlinear model for the hazard values at 130°C results in

$$\lambda_{130°\mathrm{C}}(t) = 3.68\times10^{-9}t^{1.5866}$$
$$R_{130°\mathrm{C}}(t) = e^{\frac{-t^{2.5866}}{7.02\times10^{8}}}.$$

The reliability at 100 h of operation is

$$R_{130°\mathrm{C}}(100\ \mathrm{h}) = 0.9997.$$

TABLE 6.4 Hazard Rates at Two Temperatures[a]

	Temperature	
Failure time (hours)	**130°C**	**150°C**
408	0.000054	0.001
504	0.000053	0.001
1344	0.000405	0.004
1440	0.000246	0.002
1764	0.001124	0.011
2772	0.001239	0.013
3444	0.001382	0.014
3542	0.001561	0.016
3780	0.001794	0.018
4680	0.002110	0.021
5196	0.002558	0.026

[a] Reprinted from *Reliability Engineering* 10, C. J. Dale, "Application of proportional hazard model in the reliability field," pp. 1–14, 1985, with permission from Elsevier Science Ltd, The Boulevard, Langford Lane, Kidlington OX5 1GB, UK.

■

Verification of the proportional hazards assumption can be achieved by plotting $\ln\left[-\ln\left(\hat{R}(t)\right)\right]$ versus $\ln t$ for different stress levels. Parallel lines indicate that the proportional hazards assumption is satisfied.

The PHM is also capable of modeling the hazard rates of ALT when the covariates (or applied stresses) are time dependent. Examples of time-dependent covariates include step-stressing, linear increase of the applied electric field with time, and temperature cycling.

Another variant of the PHMs is the additive hazards models (AHMs). Under the AHM, the effects of the explanatory variables (applied stresses) are assumed to be additive on the baseline hazard rather than multiplicative as is the case in the PHM approach (Wightman et al., 1994). The most common and simplest to implement form of the covariate effects in an AHM formulation is to assume a linear function (other forms are discussed in Hastie and Tibshirani, 1990):

$$\lambda(t; z_1, z_2, \ldots, z_n) = \lambda_0(t) + \alpha_1 z_1 + \alpha_2 z_2 + \ldots + \alpha_n z_n, \tag{6.39}$$

where $\lambda_0(t)$ is the baseline hazard; $z_1, z_2, \ldots, z_n$ are the covariate values, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the parameters of the model. Like the PHM, the baseline hazard, $\lambda_0(t)$ can be estimated using either parametric or nonparametric approaches. The parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ can be estimated using the linear regression approach.

Clearly, the choice between PHM and AHM depends on the effects of the covariates on the failure data. For example, Wightman et al. (1994) state that the PHM is inappropriate for modeling repairable software system. However, the AHM is more applicable, robust, and simpler to implement.

6.5.3 Proportional Odds Model

The PHM is versatile and usually produces “good” reliability estimation with failure data even when the proportional hazards assumption does not exactly hold. In many applications, however, it is often unreasonable to assume that the effects of covariates on the hazard rates remain fixed over time. Brass (1972) observes that the ratio of the death rates, or hazard rates, of two populations under different stress levels (e.g., one population for smokers and the other for nonsmokers) is not constant with age, or time, but follows a more complicated course, in particular converging closer to unity for older people. The PHM is not suitable for modeling such cases. Brass (1974) proposes a more realistic model as

$$\frac{F(t; \mathbf{z})}{1 - F(t; \mathbf{z})} = \exp\left(\boldsymbol{\beta}^t \mathbf{z}\right) \frac{F_0(t)}{1 - F_0(t)}. \tag{6.40}$$

This model is referred to as the POM since the odds functions $\theta(t) = F(t)/1) - F(t)$ at different stress levels are proportional to each other. After mathematical transformation, the POM in Equation 6.40 can be represented by

$$\lambda(t;z)=\frac{\exp(\boldsymbol{\beta}^t z)\lambda_0(t)}{1-[1-\exp(\boldsymbol{\beta}^t z)]F_0(t)}, \tag{6.41}$$

where $F_0(t)$ is the baseline CDF, $\lambda(t; z)$ is hazard-rate function associated with stress vector z, and $\boldsymbol{\beta}$ is coefficient vector of the stresses.

Murphy et al. (1997) propose profile likelihood to estimate the parameter of the general POM. The number of unknown parameters in the profile likelihood is the number of covariates plus the number of observed failure times. As the size of failure-time sample increases, the estimation procedure becomes extremely difficult.

The general POM with its property of convergent hazard functions is of considerable interest. Unlike the partial likelihood estimation for the PHMs, we set the baseline function to be an odds function instead of the hazard-rate function for the PHMs. A polynomial function is used to approximate the general form of the odds function. This facilitates the construction of the log likelihood function, and the model parameters can easily be estimated by numerical algorithms such as search method or Newton–Raphson method.

We first present the POM and its properties. The underlying assumption of the use of POM in ALT is that the odds of failure are directly proportional to the applied stresses. In other words, the odds of failure at higher stresses are higher than the odds of failure at lower stresses.

6.5.3.1 Derivations of Proportional Odds Model Let $T > 0$ be a failure time associated with stress level z with cumulative distribution $F(t; z)$, and that ratio $F(t; z)/1) - F(t; z)$ be the odds of failure by time t. The POM is then expressed as

$$\frac{F(t;z)}{1-F(t;z)}=\exp(\beta z)\frac{F_0(t)}{1-F_0(t)},$$

where $F_0(t) \equiv F(t; z = 0)$ is the baseline CDF and β is unknown regression parameter. Let $\theta(t; z)$ denote the odds function, then above POM is transformed to

$$\theta(t;z)=\exp(\beta z)\theta_0(t), \tag{6.42}$$

where $\theta_0(t) \equiv \theta(t; z = 0)$ is the baseline odds function.

For two failure-time samples with stress levels z_1 and z_2, the difference between the respective log odds functions is

$$\log[\theta(t;z_1)]-\log[\theta(t;z_2)]=\beta(z_1-z_2),$$

which is independent of the baseline odds function $\theta_0(t)$ and time t. Thus, the odds functions are constantly proportional to the each other. The baseline odds function can be expressed as any monotone increasing function of time t with the property of $\theta_0(0) = 0$. When $\theta_0(t) = t^{\varphi}$, POM presented by Equation 6.42 becomes the log-logistic accelerated failure-time model (McCullagh, 1980; Bennett, 1983), which is a special case of the general nonparametric POMs.

6.5.3.2 Properties of the Odds Function and the Proposed Baseline Odds Function In order to utilize the POM in predicting reliability at normal operating conditions, it is important that both the baseline function and the covariate parameter β be estimated accurately. Since the baseline odds function of the general POMs could be any monotone increasing function, we choose a viable baseline odds function structure to approximate most, if not all, of possible odds function. In order to find such a universal baseline odds function, we investigate the properties of odds function and its relation to the hazard-rate function.

The odds function $\theta(t)$ is denoted by

$$\theta(t)=\frac{F(t)}{1-F(t)}=\frac{1-R(t)}{R(t)}=\frac{1}{R(t)}-1. \tag{6.43}$$

From the properties of reliability function and its relation to odds function shown in Equation 6.43, we derive the following properties of odds function $\theta(t)$:

(1) $\theta(0) = 0,\ \theta(\infty) = \infty$;

(2) $\theta(t)$ is monotonically increasing function in time;

(3) $\theta(t)=\dfrac{1-\exp[-\Lambda(t)]}{\exp[-\Lambda(t)]}=\exp[\Lambda(t)]-1$, and $\Lambda(t) = \ln[\theta(t) + 1]$; and

(4) $\lambda(t)=\dfrac{\theta'(t)}{\theta(t)+1}$.

Plotting the odds functions of some common failure-time distributions as shown Figures 6.4 and 6.5 provides further clarification of the odds functions' properties.

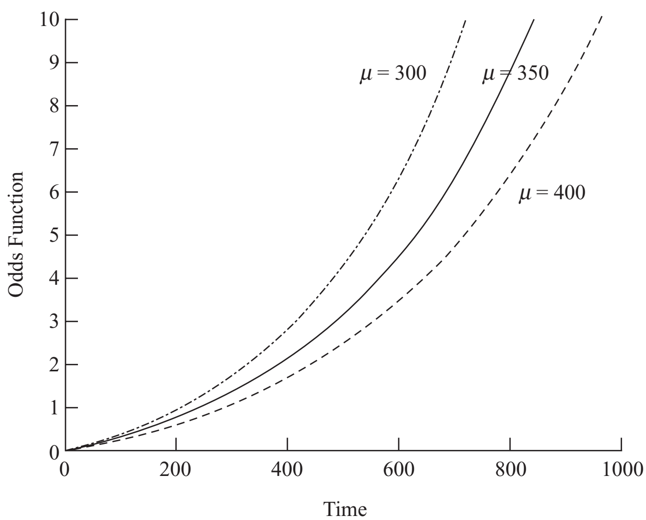

FIGURE 6.4 Odds functions for the exponential distribution ($\lambda = 1/\mu$).

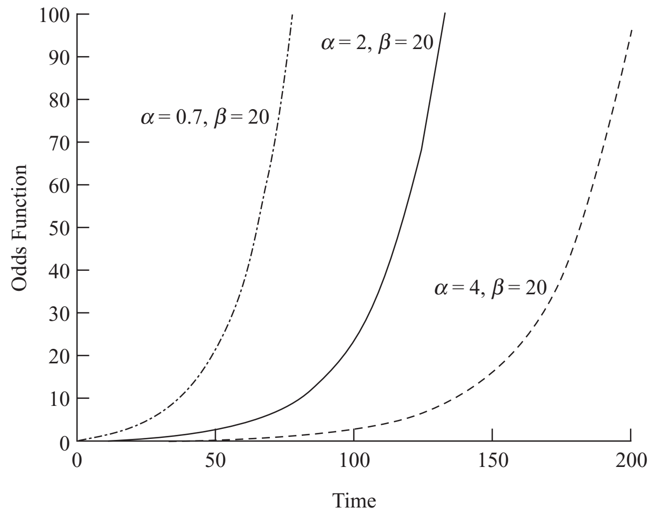

FIGURE 6.5 Odds functions for the gamma distribution.

Based on the properties of the odds function and the plots in Figures 6.4 and 6.5, we could use a polynomial function to approximate the general baseline odds function in the POMs. The proposed general baseline odds function is

$$\theta_0(t) = \gamma_1 t + \gamma_2 t^2 + \gamma_3 t^3 + \ldots$$

Usually high orders are not necessary; second- or third-order polynomial functions are sufficient to cover most of possible odds functions.

6.5.3.3 Log Likelihood Function of the POM-Based ALT Method Consider a baseline odds function is assumed to be quadratic in the form

$$\theta_0(t) = \gamma_1 t + \gamma_2 t^2, \gamma_1 \geq 0, \gamma_2 \geq 0,$$

where γ_1 and γ_2 are constants to be estimated, and the intercept parameter is zero since the odds function crosses the origin according to property (1) in Section 6.5.3.2. Therefore, the POM is represented by

$$\theta(t; z) = \exp(\beta z)\theta_0(t) = \exp(\beta z)(\gamma_1 t + \gamma_2 t^2). \tag{6.44}$$

The hazard-rate function $\lambda(t; z)$ and cumulative hazard-rate function $\Lambda(t; z)$ are

$$\lambda(t; z) = \frac{\theta'(t; z)}{\theta(t; z) + 1} = \frac{\exp(\beta z)(\gamma_1 + 2\gamma_2 t)}{\exp(\beta z)(\gamma_1 t + \gamma_2 t^2) + 1}, \tag{6.45}$$

$$\Lambda(t; z) = \ln[\theta(t; z) + 1] = \ln[\exp(\beta z)(\gamma_1 t + \gamma_2 t^2) + 1]. \tag{6.46}$$

The parameters of the POM with the proposed baseline odds function for censored failure-time data can be estimated as follows. Let t_i represent the failure time of the ith unit, z_i represent the stress level of the ith unit, and I_i represent an indicator function, which is 1 if $t_i \le \tau$ (the censoring time), or 0 if $t_i > \tau$. The log likelihood function of the proposed ALT based on POM is

$$l = \sum_{i=1}^{n} I_i \ln[\lambda(t_i; z_i)] - \sum_{i=1}^{n} \Lambda(t_i; z_i). \tag{6.47}$$

Substituting Equations 6.45 and 6.46 into Equation 6.47 results in

$$\begin{aligned} l = &\sum_{i=1}^{n} I_i\{\beta z_i + \ln(\gamma_1 + 2\gamma_2 t_i) - \ln[\exp(\beta z_i)(\gamma_1 t_i + \gamma_2 t_i^2) + 1]\} \\ &- \sum_{i=1}^{n} \ln[\exp(\beta z_i)(\gamma_1 t_i + \gamma_2 t_i^2) + 1]. \end{aligned} \tag{6.48}$$

Taking the derivatives of the log likelihood function with respect to the three unknown parameters (β, γ_1, γ_2), respectively, we obtain

$$\begin{aligned} \frac{\partial l}{\partial \beta} &= \sum_{i=1}^{n} I_i z_i - \sum_{i=1}^{n} I_i \frac{z_i \exp(\beta z_i)(\gamma_1 t_i + \gamma_2 t_i^2)}{\exp(\beta z_i)(\gamma_1 t_i + \gamma_2 t_i^2) + 1} \\ &\quad - \sum_{i=1}^{n} \frac{z_i \exp(\beta z_i)(\gamma_1 t_i + \gamma_2 t_i^2)}{\exp(\beta z_i)(\gamma_1 t_i + \gamma_2 t_i^2) + 1}, \\ \frac{\partial l}{\partial \gamma_1} &= \sum_{i=1}^{n} I_i \frac{1}{(\gamma_1 + 2\gamma_2 t_i)} - \sum_{i=1}^{n} I_i \frac{\exp(\beta z_i) t_i}{\exp(\beta z_i)(\gamma_1 t_i + \gamma_2 t_i^2) + 1} \\ &\quad - \sum_{i=1}^{n} \frac{\exp(\beta z_i) t_i}{\exp(\beta z_i)(\gamma_1 t_i + \gamma_2 t_i^2) + 1} \\ \frac{\partial l}{\partial \gamma_2} &= \sum_{i=1}^{n} I_i \frac{2t_i}{(\gamma_1 + 2\gamma_2 t_i)} - \sum_{i=1}^{n} I_i \frac{\exp(\beta z_i) t_i^2}{\exp(\beta z_i)(\gamma_1 t_i + \gamma_2 t_i^2) + 1} \\ &\quad - \sum_{i=1}^{n} \frac{\exp(\beta z_i) t_i^2}{\exp(\beta z_i)(\gamma_1 t_i + \gamma_2 t_i^2) + 1}. \end{aligned}$$

The estimates of the model parameters (β, γ_1, γ_2) are obtained by setting the above three derivatives to zero and solving the resultant equations simultaneously. There are no closed-form solutions for these equations. Therefore, the solutions can be obtained by numerical methods such as Newton–Raphson method.

6.5.4 Other ALT Models

In addition to the models discussed earlier in this chapter, there are other models that are used for reliability prediction; each has advantages and limitations. We provide brief presentations of two such models.

6.5.4.1 Extended Linear Hazards Regression Model The PHM and AFT model have different assumptions. The only model that satisfies both assumptions is the Weibull regression model (Kalbfleisch and Prentice, 2002). For generalization, the extended hazard regression (EHR) model (Ciampi and Etezadi-Amoli, 1985; Etezadi-Amoli and Ciampi, 1987; Shyur et al., 1999) is proposed to combine the PHM and AFT model into one form:

$$\lambda(t;\mathbf{z}) = \lambda_0\left(e^{\mathbf{z}'\boldsymbol{\beta}}t\right)\exp(\mathbf{z}'\boldsymbol{\alpha}). \tag{6.49}$$

The unknowns of this model are the regression coefficients $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and the unspecified baseline hazard function $\lambda_0(t)$. The model reflects that the covariate z has both the time-scale changing effect and hazard multiplicative effect. It becomes the PHM when $\boldsymbol{\beta} = 0$ and the AFT model when $\boldsymbol{\alpha} = \boldsymbol{\beta}$.

Elsayed et al. (2006) propose the extended linear hazard regression (ELHR) model which assumes those coefficients to be changing linearly with time:

$$\lambda(t;z) = \lambda_0\left(te^{(\beta_0+\beta_1 t)z}\right)\exp((\alpha_0+\alpha_1 t)z). \tag{6.50}$$

The model considers the proportional hazards effect, time-scale changing effect as well as time-varying coefficients effect. It encompasses all previously developed models as special cases. It may provide a refined model fit to failure-time data and a better representation regarding complex failure processes.

Since the covariate coefficients and the unspecified baseline hazard cannot be expressed separately, the partial likelihood method is not suitable for estimating the unknown parameters. Elsayed et al. (2006) propose the maximum likelihood method which requires the baseline hazard function to be specified in a parametric form. In the EHR model, the baseline hazards function has two specific forms; one is a quadratic function and the other is a quadratic spline. In the proposed ELHR model, we assume the baseline hazard function $\lambda_0(t)$ to be a quadratic function:

$$\lambda_0(t) = \gamma_0 + \gamma_1 t + \gamma_2 t^2. \tag{6.51}$$

Substituting $\lambda_0(t)$ into the ELHR model yields

$$\lambda(t;z) = \gamma_0 e^{\alpha_0 z+\alpha_1 zt} + \gamma_1 t e^{\theta_0 z+\theta_1 zt} + \gamma_2 t^2 e^{\omega_0 z+\omega_1 zt}, \tag{6.52}$$

where

$$\theta_0 = \alpha_0+\beta_0,\ \theta_1 = \alpha_1+\beta_1,\ \omega_0 = \alpha_0+2\beta_0,\ \omega_1 = \alpha_1+2\beta_1.$$

The cumulative hazard-rate fnction is obtained as

$$\begin{aligned}
\Lambda(t;z) &= \int_0^t \lambda(u;z)\,du = \int_0^t \gamma_0 e^{\alpha_0 z+\alpha_1 zu}du + \int_0^t \gamma_1 u e^{\theta_0 z+\theta_1 zu}du + \int_0^t \gamma_2 u^2 e^{\omega_0 z+\omega_1 zu}du \\
&= \frac{\gamma_0}{\alpha_1 z}e^{\alpha_0 z+\alpha_1 zt} - \frac{\gamma_0}{\alpha_1 z}e^{\alpha_0 z} + \frac{\gamma_1 t}{\theta_1 z}e^{\theta_0 z+\theta_1 zt} - \frac{\gamma_1}{(\theta_1 z)^2}e^{\theta_0 z+\theta_1 zt} + \frac{\gamma_1}{(\theta_1 z)^2}e^{\theta_0 z} \\
&\quad + \frac{\gamma_2 t^2}{\omega_1 z}e^{\omega_0 z+\omega_1 zt} - \frac{2\gamma_2 t}{(\omega_1 z)^2}e^{\omega_0 z+\omega_1 zt} + \frac{2\gamma_2}{(\omega_1 z)^3}e^{\omega_0 z+\omega_1 zt} - \frac{2\gamma_2}{(\omega_1 z)^3}e^{\omega_0 z}
\end{aligned}$$

The reliability function, $R(t; z)$, and the p.d.f.'s, $f(t; z)$ are obtained as

$$R(t;z) = \exp(-\Lambda(t;z))$$
$$f(t;z) = \lambda(t;z)\exp(-\Lambda(t;z)).$$

Although the ELHR model is developed based on the distribution-free concept, a close investigation of the model reveals its capability of capturing the features of commonly used failure-time distributions. The main limitation of this model is that "good" estimates of the many parameters of the model require a large number of test units.

6.5.4.2 Proportional Mean Residual Life Model Oakes and Dasu (1990) propose the concept of the proportional mean residual life (PMRL) by analogy with the PHM. Two survivor distributions $F(t)$ and $F_0(t)$ are said to have PMRL if

$$e(x) = \theta e_0(x), \tag{6.53}$$

where $e_0(x)$ is the mean residual life (MRL) at time x.

We extend the model to a more general framework with a covariate vector Z (applied stress):

$$e(t/Z) = \exp(\beta^T Z)e_o(t). \tag{6.54}$$

We refer to this model as the PMRL regression model which is used to model ALT data. Clearly, $e_0(t)$ serves as the MRL corresponding to a baseline reliability function $R_0(t)$and is called the baseline mean residual function, $e(t/Z)$is the conditional MRL function of $T - t$ given $T > t$ and $Z = z$. Where $Z^T = (z_1, z_2, \ldots, z_p)$ is the vector of covariates, $\beta^T = (\beta_1, \beta_2, \ldots, \beta_p)$is the vector of coefficients associated with the covariates, and p is the number of covariates. Typically, we can experimentally obtain $\{(t_i, z_i); i = 1, 2, \ldots, n\}$ the set of failure time and the vectors of covariates for each unit (Zhao and Elsayed, 2005). The main assumption of this model is the proportionality of mean residual lives with applied stresses. In other words, the MRL of a unit subjected to high stress is proportional to the MRL of a unit subjected to low stress.

EXAMPLE 6.10

The designer of an MOS device conducts an accelerated life test in order to estimate its reliability at normal operating conditions of $T = 30°C$ and $V = 26.7$ V. The test is conducted using two types of stresses simultaneously: temperature in centigrade and electric field expressed in volts. The failure times (in hours) for test units subjected to combinations of the two stresses are given in Table 6.5. Obtain the reliability function and determine the MTTF at normal conditions.

SOLUTION

This ALT has two types of stresses: temperature and volt. The covariate Z_1 is a transformation of the temperature as $1/T + 273.16$ while the covariate Z_2 directly represents the volt. The

TABLE 6.5 Failure Times at Different Temperature and Volt

Time	T	Z_1	Z_2
1	25	0.0033539	27
1	25	0.0033539	27
1	25	0.0033539	27
73	25	0.0033539	27
101	25	0.0033539	27
103	25	0.0033539	27
148	25	0.0033539	27
149	25	0.0033539	27
153	25	0.0033539	27
159	25	0.0033539	27
167	25	0.0033539	27
182	25	0.0033539	27
185	25	0.0033539	27
186	25	0.0033539	27
214	25	0.0033539	27
214	25	0.0033539	27
233	25	0.0033539	27
252	25	0.0033539	27
279	25	0.0033539	27
307	25	0.0033539	27
1	225	0.0020074	26
14	225	0.0020074	26
20	225	0.0020074	26
26	225	0.0020074	26
32	225	0.0020074	26
42	225	0.0020074	26
42	225	0.0020074	26
43	225	0.0020074	26
44	225	0.0020074	26
45	225	0.0020074	26
46	225	0.0020074	26
47	225	0.0020074	26
53	225	0.0020074	26
53	225	0.0020074	26
55	225	0.0020074	26
56	225	0.0020074	26
59	225	0.0020074	26
60	225	0.0020074	26
60	225	0.0020074	26
61	225	0.0020074	26
1365	125	0.0025116	25.7

TABLE 6.5 (*Continued*)

Time	T	Z_1	Z_2
1401	125	0.0025116	25.7
1469	125	0.0025116	25.7
1776	125	0.0025116	25.7
1789	125	0.0025116	25.7
1886	125	0.0025116	25.7
1930	125	0.0025116	25.7
2035	125	0.0025116	25.7
2068	125	0.0025116	25.7
2190	125	0.0025116	25.7
2307	125	0.0025116	25.7
2309	125	0.0025116	25.7
2334	125	0.0025116	25.7
2556	125	0.0025116	25.7
2925	125	0.0025116	25.7
2997	125	0.0025116	25.7
3076	125	0.0025116	25.7
3140	125	0.0025116	25.7
3148	125	0.0025116	25.7
3736	125	0.0025116	25.7

sample sizes are relatively small, and it is appropriate to use a nonparametric method to estimate the reliability and MTTF. We utilize the PHM and express it as

$$\lambda(t;T,V)=\lambda_0(t)\exp\left(\frac{\beta_1}{T}+\beta_2 V\right),$$

where T is temperature in Kelvin and V is volt. The parameters β_1 and β_2 are determined using SAS® PROC PHREG and their values are −24,538 and 30.332,739, respectively. The unspecified baseline hazard function can be estimated using any of the methods described in Kalbfleisch and Prentice (2002) and Elsayed (1996). We estimate the reliability function at design conditions by using their values with PROC PHREG.

We assume a Weibull baseline hazard function and fit the reliability values obtained at design conditions to the Weibull function which results in

$$R(t;30°\text{C},26.7)=e^{-1.55065\times10^{-8}t^2}.$$

The MTTF is obtained using the MTTF expression of the Weibull models as shown in Chapter 1:

$$MTTF=\theta^{1/\gamma}\Gamma\left(1+\frac{1}{\gamma}\right)=7166\text{ h}.$$

■

6.6 PHYSICS-STATISTICS-BASED MODELS

The physics-statistics-based models utilize the effect of the applied stresses on the failure rate of the units under test. For example, the failure rate of many ICs is accelerated by temperature, and the model that relates the failure rate with temperature should reflect the physical and chemical properties of the units. Moreover, since several units are usually tested at the same stress level and failure times are random events, the failure-rate expression should also reflect the underlying failure-time distribution. Thus, physics-statistics-based models are needed to describe the failure-rate relationships. The following sections present such models for both single and multiple stresses.

6.6.1 The Arrhenius Model

Elevated temperature is the most commonly used environmental stress for ALT of microelectronic devices. The effect of temperature on the device is generally modeled using the Arrhenius reaction rate equation given by

$$r = Ae^{-(E_a/kt)}, \tag{6.55}$$

where,

r = the speed of reaction;

A = an unknown nonthermal constant;

E_a = the activation energy (eV); energy that a molecule must have before it can take part in the reaction;

k = the Boltzmann constant (8.623×10^{-5} eV/K); and

T = the temperature in Kelvin.

Activation energy (E_a) is a factor that determines the slope of the reaction rate curve with temperature—that is, it describes the acceleration effect that temperature has on the rate of a reaction and is expressed in electron volts (eV). For most applications, E_a is treated as a slope of a curve rather than a specific energy level. A low value of E_a indicates a small slope or a reaction that has a small dependence on temperature. On the other hand, a large value of E_a indicates a high degree of temperature dependence.

Assuming that device life is proportional to the inverse reaction rate of the process, then Equation 6.55 can be rewritten as

$$L = Ae^{+(E_a/kt)}.$$

The lives of the units at normal operating temperature L_o and accelerated temperature L_s are related by

$$\frac{L_o}{L_s} = \frac{e^{(E_a/kT_o)}}{e^{(E_a/kT_s)}}$$

or

$$L_o = L_s \exp \frac{E_a}{k}\left(\frac{1}{T_o} - \frac{1}{T_s}\right). \tag{6.56}$$

When the mean life L_o at normal operating conditions is calculated and the underlying life distribution is exponential, then the failure rate at normal operating temperature is

$$\lambda_o = \frac{1}{L_o},$$

and the thermal acceleration factor is

$$A_T = \frac{L_o}{L_s}$$

or

$$A_T = \exp\left[\frac{E_a}{k}\left(\frac{1}{T_o} - \frac{1}{T_s}\right)\right]. \tag{6.57}$$

Equation 6.57 is the same as Equation 6.21 and is similar to the proportional hazard when E_a/k is replaced by β.

EXAMPLE 6.11

An accelerated test is conducted at 200°C. Assume that the mean failure time of the microelectronic devices under test is found to be 4000 h. What is the expected life at an operating temperature of 50°C?

SOLUTION

The mean life at the accelerated conditions is L_s = 4000 h, the accelerated temperature is T_s = 200 + 273 = 473 K, and the operating temperature is T_o = 50 + 273 = 323 K. Assuming an activation energy of 0.191 eV (Blanks, 1980), then

$$\begin{aligned} L_o &= 4000 \exp\left[\frac{0.191}{8.623\times 10^{-5}}\left(\frac{1}{323} - \frac{1}{473}\right)\right] \\ &= 35{,}198 \text{ h}. \end{aligned}$$

The acceleration factor is

$$A_T = \exp\left[\frac{0.191}{8.63\times 10^{-5}}\left(\frac{1}{323} - \frac{1}{473}\right)\right] = 8.78.$$

Simply, it is the ratio between L_o and L_s or 35,198/4000 = 8.78. ■

6.6.2 The Eyring Model

The Eyring model is similar to the Arrhenius model. Therefore, it is commonly used for modeling failure data when the accelerated stress is temperature. It is more general than the Arrhenius model since it can model data from temperature acceleration testing as well as data from other single stress testing such as electric field. The Eyring model for temperature acceleration is

$$L = \frac{1}{T}\exp\left[\frac{\beta}{T} - \alpha\right], \tag{6.58}$$

where α and β are constants determined from the accelerated test data, L is the mean life, and T is the temperature in Kelvin. As shown in Equation 6.58, the underlying failure-time distribution is exponential. Thus, the hazard rate λ is $1/L$. The relationship between lives at the accelerated conditions and the normal operating conditions is obtained as follows. The mean life at accelerated stress conditions is

$$L_s = \frac{1}{T_s}\exp\left[\frac{\beta}{T_s} - \alpha\right]. \tag{6.59}$$

The mean life at normal operating conditions is

$$L_o = \frac{1}{T_o}\exp\left[\frac{\beta}{T_o} - \alpha\right]. \tag{6.60}$$

Dividing Equation 6.60 by Equation 6.59, we obtain

$$L_o = L_s\left(\frac{T_s}{T_s}\right)\exp\left[\beta\left(\frac{1}{T_o} - \frac{1}{T_s}\right)\right]. \tag{6.61}$$

The acceleration factor is

$$A_F = \frac{L_o}{L_s}.$$

Equation 6.61 is identical to the result of the Arrhenius model given in Equation 6.56 with the exception that the ratio left (T_s/T_o) of the nonexponential curve in Equation 6.61 is set to equal 1. In this case, β reduces to be the ratio between E_a and k (Boltzmann's constant).

The constants α and β can be obtained through the maximum likelihood method, by solving the following two equations for l samples tested at different stress levels and r_i failures ($i = 1, 2, \ldots, l$) are observed at stress level V_i. The equations are the resultants of taking the derivatives of the likelihood function with respect to α and β, respectively and equating them to zero (Kececioglu and Jacks, 1984).

$$\sum_{i=1}^{l} R_i - \sum_{i=1}^{l}\left[R_i/\left(\hat{\lambda}_i V_i\right)\right]\exp\left[\alpha - \beta\left(V_i^{-1} - \bar{V}\right)\right] = 0 \tag{6.62}$$

$$\sum_{i=1}^{l}\left(R_i/\left(\hat{\lambda}_i V_i\right)\right)\left(V_i^{-1} - \bar{V}\right)\exp\left[\alpha - \beta\left(V_i^{-1} - \bar{V}\right)\right] = 0, \tag{6.63}$$

where

$\hat{\lambda}_i$ = the estimated hazard rate at stress V_i,

$$R_i = \begin{cases} r_i & \text{if the location of the parameter is known,} \\ r_i - 1 & \text{if the location of the parameter is unknown,} \end{cases}$$

$$\bar{V} = \frac{\sum_{i=1}^{l} \frac{R_i}{V_i}}{\sum_{i=1}^{l} R_i}.$$

V = stress variable. If temperature, then V is in Kelvin.

EXAMPLE 6.12

A sample of 20 devices is subjected to an accelerated test at 200°C. The failure times (in hours) shown in Table 6.6 are observed. Use the Eyring model to estimate the mean life at 50°C. What is the acceleration factor?

TABLE 6.6 Failure Data of 20 Devices

170.948	6124.780
1228.880	6561.350
1238.560	6665.030
1297.360	7662.570
1694.950	7688.870
2216.110	9306.410
2323.340	9745.020
3250.870	9946.490
3883.490	10187.600
4194.720	10619.100

SOLUTION

Using the data in Table 6.6, we estimate the mean life at the accelerated stress (200°C) as

$$L_s = \frac{1}{20}\sum_{i=1}^{20} t_i = 5300.32 \text{ h}.$$

The constant α is obtained by substituting in Equation 6.62 as follows,

$$20 - \frac{20 \times 5300.32}{473}\exp\left[\alpha - \beta\left(\frac{1}{473} - \frac{1}{473}\right)\right] = 0$$

or

$$20 - 224.115 e^{\alpha} = 0$$
$$\alpha = -2.416.$$

The constant β is obtained by substituting in Equation 6.58 as follows:

$$5300.32 = \frac{1}{473}\exp\left[\frac{\beta}{473} - \alpha\right]$$

or

$$\beta = 5826.706.$$

The mean life at normal operating conditions of 50°C is

$$L_o = 5300.32\left(\frac{473}{323}\right)\exp\left[5826.706\left(\frac{1}{323} - \frac{1}{473}\right)\right]$$
$$L_o = 2.368\times 10^6 \text{ h.}$$

■

The Eyring model can be used effectively when multiple stresses are applied simultaneously at the accelerated life test. For example, McPherson (1986) developed a generalized Eyring model to analyze thermally activated failure mechanisms. The general form of the Eyring model is

$$L_s = \frac{\alpha}{T_s}\exp\left(\frac{E_a}{kT_s}\right)\exp\left[\left(\beta + \frac{\gamma}{T_s}\right)s\right], \tag{6.64}$$

where E_a is the activation energy of the device under test, k is the Boltzmann's constant, T_s is the applied temperature stress in Kelvin, s is the applied physical stress (load/area), and α, β, and γ are constants. The model relates the time to failure (or life) to two different tresses—thermal and mechanical. It predicts a stress-activated energy, provided that two conditions are met: (1) the applied stress must be of the same order of magnitude as the strength of the material, and (2) a stress acceleration parameter must be a function of temperature (Christou, 1994).

6.6.3 The Inverse Power Rule Model

The inverse power rule model is derived based on the kinetic theory and activation energy. The underlying life distribution of this model is Weibull. The MTTF (life) decreases as the nth power of the applied stress (usually voltage). The inverse power law is expressed as

$$L_s = \frac{C}{V_s^n} \qquad C > 0, \tag{6.65}$$

where L_s is the mean life at the accelerated stress V_s, C and n are constants. The mean life at normal operating conditions is

$$L_o = \frac{C}{V_o^n}. \tag{6.66}$$

Thus,

$$L_o = L_s \left(\frac{V_s}{V_o} \right)^n. \tag{6.67}$$

To obtain estimates of C and n, Mann et al. (1974) amended Equation 6.65 without changing its basic character to

$$L_i = \frac{C}{\left(V_i / \dot{V} \right)^n}, \tag{6.68}$$

where L_i is the mean life at stress level V_i and $\dot{V}$ is the weighted geometric mean of the V_i's and is expressed as

$$\dot{V} = \prod_{i=1}^{k} \left(V_i \right)^{R_i / \sum_{i=1}^{k} R_i}, \tag{6.69}$$

where $R_i = \gamma_i$ (number of failures at stress V_i) or $R_i = \gamma_i - 1$ depending on whether or not the shape parameter of the failure-time distribution is known. The likelihood function of C and n is

$$\prod_{i=1}^{k} \Gamma^{-1}(R_i) \left[\frac{R_i}{C} \left(\frac{V_i}{\dot{V}} \right)^n \right]^{R_i} \left(\hat{L}_i \right)^{R_i - 1} \exp\left[-\frac{R_i \hat{L}_i}{C} \left(\frac{V_i}{\dot{V}} \right)^n \right],$$

where $\hat{L}_i$ is the estimated mean life at stress V_i. The maximum likelihood estimators of $\hat{C}$ and $\hat{n}$ are obtained by solving the following two equations:

$$\hat{C} = \frac{\sum_{i=1}^{k} R_i \hat{L}_i \left(V_i / \dot{V} \right)^{\hat{n}}}{\sum_{i=1}^{k} R_i}. \tag{6.70}$$

$$\sum_{i=1}^{k} R_i \hat{L}_i \left(\frac{V_i}{\dot{V}} \right)^{\hat{n}} \ln \frac{V_i}{\dot{V}} = 0. \tag{6.71}$$

The asymptotic variances of $\hat{n}$ and $\hat{C}$ are

$$\sigma_n^2 = \left[\sum_{i=1}^{k} R_i \left(\ln \frac{V_i}{\dot{V}} \right)^2 \right]^{-1}. \tag{6.72}$$

$$\sigma_C^2 = C^2 \left(\sum_{i=1}^{k} R_i \right)^{-1}. \tag{6.73}$$

EXAMPLE 6.13

CMOS ICs suffer from a dielectric induced instability at negative bias, which eventually causes defects and breakdown of the device. A manufacturer subjects two samples of 20 devices to two electric field stresses of 25 V and 10 V, respectively. The failure times (in hours) are listed in Table 6.7.

a. Assuming that the shape parameter of the failure-time distribution is known, use the inverse power model to estimate the mean life at 5 V. What are the variances of the model parameters?

b. Assuming that the failure times follow Weibull distribution, estimate the mean life at the normal operating condition of 5 V.

TABLE 6.7 Failure Data of 40 Devices

25 V test		10 V test	
809.10	3802.88	1037.39	9003.08
1135.93	3944.15	3218.11	9124.50
1151.03	4095.62	3407.17	9365.93
1156.17	4144.03	3520.36	9642.53
1796.53	4305.32	3879.49	10429.50
1961.23	4630.58	3946.45	10470.60
2366.54	4720.63	6635.54	11162.90
2916.91	6265.99	6941.07	12204.50
3013.68	6916.16	7849.78	12476.90
3038.61	7113.82	8452.49	23198.30

SOLUTION

a. Define the 25 V and 10 V stress levels as s_1 and s_2, respectively. Thus,

$$R_{s_1} = R_{s_2} = 20$$

$$L_{s_1} = \frac{\sum \text{failure times}}{20} = 3464.25\ \text{h}$$

$$L_{s_2} = 8298.33$$

$$\dot{V} = (25)^{1/2}(10)^{1/2} = 15.81.$$

Using Equations 6.70 and 6.71, we obtain the corresponding equations below, respectively.

$$\hat{C} = \frac{1}{2}\left[3464.25\left(\frac{25}{15.81}\right)^{\hat{n}} + 8298.33\left(\frac{10}{15.81}\right)^{\hat{n}}\right]$$

$$69{,}285\left(\frac{25}{15.81}\right)^{\hat{n}} \ln 1.5812 + 165{,}966.6\left(\frac{10}{15.81}\right)^{\hat{n}} \ln 0.6325 = 0,$$

which results in $\hat{n} = 0.953{,}18$ and $\hat{C} = 5362.25$.

The mean life at 5 V is

$$L_5 = L_{25}\left(\frac{25}{5}\right)^{0.95318} = 16{,}065 \text{ h}$$

or

$$L_5 = \frac{\hat{C}}{(5/15.81)^{\hat{n}}} = 16{,}065 \text{ h}.$$

The standard deviations of $\hat{n}$ and $\hat{C}$ are

$$\hat{\sigma}_n = 0.7946.$$
$$\hat{\sigma}_C = 847.84.$$

b. The shape parameters of the Weibull distributions at stresses s_1 and s_2 are obtained from fitting a Weibull distribution to each stress. This results in

$$\gamma_{s_1} = 1.98184, \qquad \theta_{s_1} = 3{,}916.97$$
$$\gamma_{s_2} = 1.83603, \qquad \theta_{s_2} = 9{,}343.58$$

Let $\gamma_{s_1} = \gamma_{s_2} = \gamma_o \cong 2$, where γ_o is the shape parameter at the normal operating conditions. Assume an acceleration factor of 1.5. The MTTF is 12,140 h. ■

6.6.4 Combination Model

This model is similar to the Eyring multiple stress model when temperature and another stress such as voltage are used in the accelerated life test. The essence of the model is that the Arrhenius reaction model and the inverse power rule model are combined to form this combination model. It is valid when the shape parameter of the Weibull distribution is equal to one in the inverse power model (Kececioglu and Jacks, 1984). The model is given by

$$\frac{L_o}{L_s} = \left(\frac{V_o}{V_s}\right)^{-n} \exp\left[E_a/k\left(\frac{1}{T_o} - \frac{1}{T_s}\right)\right], \tag{6.74}$$

where

L_o = the life at normal operating conditions;

L_S = the life at accelerated stress conditions;

V_o = the normal operating volt;

V_s = the accelerating stress volt;

T_s = the accelerated stress temperature; and

T_o = the normal operating temperature.

EXAMPLE 6.14

Samples of long-life bipolar transistors for submarine cable repeaters are tested at accelerated conditions of both temperature and volt. The mean lives at the combinations of temperature and volt are given in Table 6.8. Assume an activation energy of 0.2 eV; estimate the mean life at normal operating conditions of 30°C and 25 V.

TABLE 6.8 Mean Lives in Hours at Stress Conditions

	Applied volt (V)			
Temperature (°C)	**50**	**100**	**150**	**200**
60	1800	1500	1200	1000
70	1500	1200	1000	800

SOLUTION

Substitution in Equation 6.74 using two stress levels results in

$$\frac{1800}{1200} = \left(\frac{50}{100}\right)^{-n} \exp\left[\frac{0.2}{8.623\times 10^{-5}}\left(\frac{1}{333} - \frac{1}{343}\right)\right].$$

Solving the above equation, we obtain $n = 0.292$. Therefore,

$$L_o = L_s\left(\frac{V_o}{V_s}\right)^{-n} \exp\left[\frac{E_a}{k}\left(\frac{1}{T_o} - \frac{1}{T_s}\right)\right]$$

$$L_o = 1500\left(\frac{25}{50}\right)^{-0.292} \exp\left[\frac{0.2}{8.623\times 10^{-5}}\left(\frac{1}{303} - \frac{1}{343}\right)\right]$$

$$L_o = 4484.11\text{ h}$$

The acceleration factor = $\dfrac{4484.11}{1500} \cong 3.0$. ■

6.7 PHYSICS-EXPERIMENTAL-BASED MODELS

The time to failure of many devices and components can be estimated based on the physics of the failure mechanism by either the development of theoretical basis for the failure mechanisms or the conduct of experiments using different levels of the parameters, which affect the time to failure. There are many failure mechanisms resulting from the application of different stresses at different levels. For example, the time to failure of packaged silicon ICs due to the electromigration phenomenon is affected by the current density through the circuit and by the temperature of the circuit. Similarly, the time to failure of some components may be affected by relative humidity only.

The following sections present the most widely used models for predicting the time to failure as a function of the parameters that result in device or component failures.

6.7.1 Electromigration Model

Electromigration is the transport of microcircuit current conductor metal atoms due to electron wind effects. If, in an aluminum conductor, the electron current density is sufficiently high, an electron wind effect is created. Since the size and mass of an electron are small compared to the atom, the momentum imparted to an aluminum atom by an electron collision is small (Christou, 1994). If enough electrons collide with an aluminum atom, then the aluminum atom will move, gradually causing depletion at the negative end of the conductor. This will result in voids or hillocks along the conductor, depending on the local microstructure, causing a catastrophic failure. The median time to failure in the presence of electromigration is given by Black's (1969) equation:

$$MTF = AJ^{-n}e^{E_a/kT}, \tag{6.75}$$

where A, n are constants, J is the current density, k is Boltzmann's constant, T is the absolute temperature, and E_a is the activation energy (≅0.6 eV for aluminum and ≅0.9 eV for gold). The electromigration exponent n ranges from 1 to 6.

In order to determine the lives of components at normal operating conditions, we perform ALT on samples of these components by subjecting them to different stresses. In the case of electromigration, the stresses are the electric current and the temperature. Buehler et al. (1991) use linear regression and propagation-of-errors analyses of the linearized equation to select the proper levels of the currents and temperatures. They show that the electromigration parameters such as E_a and n can be obtained from three or more stress conditions.

For a fixed current, we can estimate the median life at the operating temperature as

$$\frac{t_{50}(T_o)}{t_{50}(T_s)} = \exp\left[\frac{E_a}{k}\left(\frac{1}{T_o} - \frac{1}{T_s}\right)\right], \tag{6.76}$$

where $t_{50}(T_i)$is the median life at $T_i(i = o$ or $s)$.

Similarly, we can fix the temperature and vary the current density. Thus,

$$\frac{t_{50}(J_o)}{t_{50}(J_s)} = \left(\frac{J_o}{J_s}\right)^{-n}.$$

6.7.2 Humidity Dependence Failures

Corrosion in a plastic IC may deteriorate the leads outside the encapsulated circuit or the metallization interconnect inside the circuit. The basic ingredients needed for corrosion are moisture (humidity) and ions for the formation of an electrolyte; metal for electrodes; and an electric field. If any of these is missing, corrosion will not take place.

The general humidity model is

$$t_{50} = A(RH)^{-\beta} \quad \text{or} \quad t_{50} = Ae^{-\beta(RH)},$$

where t_{50} is the median life of the device, A and β are constants, and RH is the relative humidity. However, conducting an accelerated test for only humidity requires years before meaningful results are obtained. Therefore, temperature and humidity are usually combined for life testing. Voltage stress is usually added to these stresses in order to reduce the duration of the test further. The time to failure of a device operating under temperature, relative humidity, and voltage conditions is expressed as (Gunn et al., 1983)

$$t = ve^{\frac{E_a}{kT}} e^{\frac{\beta}{RH}}, \tag{6.77}$$

where

t = the time to failure;

v = the applied voltage;

E_a = the activation energy;

k = Boltzmann's constant;

T = the absolute temperature;

β = a constant; and

RH = the relative humidity.

Let the subscripts s and o represent the accelerated stress conditions and the normal operating conditions, respectively. The acceleration factor is obtained as

$$A_F = \frac{t_o}{t_s} = \frac{v_o}{v_s} e^{\frac{E_a}{k}\left[\frac{1}{T_o} - \frac{1}{T_s}\right]} e^{-\beta\left[\frac{1}{RH_o} - \frac{1}{RH_s}\right]}. \tag{6.78}$$

Changes in the microelectronics require that the manufacturers consider faster methodologies to detect failures caused by corrosion. Some manufacturers use pressure cookers to induce corrosion failures in a few days of test time. Studies showed that pressurized humidity test environments forced moisture into the plastic encapsulant much more rapidly than other types of humidity test methods.

6.7.3 Fatigue Failures

When repetitive cycles of stresses are applied to material, fatigue failures usually occur at a much lower stress than the ultimate strength of the material due to the accumulation of damage. Fatigue loading causes the material to experience cycles of tension and compressions, which result in crack initiations at the points of discontinuity, defects in material, or notches or scratches where stress concentration is high. The crack length grows as the repetitive cycles of stresses continue until the stress on the remaining cross-section area exceeds the ultimate strength of the material. At this moment, sudden fracture occurs, causing instantaneous failure of the component or member carrying the applied stresses. It is important to recognize that the applied stresses are not only caused by applying physical load or force but also by temperature or voltage cycling. For example, creep fatigue, or the thermal expansion strains caused by thermal cycling, is the dominant failure mechanism causing breaks in surface mount technology (SMT)—solder attachments of printed circuits. Each thermal cycle produces a specific net strain energy density in the solder that corresponds to a certain amount of fatigue damage. The long-term reliability depends on the cyclically accumulated fatigue damage in the solder, which eventually results in fracture (Flaherty, 1994). The reliability of components or devices subject to fatigue failure is often expressed in number of stress cycles corresponding to a given cumulative failure probability. A typical model for fatigue failure of a solder attachment is given by (Engelmaier, 1993)

$$N_f(x\%) = \frac{1}{2}\left[\frac{2\varepsilon}{F}\frac{h}{L_D \Delta\alpha \Delta T_e}\right]^{\frac{-1}{c}}\left[\frac{\ln(1-0.01x)}{\ln(0.5)}\right]^{\frac{1}{\beta}}, \tag{6.79}$$

where

$N_f(x\%)$ = number of cycles (fatigue life) that correspond to $x\%$ failures;

ε = the solder ductility;

F = an experimental factor (Engelmaier, 1993);

h and L_D = dimensions of the solder attachment;

$\Delta\alpha$ = a factor of the differences in the thermal expansion coefficient of component and substrate (that produces the stress);

ΔT_e = the effective thermal cycling range;

c = a constant that relates the average temperature of the solder joint and the time for stress relaxation/creep per cycle; and

β = 4 for leadless surface mounted attachment.

6.8 DEGRADATION MODELS

Most reliability data obtained from ALT are time-to-failure measurements obtained from testing samples of units at different stresses. However, there are many situations where the actual failure of the units, especially at stress levels close to the normal operating conditions, may not fail catastrophically but degrade with time. For example, a component may start the test with an acceptable resistance value but during the test, the resistance reading “drifts” (Tobias and Trindade, 1986). As the test time progresses, the resistance eventually reaches an unacceptable level that causes the unit to fail. In such cases, measurements of the degradation of the characteristics of interest (those whose failure may cause catastrophic failure of the part) are frequently taken during the test. The degradation data are then analyzed and used to predict the time to failure at normal conditions. Like failure-time data, degradation data can be obtained at normal operating conditions or at accelerated conditions. We refer to the latter as ADT which requires a reliability prediction model to relate results of a test at accelerated conditions to normal operating conditions.

Proper identification of the degradation indicator is critical for the analysis of degradation data and the subsequent decisions such as maintenance, replacements, and warranty policies.

Examples of these indicators include hardness which is a measure of degradation of elastomers. This is due to the fact that elastomeric materials are critical to many applications including hoses, seals, and dampers of various types, and their hardness increases over time to a critical level at which their ability to absorb energy is severely degraded. This may lead to cracks or excessive wear and related failure modes in components (Evans and Evans, 2010). Other indicators include loss of stiffness of springs, corrosion rate of beams and pipes, crack growth in rotating machinery, and more. In some cases, the degradation indicator might not be directly observed, and destruction of the unit under test is the only alternative available to assess its degradation (Jeng et al., 2011). This type of testing is referred to as accelerated destructive degradation testing (ADDT). In this section, we present physics-based-degradation models for specific devices and units, and then we will present general statistics-based degradation modeling for both normal and accelerated degradation cases.

We begin our presentation with specific degradation models.

6.8.1 Resistor Degradation Model

The thin film IC resistor degradation mechanism can be described by (Chan et al., 1994)

$$\frac{\Delta R(t)}{R_0} = \left(\frac{t}{\tau}\right)^m, \tag{6.80}$$

where

$\Delta R(t)$ = the change in resistance at time t;

R_0 = the initial resistance;

t = time;

τ = the time required to cause 100% change in resistance; and

m = a constant.

The temperature dependence is embedded in τ as

$$\tau = \tau_0 e^{\frac{E_a}{kT}}, \tag{6.81}$$

where τ_0 is constant.

Substituting Equation 6.81 into Equation 6.80 and taking the logarithm, we obtain

$$\ln\left(\frac{\Delta R(t)}{R_0}\right) = m\left[\ln(t) - \ln(\tau_0) - \frac{E_a}{kT}\right]$$

or

$$\ln(t) = \ln(\tau_0) + \frac{1}{m}\ln\left(\frac{\Delta R(t)}{R_0}\right) + \frac{E_a}{kT}. \tag{6.82}$$

Once the constants m and τ_0 are determined, we can use Equation 6.82 to calculate the change in resistance at any time. The above equation can also be used to predict the life of a device subject to electromigration failures. Recall that the median time to failure due to electromigration is given by Equation 6.75. Taking the natural logarithm of Equation 6.75 results in

$$\ln(MTF) = \ln(A) - n\ln(J) + \frac{E_a}{kT}. \tag{6.83}$$

Note, Equations 6.83 and 6.82 are identical.

The constants m and τ_0 can be obtained using the standard multiple regression procedure as shown in the following example.

EXAMPLE 6.15

The data shown in Table 6.9 represent 16 measurements of $\Delta R(t)/R_0$ at different time intervals from one sample at two temperatures (100°C and 150°C). Both the change in resistance and the exact time of the measurements are multiplied by arbitrary scale factors. Determine the time at which $\ln(\Delta R(t)/R_0) = 8.5$ when the resistor is operating at 28°C. This change in resistor corresponds to a catastrophic failure of the device.

TABLE 6.9 Degradation Data of the Resistor[a]

Time t in seconds	$\ln t$	$\ln\left(\frac{\Delta R(t)}{R_0}\right)$	$(kT)^{-1}$
5,000	8.517193	6.3969297	31.174013
15,000	9.615805	6.6846117	31.174013
50,000	10.819778	6.9077553	31.174013
150,000	11.918390	7.3132204	31.174013
500,000	13.122363	7.5286778	31.174013
1,500,000	14.220975	8.1992672	31.174013
5,000,000	15.424948	8.5003481	31.174013
10,000,000	16.118095	8.7809009	31.174013
5,000	8.517193	7.2412823	27.489142
15,000	9.615805	7.8913435	27.489142
50,000	10.819778	8.0966259	27.489142
150,000	11.918390	8.2540092	27.489142
500,000	13.122363	8.7809009	27.489142
1,500,000	14.220975	9.2256796	27.489142
5,000,000	15.424948	9.9034876	27.489142
10,000,000	16.118095	10.308953	27.489142

[a] Reprinted from 1994 Proceedings of the Annual Reliability and Maintainability Symposium, "Analysis of parameter degradation data using life-data analysis program," C. K. Chan, M. Boulanger, and M. Tortorella, pp. 288–291. © 1994 IEEE.

SOLUTION

Using the data given in Table 6.9, we develop a multiple linear regression model of the form

$$\ln(t)=\ln(\tau_0)+\frac{1}{m}\ln\left(\frac{\Delta R(t)}{R_0}\right)+\frac{E_a}{kT}.$$

The coefficients $\ln(\tau_0)$, $1/m$, and E_a are obtained from the regression model as −15.982, 2.785 and 0.24, respectively. The time required for a device operating at 28°C to reach $\ln(\Delta R(t)/R_0) = 8.5$ is

$$\begin{aligned}\ln(t)&=-15.982+2.785\times 8.5+\left(0.24/8.623\times 10^{-5}\times 301\right)\\ \ln(t)&=16.93719\\ \text{or time}&=22.6845\times 10^{6}\text{ s or 7.6 months.}\end{aligned}$$

■

6.8.2 Laser Degradation

A laser diode is a source of radiation that utilizes simulated emissions. Through high current density, a large excess of charge carriers is generated in the conduction band of the laser so that a strong simulated emission can take place. The performance of the laser diode is greatly affected by the driving current. Therefore, the degradation parameter that should be observed is the change in current with time. We utilize the degradation model developed by Takeda and Suzuki (1983) and modified by Chan et al. (1994):

$$\frac{D(t)}{D_0}=\exp\left[-\left(\frac{t}{\tau_d}\right)^{p}\right], \tag{6.84}$$

where

$D(t)$ = the change in degradation parameter at time t;

D_0 = the original value of the degradation parameter; and

τ_d, p = constants.

Again, we linearize Equation 6.84 by taking its logarithm twice to obtain

$$\ln\left[\ln\left(\frac{D(t)}{D_0}\right)\right]=-p[\ln(t)-\ln(\tau_d)]. \tag{6.85}$$

The parameters p and τ_d can be obtained in a similar fashion as discussed above.

6.8.3 Hot-Carrier Degradation

Technological advances in very large-scale integrated (VLSI) circuits fabrication resulted in significant reductions in the device dimensions such as the channel length, the gate oxide thickness, and the junction depth without proportional reduction in the power supply voltage. This has resulted in a significant increase of both the horizontal and vertical electric fields in the channel region. Electrons and holes gaining high kinetic energies in the electric field (hot-carriers) may be injected into the gate oxide causing permanent changes in the oxide-interface charge distribution and degrading the current-voltage characteristics of the device. For example, the damage caused by hot-carrier injection affects the characteristics of the nMOS transistors by causing a degradation in transconductance, a shift in the threshold voltage, and a general decrease in the drain current capability (Leblebici and Kang, 1993).

This performance degradation in the devices leads to the degradation of circuit performance over time. Therefore, in order to estimate the reliability of a device that may fail due to hot-carrier effects, a degradation test may be conducted and changes in device characteristics with time should be recorded. A degradation model is then developed to relate the device life with the changes in a critical characteristic. The model can be used to estimate the time or life of the device when a specified value of change in the device characteristics occurs. For example, the device life τ can be defined as the time required for a 10 mV threshold voltage shift under stress bias conditions. An empirical formula that relates the amount of the normalized substrate current ($I_{\text{substrate}}/W$) is given by (Leblebici and Kang, 1993):

$$\tau = A\left[\frac{I_{\text{substrate}}}{W}\right]^{-n}, \tag{6.86}$$

where

$I_{\text{substrate}}$ = the substrate current;

W = the channel width of the transistor;

A = a process dependent constant; and

n = an empirical constant.

Taking the logarithm of Equation 6.86 results in

$$\ln\tau = \ln A - n\ln\left[\frac{I_{\text{substrate}}}{\mathrm{W}}\right]. \tag{6.87}$$

The parameters A and n can be obtained using linear regression.

The degradation model given in Equation 6.86 is simple. However, other degradation models for a hot-carrier can be quite complex as discussed in Quader et al. (1994).

6.9 STATISTICAL DEGRADATION MODELS

Statistical degradation models are often used in absence of known physics-based or engineering-based models. There are two types of degradation models; the first deals with modeling the degradation of units operating at normal conditions, and the second deals with modeling the degradation at accelerated conditions (accelerated degradation models). The main objective of degradation modeling is to estimate the time to reach a predetermined threshold degradation level using the initial degradation path. Of course, other decisions could be made such as the optimal time to conduct maintenance and the determination of the warranty policies. It is important to note that condition-based maintenance (discussed in Chapter 8) can only be used when the unit (component) exhibits degradation. We briefly discuss the formulation of the degradation path and estimate the corresponding reliability function.

6.9.1 Degradation Path as Brownian Motion

Consider that the degradation indicator is $D(t)$ at time t. The degradation path is a stochastic process $\{D(t), t \geq 0\}$, and is said to be a Brownian motion process if $D(0) = 0$; $\{D(t), t \geq 0\}$ has stationary and independent increments; that is, $[D(t_1), D(t_2) - D(t_1), D(t_3) - D(t_2), \ldots, D(t_n) - D(t_{n-1})]$ are independent; and for $t > 0$, $D(t)$ is normally distributed with mean 0 and variance $\sigma^2 t$. This Brownian motion (also referred to as Weiner process) is useful in modeling degradation path when the degradation measurements are approximated as a continuous function of t, a realistic assumption.

When $\sigma = 1$, the process is referred to as *standard Brownian motion* which can be easily accomplished for any process. In addition, the process $\{D(t), t \geq 0\}$ is said to be a Brownian motion with drift coefficient μ and variance σ^2 (Ross, 2000) if

(i) $D(0) = 0$;

(ii) $\{D(t), t \geq 0\}$has stationary and independent increments; and

(iii) $D(t)$ is normally distributed with mean μt and variance $\sigma^2 t$.

If we define $\{B(t), t \geq 0\}$ as a standard Brownian motion, then the Brownian motion with a drift and variance can be expressed as

$$D(t) = \sigma B(t) + \mu t. \tag{6.88}$$

Finally, if $\{Y(t), t \geq 0\}$is a Brownian motion with drift μ and variance σ^2, then we relate the degradation process to $Y(t)$ as $D(t) = e^{Y(t)}$, which is referred to as *geometric Brownian motion.*

6.9.2 Degradation Model

In many engineering applications, a degradation process, $\{D(t), t \geq 0\}$, can be represented by a linear process after a simple transformation (e.g., see Tseng et al., 1995; Lu et al., 1997). We consider the Brownian motion with linear drift model given by Equation 6.88 and Karlin and Taylor (1974) in the form of a stochastic differential equation (SDE):

$$dD(t) = \mu\, dt + \sigma_D\, dW(t). \tag{6.89}$$

Its solution is

$$D(t) = D(0) + \mu t + \sigma_D W(t), \tag{6.90}$$

where t represents the time; $D(0)$ is the initial value of the degradation process, assumed to be constant throughout; $\mu$is the drift parameter; σ_D is the diffusion parameter; and $W(t)$ is the standard Brownian motion. It is well known that, when the degradation path is linear, the first passage time (failure time), T, of the process to a failure threshold level, D^*, follows the inverse Gaussian (IG) distribution (Chhikara and Folks, 1989), $IG(t; \mu, \sigma_D^2)$, with MTTF, $D^* - D(0)/\mu$, and (*p.d.f.*) as given in Chapter 1.

$$IG(t; \mu, \sigma_D^2, D^*) = \frac{D^* - D(0)}{\sigma_D \sqrt{t^3}} \phi\left(\frac{D^* - D(0) - \mu t}{\sigma_D \sqrt{t}}\right), \qquad \mu > 0, D^* > D(0), \tag{6.91}$$

where $\phi(\cdot)$ is the *p.d.f.* of the standard normal distribution. The reliability function is given by,

$$R(t) = \Phi\left(\frac{D^* - D(0) - \mu t}{\sigma_D \sqrt{t}}\right) - \exp\left(\frac{2\mu\left(D^* - D(0)\right)}{\sigma_D^2}\right)\Phi\left(-\frac{D^* - D(0) + \mu t}{\sigma_D \sqrt{t}}\right), \tag{6.92}$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution.

It is straightforward to extend model (6.90) to an ADT model (e.g., see Doksum and Hóyland, 1992) by expressing each model parameter as a function of the constant-stress vector, $\underline{z}$:

$$dD(t; \underline{z}) = \mu(\underline{z})dt + \sigma_D(\underline{z})dW(t), \tag{6.93}$$

where $\mu(\underline{z}) = h(\underline{z}; \underline{\alpha})$ and $\sigma_D^2(\underline{z}) = f(\underline{z}; \underline{\beta})$ are acceleration models (e.g., Arrhenius law and Inverse Power law) with the parameter vectors $\underline{\alpha}$ and $\underline{\beta}$, which can be estimated through an ADT experiment. The product's reliability under constant operating conditions is obtained by substituting the constant-stress, $\underline{z}$, into the resulting ADT model. It should be noted that when the degradation path model is not linear, the process to failure threshold level can no longer be expressed as an IG distribution, and numerical and approximate solutions can then be used instead.

6.10 ACCELERATED LIFE TESTING PLANS

The type of the accelerated stress to be applied on the device, component, or part to be tested depends on the failure modes and the environment at which the device will normally operate. For example, if the component will be subjected to cycles of tension and compression, then an accelerated fatigue test deems appropriate to provide prediction of the life of the component at normal operating conditions. Similarly, if the part will operate in a hot and humid environment, then a test in a temperature and humidity chamber will simulate the environment at much higher stress levels.

The main questions that need to be addressed in order to effectively conduct an accelerated life test are as follows:

- What is the objective of the test?
- Do the units to be tested have degradation behavior?
- What type of stresses should be applied on the device or component?
- At what stress levels should the device or component be tested?
- How the stresses are applied (constant, step, ramp; see Section 6.2.1)?
- What is the test duration allocated?
- What is the number of units to be tested at each stress level?
- Are there equivalent tests that result in the same reliability prediction accuracy?
- What is the reliability prediction model to be used?

As mentioned above, the type of stress to be applied depends on the actual functions that the device or component will be performing at the normal operating conditions. For example, if the device has many physical connections and will be used in an airplane cockpit, then a vibration test appears appropriate to conduct. Moreover, if the relative humidity level in the cockpit is greater than 30%, then humidity acceleration should be included in the test. Furthermore, the failure mechanism may also dictate the type of stress to be applied. Table 6.10 provides a summary of some failure mechanisms for electronic devices and the corresponding stresses that induce such mechanisms (Brombacher, 1992).

In choosing the stress levels, it is necessary to establish the highest stress to be used as the one that represents the most extreme conditions where the assumed failure model can still reasonably be expected to hold (Meeker and Hahn, 1985). Moreover, the applied stresses should not be high enough to induce different failure modes that might occur at normal operating conditions. The next step is to conduct test at two or other stress conditions (including the high stress) in each case, at least $200P$ percent (P is the percentile of the time to failure distribution) of the units will fail within the duration of the test at the higher stress level. Moreover, at least $100P$ percent of the units at the lower stress level must fail within the test duration (P represents percentile). For example, if it is desired to estimate the tenth percentile of the time to failure distribution, tests should be conducted at least at two stresses (i.e., the high and middle stress

TABLE 6.10 Failure Mechanism and Corresponding Stress

Failure mechanism	Applied stress
Electromigration	Current density
	Temperature
Thermal cracks	Dissipated power
	Temperature
Corrosion	Humidity
	Temperature
Mechanical fatigue	Repeated cycles of load
	Vibration
Thermal fatigue	Repeated cycles of temperature change

conditions) at which 20% (and, as a minimum at the middle stress, 10%) of the units tested should be expected to fail within the duration of the test. Finally, conduct a test at a third stress level as close as possible to the normal operating conditions but which will result in at least five failures (Meeker and Hahn, 1985). Clearly, other test plans that minimize cost or maximize the information obtained from the test can be used; see, for example, Barton (1980), Kielpinski and Nelson (1975), Meeker and Nelson (1975), Elsayed and Zhang (2007), Liao and Elsayed (2010), and Zhu and Elsayed (2011).

The number of units to be allocated to each stress level is inversely proportional to the applied stress. In other words, more test units should be allocated to low stress levels than to the high stress levels because of the higher proportion of failures at the high stress levels. When conducting an accelerated life test, arrangements should be made to ensure that the failures of units are independent of each other and that the conditions of the test are the same for all units under test. For example, when conducting a temperature acceleration test, arrangements should be made to ensure that the temperature distribution is uniform within the test chamber.

6.10.1 Design of ALT Plans

As stated before, an ALT plan requires the determination of the type of stress, method of applying stress, stress levels, the number of units to be tested at each stress level, and an applicable ALT model that relates the failure times at accelerated conditions to those at normal conditions.

When designing an *ALT*, we need to address the following issues: (1) selection of the stress types to use in the experiment; (2) determination of the stress levels for each stress type selected; and (3) determination of the proportion of devices to be allocated to each stress level (Elsayed and Jiao, 2002). Other approaches for the design of ALT plans are given in Meeker and Escobar (1998), Escobar and Meeker (2006), and Nelson (2004, 2005a, 2005b).

We consider the selection of the stress level z_i and the proportion of units p_i to allocate for each z_i such that the most accurate reliability estimate at use conditions z_D can be achieved. There are two types of censoring: Type I censoring involves running each test unit until a prespecified time; in this case, the censoring times are fixed while the number of failures is random. Type II censoring involves testing units until a prespecified number of units fail; in this case, the censoring time is random while the number of failures is fixed. We define the following notations:

n	total number of test units;
z_H, z_{M}, z_L	high, medium, low stress levels, respectively;
z_D	design stress level;
p_1, p_2, p_3	proportion of test units allocated to z_L, z_M, and z_L, respectively;
T	prespecified period of time over which the reliability estimate is of interest at normal operating conditions;
$R(t; z)$	reliability at time t, for given z;
$f(t; z)$	*p.d.f.* at time t and stress z;
$F(t; z)$	*CDF* at time t and stress z;

$\Lambda(t; z)$ cumulative hazard function at time t and stress z; and

$\lambda_0(t)$ unspecified baseline hazard function at time t.

We assume the baseline hazard function $\lambda_0(t)$ to be linear with time

$$\lambda_0(t) = \gamma_0 + \gamma_1 t.$$

Substituting $\lambda_0(t)$ into the PHM described earlier in this chapter,

$$\lambda(t; \mathbf{z}) = (\gamma_0 + \gamma_1 t)\exp(\boldsymbol{\beta}\mathbf{z}).$$

We obtain the corresponding cumulative hazard function $\Lambda(t; \mathbf{z})$, and the variance of the hazard function as

$$\Lambda(t; \mathbf{z}) = \left(\gamma_0 t + \frac{\gamma_1 t^2}{2}\right)e^{\boldsymbol{\beta}\mathbf{z}}$$

$$Var\left[(\hat{\gamma}_0 + \hat{\gamma}_1 t)e^{\hat{\beta}z_D}\right] = \left(Var[\hat{\gamma}_0] + Var[\hat{\gamma}_1]t^2\right)e^{2\left(\beta z + Var\left[\hat{\beta}\right]z^2\right)}$$
$$+ e^{2\beta z + Var\left[\hat{\beta}\right]z^2}\left(e^{Var\left[\hat{\beta}\right]z^2} - 1\right)(\gamma_0 + \gamma_1 t)^2.$$

6.10.2 Formulation of the Test Plan

Under the constraints of available test units, test time, and the specified minimum number of failures at each stress level, the objective of the problem is to optimally allocate stress levels and test units so that the asymptotic variance of the hazard-rate estimate at normal conditions is minimized over time T. If we consider three stress levels, then the optimal decision variables ($z_L^*, z_M^*, p_1^*, p_2^*, p_3^*,$) are obtained by solving the following optimization problem with a nonlinear objective function and both linear and nonlinear constraints (Elsayed and Zhang, 2009):

$$\text{Min} \quad \int_0^T Var\left[(\hat{\gamma}_0 + \hat{\gamma}_1 t)e^{\hat{\beta}z_D}\right]dt$$

Subject to

$$\underset{\sim}{\Sigma} = F^{-1}$$
$$0 < p_i < 1, \quad i = 1, 2, 3$$
$$\sum_{i=1}^{3} p_i = 1$$
$$z_D < z_L < z_M < z_H$$
$$np_i \Pr\left[t \le \tau | z_i\right] \ge MNF, \quad i = 1,2,3,$$

where, MNF is the minimum number of failures, and $\underset{\sim}{\Sigma}$ is the inverse of the Fisher's information matrix.

Other objective functions can be formulated which result in a different design of the test plans. These functions include the D-Optimal design that provides efficient estimates of the parameters of the distribution. It allows relatively efficient determination of all quantiles of the population, but the estimates are distribution dependent.

EXAMPLE 6.16

An accelerated life test is to be conducted at three temperature levels for MOS capacitors in order to estimate its life distribution at design temperature of 50°C. The test needs to be completed in 300 h. The total number of items to be placed under test is 200 units. To avoid the introduction of failure mechanisms other than those expected at the design temperature, it has been decided, through engineering judgment, that the testing temperature should not exceed 250°C. The minimum number of failures for each of the three temperatures is specified as 25. Design a test plan that provides accurate reliability estimate over a 10-year period of time (Elsayed and Zhang, 2009).

SOLUTION

Consider three stress levels, then the formulation of the objective function and the test constraints follow the same formulation as given above. The plan that optimizes the objective function and meets the constraints is as follows:

$$z_L = 160°\text{C}, z_M = 190°\text{C}, z_H = 250°\text{C}.$$

The corresponding allocations of units to each temperature level are $p_1 = 0.5$, $p_2 = 0.4$, $p_3 = 0.1$. ■

PROBLEMS

6.1 The failure times of diode X (Schottky diode) at both the accelerated stress conditions and the normal operating conditions are found to follow gamma distributions having the same shape parameter γ. The following is the p.d.f. of the gamma distribution:

$$f(t) = \frac{t^{\gamma-1}}{\theta^{\gamma}\Gamma(\gamma)} e^{\frac{-t}{\theta}}.$$

Where θ is the scale parameter of the distribution,

a. Develop the relationship between the hazard rates at both the accelerated stress condition and the normal operating conditions.

b. Determine the reliability expression at the normal operating conditions as a function of the acceleration factor and the parameters of the gamma distribution at the accelerated conditions.

c. Develop an expression for the acceleration factor.

6.2 The failure times of a steel shaft subject to high-speed fatigue loading at both the accelerated and normal operating conditions is found to follow Birnbaum–Saunders (BS) distribution with a p.d.f. given below. Assuming an acceleration factor A_F

$$f(t;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\sqrt{\frac{\beta}{t}} + \left(\frac{\beta}{t}\right)^{3/2}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t}\right) - 2\right], 0 < t < \infty, \alpha, \beta > 0$$

a. Develop the relationship between the hazard rates at both the accelerated stress condition and the normal operating conditions.

b. Determine the reliability expression at the normal operating conditions as a function of the acceleration factor and the parameters of the BS distribution at the accelerated conditions.

6.3 The failure times (multiplied by 100,000 to obtain number of cycles) from an actual test of Alloy 1018 mild steel shafts subject to an accelerated thermal stress are given below. The accelerated stress is 250°F and normal operating stress is 120°F. Assume a linear acceleration factor of five and the data fit a Weibull distribution. Plot the reliability function at both the accelerated and normal conditions and obtain the corresponding MTTF.

1.00	2.00	6.50	33.00	41.00	42.00
47.30	52.00	57.90	58.00	58.10	59.00
65.00	65.50	75.00	76.50	77.00	84.30
85.00	94.00	95.00	99.60	106.00	108.00
117.30	130.00	155.00	161.30	165.00	169.90
198.20	206.10	206.20	206.30	207.70	208.00
208.80	235.50	240.00	257.00	274.00	280.00

6.4 Electrolytic corrosion of metallization involves the transport of metallic ions across an insulating surface between two metals. The conductivity of the surface affects the rate of material transport and hence the device life. An accelerated life test is conducted on 20 ICs by subjecting them to moisture with known relative humidity. The following failure times in years are obtained from the test.

0.0031667
0.0056359
0.0061977
0.0067325
0.0069382
0.0076820
0.0106340
0.0107340
0.0116650
0.0119780
0.0122230
0.0128110
0.0132280
0.0154420
0.0156650
0.0164890
0.0217990
0.0296007
0.0311001
0.0373216

Plotting of the hazard rate reveals that the failure times can be described by a gamma distribution. Assuming that an acceleration factor of 20 is used in the experimentation, estimate the parameters of the distribution at the normal operating conditions. What is the reliability of a device at time of 0.6 years?

6.5 In performing the analysis of the data given in Problem 6.4, the analyst realizes that there are five more observations that are censored at time 0.0395460 years (termination time of the test). Rework the above analysis and compare the results with those obtained in Problem 6.4.

6.6 Creep failure results whenever the plastic deformation in a machine member accrues over a period of time under the influence of stress and temperature until the accumulated dimensional changes cause the part not to perform its function or to rupture (part failure). It is clear that the failure of the part is due to stress-time-temperature effect. Fifteen parts made of 18-18 plus stainless steel are subjected to a mechanical acceleration method of creep testing, in which the applied stress levels are significantly higher than the contemplated design stress levels, so that the limiting design strains are reached in a much shorter time than in actual service. The times (in hours) to failure at an accelerated stress of 710 MPa are

30.80, 36.09, 65.68, 97.98, 130.97, 500.75, 530.22, 653.96, 889.91, 1173.76, 1317.08, 1490.44, 1669.33, 2057.95, and 2711.36.

Assume that the failure times can be modeled by an exponential distribution and that the acceleration factor between the mean life at normal operating conditions and the accelerated stress condition is 20. Determine the parameter of the distribution at the normal conditions and the reliability that a part will survive to 10,000 h.

6.7 Verification of the reliability of a new stamped suspension arm requires demonstration of R90 (Reliability of 0.9) at 50,000 cycles on a vertical jounce to rebound test fixture (for testing in up and down directions). Eight suspension arm units are subjected to an accelerated test and the following results are obtained (Allmen and Lu, 1994).
Assume that the acceleration factor is 1.4 and that a Weibull distribution represents the failure-time probability distribution.

Cycles to failure	Status
75,000	Failure
95,000	Failure
110,000	Failure
125,000	Failure
125,000	Censored
125,000	Censored
125,000	Censored
125,000	Censored

a. Determine the parameters of the distribution.

b. Does the test verify the reliability requirements at normal operating conditions?

6.8 Derive expressions that relate the hazard-rate functions, p.d.f.'s, and the reliability functions at the accelerated stress and at the normal operating conditions when the failure-time distribution at the two conditions is special Erlang in the form

$$f(t) = \frac{t}{\lambda^2} e^{\frac{-t}{\lambda}},$$

where λ is the parameter of the distribution.

6.9 Optical cables are subject to a wide range of temperatures. Buried and duct cables experience small temperature variations, whereas aerial cables experience a much wider range in temperature variations. For each cable type, the transmission properties of the optical fiber may only change within a limited range.

In order to determine temperature performance, the change in attenuation is measured as a function of the temperature. This is usually performed by placing the cable along with the drum on which it is wound in a computer-controlled temperature chamber with both ends attached to an attenuation measuring test set. The average attenuation change at a wavelength of 130 nm at −40°C is less than 0.1 dB/km (Mahlke and Gossing, 1987).

Two samples each having 25 cables are subjected to temperature acceleration stresses of −20°C and 70°C and the times until an attenuation change of 0.3 dB/km are recorded (an attenuation greater than 0.28 dB/km is deemed unacceptable) as shown below.

Failure times at –20°C	Failure times at 70°C
4923.01	9082.28
4937.33	9090.59
4938.33	9228.30
4957.34	9248.30
4957.42	9271.51
4960.18	9394.54
4969.82	9438.95
4971.28	9693.95
4971.64	9694.27
4977.37	9778.96
4979.76	9966.46
4983.77	10015.00
4992.01	10086.80
4994.98	10115.00
4997.83	10131.90
5003.59	10141.90
5004.30	10149.50
5005.54	10205.60
5009.98	10249.60
5017.21	10291.20
5022.17	10310.50
5027.04	10313.10
5027.89	10341.00
5032.00	10469.00
5045.87	10533.50

a. Assume that the failure times follow lognormal distributions. Estimate the parameters of the distributions at both stress levels.

b. Assume that the mean life of the cable is linearly related to the temperature of the environment. What is the mean life at 25°C?

6.10 The gate oxide in MOS devices is often the source of device failure, especially for high-density arrays that require thin gate oxides. Voltage and temperature are two main factors that cause breakdown of the gate oxide. Therefore, ALT for MOS devices usually include these two factors. An accelerated test is performed on two samples of 15 n-channel transistors each by subjecting the first sample to a voltage of 20 V and temperature of 200°C and the second sample to a voltage of 27 V and 120°C. The normal operating conditions are 9 V and 30°C. The following failure times (in seconds) are obtained.

Failure times for the first sample	Failure times for the second sample
48.7716	10.4341
48.8160	12.4544
49.1403	12.9646
49.3617	13.0883
50.0852	13.1680
50.6413	13.1984
50.7534	13.6002
50.8506	14.1088
51.1490	14.8734
51.2638	15.1088
51.3007	15.4149
51.3085	16.0556
51.4376	16.2214
51.9868	18.2557
53.5653	18.2615

a. Calculate both the electric field and thermal acceleration factors for both stress levels (the activation energy is 1.2 eV and $E_{EF} = 3$).

b. Assuming that the data at each stress level can be modeled using a lognormal distribution as given in Equation 6.20, determine the mean life at the normal operating conditions.

6.11 Use the data in Problem 6.12 to obtain a combined acceleration factor as given by Equation 6.24.

6.12 The following is a subset of actual failure times at different accelerated stress conditions of experiments conducted on samples of an electronic device. The failure times are in seconds.

Failure time	Temp. (°C)	Volt	Failure time	Temp. (°C)	Volt	Failure time	Temp. (°C)	Volt
1	25	27	1	225	26	1365	125	25.7
1	25	27	14	225	26	1401	125	25.7
1	25	27	20	225	26	1469	125	25.7
73	25	27	26	225	26	1776	125	25.7
101	25	27	32	225	26	1789	125	25.7
103	25	27	42	225	26	1886	125	25.7
148	25	27	42	225	26	1930	125	25.7
149	25	27	43	225	26	2035	125	25.7
153	25	27	44	225	26	2068	125	25.7
159	25	27	45	225	26	2190	125	25.7
167	25	27	46	225	26	2307	125	25.7
182	25	27	47	225	26	2309	125	25.7
185	25	27	53	225	26	2334	125	25.7
186	25	27	53	225	26	2556	125	25.7
214	25	27	55	225	26	2925	125	25.7
214	25	27	56	225	26	2997	125	25.7
233	25	27	59	225	26	3076	125	25.7
252	25	27	60	225	26	3140	125	25.7
279	25	27	60	225	26	3148	125	25.7
307	25	27	61	225	26	3736	125	25.7

Use a multiple linear model to develop a relationship between the failure time, temperature, and volt. What is the time to failure at normal operating conditions of 30°C and 5 V? $E_a = 0.08$ eV.

6.13 Use the proportional hazard model to estimate the failure rate at normal operating conditions of 30°C and 5 V. Compare the estimate with that obtained in Problem 6.12. If there is a difference between the two estimates, explain why.

6.14 A temperature acceleration test is performed on 12 units at 300°C, and the following failure times (in hours) are obtained:

200, 240, 300, 360, 390, 450, 490, 550, 590, 640, 680, and 730.

a. Assume that an Eyring model describes the relationship between the mean life and temperature. What is the expected life at a temperature of 40°C?

b. Assume that the activation energy of the unit's material is $E_a = 0.04$ eV. Use the Arrhenius model to estimate the mean life at 40°C.

6.15 Use the Eyring model to obtain a relationship between the mean life, temperature, and volt. Compare the estimate of mean life at normal operating conditions (30°C and 5 V) with the estimates obtained in Problems 6.12 and 6.13. Are the physics-statistics-based models more appropriate than the physics-experimental based models when estimating the failure rate for the data given in Problem 6.14?

6.16 High-voltage power transistors are used in many applications where both high voltage and high current are present. Under these conditions, catastrophic device failure can occur due to reverse-biased second breakdown (RBSB). This breakdown can occur when a power transistor switches off an inductive load. The voltage across the device can rise by several hundred volts within a few hundred nanoseconds causing the failure of the device (White, 1994). Therefore, the manufacturers of such transistors often run accelerated life and operational life testing to improve the design of the transistor and to ensure its reliability. A manufacturer conducts an accelerated test by subjecting transistor units to two voltages of 50 V and 80 V. The failure times (hours) are given below:

a. Use the inverse power rule model to estimate the mean life at the normal operating conditions of 5 V. What is the reliability of a device operating at the normal conditions at time of 10,000 h? $E_a = 0.60$ eV.

b. What are the acceleration factors used in the test? Are they proper?

c. Solve (a) above using the Eyring model. Explain the causes for the difference in results.

Failure times at 50 V	Failure times at 80 V
10.55	3.01
11.56	3.05
12.78	3.06
13.00	3.12
13.50	4.20
15.00	4.30
15.01	4.45
16.02	5.62
19.01	5.67
25.06	8.60
25.50	8.64
29.60	9.10
30.10	9.21
35.00	9.26
45.00	9.29
49.00	10.01
58.62	10.20

6.17 The manufacturer of transistors in Problem 6.16 provides the following additional information about the failure data.

- The accelerated life test using 50 V is conducted at 90°C.
- The accelerated life test using 80 V is conducted at 150°C.

a. Solve Problem 6.16 using the additional information.

b. Compare the results obtained in (a) with those obtained from the combination model.

6.18 A specific type of device is susceptible to failure due to electromigration. An accelerated life test is conducted under the following conditions:

a. Use Black's equation to obtain a relationship among the median time to failure, the current intensity, and the temperature. $E_a = 0.60$ eV.

b. What is median life at $J = 5$ and $T = 30°C$?

c. Assume you were not aware of the electromigration model. Apply the Eyring model and compare the results.

Failure time (hours)	Current intensity	Temperature (°C)	Failure time (hours)	Current intensity	Temperature (°C)
300	10	200	264	10	250
340	10	200	270	10	250
345	10	200	271	10	250
349	10	200	272	10	250
361	10	200	280	10	250
362	10	200	285	10	250
363	10	200	200	15	200
369	10	200	205	15	200
374	10	200	207	15	200
379	10	200	209	15	200
380	10	200	210	15	200
390	10	200	211	15	200
250	10	250	215	15	200
251	10	250	220	15	200
252	10	250	222	15	200
260	10	250	225	15	200
262	10	250	228	15	200
263	10	250	230	15	200

6.19 Solve Problem 6.14 if the relative humidity (RH) at test conditions changes as shown below.

Temperature	Voltage	RH
25°C	27	70%
225°C	26	50%
125°C	25.7	40%

Assume that the device's normal operating conditions are temperature = 30°C, voltage = 5 V, and RH = 30%.

6.20 Hale et al. (1986) report on the change in resistance of a Cathode Ray Tube (CRT) bleed resistor. The resistor has the function of regulating the power supply to the CRT by providing a constant impedance across it. If the bleed resistance increases, the effect will be severe front-of-screen distortion, where the outside edges of a video image are curved. If the resistance decreases significantly, the circuit will be overloaded, and the display will power down. The main criterion for the performance of the resistor is the change in its resistance with time. The following degradation model describes the relationship between the change in resistance with time and the applied temperature:

$$\frac{dR}{dt} = Ae^{\frac{-E_a}{kT}}.$$

A manufacturer observes the resistance of two resistors tested at two different temperatures of 80°C and 120°C. The results of the test, which is conducted for 1000 h, are as follows.

Temperature 80°C		Temperature 120°C	
Time (hours)	Resistance in MΩ	Time (hours)	Resistance in MΩ
0	250	0	250
100	270	100	280
200	291	200	309
300	310	300	341
400	328	400	369
500	349	500	402
600	370	600	432
700	387	700	460
800	412	800	490
900	430	900	516
1000	448	1000	547

The normal value of the resistor is 250 MΩ. The edges of a video image become unacceptably curved when the resistor's value changes to 340 MΩ, and failure of the display occurs when the resistance's value reduces to 180 MΩ. Determine the time to system failure (distorted video or failed display) if the resistor is expected to operate normally at 30°C.

6.21 The breakdown strength of electrical insulation depends on age and temperature. The dielectric strength is measured in kilovolt. Nelson (1981) reports the results of an accelerated test of 128 specimens and the strength of their electrical insulations. The test requires four specimens for each combination of four test temperatures (180, 225, 250, 275°C) and eight aging times (1, 2, 4, 8, 16, 32, 48, 64 weeks). The dielectric strengths for the 128 specimens are as follows.

Week	Temp.	Strength (kV)	Week	Temp.	Strength (kV)	Week	Temp.	Strength (kV)
1	180	15.0	4	250	13.5	32	225	11.0
1	180	17.0	4	275	10.0	32	225	11.0
1	180	15.5	4	275	11.5	32	250	11.0
1	180	16.5	4	275	11.0	32	250	10.0
1	225	15.5	4	275	9.5	32	250	10.5
1	225	15.0	8	180	15.0	32	250	10.5
1	225	16.0	8	180	15.0	32	275	2.7
1	225	14.5	8	180	15.5	32	275	2.7
1	250	15.0	8	180	16.0	32	275	2.5
1	250	14.5	8	225	13.0	32	275	2.4
1	250	12.5	8	225	10.5	48	180	13.0
1	250	11.0	8	225	13.5	48	180	13.5
1	275	14.0	8	225	14.0	48	180	16.5
1	275	13.0	8	250	12.5	48	180	13.6
1	275	14.0	8	250	12.0	48	225	11.5
1	275	11.5	8	250	11.5	48	225	10.5
2	180	14.0	8	250	11.5	48	225	13.5
2	180	16.0	8	275	6.5	48	225	12.0
2	180	13.0	8	275	5.5	48	250	7.0
2	180	13.5	8	275	6.0	48	250	6.9
2	225	13.0	8	275	6.0	48	250	8.8
2	225	13.5	16	180	18.5	48	250	7.9
2	225	12.5	16	180	17.0	48	275	1.2
2	225	12.5	16	180	15.3	48	275	1.5
2	250	12.5	16	180	16.0	48	275	1.0
2	250	12.0	16	225	13.0	48	275	1.5
2	250	11.5	16	225	14.0	64	180	13.0
2	250	12.0	16	225	12.5	64	180	12.5
2	275	13.0	16	225	11.0	64	180	16.5
2	275	11.5	16	250	12.0	64	180	16.0
2	275	13.0	16	250	12.0	64	225	11.0
2	275	12.5	16	250	11.5	64	225	11.5
4	180	13.5	16	250	12.0	64	225	10.5
4	180	17.5	16	275	6.0	64	225	10.0
4	180	17.5	16	275	6.0	64	250	7.2
4	180	13.5	16	275	5.0	64	250	7.5
4	225	12.5	16	275	5.5	64	250	6.7
4	225	12.5	32	180	12.5	64	250	7.6
4	225	15.0	32	180	13.0	64	275	1.5
4	225	13.0	32	180	16.0	64	275	1.0
4	250	12.0	32	180	12.0	64	275	1.2
4	250	13.0	32	225	11.0	64	275	1.2
4	250	12.0	32	225	9.5			

Assume that the relationship between median (50% point) log breakdown voltage $V_{50\%}$, absolute temperature T, and exposure time t is

$$\ln V_{50\%} = \alpha - \beta t \exp(-\gamma / T),$$

where α, β, and γ are constants. Estimate the time for the insulation to degrade below 2 kV breakdown strength at the normal operating temperature of 150°C.

6.22 *In situ* accelerated aging technique is based on the same idea of the ALT: most of the physical and chemical processes are thermally activated. However, in the ALT, the purpose of the application of thermal stress is to induce a number of failures, so that a failure rate can be calculated. Whereas, in an *in situ* test, the thermal stress is applied to increase the rate at which physicochemical processes occur in the system, in order to measure their effect during the aging treatment on a parameter characterizing the performance of the system (DeSchepper et al., 1994). In other words, the main characteristic of the *in situ* technique is that the effect of the accelerated physicochemical processes on the relevant parameter is measured *during* thermal stress. For example, a simple model that represents the aging of a thin film resistor can be written in the form

$$\frac{dR(t)}{R_o} = k\, t^n,$$

where

R_o the initial value of the resistor;

k, n constants; and

t the time corresponding to the change in the resistance.

a. Use the data of Example 6.15 to estimate the time to reach $dR(t)/R_o = 8.5$.

b. Compare the solution with that obtained in Example 6.15.

6.23 Use the POM for the data in Problem 6.18 to obtain a reliability estimation model for any stress conditions. Compare the reliability prediction at a current intensity of 10 and a temperature of 200°C with that estimated using Kaplan–Meier for the recorded data at the same conditions.

6.24 Solve Problem 6.23 using the PMRL model. Compare the results with POM. Which model shows estimates closer to the Kaplan–Meier model?

6.25 Use Equation 6.92 and the data for 80°C temperature shown in Problem 6.20 to obtain the hazards and reliability functions assuming that the resistance threshold level is 550 MΩ.

6.26 Repeat Problem 6.25 for 120°C temperature. Derive a reliability expression for reliability estimation at other operating temperatures.

6.27 Image fading with time is a major concern for museums and archiving facilities. In order to extend the life of printed images, the display conditions are kept at 120 lux light levels; 72 F temperature; 40% relative humidity. Of course, the life of an image (in terms of loss in density) is affected by type of print paper (or other media), color of the dye, type of dye, temperature, light level, and relative humidity. ALT is used to predict the life at display conditions. One of the commonly used tests is referred to as accelerated light fading test (ALFT) where the image is subjected to a much more intense light than encountered at display light level. It is found that the acceleration factor is linear and is estimated as follows: for a given amount of fading, a print displayed in normal indoor lighting conditions has an expected life of 20 times that of a print subjected to ALFT using 120 times more intense lighting level (Wilhelm, 1993). For simplicity, we ignore other factors and consider lighting level only expressed in lux. An ALFT is conducted at using 1.35 klux and the fading of the image colors (expressed in density loss) is recorded for three different dyes: magenta, yellow, and cyan as shown below.

Time (years)	Magenta	Yellow	Cyan
0	0.000	0.000	0.000
1	–0.050	–0.040	–0.035
2	–0.150	–0.120	–0.060
3	–0.220	–0.180	–0.075
4	–0.300	–0.260	–0.100
5	–0.350	–0.320	–0.125
6	–0.380	–0.350	–0.150

Obtain an expression for image degradation at 120 lux.

6.28 Use the data in Problem 6.12 to obtain the parameters of PHM for the design of a test plan. The number of available units for the test is 300, and the maximum temperature and volt are 300 and 30, respectively. The duration of the test is constrained to be 500 h. The objective of the test plan is to minimize the asymptotic variance over 10 years of operation at normal conditions.

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CHAPTER 7

RENEWAL PROCESSES AND EXPECTED NUMBER OF FAILURES

7.1 INTRODUCTION

One of the most frequently sought-after quantities is the expected number of failures of a system during a time interval (0, t]. This quantity is used to determine the optimal preventive maintenance schedule, design of warranty policies, and as a criterion for reliability acceptance tests. The last exemplifies a typical reliability test when n units (components or systems) are drawn at random from a production lot. They are subjected to specified test conditions, and the entire production lot is accepted if x ($x < n$) or more units survive the test by time t.

More importantly, this quantity is extremely useful for manufacturers, suppliers, and service providers in estimating the cost of a warranty. As an example, consider the case when a manufacturer agrees to replace, free of charge, the product when it fails before a time period T is expired (warranty period). Suppose that $M(T)$ is the expected number of replacements (renewals) during the warranty period. Then the expected warranty cost $C(T)$ is

$$C(T) = c \cdot M(T), \tag{7.1}$$

where c is the fixed cost per replacement. Clearly, the cost of warranty is greatly affected by the number of replacements, and if the manufacturer produces a very large number of units, it becomes crucial for the manufacturer to determine $M(T)$ with a much greater accuracy. The cases of computer battery recalls and auto recalls because of the high number of failures due to gas-pedal failures are typical examples of the effect of the expected number of failures on warranty and recalls.

The role of $M(T)$ in estimating the warranty cost for a given warranty policy and in determining the optimal preventive replacement periods for repairable systems is emphasized in Chapters 8 and 9. The following sections present two different approaches for determining $M(T)$. The first approach, a *parametric approach*, is used when the failure-time distribution of the units is known. The second approach, a *nonparametric approach*, is used when the failure-time distribution is unknown or when the mean and the standard deviation of the failure times are the only known parameters. Because the estimation of $M(T)$ may be quite difficult to obtain, we also present approximate methods for estimating $M(T)$.

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7.2 PARAMETRIC RENEWAL FUNCTION ESTIMATION

When the failure-time distribution is known, we can determine the expected number of failures (or renewals) during any time interval $(0, t]$ by using either the continuous-time or the discrete-time approaches given below.

7.2.1 Continuous Time

This approach is also referred to as the *renewal theory approach*. Consider the case when a unit is operating until it fails. Upon failure, the unit is either replaced by a new identical unit or repaired to its original condition. This is considered a renewal process and can be formally defined as a nonterminating sequence of independent and identically distributed (i.i.d.) nonnegative random variables. To determine the expected number of failures in interval $(0, t]$, we follow Jardine (1973) and Jardine and Tsang (2005) and define the following notations as shown in Figure 7.1. Let

$N(t)$ = the number of failures in interval $(0, t]$,

$M(t)$ = the expected number of failures in interval $(0, t] = E[N(t)]$, where $E[]$ denotes expectations,

t_i = length of the time interval between failures $i - 1$ and i,

S_r = total time up to the rth failure $S_r = t_1 + t_2 + \ldots + t_r = \sum_{i=1}^{r} t_i$

The probability that the number of failures $N(t) = r$ is the same as the probability that t lies between the rth and $(r + 1)$th failure. Thus,

$$P[N(t) < r] = 1 - F_r(t),$$

where $F_r(t)$ is the cumulative distribution function (CDF) of S_r—that is, $F_r(t) = P[S_r \leq t]$, and

$$P[N(t) > r] = F_{r+1}(t).$$

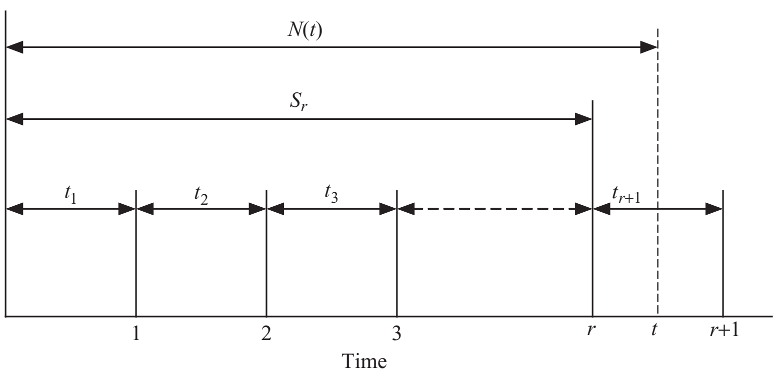

FIGURE 7.1 Failures in $(0, t]$.

But

$$P[N(t) < r] + P[N(t) = r] + P[N(t) > r] = 1.$$

Thus,

$$P[N(t) = r] = F_r(t) - F_{r+1}(t).$$

The expected value of $N(t)$ is then

$$\begin{aligned} M(t) &= \sum_{r=0}^{\infty} rP[N(t) = r] \\ &= \sum_{r=0}^{\infty} r[F_r(t) - F_{r+1}(t)] \end{aligned}$$

or

$$M(t) = \sum_{r=1}^{\infty} F_r(t). \tag{7.2}$$

$M(t)$ is referred to as the *renewal function*. Equation 7.2 can be written as

$$M(t) = F(t) + \sum_{r=1}^{\infty} F_{r+1}(t),$$

where $F_{r+1}(t)$ is the convolution of $F_r(t)$ and F. Let f be the probability density function (p.d.f.) of F, then

$$F_{r+1}(t) = \int_0^t F_r(t-x)f(x)dx$$

and

$$\begin{aligned} M(t) &= F(t) + \sum_{r=1}^{\infty} \int_0^t F_r(t-x)f(x)dx \\ &= F(t) + \int_0^t \left[\sum_{r=1}^{\infty} F_r(t-x)\right] f(x)dx, \end{aligned}$$

—that is,

$$M(t) = F(t) + \int_0^t M(t-x)f(x)dx. \tag{7.3}$$

We refer to Equation 7.3 as the *fundamental renewal equation*.

By taking Laplace transforms of both sides of Equation 7.3, we obtain

$$M^*(s) = \frac{f^*(s)}{s[1 - f^*(s)]}, \tag{7.4}$$

where

$$f^*(s) = E[e^{-sS_{N(t)}}] = \int_0^\infty e^{-st} f(t)dt$$

and $M(t) = \mathfrak{L}^{-1}M^*(s)$ is the Laplace inverse of $M^*(s)$. The renewal density $m(t)$ is the derivative of $M(t)$ or

$$m(t) = \frac{dM(t)}{dt},$$

where $m(t)$ is interpreted as the probability that a renewal occurs in the interval $[t, t + \Delta t]$. Thus, in the case of a Poisson process, renewal density $m(t)$ is the Poisson rate λ.

We can also write

$$m(t) = \sum_{r=1}^{\infty} f_r(t)$$

or

$$m(t) = f(t) + \int_0^t m(t - x) f(x)dx. \tag{7.5}$$

Equation 7.5 is known as the *renewal density equation*. To solve the renewal equation, we use Laplace transforms

$$\mathfrak{L}f(t) = \int_0^\infty e^{-st} f(t)dt$$

and

$$\mathfrak{L}m(t) = \int_0^\infty e^{-st} m(t)dt.$$

Using the convolution property of transforms,

$$\mathfrak{L}m(t) = \mathfrak{L}f(t) + \mathfrak{L}m(t)\mathfrak{L}f(t).$$

Let $m^*(s) = \mathfrak{L}m(t)$ and $f^*(s) = \mathfrak{L}f(t)$, then

$$m^*(s) = f^*(s) + m^*(s)f^*(s)$$

$$m^*(s) = \frac{f^*(s)}{1 - f^*(s)}$$

$$M^*(s) = \frac{f^*(s)}{s\left[1 - f^*(s)\right]} \tag{7.6}$$

and

$$f^*(s) = \frac{m^*(s)}{1 + m^*(s)}.$$

EXAMPLE 7.1

A component that exhibits constant failure rate is replaced upon failure by an identical component. The p.d.f. of the failure-time distribution is

$$f(t) = \lambda e^{-\lambda t}.$$

What is the expected number of failures during the interval $(0, t]$?

SOLUTION

Taking the Laplace transform of the p.d.f. results in

$$f^*(s) = \frac{\lambda}{s + \lambda}.$$

The Laplace transform of the renewal density becomes

$$m^*(s) = \frac{\lambda}{s + \lambda - \lambda} = \frac{\lambda}{s}.$$

The inverse of the above expression is

$$m(t) = \lambda \quad t \geq 0.$$

Thus,

$$M(t) = \lambda t \quad t \geq 0$$

—that is, the number of failures in $(0, t]$ is λt. ■

We now consider a numerical example for the constant-failure-rate case.

EXAMPLE 7.2

A system is found to exhibit a constant failure rate of 6×10^{-6} failures per hour. What is the expected number of failures after 1 year of operation? Note that the system is repaired upon failure and is returned to its original condition.

SOLUTION

The Laplace transform of the p.d.f. of the constant-hazard rate λ is

$$f^*(s) = \int_0^\infty \lambda e^{-\lambda t} e^{-st}\,dt = \frac{\lambda}{s+\lambda}.$$

Substituting in Equation 7.4, we obtain

$$M^*(s) = \frac{\lambda/(\lambda+s)}{s[1-\lambda/(\lambda+s)]} = \frac{\lambda}{s^2}.$$

The inverse of $M^*(s)$ to $M(t)$ is

$$M(t) = \mathfrak{L}^{-1}\frac{\lambda}{s^2} = \lambda t.$$

The expected number of failures after 1 year of service (10^4 h) is $6 \times 10^{-6} \times 10^4 = 0.06$ failures. ■

Let us consider another example. If $X_1, X_2, \ldots, X_n$ are n exponentially i.i.d. random variables having a mean of $1/\lambda$, then $X_1 + X_2 + \ldots + X_n$ is a gamma distribution with parameters n and λ. Its p.d.f. is given by

$$f(t) = \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}$$

or

$$f(t) = \frac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{\Gamma(n)}.$$

The Laplace transform of $f(t)$ is

$$f^*(s) = \frac{\lambda^n}{(\lambda+s)^n}.$$

The Laplace transform of the expected number of failures is

$$M^*(s) = \frac{f^*(s)}{s[1-f^*(s)]} = \frac{\lambda^n}{s[(\lambda+s)^n - \lambda^n]}.$$

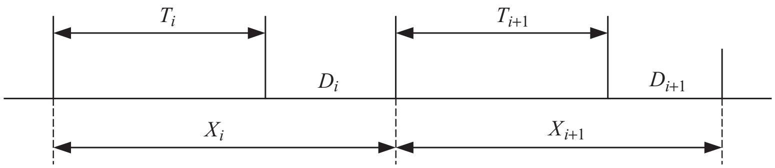

FIGURE 7.2 Renewal processes using repairs.

The inverse of the above transform is difficult to obtain. Therefore, we may obtain $M(t)$ by numerically computing it for discrete time intervals, by using nonparametric approaches, or by using approximate methods as discussed later in this chapter.

Availability Analysis under Renewals Consider the case when a failure occurs, it is repaired, and the component becomes "as good as new." Let T_i be the duration of the ith functioning period and D_i be the system downtime for the ith repair or replacement. We have a sequence of random variables $\{X_i = T_i + D_i\}$ i =1, 2, . . . as shown in Figure 7.2.

Assume T_i's are i.i.d. with CDF $W(t)$ and p.d.f. $w(t)$. D_i's are i.i.d. with CDF of $G(t)$ and p.d.f. $g(t)$. Then X_i's are i.i.d. The underlying density $f(t)$ of the renewal process is the convolution of w and g. Thus,

$$\mathfrak{L}f(t) = \mathfrak{L}w(t)\mathfrak{L}g(t)$$

or

$$f^*(s) = w^*(s)g^*(s).$$

Therefore,

$$m^*(s) = \frac{w^*(s)g^*(s)}{1 - w^*(s)g^*(s)}. \tag{7.7}$$

As shown later in Section 7.4.1, $M^*(s)$ is obtained as

$$M^*(s) = \frac{w^*(s)g^*(s)}{s\left[1 - w^*(s)g^*(s)\right]}. \tag{7.8}$$

We define the availability $A(t)$ as the probability that the component is properly functioning at time t. If no repair is performed, then $R(t) = A(t) = 1-W(t)$.

The component may be functioning at time t by reason of two mutually exclusive cases: either the component has not failed from the beginning (no renewals in (0, t]) with probability $R(t)$, or the last renewal (repair) occurred at time x, $0 < x < t$, and the component has continued to function since that time (Trivedi, 1982). The probability associated with the second case is

$$\int_0^t R(t-x)m(x)dx.$$

Thus,

$$A(t) = R(t) + \int_0^t R(t-x)m(x)dx.$$

Taking Laplace transforms we obtain

$$A^*(s) = R^*(s) + R^*(s)m^*(s)$$
$$A^*(s) = R^*(s)\left[1 + m^*(s)\right]. \tag{7.9}$$

Substituting Equation 7.7 into Equation 7.9 results in

$$A^*(s) = R^*(s)\left[1 + \frac{w^*(s)g^*(s)}{1 - w^*(s)g^*(s)}\right]$$

or

$$A^*(s) = \frac{R^*(s)}{1 - w^*(s)g^*(s)}.$$

But $R(t) = 1 - W(t)$ and its Laplace transform is

$$R^*(s) = \frac{1}{s} - W^*(s)$$

or

$$R^*(s) = \frac{1}{s} - \frac{w^*(s)}{s}$$
$$= \frac{1 - w^*(s)}{s}.$$

Thus,

$$A^*(s) = \frac{1 - w^*(s)}{s\left[1 - w^*(s)g^*(s)\right]}.$$

The steady-state availability A is

$$A = \lim_{t \to \infty} A(t) = \lim_{s \to 0} sA^*(s).$$

When s is small, we approximate $e^{-st} \simeq 1 - st$
or

$$w^*(s) = \int_0^\infty e^{-st} w(t)\,dt$$
$$\simeq \int_0^\infty w(t)\,dt - s\int_0^\infty tw(t)\,dt$$
$$\simeq 1 - \frac{s}{\alpha},$$

where $1/\alpha$ is the mean time to failure (MTTF). Also

$$g^*(s) \simeq 1 - \frac{s}{\beta},$$

where $1/\beta$ is the mean time to repair (MTTR):

$$A = \lim_{x \to \infty} \frac{1 - \left[1 - \frac{s}{\alpha}\right]}{1 - \left[1 - \frac{s}{\alpha}\right]\left[1 - \frac{s}{\beta}\right]} = \frac{\frac{1}{\alpha}}{\frac{1}{\alpha} + \frac{1}{\beta}}$$

$$A = \frac{MTTF}{MTTF + MTTR}.$$

EXAMPLE 7.3

Consider the case of exponential failure and repair-time distributions. Derive an expression for the renewal density $m(t)$. What are the availability $A(t)$ and the steady-state availability $A(\infty)$?

SOLUTION

Let $w(t)$ and $g(t)$ represent the p.d.f.'s of the failure-time and repair-time distributions, respectively. Then,

$$w(t) = \lambda e^{-\lambda t}$$
$$g(t) = \mu e^{-\mu t}.$$

The Laplace transforms of these two functions are

$$w^*(s) = \frac{\lambda}{s + \lambda}$$
$$g^*(s) = \frac{\mu}{s + \mu}.$$

Using Equation 7.7, the renewal density is obtained as

$$m^*(s) = \frac{w^*(s) g^*(s)}{1 - w^*(s) g^*(s)}$$
$$= \frac{\lambda\mu}{s[s + (\lambda + \mu)]}$$

or

$$m^*(s) = \frac{\lambda\mu}{(\lambda + \mu)s} - \frac{\lambda\mu}{(\lambda + \mu)^2} \frac{\lambda + \mu}{s + \lambda + \mu}.$$

The renewal density function is

$$m(t) = \frac{\lambda\mu}{\lambda+\mu} - \frac{\lambda\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

$$\lim_{t\to\infty} m(t) = \frac{\lambda\mu}{\lambda+\mu}$$

or

$$\lim_{t\to\infty} m(t) = \frac{1}{MTTF + MTTR}.$$

The availability of the system at time t is obtained as

$$A^*(s) = \frac{1 - \dfrac{\lambda}{s+\lambda}}{s\left[1 - \dfrac{\lambda\mu}{(s+\lambda)(s+\mu)}\right]}$$

or

$$A^*(s) = \frac{s+\mu}{s[s+(\lambda+\mu)]}.$$

No transform for this function exists in Laplace transform tables. However, a partial-fraction algebra reduces the above expression to known results.

$$A^*(s) = \frac{\dfrac{\mu}{\lambda+\mu}}{s} + \frac{\dfrac{\lambda}{\lambda+\mu}}{s+(\lambda+\mu)}.$$

The Laplace inverse is

$$A(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

and

$$A = \lim_{t\to\infty} A(t) = \frac{\mu}{\lambda+\mu}.$$ ■

EXAMPLE 7.4

Permanent magnet synchronous motor (PMSM) brushless DC (BLDC) servos are becoming attractive replacements for DC motors in industrial servo motors. The PMSM BLDC servo has higher torque and velocity bandwidth and does not require the regular brush and maintenance requirements of conventional motors.

A producer of the PMSMs designs a reliability test by subjecting a motor to a continuous load. Upon failure, the motor is immediately repaired and restored to its initial condition. The test is then continued and the above procedure is repeated. The failure and the repair time distributions are exponential with rates λ and μ with estimates of 6×10^{-5} failures per hour and 4×10^{-2} repairs per hour, respectively.

Determine the expected number of motor failures during $(0, 2 \times 10^4$ h) and the availability of the motor at the end of 2 years of testing. Plot $M(t)$ and $A(t)$ for different values of λ and μ.

SOLUTION

Since failure and repair times are exponential, then we use the results in Example 7.3 to obtain

$$m^*(s) = \frac{\lambda\mu}{(\lambda+\mu)s} - \frac{\lambda\mu}{(\lambda+\mu)^2} \cdot \frac{1}{(s+\lambda+\mu)}$$

and

$$m(t) = \frac{\lambda\mu}{\lambda+\mu} - \frac{\lambda\mu}{(\lambda+\mu)} e^{-(\lambda+\mu)t}.$$

The expected number of renewals in $(0, t]$ is

$$M(t) = \frac{\lambda\mu}{(\lambda+\mu)} t - \frac{\lambda\mu}{(\lambda+\mu)^2} + \frac{\lambda\mu}{(\lambda+\mu)^2} e^{-(\lambda+\mu)t}.$$

Substitution of the values of λ and μ in the above expression results in

$$M(t) = 5.991 \times 10^{-5} t - 0.001495 + 1.495 \times 10^{-3} e^{-4.006\times 10^{-2} t}.$$

The expected number of failures in a 2×10^4 h interval is

$$M(2\times 10^4) = 1.197 \text{ failures.}$$

The availability of the motor is

$$A(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t},$$

and the availability at the end of 2 years of testing is

$$A(2\times 10^4) = \frac{4\times 10^{-2}}{4.006\times 10^{-2}} + \frac{6\times 10^{-5}}{4.006\times 10^{-2}} e^{-(4.006)\times 2\times 10^{2}}$$

or

$$A(2 \times 10^4) = 0.9985.$$

The plots of $M(t)$ and $A(t)$ for different values of λ and μ are shown in Figures 7.3 and 7.4, respectively.

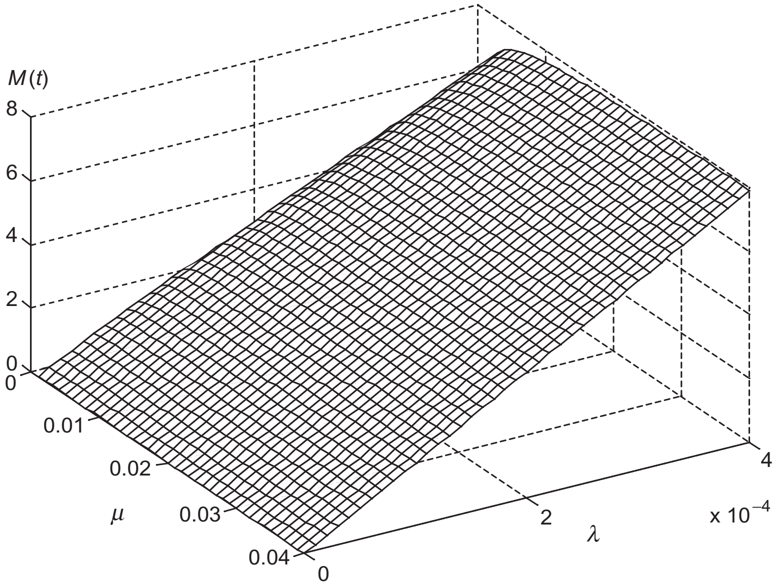

FIGURE 7.3 $M(t)$ for different λ and μ ($t = 2 \times 10^4$).

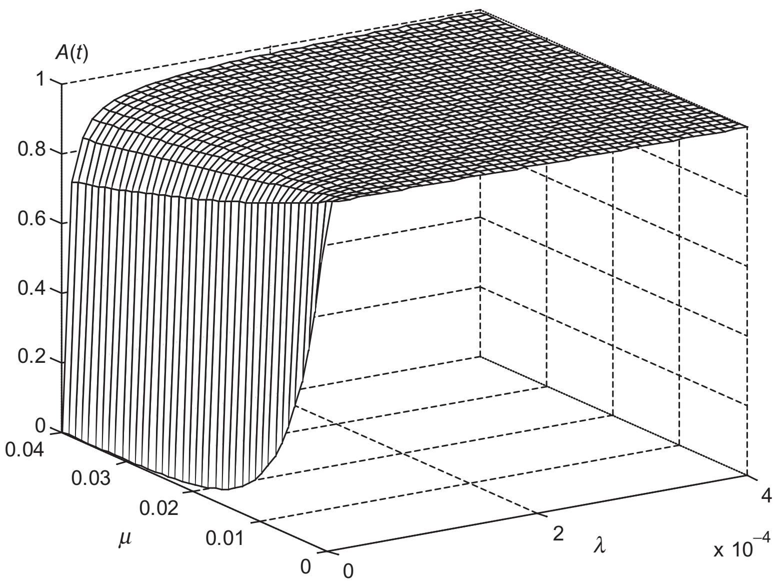

FIGURE 7.4 $A(t)$ for different λ and μ. ■

7.2.2 Discrete Time

Let us consider the situation where the time scale is discrete—that is, the system (or a component) is observed at discrete time intervals such as once a week, once a month, or once a year. If a failure is observed, the system is repaired, and the process is repeated. We are interested in determining the number of failures at the end of a discrete time interval—say, the third week. There are three possible ways that result in having failures in the third week. The expected number of failures at the end of the third week, $M(3)$, is obtained as

$$\begin{aligned} M(3) = {} & \text{number of expected failures which occur in interval } (0,3) \text{ when the} \\ & \text{first failure occurs in the first week} \times \text{probability of the first failure occurring in} \\ & \text{interval } (0,1) + \text{number of expected failures that occur in interval } (0,3) \text{ when} \\ & \text{the first failure occurs in the second week} \times \text{probability of the first failure} \\ & \text{occurring in interval } (1,2) + \text{number of expected failures that occur in interval} \\ & (0,3) \text{ when the first failure occurs in the third week} \times \text{probability of the first} \\ & \text{failure occurring in interval } (2,3) \end{aligned}$$

The expected number of failures that occur in the interval (0, 3) when the first failure occurs in the first week can be written as

$$\begin{aligned} M_1(3) = {} & \text{the number of failures that occurred in the first week} + \text{the expected} \\ & \text{number of failures in the remaining 2 weeks} \end{aligned}$$

or

$M_1(3) = 1 + M(2)$, where $M_1(3)$ is the expected number of failures at the end of the 3 weeks, provided that the first failure occurred at the first week.

By definition, the expected number of failures in the remaining 2 weeks is $M(2)$, starting with a new component or replacing the failed one after the failure that occurred in the first week. In order to calculate the expected number of failures in any interval, we need to calculate the probability that the first failure occurs in an interval (t_1, t_2) as follows.

Probability that the first failure occurs in the interval $(t_1, t_2) = \int_{t_1}^{t_2} f(t)dt$.

Thus, the expected number of failures by the third week is

$$M(3) = [1 + M(2)]\int_0^1 f(t)dt + [1 + M(1)]\int_1^2 f(t)dt + [1 + M(0)]\int_2^3 f(t)dt.$$

Since $M(0) = 0$, then the above equation can be rewritten as

$$M(3) = \sum_{i=0}^{2} [1 + M(2-i)]\int_i^{i+1} f(t)dt.$$

In general, the number of failures at time period T is obtained as

$$M(T) = \sum_{i=0}^{T-1} [1 + M(T-i-1)] \int_{i}^{i+1} f(t)dt \quad T \geq 1, \tag{7.10}$$

with $M(0) = 0$.

EXAMPLE 7.5

The manufacturer of five-volt electric bulbs estimates the expected number of failures during a 20-week period by subjecting a bulb to 10 V. Upon failure, the bulb is replaced by a new one, and the process is repeated. The failure time for the bulbs is found to follow a uniform distribution between $0 \leq t \leq 20$ weeks with a $f(t) = 1/20$. Determine the expected number of failures in the 4-week period.

SOLUTION

Using Equation 7.10, we obtain $M(4)$ as follows:

$$M(4) = \sum_{i=0}^{3} [1 + M(3-i)] \int_{i}^{i+1} f(t)dt$$

$$M(4) = [1 + M(3)] \int_{0}^{1} \frac{1}{20} dt + [1 + M(2)] \int_{1}^{2} \frac{1}{20} dt + [1 + M(1)] \int_{2}^{3} \frac{1}{20} dt + [1 + M(0)] \int_{3}^{4} \frac{1}{20} dt$$

$$M(0) = 0$$

$$M(1) = [1 + M(0)] \int_{0}^{1} \frac{1}{20} dt = \frac{1}{20}$$

$$M(2) = [1 + M(1)] \int_{0}^{1} \frac{1}{20} dt + [1 + M(0)] \int_{1}^{2} \frac{1}{20} dt = \frac{41}{400}$$

$$M(3) = [1 + M(2)] \int_{0}^{1} \frac{1}{20} dt + [1 + M(1)] \int_{1}^{2} \frac{1}{20} dt + [1 + M(0)] \int_{2}^{3} \frac{1}{20} dt$$

$$= \frac{1261}{8000}$$

$$M(4) = 0.2155 \text{ failures.}$$

■

The following example illustrates the estimation of the expected number of failures when the failure time is normally distributed.

EXAMPLE 7.6

Consider the case when the system is observed every 2 weeks and the failure-time distribution is normal with a mean = 4 and a standard deviation = 1 week. Determine the expected number of failures at the end of 2 weeks.

SOLUTION

$$M(2) = [1+M(1)]\frac{1}{\sqrt{2\pi}}\int_0^1 \exp\left[\frac{-(t-4)^2}{2}\right]dt + [1+M(0)]\frac{1}{\sqrt{2\pi}}\int_1^2 \exp\left[\frac{-(t-4)^2}{2}\right]dt,$$

but

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_0^1 \exp\left[-\frac{(t-4)^2}{2}\right]dt &= \Phi(1-4)-\Phi(0-4)\\ &= \Phi(-3)-\Phi(-4).\end{aligned}$$

where

$$\Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} \exp\left[-\frac{t^2}{2}\right]dt$$

is the CDF of the standard normal distribution. From the standard tables, we obtain

$$\Phi(-3)-\Phi(-4) \simeq 0.0014.$$

Meanwhile,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_1^2 \exp\left[\frac{-(t-4)^2}{2}\right]dt &= \Phi(-2)-\Phi(-3)\\ &= 0.0228-0.0014 = 0.0214\end{aligned}$$

$$M(0) = 0$$

$$\begin{aligned}M(1) &= [1+M(0)]\frac{1}{\sqrt{2\pi}}\int_0^1 \exp\left[\frac{-(t-4)^2}{2}\right]dt\\ &= [1+0]0.0014 = 0.0014\end{aligned}$$

$$M(2) = (1+0.0014)0.0014 + (1+0)(0.0214) = 0.0228 \text{ failures.}$$ ■

7.3 NONPARAMETRIC RENEWAL FUNCTION ESTIMATION

When it is difficult to determine the Laplace transform or its inverse for complex p.d.f.'s or when the failure-time distribution is unknown but the mean and standard deviation of the failure times are known, one may estimate the expected number of failures (or renewals) in interval $(0, t]$ when the time horizon is continuous or when the time interval is discrete as described below.

7.3.1 Continuous Time

When the time horizon is continuous, one may use a general expression to determine the expected number of failures at time t. This expression is developed by Cox (1962), and its derivation is given below.

Consider the form of $M(t)$ as $t \to \infty$. Let us examine the behavior of $M^*(s)$ for small s. The Laplace transform of the p.d.f. of the failure time $f(t)$ is $f^*(s)$. From the properties of the Laplace transform, the mean (μ) and standard deviation (σ) of the failure time can be determined by using the following equations

$$\left.\frac{df^*(s)}{ds}\right|_{s=0} = -\mu$$

$$\left.\frac{d^2 f(s)}{ds^2}\right|_{s=0} = \sigma^2 + \mu^2, \quad f^*(0) = 1.$$

From the above equations, we can express $f^*(s)$ as a Taylor series expansion around the point $s = 0$:

$$f^*(s) = 1 - s\mu + \frac{1}{2}s^2(\mu^2 + \sigma^2) + O(s^2), \tag{7.11}$$

where $O(s^2)$ denotes a function of s tending to zero as $s \to 0$ faster than s^2.

Substituting Equation 7.11 into Equation 7.8 we obtain

$$M^*(s) = \frac{1 - s\mu + \frac{1}{2}s^2(\mu^2 + \sigma^2) + O(s^2)}{s^2\mu - \frac{1}{2}s^3\left(\mu^2 + \sigma^2\right) + O(s^3)}.$$

The above equation can be simplified by using the partial-fraction-expansion formula given below:

$$g(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_{i=1}^{n}(s + r_i)} = \frac{A_1}{s + r_1} + \frac{A_2}{s + r_2} + \ldots + \frac{A_n}{s + r_n},$$

where r_i, $i = 1, 2, \ldots, n$ are the roots of $D(s)$

$$A_i = \left[\frac{N(s)}{D(s)} \cdot (s + r_i)\right]_{s=-r_i}.$$

The above equation is valid when the roots of the $D(s)$ expression are all real and different. Clearly, the solution should be modified when we have repeated roots or some of the roots are imaginary (see Muth 1977; Beerends et al. 2003).

The denominator of the $M^*(s)$ equation has two repeated roots $r_1 = r_2 = 0$ and a third real root r_3. Thus, we can rewrite $M^*(s)$ as

$$M^*(s) = \frac{A_1}{s^2 + 0} + \frac{A_2}{s + 0} + \frac{A_3}{s + r_3}.$$

The coefficients A_1, A_2, A_3 can be obtained as follows:

$$A_1 = \left[\frac{\left(1 - s\mu + \frac{1}{2}s^2\left(\mu^2 + \sigma^2\right)\right) + O(s^2)}{s^2\mu - \frac{1}{2}s^3(\mu^2 + \sigma^2) + O(s^3)} \cdot s^2\right]_{s=0} = \frac{1}{\mu}.$$

Since the first two roots s_1 and s_2 are repeated, A_2 is obtained as

$$A_2 = \frac{d}{ds}\left[\frac{(1 - s\mu + \frac{1}{2}s^2(\mu^2 + \sigma^2)) + O(s^2)}{s^2\mu - \frac{1}{2}s^3(\mu^2 + \sigma^2) + O(s^3)} \cdot s^2\right]_{s=0} = \frac{-\mu^2 + \frac{1}{2}(\mu^2 + \sigma^2)}{\mu^2}$$

or

$$A_2 = \frac{(\sigma^2 - \mu^2)}{2\mu^2},$$

and the last term is a function of the order $O(1/s)$. This results in

$$M^*(s) = \frac{1}{s^2\mu} + \frac{1}{s}\frac{\sigma^2 - \mu^2}{2\mu^2} + O\left(\frac{1}{s}\right). \tag{7.12}$$

The inverse of Equation 7.12 as $t \to \infty$ is

$$M(t) = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + O(1). \tag{7.13}$$

The above equation holds true as σ^2 is finite. One should note that

- If $\sigma = \mu$, then $M(t) = t/\mu + O(1)$. For the exponential failure time, $M(t) = t/\mu$ or λt.
- If $\sigma < \mu$, then $(\sigma^2 - \mu^2)/2\mu^2$ in Equation 7.13 becomes negative, and if $\sigma << \mu$, then

$$M(t) \approx \frac{t - \frac{1}{2}\mu}{\mu} + O(1). \tag{7.14}$$

  This implies that to start with a new component rather than an *average* component is equivalent to saving one-half a failure (Cox, 1962).
- If $\sigma > \mu$, the second term in Equation 7.13 becomes positive. This implies that when the coefficient of variation σ^2/μ^2 is greater than one, it is likely to have appreciable probability near zero failure-time and that to start with a new component is therefore worse than to start with an a*verage* component (Bartholomew, 1963).

The following two examples illustrate the use of the general equation to determine the expected number of failures (renewals) when the parameters of the failure-time distribution are known and when the inverse of the Laplace transform is difficult to obtain.

EXAMPLE 7.7

Consider a component that fails according to a distribution with $\mu = 5$, $\sigma^2 = 1$. When the component fails, it is immediately repaired and placed in service. Moreover, a preventive replacement is performed every 1000 weeks. How many failures would have occurred before a preventive replacement is made?

SOLUTION

Using Equation 7.13, we obtain the expected number of failures as

$$M(t) = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2}$$

$$M(1000) = \frac{1000}{5} + \frac{1 - 25}{2 \times 25} = 199.5 \text{ failures.}$$

■

We now determine the expected number of failures for components whose failure time follows a gamma distribution.

EXAMPLE 7.8

It is found that the failure-time distribution of a complex system with a large number of units, each has an exponential failure-time distribution, can be described by a gamma distribution. The parameters of the distribution are $n = 50$, and $\lambda = 0.001$. Determine the expected number of failures after 10^5 h of operation.

SOLUTION

The gamma distribution has the following $f(t)$:

$$f(t) = \frac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{\Gamma(n)}.$$

The Laplace transform of the expected number of failures is

$$M^*(s) = \frac{1}{s\left[\left(1+\frac{s}{\lambda}\right)^n - 1\right]}.$$

The inverse of the Laplace transform of $M^*(s)$ is complex and difficult to obtain. Therefore, we utilize the general expression for the expected number of failures as shown below.

The mean is n/λ and the variance is n/λ^2. Thus, using Equation 7.13, we obtain

$$\begin{aligned} M(10^5) &= \frac{10^5}{n/\lambda} + \frac{(n/\lambda^2) - (n/\lambda)^2}{2(n/\lambda)^2} \\ &= \frac{10}{5} - \frac{2450}{5000} \\ &= 1.51 \text{ failures.} \end{aligned}$$

■

Systems with Two Stages of Failure: Consider a system whose components fail if they enter either of two stages of failure mechanisms. The first mechanism is due to excessive voltage, and the second is due to excessive temperature. Suppose that the failure mechanism enters the first stage with probability θ, and the p.d.f. of the failure time is $\lambda_1 e^{-\lambda_1 t}$. It enters the second stage with probability $(1 - \theta)$, and the p.d.f. of its failure time is $\lambda_2 e^{-\lambda_2 t}$. The failure of a component occurs at the end of either stage. Hence, the p.d.f. of the failure time is

$$f(t) = \theta\lambda_1 e^{-\lambda_1 t} + (1-\theta)\lambda_2 e^{-\lambda_2 t}.$$

The Laplace transform of $f(t)$ is

$$f^*(s) = \frac{\theta\lambda_1}{\lambda_1 + s} + \frac{(1-\theta)\lambda_2}{\lambda_2 + s}$$

$$f^*(s) = \frac{\lambda_1\lambda_2 + \theta\lambda_1 s + (1-\theta)\lambda_2 s}{(\lambda_1 + s)(\lambda_2 + s)},$$

but

$$M^*(s) = \frac{f^*(s)}{s\left[1 - f^*(s)\right]}.$$

Substitution of $f^*(s)$ into the above equation yields

$$M^*(s) = \frac{s(\theta\lambda_1 + (1-\theta)\lambda_2) + \lambda_1\lambda_2}{s^2(s + (1-\theta)\lambda_1 + \theta\lambda_2)}. \tag{7.15}$$

Equation 7.15 has double poles $r_1 = 0$ and $r_2 = 0$ and the root $r_3 = -[(1-\theta)\lambda_1 + \theta\lambda_2]$. Thus, one can rewrite Equation 7.15 as

$$M^*(s) = \frac{A_1}{s^2} + \frac{A_2}{s} + \frac{A_3}{s - r_3},$$

where A_1, A_2, and A_3 are obtained as follows

$$A_1 = \left[\frac{s(\theta\lambda_1 + (1-\theta)\lambda_2) + \lambda_1\lambda_2}{s^2(s + (1-\theta)\lambda_1 + \theta\lambda_2)} \cdot s^2\right]_{s=0} = \frac{\lambda_1\lambda_2}{(1-\theta)\lambda_1 + \theta\lambda_2}.$$

Since $f'^*(s = 0) = -\mu$, then

$$\frac{\lambda_1\lambda_2}{(1-\theta)\lambda_1 + \theta\lambda_2} = A_1 = \frac{1}{\mu}.$$

Similarly, A_2 can be obtained as

$$A_2 = \frac{\sigma^2 - \mu^2}{2\mu^2}.$$

Finally, A_3 is obtained as

$$A_3 = \left[\frac{s(\theta\lambda_1 + (1-\theta)\lambda_2) + \lambda_1\lambda_2}{s^2(s + (1-\theta)\lambda_1 + \theta\lambda_2)} \cdot (s + (1-\theta)\lambda_1 + \theta\lambda_2)\right]_{s=-[(1-\theta)\lambda_1 + \theta\lambda_2]}$$

$$A_3 = \frac{-\theta(\lambda_1 - \lambda_2)^2 + \theta^2(\lambda_1 - \lambda_2)^2}{[(1-\theta)\lambda_1 + \theta\lambda_2]^2}.$$

Using the above A_i $i = 1$, 2, and 3 in Equation 7.15, M*(s) becomes

$$M^*(s) = \frac{1}{s^2\mu} + \frac{1}{s}\frac{\sigma^2 - \mu^2}{2\mu^2} - \frac{\theta(1-\theta)\left(\lambda_1 - \lambda_2^2\right)}{[(1-\theta)\lambda_1 + \theta\lambda_2]^2(s - r_3)}.$$

The inverse of the above equation is

$$M(t)=\frac{t}{\mu}+\frac{\sigma^2-\mu^2}{2\mu^2}-\frac{\theta(1-\theta)(\lambda_1-\lambda_2)^2}{[(1-\theta)\lambda_1+\theta\lambda_2]^2}\exp\{-[(1-\theta)\lambda_1+\theta\lambda_2]t\}. \tag{7.16}$$

EXAMPLE 7.9

The laser diodes (LD) used in a submarine optical fiber transmission system are composed of many kinds of circuit elements such as lenses, dielectric multilayer thin-film filters, and optical fibers as well as their holders, which are composed of various materials such as metals, ceramics, and organic matter. The failure of the system is caused by the failure of the individual components as well as the thermal stresses in their joints. Suppose that the dielectric filters fail due to thermal stresses with a probability of 0.2 and the failure time follows a p.d.f. $\lambda_1 e^{-\lambda_1 t}$ with $\lambda_1 = 0.00001$. It may also fail due to voltage stresses with a probability of 0.8, and the failure-time distribution follows a p.d.f. $\lambda_2 e^{-\lambda_2 t}$ with $\lambda_2 = 0.00006$. The filter fails when either of the two events occur. Determine the expected number of failures during the first year of operation.

SOLUTION

The parameters of the system are

$$\begin{aligned}
\theta &= 0.20\\
1-\theta &= 0.80\\
\lambda_1 &= 0.00001\\
\lambda_2 &= 0.00006\\
\mu &= \frac{(1-\theta)\lambda_1+\theta\lambda_2}{\lambda_1\lambda_2}=3.3333\times10^4\\
\sigma^2 &= \frac{(2-\theta)\theta\lambda_2^2+(1-\theta^2)\lambda_1^2-2\theta(1-\theta)\lambda_1\lambda_2}{\lambda_1^2\lambda_2^2}=3.3333\times10^9.
\end{aligned}$$

At time $t = 10^4$, substituting the above parameters into Equation 7.16, we obtain

$$\begin{aligned}
M(10^4) &= \frac{10^4}{3.3333\times10^4}+\frac{33.3333-11.1111}{22.2222}-e^{-0.20}\\
M(10^4) &= 0.3000+1.0000-0.8187=0.4812 \text{ failures.}
\end{aligned}$$

■

7.3.2 Discrete Time

The asymptotic result of Equation 7.13 is very useful when large values of t are used. The meaning of *large* depends on the distribution of the failure times and the accuracy required, but the approximation is not usually acceptable unless at least, say, $t \geq 2\mu$. The smaller values of t necessitate initially fitting some p.d.f., $f(t)$, to the data. However, the choice of the most appropriate p.d.f. is not always easy, especially when some of the data are censored or when the sample size is small. Therefore, it is more appropriate to consider a nonparametric approach for determining $M(t)$. The development of a nonparametric approach for $M(t)$ for discrete time intervals is now discussed.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. nonnegative random variates, having a p.d.f. $f(x)$ with mean μ and variance σ^2. If $S_k = \sum_{i=1}^{k} X_i$, the renewal function may be written as

$$M(t) = \sum_{k=1}^{\infty} P(S_k \le t). \tag{7.17}$$

Frees's (1986b) estimator of $M(t)$ is

$$\hat{M}_p(t) = \sum_{k=1}^{\infty} \hat{F}_n^{(k)}(t), \tag{7.18}$$

where p is the cutoff, and $\hat{F}_n^{(k)}(t)$ is the distribution function of the permutation distribution of the sum of any k of the random variates X_i. $\hat{F}_n^{(k)}(t)$ can be written in terms of an indicator function I as

$$\hat{F}_n^{(k)}(t) = \left(C_k^n\right)^{-1} \sum_i I(X_{i1} + \ldots + X_{ik} \le t), \tag{7.19}$$

where the sum is over all C_k^n choices of k out of the n values of X_i, and $I(A)$ is 1 if event A occurs, and 0 otherwise. The choice of p is somewhat arbitrary, typically 5 or 10 (Baker, 1993). The following example illustrates the use of Equations 7.18 and 7.19 to obtain the expected number of failures in an interval $(0, T]$.

EXAMPLE 7.10

This example illustrates the use of the nonparametric discrete time approach to determine the renewal function. We use some of the data in Juran and Gryna (1993), which can also be found in Kolb and Ross (1980). Use the nonparametric expression of Equation 7.13 to estimate $M(t)$ at $t = 20$ h and 100 h. The data are given in Table 7.1.

TABLE 7.1 Failure Data of Electronic Ground Support

1.0	1.2	1.3	2.0	2.4	2.9	3.0	3.1	3.3	3.5
3.8	4.3	4.6	4.7	4.8	5.2	5.4	5.9	6.4	6.8
6.9	7.2	7.9	8.3	8.7	9.2	9.8	10.2	10.4	11.9
13.8	14.4	15.6	16.2	17	17.5	19.2	28.1	28.2	29.0
29.9	30.6	32.4	33.0	35.3	36.1	40.1	42.8	43.7	44.5
50.4	51.2	52.0	53.3	54.2	55.6	56.4	58.3	60.2	63.7
64.6	65.3	66.2	70.1	71.0	75.1	75.6	78.4	79.2	84.1
86.0	87.9	88.4	89.9	90.8	91.1	91.5	92.1	97.9	100.8
102.6	103.2	104.0	104.3	105.0	105.8	106.5	110.7	112.6	113.5
114.8	115.1	117.4	118.3	119.7	120.6	121.0	122.9	123.3	124.5
125.8	126.6	127.7	128.4	129.2					

Source: The data are adapted from Juran and Gryna (1993).

SOLUTION

The sample mean is $\overline{X} = 55.603$ h and the sample standard deviation is $s = 43.926$ h. Using Equation 7.13, we obtain

$$M(20) = 0.17173 \text{ and } M(100) = 1.6105.$$

We now use Frees's estimator for M(t),

$$M_p(t) = \sum_{k=1}^{p} F_n^{(k)}(t),$$

where $F_n^{(k)}(t)$ is defined by Equation 7.19.

The values of $\sum_{k=1}^{p} F_{105}^{(k)}(t)$ for t = 20 and 100, and p = 1, 2, . . . , 8 appear in Table 7.2. These values are based on Frees (1988). We now show how the values in column 2 for p = 1 and p = 2 are obtained.

TABLE 7.2 Calculations of $\sum_{k=1}^{p} F_{105}^{(k)}(t)$

p	$\sum_{k=1}^{p} F_{105}^{(k)}(20)$	$\sum_{k=1}^{p} F_{105}^{(k)}(100)$
1	0.35238	0.75238
2	0.44286	1.1775
3	0.4597	1.3729
4	0.46178	1.4514
5	0.46194	1.4798
6	0.46194	1.4890
7	0.46194	1.4917
8	0.46194	1.4924

For p = 1

For this case, the $\sum_i I(X_{i1} \le 20)$ is the total number of observations whose individual values are ≤ 20. Thus,

$$F_{105}^{(1)}(20) = \frac{1}{C_1^{105}} \sum_i I(X_{i1} \le 20) = \frac{37}{105} = 0.35238.$$

Therefore, $M_1(20)$ = 0.35238.

For p = 2

In this case, the $\sum_i I(X_{i1} + X_{i2} \le 20)$ is the total number of cases where the sum of any two observations is ≤ 20. Thus,

$$F_{105}^{(2)}(20) = \frac{1}{C_2^{105}} \sum_i I(X_{i1} + X_{i2} \le 20) = \frac{494 \times 2}{104 \times 105} = 0.09048.$$

Therefore, $M_2(20)$ = 0.35238 + 0.09048 = 0.44286. ■

From Table 7.2, using the recommended value of $p = 5$, the expected number of failures for times 20 and 100 h are 0.461,94 and 1.4798, respectively.

Baker (1993) develops a discretization approach to calculate $\hat{M}_n(t)$ by scaling up the X_i's and approximating them by integers. The p.d.f. of the permutation distribution of the sum of any k of the X_i—whose distribution function is $\hat{F}_n^{(k)}(t)$—is represented as a histogram, and the kth histogram has C_k^n partial sums of k of the X_i contributing to it. The set of n histograms is built up by adding the X_i successively. The error in estimating $\hat{M}(t)$ decreases as the number N of histogram intervals increases. The choice of N can be accomplished by running simulation experiments and choosing N that reduces the discretization error to an acceptably low value without increasing the computational difficulty. The following algorithm is developed by Baker, (1993).

1. Sort the X_i's into ascending order—that is, work with the order statistics $X_{(i)}$.
2. Find the sums of the first p order statistics and hence the largest value of p such that $\sum_{i=1}^{p} X_{(i)} < t_0$, where t_0 is the largest value of t for which M must be estimated.
3. Zero the p histograms, and for i from 1 to n, add each $X_{(i)}$ in turn. All translations that add to array elements $>N$ are discarded.
4. Normalize each of the p histograms to unity by dividing the kth histogram by C_k^n.
5. Add the p calculated histograms (distribution functions) together to give $\hat{M}(t)$ for each of the N times, jh, where h is the step size, and $j = 1, 2, \ldots, N$.
6. Make a continuity correction to each value of $\hat{M}(t)$ by averaging it with the corresponding value for the previous time-point.

EXAMPLE 7.11

Use Baker's algorithm to determine $M(20)$ and $M(100)$ for the data given in Example 7.10.

SOLUTION

The algorithm was coded by Baker in a computer program listed in Appendix J. The results obtained from the program are listed in Table 7.3. Note that we only listed those results in the neighborhood of $y = 20$ and $t = 100$. The results obtained from the program are $M(20) = 0.462972$ and $M(100) = 1.493999$, which approximately equal those obtained from Frees's estimator.

TABLE 7.3 Expected Number of Failures Using Baker's Approach

Time	Expected number of failures
********	********
********	********
16.799999	0.415758
17.849999	0.432715
18.899999	0.449924
19.949999	0.462617
20.999999	0.470082

(*Continued*)

TABLE 7.3 (*Continued*)

Time	Expected number of failures
********	********
********	********
96.599996	1.443323
97.649996	1.460376
98.699996	1.477292
99.749995	1.489561
100.799995	1.506470

■

EXAMPLE 7.12

Fifty n-channel metal oxide semiconductor (MOS) transistor arrays are subjected to a voltage stress of 27 V and 25°C to investigate the time-dependent dielectric breakdown (TDDB) behavior of such transistors (Swartz, 1986). The times to failure in minutes are given in Table 7.4. Use Baker's algorithm to determine the expected number of failures starting from $t = 100$ to $t = 150$ min. Compare these estimates with those obtained using Equation 7.13.

TABLE 7.4 Failure-Time Data of MOS Transistor Arrays

1.0	60.0	73.0	74.0	75.0	90.0	101.0	103.0	113.0	117.0
131.0	132.0	135.0	148.0	149.0	150.0	152.0	153.0	155.0	159.0
160.0	161.0	163.0	167.0	171.0	176.0	182.0	185.0	186.0	194.0
197.0	211.0	214.0	215.0	220.0	233.0	235.0	236.0	237.0	241.0
252.0	268.0	278.0	279.0	292.0	307.0	344.0	379.0	445.0	465.0

SOLUTION

We use Baker's algorithm and Equation 7.13 to obtain the results shown in Table 7.5. The first column in the table is the time, the second is the estimate of $M(t)$ using Baker's algorithm, and the third column is the estimate of $M(t)$ obtained using Equation 7.13. As shown from Table 7.5, the estimates obtained using Equation 7.13 are higher than those obtained by Baker's algorithm and that the difference between the two approaches increases with time.

TABLE 7.5 Estimates of $M(t)$

Time	Baker's estimate	Equation 7.13 estimate
100	0.124082	0.138927
102	0.134082	0.149383
104	0.154490	0.159839
106	0.165306	0.170295
108	0.165714	0.180751
110	0.165714	0.191207

TABLE 7.5 (*Continued*)

Time	Baker's estimate	Equation 7.13 estimate
112	0.165714	0.201663
114	0.175714	0.212119
116	0.186122	0.222574
118	0.196531	0.233030
120	0.206939	0.243486
122	0.207347	0.253942
124	0.207347	0.264398
126	0.207347	0.274854
128	0.207347	0.285310
130	0.207347	0.295766
132	0.227347	0.306222
134	0.248980	0.316677
136	0.261071	0.327133
138	0.271964	0.337589
140	0.272398	0.348045
142	0.272398	0.358501
144	0.272398	0.368957
146	0.272398	0.379413
148	0.282806	0.389869
150	0.314872	0.400324

■

7.4 ALTERNATING RENEWAL PROCESS

Suppose a machine breaks down and is repaired as exhibited in Figure 7.2. The breakdown (or failure) of the machine and the repair of the failure are two processes that do not occur simultaneously but alternately with time. This process is called an *alternating renewal process*. Similarly, suppose a component of a system can be replaced upon failure by either a type A component or a type B component. If the replacement is done in such a way that when a type A component fails it is replaced by a type B component and vice versa, then we have an alternating renewal process. It should be clear that the alternating renewal process is not limited to two types of replacements. In fact, the above example can be generalized to k types of components following one another in a strict cyclic order, or it can be generalized by having a probability transition matrix with element p_{ij} specifying the probability that a type i component is replaced upon failure by a type j component. Such a system is called a semi-Markov process (Cox, 1962).

7.4.1 Expected Number of Failures in an Alternating Renewal Process

Consider an alternating renewal process as depicted in Figure 7.2. Instead of T_i and D_i, which represent uptime and downtime of the machine, we consider the case when a type A component is replaced by a type B component upon failure and vice versa. Let $X_i = X_{Ai} + X_{Bi}$, $i = 1, 2 \ldots$ be the random variables that represent the renewal process, with X_{Ai} and X_{Bi} as the sequence of times during which the machine is up when type A component and type B component are used, respectively. Thus, if type A component was used at time $t = 0$, the first breakdown occurs at time X_{A1}, the second breakdown occurs at time $X_{A1} + X_{B1}$, and so on. Also, the first breakdown when type B component is in use occurs at time $X_{A1} + X_{B1}$, and the second failure, when type B component is in use, occurs at time $X_{A1} + X_{B1} + X_{A2} + X_{B2}$, and so on. Therefore, we can apply the results in Section 7.2.1 taking the distribution of failure time as the convolution of $f_A(x)$ and $f_B(x)$, with Laplace transform $f_A^*(s)f_B^*(s)$. The expected number of type B component failures in the interval $(0, t]$, $M_B(t)$, is obtained as a result of the Laplace inverse of

$$M_B^*(s) = \frac{f_A^*(s)f_B^*(s)}{s\left[1 - f_A^*(s)f_B^*(s)\right]}. \tag{7.20}$$

The expected number of type A component failures is obtained by modifying the renewal process such that the p.d.f. of the first failure time is $f_A(x)$, and the p.d.f. of the subsequent failure times is the convolution of $f_A(x)$ and $f_B(x)$ (Cox, 1962). Hence,

$$M_A^*(s) = \frac{f_A^*(s)}{s\left[1 - f_A^*(s)f_B^*(s)\right]}. \tag{7.21}$$

The renewal densities corresponding to $M_A^*(s)$ and $M_B^*(s)$ are

$$m_j^*(s) = sM_j^*(s), \quad j = A, B. \tag{7.22}$$

7.4.2 Probability That Type *j* Component Is in Use at Time *t*

One of the important criteria of component performance is the probability that it is in use at a specified time. For example, one may be interested in determining the probability $P_A(t)$ that type A component is in use at time t (when the machine is observed). This probability is obtained as the sum of the probabilities of two mutually exclusive events: in the first event, the initial type A component has a failure time greater than t, and in the second event, type B component fails in the time interval $(u, u + \delta u)$, for some time $u < t$, and is replaced by a type A component that does not fail during the interval $t - u$. Thus,

$$P_A(t) = R_A(t) + \int_0^t m_B(u)R_A(t-u)du, \tag{7.23}$$

where $R_A(t)$ is the reliability of component A at time t. By taking the Laplace transform of Equation 7.23, we obtain

$$P_A^*(s) = \left[1 - f_A^*(s)\right]\left[1 + m_B^*(s)\right] / s. \tag{7.24}$$

Substituting Equation 7.20 into the above equation, we have

$$P_A^*(s) = \frac{1 - f_A^*(s)}{s\left[1 - f_A^*(s)f_B^*(s)\right]}. \tag{7.25}$$

Equation 7.25 implies that $P_A^*(s) = M_B^*(s) - M_A^*(s) + 1/s.$ Thus,

$$P_A(t) = M_B(t) - M_A(t) + 1. \tag{7.26}$$

EXAMPLE 7.13

Microcasting is a droplet-based deposition process. The droplets of the molten material to be cast are relatively large (1–3 mm in diameter). They contain sufficient heat to remain significantly superheated until inspecting the substrate and rapidly solidify due to significantly low substrate temperatures. By controlling the superheat of the droplets and the substrate temperature, conditions can be attained, such that the impacting droplets superficially remelt the underlying material, leading to metallurgical interlayer bonding (Merz et al., 1994).

The apparatus used for microcasting usually fails due to the clogging of the nozzle that controls the size of the droplets. Therefore, the manufacturer of such an apparatus includes two nozzles, A and B, which are alternatively changed. Nozzle A is made of material with a higher melting temperature than that of nozzle B. The failure times of nozzles A and B follow exponential distributions with parameters 1×10^{-5} and 0.5×10^{-5} failures per hour, respectively. What is the probability that nozzle A is in use at $t = 10^4$ h?

SOLUTION

The p.d.f.'s of nozzles A and B are

$$f_A(t) = \lambda_A e^{-\lambda_A t} \text{ and } f_B(t) = \lambda_B e^{-\lambda_B t}.$$

Using Equations 7.20 and 7.21, we obtain the expected number of failures as

$$M_B(t) = \frac{\lambda_A \lambda_B}{\lambda_A + \lambda_B} t + \left(-\frac{\lambda_A \lambda_B}{(\lambda_A + \lambda_B)^2} \right) + \frac{\lambda_A \lambda_B}{(\lambda_A + \lambda_B)^2} e^{-(\lambda_A + \lambda_B)t}.$$

$$M_A(t) = \frac{\lambda_A \lambda_B}{\lambda_A + \lambda_B} t + \frac{\lambda_A^2}{(\lambda_A + \lambda_B)^2} - \frac{\lambda_A^2}{(\lambda_A + \lambda_B)^2} e^{-(\lambda_A + \lambda_B)t}.$$

Substituting the above expressions into Equation 7.26, we obtain

$$P_A(t) = M_B(t) - M_A(t) + 1$$

or

$$P_A(t) = \frac{\lambda_B}{\lambda_A + \lambda_B} + \frac{\lambda_A}{\lambda_A + \lambda_B} e^{-(\lambda_A + \lambda_B)t}.$$

Thus, the probability that nozzle A is in use at $t = 10^4$ h is

$$P_A(10^4) = \frac{0.5}{1.5} + \frac{1}{1.5} e^{-0.15} = 0.907.$$

■

7.5 APPROXIMATIONS OF *M*(*t*)

Estimating the expected number of renewals $M(t)$ using Equation 7.3 is difficult since $M(t)$ appears on both sides of the equation. Therefore, researchers investigated approximate methods for the integral of Equation 7.3 in order to obtain $M(t)$ by direct substitutions. In this section we summarize three approximations.

The first approximation is proposed by Bartholomew (1963) and is given by

$$M_b(t) = F(t) + \lambda \int_0^t [1 - F_e(t-x)]dx, \tag{7.27}$$

where

$$F_e(t) = \lambda \int_0^t [1 - F(x)]dx,$$

and $\lambda = 1/\mu$, μ is the expected value of the time between renewals.

The second approximation is proposed by Ozbaykal (1971), $M_o(t)$, which is given by

$$M_o(t) = \lambda t - F_e(t) + \int_0^t [1 - F_e(t-x)]dx. \tag{7.28}$$

The third approximation is proposed by Deligönül (1985), $M_d(t)$, and is derived as follows. The renewal function $M(t)$ is

$$M(t) = F(t) + \int_0^t M(t-x) f(x)dx.$$

The renewal density $m(t)$ is obtained as

$$m(t) = \frac{dM(t)}{dt}. \tag{7.29}$$

An equivalent equation to the renewal density can be written as

$$m(t) = f(t) + \int_0^t m(t-x) f(x)dx. \tag{7.30}$$

Karlin and Taylor (1975) provide an alternative expression of the renewal function $M(t)$ as

$$M(t) = \lambda t - F_e(t) + \int_0^t [1 - F_e(t-x)]dM(x). \tag{7.31}$$

As defined earlier, $F_e(t) = \lambda \int_0^t [1 - F(x)]dx$ and $\lambda = 1/\mu$ where μ is the expected value of the time between renewals.

Since $M(t)$ satisfies the renewal equation, it also satisfies the equation

$$F(x) = \int_0^x m(x-t)[1 - F(x)]dx. \tag{7.32}$$

Combining Equation 7.29 through Equation 7.32 yields

$$M(t) = \lambda t - F_e(t) + \int_0^t [1 - F_e(t-x)] \left[\frac{F(x)\int_0^x m(x-t)f(t)dt}{\int_0^x m(x-t)[1-F(t)]dt} + f(x) \right] dx. \tag{7.33}$$

Deligönül (1985) approximates Equation 7.33 by dropping out $m(x - t)$'s to obtain the following estimate of $M_d(t)$:

$$M_d(t) = \lambda t - F_e(t) + \int_0^t [1 - F_e(t-x)] \left[f(x) + \frac{\lambda F^2(x)}{F_e(x)} \right] dx. \tag{7.34}$$

A comparison between $M_b(t)$ and $M_d(t)$ estimates for an increasing hazard-rate gamma distribution of the form $F(t) = 1 - (1 + 2t)e^{-2t}$ with mean = 1 and Var = 1/2 is shown in Table 7.6.

Before we present other important characteristics of the expected number of renewals such as the variance, the confidence interval for $M(t)$, and the residual life, we briefly discuss other types of renewal processes.

7.6 OTHER TYPES OF RENEWAL PROCESSES

So far, we have only considered the case where the times to renewal (failure or failure and repair) are nonnegative identical and independent. We refer to this type of renewal processes as the *ordinary renewal process*. There are two slightly different renewal processes: the *modified renewal process* (or the *delayed renewal process*) and the *equilibrium renewal process*. In the modified renewal process, the time to the first failure T_1 has a p.d.f., $f_1(t)$, and failure times between two successive failures, beyond the first, all have the same p.d.f. $f(x)$. In other words, the conditions for the modified renewal process are the same as those of the ordinary renewal process, except that the time from the origin to the first failure has a different distribution from the other failure times (Cox, 1962).

The equilibrium renewal process is a special case of the modified renewal process where the time to the first failure has the p.d.f. $R(t)/\mu$, where $R(t)$ is the reliability function at time t and μ is the mean failure time.

We now give typical examples of these three types of renewal processes. The ordinary renewal process is exemplified by replacements of an electric light bulb upon failure, the air filter and the brake pads of an automobile, and the spark plugs of an engine. The modified renewal process arises when the life of the original part or component is significantly different from that of its replacements. For example, when a customer acquires a new vehicle, the oil and air filters are usually replaced after 1000 mi, whereas subsequent replacements of the oil filters occur at approximately equal intervals of 3000 mi. The equilibrium renewal process can be regarded as an ordinary renewal process in which the system or component has been operating for a long time before it is first observed.

It should be noted that the renewal density for the modified renewal process is similar to that of the ordinary renewal process given by Equation 7.5. It is expressed as

TABLE 7.6 A Comparison between $M_b(t)$ and $M_d(t)$ for Gamma Distribution (Mean = 1, Var = 1/2)

t	$M_b(t)$	$M_d(t)$	$M(t)$,exact
0.1	0.0176	0.0176	0.0176
0.2	0.0626	0.0626	0.0623
0.3	0.1264	0.1263	0.1253
0.4	0.2031	0.2029	0.2005
0.5	0.2888	0.2882	0.2838
0.6	0.3807	0.3794	0.3727
0.7	0.4768	0.4746	0.4652
0.8	0.5758	0.5723	0.5602
0.9	0.6766	0.6717	0.6568
1.0	0.7787	0.7718	0.7546
1.5	1.2946	1.2745	1.2506
2.0	1.8074	1.7720	1.7501
2.5	2.3148	2.6624	2.2500
3.0	2.8185	2.7605	2.7500
3.5	3.3202	3.2562	3.2500
4.0	3.8210	3.7534	3.7500
5.0	4.8215	4.7509	4.7500
6.0	5.8215	5.7502	5.7500
7.0	6.8215	6.7500	6.7500
8.0	7.8215	7.7500	7.7500
9.0	8.8215	8.7500	8.7500
10.0	9.8215	9.7500	9.7500
11.0	10.8215	10.7500	10.7500
12.0	11.8215	11.7500	11.7500
13.0	12.8215	12.7500	12.7500
14.0	13.8215	13.7500	13.7500
15.0	14.8215	14.7500	14.7500

Note: $F(t) = 1 - (1 + 2t)e^{-2t}$ (Deligönül, 1985).

$$m_m(t) = f_1(t) + \int_0^t m_m(t-x)f(x)dx, \tag{7.35}$$

where $m_m(t)$ is the renewal density of the modified renewal process and $f_1(t)$ is the p.d.f. of the time to the first failure (or renewal).

Finally, the expected number of renewals of the equilibrium renewal process is

$$M_e(t) = \frac{t}{\mu}, \tag{7.36}$$

where μ is the mean failure (renewal) time.

7.7 THE VARIANCE OF NUMBER OF RENEWALS

As shown later in Chapters 8 and 9, the warranty cost, the length of a warranty policy, and the optimal maintenance schedule for a component (replacement or repair) are dependent on the expected number of failures (or renewals) during the warranty period and length of the maintenance schedule. Moreover, the variance of the number of renewals has a more significant impact on the choice of the appropriate warranty policy. Indeed, when two warranty policies have the same expected warranty cost for the same warranty length, the variance of the warranty cost would be the deciding factor in preferring one policy to another. Hence, it is important to determine the variance of the number of renewals Var[$N(t)$] in the interval (0, t].

From the definition of the variance, we obtain

$$\mathrm{Var}[N(t)] = E\left[N^2(t)\right] - E\left[N(t)\right]^2. \tag{7.37}$$

But

$$E\left[N(t)\right] = M(t) = \sum_{r=0}^{\infty} rP\left[N(t) = r\right],$$

which is expressed in Equation 7.2 as

$$E\left[N(t)\right] = M(t) = \sum_{r=1}^{\infty} F_r(t). \tag{7.38}$$

Similarly,

$$\begin{aligned} E\left[N^2(t)\right] &= \sum_{r=0}^{\infty} r^2 P\left[N(t) = r\right] \\ &= \sum_{r=0}^{\infty} r^2 \left[F_r(t) - F_{r+1}(t)\right] \end{aligned}$$

or

$$E\left[N^2(t)\right] = \sum_{r=1}^{\infty} (2r-1) F_r(t). \tag{7.39}$$

Substituting Equations 7.38 and 7.39 into Equation 7.35 results in

$$\mathrm{Var}\left[N(t)\right] = \sum_{r=1}^{\infty} (2r-1) F_r(t) - \left[M(t)\right]^2. \tag{7.40}$$

Equation 7.40 is computationally difficult to evaluate. Therefore, we follow Cox's (1962) work and obtain a simpler algebraic form of Var[N(t)] by using $\psi(t)$, which is defined as

$$\psi(t) = E\left[N(t)(N(t)+1)\right]. \tag{7.41}$$

Equation 7.41 represents the sum of $E[N^2(t)] + E[N(t)]$. Thus, the variance of $N(t)$ can be expressed in terms of $\psi(t)$ as

$$\mathrm{Var}[N(t)] = \psi(t) - M(t) - M^2(t). \tag{7.42}$$

Equation 7.41 can be written as

$$\psi(t) = \sum_{r=0}^{\infty} r(r+1)P[N(t)=r]. \tag{7.43}$$

But

$$P[N(t)=r] = F_r(t) - F_{r+1}(t). \tag{7.44}$$

Substituting Equation 7.44 into Equation 7.43 results in

$$\psi(t) = \sum_{r=0}^{\infty} r(r+1)[F_r(t) - F_{r+1}(t)]. \tag{7.45}$$

Taking the Laplace transform of Equation 7.45 yields

$$\psi^*(s) = \frac{1}{s}\sum_{r=0}^{\infty} r(r+1)[f_r^*(s) - f^*{}_{r+1}(s)]$$

or

$$\begin{aligned}\psi^*(s) &= \frac{1}{s}[0 + 2f_1^*(s) - 2f_2^*(s) + 6f_2^*(s) - 6f_3^*(s) + 12f_3^*(s) - 12f_4^*(s) + \ldots]\\ \psi^*(s) &= \frac{2}{s}\sum_{r=1}^{\infty} rf_r^*(s).\end{aligned} \tag{7.46}$$

For an ordinary renewal process, $f_r^*(s) = [f^*(s)]^r$. Thus,

$$\psi_o^*(s) = \frac{2f^*(s)}{s[1-f^*(s)]^2}. \tag{7.47}$$

For an equilibrium renewal process, $F_r^*(s) = [f^*(s)]^{r-1}[1-f^*(s)]/s\mu$, or

$$\psi_e^*(s) = \frac{2}{s^2\mu[1-f^*(s)]}. \tag{7.48}$$

Cox (1962) shows that there is a relationship between Equation 7.48 and the renewal function of the ordinary renewal process as

$$\psi_e^*(s) = \frac{2}{s\mu}\left[M_0^*(s) + \frac{1}{s}\right] \tag{7.49}$$

or

$$\psi_e(t) = \frac{2}{\mu}\int_0^t M_0(x)dx + \frac{2t}{\mu}. \tag{7.50}$$

Substituting Equations 7.36 and 7.50 into Equation 7.42, we obtain the variance of the number of renewals of the equilibrium process as

$$\mathrm{Var}[\mathrm{N_e}(t)] = \frac{2}{\mu}\int_0^t \left[M_o(x) - \frac{x}{\mu} + \frac{1}{2}\right]dx. \tag{7.51}$$

EXAMPLE 7.14

Most machine parts are subjected to fluctuating or cyclic loads that induce fluctuating or cyclic stresses that often result in failure by fatigue. Fatigue may be characterized by a progressive failure phenomenon that proceeds by the *initiation* and propagation of cracks to an unstable size. Thus, the time to failure can be represented by a two-stage process that can be modeled as a special two-stage Erlang distribution with a parameter λ failures per hour.

1. Assuming an ordinary renewal process, graph $M_o(t)$ and Var[$N_o(t)$] for different values of λ and t.
2. Repeat 1 under the equilibrium renewal process assumption.

SOLUTION

The p.d.f. of the special Erlang distribution is

$$f(t) = \frac{t}{\lambda^2}e^{-t/\lambda}.$$

The Laplace transform of $f(t)$ is

$$f^*(s) = \frac{1}{(1+s\lambda)^2}. \tag{7.52}$$

1. The expected number of renewals of the ordinary renewal process is obtained by substituting Equation 7.52 into Equation 7.6. Thus,

$$M_o^*(s) = \frac{f^*(s)}{s[1-f^*(s)]}$$
$$M_0^*(s) = \frac{\frac{1}{(1+s\lambda)^2}}{s\left[1-\frac{1}{(1+s\lambda)^2}\right]^2} = \frac{1/\lambda^2}{s^2(s+2/\lambda)}. \tag{7.53}$$

To obtain the inverse, we rewrite Equation 7.53 as

$$M_0^*(s) = \frac{1}{2\lambda s^2} - \frac{1}{4s} + \frac{1}{4\left(s + \frac{2}{\lambda}\right)}. \tag{7.54}$$

Thus, the inverse is

$$M_o(t) = \frac{t}{2\lambda} - \frac{1}{4} + \frac{1}{4}e^{-2t/\lambda}. \tag{7.55}$$

To obtain the variance of the number of renewals of the ordinary renewal process, we utilize Equation 7.42

$$\mathrm{Var}[N_o(t)] = \psi_o(t) - M_o(t) - M_0^2(t). \tag{7.56}$$

We first estimate $\psi_o(t)$ by obtaining the inverse of $\psi_0^*(s)$:

$$\psi_o^*(s) = \frac{2f^*(s)}{s[1 - f^*(s)]^2}$$

or

$$\psi_o^*(s) = \frac{2\left[\frac{1}{(1+s\lambda)^2}\right]}{s\left[1 - \frac{1}{(1+s\lambda)^2}\right]} = \frac{2(1+s\lambda)^2}{\lambda s^3(2+s\lambda)^2}$$

and

$$\psi_o(t) = \frac{t^2}{4\lambda^2} + \frac{t}{2\lambda} - \frac{1}{8} + \frac{1}{8}e^{\frac{-2t}{\lambda}} - \frac{t}{4\lambda}e^{\frac{-2t}{\lambda}}. \tag{7.57}$$

Substituting Equations 7.55 and 7.57 into Equation 7.56, we obtain

$$\mathrm{Var}[\mathrm{N_o}(t)] = \frac{t}{4\lambda} + \frac{1}{16} - \frac{t}{2\lambda}e^{\frac{-2t}{\lambda}} - \frac{1}{16}e^{\frac{-4t}{\lambda}}. \tag{7.58}$$

Figures 7.5 and 7.6 show the effect of λ and t on $M_o(t)$ and on Var[$N_o(t)$], respectively.

2. The expected number of renewals of the equilibrium renewal process is given by Equation 7.36 as

$$M_e(t) = \frac{t}{\mu},$$

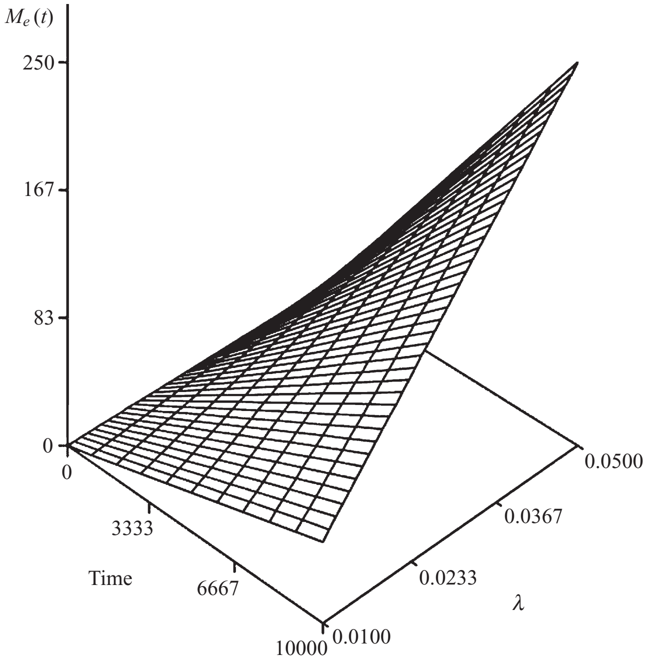

FIGURE 7.5 Relationship between $M_0(t)$, λ, and t.

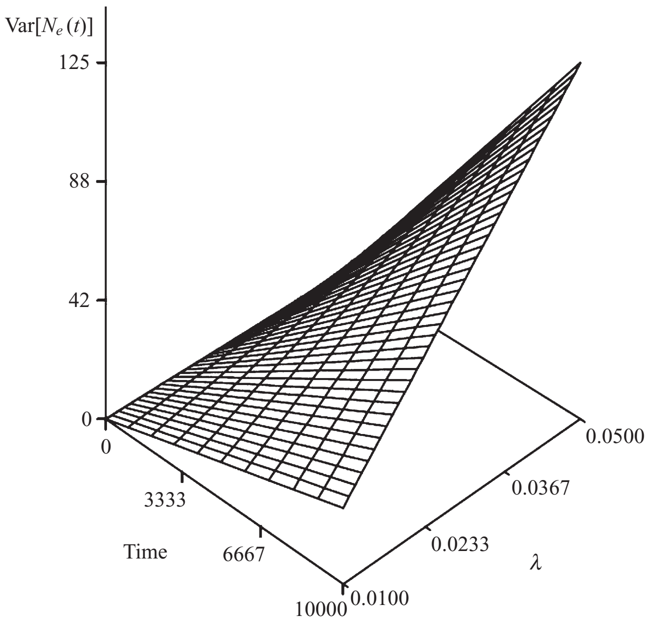

FIGURE 7.6 Effect of λ and t on Var[$N_o(t)$].

where μ of the special Erlang distribution is 2λ, thus

$$M_e(t) = \frac{t}{2\lambda}. \tag{7.59}$$

The variance of the number of renewals of the equilibrium renewal process is obtained by substitution of $M_o(t)$ in Equation 7.51 to obtain

$$\begin{aligned}
\mathrm{Var}[N_e(t)] &= \frac{2}{\mu}\int_0^t \left[\frac{x}{2\lambda} - \frac{1}{4} + \frac{1}{4}e^{\frac{-2x}{\lambda}} - \frac{x}{2\lambda} + \frac{1}{2}\right]dx \\
&= \frac{2}{\mu}\int_0^t \left(\frac{1}{4} + \frac{1}{4}e^{\frac{-2x}{\lambda}}\right)dx \\
\mathrm{Var}[N_e(t)] &= \frac{t}{4\lambda} + \frac{1}{8} - \frac{1}{8}e^{\frac{-2t}{\lambda}}.
\end{aligned} \tag{7.60}$$

Figures 7.7 and 7.8 show the effect of λ and t on $M_e(t)$ and $\mathrm{Var}[N_e(t)]$, respectively.

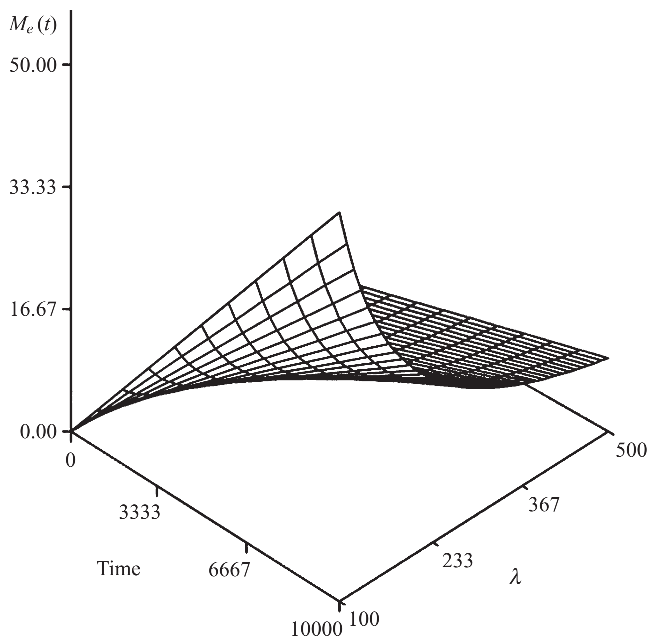

FIGURE 7.7 Effect of λ and t on $M_e(t)$.

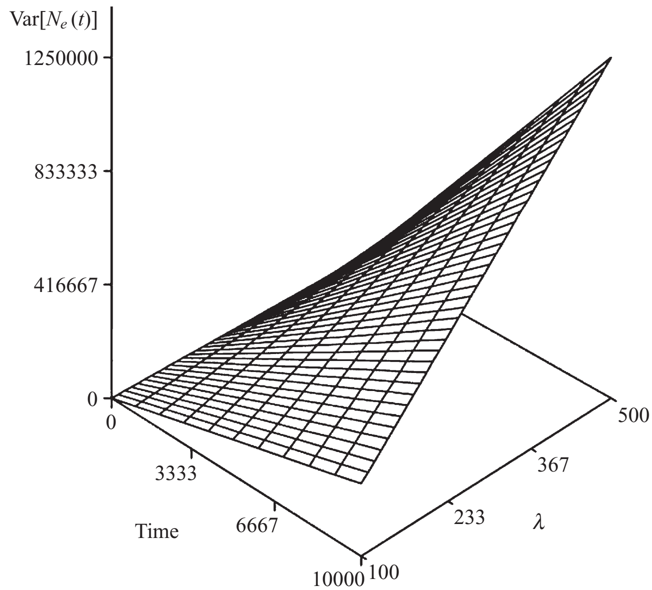

FIGURE 7.8 Effect of λ on t on Var[$N_e(t)$]. ■

7.8 CONFIDENCE INTERVALS FOR THE RENEWAL FUNCTION

The point estimate of $M(t)$ was derived earlier in this chapter. Approximate confidence intervals may be calculated when the parameter estimates are asymptotically normally distributed. If the functional forms of the underlying distribution functions are unknown, a nonparametric approach is required. Frees (1986a, 1986b, 1988) presents nonparametric estimators of the renewal function and constructs a nonparametric confidence interval for $M(t)$.

In this section, we present an alternative nonparametric confidence interval, based on Baxter and Li (1994), for the renewal function, which is easier to compute and appreciably narrower than that of Frees (1986a). The approach is based on the assumption that the empirical renewal function converges weakly to a Gaussian process as the sample size increases.

When $F(t)$ is known, the renewal function is given by Equation 7.2. When $F(t)$ is unknown, we follow the same derivations given in Section 7.3.2 to obtain an alternative estimate of $M(t)$.

Suppose that F is unknown and we wish to calculate a confidence interval for $M(t)$ for a fixed t given $x1, x2, \ldots, xn$, a random sample of n observations of a random variable with distribution function F. Baxter and Li (1994) utilize Equations 7.18 and 7.19 to obtain a nonparametric maximum likelihood estimator (MLE) of F as the empirical distribution function (EDF):

$$\hat{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} I_{\{xi \le t\}}, \tag{7.61}$$

where I_A denotes the indicator of the event A (see Section 7.3.2). Thus, a natural estimator of $M(t)$ is the empirical renewal function

$$\hat{M}_n(t) = \sum_{k=1}^{\infty} \hat{F}_n^{(k)}(t), \tag{7.62}$$

where $\hat{F}_n^{(k)}$ is the k-fold recursive Stieltjes convolution of $\hat{F}_n$.

Baxter and Li (1994) prove that as $n \to \infty$,

$$\frac{\sqrt{n}}{\hat{\sigma}_n(t)}\left[\hat{M}_n(t) - M(t)\right]$$

converges in distribution to a standard normal variate. Hence, for $\alpha \in (0,1)$, an approximate $100(1 - \alpha)$ confidence interval for $M(t)$ is

$$\hat{M}_n(t) - z_{\alpha/2}\frac{\hat{\sigma}_n(t)}{\sqrt{n}} \le M(t) \le \hat{M}_n(t) + z_{\alpha/2}\frac{\hat{\sigma}_n(t)}{\sqrt{n}}, \tag{7.63}$$

where $z_{\alpha/2}$ denotes the upper $\alpha/2$ quantile of the standard normal distribution. An alternative procedure for calculating $\hat{M}_n(t)$ rather than using Equation 7.62 requires the partitioning of the interval $(0, t]$ into k subintervals of equal width—say, $0 = t_o < t_1 < \ldots < t_k = t$—where the value of k depends on t and on the actual observations. $\hat{M}_n(t_i)(i = 1, 2, \ldots, k)$ can then be recursively calculated as

$$\hat{M}_n(t_i) = \hat{F}_n(t_i) + \sum_{j=1}^{i} \hat{M}_n(t_i - t_j)\left[\hat{F}_n(t_j) - \hat{F}_n(t_{j-1})\right].$$

Clearly, if F is known, we utilize Equation 7.4 or its approximations (Eqs. 7.27, 7.28, and 7.34) for the ordinary renewal process or Equations 7.35 and 7.36 for the modified renewal and equilibrium renewal process, respectively, to estimate $M(t)$. We then use Equation 7.42 to obtain the corresponding estimate of the variance. Finally, assuming $n = 25$, we substitute these estimates in Equation 7.63 to obtain the confidence interval for $M(t)$.

EXAMPLE 7.15

Determine the 95% confidence intervals for $M(t)$ of the ordinary renewal process and the equilibrium renewal process for $t = 100$ to 1000 (increments of 100) and for $t = 2000$ to 10,000 (increments of 1000) for the machine parts given in Example 7.14. Assume $\lambda = 5 \times 10^3$ h between failures.

SOLUTION

Substitute $\lambda = 5 \times 10^3$ in Equations 7.55, 7.58–7.60 to obtain $M_o(t)$, Var$[N_o(t)]$, $M_e(t)$, and Var$[N_e(t)]$, respectively. The confidence intervals for $M_o(t)$ and $M_e(t)$ are obtained by using Equation 7.63 and substituting $n = 30$. The results are shown in Table 7.7.

TABLE 7.7 $M_o(t)$, $M_e(t)$, Var$[N_o(t)]$, Var$[N_e(t)]$, and Bounds for $M(t)$

Time	Var $[N_o(t)]$	Lower $M_o(t)$	$M_o(t)$	Upper $M_o(t)$	Var $[N_e(t)]$	Lower $M_e(t)$	$M_e(t)$	Upper $M_e(t)$
100	0.0002	0.0001	0.0002	0.0003	0.0099	0.0065	0.0100	0.0135
200	0.0008	0.0005	0.0008	0.0011	0.0196	0.0130	0.0200	0.0270
300	0.0017	0.0011	0.0017	0.0023	0.0291	0.0196	0.0300	0.0404
400	0.0030	0.0020	0.0030	0.0041	0.0385	0.0262	0.0400	0.0538
500	0.0047	0.0030	0.0047	0.0064	0.0477	0.0329	0.0500	0.0671
600	0.0066	0.0043	0.0067	0.0090	0.0567	0.0397	0.0600	0.0803
700	0.0089	0.0058	0.0089	0.0121	0.0655	0.0466	0.0700	0.0934
800	0.0115	0.0074	0.0115	0.0156	0.0742	0.0534	0.0800	0.1066
900	0.0143	0.0093	0.0144	0.0195	0.0828	0.0604	0.0900	0.1196
1000	0.0174	0.0114	0.0176	0.0238	0.0912	0.0674	0.1000	0.1326
2000	0.0600	0.0409	0.0623	0.0838	0.1688	0.1396	0.2000	0.2604
3000	0.1165	0.0836	0.1253	0.1670	0.2374	0.2151	0.3000	0.3849
4000	0.1792	0.1364	0.2005	0.2646	0.2998	0.2927	0.4000	0.5073
5000	0.2437	0.1966	0.2838	0.3710	0.3581	0.3719	0.5000	0.6281
6000	0.3076	0.2626	0.3727	0.4827	0.4137	0.4520	0.6000	0.7480
7000	0.3697	0.3329	0.4652	0.5975	0.4674	0.5327	0.7000	0.8673
8000	0.4298	0.4064	0.5602	0.7140	0.5199	0.6140	0.8000	0.9860
9000	0.4879	0.4823	0.6568	0.8314	0.5716	0.6955	0.9000	1.1045
10000	0.5442	0.5599	0.7546	0.9493	0.6227	0.7772	1.0000	1.2228

■

7.9 REMAINING LIFE AT TIME *T*

In Section 7.2.1, we defined the total time up to the rth failure as $S_r = t_1 + t_2 + \ldots + t_r = \sum_{i=1}^{r} t_i$ where t_i is the interval between failures i–1 and i. In this section, we are interested in estimating the time from t until the next renewal—that is, the excess life or remaining life at time t. Let $L(t)$ represent the remaining life at t—that is,

$$L(t) = S_{N(t)+1} - t. \tag{7.64}$$

The distribution function of $L(t)$ is given for $x \geq 0$ by

$$P\big(L(t) \leq x\big) = F(t+x) - \int_0^t \big[1 - F(t+x-y)\big]\, dM(y). \tag{7.65}$$

In general, it is difficult to solve Equation 7.65 analytically. When $t \to \infty$, then

$$\lim_{t\to\infty} P\big(L(t) \leq x\big) = \frac{\int_0^x \big[1 - F(y)\big]\, dy}{\mu}, \tag{7.66}$$

where $\mu = E[X]$.

Consider a renewal process that has been "running" for a very long time and was observed beginning at time $t = 0$. Let T_1 denote the time to the first renewal after time $t = 0$. Then T_1 is the remaining life of the unit which was in operation at time $t = 0$ (Hoyland and Rausand, 1994, 2003). From Equation 7.66, the distribution of T_1 is

$$F_{T_1}(t) = \frac{1}{\mu}\int_0^t [1 - F_T(y)]dy. \tag{7.67}$$

When the age of the unit (or component) that is in operation at time $t = 0$ is greater than 0, we have a modified renewal process. The distribution of the remaining lifetime $L(t)$ becomes

$$P(L(t) \le x) = F_{T_1}(t + x) - \int_0^t [1 - F_T(t + x - y)]dM_m(y), \tag{7.68}$$

where $M_m(y)$ is the renewal function of the modified renewal process.

The mean remaining lifetime in a stationary renewal process is

$$E[L(t)] = \int_0^\infty [1 - P(L(t)) \le x]dx, \tag{7.69}$$

and

$$\lim_{t\to\infty} E[L(t)] = \frac{E[X^2]}{2\mu},$$

where X is the time between renewals.

EXAMPLE 7.16

Communications cables are drawn in cable duct plants (conduits), either by using a cable grip when the diameter of the cable beneath the sheath is less than 50 mm or by a drawing ring applied to the cable when the diameter is greater than 50 mm. A cable production facility has a drawing machine with two identical rings. The machine stops production only when the two rings fail. The failure time follows an exponential distribution with parameter $\lambda = 0.002$ failures per hour. Determine the mean remaining lifetime of the drawing machine.

SOLUTION

Since the drawing machine stops production only when the two rings fail, the machine reliability is therefore estimated as

$$R(t) = 2e^{-\lambda t} - e^{-2\lambda t}.$$

The MTTF, μ, is

$$\mu = \int_0^\infty R(t)dt = \frac{3}{2\lambda}.$$

Using Equation 7.66, we obtain

$$P(L(t) \le x) = \frac{1}{\mu}\int_0^x R(t)dt$$
$$= \frac{2\lambda}{3}\left[\frac{-2}{\lambda}e^{-\lambda t} + \frac{2}{2\lambda}e^{-2\lambda t}\right]_0^x$$

or

$$P(L(t) \le x) = \frac{2\lambda}{3}\left[\frac{-2}{\lambda}e^{-\lambda x} + \frac{1}{2\lambda}e^{-2\lambda x} + \frac{3}{2\lambda}\right].$$

The mean remaining lifetime is

$$E[L(t)] = \int_0^\infty [1 - P(L(t) \le x)]dx$$
$$= \int_0^\infty \left(\frac{4}{3}e^{-\lambda x} - \frac{1}{3}e^{-2\lambda x}\right)dx$$

or

$$E[L(t)] = \frac{7}{6\lambda} = 583\text{ h.}$$

■

7.10 POISSON PROCESSES

Two important point processes are commonly used in modeling repairable systems. A *repairable system* is a system that can be repaired when failures occur, such as cars, airplanes, and computers. A *nonrepairable system* is a system that is discarded or replaced upon failure, such as electronic chips, cell phones, and inexpensive calculators. The point processes to be discussed are the homogeneous Poisson process (HPP) and the nonhomogeneous Poisson process (NHPP).

7.10.1 Homogeneous Poisson Process (HPP)

Before defining the homogeneous Poisson process (HPP), we introduce the counting process $N(t)$, $t \ge 0$. It represents the total number of events (such as failures and repairs) that have occurred up to time t. The counting process $N(t)$ must satisfy the following:

1. $N(t) \ge 0$,
2. $N(t)$ is integer valued,
3. If $t_1 < t_2$, then $N(t_1) \le N(t_2)$, and
4. The number of events that occur in the interval $[t_1, t_2]$ where $t_1 < t_2$ is $N(t_2) - N(t_1)$.

For an HPP, condition 4 is modified such that the number of events (failures) in the interval $[t_1, t_2]$ has a Poisson distribution with mean $\lambda(t_2 - t_1)$ where λ is the failure rate and as additional conditions $N(0) = 0$ and the numbers of events in nonoverlapping intervals are independent—that is, the process has independent increments. Thus, for $t_2 > t_1 \geq 0$, the probability of having n failures in the interval $[t_1, t_2]$ is

$$P\{N(t_2) - N(t_1) = n\} = \frac{e^{-\lambda(t_2 - t_1)}[\lambda(t_2 - t_1)]^n}{n!}$$

for $n \geq 0$.

It follows from condition 4 that a Poisson process has an expected number of failures (events) as

$$E[N(t_2 - t_1)] = \lambda(t_2 - t_1).$$

The Poisson process is referred to as homogeneous when λ is not time dependent—that is, the number of events in an interval depends only on the length of the interval (process has stationary increments). Hence, the reliability function $R(t_1,t_2)$, for the interval $[t_1, t_2]$ is

$$R(t_1, t_2) = e^{-\lambda(t_2 - t_1)}.$$

7.10.2 Nonhomogeneous Poisson Process (NHPP)

This nonhomegeneous Poisson Process (NHPP) is similar to the HPP with the exception that the failure rate (occurrence rate of the event) is time dependent. Thus, the process is nonstationary. In other words, we modify condition 4 as follows:

4. The number of events that occur in the interval $[t_1, t_2]$ where $t_1 < t_2$ has a Poisson distribution with mean $\int_{t_1}^{t_2} \lambda(t)dt$.

Therefore, the probability of having n failures in the interval $[t_1, t_2]$ is

$$P[N(t_2) - N(t_1) = n] = \frac{e^{-\int_{t_1}^{t_2} \lambda(t)dt} \left[\int_{t_1}^{t_2} \lambda(t)dt\right]^n}{n!},$$

and the expected number of failures in $[t_1, t_2]$ is

$$E[N(t_2) - N(t_1)] = \int_{t_1}^{t_2} \lambda(t)dt.$$

The reliability function of the NHPP for the interval $[t_1, t_2]$ is

$$R(t_1, t_2) = e^{-\int_{t_1}^{t_2} \lambda(t)dt}.$$

EXAMPLE 7.17

Determine $M(t)$ when

1. $F(t) = 1 - e^{-t/\mu_1}$
2. $f(t)$ is a gamma density of order k

$$f(t) = \frac{\lambda(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t}.$$

SOLUTION

1. The distribution function

$$F(t) = 1 - e^{-t/\mu_1}$$

has a p.d.f.

$$f(t) = \frac{1}{\mu_1} e^{-t/\mu_1}$$

$$\text{Set } \lambda = \frac{1}{\mu_1}.$$

But,

$$m^*(s) = \frac{f^*(s)}{1 - f^*(s)}$$

or

$$m^*(s) = \frac{\dfrac{\lambda}{s+\lambda}}{1 - \dfrac{\lambda}{s+\lambda}} = \frac{\lambda}{s}.$$

The inverse of $m^*(s)$ is

$$m(t) = \lambda$$

and

$$M(t) = \int_0^t \lambda dt = \lambda t$$

or

$$M(t) = \frac{t}{\mu_1}.$$

Hence, for the HPP, the expected number of renewals in an interval of length t is simply t divided by the mean life.

2. It is known that the p.d.f. of a gamma distribution of order k is the convolution of k exponentials with parameter λ. Therefore, the probability of n renewals in (0, t] for a renewal process defined by f(t) is equal to the probability of either nk, $nk + 1$, . . . or $nk + k$–1 events occurring in (0, t] for a Poisson process with parameter λ. Therefore,

$$P[N(t)=n]=\frac{(\lambda t)^{nk}}{(nk)!}e^{-\lambda t}+\frac{(\lambda t)^{nk+1}}{(nk+1)!}e^{-\lambda t}+\ldots+\frac{(\lambda t)^{nk+k-1}}{(nk+k-1)!}e^{-\lambda t}.$$

Let $m(t)$ be the renewal density for a gamma density of order k. For $k = 1$ (exponential density), $m(t) = \lambda$ as shown in part (1) of this example. Since $m(t)\ dt$ is the probability of a renewal in [t, $t + dt$], we can interpret this probability for the gamma density of order k as (Barlow et al., 1965)

$$m(t)dt=\sum_{j=1}^{\infty}\left[\frac{(\lambda t)^{kj-1}}{(kj-1)!}e^{-\lambda t}\right]\lambda dt. \tag{7.70}$$

The right side is the probability of $kj - 1$ events occurring in (0, t] from a Poisson process with parameter λ times the probability of an additional event occurring in [t, $t + dt$] and summed over all permissible j.

When $k = 2$,

$$m(t)=\frac{\lambda}{2}-\frac{\lambda}{2}e^{-2\lambda t}$$

and

$$M(t)=\frac{\lambda t}{2}-\frac{1}{4}+\frac{1}{4}e^{-2\lambda t}.$$

The expected number of failures during an interval (0, t] for a component whose failure time exhibits an Erlang distribution with k stages is obtained by integrating Equation 7.67 to obtain (Parzen, 1962)

$$M(t)=\frac{\lambda t}{k}+\frac{1}{k}\sum_{j=1}^{k-1}\frac{\theta^j}{1-\theta^j}\left[1-e^{-\lambda t(1-\theta^j)}\right],$$

where

$$\theta=e^{(2\pi i/k)}$$

and

$$i=\sqrt{-1}.$$

Details of the above derivation are given in Barlow et al. (1965). ■

7.11 LAPLACE TRANSFORM AND RANDOM VARIABLES

Laplace transform is one of the efficient approaches for studying the characteristics of random variables and in solving convolutions of functions. In this section, we provide a brief discussion of the use of Laplace transform in obtaining the expectations of random variable, expected number of renewals, and solving convolutions of function.

7.11.1 Laplace Transform and Expectations

Laplace transform of a random variable with p.d.f. $f_X(t)$ is defined as

$$f_X^*(s) = \mathfrak{L} f_X(t) = \int_0^\infty e^{-st} f_X(t)dt = E[e^{-sX}], \tag{7.71}$$

where E [.] is the expectation operator. The derivatives of Equation 7.71 are

$$f^{*\prime}(s) = \frac{d}{ds}E[e^{-sX}] = E[-Xe^{-sX}]$$

and the nth derivative is

$$f^{*(n)}(s) = \frac{d^n}{ds^n}E[e^{-sX}] = E[(-X)^n e^{-sX}].$$

Evaluation of these derivatives at $s = 0$ results in

$$\begin{aligned} E[X] &= -f^{*\prime}(0) \\ E[X^2] &= +f^{*\prime\prime}(0) \\ &\vdots \\ E[X^n] &= (-1)^n f^{*(n)}(0). \end{aligned}$$

Consider, for example, the p.d.f. of the exponential distribution

$$f_X(t) = \lambda e^{-\lambda t}.$$

Its Laplace transform is

$$f_X^*(s) = \int_0^\infty e^{-st} \cdot \lambda e^{-\lambda t} dt = \frac{\lambda}{\lambda + s}.$$

Using the derivatives above, we obtain

$$E[X] = -f_X^{*\prime}(0) = \frac{\lambda}{(\lambda + s)^2}\bigg|_{s=0} = \frac{1}{\lambda}$$

and

$$E[X^2] = +f_X^{*}{''}(0) = \left.\frac{2\lambda}{(\lambda+s)^3}\right|_{s=0} = \frac{2}{\lambda^2}.$$

The variance is obtained as

$$Var(X) = E[X^2] - E[X]^2 = \frac{1}{\lambda^2}.$$

The above procedure can be extended to include the case of competing risk models (competing failure modes). This is accomplished via the following property.

If X and Y are two independent random variables with $\mathfrak{L}f_X(t) = f_X^*(s)$ and $\mathfrak{L}f_Y(t) = f_Y^*(s)$, then

$$\begin{aligned} f_{X+Y}^*(s) &= E[e^{-s(X+Y)}] \\ &= E[e^{-sX}]E[e^{-sY}] \\ f_{X+Y}^*(s) &= f_X^*(s)f_Y^*(s). \end{aligned}$$

Likewise $\mathfrak{L}\left[af_X(t) + bf_Y(t)\right] = af_X^*(s) + bf_Y^*(s)$.

7.11.2 Laplace Transform and Renewals

As shown in Section 3.4, we consider a repairable system that has a failure-time distribution with a p.d.f. $w(t)$, and a repair-time distribution with a p.d.f. $g(t)$. When the system fails, it is repaired, and the process is continuously repeated. The density function of this renewal process and the density function of the number of renewals are $f(t)$ and $n(t)$, respectively. The underlying density function $f(t)$ of the renewal process is the convolution of w and g. In other words,

$$f(t) = \int_0^t w(t)g(t-\tau)d\tau. \tag{7.72}$$

Equation 7.72 shows that the convolution of two functions of t is another function of t. In general, such an integral is difficult to obtain. However, in most cases, it is simpler to obtain the solution in the s domain then obtain its inverse in time domain. Taking Laplace transform of Equation 7.72 results in

$$f^*(s) = w^*(s)g^*(s), \tag{7.73}$$

where $f^*(s)$, $w^*(s)$, and $g^*(s)$ are the Laplace transforms of the corresponding density functions. The renewal density equation is

$$m^*(s) = \frac{f^*(s)}{1 - f^*(s)}. \tag{7.74}$$

Numerical solutions of Equation 7.74 and the corresponding expected number of renewals have been proposed by Tortorella (2008, 2010) and others as listed in Chapter 3. Standard software such as Matlab® and Mathematica® include methods for Laplace inversion. For example, in Mathematica, a simple line command such as

InverseLaplaceTransform [1/(1 + s), s, t] returns the solution e^{-t}.

These approaches are useful in estimating the expected number of renewals during a time period, such as a warranty period.

PROBLEMS

7.1 Assume $f(t) = 1/4$, $0 \le t \le 4$ (uniform distribution). Determine the expected number of failures if a replacement occurs every 2 weeks.

7.2 Capacitors are used in electrical circuits whenever radio interference needs to be suppressed. Most radio frequency interference (RFI) capacitors are made from either a metalized plastic film or metalized paper. Metalized paper capacitors often fail after short circuits. But the major advantage of metalized paper over metalized plastic capacitors is their superior self-healing capability under dry conditions. A manufacturer develops capacitors with different structures that result in better performance than both the metalized paper and the metalized plastic capacitors. The manufacturer subjects a capacitor to a transient voltage of 1.2 kV and 10 pulses per day. The duration of each pulse is 10 μs. When a capacitor fails, it is repaired and immediately placed under the same test conditions. Assume that the failure time of the capacitor follows an Erlang distribution with a p.d.f. of

$$f(t) = \frac{t}{\lambda^2} e^{-t/\lambda},$$

where λ is 2×10^3 h. The repair time is exponentially distributed with a repair rate of 4×10^2 repairs per hour.

a. Determine the expected number of failures during 1 year of testing (0, 10^4 h].

b. Determine the availability of the capacitor at the end of the testing period.

7.3 Consider a component whose failure time exhibits a shifted exponential distribution as shown below.

$$F(t) = \begin{cases} 0, & \text{t} < \beta \\ 1 - e^{-\lambda(t-\beta)}, & \text{t} \ge \beta \end{cases}.$$

Determine the expected number of failures in the interval (0, t].

7.4 Given the following failure times in hours,

1.20, 3.5, 4.5, 6.0, 7.9, 12.8, 15.9, 17.9, 22.7, 26.9, 29.8, 30.5, 37.8, 39.0, 48.0, 58.0, 67.0, and 75.0,

a. What is the expected number of failures during the periods (0, 20] and (0, 40]?

b. Compare your results with those obtained using the asymptotic equation of the expected number of failures.

7.5 Use the following Laplace expression for the expected number of failures in the interval (0, t]—

$$M^*(s) = \frac{f^*(s)}{s\left[1 - f^*(s)\right]}$$

—to estimate the expected number of failures at time t for n components in operation beginning at time 0 when

a. The failure-time distribution follows the special Erlang given by

$$f(t) = \frac{t}{\lambda^2} e^{-\frac{t}{\lambda}} \quad \lambda > 0.$$

b. The failure-time distribution follows the normal distribution given by

$$f(t) = \frac{1}{\sigma\sqrt{(2\pi)}} \exp\left[-(t-\mu)^2 / 2\sigma^2\right].$$

7.6 Given the following failure data,

1.0, 1.5, 2.0, 2.3, 2.5, 3.1, 3.7, 4.2, 4.8, 5.6, 5.9, 6.2, 6.7, 8.9, 10.0, 12.0, 15.2, 17.0, 18.9, 20.3, 21.5, 24.5, 26.8, 29.1, 34.6, 44.5, 47.8, 50, 52.7, 55.5, 59.3,

a. Use Frees's method to estimate the expected number of failures at time $t = 25$.

b. Use Baker's approach to obtain the expected number of failures at times $t = 25, 40, 50$.

c. Compare the results of (a) and (b). What is your conclusion?

7.7 Electromagnetic (EM) sensors and actuators are replacing many of the mechanical components in automobiles. An example of such replacements is the antilock braking systems that replace traditional hydraulic components with EM sensors and actuators. As a result, an accurate estimate and prediction of the reliability of the EM components is of a high importance for the automobile's manufacturer. A producer of EM sensors subjects 50 units to an electric field and obtains the following failure times:

Failure times × 100				
0.076196	0.480874	0.745838	1.085770	1.575040
0.145768	0.512149	0.774938	1.126840	1.627090
0.248490	0.547918	0.832483	1.128630	1.674570
0.268816	0.556499	0.863123	1.205600	1.686560
0.278996	0.599449	0.926084	1.205750	1.737600
0.292879	0.614937	0.926734	1.312090	1.807160
0.322036	0.633408	0.973047	1.401850	1.946720
0.371150	0.636191	0.988017	1.435250	2.081600
0.393230	0.680449	1.022900	1.444370	2.235920
0.462698	0.719642	1.057490	1.490110	2.400730

a. Determine the expected number of failures at $t = 200$ h.

b. Solve (a) using Baker's approximation.

c. Fit the above data to a Weibull distribution and determine the expected number of failures at $t = 200$ h.

d. Compare the results obtained from a, b, and c. What do you conclude?

7.8 Recent advances in semiconductor integration, motor performance, and reliability have resulted in the development of inexpensive electronics that have BLDC motors. Unlike a brush-type motor, the BLDC motor has a wound stator, a permanent-magnet rotor, and internal or external devices to sense rotor position. The sensing devices can be optical encoders or resolvers providing signals for electronically switching the stator windings in the proper sequence to maintain the rotation of the magnet. The elimination of brushes reduces maintenance due to arcing and dust, reduces noise, and increases life and reliability. A manufacturer of motors wishes to replace its product from brush-type to brushless motors if it is shown to be economically feasible. Assume that the current facility and equipment can be used to produce either type of motor and that the cost of producing a brushless motor is $\$x$ higher than the cost of a brush-type motor, but the warranty cost will decrease by $\$y$ per motor per year. The manufacturer's experience with brush-type motors reveals that their failure-time distribution is given by a mixture of two exponential distributions,

$$f(t) = \theta\lambda_1 e^{-\lambda_1 t} + (1-\theta)\lambda_2 e^{-\lambda_2 t},$$

where $\theta = 0.2$, $\lambda_1 = 0.6 \times 10^{-4}$, and $\lambda_2 = 1.8 \times 10^{-4}$ failures per hour. On the other hand, the failure time of the BLDC motors can be expressed by a special Erlang distribution with the following p.d.f.:

$$f(t) = \frac{t}{\lambda^2} e^{-t/\lambda},$$

where

$$\lambda = 0.45 \times 10^4.$$

Determine the relationship between x and y that will make the production of the brushless motors feasible.

7.9 Solve Problem 7.8 using Frees's estimator and Baker's estimator. Compare the relationships obtained using these estimators with the relationship obtained from the exact solution of the renewal density function. What are your conclusions?

7.10 A telephone switching system uses two types of exchangeable modules A and B. When module A fails, it is instantaneously replaced by module B and module A undergoes repair. When B fails, it is instantaneously replaced by A and B undergoes repair. Assume that the repair time is significantly less than the time to failure and that the p.d.f. of the failure-time distributions for modules A and B are as follows. The p.d.f. for A is

$$f_A(t) = \frac{\beta t^{\beta-1}}{\lambda^\beta} \exp\left[-\left(\frac{t}{\lambda}\right)^\beta\right], \quad \text{where } t \ge 0, \beta, \lambda > 0.$$

The p.d.f. for B is

$$f_B(t) = \frac{t^{\beta-1}}{\lambda^\beta \Gamma(\beta)} \exp\left[-\frac{t}{\lambda}\right], \quad \text{where } t \ge 0, \beta, \lambda > 0.$$

It is found that $\lambda_A = 1000$, $\beta_A = 3$, $\lambda_B = 2000$, and $\beta_B = 2$. Note that λ_A and λ_B are the parameters of module A, whereas λ_B and β_B are the parameters of module B. Determine the following:

a. Expected number of failures in the interval (0, 200 h) for both modules A and B.

b. What is the probability that module A is functioning at $t = 200$ h?

Also note that

$$f_A^*(s) = \sum_{j=0}^{\infty} (-1)^j \frac{(\lambda s)^j}{j!} \Gamma\left(\frac{j+\beta}{\beta}\right)$$

and

$$f_B^*(s) = \frac{1}{(1+\lambda s)^\beta}.$$

7.11 Recent developments in the area of microelectromechanical systems (MEMS) have resulted in the construction of microgrippers, which are capable of handling microsized objects and have wide applications in biomedical engineering and microtelerobotics. A typical microgripper consists of a fixed closure driver and two movable jaws that are closed by an electrostatic voltage applied across them and the closure driver. A typical gripper can exert 40 *nN* of force on the object between its jaws, with an applied voltage of 40 V. A microtelerobot (MT) is used in experimental medical applications where microgrippers are attached to the MT. The microgrippers exert repeated forces on objects clogging a pathway. The time to failure of the grippers follows a Weibull distribution with a p.d.f. of the form

$$f(t) = \frac{\beta t^{\beta-1}}{\lambda^\beta} \exp\left[-\left(\frac{t}{\lambda}\right)^\beta\right] \quad t \geq 0, \beta > 0, \lambda > 0,$$

and its Laplace transform is

$$f^*(s) = \sum_{j=0}^{\infty} (-1)^j \frac{(\lambda s)^j}{j!} \Gamma\left(\frac{j+\beta}{\beta}\right).$$

The parameters β and λ are 2 and 2000, respectively. When the grippers fail, they are replaced by a new set of grippers, and the replacement time follows an exponential distribution of the form

$$f(t) = \theta \exp(-\theta t) \quad t \geq 0,$$

with a parameter $\theta = 500$.

Assuming that the sequence of the grippers' failure and replacement follows an alternating renewal process, determine the following:

a. The expected number of the grippers' failures in 10,000 h.

b. The availability of the grippers at 10,000 h.

c. The steady-state availability of the grippers.

d. The probability that the grippers will fail during a medical operation of an expected length of 8 h.

e. A way to improve the availability of such grippers.

7.12 Paper stock consists of cellulose fibers suspended in water. Once the stock has been washed and screened to remove unwanted chemicals and impurities, it is refined to improve the quality of the paper sheets. Additives such as starch, alum, and clay fillers are then introduced to develop required characteristics of the paper product. The paper stock is then pumped to different tanks and processed. There are two preferred types of pumps for that purpose: the reciprocating suction pumps and the centrifugal pumps. The latter are frequently clogged with high-density paper stock. A paper producer uses a centrifugal pump in order to pump the paper stock from the main tank to the next process. When the pump fails (mainly due to clogging), it is replaced by a reciprocating suction pump and vice versa. The following failure times (in hours) are observed for the centrifugal pump.

38.93	443.61	1352.84	2728.80	4064.14
79.38	447.44	1375.61	2755.42	4074.49
89.63	558.08	1492.33	2890.38	4335.69
117.39	682.27	1525.85	2891.07	4337.81
274.81	898.10	1559.99	2999.77	5078.74
299.70	946.85	1662.41	3108.44	5418.34
326.80	1013.81	1763.87	3458.03	6659.95
417.36	1157.73	2060.99	3529.30	7038.81
421.82	1285.96	2122.60	3754.15	7762.72
432.78	1326.85	2297.35	3780.32	7859.20

Similarly, the following failure times (in hours) are observed for the reciprocating pump.

9.87	259.64	592.30	934.95	1630.92
67.20	330.22	592.44	1018.67	1661.09
77.58	337.38	643.88	1140.36	1821.21
80.43	366.20	649.49	1153.35	1885.12
85.48	381.19	657.77	1260.16	2470.53
127.74	412.98	672.60	1361.80	2697.63
142.53	457.82	674.74	1421.64	2862.29
146.49	538.24	679.83	1425.61	3356.10
157.99	553.35	710.15	1488.32	3372.39
206.87	565.57	783.42	1493.67	3878.58

a. What is the expected number of failures for each type of pump in the interval $(0, 10^4$ h]?

b. What is the probability that the centrifugal pump is in use at time $t = 10^4$ h?

c. Graph the above probability over the interval $(0, 10^4$ h].

7.13 A producer of motor control boards uses a surface-mount chip resistor subassembly as a part of the board's assembly. The chip substrate is high-purity alumina, and the resistive element is a sintered thick film that is coated with a protective glass film after laser trimming and is finished with an epoxy coating. Continuity through the resistive element is established by solder attachments to the subassembly lead frame through edge terminations. Field results show that the resistor exhibits a failure mode characterized by an increase in resistance beyond the system's tolerance. Therefore, the producer develops a thermal shock test (from −85 to 200°F) and obtains the following failure times (in hours).

0.90	25.95	59.25	93.50	163.00
6.75	33.05	59.45	102.00	166.20
7.80	33.80	64.40	115.20	182.20
8.00	36.60	64.90	115.80	188.50
8.65	38.15	65.75	126.50	247.00
12.80	41.30	67.25	136.20	269.75
14.26	45.80	67.45	142.60	286.30
14.61	53.85	68.00	142.90	335.40
15.80	55.00	71.05	148.90	336.90
20.65	56.60	78.30	150.20	390.00

a. Compare the estimates of the expected number of failures in the interval (0, 5000 h] using M_b, M_d, and M_o.

b. Fit an exponential distribution to the failure data and obtain its parameter.

c. Compare the results obtained from (a) and (b).

d. Assume that when the resistor subassembly fails, it is replaced by a new one. Thus, the failure replacement sequence can be represented by an ordinary renewal process. Calculate its variance and the 95% confidence interval for the number of renewals in the interval (0, 5000 h].

7.14 The producer of the resistor subassembly in Problem 7.13 modifies the resistor but observes that the time to the first failure has a distinct distribution of the form

$$f_1(t) = \lambda_1 e^{-\lambda_1 t},$$

where $\lambda_1 = 0.0133$ failures per hour.

The failure times between any two successive failures beyond the first have the same p.d.f. as the p.d.f. obtained from the failure data given in Problem 7.13. When a resistor subassembly fails, it is replaced by a new subassembly and subsequent failures are replaced accordingly.

a. Estimate the expected number of replacements during the interval (0, 10^4 h]. What is its variance?

b. Construct a 90% confidence interval for the expected number of replacements obtained in (a).

7.15 Consider an NHPP with the following hazard function

$$h(t) = \frac{t^{\beta-1}}{\lambda^{\beta}\Gamma(\beta)\sum_{j=0}^{\beta-1}\left(\frac{t}{\lambda}\right)^{j}\frac{1}{\Gamma(j+1)}}.$$

Assume $\beta = 3$ and $\lambda = 500$.

a. Determine the probability that five failures occur in the interval (0, 6000 h].

b. What is the expected number of failures during the same interval? Plot the expected number of failures in the interval (0, t] versus t.

c. What are your conclusions regarding the hazard-rate function?

7.16 A system composed of 100 identical components and its operation is independent of the number of failed units at any time t. Assume the failures constitute a nonhomogeneous Poisson process with failure-rate function $\lambda(t)$ given by

$$\lambda(t)=\begin{cases}0.001+0.0002t & 0\le t\le 200\text{ h}\\ 0.041 & 200\le t\le 300\text{ h}\\ 0.044-0.00001t & 300\le t\le 400\text{ h}\end{cases}$$

and

$$\lambda(t)=0.007t \quad t>400.$$

a. What is the expected number of failures in the interval (200, 600]?

b. What is the reliability of the system at time $t = 600$ h?

7.17 A manufacturing center is composed of 50 machine tools and the tool for each machine is continuously monitored until its wearout is unacceptable. At that time, the tool is replaced with a new one and the monitoring begins again. The tool's failure rate is monotonically increasing and follows a gamma model. The p.d.f. of the failure times is expressed as

$$f(t)=\frac{t^{\gamma-1}}{\theta^{\gamma}\Gamma(\gamma)}e^{-\frac{t}{\theta}}.$$

The replacement rate follows an exponential distribution with a rate of μ.

a. Derive the expression for the renewal density.

b. Plot the expected number of renewals for different values of μ, γ, and θ. Analyze the effects of these parameters on the expected number of renewals. What are the conclusions?

c. What is the variance of the number of renewals?

7.18 Due to the criticality of the data in a data processing center, it is important to maintain the electric power requirements without interruption. A standby generator is placed into service as soon as the main power source fails. When the standby generator fails, the main source assumes the delivery of power. This cycle is repeated. The standby generator does not fail when it is not providing power. The failure times of the main source follow a Weibull distribution with parameters γ and θ whereas the failure times of the standby generator follow an exponential distribution with parameter λ.

a. Derive the expression for the renewal density.

b. What is the expected number of renewals (a renewal consists of a complete cycle of alternating the main power source with the standby generator or vice versa)?

c. What is the variance of the number of renewals?

d. What is the probability that the standby generator is providing the power at time t?

7.19 Multimodal failure is common in very high cycle fatigue (VHCF) testing where failure is induced from either surface damage or subsurface inclusion (Höppel et al., 2011). Methods for modeling multimodal data are divided into three classes: mixtures, competing risk, and dominant mode. The competing risk model is equivalent to the weakest link model. It is characterized by the minimum of the random variables that represent the failure modes where the dominant mode model statistically is diametrically opposite from the competing risk model (Harlow, 2011). For multimodal data of VHCF, each component tested is assumed to fail by the maximum of the statistically independent and distinct mechanisms. Assume that there are n failure modes for each component under test; then, the time to failure $T_{\max}$ for the domain mode model is given by

$$T_{\max}=\max_{l\le k\le n}T_k.$$

This is equivalent to the parallel component concept. Assuming that the failure modes are i.i.d, the CDF for $T_{\max}$ is

$$F_n(t) = P\{T_{\max} \le t\} = P\left\{\max_{l\le k\le n} T_k \le t\right\} = \prod_{l\le k\le n} F_k(t),$$

a. Derive the expression for the renewal density when $n = 2$ and $F_i(t) = 1 - e^{-\lambda_i t}$, $i = 1, 2$.

b. What is the variance of the number of renewals?

c. Derive the expression for the renewal density when $n = 2$ and $F_i(t) = 1 - e^{-\left(\frac{t}{\theta_i}\right)^{\gamma_i}}$, $i = 1, 2$.

d. What are the effective hazard rates of this component for both assumptions in (a) and (d)?

7.20 Assume the analyst of the multimodal Problem 7.19 considered a competing risk model instead. Solve items (a) through (d) above and compare the results. Does the competing risk model underestimate (overestimate) these items? What are the conclusions?

7.21 Solve Problems 7.19 and 7.20 using the following parameters: $\lambda_1 = 0.00005$, $\lambda_2 = 0.00009$, $\theta_1 = 500$, $\theta_2 = 1000$, and $\gamma_1 = 1.2$, $\gamma_2 = 2.1$.

7.22 Plot the effective hazard rates for both the competing risk model and the dominant mode model for different values of the parameters in Problem 7.21. State the conditions that make the two models equivalent to each other.

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CHAPTER 8

PREVENTIVE MAINTENANCE AND INSPECTION

A component's degree of reliability is directly proportional to its ease of accessibility; that is, the harder it is to get to, the more often it breaks down.

—*Jonathan Waddell, crew member of the oil tanker Exxon New Orleans*

8.1 INTRODUCTION

Reliability of a system is greatly affected by its structural design, quality, and reliability of its components, whereas its availability is affected, in addition to these factors, by the implementation of an effective maintenance and inspection program, when applicable. In the previous chapters of this book we presented methods for estimating reliability of different structural designs such as series, parallel, parallel-series, series-parallel, k-out-of-n, and complex networks. We also presented methods for estimating reliability of components using accelerated and operational life testing. In this chapter, we will present models for optimum preventive maintenance, replacements, and inspection (PMRI) schedules. We will also emphasize the role of the advancement in sensor technologies and its impact on a new type of maintenance, namely: condition-based maintenance (CBM). The term *optimum* arises from the fact that high frequency of PMRI increases the total cost of maintenance and reduces the cost due to the downtime of the system, whereas low frequency of PMRI decreases the cost of maintenance but increases the cost due to the downtime of the system. Hence, depending on the type of failure-time distribution, an optimum PMRI may exist. In other situations the availability of the system, not cost, is the criterion for determining the optimum maintenance schedule. Preventive maintenance may imply minimal repairs, replacements, or inspection of the components. Obviously, there are systems where minimal repairs or inspections are not applicable such as a microprocessor of a programmable logical controller. Similarly, there are systems where their status can only be determined by inspection or partial testing—as in "one shot" devices, such as military explosives, missiles, and others. In this case, replacement might be the only possible alternative.

The primary function of the preventive maintenance and inspections is to "control" the condition of the equipment and ensure its availability. Doing so requires the determination of the following:

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- Frequency of the PMRI,
- Replacement rules for components,
- Effect of technological changes on the replacement decisions,
- The size of the maintenance crew,
- Optimum inventory levels of spare parts,
- Sequencing and scheduling rules for maintenance jobs, and
- Number and type of machines available in the maintenance workshop.

The above topics are a partial list of what constitutes a comprehensive PMRI system. In this chapter, we will present analytical models that address some of these topics. More specifically, we will present different approaches for determining the optimum frequency to perform PMRI for systems operating under different conditions. Methods for determining the optimum inventory levels of spare parts also are discussed.

8.2 PREVENTIVE MAINTENANCE AND REPLACEMENT MODELS: COST MINIMIZATION

Preventive maintenance and replacements are maintenance actions that are performed on the system by making minimal repairs or full replacements of some of the system's components or the entire system. Before presenting the analytical models for preventive maintenance and replacements, it is important to note that most, if not all, models available in the literature reasonably assume the following:

- The total cost associated with failure replacement is greater than that associated with a preventive maintenance action whether it is a repair or replacement. In other words, the cost to repair the system after its failure is greater than the cost of maintaining the system before its failure. For example, replacement of a cutting tool in a milling operation before the breakage of the tool may result in a reduced total cost of the milling operation since a sudden tool breakage may cause damage to the workpiece.
- The system's failure-rate function is monotonically increasing with time. Clearly, if the system's failure rate is decreasing with time, then the system is likely to improve with time, and any preventive maintenance action or replacement is considered a waste of resources. Likewise, if the equipment or system has a constant failure rate, then any preventive maintenance action is also a waste of resources. This can be attributed to the fact that when the failure rate is constant, replacing equipment before failure does not affect the probability that the equipment will fail in the next instant, given that it is now operational (Jardine and Buzacott, 1985).
- Minimal repairs do not change the failure rate of the system. Even though a component in a system may be replaced with a new component, the complexity of the system and the large number of components in the system make the effect of such replacement negligible or nonexistent.

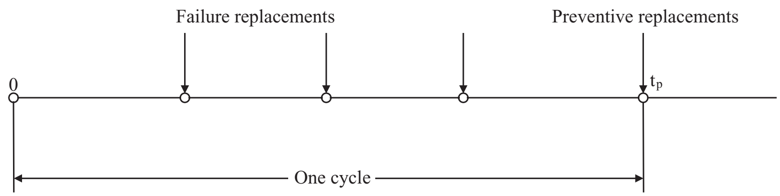

FIGURE 8.1 Constant interval replacement policy.

In the following sections, we examine common policies for preventive maintenance and replacements and will consider cases with realistic assumptions.

8.2.1 The Constant Interval Replacement Policy (CIRP)

The constant interval replacement policy (CIRP) is the simplest preventive maintenance and replacement policy. Under this policy, two types of actions are performed. The first type is the preventive replacement that occurs at fixed intervals of time. Components or parts are replaced at predetermined times regardless of the age of the component or the part being replaced. The second type of action is the failure replacement where components or parts are replaced upon failure. This policy is illustrated in Figure 8.1 and is also referred to as *block replacement policy*.

As mentioned earlier, the objective of the PMRI models is to determine the parameters of the preventive maintenance policy that optimize some criterion. The most widely used criterion is the total expected replacement cost per unit time. This can be accomplished by developing a total expected cost function per unit time as follows.

Let $c(t_p)$ be the total replacement cost per unit time as a function of t_p. Then,

$$c(t_p) = \frac{\text{Total expected cost interval } (0, t_p]}{\text{Expected length of the interval}}. \tag{8.1}$$

The total expected cost in the interval $(0, t_p]$ is the sum of the expected cost of failure replacements and the cost of the preventive replacement. During the interval $(0, t_p]$, one preventive replacement is performed at a cost of c_p and $M(t_p)$ failure replacements at a cost of c_f each, where $M(t_p)$ is the expected number of replacements (or renewals) during the interval $(0, t_p]$. The expected length of the interval is t_p. Equation 8.1 can be rewritten as

$$c(t_p) = \frac{c_p + c_f M(t_p)}{t_p}. \tag{8.2}$$

The expected number of failures, $M(t_p)$, during $(0, t_p]$ may be obtained using any of the methods discussed earlier in this book (see Chapter 7).

EXAMPLE 8.1

A critical component of a complex system fails when its failure mechanism enters one of two stages. Suppose that the failure mechanism enters the first stage with probability θ and that it enters the second stage with probability $1 - \theta$. The probability density functions (p.d.f.'s) of failure time for the first and second stage are $\lambda_1 e^{-\lambda_1 t}$ and $\lambda_2 e^{-\lambda_2 t}$, respectively. Determine the optimal preventive replacement interval of the component for different values of λ_1, λ_2, and θ.

SOLUTION

The p.d.f. of the failure time of the component is

$$f(t) = \theta\lambda_1 e^{-\lambda_1 t} + (1-\theta)\lambda_2 e^{-\lambda_2 t}, \tag{8.3}$$

and the Laplace transform is

$$f^*(s) = \frac{\lambda_1\lambda_2 + \theta\lambda_1 s + (1-\theta)\lambda_2 s}{(\lambda_1+s)(\lambda_2+s)}. \tag{8.4}$$

The Laplace transform equation of the expected number of failures is

$$M^*(s) = \frac{s[\theta\lambda_1 + (1-\theta)\lambda_2] + \lambda_1\lambda_2}{s^2[s+(1-\theta)\lambda_1 + \theta\lambda_2]}.$$

The above equation has roots 0 and $s_1 = -[(1-\theta)\lambda_1 + \theta\lambda_2]$. The expansion of the Laplace transform equation of the expected number of failures is

$$M^*(s) = \frac{1}{s^2\mu} + \frac{1}{s}\frac{\sigma^2-\mu^2}{2\mu^2} - \frac{\theta(1-\theta)(\lambda_1-\lambda_2)^2}{[(1-\theta)\lambda_1+\theta\lambda_2]^2(s-s_1)}. \tag{8.5}$$

The expected number of failures at time t is obtained as the inverse of Laplace transform of Equation 8.5 as

$$M(t) = \frac{t}{\mu} + \frac{\sigma^2-\mu^2}{2\mu^2} - \frac{\theta(1-\theta)(\lambda_1-\lambda_2)^2}{[(1-\theta)\lambda_1+\theta\lambda_2]^2} e^{-[(1-\theta)\lambda_1+\theta\lambda_2]t}, \tag{8.6}$$

where μ and σ^2 are

$$\mu = E(t) = \int_0^\infty tf(t)\,dt$$
$$\sigma^2 = E(t^2) - [E(t)]^2.$$

Without solving this example numerically, it is clear that when $\lambda_1 = \lambda_2$ the component has a constant failure rate. Therefore, the optimal preventive replacement is to replace the component upon failure. ■

When the preventive replacement is performed at discrete time intervals, such as every 4 weeks, it is then more appropriate to estimate the expected number of failures during the time interval by using the discrete time approach discussed earlier in this book.

EXAMPLE 8.2

A sliding bearing of a high-speed rotating shaft wears out according to a normal distribution with mean of 1,000,000 cycles and standard deviation of 100,000 cycles. The cost of preventive replacement is \$50 and that of the failure replacement is \$100. Assuming that the preventive replacements can be performed at discrete time intervals equivalent to 100,000 cycles per interval, determine the optimum preventive replacement interval.

SOLUTION

Substituting the cost elements in Equation 8.2 results in

$$c(t_p) = \frac{50 + 100M(t_p)}{t_p}.$$

Using the discrete time approach for calculating the expected number of failures and equating 100,000 cycles to one time interval, then

$$\begin{aligned}
M(0) &= 0 \\
M(1) &= [1 - M(0)]\frac{1}{\sqrt{2\pi}}\int_0^1 \exp\left[\frac{-(t-10)^2}{2}\right]dt \\
M(1) &= [1 + M(0)][\Phi(1-10) - \Phi(-10)] = [1+0]0 = 0,
\end{aligned}$$

where

$$\Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} \exp\left[\frac{-t^2}{2}\right]dt$$

is the cumulative distribution function (CDF) of the standardized normal distribution with mean = 0 and standard deviation = 1.

$$\begin{aligned}
M(2) &= [1 + M(1)]\frac{1}{\sqrt{2\pi}}\int_0^1 \exp\left[\frac{-(t-10)^2}{2}\right]dt + [1 + M(0)]\frac{1}{\sqrt{2\pi}}\int_1^2 \exp\left[\frac{-(t-10)^2}{2}\right]dt \\
M(2) &= [1+0]0 + [1+0][\Phi(-8) - \Phi(-9)] = 0.
\end{aligned}$$

Similarly,

$$M(3)=0,\quad M(4)=0,\quad M(5)=0,\quad M(6)=0$$

$$\begin{aligned}M(7)=&[1+M(6)]\frac{1}{\sqrt{2\pi}}\int_0^1 \exp\left[\frac{-(t-10)^2}{2}\right]dt+[1+M(5)]\frac{1}{\sqrt{2\pi}}\int_1^2 \exp\left[\frac{-(t-10)^2}{2}\right]dt\\&+[1+M(4)]\frac{1}{\sqrt{2\pi}}\int_2^3 \exp\left[\frac{-(t-10)^2}{2}\right]dt+[1+M(3)]\frac{1}{\sqrt{2\pi}}\int_3^4 \exp\left[\frac{-(t-10)^2}{2}\right]dt\\&+[1+M(2)]\frac{1}{\sqrt{2\pi}}\int_4^5 \exp\left[\frac{-(t-10)^2}{2}\right]dt+[1+M(1)]\frac{1}{\sqrt{2\pi}}\int_5^6 \exp\left[\frac{-(t-10)^2}{2}\right]dt\\&+[1+M(0)]\frac{1}{\sqrt{2\pi}}\int_6^7 \exp\left[\frac{-(t-10)^2}{2}\right]dt\end{aligned}$$

$$M(7)=[1+0][\Phi(-3)-\Phi(-4)]=0.0014$$

$$\begin{aligned}M(8)=&0+[1+M(1)]\frac{1}{\sqrt{2\pi}}\int_6^7 \exp\left[\frac{-(t-10)^2}{2}\right]dt\\&+[1+M(0)]\frac{1}{\sqrt{2\pi}}\int_7^8 \exp\left[\frac{-(t-10)^2}{2}\right]dt=0.00275\end{aligned}$$

$$M(9)=0.15875$$
$$M(10)=0.50005$$
$$M(11)=0.84135.$$

The summary of the calculations is shown in Table 8.1. From the table, the minimum cost per cycle corresponds to 800,000 cycles. Therefore, the optimum preventive replacement length is equivalent to 800,000 cycles of the sliding bearing.

TABLE 8.1 Calculations for the Optimal Preventive Interval

Interval, t_p	$M(t_p)$	$c(t_p)$
100,000	0	0.000500
200,000	0	0.000250
300,000	0	0.000166
400,000	0	0.000125
500,000	0	0.000100
600,000	0	0.000083
700,000	0.00140	0.000072
800,000	0.00275	0.000063[a]
900,000	0.15875	0.000073
1,000,000	0.50005	0.000100
1,100,000	0.84135	0.000121

[a] Indicates minimum cost. ■

8.2.2 Replacement at Predetermined Age

The disadvantage of the CIRP is that the units or components are replaced at failures and at a constant interval of time since the last preventive replacement. This may result in performing preventive replacements on units shortly after failure replacements. Under the replacement at predetermined age policy, the units are replaced upon failure or at age t_p, whichever occurs first. The models of Barlow and Hunter (1960), Senju (1957), and Jardine (1973), *inter alia*, apply. Following Blanks and Tordan (1986), Jardine (1973), and Jardine and Tsang (2005), if the component's operating cost is independent of time, the cost per unit time is

$$c(t_p) = \frac{\text{Total expected replacement cost per cycle}}{\text{Expected cycle length}}. \tag{8.7}$$

To calculate both numerator and denominator of Equation 8.7 we need first to discuss a typical cycle. There are two possible cycles of operation. The first is when the equipment reaches its planned preventive replacement age t_p, and the second is when the equipment fails before the planned replacement age. Hence, the numerator of the above equation can be calculated as (Jardine and Buzacott, 1985).

Numerator: Cost of preventive replacement × probability the component survives to the planned replacement age + cost of failure replacement × probability of component failure before t_p

$$= c_p R(t_p) + c_f\left[1 - R(t_p)\right]. \tag{8.8}$$

Similarly, the denominator is obtained as

Denominator: Length of a preventive cycle × probability of a preventive cycle + expected length of a failure cycle × probability of a failure cycle

$$\begin{aligned} &= t_p R(t_p) + \frac{\int_{-\infty}^{t_p} tf(t)dt}{\left[1 - R(t_p)\right]} \times \left[1 - R(t_p)\right] \\ &= t_p R(t_p) + \int_{-\infty}^{t_p} tf(t)dt. \end{aligned} \tag{8.9}$$

Dividing Equation 8.8 by Equation 8.9, we obtain

$$c(t_p) = \frac{c_p R(t_p) + c_f\left[1 - R(t_p)\right]}{t_p R(t_p) + \int_{-\infty}^{t_p} tf(t)dt}. \tag{8.10}$$

The optimum value of the length of the preventive replacement cycle is obtained by determining t_p that minimizes Equation 8.10. This can be achieved by taking the partial derivative of Equation 8.10 with respect to t_p and equating the resultant equation to zero as shown below:

$$\frac{\partial c(t_p)}{\partial t_p} = \frac{\left[-c_p f(t_p) + c_f f(t_p)\right] \int_0^{t_p} R(t)dt - \left[c_p R(t_p) + c_f F(t_p)\right] R(t_p)}{\left[\int_0^{t_p} R(t)dt\right]^2} = 0.$$

The optimal preventive replacement cycle t_p^* is obtained by simple algebraic manipulations of the above expression as follows:

$$f(t_p^*)\left[c_f - c_p\right] \int_0^{t_p^*} R(t)dt = \left[c_p R(t_p^*) + c_f F(t_p^*)\right] R(t_p^*)$$

$$\frac{f(t_p^*)}{R(t_p^*)} \int_0^{t_p^*} R(t)dt = \frac{1}{c_f - c_p}\left[c_p R(t_p^*) + c_p F(t_p^*) + c_f F(t_p^*) - c_p F(t_p^*)\right]$$

$$h(t_p^*) \int_0^{t_p^*} R(t)dt = \frac{1}{c_f - c_p}\left[c_p + (c_f - c_p) F(t_p^*)\right]$$

or

$$h(t_p^*) \int_0^{t_p^*} R(t)dt = \frac{c_p}{c_f - c_p} + F(t_p^*).$$

EXAMPLE 8.3

Assume that CIRP in Example 8.2 is to be compared with an age replacement policy (ARP) using the same cost values. Determine the optimum preventive replacement interval for the ARP. Which policy is preferred?

SOLUTION

We evaluate Equation 8.10 for different values of t_p:

$$c(t_p) = \frac{100 - 50R(t_p)}{t_p R(t_p) + \int_{-\infty}^{t_p} t f(t)dt}, \tag{8.11}$$

where

$$R(t_p) = 1 - \int_{-\infty}^{t_p} f(t)dt = \int_{t_p}^{\infty} f(t)dt$$

or

$$R(t_p) = \frac{1}{\sqrt{2\pi}} \int_{t_p}^{\infty} \exp\left[\frac{-(t-10)^2}{2}\right] dt.$$

Equivalently,

$$R(t_p) = \frac{1}{\sqrt{2\pi}} \int_{t_p-10}^{\infty} \exp\left[\frac{-t^2}{2}\right] dt. \tag{8.12}$$

The second term in the denominator of Equation 8.11 is obtained as follows:

$$\begin{aligned}
\int_{-\infty}^{t_p} tf(t)dt &= \int_{-\infty}^{t_p} \frac{t}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(t-10)^2}{2\sigma^2}\right] dt \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t_p} [(t-10)+10] \exp\left[\frac{-(t-10)^2}{2\sigma^2}\right] dt \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t_p} (t-10) \exp\left[\frac{-(t-10)^2}{2\sigma^2}\right] dt + \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t_p} 10 \left[\frac{-(t-10)^2}{2\sigma^2}\right] dt \\
&= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{t_p} -d\left(\exp\left[\frac{-(t-10)^2}{2\sigma^2}\right]\right) + 10\,\Phi\left(\frac{t_p-10}{\sigma}\right) \\
&= \frac{-\sigma}{\sqrt{2\pi}} \exp\left[\frac{-(t-10)^2}{2\sigma^2}\right] + 10\,\Phi\left(\frac{t_p-10}{\sigma}\right).
\end{aligned}$$

or

$$\int_{-\infty}^{t_p} tf(t)dt = -\sigma\phi\left(\frac{t_p-\mu}{\sigma}\right) + \mu\Phi\left(\frac{t_p-\mu}{\sigma}\right), \tag{8.13}$$

where

$$\phi(t) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-t^2}{2}\right] \quad \text{and} \quad \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left[\frac{-t^2}{2}\right] dt$$

are referred to as the ordinate and CDFs of the standard normal distribution $N(0, 1)$, respectively, and their values are shown in Appendix K. The summary of the calculations for different values of t_p is shown in Table 8.2. The optimal preventive replacement interval for the age replacement is 800,000 cycles. The results of this policy are identical to the CIRP. This is due to the fact that the time is incremented by 100,000 cycles. Smaller increments of time may result in a significant difference between these two policies.

TABLE 8.2 Optimal Age Replacement Policy

t_p	$R(t_p)$	$\phi\left(\frac{t_p-\mu}{\sigma}\right)$	$\Phi\left(\frac{t_p-\mu}{\sigma}\right)$	$c(t_p)$ per cycle
100,000	1.00	0	0	0.000500
200,000	1.00	0	0	0.000250
300,000	1.00	0	0	0.000166
400,000	1.00	0	0	0.000125
500,000	1.00	0	0	0.000100
600,000	1.00	0	0	0.000083
700,000	0.9987	0.004	0.0013	0.000072
800,000	0.9773	0.054	0.0227	0.000064[a]
900,000	0.8413	0.242	0.1587	0.000065
1,000,000	0.5000	0.398	0.5000	0.000160
1,100,000	0.1587	0.242	0.8413	0.000090

[a] Indicates minimum cost. ■

It is important to note that both the preventive maintenance policy and ARP, when the failure rate of the component (unit) is constant, are to maintain or replace the unit upon failure, respectively.

Variants of these two policies include the opportunity-based age replacement where a unit is replaced upon failure or upon the first opportunity after reaching a predetermined threshold age, whichever occurs first. In this case, the preventive replacements are only possible at opportunities (perhaps at the time of preventive maintenance or at time when a "more reliable" unit is introduced). Assume such opportunities occur according to a Poisson distribution with rate α independent of the failure time (Coolen-Schrijner et al., 2006). Therefore, the residual time Y to the next opportunity after a preventive maintenance or replacement is exponentially distributed with mean $1/\alpha$. Following Equation 8.1, we estimate the expected cost per cycle and the expected cycle length as

$$\begin{aligned}\text{Expected cost per cycle} &= c_p E[P(\text{time to failure} \geq t_p + Y)] + c_f E[P(\text{time to failure} < t_p + Y)] \\ &= c_f - (c_f - c_p) E[P(\text{time to failure} > t_p + Y)]. \qquad (8.14)\end{aligned}$$

$$\begin{aligned}\text{Expected cycle lenght} &= E[\text{Min (time to failure}, t_p + Y)] \\ &= \int_0^{t_p} R(t)dt + E[Y]E[P(\text{time to failure} > t_p + Y)]. \qquad (8.15)\end{aligned}$$

Dividing Equation 8.14 by Equation 8.15 results in the long-run cost per unit time which can be solved to obtain the optimum value of t_p.

8.3 PREVENTIVE MAINTENANCE AND REPLACEMENT MODELS: DOWNTIME MINIMIZATION

The models discussed in Section 8.2 determine the optimum preventive maintenance interval that minimizes the total cost per unit time. There are many situations where the availability of the equipment is more important than the cost of repair or maintenance. Indeed, the consequences of the downtime of equipment may exceed any measurable cost. In such cases, it is more appropriate to minimize the downtime per unit time than to minimize the total cost per unit time. In the following section, we present two preventive replacement policies with the objective of minimizing the total downtime per unit time.

8.3.1 The Constant Interval Replacement Policy (CIRP)

This is the simplest preventive maintenance and replacement policy. It is identical to the policy discussed in Section 8.2.1 with the exception that the objective is to minimize the total downtime per unit time, that is, minimize the unavailability of the equipment. Under this policy, replacements are performed at predetermined times regardless of the age of the equipment being replaced. In addition, replacements are performed upon failure of the equipment. Following Jardine (1973) and Blanks and Tordan (1986), we rewrite Equation 8.1 as follows:

$$D(t_p) = \frac{\text{Total downtime per cycle}}{\text{Cycle length}}, \tag{8.16}$$

where

$$\begin{aligned}\text{Total downtime} &= \text{Downtime due to failure} + \text{Downtime due to preventive replacement} \\ &= \text{Expected number of failures in } (0, t_p] \\ &\quad \times \text{Time to perform a failure replacement} + T_p\end{aligned}$$

or

$$\text{Total downtime} = M(t_p)T_f + T_p,$$

where

T_f = time to perform a failure replacement,

T_p = time to perform a preventive replacement, and

$M(t_p)$ = expected number of failures in the interval $(0, t_p]$.

The cycle length is the sum of the time to perform preventive maintenance and the length of the preventive replacement cycle = $T_p + t_p$. Thus, Equation 8.16 becomes

$$D(t_p) = \frac{M(t_p)T_f + T_p}{T_p + t_p}. \tag{8.17}$$

8.3.2 Preventive Replacement at Predetermined Age

Again, this policy is similar to that discussed in Section 8.2.2. Under this policy, preventive replacements are performed upon equipment failure or when the equipment reaches age t_p. The objective is to determine the optimal preventive replacement age t_p that minimizes the downtime per unit time

$$D(t_p) = \frac{\text{Total expected downtime per cycle}}{\text{Expected cycle length}}. \tag{8.18}$$

Total expected downtime per cycle is the sum of the downtime due to a preventive replacement × the probability of a preventive replacement and the downtime due to a failure cycle × the probability of a failure cycle. The numerator of Equation 8.18 is

$$T_p R(t_p) + T_f \left[1 - R(t_p)\right].$$

Similarly, the expected cycle length (Jardine, 1973) is

$$(t_p + T_p)R(t_p) + \left[\int_{-\infty}^{t_p} tf(t)dt + T_f\right]\left[1 - R(t_p)\right].$$

Therefore,

$$D(t_p) = \frac{T_p R(t_p) + T_f \left[1 - R(t_p)\right]}{(t_p + T_p)R(t_p) + \left[\int_{-\infty}^{t_p} tf(t)dt + T_f\right]\left[1 - R(t_p)\right]}. \tag{8.19}$$

It is important to note that the conditions for the cost minimization models are also applicable to the downtime minimization models. Moreover, we replace the cost constraint by a replacement time constraint—that is, the time to perform failure replacements is greater than the time to perform preventive replacements or $T_f > T_p$.

EXAMPLE 8.4

Assume that $T_f = 50{,}000$ cycles and $T_p = 25{,}000$ cycles. Determine the parameters of the constant preventive replacement interval policy and the ARP for the equipment given in Example 8.2.

SOLUTION

We calculate $M(t_p)$, $\int_{-\infty}^{t_p} f(t)dt$ and $R(t_p)$ as shown in Table 8.3. For the CIRP, we substitute the known parameter of the policy into Equation 8.17 to obtain

$$D_{CIRP}(t_p) = \frac{25{,}000\left[1 + 2M(t_p)\right]}{25{,}000 + t_p}. \tag{8.20}$$

TABLE 8.3 $M(t_p)$, $R(t_p)$, and $\int_{-\infty}^{t_p} tf(t)dt$

t_p	$M(t_p)$	$R(t_p)$	$1 - R(t_p)$	$\phi(t_p)$	$\Phi(t_p)$	$\int_{-\infty}^{t_p} tf(t)dt$
100,000	0	1.00	0	0	0	0
200,000	0	1.00	0	0	0	0
300,000	0	1.00	0	0	0	0
400,000	0	1.00	0	0	0	0
500,000	0	1.00	0	0	0	0
600,000	0	1.00	0	0	0	0
700,000	0.00140	0.9987	0.0013	0.004	0.0013	900
800,000	0.00275	0.9773	0.0227	0.054	0.0227	17300
900,000	0.15875	0.8413	0.1587	0.242	0.1587	134500
1,000,000	0.50050	0.5000	0.5000	0.398	0.5000	460110
1,100,000	0.84135	0.1587	0.8413	0.242	0.8413	817100

TABLE 8.4 Summary of $D(t_p)$ Calculations

t_p	$D_{CIRP}(t_p)$	$D_{ARP}(t_p)$
100,000	0.2000	0.2000
200,000	0.1111	0.1111
300,000	0.0769	0.0769
400,000	0.0588	0.0588
500,000	0.0476	0.0476
600,000	0.0400	0.0400
700,000	0.0346	0.0346
800,000	0.0305[a]	0.0316[a]
900,000	0.0356	0.0362
1,000,000	0.0488	0.0505
1,100,000	0.0596	0.0532

[a] Indicates minimum downtime.

The downtime policy per cycle for the ARP is obtained using

$$D_{ARP}(t_p) = \frac{25{,}000\,[2 - R(t_p)]}{(25{,}000 + t_p)R(t_p) + \left[\int_{-\infty}^{t_p} tf(t)dt + 50{,}000\right][1 - R(t_p)]}. \tag{8.21}$$

The summary of the calculations is shown in Table 8.4. The two policies result in the same optimal preventive replacement interval of 800,000 cycles. ■

8.4 MINIMAL REPAIR MODELS

Maintaining a complex system, which is composed of many components may be achieved by replacing, repairing, or adjusting the components of the system. The replacements, repairs, or adjustments of the components usually restore function to the entire system, but the failure rate of the system remains unchanged, as it was just before failure. This type of repair is called *minimal repair*. Since the failure rate of complex systems increases with age, it would become increasingly expensive to maintain operation by minimal repairs (Valdez-Flores and Feldman, 1989). The main decision variable is the optimal time to replace the entire system instead of performing minimal repairs.

Minimal repair models generally assume that the system's failure-rate function is increasing and that the minimal repairs do not affect the failure rate. Like the preventive replacement models, the cost of minimal repair c_f is less than the cost of replacing the entire system c_r. The expected cost per unit time at age t is

$$c(t) = \frac{c_f M(t) + c_r}{t}, \tag{8.22}$$

where $M(t)$ is the expected number of minimal repairs during the interval $(0, t]$. This model is similar to the preventive replacement model given by Equation 8.1.

Tilquin and Cléroux (1975, 1985) add cost of adjustments to the numerator of Equation 8.22. The adjustment cost $c_a(ik)$ at age ik, $i = 1, 2, 3, \ldots$, and $k > 0$ (k represents the kth minimal repair) is added to the cost of minimal repair and the cost of system replacement. This model is closer to reality since the adjustment cost $c_a(ik)$ can be used to reflect the actual operating cost of the system such as periodic adjustment costs, depreciation costs, or interest charges. Rewriting Equation 8.22 to include the adjustment costs, we obtain

$$c(t) = \frac{c_f M(t) + c_r + c_a^*(v(t))}{t}, \tag{8.23}$$

where $c_a^*(v(t)) = \sum_{i=0}^{v(t)} c_a(ik)$ and $v(t)$ is the number of adjustments in the interval $(0, t)$. This model can be extended to modify the minimal repair cost c_f to include two parts: the first part represents a fixed charge or setup a, and the second part represents a variable cost that depends on the number of minimal repairs that occurred since the last replacement—that is, $c_f = a + bk$, where $a > 0$ and $b \geq 0$ are constants, and k represents the kth minimal repair.

8.4.1 Optimal Replacement under Minimal Repair

As mentioned earlier, most repair models assume that repairs result in making the system function "as good as new." In other words, the system is renewed after each failure. Although this is true for some situations, as in the case of replacing the entire brake system of a vehicle with a new one, there are situations where the failed system will function again after repair but will have the same failure rate and the same effective age at the time of failure. Clearly, when a machine has an increasing failure rate (IFR), the duration of its function after repairs will become shorter and shorter resulting in a finite functioning time. Similarly, as the system ages, its repair time will become longer and longer and will tend to infinity—that is, the system becomes nonrepairable. Thus, in an appropriate model for such systems, successive survival times which are stochastically decreasing, each survival time is followed by a repair time, and the repair times are stochastically increasing.

This problem can be modeled using the nonhomogeneous Poisson process as described in Ascher and Feingold (1984), Barlow et al. (1965), Downton (1971), and Thompson (1981). Lam (1988, 1990), modeled the problem using geometric processes. More recently, Liao et al. (2006) consider a CBM model for continuously degrading systems under continuous monitoring. After maintenance, the states of the system are randomly distributed with residual damage. We consider a more general replacement model based on Stadje and Zuckerman (1990) and Lam (1990). Assume that the successive survival (operational) times of the system (X_n, $n = 1, 2, \ldots$) form a stochastically decreasing process and each survival time has an IFR and the consecutive repair times (Y_n, $n = 1, 2, \ldots$) constitute a stochastically increasing process and each repair time has the property that new is better than used in expectation (NBUE). A replacement policy T is considered. Under this policy, the system is replaced (repaired) after the elapse of time T from the last replacement. Assume that the repair cost rate is c and the replacement cost during an operating interval is c_o. We also assume that the replacement cost is c_f if the system is replaced upon failure or during repair and $c_f \geq c_o$. The reward or profit per unit time of system is R.

Theorem 1 (Stadje and Zuckerman, 1990), if

1. X_n and Y_n are both nonnegative random variables, $\forall n \geq 1$,
   $\lambda_n = E(X_n)$ is nonincreasing and $\mu_n = E(Y_n)$ is nondecreasing,
2. $\lim_{n\to\infty}\lambda_n = 0$, or $\lim_{n\to\infty}\mu_n = \infty$,
3. ($X_n = 1, 2, \ldots$) and ($Y_n = 1, 2, \ldots$) are two independent sequences of independent random variables, also X_n has IFR and Y_n is NBUE, $\forall n \geq 1$, and
4. $c_o = c_f$,

then the optimum replacement policy is

$$T^* = \sum_{i=1}^{n_0} X_i + \sum_{i=1}^{n_0-1} Y_i, \tag{8.24}$$

where

$$n_0 = \min\left\{n \geq 1 \mid \left(c+\phi^*\right)\mu_n \geq \left(R-\phi^*\right)\lambda_{n+1}\right\} \tag{8.25}$$

and ϕ^* is the optimal value of the long-run average reward (profit).

An equivalent replacement policy is that to replace the system after the Nth failure where the time at which the Nth failure occurs is near time T^*. We refer to this policy as policy N. Once N^* is determined, we can immediately evaluate the corresponding T^* and ϕ^* as shown below.

We now consider a policy N that operates under the following assumptions:

1. When the system fails, it is either repaired or replaced by a new and identical system.
2. Similar to policy T, the survival (or operating) time X_k after the $(k-1)$th repair forms a sequence of nonnegative random variables with nonincreasing means $E[X_k] = \lambda_k$, and

the repair time Y_k after the kth failure forms a sequence of nonnegative random variables with nondecreasing means $E[Y_k] = \mu_k$.

3. The repair cost rate is c, the replacement cost under this policy is c_f, and the reward (or profit) per unit time of system operation is R.

From renewal theory, the average reward per unit time until the Nth failure is (Lam, 1990)

$$R(N) = \frac{R\sum_{k=1}^{N}\lambda_k - c\sum_{k=1}^{N-1}\mu_k - c_f}{\sum_{k=1}^{N}\lambda_k + \sum_{k=1}^{N-1}\mu_k} \tag{8.26}$$

or

$$\mathrm{R(N) = R - c(N)}, \tag{8.27}$$

where R is the reward per unit time of system operation and $c(N)$ is the cost per unit time and is given by

$$c(N) = \frac{(c+R)\sum_{k=1}^{N-1}\mu_k + c_f}{\sum_{k=1}^{N}\lambda_k + \sum_{k=1}^{N-1}\mu_k}, \quad N = 1, 2, \ldots \tag{8.28}$$

The optimum replacement policy N^* is obtained by finding N that maximizes $R(N)$ or minimizes $c(N)$. This can be accomplished by using Equation 8.28 and subtracting $c(N^*)$ from $c(N^* + 1)$ as shown

$$c(N^*+1) - c(N^*) = \left\{(c+R)f_N - c_f(\lambda_{N+1} + \mu_N)\right\}/\Delta_N,$$

where

$$\Delta_N = \left(\sum_{k=1}^{N+1}\lambda_k + \sum_{k=1}^{N}\mu_k\right)\left(\sum_{k=1}^{N}\lambda_k + \sum_{k=1}^{N-1}\mu_k\right)$$

and

$$f_N = \mu_N\sum_{k=1}^{N}\lambda_k - \lambda_{N+1}\sum_{k=1}^{N-1}\mu_k. \tag{8.29}$$

Note that $\Delta_N > 0$ and $f_N \geq \lambda_1\mu_N > 0$ for all $N \geq 1$.

Define

$$g_N = (\lambda_{N+1} + \mu_N)/f_N. \tag{8.30}$$

Hence,

$$g_{N+1} - g_N = (\lambda_{N+2}\mu_N - \lambda_{N+1}\mu_{N+1})\left(\sum_{k=1}^{N+1}\lambda_k + \sum_{k=1}^{N}\mu_k\right)\Big/(f_N f_{N+1}) \le 0.$$

But g_N is a nonincreasing sequence from $g_1 = (\lambda_2 + \mu_1)/(\lambda_1\mu_1)$ to $g_\infty = \lim_{N\to\infty} g_N$ (Lam, 1990) and

$$c(N+1)\begin{cases} \le c(N), & \text{if } g_N \ge (c+R)/c_f \\ \ge c(N), & \text{if } g_N \le (c+R)/c_f \end{cases}.$$

Therefore, the optimal policy N^* is determined by

$$N^* = \min\{N \ge 1 \mid g_N \le (c+R)/c_f\}. \tag{8.31}$$

The optimal replacement policy N^* given by Equation 8.31 is exactly the same as policy T^* given by Equation 8.24, and the optimal ϕ^* is

$$\phi^* = R(N^*) = R - c(N^*). \tag{8.32}$$

EXAMPLE 8.5

A turbine is used to derive power at a natural gas letdown station. The impeller is the most critical component of the turbine. To reduce the effect of impeller fracture and crack propagation, a replacement policy N is implemented. When the impeller fails, it is replaced or repaired at a cost c = \$100. After N replacements (repairs) of the impeller, the turbine is replaced by a new one at a cost c_f = \$12,000. Assume that the reward rate is \$20/h. The operating time λ_k and the repair time μ_k after the kth failure are, respectively,

$$\lambda_k = \frac{5000}{2^{k-1}}$$
$$\mu_k = 100 \times 2^{k-1}.$$

Determine the parameters of the optimum replacement policy N^*.

SOLUTION

We calculate

$$\frac{c+R}{c_f} = \frac{100+20}{12{,}000} = 0.01.$$

We also calculate g_N and f_N for different N as shown in Table 8.5. From Table 8.5, the optimal replacement policy is $N^* = 2$. In other words, the turbine should be replaced after two failures or 7500 h. The optimal ϕ^* is obtained by substituting corresponding values in Equation 8.30

TABLE 8.5 Calculations for N^*

N	λ_N	μ_N	f_N	g_N
1	5000	100	5×10^5	0.01020
2	2500	200	13.75×10^5	0.00196

$$\phi^* = R - c(N^*)$$

or

$$\phi^* = 20 - 3.077 = \$16.923.$$

■

8.5 OPTIMUM REPLACEMENT INTERVALS FOR SYSTEMS SUBJECT TO SHOCKS

The preventive maintenance and replacement models discussed so far assume that the components or the systems exhibit wear or gradual deterioration—that is, IFR.

There are many situations where the system is subject to shocks that cause it to deteriorate. For example, the hydraulic and electrical systems of airplanes are subject to shocks that occur at takeoff and landing. Likewise, a breakdown of an insert of a multi-insert cutting tool may subject the tool to a sudden shock. Clearly, systems that are subject to such shocks will eventually deteriorate and fail (due to the cumulative damage from every shock). In this section, we discuss an optimum replacement policy for such systems.

Assume that the normal cost of running the system is a per unit time and that each shock increases the running cost by c per unit time. The system is entirely replaced at times T, $2T$, . . . at a cost of c_0 per complete system replacement. This is referred to as a periodic replacement policy of length T. The only parameter of such policy is T, and reliability engineers usually seek the optimal value of T that optimizes some criterion such as the minimization of the long-run average cost per unit of time or the maximization of the system availability during T. Clearly, if the length of the period T is rather long, then the cost of system operation and replacement will vary from one period to another due to the changes in labor and material cost with time. Therefore, we present two periodic replacement policies where the first policy considers all cost components of the system to be time independent and the second policy considers some of the cost components to be time dependent. These policies are based on Abdel-Hameed's (1986) work.

8.5.1 Periodic Replacement Policy: Time-Independent Cost

As mentioned in the previous section, the system is subject to repeated shocks and is entirely replaced after a fixed period of time T has elapsed. Let $N(t)$ be the number of shocks that the system is subject to during the interval $(0, t)$, and let $n = (N(t), t \geq 0)$. To simplify the analysis, we assume that the jumps of N are of one unit magnitude and that τ_n is the sequence of the jump times of the process N.

The total cost of running the system per period T for a given realization of the sequence τ_n is

$$aT + c(\tau_2 - \tau_1) + \ldots + c(N(T) - 1)(\tau_{N(T)} - \tau_{N(T)-1}) + cN(T)(T - \tau_{N(T)}) + c_0. \tag{8.33}$$

The above expression can be rewritten as

$$aT + c\int_0^T N(t)dt + c_0. \tag{8.34}$$

Utilizing Fubini's theorem (Heyman and Sobel, 1982), the expected total cost of running the system per period is given by

$$aT + c\int_0^T M(t)dt + c_0, \tag{8.35}$$

where $M(t)$ is the expected number of shocks in $(0, t]$. The long-run average cost per unit time, $C(T)$, is obtained by dividing Equation 8.33 by the length of the replacement period T. In other words,

$$C(T) = \left[aT + c\int_0^T M(t)dt + c_0 \right] \Big/ T. \tag{8.36}$$

The objective is to determine T that minimizes Equation 8.36. Since $C(T)$ is a differential function of T and that the first-order derivative of $C(T)$ is given by

$$C'(T) = \left[c\int_0^T [M(T) - M(t)]dt - c_0 \right] \Big/ T^2,$$

and since $\int_0^T [M(T) - M(t)]dt$ is positive and increasing, then the optimal value of the periodic replacement time always exists and is equal to the unique solution of

$$\int_0^T [M(T) - M(t)]dt = \frac{c_0}{c}. \tag{8.37}$$

Moreover, the value of T^* is finite if and only if (Abdel-Hameed, 1986)

$$\lim_{T\to\infty} \int_0^T [M(T) - M(t)]dt > c_0 / c. \tag{8.38}$$

Abdel-Hameed (1986) also develops an expression to estimate $M(t)$ when the shocks occur according to a nonstationary pure birth process. If the shock occurrence rate is $\lambda(t)$, the probability of a shock occurring in $[t, t + \Delta)$ given that k shocks occurred in $(0, t)$ is $\lambda_k(t)$.

Assume that $\{N(t);\ t > 0\}$ counts the number of shocks. When $\lambda_k(t) = k\lambda(t)$ and $N(0) > 0$, the counting process is called *Yule process* (Heyman and Sobel, 1982); when $\lambda_k(t) \equiv \lambda$, it is called a Poisson process. Since $N(t)$ is a nonstationary Yule process—that is, for $k = 1, 2, \ldots$

$$\lambda_k(t) = k\lambda(t)$$

—then we have

$$M(t) - 1 = \int_0^t \lambda(x) M(x) dx. \tag{8.39}$$

Equation 8.39 has the solution

$$M(t) = e^{\int_0^t \lambda(x) dx}. \tag{8.40}$$

As discussed earlier in this chapter, the optimal value of the periodic replacement policy (T^*) exists only if $\lambda(t)$ is an increasing function of t.

EXAMPLE 8.6

Consider a component whose $\lambda(t) = \lambda$. Assume that the normal cost of running the system is \$0.50/h, the increase in running cost due to each shock is \$0.055/h, and the cost of replacing the entire component is \$15,000. Determine the optimal value of T and the corresponding long-run average cost of replacement per hour for different values of shock rates.

SOLUTION

From the above description we list

$$a = \$0.5,$$
$$c = \$0.055,$$

and

$$c_0 = \$15{,}000.$$

Using Equation 8.40, we obtain

$$M(t) = e^{\lambda t}.$$

Substituting in Equation 8.37, we obtain

$$e^{\lambda T}[\lambda T - 1] = \frac{\lambda c_0}{c} - 1.$$

Solving the above equation for different values of λ, we obtain the optimum replacement interval and the long-run average cost of replacement per unit time $C(T)$ as shown in Table 8.6.

TABLE 8.6 Optimum Replacement Interval

λ	T^*	$C(T^*)$
1×10^{-4}	27239.267	1.0507
2×10^{-4}	15972.763	1.4391
4×10^{-4}	9231.254	2.1251
6×10^{-4}	6657.679	2.7534
8×10^{-4}	5266.755	3.3487
1×10^{-3}	4385.343	3.9214
6×10^{-3}	970.945	15.9680
8×10^{-3}	758.135	20.3165
1×10^{-2}	625.206	24.5376
6×10^{-2}	129.794	117.0886
8×10^{-2}	100.487	151.4683
1×10^{-1}	82.347	185.1727

■

8.5.2 Periodic Replacement Policy: Time-Dependent Cost

This policy is similar to that presented in Section 8.5.1 with the exception that the replacement cost per unit time is dependent on the number of shocks and the time at which shocks occur. Following the policy in Section 8.5.1, we assume that the shock process $N = (N(t), t \geq 0)$ has jumps of size 1. Let τ_n be the sequence describing the jump times of the shock process N. The additional cost of operating the system per unit time due to every additional shock in the interval $[\tau_i, \tau_i + 1]$ is $c_i(u)$, $i = 0, 1, \ldots,$ and u is the state space at which the periodic replacement can be performed. We assume that $\tau_0 = 0$. The normal cost of running the system is a per unit time, and the cost of completely replacing the system is c_0 (Abdel-Hameed, 1986). Similar to the periodic replacement policy with constant cost structure, the system is completely replaced at T, $2T$, . . . and the process is reset to time zero at every replacement. The expected total cost of running the system per period is the sum of the operating cost (aT), expected additional cost due to shock $\int_0^T Ec_{N(t)}(t)dt$, and the cost of completely replacing the system. This is expressed as

$$aT + \int_0^T Ec_{N(t)}(t)dt + c_0, \tag{8.41}$$

where $Ec_{N(t)}(t)$ is the expectation of the additional cost function due to shocks at time t. The long-run average cost per unit time is obtained by dividing Equation 8.41 by the length of the period T. Abdel-Hameed (1986) defines $h(t) = E(c_{N(t)}(t))$ and shows that if h is continuous and increasing, then the optimal value of the periodic replacement time exists and is the unique solution of Equation 8.42

$$\int_0^T [h(T) - h(t)]dt = \frac{c_0}{c}. \tag{8.42}$$

The period T is finite if and only if

$$\lim_{T\to\infty}\int_0^T [h(T)-h(t)]dt > \frac{c_0}{c}. \tag{8.43}$$

8.6 PREVENTIVE MAINTENANCE AND NUMBER OF SPARES

In Section 8.2.1, the total expected cost per unit time is expressed as

$$c(t_p)=\frac{c_p+c_f M(t_p)}{t_p}, \tag{8.44}$$

where c_p and c_f are the cost per preventive replacement (or planned replacement) and the cost per failure replacement, respectively. $M(t_p)$ is the renewal function and is related to the failure-time distribution $F(t)$ by the integral renewal equation given in Chapter 7.

One of the most important decisions regarding preventive maintenance function is the determination of the number of spare units to carry in the spares' inventory. Clearly, if the number of spares on hand is less than what is needed during the preventive maintenance cycle, the system to be repaired will experience unnecessary downtime until the spares become available. On the other hand, if more spares are carried in the spares' inventory than what is needed during the preventive maintenance cycle, an unnecessary inventory carrying cost will incur. Ideally, the number of spares carried in the inventory should equal the number of repairs during the preventive cycle. However, the number of repairs (failures) is a random variable that makes the determination of the exact number of spares a difficult task. In this section, we determine the optimal number of spares to hold at the beginning of the preventive maintenance cycle such that the total cost during the cycle is minimized.

The number of spares needed per preventive maintenance cycle is $[1 + N_1(t_p)]$, where $N_1(t_p)$ is a random variable that represents the number of replacements due to failure in the cycle t_p. Assume that the initial inventory of the spares is L units. At the end of the preventive cycle, the inventory cost equals zero if $L = N_1(t_p) + 1$; otherwise a carrying cost or shortage cost (cost due to unavailability of spares when needed) will incur. Define $g(L, N_1(t_p))$ as the penalty function that increases as $[L - (N_1(t_p) + 1)]$ deviates from zero. Following Taguchi et al.'s (1989) loss function and Murthy's (1982) function, and assuming that the inventory carrying cost per excess unit equals the shortage cost per unit, we express the penalty function g as

$$g[L, N_1(t_p)]=[L-(N_1(t_p)+1)]^2. \tag{8.45}$$

Since $N_1(t_p)$ is a random variable, then the expected value of the penalty function $D(L, t_p)$ is

$$D(L,t_p)=E\{g[L,N_1(t_p)]\}$$

or

$$D(L, t_p) = \sum_{n=0}^{\infty} g(L, n) p_n, \tag{8.46}$$

where p_n is the probability that $N_1(t_p) = n$. The overall cost function is the sum of two terms: the first is the average cost of the system per unit time given by Equation 8.44, and the second is the penalty cost associated with the number of spares. Thus, we express the overall cost as

$$TC = c(t_p) + \alpha D(L, t_p), \tag{8.47}$$

where α is a scaling factor. If $\alpha = 0$, then there is no penalty cost, and a large value of α implies very high penalty cost for both shortage and excess inventory. It is obvious that the optimum preventive maintenance cycle t_p is a function of α. The optimal values of t_p^* and L^* are obtained by minimizing TC with respect to t_p and L.

We rewrite $D(L, t_p)$ as

$$\begin{aligned} D(L, t_p) &= E[L - (M(t_p) + 1) + M(t_p) - N_1(t_p)]^2 \\ &= [L - (M(t_p + 1))]^2 + E[N_1(t_p) - M(t_p)]^2 \end{aligned}$$

or

$$D(L, t_p) = [L - (M(t_p) + 1))]^2 + \mathrm{Var}[N_1(t_p)], \tag{8.48}$$

where $M(t_p) = E[N_1(t_p)]$ and Var $[N_1(t_p)]$ is the variance of $N_1(t_p)$. Substituting Equations 8.48 and 8.44 into Equation 8.47, we obtain

$$TC = [c_f M(t_p) + c_p]/t_p + \alpha \mathrm{Var}[N_1(t_p)] + \alpha[L - (M(t_p) + 1)]^2. \tag{8.49}$$

The optimal number of spares at the beginning of the preventive cycle is obtained by setting $\partial TC/\partial L = 0$ as follows:

$$\frac{\partial TC}{\partial L} = 2\alpha[L - (M(t_p) + 1)] = 0$$

or

$$L^* = 1 + M(t_p). \tag{8.50}$$

This is an intuitive and expected result, since it states that the optimal number of spares must equal the expected number of repairs (failures) during the preventive maintenance cycle.

Similarly, the optimal length of the preventive maintenance cycle for a given α is obtained by setting $\partial TC/\partial t_p = 0$, which results in

$$\frac{\partial c(t_p)}{\partial t_p} + \alpha \frac{\partial V(t_p)}{\partial t_p} = 0, \tag{8.51}$$

where

$$V(t_p) = \mathrm{Var}(\mathrm{N}_1(t_p)).$$

The optimal value of t_p^* is obtained by solving Equation 8.51.

EXAMPLE 8.7

The blades of a high-pressure compressor used in the first stage of an aeroengine are subject to fatigue cracking. The fatigue life of a blade is evaluated as the product of the amplitude and frequency (AF) value during vibratory fatigue testing. If a blade exhibits unusually low AF values, it is replaced in order to avoid the fatigue cracking of the blade. Since the cost of conducting the fatigue test on a regular basis is high, the users of such compressors usually use a preventive maintenance schedule to replace the blades based on the number of operating hours. The cost of replacing a blade at the end of the preventive maintenance cycle is \$250 whereas the cost of replacing a blade during the cycle is \$1000. The time between successive failures is expressed by a two-stage Erlang distribution with a parameter of $\lambda = 0.005$ failures per hour. Assuming $\alpha = 0.8$, what are the optimal number of spares and preventive maintenance intervals that minimize the total cost?

SOLUTION

The p.d.f. of the two-stage Erlang distribution is

$$f(t;\lambda) = \lambda^2 t e^{-\lambda t}.$$

The expected number of failures during the preventive maintenance cycle t_p is

$$M(t_p) = \frac{1}{2}\lambda t_p - \frac{1}{4} + \frac{1}{4}e^{-2\lambda t_p}. \tag{8.52}$$

Consequently, $c(t_p)$ is

$$c(t_p) = \frac{c_p}{t_p} + \frac{1}{2}\lambda c_f - \frac{1}{4}\frac{c_f}{t_p} + \frac{c_f}{4t_p}e^{-2\lambda t_p}. \tag{8.53}$$

The variance of the number of failures is obtained as discussed in Cox (1962):

$$\mathrm{Var}(N(t_p)) = \frac{1}{4}\lambda t_p + \frac{1}{16} - \frac{1}{2}\lambda t_p e^{-2\lambda t_p} - \frac{1}{16}e^{-4\lambda t_p}. \tag{8.54}$$

Substituting Equations 8.53 and 8.54 into Equation 8.51, we obtain

$$\frac{-c_p}{t_p^2} + \frac{c_f}{4t_p^2} - \frac{c_f}{4}e^{-2\lambda t_p}\left(\frac{2\lambda t_p + 1}{t_p^2}\right) + \alpha\left[\frac{\lambda}{4} - \frac{1}{2}\lambda e^{-2\lambda t_p}(1 - 2\lambda t_p) + \frac{1}{4}\lambda e^{-4\lambda t_p}\right] = 0$$

or

$$e^{-0.001t_p}[t_p^3 - 10t_p^2 - 12{,}500t_p - 1{,}250{,}000] + 5t_p^2(1 + e^{-0.02t_p}) = 0.$$

The solution of the above equation is

$$t_p^* = 140.568 \text{ h}.$$

The corresponding expected number of failures during the preventive maintenance cycles is 0.163. Therefore, the optimal number of spares is 1.163. This value represents the number of spares for every operating unit. ■

The difficulty in using Equation 8.51 is due to the estimation of the variance of the expected number of failures during the preventive cycle t_p. Cox (1962) derives asymptotic results for the variance as a function of the mean and standard deviation of the failure-time distribution. The asymptotic variance is

$$\text{Var}[N(t_p)] = \frac{\sigma^2}{\mu^3} t_p. \tag{8.55}$$

We now illustrate the use of Equation 8.55 when closed-form expressions for $M(t_p)$ and Var $(N(t_p))$ are difficult to attain.

EXAMPLE 8.8

High-pressure ball valves made from martensitic stainless steel are usually used in chemical plants to control the flow of dry synthetic gas (a three to one mixture of nitrogen and hydrogen with 4% ammonia). Assume that the failure times follow a Weibull distribution of the form

$$f(t) = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1} e^{-\left(\frac{t}{\theta}\right)^{\gamma}} \quad t > 0, \theta > 0, \gamma > 0.$$

The estimated values of γ and θ are 2 and 50, respectively (measurements in hundreds). The cost of a failure replacement is \$500 and that of a preventive replacement is \$300. The scale of the penalty function is 3.5. Determine the optimal preventive maintenance interval and the optimal number of spares during the maintenance interval.

SOLUTION

Since it is difficult to estimate the expected number of failures during t_p, we utilize the asymptotic form of the renewal function:

$$M(t_p) = \frac{t_p}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2},$$

where μ and σ are the mean and the standard deviation of the failure-time distribution. The mean and variance of the Weibull distribution are

$$\mu = \theta\Gamma\left(1+\frac{1}{\gamma}\right) = 50\Gamma\left(\frac{3}{2}\right) = 44.33$$

$$\sigma^2 = \theta^2\left[\Gamma\left(1+\frac{2}{\gamma}\right) - \left(\Gamma\left(1+\frac{1}{\gamma}\right)\right)^2\right]$$

or

$$\sigma^2 = 50^2[1-0.7854] = 536.50$$

and

$$\sigma = 23.162 \text{ h}.$$

Thus,

$$M(t_p) = \frac{t_p}{44.33} - 0.363. \tag{8.56}$$

The variance of the expected number of failures is

$$\text{Var}[N(t_p)] = 0.006156 t_p. \tag{8.57}$$

Substituting Equation 8.56 into Equation 8.44, we obtain

$$\begin{aligned} c(t_p) &= \frac{c_p}{t_p} + \frac{c_f}{t_p}\left[\frac{t_p}{44.33} - 0.363\right] \\ c(t_p) &= \frac{118.5}{t_p} + 11.279. \end{aligned} \tag{8.58}$$

Substituting Equations 8.57 and 8.58 into Equation 8.51 results in

$$\frac{-118.5}{t_p^2} + 3.5 \times 0.006158 = 0$$

or

$$t_p^* = 74.149 \text{ or } 74{,}149 \text{ h}.$$

The optimal number of spares is

$$L^* = 1 + M(t_{p\alpha}^*) = 1 + 1.3096 = 2.3096 \text{ valves}.$$ ■

Number of Spares and Availability When systems provide critical services such as the computer systems of the Federal Reserve Bank, it is important to stock spare parts on hand to ensure a specified availability level of the system. In this case, we utilize Erlang's loss formula to estimate the number of spares during the constant failure-rate region. The formula is given as follows (Cooper, 1972):

$$\bar{A}(s, a) = \frac{a^s / s!}{\sum_{k=0}^{s} (a_k / k!)},$$

where

$\bar{A}(s, a)$ = steady state unavailability of the system when the number of spares on hand is s and the number of units under repair is a,

a = number of units under repair,

s = number of units on hand, and

$A(s, a) = 1 - \bar{A}(s, a)$ = availability of the system.

The number of units under repair depends on the total number of units in service, the repair rate, and the average lead time for obtaining spares. Let N be the total number of units for which we need to provide spares in order to maintain a specified availability level. Assume that the repair rate is R units per unit time and l is the lead time. Then a, number of units under repair, is in effect the product NRl. Thus, the number of spares s, required to obtain a specified availability level, can be obtained by substituting the values of a and $A(s, a)$ into the Erlang's loss formula and solving for s.

EXAMPLE 8.9

A telephone company maintains a large communication network that contains 2200 repeaters (devices capable of receiving one or two communication signals and delivering corresponding signals). Each repeater experiences a constant failure rate of 3000 FITs (one FIT is 10^{-9} failure per hour). Assume that the company's standard repair rate is 1.70 times the failure rate and the average lead time is 48 h. Determine the required number of spares that maintains steady-state availability of 0.998.

SOLUTION

The repair rate is $1.7 \times 3000 \times 10^{-9} = 5100 \times 10^{-9}$ replacements per hour. The product NRl = $2200 \times 5100 \times 10^{-9} \times 48 = 0.539$.

Using Erlang's loss function,

$$\bar{A}(s,a) = \frac{a^s/s!}{\sum_{k=0}^{s}(a^k/k!)}$$

$$\bar{A}(s,a) = 0.002.$$

For $s = 2$,

$$\bar{A}(2,a) = \frac{0.1450}{1+0.539+0.145} = 0.08 > 0.002.$$

For $s = 3$,

$$\bar{A}(3,a) = \frac{0.02609}{1+0.539+0.145+0.02609} = 0.015 > 0.002.$$

For $s = 4$,

$$\bar{A}(4,a) = \frac{0.003516}{1+0.539+0.145+0.02609+0.00351} = 0.00208 \cong 0.002.$$

Therefore, the number of spares required to achieve an availability level of 0.998 is 4. ■

The relationship among the availability, the product NR*l*, and the number of spares is shown in Figure 8.2. The number of spares can be obtained from this figure by observing where the intersection of NR*l* and $A(s, a)$ occurs. Similar graphs can be developed for different ranges of NR*l* and availabilities (AT&T, 1983).

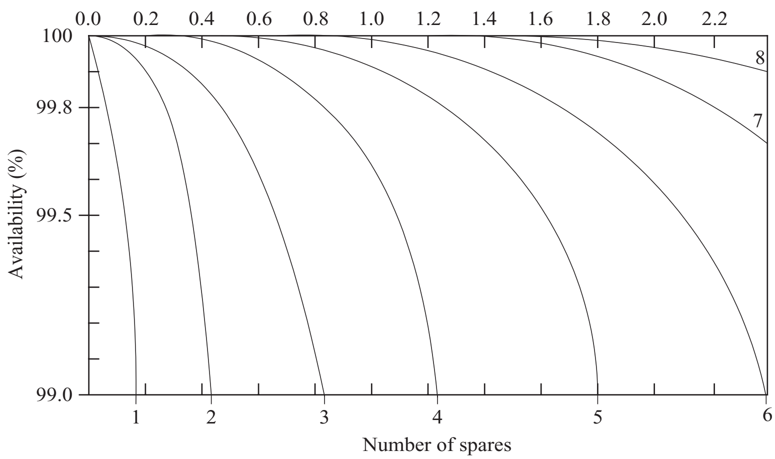

FIGURE 8.2 Relationship between availability and number of spares.

8.7 GROUP MAINTENANCE

The preventive maintenance schedules presented so far in this chapter are limited to the replacement of the components of the system one at a time at predetermined time intervals that either minimize the cost of maintenance or maintain an acceptable level of system availability. By studying the trade-off between preventive maintenance and corrective maintenance (failure replacements) costs, it is shown (Barlow et al., 1965) that the optimum scheduled time for preventive maintenance is nonrandom, and there exists a unique optimum policy if the distribution of time to failure has IFR.

In situations where many similar products or machines perform the same function—such as copiers in a copy center or a fleet of identical vehicles that transport passengers from one location to another regularly—it is perhaps more economical to perform preventive maintenance on a group of the products, machines, or vehicles at the same time. Similar to the single component or product preventive maintenance models, we are interested in determining the optimum preventive maintenance interval for the group.

Consider a group of N independent operating machines that are subject to failure. The repair cost is composed of a fixed cost for each repair and a variable cost per machine. The repair cost per machine decreases as the number of machines requiring repairs increases while the production loss (or service provided) due to machine breakdowns increases. Let $N(t)$ represent the number of machines operating at time $t(0 \le N(t) \le N)$, and the machines have identical failure distributions $F(t)$. The distribution of $N(t)$ is

$$P[N(t)=n]=\binom{N}{n}[1-F(t)]^{n}[F(t)]^{N-n}, \tag{8.59}$$

where $P\ [N(t) = n]$ is the probability that the number of machines operating at time t equals n. The distribution of $N(t)$ is binomial with a mean of

$$E[N(t)]=N[1-F(t)]. \tag{8.60}$$

Suppose that the failed machines are repairable at a fixed c_0 and a variable cost c_1 per machine. If a failed machine is not repaired upon failure, then a production loss of c_2 per unit time per machine is incurred. Since the production will increase as the scheduled time for repair increases and the repair cost per machine decreases, then there exists an optimum scheduled time (maintenance time) that minimizes the expected total cost per unit time.

Following Okumoto and Elsayed (1983), we consider a random maintenance scheduling policy that states that repairs are undertaken whenever the number of operating machines reaches a certain level n. The time to reach this level is a random variable T with CDF of $G(t)$. T represents the nth-order statistic of N random variables. The expected repair cost per cycle R_c is

$$R_c = c_0 + c_1\int_0^{\infty}[N-E[N(t)]]dG(t)$$

or

$$R_c = c_0 + c_1 N \int_0^{\infty} F(t) dG(t). \tag{8.61}$$

The expected production loss per cycle P_c is

$$P_c = c_2 \int_0^{\infty} [N - E[N(t)]] \bar{G}(t) dt$$

or

$$P_c = c_2 N \int_0^{\infty} F(t) \bar{G}(t) dt, \tag{8.62}$$

where $\bar{G}(t) = 1 - G(t)$. The total expected cost per unit time is

$$c[G(t)] = \frac{R_c + P_c}{\int_0^{\infty} t dG(t)}. \tag{8.63}$$

Barlow et al. (1965) show that the optimum scheduling policy that minimizes Equation 8.63 is deterministic. In other words,

$$G(t) = \begin{cases} 0 & \text{if} \quad t \le t_0 \\ 1 & \text{if} \quad t > t_0, \end{cases}$$

where t_0 is the scheduled time for group maintenance. Thus, Equation 8.63 can be rewritten as

$$c(t_0) = \frac{1}{t_0} \left[c_0 + c_1 N F(t_0) + c_2 N \int_0^{t_0} F(t) dt \right]. \tag{8.64}$$

From Equation 8.64, $c(0) = \infty$ and $c(\infty) = c_2 N$. This implies that the cost per unit time for the optimum schedule is less than $c_2 N$. The optimum schedule policy is summarized as follows.

Assume $F(t)$ is continuous, the derivative of $f(t)$ exists, and that the failure rate per machine is λ. Suppose $-f'(t)/f(t) < c_2/c_1$ for $t \ge 0$. Then,

1. If $c_2/\lambda > c_0/N + c_1$, then there exists a unique and finite optimum scheduling time t_0^* that satisfies the following equation:

$$c_1 t_0 f(t_0^*) + c_2 t_0 F(t_0^*) - c_1 F(t_0^*) - c_2 \int_0^{t_0^*} F(t) dt = c_0 / N. \tag{8.65}$$

The minimum cost per unit time is obtained by

$$c(t_0^*) = c_1 N f(t_0^*) + c_2 N F(t_0^*). \tag{8.66}$$

2. Otherwise, $t_0^* = \infty$.

The condition in (1) is realistic. For instance, if the failure-time distribution is assumed to be exponential with a rate of λ, then the condition $-f'(t)/f(t) < c_2/c_1$ becomes $(c_2/\lambda) > c_1$, which translates to the average production loss per machine. Furthermore, the condition $c_2/\lambda > c_0/N + c_1$ implies that the average production loss per machine is more than the total repair cost per machine when group repair of N machines is performed. Therefore, it is reasonable to schedule the repair before the failure of all machines (Okumoto and Elsayed, 1983).

We have shown earlier in this chapter that when a machine exhibits constant failure rate, the optimal preventive maintenance policy is to replace the component upon failure. However, when N identical machines each exhibit a constant failure rate λ, the condition for the existence of an optimum policy is given by $c_2/\lambda > c_0/N + c_1$. By substituting the p.d.f. of the exponential distribution into Equation 8.65, we obtain

$$(X^* + 1)e^{-X^*} = A, \tag{8.67}$$

where

$$X^* = \lambda t_0^* \tag{8.68}$$

and

$$A = \frac{c_2 / \lambda - (c_0 / N + c_1)}{c_2 / \lambda - c_1}. \tag{8.69}$$

Equation 8.67 represents the expected number of machines to be repaired under the optimum policy. The expected cost for the optimum policy is

$$c(t_0^*) = c_2 N + N\lambda(c_1 - c_2 / \lambda)e^{-\lambda t_0^*}. \tag{8.70}$$

EXAMPLE 8.10

The milling department in a large manufacturing facility has 10 identical computer numerically controlled (CNC) milling machines. Each machine exhibits a constant failure rate of 0.0005 failures per hour. The cost of lost production is \$200 per machine per hour. The repair cost consists of two components: a fixed cost of \$150 and a variable cost of \$100 per machine. Determine the optimum preventive maintenance schedule and the corresponding cost.

SOLUTION

The machines exhibit constant failure rates that follow exponential distributions. The p.d.f. is

$$f(t) = \lambda e^{-\lambda t}.$$

The repair cost has two elements $c_0 = \$150$ and $c_1 = \$100$. The production loss $c_2 = \$200$/h. We check the condition$(c_2/\lambda) > c_0/N + c_1$ for the existence of an optimum policy.

Since

$$\frac{200}{0.0005} > \frac{150}{10} + 100,$$

then an optimum policy exists. The optimum repair time of the machines is obtained by using Equation 8.67 through Equation 8.69. From Equation 8.69, we obtain

$$A = \frac{c_2 / \lambda - (c_0 / N + c_1)}{c_2 / \lambda - c_1}$$

or

$$A = \frac{400{,}000 - 115}{400{,}000 - 100} = \frac{399{,}885}{399{,}900} = 0.9999624.$$

From Equation 8.68, $X^* = 0.0005\, t_0^*$. Thus, Equation 8.67 becomes

$$(0.0005 t_0^* + 1)e^{-0.0005 t_0^*} = 0.9999624,$$

and the optimal time to repair the failed machines is 17.5 h. The corresponding cost is

$$\begin{aligned} c(17.5) &= c_2 N + N\lambda(c_1 - c_2 / \lambda)e^{-\lambda t_0^*} \\ &= 2000 + 10 \times 0.0005\left(100 - \frac{200}{0.0005}\right)e^{-0.0005 \times 17.5} \end{aligned}$$

or

$$c(17.5) = \$17.92.$$ ■

8.8 PERIODIC INSPECTION

Performing preventive maintenance, when applicable, at specified schedules will certainly improve the reliability of the system. To further improve reliability, especially for critical systems, a preventive maintenance schedule is usually coupled with a periodic inspection schedule. In such situations, the status of the systems is determined by inspection, such as the case of bridges and structures. Continuous monitoring of the system is an alternative to periodic inspection. Methods for condition monitoring are presented in Section 8.9. In this section, we discuss two inspection policies.

8.8.1 An Optimum Inspection Policy

Under this policy, the state of the equipment is determined by inspection. For example, the quality of the products produced by a machine may fall outside the acceptable control limits indicating machine degradation (assuming other factors affecting product quality have not been changed). When a failure or degradation is detected, the equipment is repaired or adjusted and is returned to its original condition before the failure or the degradation reaches a critical threshold level. If the inspection fails to detect the failure or the degradation of the equipment, then unnecessary cost associated with equipment failure will incur. We refer to this cost as *nondetection cost*. The objective is to determine an optimum inspection schedule which minimizes the total cost per unit time associated with inspection, repair, and the nondetection cost.

The inspection policy is to perform inspections at times x_1, x_2, x_3, . . . until a failed or degraded equipment is detected. Repairs are immediately performed upon the detection of a failure or degradation. The inspection intervals are not necessarily equal but may be reduced as the probability of failure increases (Jardine, 1973; Jardine and Tsang (2005). We use the following definitions:

c_i = the inspection cost per inspection,

c_u = the cost per unit time of undetected failure or degradation,

c_r = the cost of a repair,

T_r = the time required to repair a failure (or degradation), and

$f(t)$ = the p.d.f. of the equipment's time to failure.

The expected total cost per unit time is

$$c(x_1, x_2, x_3, \ldots) = \frac{E_c}{E_l}, \tag{8.71}$$

where E_c and E_l are the total expected cost per cycle and expected cycle length, respectively.

We now illustrate the estimation of E_c and E_l as follows. Assume that failure of the equipment occurs between any pair of inspection times. If the failure occurs at time t_1 between time 0 and x_1, then the cost of the cycle would be

$$c_i(1) + c_u(x_1 - t_1) + c_r.$$

The expected value of this cost is

$$\int_0^{x_1} [c_i(1) + c_u(x_1 - t) + c_r] f(t)dt. \tag{8.72}$$

Similarly, if the failure occurs between the inspection times x_1 and x_2, the expected value of the cost would be

$$\int_{x_1}^{x_2} [c_i(2) + c_u(x_2 - t) + c_r] f(t)dt. \tag{8.73}$$

Thus, the total expected cost per cycle is

$$E_c = \int_0^{x_1} [c_i(0+1) + c_u(x_1 - t) + c_r] f(t)dt + \int_{x_1}^{x_2} [c_i(1+1) + c_u(x_2 - t) + c_r] f(t)dt$$
$$+ \int_{x_2}^{x_3} [c_i(2+1) + c_u(x_3 - t) + c_r] f(t)dt + \ldots + \int_{x_j}^{x_{j+1}} [c_i(j+1) + c_u(x_{j+1} - t) + c_r] f(t)dt + \ldots \tag{8.74}$$

Equation 8.74 can be written as

$$E_c = \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} [c_i(k+1) + c_u(x_{k+1} - t) + c_r] f(t)dt$$

or

$$E_c = c_r + \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} [c_i(k+1) + c_u(x_{k+1} - t)] f(t)dt. \tag{8.75}$$

We estimate the expected cycle length E_l by following the same steps of estimating the expected total cost per cycle. Hence, the expected cycle length is

$$\int_0^{x_1} [t + (x_1 - t) + T_r] f(t)dt + \int_{x_1}^{x_2} [t + (x_2 - t) + T_r] f(t)dt + \ldots + \int_{x_j}^{x_{j+1}} [t + (x_{j+1} - t) + T_r] f(t)dt + \ldots$$

or

$$E_l = \mu + T_r + \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} (x_{k+1} - t)] f(t)dt, \tag{8.76}$$

where μ is the mean time to failure of the equipment.

Substituting Equations 8.75 and 8.76 into Equation 8.71, we obtain

$$c(x_1, x_2, x_3, \ldots) = \frac{c_r + \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} [c_i(k+1) + c_u(x_{k+1} - t)] f(t)dt}{\mu + T_r + \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} (x_{k+1} - t) f(t)dt}. \tag{8.77}$$

The optimal inspection schedule is obtained by taking the derivatives of Equation 8.77 with respect to x_1, x_2, x_3, . . . and equating the resulting equations to zero, and then solving the resulting equations simultaneously.

Following Brender (1962), Barlow et al. (1965), and Jardine (1973), we present the following procedure to determine the optimum inspection schedule.

Define a residual function as

$$R(L; x_1, x_2, x_3, \ldots) = LE_l - E_c, \tag{8.78}$$

where L represents either an initial estimation of the minimum cost $c(x_1, x_2, x_3, \ldots)$ or a value of $c(x_1, x_2, x_3, \ldots)$ obtained from a previous cycle of an iteration process. The schedule that minimizes $R(L; x_1, x_2, x_3, \ldots)$ is the same schedule that minimizes $c(x_1, x_2, x_3, \ldots)$. The following procedure determines $x_1, x_2, x_3, \ldots$

Step 1. Choose a value of L.

Step 2. Choose a value of x_1.

Step 3. Generate a schedule $x_1, x_2, x_3, \ldots$ using the following relationship:

$$x_{i+1} = x_i + \frac{F(x_i) - F(x_{i-1})}{f(x_i)} - \frac{c_i}{c_u - L}. \tag{8.79}$$

Step 4. Compute R using Equation 8.78.

Step 5. Repeat Steps 2 through 4 with different values of x_1 until $R_{\max}$ is obtained.

Step 6. Repeat Steps 1 through 5 with different values of L until $R_{\max} = 0$.

A procedure for adjusting L until it is identical with the minimum cost can be obtained from

$$c(L; x_1, x_2, x_3, \ldots) = L - \frac{R_{\max}}{E_l}.$$

EXAMPLE 8.11

The strain-gauge technique is used to measure the stress-strain fields in the pipes of chemical plants. The stress data measured by the strain-gauge technique are compared with the results of the thermomechanical and sectional flexibility analyses. If inconsistencies exist between the measured data and the results of the analyses, then further examinations of the pipes using ultrasonic or acoustic emission (AE) techniques are warranted. This, of course, will eliminate possible leaks in the pipes or ruptures of their walls.

The reliability engineer of this plant recommends an inspection policy where inspections of the strain-gauge's measurements are analyzed at times $x_1, x_2, x_3, \ldots$ until an inconsistency exists between the results of thermomechanical analysis and the data from the strain gauges. At that time, preventive maintenance or repair is performed on the pipes. The inspection cost per inspection is \$80, the cost per unit time of undetected failure or degradation of the pipe is \$9.0/h, and the cost of repair is \$2000. The time required to repair a failure is 90 h. The time to failure of the pipes follows a gamma distribution with the following p.d.f.:

$$f(t) = \frac{\alpha(\alpha k)^{k-1} \exp[-\alpha t]}{(k-1)!},$$

where $\alpha = 1/b$, b is a scale parameter. The mean of the distribution, $\mu = kb$. Assuming $k = 3$ and $\mu = 1000$ h, determine the first four points of the optimal schedule.

SOLUTION

By substituting the distribution parameters into the p.d.f. of the gamma distribution, we obtain

$$f(t) = \frac{12.65}{10^9} e^{\frac{-3}{1000}t}.$$

Using Brender's algorithm, we obtain the results shown in Table 8.7. No schedule exists for $x_1 < 20$. The optimum inspection schedule for the pipes is $x_1 = 20$, $x_2 = 41$, $x_3 = 64$, and $x_4 = 89$, and the corresponding cost is $95.50.

TABLE 8.7 Optimum Inspection Schedule

x_1	x_2	x_3	x_4	Total cost
100	216	356	530	1581.5
80	170	274	396	1083.5
60	125	198	280	648.5
40	82	128	178	312.5
20	41	64	89	95.5*

■

8.8.2 Periodic Inspection and Maintenance

Periodic inspection coupled with preventive maintenance is usually performed on critical components and systems such as airplane engines, standby power generators for hospitals, missiles and weaponry systems, and backup computers for banking and airline passenger reservation systems. In all these cases, inspection is periodically performed. When failures of the components are detected, the components are repaired or replaced with new ones. If the failed components are not detected during inspection and the system fails after the inspection is performed, a nondetection cost (or loss) will incur. Obviously, more frequent inspections will reduce the nondetection cost but will increase the cost of inspections. Therefore, an inspection schedule that minimizes the expected cost until detection of failure while minimizing the expected cost assuming renewal at detection of failure is desirable.

We consider a system which is periodically inspected to determine whether or not it requires repair (or replacement), and at the same time provides preventive maintenance if needed. Let us assume that after inspection, the unit (or component) has the same age as before with probability p and is as good as new with probability q (Nakagawa, 1984). We are interested in estimating the mean time to failure and the expected number of inspections before failure. We are also interested in estimating the total expected cost and the expected cost per unit time until detection of failure. Furthermore, we seek the optimum number of checks that minimize the expected cost.

Let us assume that the system to be inspected begins operating at time $t = 0$ and that inspection is performed at times kT ($k = 1, 2, \ldots$), where $T > 0$ is constant and previously determined. The CDF of the time to failure of the system is $F(t)$ with a finite mean μ. The failure of the system is detected only by inspection, and the time to perform inspection is negligible when compared with the length of time between two successive inspections. Following Nakagawa (1984), we estimate the mean time to failure of the system $\gamma(T, p)$ as

$$\gamma(T, p) = \sum_{j=1}^{\infty} \left\{ p^{j-1} \int_{(j-1)T}^{jT} t dF(t) + p^{j-1} q \bar{F}(jT)[jT + \gamma(T, p)] \right\}, \tag{8.80}$$

where $\bar{F}(t) = 1 - F(t)$. The first term in Equation 8.80 represents the mean time until the system fails between the $(j - 1)$th and the jth inspections. The second term represents the mean time until the system becomes new by the jth check; after that it fails. By solving and rearranging terms of Equation 8.80, we obtain

$$\gamma(T, p) = \frac{\sum_{j=0}^{\infty} p^j \int_{jT}^{(j+1)T} \bar{F}(t) dt}{\sum_{j=0}^{\infty} p^j \left\{ \bar{F}(jT) - \bar{F}[(j+1)T] \right\}}. \tag{8.81}$$

If $p = 0$, then the system is as good as new after each inspection and

$$\gamma(T, 0) = \frac{\int_0^T \bar{F}(t) dt}{F(t)}. \tag{8.82}$$

On the other hand, if $p = 1$, then the system has the same age as before inspection and

$$\gamma(T, 1) = \mu. \tag{8.83}$$

Following Equation 8.81, the expected number of inspections before failure $M(T, p)$ is obtained as follows:

$$M(t, p) = \frac{\sum_{j=0}^{\infty} p^j \bar{F}[(j+1)T]}{\sum_{j=0}^{\infty} p^j \left\{ \bar{F}(jT) - \bar{F}[(j+1)T] \right\}}. \tag{8.84}$$

When $p = 0$, then

$$M(T, 0) = \frac{\bar{F}(T)}{F(T)}. \tag{8.85}$$

When $p = 1$, then

$$M(T, 1) = \sum_{j=0}^{\infty} \bar{F}[(j+1)T]. \tag{8.86}$$

Assume that c_1 is the cost of each inspection and c_2 is the cost of nondetection of failure—that is, the cost associated with the elapsed time between failure and its detection per unit time. The total expected cost until a failure is detected $c(T; p)$ can be expressed as

$$c(T; p) = (c_1 + c_2T)[M(T, p) + 1] - c_2\gamma(T, p). \tag{8.87}$$

Substituting Equations 8.81 and 8.84 into Equation 8.87 results in

$$c(T; p) = \frac{(c_1 + c_2T)\left\{\sum_{j=0}^{\infty} p^j \bar{F}[(j+1)T] + \sum_{j=0}^{\infty} p^j \bar{F}(jT) - \sum_{j=0}^{\infty} p^j \bar{F}(j+1)T\right\}}{\sum_{j=0}^{\infty} p^j \left\{\bar{F}(jT) - \bar{F}[(j+1)T]\right\}} - \frac{c_2 \sum_{j=0}^{\infty} p^j \int_{jT}^{(j+1)T} \bar{F}(t)dt}{\sum_{j=0}^{\infty} p^j \left\{\bar{F}(jT) - \bar{F}[(j+1)T]\right\}}$$

or

$$c(T; p) = \frac{(c_1 + c_2T)\sum_{j=0}^{\infty} p^j \bar{F}(jT) - c_2 \sum_{j=0}^{\infty} p^j \int_{jT}^{(j+1)T} \bar{F}(t)dt}{\sum_{j=0}^{\infty} p^j \left\{\bar{F}(jT) - \bar{F}[(j+1)T]\right\}}. \tag{8.88}$$

From Equation 8.88, $\lim_{T\to 0} c(T; p) = \lim_{T\to\infty} c(T; p) = \infty$, which implies that there exists a finite optimal value T^* that minimizes the total expected cost $c(T; p)$. Nakagawa (1984) shows that

$$M(T, p) \le \frac{\gamma(T, p)}{T} \le [1 + M(T, p)].$$

Consequently,

$$c_1 \frac{\gamma(T, p)}{T} \le c(T, p) \le c_1[1 + M(T, p)] + c_2T. \tag{8.89}$$

It may be of interest to seek the optimum inspection interval that minimizes the expected cost per unit time until detection of failure; $c_d(T, p)$. In this case, we follow the same derivation as that of Equation 8.87 to obtain

$$c_d(T; p) = \frac{(c_1 + c_2 T)[M(T, p) + 1] - c_2 \gamma(T, p)}{T[M(T, p) + 1]}$$

or

$$c_d(T; p) = \frac{c_1}{T} + c_2 \left\{ 1 - \frac{\sum_{j=0}^{\infty} p^j \int_{jT}^{(j+1)T} \bar{F}(t)dt}{T \sum_{j=0}^{\infty} p^j \bar{F}(jT)} \right\}. \tag{8.90}$$

The bounds of c_d $(T; p)$ are $\lim_{T \to 0} c_d(T; p) = \infty$ and $\lim_{T \to \infty} c_d(T; p) = c_2$. From Equation 8.90 and for a given value of T, the expected cost $c_d(T; p)$ is an increasing function of p for IFR. Therefore,

$$\frac{c_1 + c_2 \int_0^T F(t)dt}{T} \le c_d(T; p) \le \frac{(c_1 + c_2 T) \sum_{j=0}^{\infty} \bar{F}(jT) - c_2 \mu}{T \sum_{j=0}^{\infty} \bar{F}(jT)}. \tag{8.91}$$

EXAMPLE 8.12

In a copper mining operation, the copper ore bodies are mined about a half-mile below the surface. The mining operation consists of drilling, blasting, and transportation processes. The diesel loaders are considered an essential part of the mining operation since they carry both drilling equipment and the exploded ore. Because of the high cost of lost production, the loaders are subject to an extensive inspection program; more specifically, the wheel axles of the loaders are inspected for possible cracks. The failure time of the axles is exponential with a failure rate of 0.000,5 failures per hour. The cost per inspection is \$120, and the cost of an undetected failure is \$80. Determine the following:

1. The optimum inspection interval that minimizes the total expected cost until a failure is detected.
2. The optimum inspection interval that minimizes the expected cost per unit time until detection of failure.

SOLUTION

Since the failure time is exponentially distributed, $[F(t) = 1 - e^{-\lambda t}]$, then for any value of p, Equation 8.88 reduces to

$$c_d(T;\, p) = \frac{c_1 + c_2 T}{1 - e^{-\lambda T}} - \frac{c_2}{\lambda}, \tag{8.92}$$

and Equation 8.90 reduces to

$$c_d(T;\, p) = \frac{c_1 + c_2 T - (c_2/\lambda)(1 - e^{-\lambda T})}{T}. \tag{8.93}$$

1. The optimum inspection interval that minimizes the total expected cost until a detected failure is the value of T^* that minimizes Equation 8.92. In other words,

$$c(T^*;\, p) = \frac{120 + 80T^*}{1 - e^{-0.0005T^*}} - \frac{80}{0.0005}$$

   and $T^* = 78$ h.

2. The optimum inspection interval that minimizes the expected cost per unit time until detection of failure is obtained using Equation 8.93 as given below

$$c_d(T^*;\, p) = \frac{120 + 80T^* - \left(\dfrac{80}{0.0005}\right)(1 - e^{-0.0005T^*})}{T^*}$$

   or

$$T^* = 77 \text{ h.}$$

In other words, the axles should be inspected periodically at fixed intervals of 77 h. ■

8.9 CONDITION-BASED MAINTENANCE

As shown earlier in this chapter, maintenance policies can be classified as corrective maintenance (CM), preventive maintenance (PM), and condition-based maintenance (CBM). The corrective maintenance is performed when failures occur or when the degradation level of the system reaches an unacceptable level. The preventive maintenance is performed at predetermined time intervals which are estimated based on historical data, failure-time distributions of the systems, and economic or availability models. The CBM is performed when an indicator of the condition of the system reaches a predetermined level. We presented models that determine the optimum preventive maintenance schedule under different criteria such as minimization of the cost per unit time, maximization of the system availability, and others. In general, these two policies depend on several factors including the failure rate of the system, the cost associated with downtime, the cost of repair, the expected life of the system, and the desired availability level. For example, a maintenance policy which requires no repairs, replacements, or preventive maintenance until failure, allows for maximum run-time between repairs. Although it allows for maximum run-time between repairs, it is neither economical nor efficient as it may result in a catastrophic failure that requires extensive repair time and cost. Another widely used maintenance policy is to maintain the system (equipment, unit) according to a predetermined schedule, whether a problem is apparent or not. On a scheduled basis, equipment are removed from operation, disassembled, inspected for defective parts, and repaired accordingly. Actual repair costs can be reduced in this manner, but production loss may increase if the equipment is complex and requires days or even weeks to maintain. This preventive maintenance may also create equipment problems where none existed before Liao et al. (2006).

Obviously, if an equipment failure can be predicted and the equipment can be taken offline to make only the necessary repairs, a tremendous cost saving can be achieved. Predictive maintenance can also be done when failure modes for the equipment can be identified and monitored for increased intensity and when the equipment can be shut down at a fixed control limit before critical fault levels are reached (Jeong and Elsayed, 2000). This is the underlying principle of CBM. Predicting potential failure of a component or a system is achieved by continuous monitoring of a degradation indicator of the system status. These indicators include crack length in components subject to fatigue loading, acoustic signal for corrosion of pipes and bridge beams, particle count in oil lubricants, and change in pressure in pneumatically operating units. The indicator could provide direct or indirect measurements of the system's status. These measurements are then used to describe the degradation path with time. Careful modeling of the degradation process will enable the user to determine an optimum threshold level of the degradation level that minimizes the cost per unit time or maximizes the system's availability. Low levels of the threshold result in more frequent maintenance (higher cost and less availability) whereas high levels will result in potential failure of the system before it reaches the threshold which incurs high repair cost.

The recent advances in sensors technology, chemical and physical nondestructive testing (NDT), and sophisticated measurement techniques, information processing, wireless communications, and Internet capabilities, have significantly impacted the CBM approach by providing dynamic maintenance schedules that minimize the cost, downtime, and increase system availability. More importantly, the sensors provide indicators about the system's operating conditions and potential failures. In addition, sensors for monitoring the equipment eliminate the time for diagnostics thus reducing the time to perform the actual repair. A major international elevator company is using this approach to remotely monitor the braking system of elevators in high-rise buildings; when the deceleration of the elevator reaches a specific value, action is taken immediately to repair or replace the braking system. Likewise, the conditions of aircraft engines are continuously monitored by the operating companies during flying (messages are referred to as AOC or Aircraft Operational Communication), and the aircraft crew is provided with decisions, through ground stations, to change destination and proceed to another destination where spare parts and repairs can be performed if these specific maintenance actions are not available at the original destination. Furthermore, recent inspection technologies that require no human entry into underground structures have been developed; they are now fully automated, from data acquisition to data analysis, and eventually to condition assessment, which can be used during the manufacturing as well as maintenance actions (Kumar et al., 2005). Other motivational examples for monitoring degradation of the system and performing maintenance actions when the degradation threshold reaches a specified value is the degradation of semiconductor lasers during operation which is characterized by an increase in the threshold current, accompanied by decrease in external differential quantum efficiency. This in turn promotes defect formation in the active region of the laser (Ng, 2006).

In general, CBM requires three main tasks: (1) determining the condition indicator which can describe the condition of the unit. As stated above, a condition indicator could be a characteristic such as corrosion rate, crack growth, wear and lubricant condition such as its viscosity; (2) monitoring the condition indicator and assessing the condition of the unit using the collected measurements, and (3) determining the limit value of the condition indicator and its two components: the alarm limit and the failure or breakdown limit. Determination of these limits as well as the maintenance and/or inspection strategies that optimize one or more criteria such as cost or system availability have been the subject of investigation.

In Chapter 6, we presented methodologies for modeling the degradation path which can be used to determine the level crossing of the degradation threshold level. In the following section, we present approaches for monitoring the degradation indicator.

8.10 ONLINE SURVEILLANCE AND MONITORING

Data needed to diagnose the condition of equipment or a system include noise level, speed, flow rate, temperature, differential expansion, vibration, position, accuracy, repeatability, and others. Majority of sensors and monitoring devices are based on vibration, acoustic, electrical, hydraulic, pneumatic, corrosion, wear, vision, and motion patterns. In this section, we briefly discuss some of the most commonly used diagnostic systems for component and/or system monitoring.

8.10.1 Vibration Analysis

Machines or equipment produce vibration when in operation. Each machine has a characteristic vibration or "signature" composed of a large number of harmonic vibrations of different amplitudes. The effects of component wear and failure on these harmonic vibrations differ widely depending on the contribution made by a particular component to the overall "signature" of the machine. For example, in reciprocating engines and compressors, the major force is produced by harmonic gas forcing torques, which are functions of the thermodynamic cycle on which the machine operates. In a multicylinder engine, the major harmonic vibrations are calculated from the number of working strokes per revolution. Thus, misfiring of one or more cylinders would produce significantly different vibrations than the original "signature" of the engine, which can be easily detected by an accelerometer. The accelerometer is an electromechanical transducer that produces an electric output proportional to the vibratory acceleration to which it is exposed (Niebel, 1994).

The vibration "signature" of equipment is dependent on the frequency, amplitude, velocity, and acceleration or wave slope of the vibration wave. In general, there is no single transducer that is capable of the extreme wide range of signatures. Therefore, transducers are developed for different frequency, amplitude, velocity, and acceleration ranges. For example, the bearing probe, a displacement measuring device, is sensitive only to low-frequency large-amplitude vibrations. This makes it only useful as a vibration indicator for gear box vibrations and turbine blades.

On the other hand, seismic accelerometers can be used for monitoring vibrations of electrical machines that are characterized by usual bearing and balance vibrations, in addition to high frequency vibrations which are functions of the electrical geometry via number of starter and rotor poles (Downham, 1975).

Degradation due to vibration can be assessed by analyzing the vibration time series (vibration signal). One of the most commonly used method for monitoring the vibration signal is its root mean square (RMS) average or peak levels. These values are then plotted against time to observe the degradation trend. This trend can be observed using statistical process control charts such as cumulative sum (CUSUM) and the exponentially weighted moving average (EWMA) chart which are sensitive to small changes in the time series. The root mean square is defined as

$$\mathrm{RMS} = \sqrt{\frac{1}{N}\sum_{k=1}^{N} x_k^2}, \tag{8.94}$$

where N is the total number of observations of the time series window. The window size should not be too small or too long in order to detect the changes in the RMS values. Choice of the window size is discussed in Fahmy and Elsayed (2006a, 2006b). The crest factor (CF) is an immediate extension of RMS for monitoring degradation due to vibration. It is the ratio of the peak to peak value signal to its RMS value and is expressed as

$$CF = \frac{peak\ to\ peak\ value}{RMS}. \tag{8.95}$$

CF is much more sensitive than RMS value alone to changes in the spiky nature of a vibration signal (Williams et al., 1994).

Kurtosis is another commonly used method for degradation due to vibration. The kurtosis (K) of a random variable x is the fourth standard moment of the random variable and is defined as

$$K = \int_{-\infty}^{\infty} \frac{(x-\overline{x})^4 f(x)}{\sigma^4} dx \tag{8.96}$$

where:

$f(x)$ is the p.d.f. of x
σ is the standard deviation of x
$\bar{x}$ is the average of x.

Since the observed vibration signals (usually in volt, depending on the configuration of the data acquisition system) are not continuous, we may estimate K for a sample size N as

$$K = \frac{1}{N\sigma^4}\sum_{k=1}^{N}(x_k - \bar{x})^4. \tag{8.97}$$

It is important to note that the standard normal distribution has $K = 3$; therefore, another definition of the kurtosis (named excess kurtosis) is used as an alternative. It is expressed as

$$K = \frac{1}{N\sigma^4}\sum_{k=1}^{N}(x_k - \bar{x})^4 - 3. \tag{8.98}$$

Use of Equation 8.97 or Equation 8.98 does not affect the main objective of the kurtosis. Its value increases as the damage of the unit increases, and positive excess kurtosis means that distribution has fatter tails than a normal distribution. Fat tails means there is a higher than normal probability of large positive and negative signals. When calculating the excess kurtosis, a result of +3.00 indicates the absence of damage. It is interesting to note that practical experience shows that K is insensitive to changes in machine speed, loading, and geometry, and is least affected by temperature. This explains why it is commonly used as an indicator for damage due to vibration.

8.10.2 Acoustic Emission and Sound Recognition

Acoustic emission (AE) can be defined as the transient elastic energy spontaneously released from materials undergoing deformation, fracture, or both. The released energy produces high-frequency acoustic signals. The strength of the signals depends on parameters such as the rate of deformation, the volume of the participating material, and the magnitude of the applied stress. The signals can be detected by sensors often placed several feet away from the source of signal generation.

Most of the AE sensors are broadband or resonant piezoelectric devices. Optical transducers for AE are in the very early stage of development. They have the advantages of being used as contacting and noncontacting measurement probes and of the flat frequency response over a large bandwidth. Acoustic emission is used in many applications such as tool wear monitoring, material fatigue, and weld defects.

Sound recognition is used to detect a wide range of abnormal occurrences in manufacturing processes. The sound recognition system recognizes various operational sounds, including stationary and shock sounds, using a speech recognition technique, then compares them with the expected normal operational sounds (Takata and Ahn, 1987).

The operational sound is collected by a unidirectional condenser microphone that is set near the component or machine to be monitored. When the sound of an abnormal operation is generated, such as the sound of tool breakage or the sound of worn-out motor bearings, the features of the sound signal are extracted, and a sound pattern is formed. The sound pattern is then compared with the standard patterns through pattern matching techniques, and the most similar standard pattern is selected. The failure or fault corresponding to this category is then recognized and diagnosed.

8.10.3 Temperature Monitoring

Elevation in component or equipment temperature is frequently an indication of potential problems. For example, most of the failures of electric motors are attributed to excessive heat that is generated by antifriction bearings. The bearing life is dependent on its preventive maintenance schedule and their operating conditions. Similarly, hot spots in electric boards indicate that failure is imminent. The hot spots are usually caused by excessive currents.

Therefore, a measure of temperature variation can be effectively used in monitoring components and equipment for preventive maintenance purposes. There is a wide range of instruments for measuring variations in temperature—such as mercury thermometers, which are capable of measuring temperatures in the range of −35 to 900°F, and thermocouples, which can provide accurate measurements up to about 1400°F. Optical pyrometers, where the intensity of the radiation is compared optically with a heated filament, are useful for the measurements of very high temperatures (1000–5000°F).

Recent advances in computers made the use of the infrared temperature measure possible for many applications that are difficult and impractical to contact with other instruments (Niebel, 1994).

The infrared emissions are the shortest wavelengths of all radiant energy and are visible with special instrumentation. Clearly, the intensity of the infrared emission from an object is a function of its surface temperature. Therefore, when a sensing head is aimed at the object whose surface temperature is being measured, the computer calculates the surface temperature and provides a color graphic display of temperature distribution. This instrument is practical and useful in monitoring temperature of controllers and detecting heat loss in pipes.

8.10.4 Fluid Monitoring

Analysis of equipment fluids such as oil can reveal important information about the equipment wear and performance. It can also be used to predict the reliability and expected remaining life of parts of the equipment. As the equipment operates, minute particles of metal are produced from the oil-covered parts. The particles remain in suspension in the oil and are not removed by the oil filters due to their small size. The particle count will increase as equipment parts wear out. There are several methods that can identify the particle count and the types of particles in the oil. The two most commonly used methods are *atomic absorption* and *spectrographic emission*.

With the atomic absorption method, a small sample of oil is burnt and the flame is analyzed through a light source that is particular for each element. The method is very accurate and can obtain a particle count as low as 0.1 parts per million (ppm). However, the analysis is tedious and time-consuming except when the type of particle is known (Cumming, 1990).

Spectrographic emission is similar to the atomic absorption method in burning a small sample of oil. It has the advantage that all quantities of all the materials can be read at one burn. However, it is only capable of detecting particle counts of 1 ppm or higher. Moreover, spectrometry is unable to give adequate warning in situations when the failure mode is characterized by the generation of large particles from rapidly deteriorating surfaces (Eisentraut et al., 1978).

8.10.5 Corrosion Monitoring

Corrosion is a degradation mechanism of many metallic components. Clearly, monitoring the rate of degradation—the amount of corrosion—has a major impact on the preventive maintenance schedule and the availability of the system. There are many techniques for monitoring corrosion such as visual, ultrasonic thickness monitoring, electrochemical noise, impedance measurements, and thin layer activation (TLA). We briefly describe one of the most effective online corrosion monitoring techniques—TLA. The principle of TLA is that trace quantities (1 in 10^{10}) of a radioisotope are generated in a thin surface layer of the component under study by an incident high-energy ion beam. Loss of the material (due to corrosion) from the surface of the component can be readily detected by a simple γ-ray monitor (Asher et al., 1983). The reduction in activity is converted to give a depth of corrosion directly, and, provided that the corrosion is not highly localized, this gives a reliable measurement of the average loss of material over the surface.

8.10.6 Other Diagnostic Methods

Components and systems can be monitored in order to perform maintenance and replacements by observing some of the critical characteristics using a variety of sensors or microsensors. For example, pneumatic and hydraulic systems can be monitored by observing pressure, density of the flow, rate of flow, and temperature change. Similarly, electrical components or systems can be monitored by observing the change in resistance, capacitance, volt, current, temperature, and magnetic field intensity. Mechanical components and systems can be monitored by measuring velocities, stress, angular movements, shock impulse, temperature, and force.

Recent technological advances in measurements and sensors resulted in observing characteristics that were difficult or impossible to observe, such as odor sensing. At this point of time, silicon microsensors have been developed that are capable of mimicking the human sense of sight (such as a charge coupled device [CCD]), touch (such as a tactile sensor array), and hearing (such as silicon microphone). Sensors to mimic the human sense of smell to discriminate between different odor types or notes are at the early stage of development.

Nevertheless, some commercial odor discriminating sensors are now available such as the Fox 2000 or Intelligent Nose (Alpha MOS, France). The instrument is based upon an array of six sintered metal oxide gas sensors that respond to a wide range of odorants. The array signals are processed using an artificial neural network (ANN) technique. The electronic nose is first trained on known odors; then the ANN can predict the nature of the unknown odors with a high success rate (Gardner, 1994).

The improvements in sensors' accuracy and the significant reduction in their cost have resulted in their use in a wide variety of applications. For example, most of the automobiles are now equipped with electronic diagnostic systems that provide signals indicating the times to service the engine, replace the oil filter, and check engine fluids.

Most importantly, the advances in microcomputers, microprocessors, and sensors can now offer significant benefits to the area of preventive maintenance and replacements. Many components, systems, and entire plants can now be continuously monitored for sources of disturbances and potential failures. Moreover, online measurements, analysis, and control of properties and characteristics, which have been traditionally performed offline, result in monitoring of a wider range of components and systems than ever before.

PROBLEMS

8.1 In a block replacement policy, the cost of a failure replacement is \$150 while the cost of a preventive replacement is \$80. Assume that the failure times of a component that is replaced, based on the block replacement policy, follow a beta distribution. The parameters of the distribution are $\alpha = 4$ and $\beta = 3$ h. The p.d.f. of the beta distribution is

$$f(t) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1}, \quad 0 < t < 1.$$

a. What is the optimal replacement interval?

b. Assume that the penalty factor for surplus or shortage inventory (amount of stock that meets the expected failures during the replacement interval) is \$20. Assuming that there are 200 units in operation, what are the optimal interval T and stock level L?

8.2 Consider a block replacement policy, where the failing unit is replaced by a new one at time instants $t = KT$, $K = 1, 2, \ldots$ and at failure. The cost of replacing a failed unit is \$80 and the cost of replacing nonfailed units is \$50. The penalty for both shortage or excess spares in the inventory is \$25 per unit. There are 100 units in operation at the beginning of a replacement cycle, and the failure density function is given by

$$f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} t^{\alpha-1} e^{-t/\beta} & t > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where α and β are parameters that determine the specific shape of the curve. For $\alpha = 2$ and $\beta = 400$.

a. Determine the optimal replacement period, and

b. Determine the optimal inventory level of the spares.

8.3 Consider a replacement policy where the time at which preventive replacement occurs depends on the age of the equipment; failure replacements are made when failures occur. Let t_p be the age of the equipment at which a preventive replacement is made, c_p the cost of a preventive replacement, c_f the cost of a failure replacement. It is given that $c_p = \$50$, $c_f = \$100$, $\lambda = 1$, and the p.d.f. of the failure-time distribution is given by the special Erlang function

$$f(t) = \frac{t}{\lambda^2}\exp(-t/\lambda) \quad t \geq 0.$$

a. What is the optimal replacement interval?

b. Repeat part (a) for $f(t) = \lambda e^{-\lambda t}$.

8.4 A typical twin-turboprop transport aircraft has an average gross landing weight of 40,000 lb and a tricycle landing gear. The main landing gear is equipped with two wheels on each side. It is the principal support for the aircraft and has many components including air/oil shock struts to absorb landing impact and taxing loads, alignment and support units, retraction mechanisms and safety devices, auxiliary gear protective devices, wheels, tires, tubes, and braking systems.

The landing gear system (main nosewheel, left landing gear, right landing gear) is periodically inspected, and preventive maintenance or replacements are carried out. The cost of preventive maintenance or replacement is \$5000, whereas the cost of a failure replacement depends on where the failure occurs. The failure of either the left or right landing gear may result in right- or left-wing tipping, which causes the propeller blades of the engines and the lower portion of the rear fuselage to scrape along the runway, resulting in a substantial damage of \$25,000. The failure of nosewheel landing gear results in a damage worth \$45,000. The landing gears have equal probabilities of failure. The time to failure follows a normal distribution with mean of 500 aircraft landings and standard deviation of 20 landings. Determine the optimum preventive constant replacement intervals.

8.5 Assume that the CIRP in Problem 8.4 is to be compared with an ARP using the same cost values. Determine the optimum preventive replacement interval for the ARP.

8.6 Transport aircraft with a steel piston engine crankshaft may fail catastrophically during flight if the crankshaft fails. The massive and complex-shaped crankshaft is usually produced by forging a triple alloy steel. During flight, the crankshaft experiences significant and complex stresses including bending and torsion. It is periodically checked for crack indications and defects using the magnetic particle nondestructive test method. If cracks or defects are found, the crankshaft is repaired or replaced. The time to perform preventive replacement, T_p, is 20 h, and the time to perform failure replacement is 50 h. Assume that failure times of the crankshaft follow a Weibull distribution having a p.d.f. as

$$f(t) = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1} e^{-\left(\frac{t}{\theta}\right)^{\gamma}} \quad t > 0.$$

The shape parameter γ is found to be 2.7, and scale parameter θ is 250. Determine the optimum preventive replacement age t_p that minimizes the downtime per unit time when the following preventive replacement policy is implemented: perform preventive replacement or when the equipment reaches age t_p.

8.7 In the optimal replacement policy under minimal repair (Lam, 1990), a critical component is observed. When the component fails, it is replaced or is minimally repaired at a cost of \$1200. After N minimal repairs (or replacements) of the component, the entire system is replaced by a new one at a cost of \$42,000. Assume that the reward rate is \$100/h and the operating time λ_k and the repair time μ_k after the kth failure are

$$\lambda_k = \frac{9000}{3^{k-1}}$$
$$\mu_k = 50 \times 3^{k-1}.$$

Determine the parameters of the optimum replacement policy N^*.

8.8 Consider a periodic replacement policy where the component is subject to shock. Assume that the normal cost of running the system is a per unit of time, and that each shock to the system increases its running cost by c per unit of time and the cost of completely replacing the system is c_0. Prove that, when the shock rate is

$$\lambda(t) = \begin{cases} 0, & t < 1 \\ \frac{1}{t} & t \geq 1, \end{cases}$$

the optimal value of the period replacement time is

$$T^* = \sqrt{\frac{2c_0}{c} + 1}.$$

8.9 Determine the optimal preventive maintenance interval and the optimal number of spares when the penalty function g is expressed as

$$g(L, N_1(t_p)) = \left| L - (N_1(t_p) + 1) \right|.$$

8.10 Consider a group maintenance policy for N machines each having a failure rate λ. Assume that $F(t)$ is continuous, $f(t)$ exists, and $-f'(t)/f(t) < c_2/c_1$ for $t \geq 0$. Prove that if $c_2/\lambda > c_0/N + c_1$, then there exists a unique and finite optimum scheduling time t_0^* that satisfies Equation 8.63.

8.11 A group of production machines consists of N machines. Each machine exhibits the same failure-time distribution with the following p.d.f.

$$f(t) = kte^{\frac{-kt^2}{2}},$$

where $k = 0.01$. Assume that $c_0 = \$300$, $c_1 = \$150$, and $c_2 = \$250$. Determine the optimum group replacement interval.

8.12 Consider an inspection policy where inspections of a power generator unit are performed at times x_1, x_2, x_3, and x_4. The power generator unit exhibits a failure-time distribution with a p.d.f. given by

$$f(t) = \frac{\alpha(\alpha k)^{k-1} \exp(-\alpha t)}{(k-1)!},$$

where $\alpha = 1/b$, b is a scale parameter. The mean of the distribution, $\mu = kb$. Assume $k = 3$, $\mu = 1000$ h, cost of inspection = \$150, cost of undetected failure $c_u = \$300$, and the cost of repair is $c_r = \$2000$. Determine the first four points of the optimal inspection schedule.

8.13 An aircraft maintenance group uses a thermographic detection method to detect corrosion over a large surface area of the aircraft. The detection method uses a noncontact device. The inspection involves heating the surface with flash or quartz lamps and then measuring the temperature with an infrared camera over a set time span. Heating the surfaces creates temperature differences that indicate disbonds or corrosion.

The aircraft is periodically inspected to determine whether or not corrosion is formed. The corroded components or surfaces are repaired or replaced, and at the same time, preventive maintenance is provided if needed. We assume that after inspection the component (or surface) has the same age as before with probability p and that the component is as good as new with probability q. The times to the formation of corrosion follow a log logistic distribution with the following p.d.f.

$$f(t) = \frac{\lambda k(\lambda t)^{k-1}}{[1+(\lambda t)^k]^2} \quad 0 \le t < \infty,$$

where $k = 2$ and $\lambda = 0.008$ failures per hour. The cost per inspection is \$2500 and the cost of undetected corrosion is \$500. Determine the following:

a. The optimum inspection interval that minimizes the total expected cost per unit time.

b. The optimum inspection interval that minimizes the expected cost per unit time until detection of failure.

c. Solve (a) and (b) if the true cost of an undetected failure is \$3000.

8.14 Leaf springs are attached to the undercarriage assemblies of trains in order to provide a smooth ride to the passengers. The repeated loads on a leaf spring result in subjecting both surfaces of the spring to cycles of tension and compression stress. Such repeated loads coupled with crack initiation and shocks may result in the spring failure.

Consider a leaf spring whose $\lambda(t) = \lambda t$ with $\lambda = 10 \times 10^{-5}$. Assume that the normal cost of running the system is \$50/h, the increase in running cost due to each shock is \$6/h, and the cost of replacing the entire spring system is \$20,000. Determine the optimal value of T (the length of the periodic replacement policy) that minimizes the long-run average cost per unit time.

8.15 Assume that the replacement cost of the leaf springs in Problem 8.14 is a function of time (or number of shocks). In other words, the operating cost per unit time due to every additional shock in the interval $[\tau_i, \tau_{i+1})$ is $c_i(u)$, $i = 0, 1, \ldots,$ and u is the state space at which the periodic replacement can be performed. Let $c_{N(t)}$ represent the additional operating cost per unit as a function of the number of shocks at time t; $N(t)$

$$c_{N(t)} = 5.0 + 2N(t) + 1.5(N(t))^2.$$

Determine the optimum replacement interval T^*.

8.16 Consider a maintenance policy in which a component (or a system) is minimally repaired at equal intervals of time. The minimal repairs involve adjustments, cleaning, and replacement of nonessential parts. After a minimum repair, the component (or system) has the same age as before the repair. Moreover, the component is replaced by a new one upon failure and when the ratio between the failure rates $r + 1$ minimal repairs and r is greater than $\phi(\phi > 1)$. Assume that the hazard rate of the component is given by

$$h(t) = \delta k^t,$$

where δ and k are positive constants. The cost per minimal repair is c_m, the cost of the scheduled replacement is c_r, and the cost due to the failure replacement is c_f. It should be noted that $c_f > c_r > c_m$.

a. Determine the optimum minimal repair interval and

b. Determine the optimum replacement time that minimizes the total expected cost per unit time.

8.17 Define, e_π, the effectiveness of a preventive maintenance or replacement policy π, as the ratio between the expected number of failures avoidable by the implementation of the policy and the total expected number of failures under the failure replacement policy (FRP). Under the FRP, components are only replaced upon failure (Al-Najjar, 1991). Derive e_π for the policy stated in Problem 8.16.

8.18 The efficiency of a preventive maintenance or replacement policy, η, can be measured as the ratio between the total expected cycle cost when the policy is in effect and the total expected cycle cost when replacements are made only when failures occur. Derive an expression for η for the policy stated in Problem 8.16.

8.19 The kurtosis method is a statistical means of studying the time domain signal generated due to vibrations generated from running a machine. The principle of the kurtosis method is to take observations in a suitable frequency range. The kurtosis of the signal is calculated as

$$K = \frac{1}{\sigma^4} \sum_{i=0}^{N} \frac{(x_i - \overline{x})^4}{N},$$

where σ^2 is the variance, N is the number of observations, $\overline{x}$ is the mean value of the observations, and x_i is the observed value i. The use of K to monitor the condition of rotating machinery is based on the fact that a rolling bearing in normal operating conditions exhibits a kurtosis value of about 3. However, the value of K increases rapidly as the wear of the parts increases.

Reliability engineers use the kurtosis method to determine whether or not to perform preventive maintenance or replacement of the sliding bearings of high-pressure presses. In other words, the kurtosis method is used as an inspection tool. Under this policy, inspections are performed at times x_1, x_2, x_3, . . . until a failed or degraded bearing is detected. Repairs are immediately performed upon the detection of a failure or degradation. The inspection intervals are not necessarily equal as they may be reduced as the probability of failure increases.

In order to accurately predict the optimal inspection intervals, observations from the last 11 inspection intervals are shown in Table 8.8.

a. Using the information in Tables 8.8 and 8.9, obtain the p.d.f. of the failure times

b. Given the cost per inspection is \$200, cost of undetected failure is \$20, and the cost of repair is \$1800. The repair time is 30 h, Determine the first three optimum inspection intervals that minimize the expected total cost per unit time.

8.20 Exhaust gases from paper plants usually contain fine particles that are removed by electrostatic precipitators. A typical precipitator contains thin wires that are charged to several thousands of volts. When the gases pass through the wires, the fine particles are attracted to the wires and to the dust-collector plates. The wires are vibrated periodically to remove the particles, which are then concentrated in receptacles, collected, and disposed of. Since the wires are subject to thermal stresses, they may experience breaks that reduce their effectiveness in removing the particles. Therefore, a periodic inspection policy is implemented. Under this policy, the wires are inspected at times x_1, x_2, x_3, x_4, . . . until a broken wire is detected. Repairs are immediately performed when breaks are found.

The cost per inspection is \$150; the cost of undetected failure is \$12; and the cost of repair is \$225. The repair time is 12 h. The p.d.f. of the wire breaks follow

$$f(t) = kte^{\frac{-kt^2}{2}},$$

where $k = 0.000,04$. Determine the first five inspection intervals that minimize the total cost.

TABLE 8.8 Observations from 11 Inspection Intervals

Obs. A	Obs. B	Obs. C	Obs. D	Obs. E	Obs. F	Obs. G	Obs. H	Obs. I	Obs. J	Obs. K
830	850	830	835	855	860	855	860	864	860	834
853	863	853	853	863	873	873	873	874	874	844
880	870	885	886	88	880	886	883	887	884	857
892	882	892	893	893	892	893	895	898	896	878
980	900	984	985	985	990	995	996	997	997	987
999	932	995	996	996	992	997	997	998	997	998
999	936	999	999	999	1000	999	1002	999	1004	999
1000	953	10	1001	1001	1001	1011	1003	1001	1002	1001
1002	969	1002	1002	1002	1002	1002	1004	1012	1005	1012
1004	972	1005	1004	1003	1003	1003	1005	1013	1005	1023
1005	996	1007	1007	1006	1004	1016	1006	1015	1006	1025
1010	1005	1010	1011	1015	1005	1015	1007	1016	1008	1016
1019	1009	1019	1018	1018	1009	1018	1009	1017	1009	1017
1023	1013	1025	1023	1022	1013	1032	1012	1021	1011	1021
1028	1018	1028	1026	1023	1016	1023	1015	1024	1013	1034
1035	1025	1035	1038	1034	1027	1025	1025	1025	1023	1035
1036	1026	1036	1034	1035	1029	1036	1029	1026	1025	1026
1044	1034	1043	1042	1042	1034	1044	1035	1031	1032	1041
1054	1034	1054	1050	1051	1039	1055	1038	1054	1036	1054
1064	1044	1064	1064	1064	1044	1044	1046	1044	1048	1064
1078	1058	1078	1078	1075	1053	1065	1056	1064	1055	1074
1099	1069	1099	1099	1094	1060	1074	1061	1075	1063	1095
1106	1076	1106	1106	1104	1066	1100	1064	1098	1067	1098
1115	1089	1115	1115	1116	1069	1110	1065	1100	1066	1110
1135	1099	1135	1135	1125	1079	1115	1076	1105	1073	1115

TABLE 8.9 Times at Which the Observations Are Taken

Obs. A	Obs. B	Obs. C	Obs. D	Obs. E	Obs. F	Obs. G	Obs. H	Obs. I	Obs. J	Obs. K
58	60	61	81	82	94	107	123	127	134	146

8.21 The main cause of failure of cell phones is the cumulative damage due to shock and vibration. A test is conducted by subjecting a cell phone to multiple drops and recording the acceleration against frequency (we normalize it to be 1, 2, 3 . . .). The following accelerations versus frequency are recorded for five drops of the cell phone from the same height of 10 ft. When the acceleration level reaches 3 g, the cell phone is removed from the test and critical components are replaced. This is in effect a CBM policy. Analyze the data in Table 8.10 and determine when such maintenance needs to be performed.

TABLE 8.10 Accelerations from Five Drops

Drop 1	Drop 2	Drop 3	Drop 4	Drop 5
2.333	2.533	3.444	4.666	4.987
3.333	4.225	4.321	4.335	4.887
3.667	3.876	3.254	3.888	4.654
2.667	5.555	4.256	3.989	4.765
4.000	6.555	4.126	3.778	3.997
5.000	4.556	5.123	3.876	3.765
2.667	3.899	6.211	4.11	3.456
3.000	2.999	5.888	4.118	5.789
1.667	3.675	5.998	4.228	5.987
3.333	4.123	5.997	4.666	5.985
4.000	4.235	6.012	4.887	5.765
4.667	4.568	6.213	4.987	6.126
4.333	4.897	6.333	5.998	6.125
2.000	3.889	4.999	5.989	6.235
2.667	3.976	5.889	6.213	6.435
6.333	4.768	7.465	6.342	6.667
5.667	5.991	6.448	6.278	7.256
6.000	5.444	5.677	6.487	7.389
5.333	5.789	6.895	6.666	7.123
5.000	6.737	5.999	7.112	6.897

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CHAPTER 9

WARRANTY MODELS

Behold the warranty: the bold print giveth and the fine print taketh away.

—*Anonymous*

9.1 INTRODUCTION

The increasing worldwide competition is prompting manufacturers to introduce innovative approaches in order to increase their market shares. In addition to improving quality and reducing prices, they also provide attractive warranties for their products. In other words, warranties are becoming an important factor in the consumer's decision-making process. For example, when several products that perform the same functions are available in the market and their prices are essentially equal, the customer's deciding factor of preference of one product over the others includes the manufacturer's reputation and the type and length of the warranty provided with the product. Because of the impact of the warranty on future sales, manufacturers who traditionally do not provide warranties for some products and services are now providing or are required to provide some type of warranty. For example, there were no warranties on the weapon systems until the Defense Procurement Reform Act was established in 1985, requiring the prime contractor for the production of weapon systems to provide written guarantees for such systems. The Defense Procurement Reform Act also delineates the types of coverage required, lists the possible remedies and specific reasons for securing a waiver, and actions to be taken in the event a waiver is sought. Thus, the warranty is becoming increasingly important for both consumers and military products.

A warranty is a contract or an agreement under which the manufacturer of a product or service must agree to repair, replace, or provide service when the product fails or the service does not meet the customer's requirements before a specified time life (length of warranty). This specified "time" may be measured in calendar time units such as hours, months, and years, or in usage units such as miles, hours of operation, number of times the product has been used (number of copies made by a copier, number of pages printed by a printer . . .), or both. Other warranties have no specified "times" and are referred to as lifetime warranties.

Three types of warranties are commonly used for consumer goods: the *ordinary free replacement* warranty, the *unlimited free replacement* warranty, and the *pro rata* warranty. Under an *ordinary free replacement* warranty, if an item fails before the end of the warranty

Reliability Engineering, Second Edition. Elsayed A. Elsayed.

length, it is replaced or repaired at no cost to the consumer. The repaired or replaced item is then covered by an ordinary free replacement warranty with a length equal to the remaining length of the original warranty. Such warranty assures that the consumer will receive as many free repairs or replacements as needed during the original length of the warranty. The ordinary free replacement warranty is the most common type of warranties; it is most often used to cover consumer durables such as cars and kitchen appliances (Mamer, 1987).

The second type of warranty, the *unlimited free replacement*, is identical to the ordinary replacement warranty except that each replacement item carries an identical warranty to the original purchase warranty. Such warranties are only used for small electronic appliances that have high early failure rates and are usually limited to very short periods of time.

Thus, under the free replacement warranty, whether it is ordinary or unlimited, a long warranty period will result in a very large warranty cost. Furthermore, an increase in the warranty period will reduce the number of replacement purchases over the life cycle of the product, which consequently reduces the total profit of the manufacturer. Clearly, the free replacement policy is more beneficial to the consumer than to the manufacturer. Therefore, it is extremely important for the manufacturer to determine the optimal price of the product and the optimal warranty length such that the total cost over the product life cycle is minimized.

The third type of warranty is the *pro rata warranty*. Under this warranty, if the product fails before the end of the length of the warranty, it is replaced at a cost that depends on the age of the item at the time of failure and the replacement item is covered by an identical warranty. Typically, a discount proportional to the remaining length of the warranty is given on the purchase price of the replacement item. For example, if the length of the warranty is w and the item fails at a time $t < w$, the consumer pays the proportion t/w of the cost of the replacement items (automobile tires represent an ideal product for which the pro rata warranty policy is appropriate), and the manufacturer covers the remaining cost of the product replacement (the proportion is not necessarily linear). Unlike the free replacement warranty, the pro rata warranty is more beneficial to the manufacturer than to the consumer.

Of course, other warranty policies can be derived by combining the terms of the above policies and modifying them. Indeed, it is interesting to note that the manufacturer can provide several warranty policies that appear different to the consumer, but they have the same cost to the manufacturer.

As presented above, a pure free replacement warranty favors the consumer whereas a pure pro rata warranty favors the manufacturer. Therefore, a mix of these policies may present an alternative warranty, which is fair to both the consumer and manufacturer. For example, a policy that provides a replacement free of charge up to time w_1 from the initial purchase; any failure in the interval w_1 to w ($w > w_1$) is replaced at a prorated cost is fair to both the manufacturer and the consumer (Nguyen and Murthy, 1984b; Blischke and Murthy, 1994). It has a promotional appeal to attract consumers and at the same time keeps the warranty cost for the manufacturer within a reasonable amount. There are other types of warranty policies including a reliability improvement warranty where the manufacturer provides guaranteed mean time between failures (MTBFs) or provides support for engineering changes during the warranty period (Blischke and Murthy, 1994). Another example of warranty policies that combine the three main ones described above is the case where replacements or repairs are performed free of charge up to time w_1 after the initial purchase and at a cost c_1 if the failure occurs in the interval $[w_1, w_2]$ and at a cost c_2 if the failure occurs in the interval $[w_2, w_3]$, and so forth.

These are referred to as *one-dimensional* warranties as the warranty period is the main decision variable to be determined. In other warranty policies, a warranty is characterized by two variables: warranty period and usage of the product. For example, many auto manufacturers provide a warranty for 3 years or 36,000 mi. In other words, the warranty ends when either one of these conditions is reached. We refer to such policies as *two-dimensional* warranty policies.

In deciding which warranty policy should be used, the manufacturer usually considers the type of repair or replacements to be made. We classify the products (items) into two types: repairable and nonrepairable. Repairable products are those for which repair cost is significantly less than the cost of replacing the products with new ones, such as copying machines, printers, computers, large appliances, and automobiles. Nonrepairable products are those for which the repair cost is close to the replacement cost or those that cannot be repaired due to the difficulty of accessing the components of the product or accessing the product itself. Typical examples of nonrepairable products are small appliances, radios, inexpensive watches, and satellites (not accessible for repairs). The rebate warranty policy is commonly used for nonrepairable products. Under the rebate policy, the consumer is refunded some proportion of the sale price if the product fails before the warranty period expires (Nguyen and Murthy, 1984b).

Manufacturers face two main problems in planning a warranty program. These problems require the determination of

- The type of warranty policy, length of warranty period, and its cost; and
- The amount of capital that must be allocated to cover future expenses for failures during a specified warranty period, that is, determination of the allocation for future warranty expenses. Too large a warranty reserve might make the sale price noncompetitive, thus reducing sales volume and profit. On the other hand, too little warranty reserve results in hidden losses that impact future profits.

In this chapter, we discuss different warranty policies and address the above two problems for both repairable and nonrepairable products.

9.2 WARRANTY MODELS FOR NONREPAIRABLE PRODUCTS

In this section, we determine the optimal warranty reserve fund for nonrepairable products when a pro rata warranty policy is used. Under this policy, a product is replaced by a new product when the warranty is invoked and a rebate, which decreases linearly with time or product use, is subtracted from the replacement price of the product. As mentioned in Section 9.1, this type of rebate is applicable to consumer products like batteries and tires.

9.2.1 Warranty Cost for Nonrepairable Products

Consider a product whose hazard-rate function, $h(t)$, is constant—that is,

$$h(t) = \lambda, \tag{9.1}$$

where λ is the failure rate. The probability of failure at any time less than or equal to t, $F(t)$, is

$$F(t) = 1 - e^{-\lambda t}. \tag{9.2}$$

The mean time to failure (MTTF), m, obtained from Equation 9.1 is $1/\lambda$. We rewrite Equation 9.2 as

$$F(t) = 1 - e^{-\frac{t}{m}}. \tag{9.3}$$

The procedure for determining the warranty reserve fund is as follows. Knowing the MTTF, determine the expected number of products that will fail in any small time interval dt. Then multiply the expected number of failures by the cost of replacement at time t to estimate the increment of warranty reserve that must be set aside for failures during the interval. Finally, add the incremental warranty cost for all increments dt from $t = 0$ to $t = w$ (end of warranty period). This results in the total warranty reserve fund. It is assumed that all failures during the warranty period are claimed. To transform this procedure into mathematical expressions, we define the following notations (Menke, 1969).

c	= constant unit product price, including warranty cost,
t	= time,
m	= MTTF of the product,
w	= duration of warranty period,
L	= product lot size for warranty reserve determination,
R	= total warranty reserve fund for L units,
$C(t)$	= pro rata customer rebate at time t,
r	= warranty reserve cost per unit product, and
E	= expected number of failures at time t.

Using Equation 9.3, we obtain the expected number of failures occurring at any time t,

$$E[N(t)] = L \times \text{P[product failure before or at time } t] = L\left[1 - e^{\frac{-t}{m}}\right].$$

The total number of failures in the interval t and $t + dt$ is

$$dE[N(t)] = \frac{\partial E[N(t)]}{\partial t}dt = (L/m)e^{-t/m}dt.$$

The cost for the failures in t and $t + dt$ is

$$d(R) = C(t)dE[N(t)] = c\left(1 - \frac{t}{w}\right)(L/m)e^{-t/m}dt. \tag{9.4}$$

The total cost for all failures occurring in $t = 0$ to $t = w$ is

$$R = \int_0^w \frac{Lc}{m}\left(1 - \frac{t}{w}\right)e^{-t/m}dt$$

or

$$R = Lc\left[1 - \left(\frac{m}{w}\right)(1 - e^{-w/m})\right]. \tag{9.5}$$

The warranty reserve fund per unit is

$$r = \frac{R}{L} = c\left[1 - \left(\frac{m}{w}\right)(1 - e^{-w/m})\right].$$

Thus,

$$\frac{r}{c} = 1 - \left(\frac{m}{w}\right)(1 - e^{-w/m}). \tag{9.6}$$

Let c' be the unit price before warranty cost is added. Then,

$$c = c' + r$$

and

$$c' = c\left(1 - \frac{r}{c}\right)$$

or

$$c = \frac{c'}{1 - \frac{r}{c}}. \tag{9.7}$$

The total warranty reserve fund to be allocated for L units of production is obtained from Equation 9.5.

The ratio between the warranty reserve cost and the product cost increases as the ratio between the warranty length and the MTTF increases as shown in Figure 9.1.

EXAMPLE 9.1

Assume the manufacturer of short wave radios wishes to extend a 12-month warranty on new types of radios. An accelerated life test was performed which indicated that the failure time of the radio follows an exponential distribution with parameter $\lambda = 0.01$ failures per month. The manufacturer's cost of a radio (not including warranty cost) is \$45. Assuming a total production run of 4000 radios, determine the warranty reserve fund and the adjusted price of the radio.

SOLUTION

Following are the data for the radios:

w = 12 months
c' = \$45
$m = 1/\lambda$ = 100 months
L = 4000 units.

Using Equation 9.6 we obtain r/c as

$$\frac{r}{c} = 1 - \left(\frac{100}{12}\right)\left(1 - e^{-\frac{12}{100}}\right) = 0.0576.$$

Using Equation 9.7 we obtain the adjusted price of the radio

$$c = \frac{c'}{1 - \frac{r}{c}} = \frac{45}{1 - 0.0576} = \$47.75.$$

The warranty reserve fund for the 4000 radios is

$$R = 0.05767 \times 4{,}000 \times 47.75 = \$11{,}014.$$ ■

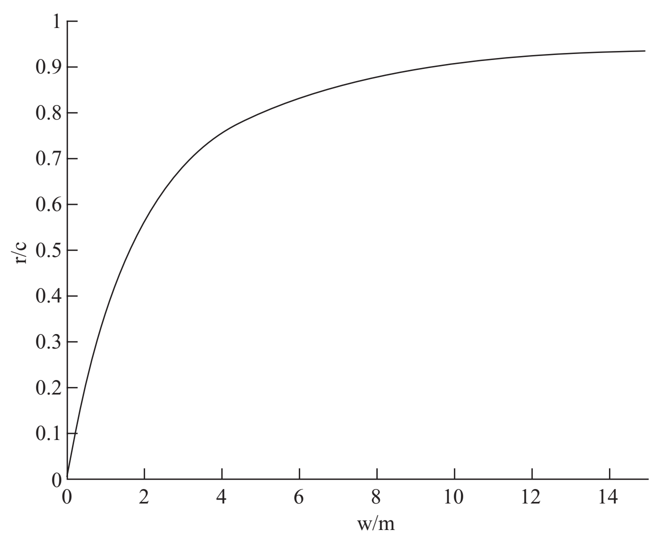

FIGURE 9.1 Relationship between *r/c* and *w/m*.

If the replacement product has the same warranty as the original product, then the expected number of failures occurring before or at time t is

$$E[N(t)] = LM(t) = L\frac{t}{m}.$$

The total number of failures in the interval t and $t + dt$ is

$$dE[N(t)] = \frac{\partial E[N(t)]}{\partial t}dt = \frac{L}{m}dt.$$

The cost for the failures in t and $t + dt$ is

$$d(R) = C(t)dE[N(t)] = \frac{Lc}{m}\left(1 - \frac{t}{w}\right)dt.$$

The total cost for all failures occurring in $t = 0$ to $t = w$ is

$$R = \int_0^w \frac{Lc}{m}\left(1 - \frac{t}{w}\right)dt$$

or

$$R = \frac{Lcw}{2m}.$$

The warranty reserve fund per unit is

$$r = \frac{R}{L} = \frac{cw}{2m}.$$

Thus,

$$\frac{r}{c} = \frac{w}{2m}.$$

For Example 9.1,

$$\frac{r}{c} = \frac{w}{2m} = \frac{12}{2 \times 100} = 0.06.$$

The adjusted price is

$$c = \frac{c'}{1 - \frac{r}{c}} = \frac{45}{1 - 0.06} = \$47.87.$$

The warranty reserve fund for the 4000 radios is

$$R = 0.06 \times 4000 \times 47.87 = \$11{,}489.$$

EXAMPLE 9.2

Suppose that the manufacturer in Example 9.1 approximated the failure-time distribution from being Weibull with shape parameter $\gamma = 1.8$ and scale parameter $\theta = 20$ to be the constant failure-rate model. Determine the true adjusted price and warranty reserve.

SOLUTION

Estimation of expected number of failures for Weibull distribution, $M(t)$, cannot be expressed analytically in a closed form but has been extensiviely tabulated for different values of γ and θ by Baxter et al. (1981, 1982) and Giblin (1983). More recently, Constantine and Robinson (1997) developed an approach to obtain $M(t)$ for moderate and large values of the Weibull parameters. We utilize the approximation of $M(t)$ given by Equation 7.13 which is

$$M(t) = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2}.$$

The mean and variance of the Weibull model are

$$\mu = \theta \Gamma\left(1 + \frac{1}{\gamma}\right) = 20\ \Gamma(1.5555) = 17.778$$

$$Var = \theta^2 \left\{ \Gamma\left(1 + \frac{2}{\gamma}\right) - \left(\Gamma\left(1 + \frac{1}{\gamma}\right)\right)^2 \right\} = 400 \times 0.2614 = 104.58$$

$$\sigma = 10.22.$$

Therefore, $M\ (12) = 0.34044$ failures.

$c' = \$45$

$L = 4000$ units.

Using Equation 9.6, we obtain r/c as $r/c = 0.34044$.

Thus,

$$c = \frac{c'}{1 - \frac{r}{c}} = \frac{45}{1 - 0.34044} = \$68.2273.$$

The warranty reserve fund for the 4000 radios is

$$R = 0.34044 \times 4000 \times 68.2273 = \$92{,}909.$$ ■

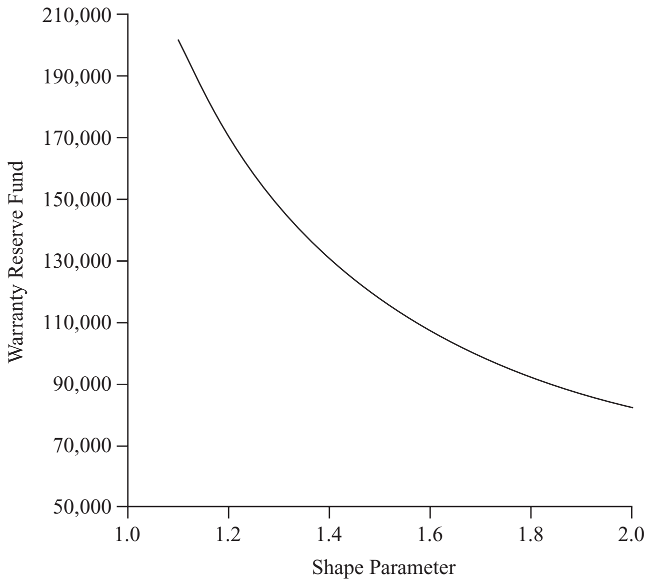

FIGURE 9.2 Effect of shape parameter on the warranty reserve fund.

TABLE 9.1 Values of c and R for Different Shape Parameters

γ	μ	σ	$M(t)$	c	R
1.1	19.30	17.43	0.5290	95.55	202198
1.2	18.79	15.67	0.4860	87.55	170216
1.3	18.43	14.27	0.4510	81.96	147850
1.4	18.21	13.11	0.4178	77.30	129192
1.5	18.03	12.25	0.3961	74.52	118064
1.6	17.93	11.48	0.3741	71.90	107597
1.7	17.85	10.78	0.3548	69.75	98987
1.8	17.78	10.23	0.3404	68.23	92909
1.9	17.74	9.70	0.3261	66.77	87085
2.0	17.72	9.27	0.3137	65.57	82278

It is important to accurately estimate the parameters of the failure-time distribution as a small error in the shape parameter might result in a significant allocation of the reserve fund as shown in Figure 9.2. It is interesting to note that the increase in the shape parameter results, as expected, in a decrease in the characteristic life of the unit and a faster reduction in its variance. This results in a reduction in the expected number of failures during the warranty period and reduction in the warranty reserve as a consequence as shown in Table 9.1.

This might appear counterintuitive, which is attributed to the approximation of the equation for the expected number of failures as well as the fact that the variance decreases as gamma increases.

9.2.2 Warranty Reserve Fund: Lump-Sum Rebate

If the administrative cost and the errors in estimating pro rata claims are too expensive, the manufacturer may wish to consider an alternative warranty plan by paying a fixed or lump-sum rebate to the customer for any failure occurring before the warranty expires. Again, we are interested in determining the adjusted price of the product and the warranty reserve fund that meets customer claims.

Let k be the proportion of the unit cost to be refunded as a lump-sum rebate and S be the unit lump-sum rebate ($S = kc$).

Substituting $C(t) = kc$ in the pro rata warranty model, we obtain

$$r_s = kc\left(1 - e^{-\frac{w}{m}}\right),$$

where r_s is the warranty reserve cost per unit under the lump-sum warranty plan.

If it is desirable to make the warranty cost per unit of production equal for both the pro rata and the lump-sum plan, then

$$1 - \frac{m}{w}\left(1 - e^{-\frac{w}{m}}\right) = k\left(1 - e^{-\frac{w}{m}}\right)$$

or

$$k = \frac{1}{1 - e^{-w/m}} - \frac{m}{w}. \tag{9.8}$$

The proportion of the unit cost to be refunded as a lump-sum rebate as a function of ratio w/m is shown in Figure 9.3. The lump-sum rebate per unit becomes

$$S = kc.$$

The total warranty reserve fund, R_s is (Menke, 1969)

$$R_s = LS\left(1 - e^{-\frac{w}{m}}\right). \tag{9.9}$$

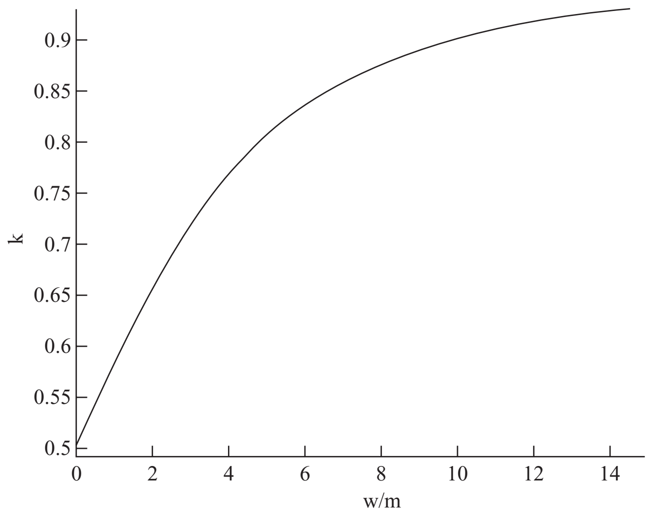

FIGURE 9.3 Plot of *k* versus *w*/ *m*.

EXAMPLE 9.3

Assume that the radio manufacturer in Example 9.1 wishes to adopt a lump-sum warranty plan equivalent to the pro rata plan by providing a lump sum of the initial price to customers whose radios fail during the warranty period. Determine the portion of the price to be refunded upon failure before the warranty expires.

SOLUTION

Using the same data for the radio,

w = 12 months

c' = \$45

m = 100 months, and

L = 4000 units.

We first determine the proportion of the unit cost to be returned as a lump sum by using Equation 9.8.

$$\begin{aligned} k &= \frac{1}{1-e^{-w/m}} - \frac{m}{w} \\ &= \frac{1}{1-e^{-\frac{12}{100}}} - \frac{100}{12} = 0.5099. \end{aligned}$$

In other words, the manufacturer should pay 51% of the initial price to customers whose radios fail before the expiration of the warranty period. The warranty cost per unit is

$$c = c' + r = \$47.75$$

and

$$S = ck = \$24.35.$$

The total warranty reserve fund is

$$R_s = 4{,}000 \times 24.35\left(1 - e^{-\frac{12}{100}}\right)$$

$$R_s = \$11{,}014. \qquad \blacksquare$$

It is important to consider the situations for which a pro rata or a lump-sum warranty policy can be used. For example, a manufacturer should consider the use of a lump-sum plan when it is possible to determine that the product failed before its warranty period, but the exact time of failure is not possible to determine (Menke, 1969).

The model discussed in Sections 9.2.1 and 9.2.2 overestimates the required warranty reserve fund since the discounting of future warranty claim costs for the time value of money, and changes in the general price level due to inflation (or deflation) are ignored.

Let θ be the rate of return earned through the investment of the warranty reserve fund and ϕ be the expected change per period in the general price level. Then, the real present value of the warranty claims (Amato and Anderson, 1976) in t and $t + dt$ period is obtained by rewriting Equation 9.4 as

$$d(R^*) = c^*\left(1 - \frac{t}{w}\right)(1+\theta+\phi)^{-t}\left(\frac{L}{m}\right)e^{-\frac{t}{m}}dt, \tag{9.10}$$

where R^* and c^* are the present values of warranty reserve fund and the price of the product, respectively, $(1 + \theta)^n(1 + \phi)^n \simeq (1 + \theta + \phi)^n$ for small θ and ϕ. Equation 9.10 can be rewritten as

$$\begin{aligned} R^* &= \frac{Lc}{m}\int_0^w \left(1 - \frac{t}{w}\right)\left[(1+\theta+\phi)e^{\frac{1}{m}}\right]^{-t} dt \\ &= \left\{\frac{Lc^*}{1 + m\ln(1+\theta+\phi)}\right\} \times \left\{1 - \left(\frac{m}{w}\right)\frac{1}{1+m\ln(1+\theta+\phi)} \times \left[1 - (1+\theta+\phi)^{-w}e^{-\frac{w}{m}}\right]\right\}. \end{aligned} \tag{9.11}$$

Again, the manufacturer can assign the following per unit price to its product in order to incorporate the warranty cost

$$c^* = c' + r^*,$$

where

$$\frac{r^*}{c^*} = [1 + m\ln(1+\theta+\phi)]^{-1} \times \left\{1 - \frac{m}{w}[1 + m\ln(1+\theta+\phi)]^{-1}\left[1 - (1+\theta+\phi)^{-w}e^{-\frac{w}{m}}\right]\right\}. \tag{9.12}$$

and

$$r^* = \frac{R^*}{L}. \tag{9.13}$$

EXAMPLE 9.4

The manufacturer of the radios in Example 9.1 intends to invest the warranty reserve fund to earn an interest rate of 5% and to increase the price of the radio in the following year by 6%. Determine the warranty reserve fund and the price of the radio after adjustments.

SOLUTION

The following data were provided in Example 9.1:

m = 100 months,
w = 12 months,
c' = \$45, and
L = 4000 units.

We now include the effect of θ and ϕ

$$(1+\theta+\phi) = 1+0.05+0.06 = 1.11.$$

We obtain r^*/c^* from Equation 9.12

$$\begin{aligned}\frac{r^*}{c^*} &= (1+100\ln 1.11)^{-1}\left\{1-\frac{100}{12}(1+100\ln 1.11)^{-1}\left[1-(1.11)^{-12}e^{-\frac{12}{100}}\right]\right\}\\ &= 0.09495\{1-0.791[1-0.2858\times 0.8869]\} = 0.03888.\end{aligned}$$

But,

$$c^* = \frac{c'}{1-\frac{r^*}{c^*}} = \frac{45}{1-0.03888} = \$46.82$$

and

$$R^* = L\times\frac{r^*}{c^*}\times c^* = 4{,}000\times 0.0388\times 46.82 = \$7{,}281.$$

These two estimates are smaller than those of Example 9.1 due to the return on investment. ■

9.2.3 Mixed Warranty Policies

In the previous two sections we presented the pro rata and the lump-sum rebate warranty policies. In this section, we present and compare two warranty policies. The first policy, which we refer to as *full rebate policy*, occurs when a manufactured product is sold under full warranty. If a failure occurs within w_0 units of time, the product is replaced at no cost to the consumer, and a new warranty is issued. The second policy is a *mixed warranty policy* where a full compensation is provided to the consumer if the product fails before time w_1, followed by a linear prorated compensation up to the end of the warranty coverage period, w_2. Other mixed warranty policies are developed such as the mixed policy that considers the *warranty of malfunctioning* (it is related to the product's failure to perform the functions as specified in its description for a predetermined (warranty) period of time) and *warranty of misinforming* (it is related to a failure in the communication process during the course of the product sale, which leads to customers being misinformed regarding the product's features and scope of usage) as discussed in Christozov et al. (2010). In this section we consider the former mixed warranty policy. In order to simplify the analysis, we define the following notations (Ritchken, 1985):

w_0 = warranty length of the full rebate policy,

w_1 = length of the full compensation period for the mixed policy,

$w_2 - w_1$ = length of the prorated period for the mixed policy,

c_0 = unit cost of replacement,

ϕ = notation for the full rebate policy,

ψ = notation for the mixed policy,

X_i = time between failures i and $i - 1$; $X_i > 0$,

$I(X_i)$ = cost of a failure to the manufacturer under a given policy (ϕ or ψ),

V = random variable representing the total warranty cost accumulated per product,

$F(X_i)$, $R(X_i)$ = cumulative distribution function (CDF) and the reliability function of X; $R(X_i) = 1 - F(X_i)$,

$$F^{(2)}(x) = \int_0^x F(w)dw,$$

$$F^{(3)}(x) = \int_0^x F^{(2)}(w)dw, \text{ and}$$

N' = number of failures that occur until a failure time exceeds the warranty period.

The two warranty policies are shown in Figure 9.4. The cost associated with full rebate policy is

$$I_\phi(X_i) = \begin{cases} c_0 & 0 \le X_i \le w_0 \\ 0 & \text{otherwise.} \end{cases} \tag{9.14}$$

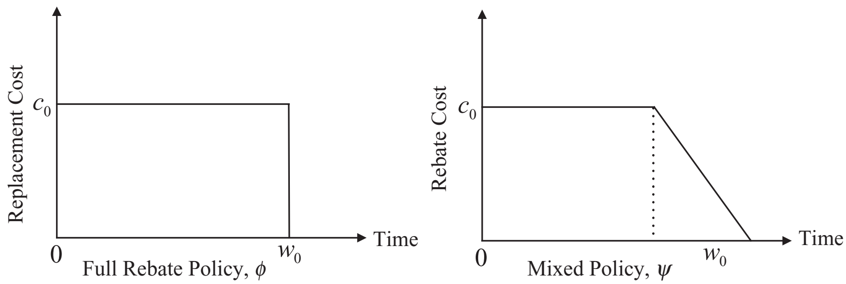

FIGURE 9.4 Two warranty policies.

The cost associated with the mixed policy is

$$I_\psi(X_i) = \begin{cases} c_0 & 0 \le X_i \le w_1 \\ c_0(w_2 - X_i)/(w_2 - w_1) & w_1 \le X_i \le w_2. \\ 0 & \text{otherwise} \end{cases} \tag{9.15}$$

The total warranty expenses accumulated per product sold is

$$V = \sum_{i=1}^{N'} I(X_i). \tag{9.16}$$

The decision variable for the full rebate policy ϕ is w_0, whereas the decision variables for the mixed policy ψ are the time of full compensation and the total time of the warranty length.

We now derive the expected total warranty cost for a product under policy ψ

$$E[V_\psi] = E\left\{E\left[\sum_{i=1}^{N'} I_\psi(X_i)\right]\right\}.$$

Since N' is a stopping time, then

$$E[V_\psi] = E[N']E[I_\psi(X_i)]. \tag{9.17}$$

But N' is geometric—that is,

$$P[N' = k] = [1 - F(w_2)][F(w_2)]^k \quad k = 0, 1, 2, \ldots$$

or

$$P[N' = k] = R(w_2)F(w_2)^k.$$

Thus,

$$E[N'] = \sum_{k=0}^{\infty} kR(w_2)F(w_2)^k = F(w_2)/R(w_2). \tag{9.18}$$

Moreover, from Ritchken (1985) and Thomas (1983, 2006)

$$\begin{aligned} E[I_\psi(X_i)] &= c_0 \int_0^{w_1} f(x)dx + \frac{c_0}{w_2 - w_1} \int_{w_1}^{w_2} (w_2 - x) f(x)dx \\ &= \frac{c_0}{w_2 - w_1} \int_{w_1}^{w_2} F(u)du, \end{aligned} \tag{9.19}$$

where $f(x)$ is the probability density function (p.d.f.) of the failure-time distribution.

Substituting Equations 9.18 and 9.19 into Equation 9.17, we obtain the expected warranty cost of a product under policy ψ as

$$E[V_\psi] = \frac{c_0 F(w_2)}{(w_2 - w_1)R(w_2)} \int_{w_1}^{w_2} F(u)du. \tag{9.20}$$

EXAMPLE 9.5

Consider a product that exhibits a constant failure rate with MTTF of 60 months. It is intended to use a mixed policy with $w_1 = 3$ months, and $w_2 = 12$ months. The cost of a replacement is \$120. What is the expected warranty cost?

SOLUTION

Since the product exhibits constant failure rate, then

$$\begin{aligned} F(x_i) &= 1 - e^{\frac{-x_i}{60}} \quad x_i \geq 0 \\ R(x_i) &= e^{\frac{-x_i}{60}}. \end{aligned}$$

Using Equation 9.20, we obtain

$$\begin{aligned} E[V_\psi] &= \frac{120F(w_2)}{(12-3)R(w_2)} \int_3^{12} 1 - e^{-\frac{x}{60}} dx \\ &= \$3.0997 \end{aligned}$$

■

For each mixed policy ψ, there is a full rebate policy ϕ that yields the same cost. The expected cost of a failure under a full rebate policy is

$$E[I_\phi(X_i)] = c_0 F(w_0). \tag{9.21}$$

Using Equations 9.17, 9.18, and 9.21 we obtain

$$E[V_\phi] = \frac{c_0 F(w_o)^2}{R(w_0)}. \tag{9.22}$$

We equate Equations 9.20 and 9.22 so that the two policies will have the same cost. Thus,

$$\frac{F(w_0)^2}{R(w_0)} = \frac{F(w_2)}{R(w_2)} \frac{1}{(w_2 - w_1)} \int_{w_1}^{w_2} F(u)du,$$

which is reduced to

$$\frac{F(w_0)^2}{R(w_0)} = \frac{F(w_2)}{R(w_2)} \frac{F^{(2)}(w_2) - F^{(2)}(w_1)}{(w_2 - w_1)}. \tag{9.23}$$

If we consider a linear pro rata warranty policy only ($w_1 = 0$), then the above equation can be rewritten as

$$\frac{F(w_0)^2}{R(w_0)} = \frac{F(w_2)}{R(w_2)} \frac{F^{(2)}(w_2)}{w_2}. \tag{9.24}$$

In other words, the two policies are equivalent if w_0 is chosen such that Equation 9.24 is satisfied.

EXAMPLE 9.6

Using the data of Example 9.5, determine the warranty length (w_0) for the full rebate policy, which makes it equivalent to the mixed policy.

SOLUTION

Substituting in Equation 9.23, we obtain

$$\frac{F(w_0)^2}{R(w_0)} = \frac{0.22140(1.12384 - 0.07376)}{12 - 3} = 0.0258$$

$$\frac{\left(1 - e^{-\frac{w_0}{60}}\right)^2}{e^{-\frac{w_0}{60}}} = 0.0258$$

$$w_0 = 9.6 \text{ months}$$

■

EXAMPLE 9.7

Determine the warranty length (w_0) for the full rebate policy that makes it equivalent to a pro rata policy with $w_2 = 12$.

SOLUTION

Using $w_2 = 12$ and substituting into Equation 9.24, we obtain

$$\frac{F(w_0)^2}{R(w_0)} = \frac{F(w_2)}{R(w_2)}\frac{F^{(2)}(w_2)}{w_2}$$
$$w_0 = 8.63 \text{ months.}$$ ■

It is not sufficient to compare two policies based only on the expected cost since the variance of the cost (or the distribution of the cost) may influence the choice of the warranty policy. For example, the manufacturer may prefer a warranty policy with a smaller cost variance. The manufacturer may also compare different warranty policies using the mean-variance orderings. Therefore, the variances of the warranty cost need to be determined. Ritchken (1985) derives the following expressions for the variances of total warranty cost:

- For the linear pro rated policy, ψ with $w_1 = 0$,

$$\mathrm{Var}(V_\psi) = c_0^2 F(w_2)[2R(w_2)F^{(3)}(w_2) + F^{(2)}(w_2)^2 F(w_2)] / w_2^2 R(w_2)^2. \tag{9.25}$$

- For the full rebate policy, ϕ the variance is

$$\mathrm{Var}(V_\phi) = c_0^2 F(w_0)^2 [R(w_0)^2 + F(w_0)] / R(w_0)^2 \tag{9.26}$$

—when

$$F(w) = 1 - e^{-\lambda w} \quad w \geq 0,$$

and λ is the failure rate, then

$$F^{(2)}(w) = [\lambda w - F(w)] / \lambda \tag{9.27}$$

$$F^{(3)}(w) = (\lambda w - 1)^2 + (2F(w) - 1) / 2\lambda^2. \tag{9.28}$$

Substituting Equations 9.27 and 9.28 into Equation 9.25 results in

$$\mathrm{Var}(V_\psi) = \frac{c_0^2 F(w_2)}{[\lambda w_2 R(w_2)]^2}[(\lambda w_2 - 1)^2 + 2F(w_2) - 1]R(w_2). \tag{9.29}$$

The expected time to the first failure that is not covered by the warranty cost is

$$E[T] = E[X_i] / R(w_2), \tag{9.30}$$

where $E[X_i]$ is the MTTF.

EXAMPLE 9.8

Using the warranty lengths of $w_2 = 12$ and $w_0 = 8.63$ that make the pro rata policy equivalent to the full rebate policy, determine the variances of the total warranty cost for each policy. Which policy do you prefer?

SOLUTION

From Equation 9.29, we obtain the variance for the pro rata policy as

$$\begin{aligned} Var(V_\psi) &= \frac{120^2\left(1-e^{-\frac{12}{60}}\right)}{\left(\frac{12}{60}\times e^{-\frac{12}{60}}\right)^2}\left[\left(\frac{12}{60}-1\right)^2+2\left(1-e^{-\frac{12}{60}}\right)-1\right]e^{-\frac{12}{60}} \\ &= 167.44. \end{aligned}$$

Using Equation 9.26, the variance for the full rebate policy is

$$Var(V_\phi) = \frac{120^2\left[1-e^{-\frac{8.63}{60}}\right]^2}{\left(e^{-8.63/60}\right)^2}\left[\left(e^{-8.63/60}\right)^2+\left(1-e^{-8.63/60}\right)\right] = 304.61.$$

Since the two policies are equivalent, the manufacturer should adopt the pro rata policy in order to reduce the variability in the total warranty cost. ■

9.2.4 Optimal Replacements for Items under Warranty

A typical age replacement policy of items calls for an item replacement upon failure or at a fixed time, whichever comes first. Clearly, such a policy is only applicable for items whose failure rates increase with age. In this section, we develop a model for the determination of the optimal age replacement policies for warrantied items, such that the average cost is minimized. We summarize an age replacement policy as follows (Ritchken and Fuh, 1986).

Assume a nonrepairable item is installed at time zero and is provided with a warranty policy. If the item fails during its warranty period, it is replaced at a cost shared by both the manufacturer and the customer (such as a linear pro rata policy) in accordance with a rebate policy. After the warranty expires, an age replacement policy is followed, with the item being replaced after an additional fixed time or upon failure—whichever comes first. We are interested in determining the parameters of the optimum age replacement warranty. We define the following notation.

ϕ	= warranty policy,
w	= length of the warranty period,
X_i	= time to the ith failure,
$F(t)$	= CDF of time to failure,
$R(t)$	= reliability function up to time t,
$h(t)$	= hazard rate at time t,
$F_r()$	= CDF of the residual life time beyond w,
$f_r()$	= p.d.f. of the residual life time beyond w,
$[N(t),\ t > 0]$	= number of times an item fails in the time interval $(0, t)$,
$M(t) = E[N(t)]$	= renewal function,
Y	= time for which first failure occurs outside warranty period—that is, $Y \equiv \inf[t \mid N(t) = N(w) + 1]$,
$r(w)$	= residual life of the functioning item at time w; $r(w) = Y - w$,
T	= age replacement parameter measured from the end of the warranty period,
T_r	= time between replacements outside the warranty interval, $T_r \equiv \min\{T + w, Y\}$,
$G(T)$	= $\int_0^T x f_r(x)dx$ partial mean of time to replacement beyond w,
$\hat{C}(t)$	= mean cost incurred between replacements outside the warranty interval,
c_1	= cost of replacing a failed item,
c_2	= cost of replacing a functioning item,
$I_j(\phi)$	= cost to the consumer for replacement j under the warranty policy ϕ, and
W_ϕ	= total mean cost of replacements over the warranty period.

The age replacement policy under warranty is illustrated in Figures 9.5 and 9.6.

Consider a linear pro rata policy, then the cost of replacing an item j that fails before the warranty period w is

$$I_j(\phi) = X_j c_1 / w \quad \text{for } X_j < w. \tag{9.31}$$

The mean cost of replacements to the customer over the warranty period is

$$W_\phi = E\left[\sum_{j=1}^{N(w)} I_j(\phi)\right]. \tag{9.32}$$

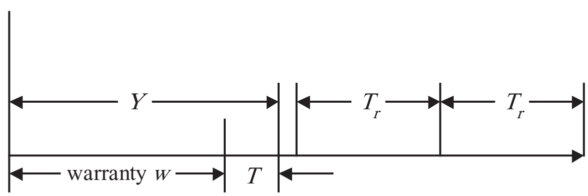

FIGURE 9.5 Replacement of an item that fails at Y.

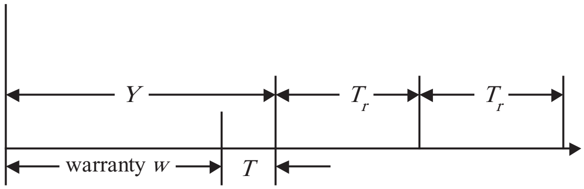

FIGURE 9.6 Replacement of an item that survives until the replacement interval.

If the item survives beyond the warranty period, the residual life of the item, $r(w)$, is a random variable with CDF given by (Ross, 1970)

$$F_r(t) = F(w+t) - \int_0^w R(w+t-x)dm(x). \tag{9.33}$$

The cost incurred over the full replacement cycle, T_r, is the sum of the warranty expenses over the warranty period w, together with the replacement cost of either a failed item at time Y (as shown in Figure 9.5) or a functioning item at time $T + w$. The expected cost over the cycle is

$$\begin{aligned}\hat{C}(T) &= W_\phi + c_1\int_0^T f_r(x)dx + c_2\int_T^\infty f_r(x)dx \\ &= W_\phi + c_1F_r(T) + c_2\bar{F}_r(T).\end{aligned} \tag{9.34}$$

where

$$\bar{F}_r(T) = 1 - F_r(T).$$

Similarly, the mean time between replacements is

$$\begin{aligned}E[T_r] &= w + \int_0^T tf_r(t)dt + T\bar{F}_r(T). \\ &= w + G(T) + T\bar{F}_r(T).\end{aligned} \tag{9.35}$$

The steady state average cost is obtained by dividing Equation 9.34 by Equation 9.35 as follows:

$$\bar{C}(T) = \hat{C}(T)/E(T_r). \tag{9.36}$$

The objective is to determine T^*, which minimizes $\bar{C}(T)$. Ritchken and Fuh (1986) prove that if $h(t)$ is continuous and monotonically nondecreasing, then a unique solution exists that minimizes Equation 9.36.

EXAMPLE 9.9

Consider an item that exhibits a constant failure rate λ. What is the optimal replacement interval?

SOLUTION

Since the failure rate is constant, then

$$h(t) = \lambda$$

and

$$F_r(t) = 1 - e^{-\lambda t} \quad 0 \le t < \infty.$$

Substituting in Equation 9.36,

$$\bar{C}(T) = \frac{W_\phi + c_1(1 - e^{-\lambda T}) + c_2 e^{-\lambda T}}{w + (1 - e^{-\lambda T})/\lambda}.$$

$\bar{C}(T)$ is monotonically decreasing in T. Hence, the optimal policy is that items should not be replaced before failure. This is in agreement with the preventive maintenance policy of units with constant failure rates which is discussed in Chapter 8. ■

EXAMPLE 9.10

A hot standby system consists of two components in parallel (1-out-of-2 system). The cost of replacing a failed unit is \$11, the failure rates of the components are identical, $\lambda = 0.2$ failures per month, $W_\phi = 1$, and the warranty length is 5 months. What is the optimal replacement interval?

SOLUTION

The failure distribution of two components in parallel is

$$F(x) = \prod_{i=1}^{2} F_i(x).$$

Let

$$F_i(x) = 1 - e^{-\lambda x} \quad i = 1, 2.$$

Then

$$f(x) = F'(x) = 2\lambda e^{-\lambda x} - 2\lambda e^{-2\lambda x}.$$

The Laplace transform of $f()$ is

$$f^*(s) = \int_0^\infty e^{-sx} dF(x) \quad x > 0$$

and

$$M(x) = \sum_{n=1}^{\infty} F_n(x).$$

The Laplace transform of $M()$ is

$$M^*(s) = \sum_{n=1}^{\infty} F_n^*(s) = \sum_{n=1}^{\infty} [F^*(s)]^n$$

$$M^*(s) = \frac{f^*(s)}{s[1 - f^*(s)]}.$$

$$m^*(s) = \frac{f^*(s)}{1 - f^*(s)}.$$

From the above equations, we obtain

$$f^*(s) = \frac{2\lambda^2}{(s+\lambda)(s+2\lambda)}$$

$$m^*(s) = \frac{2\lambda}{3s} - \frac{2\lambda}{3s + 9\lambda}.$$

Hence,

$$m(x) = M'(x) = \frac{2\lambda}{3} - \frac{2\lambda}{3} e^{-3\lambda x}.$$

From Equation 9.33, we obtain the residual life distribution after time t as

$$F_r(t) = 1 - 2e^{-\lambda(w+t)} + e^{-2\lambda(w+t)} + \frac{4}{3} e^{-\lambda(w+t)} \left[e^{\lambda w} - 1\right] - \frac{1}{3} e^{-2\lambda(w+t)} - \frac{2}{3} e^{-\lambda(w+t)} \left[1 - e^{-2\lambda w}\right] + \frac{2}{3} e^{-2\lambda(w+t)} \left[1 - e^{-\lambda w}\right]$$

or

$$F_r(t) = 1 - 1.25e^{-0.2t} + 0.15e^{-0.4t}$$

and

$$h(t) = \frac{0.25e^{-0.2t} - 0.06e^{-0.4t}}{1.25e^{-0.2t} - 0.15e^{-0.4t}}.$$

$h(t)$ is monotonically nondecreasing. Hence, there exists a finite T^* that minimizes $\bar{C}(t)$. Assuming c_2 = \$5 and substituting in Equation 9.36, we obtain

$$C(T) = \frac{e^{-0.4T} - 8\frac{1}{3}e^{0.2T} + 13\frac{1}{3}}{e^{0.4T} - 16\frac{2}{3}e^{-0.2T} + 2\frac{1}{3}}$$

$$T^* = 3.849 \text{ months.}$$ ■

9.3 WARRANTY MODELS FOR REPAIRABLE PRODUCTS

Most products are repairable upon failure. A warranty for such products may have a fixed duration in terms of calendar time or other measures of usage. Such products may also have a lifetime warranty, which means that the manufacturer must repair or replace the failed product during the consumer ownership of the product. The lifetime of the product may terminate due to technological obsolescence, changes in design, change in the ownership of the product, or failure of a critical component, which is not under warranty. In this section, we present warranty models for repairable products.

9.3.1 Warranty Cost for Repairable Products

Consider a product that is subject to failure and minimum repair is performed to return the product to an average condition for a working product of its age. In other words, repair is performed to restore the unit to its operational conditions (Park, 1979). The product is warranted for a warranty length w. If the product fails at any time before w, it is minimally repaired to bring it to an operational condition comparable to other products having the same age. No warranty extension beyond w is provided after repair. We now develop a model to determine the present worth of the repairs during w.

We define the following notation after Park and Yee (1984).

$R(t)$	= reliability of the product at time t,
λ	= Weibull scale parameter,
β	= Weibull shape parameter,
$h(t)$	= hazard rate of the product at time t,
$H(t)$	= $\int_0^t h(t)dt$, cumulative hazard function,
$f_n(t)$	= p.d.f. of failure n,
r	= average cost per repair,
i	= nominal interest rate for discounting the future cost,
C_w	= present worth of repair during w,
C_∞	= present worth of repair for a product with lifetime warranty, and
poim(k; μ)	= Poisson *p.m.f.*; $\mu^k e^{-\mu}/k!$

Since minimal repair is performed upon failure, and the hazard rate resumes at $h(t)$ instead of returning to $h(0)$, the system failure times are not renewal points but can be described by a nonhomogeneous Poisson process (NHPP). The probability density of the time to the nth failure is (Park, 1979)

$$f_n(t) = h(t)\mathrm{poim}(n-1; H(t))$$

or

$$f_n(t) = \lambda\beta(\lambda t)^{\beta-1}\left\{\exp[-(\lambda t)^{\beta}](\lambda t)^{(n-1)\beta} / \Gamma(n)\right\} \tag{9.37}$$

for a Weibull distribution, where $H(t) = (\lambda t)^{\beta}$.

The present worth of the repairs during the warranty period is

$$\begin{aligned} C_w &= \sum_{n=1}^{\infty}\int_0^w re^{-it} f_n(t)dt \\ &= r\beta\int_0^{\lambda w} \exp[-iu/\lambda]u^{\beta-1}du \\ &= r\beta(\lambda w)^{\beta}\exp(-iw)\sum_{k=0}^{\infty}\frac{(iw)^k}{\beta(\beta+1)\ldots(\beta+k)} \end{aligned} \tag{9.38}$$

or

$$C_w = r\beta(\lambda/i)^{\beta}\exp(-iw)\sum_{k=0}^{\infty}\frac{(iw)^{\beta+k}}{\beta(\beta+1)\ldots(\beta+k)}. \tag{9.39}$$

For a lifetime warranty, the cost is obtained as

$$\begin{aligned} C_\infty &= r\int_0^{\infty} h(t)e^{-it}dt \\ C_\infty &= r\left(\frac{\lambda}{i}\right)^{\beta}\Gamma(\beta+1). \end{aligned} \tag{9.40}$$

EXAMPLE 9.11

The major component of a product experiences a constant failure rate of 0.4 failures per year. The average repair cost is r = \$12, and the nominal interest rate is 5% per year. What is the expected warranty cost for 1 year?

SOLUTION

Since the component exhibits a constant failure rate, then

$$R(t) = e^{-\lambda t}.$$

Substituting $\beta = 1$ in Equation 9.39, we obtain

$$C_w = \frac{r\lambda}{i}[e^{-iw}(e^{iw} - 1)].$$

Set $w = 1$, then

$$C_1 = \frac{12 \times 0.4}{0.05}[1 - e^{-0.05\times 1}] = \$4.68.$$

The lifetime warranty cost is obtained by

$$C_\infty = \frac{r\lambda}{i} = \frac{12 \times 0.4}{0.05} = \$96.$$

■

EXAMPLE 9.12

A producer of nondestructive testing equipment is manufacturing a new ultrasonic testing unit that assesses the quality of concrete. Testing is confined to measurement of the time-of-flight of an ultrasonic pulse through the concrete from a transmitting to a receiving transducer. The pulse velocity value represents the quality of the concrete between the two transducers. Analysis of the measurements can detect the number and size of the voids in the concrete.

The producer warrants the product for a period of 2 years. If the product fails at any time before 2 years, it is minimally repaired to bring it to an age comparable to other products produced at the same time. The producer does not provide any warranty beyond the 2 years. The average cost per repair is \$80 and the interest rate is 5% per year. The cumulative hazard function of the products is expressed as

$$H(t) = (\lambda t)^\beta,$$

where

$$\lambda = 3 \text{ years, and}$$
$$\beta = 2.5.$$

Determine the expected value of the repair cost during the warranty period. Also, determine the expected value of the repair cost if the producer extends the lifetime warranty for the product.

SOLUTION

The parameters of the product and the warranty policy are

$$\beta = 2.5,$$
$$\lambda = 3,$$
$$w = 2,$$
$$r = \$80, \text{ and}$$
$$i = 0.05.$$

Using Equation 9.39, we obtain the present value of the repair cost during the warranty period as

$$C_2 = 5.046 \times 10^6 \sum_{k=0}^{\infty} \frac{(0.1)^{3.5+k}}{(2.5)(3.5)\ldots(2.5+k)}$$

or

$$C_2 = 5.046 \times 10^6 \times 15.56 \times 10^{-6} = \$78.52.$$

The repair cost for the lifetime warranty is

$$C_\infty = r\left(\frac{\lambda}{i}\right)^{\beta} \Gamma(\beta+1)$$

or

$$C_\infty = 80\left(\frac{3}{0.05}\right)^{2.5} \Gamma(3.5) = \$7{,}413{,}847.$$

Clearly, this warranty cost is excessive, due to the increasing failure rate of the product. In such a situation, the producer may wish to redesign the product in order to reduce its failure rate. ■

The failure-free warranty policy is commonly used for repairable products. Under this policy the manufacturer agrees to pay the repair cost for all failures occurring during the warranty period. We first develop a general warranty model which estimates the expected warranty cost per product for a warranty length, w, when the failure-time distribution is arbitrary and the repair cost depends on the number of repairs carried out. We then develop models for different repair policies.

9.3.2 Warranty Models for a Fixed Lot Size: Arbitrary Failure-Time Distribution

Again, when a product fails, it is restored to its operating condition by repair. In this model, we consider the case when the failure-time distribution is arbitrary. To simplify the analysis, we assume that the repair time is negligible. In addition to other notations presented earlier in this chapter, we define

S_n	= total time to the nth failure (random),
$f^{(n)}(t)$	= p.d.f. of S_n,
$F^{(n)}(t)$	= CDF of S_n,
$N(t)$	= number of failures in [0, t],
$M(t)$	= expected number of failures in [0, t] (renewal function),
c_i	= expected cost of the ith repair,
C_w	= expected warranty cost per product for a warranty period w,
C_n	= warranty cost given when there are exactly n failures in [0, w], $C_n = \sum_{i=1}^{n} c_i$,
$\sigma^2(w)$	= variance of the warranty cost per product for a warranty period w, and
L	= number of products sold.

The expected warranty cost per product is

$$C_w = \sum_{n=0}^{\infty} C_n P[N(w) = n]. \tag{9.41}$$

where $P[N(w) = n]$ is the probability of having n failures during the warranty period [0, w]. Also,

$$P[N(t) = n] = F^{(n)}(t) - F^{(n+1)}(t), \tag{9.42}$$

with $F^{(0)}(t) = 1$. Substituting Equation 9.42 into Equation 9.41, we obtain

$$C_w = \sum_{n=0}^{\infty} C_n F^{(n)}(w). \tag{9.43}$$

The total warranty cost is equal to LC_w. Similarly, $\sigma^2(w)$ is given as (Nguyen and Murthy, 1984a)

$$\sigma^2(w) = \sum_{n=1}^{\infty} C_n^2 P[N(w) = n] - [C_w]^2$$

or

$$\sigma^2(w) = \sum_{n=1}^{\infty} [C_n^2 - C_{n-1}^2] F^{(n)}(w) - [C_w]^2. \tag{9.44}$$

The expected number of failures during the warranty period is

$$M(w) = \sum_{n=0}^{\infty} nP[N(w) = n].$$

Using Equation 9.42, we rewrite $M(w)$ as

$$M(w) = \sum_{n=1}^{\infty} F^{(n)}(w), \tag{9.45}$$

which is the definition of the renewal function.

If the repair cost is independent of the number of failed units, $C_n = C$, then the repair cost during the warranty period is

$$C_w = CM(w),$$

and the variance of the warranty cost per product becomes

$$\sigma^2(w) = C^2 \mathrm{Var}[N(w)],$$

where

$$\mathrm{Var}[N(w)] = \sum_{n=1}^{\infty} n^2 [F^{(n)}(w) - F^{(n+1)}(w)]$$

or

$$\mathrm{Var}[N(w)] = \sum_{n=1}^{\infty} (2n-1)F^{(n)}(w) - [M(w)]^2.$$

The variance for the total warranty cost is $L^2\sigma(w)$.

We next consider three different repair policies: The first is the minimal repair policy; the second is the "good-as-new" repair policy; and the third is a mixture of these two policies. We use the subscripts 1 and 2 to refer to the first and second repair policies, respectively.

9.3.3 Warranty Models for a Fixed Lot Size: Minimal Repair Policy

Under this repair policy, when an item fails, it is repaired and restored to the same failure rate at the time of failure. This is the case of repairing components of large and complex systems. Clearly, repairing one or more components will not affect the total failure rate of the system since the aging of the other components will ensure that the system failure rate remains unchanged.

The model can be characterized by a counting process $[N(t), t \geq 0]$ and the probability of having exactly one failure in $[t, t + dt]$ is $h(t)dt$. Ross (1970) shows that this process is an NHPP since the failure rate changes with time, and

$$M_1(w) = \int_0^w h(t)dt = -\ln R(w) = -\ln[1 - F(w)] \tag{9.46}$$

and

$$P[N_1(w) = n] = \frac{[M_1(w)]^n e^{-M_1(w)}}{n!}. \tag{9.47}$$

Using Equations 9.45 and 9.46 we show that $F_1^{(1)}(w) = F(w)$ and

$$F_1^{(n)}(w) = 1 - \sum_{i=0}^{n-1} \frac{[M_1(w)]^i e^{-M_1(w)}}{i!} \quad n > 1. \tag{9.48}$$

9.3.4 Warranty Models for a Fixed Lot Size: Good-as-New Repair Policy

This type of repair is usually performed for simple products where the product is completely overhauled after a failure. It is assumed that the repair will return the product to its "new" condition, that is, the failure rate after repair is significantly lower than the failure rate at the time of failure. Unlike the minimal repair policy, the good-as-new repair policy is a renewal process $\{N_2(t), t \geq 0\}$. Therefore,

$$\begin{aligned} F_2^{(1)}(w) &= F(w) \\ F_2^{(n)}(w) &= \int_0^w F_2^{(n-1)}(w-t) f(t)dt, \quad n > 1, \end{aligned} \tag{9.49}$$

and $M_2(w)$ is given by the standard renewal function

$$M_2(w) = F(w) + \int_0^w M_2(w-t) f(t)dt. \tag{9.50}$$

The values of $M_2(w)$ and $F_2^{(n)}(w)$ can be analytically obtained for the mixed exponential and the Erlang distributions. However, their values for a general failure-time distribution can only be obtained by numerical methods.

EXAMPLE 9.13

The failure time of a product follows an Erlang distribution with k stages, and its CDF is given by

$$F(t) = 1 - e^{-\lambda t} \sum_{i=0}^{k-1} \frac{(\lambda t)^i}{i!}.$$

Assume $\lambda = 2$ failures per year and the repair cost $C_n = n$. What is the total warranty cost for a fixed production lot of 1000 products assuming either repair policies (minimal repair or good-as-new repair)? Assume $w = 0.5$ and 2 years.

SOLUTION

For $k = 2$, the CDF of the Erlang distribution becomes

$$F(t) = 1 - e^{-\lambda t}(1 + \lambda t).$$

We obtain the expected number of failures during the warranty period for the minimum repair policy by using Equation 9.46

$$M_1(t) = \lambda t - \ln(1 + \lambda t).$$

For $w = 0.5$ years

$$M_1(0.5) = 1 - \ln 2 = 0.307.$$

Using $F_1^{(1)}(w) = F(w)$ and Equation 9.48, we obtain the following $F_1^{(n)}(0.5)$ for different values of n:

$$\begin{aligned}
F_1^{(1)}(0.5) &= 1 - 2e^{-1} = 0.2642 \\
F_1^{(2)}(0.5) &= 1 - [e^{-0.307} + 0.307e^{-0.307}] = 0.0385 \\
F_1^{(3)}(0.5) &= 1 - \left[e^{-0.307} + 0.307e^{-0.307} + \frac{(0.307)^2}{2!}e^{-0.307}\right] = 0.0038 \\
F_1^{(4)}(0.5) &= 0.000252.
\end{aligned}$$

Higher orders of $F_1^{(n)}(0.5)$ will rapidly approach zero. Therefore, without significant loss in accuracy, we stop at $F_1^{(4)}(w)$.

$$\begin{aligned}
C_{0.5} &= \sum_{n=0}^{\infty} F_1^{(n)}(0.5) \\
C_{0.5} &= 0.307.
\end{aligned}$$

The total warranty cost at $w = 0.5$ years is $0.307 \times 1000 = \$307$. Similarly, for $w = 2$, we obtain

$$M_1(2) = 4 - \ln 5 = 2.39.$$

and

$$C_2 = \$2,390.$$

The total warranty cost for $w = 2$ is \$2390.

For the good-as-new repair policy the CDF of the Erlang distribution is

$$F(t) = 1 - e^{-\lambda t}(1 + \lambda t),$$

and $F_2^{(n)}(t)$ and $M_2(t)$ are given in Barlow and Proschan (1965) as follows:

$$F_2^{(1)}(t) = F(t)$$

$$F_2^{(n)}(t) = 1 - e^{-\lambda t} \sum_{i=0}^{nk-1} \frac{(\lambda t)^i}{i!}$$

$$M_2(t) = \frac{\lambda t}{k} + \frac{1}{k} \sum_{j=1}^{k-1} \frac{\theta^j}{1-\theta^j}[1 - \exp[-\lambda t(1-\theta^j)]],$$

where $\theta = \exp(2\pi i/k)$ is a kth root of unit.

For $k = 2$, $w = 0.5$, and $\lambda = 2$ failures per year

$$F(t) = 1 - e^{-\lambda t}(1 + \lambda t)$$

$$M_2(t) = [2\lambda t - 1 + e^{-2\lambda t}]/4.$$

Therefore,

$$F_2^{(1)}(0.5) = 0.2642$$

$$F_2^{(2)}(0.5) = 1 - e^{-1.0}\left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}\right]$$

$$= 0.018988$$

$$F_2^{(3)}(0.5) = 1 - e^{-1.0}\left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}\right]$$

$$= 0.0006216$$

Higher values of $F_2^{(n)}$ will rapidly approach zero, and the warranty cost per product at $w = 0.5$ for good-as-new repair policy is

$$C_{0.5} = \sum_{n=1}^{\infty} F_2^{(n)}(0.5)$$

$$= 0.2838.$$

For a lot of 1000 units, the total warranty cost is \$283.8.

The expected number of failures during the warranty period is

$$M_2(0.5) = [2 \times 1 - 1 + e^{-2}]/4 = 0.2838.$$

Similarly, the warranty cost per product for $w = 2$ years is

$$C_2 = \sum_{n=1}^{\infty} F_2^{(n)}(2)$$

$$= 1.7501.$$

The total warranty cost for a lot size of 1000 products is \$1750. ■

It is clear that the expected warranty cost for the minimal repair policy is always higher than the cost for the good-as-new repair policy. This is to be expected, since the product has an increasing failure-rate distribution when the minimal repair policy is used. Moreover, the rate of increase of the warranty cost as the warranty length increases is significantly much higher for the minimal repair policy when compared with the rate of increase for the good-as-new policy.

9.3.5 Warranty Models for a Fixed Lot Size: Mixed Repair Policy

We now consider the case where a repair can be either minimal or good-as-new depending on the type of failure of the product. For example, there are many components in large systems that must be replaced by new components upon failure (i.e., modular electronic components). However, there are other components that require minimum repair upon failure. Indeed, these types of components are commonplace. In other situations, the same component, depending on its age, may require minimal repair or may require a replacement with a new component. In this section, we discuss the latter situation and assume that a component may experience two types of failures: Type 1 requires good-as-new repair, and Type 2 requires minimal repair. A product of age t experiences Type 1 failure with probability $p(t)$ and Type 2 with probability $1 - p(t)$. We now derive expressions for the expected number of failures of each type.

Since the failure rate after a minimal repair remains unchanged, and for an age t, the probability of good-as-new repair at failure is $p(t)$, we can define a good-as-new repair rate of the product as $p(t)h(t)$. After repair, the product continues to function until the next failure. The process is repeated and the intervals between good-as-new repairs are independent and identically distributed with distribution function $\Re(t)$ given by Nguyen and Murthy (1984a) as

$$\Re(t) = 1 - e^{-\int_0^t p(x)h(t)dx} \tag{9.51}$$

and

$$\Re'(t) = p(t)h(t)\bar{\Re}(t),$$

where

$$\bar{\Re}(t) = 1 - \Re(t).$$

The sequence of good-as-new repairs is a renewal process whose expected number of repairs during the warranty period $[0, w]$ is $M_1(w)$

$$M_1(w) = \Re(w) + \int_0^w M_1(w-t)d\Re(t). \tag{9.52}$$

The expected number of minimal repairs at time t given that the age of the product is x can be expressed as

$$m_2(t) = \bar{p}(x)h(x). \tag{9.53}$$

where

$$\overline{p}(t) = 1 - p(t).$$

Using the distribution function of x, we rewrite Equation 9.53 as

$$m_2(t) = \overline{p}(t)h(t)\overline{\mathfrak{R}}(t) + \int_0^t \overline{p}(x)h(x)\overline{\mathfrak{R}}(x)dM_1(t-x). \tag{9.54}$$

The expected number of minimal repairs during the warranty period [0, w] is obtained by integrating Equation 9.54 with respect to t over the warranty period—that is,

$$M_2(w) = \int_0^w m_2(t)dt$$

or

$$M_2(w) = \int_0^w [1 + M_1(w-t)]h(t)\overline{\mathfrak{R}}(t)dt - M_1(w). \tag{9.55}$$

Now we can determine the total expected number of repairs during the warranty period by adding the expected number of each type of repair.

$$M(w) = M_1(w) + M_2(w).$$

Add Equation 9.52 and Equation 9.55 to obtain

$$M(w) = \int_0^w [1 + M_1(w-t)]h(t)\overline{\mathfrak{R}}(t)dt. \tag{9.56}$$

Assuming that c_1 and c_2 are the expected repair costs for the good-as-new and the minimal repair policies, respectively, then

$$\begin{aligned} C_w &= c_1 M_1(w) + c_2 M_2(w) \\ &= (c_1 - c_2)M_1(w) + c_2 \int_0^w [1 + M_1(w-t)]h(t)\overline{\mathfrak{R}}(t)dt. \end{aligned} \tag{9.57}$$

When $p(t) = 1$, then the repair policy is good-as-new only and when $p(t) = 0$, it becomes a minimal repair policy only. Also, if $p(t)$ = constant, p, then $\overline{\mathfrak{R}}(t) = [\overline{F}(t)]^p$, and Equations 9.56 and 9.57 reduce to (Nguyen and Murthy, 1984a)

$$M(w) = M_1(w) / p \tag{9.58}$$

$$C_w = (pc_1 + \overline{p}c_2)M_1(w) / p. \tag{9.59}$$

In some situations, a repair may result in an increase, a decrease, or a constant failure rate of the product. Under such situations, we consider the repair to be imperfect—that is, the failure-time distribution changes after each repair, and the failure-time distribution of a product depends on the number of repairs performed. It is possible that the mixed repair policy discussed earlier in this chapter may experience an imperfect repair that impacts the warranty cost of the product. Therefore, the failure-time distribution of the nth failure needs to be modified, to reflect the effect of imperfect repairs, as follows.

As presented earlier, $F^{(n)}(t)$ and $f^{(n)}(t)$ are the failure-time distribution function and failure-time density function for the nth failure, respectively, and

$$\begin{aligned} F^{(1)}(w) &= F(w) \\ F^{(n)}(w) &= \int_0^w F^{(n-1)}(w-t) f_n(t)dt, \quad \text{for } n > 1. \end{aligned} \tag{9.60}$$

Consider the situation where the failure-time distributions are exponential with different means—that is, $F_i(t) = 1 - e^{-\lambda_i t}$ with $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$, indicating a decrease in the MTTF. Nguyen and Murthy, (1984a) illustrate that by taking Laplace transform of Equation 9.60 and solving for $F^{(n)}(w)$, we obtain

$$F^{(1)}(w) = 1 - e^{-\lambda_1 w} \tag{9.61}$$

$$F^{(n)}(w) = \sum_{i=1}^{n} \left[\prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{\lambda_j}{\lambda_j - \lambda_i} \right) \right] [1 - e^{-\lambda_i w}]. \tag{9.62}$$

Once $F^{(n)}(w)$ is obtained, we can easily obtain the total warranty cost using Equation 9.43.

EXAMPLE 9.14

A manufacturer wishes to estimate the warranty cost for 2000 products. Assume that every time a repair is performed, it decreases the mean time to the next failure. The field data show that the failure-time distribution function is exponential with different means—that is, $F_i(t) = 1 - e^{-\lambda_i t}$ with $\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_w$, where λ_i is the failure rate of the ith failure and λ_w is the failure rate of the last failure before the expiration of the warranty length w.

The manufacturer wishes to extend the warranty for 2 years. The failure rates of the first five failures are 0.5, 0.8, 1, 1.2, and 3 failures per year, respectively. The corresponding repair costs are 20, 19, 18, 18, and 18. Determine the total warranty cost.

SOLUTION

Since $F_i(t) = 1 - e^{-\lambda_i t}$ with $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$, then by using Laplace transform of the following expression,

$$F^{(n)}(w) = \int_0^w F^{(n-1)}(w-t) f_n(t)dt, \quad \text{for } n < 1$$

we obtain

$$F^{(1)}(w) = 1 - e^{-\lambda_1 w}$$

and

$$F^{(n)}(w) = \sum_{i=1}^{n} \left[\prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{\lambda_j}{\lambda_j - \lambda_i} \right) \right] [1 - e^{-\lambda_i w}]$$

λ_1 = 0.5 failures per year,
λ_2 = 0.8 failures per year,
λ_3 = 1 failures per year,
λ_4 = 1.2 failures per year,
λ_5 = 3 failures per year,
w = 2 years,
$F^{(1)}(2)$ = 0.6321,
$F^{(2)}(2)$ = 0.3554,
$F^{(3)}(2)$ = 1.2869,
$F^{(4)}(2)$ = 2.9383, and
$F^{(5)}(2)$ = 5.8716.

$$C_w = \sum_{n=0}^{\infty} C_n F^{(n)}(w)$$
$$C_w = \$201.$$ ■

Manufacturers usually perform burn-in on new products to ensure that the product, when acquired by the customer, has already survived beyond the "infant mortality" or the decreasing failure-rate region. Thus, the number of repairs and the warranty cost during the early period of the customer's ownership of the product are minimized. However, ensuring that all products marketed have survived beyond this failure-rate region is a difficult, if not impossible, task to achieve. Moreover, most warranty periods for nonrepairable products are short. They are indeed shorter than the "infant mortality" region. Therefore, manufacturers place more emphasis on the warranty cost during the decreasing failure-rate region.

The Weibull distribution is often used to model the failure times during this region. However, it is not an analytically tractable model. Researchers hypothesize that a failure distribution of a mixture of two or more exponential densities would exhibit the desired failure-rate characteristics. The following example shows how the warranty cost is estimated during the decreasing failure-rate region for a mixture of exponential densities.

EXAMPLE 9.15

This problem is based on the warranty model developed by Karmarkar (1978). The failure distribution of a product consists of a mixture of two exponential densities that exhibits the desired failure-rate characteristics. This can be interpreted as having a mixture of two kinds of units—a proportion p of defectives with a high failure rate λ_1, and a proportion $1 - p$ of "normal" units with a lower failure rate λ_2. The p.d.f. of the model (see Chapter 1) is

$$f(t) = p\lambda_1 e^{-\lambda_1 t} + (1-p)\lambda_2 e^{-\lambda_2 t}.$$

a. Show that the failure rate is monotone decreasing in t.

b. Assume $\lambda_1 = 4$ failures per year, $\lambda_2 = 2$ failures per year, the cost per repair is \$100, and $p = 0.4$. Determine the warranty cost for a warranty length of 5 years.

SOLUTION

a. The product has a mixed failure rate with

$$f(t) = p\lambda_1 e^{-\lambda_1 t} + (1-p)\lambda_2 e^{-\lambda_2 t}$$
$$F(t) = 1 - pe^{-\lambda_1 t} - (1-p)e^{-\lambda_2 t},$$

and

$$R(t) = pe^{-\lambda_1 t} + (1-p)e^{-\lambda_2 t}.$$

The failure rate is

$$h(t) = \frac{f(t)}{1-F(t)} = \frac{p\lambda_1 e^{-\lambda_1 t} + (1-p)\lambda_2 e^{-\lambda_2 t}}{pe^{-\lambda_1 t} + (1-p)e^{-\lambda_2 t}}.$$

At $t = 0$, $h(0) = p\lambda_1 + (1 - p)\lambda_2$

and

$$\frac{dh(t)}{dt} = \frac{-(\lambda_1 - \lambda_2)^2 p(1-p)e^{-(\lambda_1+\lambda_2)t}}{[pe^{-\lambda_1 t} + (1-p)e^{-\lambda_2 t}]^2} < 0.$$

Therefore, $h(t)$ is monotone decreasing in t.

b. In order to determine the warranty cost for a 5-year warranty period, we first determine the expected number of failures during the warranty period as follows. The Laplace transform of the density function is

$$f^*(s) = [p\lambda_1/(\lambda_1 + s)] + [(1-p)\lambda_2/(\lambda_2 + s)].$$

The Laplace transform of the renewal function is

$$M(s) = \frac{f^*(s)}{s[1 - f^*(s)]}$$
$$M(s) = \frac{\lambda_1\lambda_2 + s[p\lambda_1 + (1-p)\lambda_2]}{s^2\{s + [(1-p)\lambda_1 + p\lambda_2]\}}.$$

Following Karmarkar (1978), we define

$$\Lambda_1 = p\lambda_1 + (1-p)\lambda_2,\ \Lambda_2 = (1-p)\lambda_1 + p\lambda_2 = (\lambda_1 + \lambda_2) - \Lambda_1.$$

Taking the inverse transformation of $M(s)$, we obtain

$$M(t) = [(\Lambda_1\Lambda_2 - \lambda_1\lambda_2)/\Lambda_2^2](1 - e^{-\Lambda_2 t}) + (\lambda_1\lambda_2/\Lambda_2)t.$$

Substituting $\lambda_1 = 4$, $\lambda_2 = 2$, $p = 0.4$, $t = 5$ in the above expression, we obtain

$$\Lambda_1 = 2.8,\ \Lambda_2 = 3.2, \text{ and}$$
$$M(5) = [(8.96 - 80)/3.2^2](0.9999998875) + 12.5$$
$$M(5) = 12.5937.$$

The expected warranty cost = 100 × 12.5937 = \$1259.37. ■

9.4 TWO-DIMENSIONAL WARRANTY

Two-dimensional warranties are common for many products such as copiers and automobiles where both usage and time of ownership are considered simultaneously. In this case, we have two key parameters of the warranty that need to be determined: the length of warranty (time) and usage (number of cycles, miles, etc.). This problem is much more difficult to solve analytically than one-dimensional warranties. The policies discussed above (pro rata, full replacement, mixtures, and others) are also applicable to the two-dimensional warranty policies. There are several methods for modeling these policies. However, in this section, we present a simple two-dimensional warranty policy and demonstrate its formulation and analysis.

Let (T, U) represent the age (time) and usage of a unit at first failure. Both T and U are random and can be used to model the two-dimensional warranty policy. Therefore, we express the CDF of the T and U as $F(t, u)$, defined by

$$F(t,u) = P(T \le t, U \le u).$$

The corresponding density function $f(t, u)$ and hazard-rate function $h(t, u)$ are respectively expressed as

$$f(t,u) = \frac{\partial^2 F(t,u)}{\partial t \partial u}$$

and

$$h(t, u) = \frac{f(t, u)}{\bar{F}(t, u)}.$$

$\bar{F}(t, u)$ is the probability that $T > t$ and $U > u$. The hazard rate is interpreted as follows: $h(t, u)\delta t\ \delta u$ is the probability that the first failure occurs with $(T, U) \in [t, t + \delta t) \times [u, u + \delta u)$ given that $T > t$ and $U > u$ (Blischke and Murthy, 1994).

We follow Kim and Rao (2000) and demonstrate the development of the warranty policy when the T and U follow exponential distribution. We utilize the bivariate exponential (BVE) distribution to describe the relationship between the two warranty variables T and U.

Consider that the warranty time T and the warranty usage U have exponential marginal density functions with parameters λ_1 and λ_2, respectively. The joint probability density function BVE distribution is given by Downton (1970) as

$$f(t, u) = \frac{\lambda_1 \lambda_2}{1-\rho} \exp\left\{-\frac{\lambda_1 t + \lambda_2 u}{1-\rho}\right\} I_0 \left\{\frac{2(\rho \lambda_1 \lambda_2 t u)^{1/2}}{1-\rho}\right\}, \tag{9.63}$$

where ρ is the correlation coefficient between T and U and $I_0(.)$ is the modified Bessel function of the first kind of nth order.

The Laplace transform of Equation 9.63 is obtained by Downton (1970) as

$$L\{f(t, u)\} = f^*(s_1, s_2) = \frac{\lambda_1 \lambda_2}{(\lambda_1 + s_1)(\lambda_2 + s_2) - \rho s_1 s_2}. \tag{9.64}$$

The corresponding distribution function is

$$F^*(s_1, s_2) = \frac{f^*(s_1, s_2)}{s_1 s_2} \quad \text{and}$$

The *n-fold* convolution is

$$F^{*(n)}(s_1, s_2) = \frac{[f^*(s_1, s_2)]^n}{s_1 s_2}.$$

Downton (1970) shows that the *n-fold* convolution when $\rho = 0$ is

$$F^{(n)}(t, u) = P_n(\lambda_1 t) P_n(\lambda_2 u), \tag{9.65}$$

where $P_n(x)$ is the incomplete gamma function defined as

$$P_n(x) = \int_0^x \frac{g^{n-1} e^{-g}}{\Gamma(n)} dg.$$

The expectation and variance of the random variable T for a given value u of the random variable U are

$$E(T/U=u)=\frac{1-\rho}{\lambda_1}+\rho\frac{\lambda_2}{\lambda_1}u$$
$$Var(T/U=u)=\frac{1-\rho}{\lambda_1}\left(\frac{1-\rho}{\lambda_1}+2\rho\frac{\lambda_2}{\lambda_1}u\right).$$

We now show how to estimate the expected number of renewals for the two-dimensional warranty policies.

Let $N(t, u)$ denote the number of renewals over the time and usage rectangle $[(0, t) \times (0, u)]$. Let $N_1(t)$ and $N_2(u)$ be the univariate renewal counting processed for time and usage, respectively. Therefore,

$$N(t,u)=\min\{N_1(t), N_2(u)\}. \tag{9.66}$$

Using Equation 9.65, we obtain the two-dimensional renewal function of T and U as

$$M_\rho(t,u)=E[N(t,u)]=\sum_{n=1}^{\infty}F^{(n)}(t,u).$$

After further derivations, Kim and Rao (2000) show that

$$M_\rho(t,u)=(1-\rho)M_0\left(\frac{t}{1-\rho},\frac{u}{1-\rho}\right), \tag{9.67}$$

where

$$M_0(t,u)=\sum_{n=1}^{\infty}P_n(\lambda_1 t)P_n(\lambda_2 u). \tag{9.68}$$

Similar derivations can be obtained when the probability density functions for both time (age) and usage follow different distributions. In most cases, closed-form expressions might not exist and the warranty policies are then obtained numerically.

9.5 WARRANTY CLAIMS

In the preceding sections we presented different warranty policies and methods for determining warranty length, warranty cost per unit, and warranty reserve fund that the manufacturer should allocate to cover the warranty claims during the service life of the product. It is more beneficial to the manufacturer to allocate warranty cost as a function of the age of the products, the number of claims during any time, and the number of products in service at that time. Therefore, continuous analysis of the claims data enables the manufacturer to more accurately predict the future warranty claims, compare claim rates and cost for different product lines, different components of a product, and units from the same product that are manufactured at different times. Continuous analysis of claims data may also enable the manufacturer to assess product performance that may possibly lead to product improvement.

In this section, we discuss methods for analyzing warranty claims in order to estimate the expected number of warranty claims per unit in service as a function of the time in service. Moreover, forecasts of the number and cost of claims on the population of all units in service along with standard error of the forecasts are also presented (Kalbfleisch et al., 1991).

We first determine the number of claims at time t. Assume that units are sold to the customers on day $x(0 \le x \le \tau)$. The number of claims for a unit at t days later is assumed to be Poisson with mean $\lambda_t(t = 0, 1, \ldots)$. Since the expected number of claims λ_t is small for most situations, λ_t can be interpreted as the probability of a claim at age t. The prediction of cost of claims requires that any repair claim to be immediately entered into the claims database and momentarily used in the analysis. However, repair claims are usually entered using one or more of the following procedures: (1) claims are entered as soon as they occur, (2) claims are individually entered after the lapse of time l, and (3) claims are accumulated and entered as a group at a later date.

Suppose N_x identical products are sold on day x. Repair claims enter the database after a time lag l. We define N_{xtl} to be the number of claims for products sold on day x having an age t and repair claims time lag l. The distribution of N_{xtl} is Poisson with mean $\mu_{xtl} = N_x\lambda_t f_l$, where f_l is the probability that a repair claim enters the database after a time lag l. The expected number of claims for a product up to and including time t is $\Lambda_t = \sum_{u=0}^{t} \lambda_u$.

Thus, the average number of claims at time t for products sold (or put in service) over the period $(0, \tau)$ is

$$m(t) = \frac{\sum_{x=0}^{\tau}\sum_{l=0}^{\infty} N_{xtl}}{\sum_{x=0}^{\tau} N_x}, \quad t = 0, 1, \ldots \tag{9.69}$$

and

$$M(t) = \sum_{u=0}^{t} m(u). \tag{9.70}$$

We follow Kalbfleisch et al. (1991) and assume that the data are available over the calendar time 0 to T. All the claims that entered into the database by time T are included in the analysis and the counts N_{xtl} for x, t, l, such that $0 < x + t + l \le T$ are observed. This makes the estimation of $m(t)$ and $M(t)$ a prediction problem that requires the prediction of N_{xtl}'s. Once $m(t)$ and $M(t)$ are estimated, an estimate of the cost of warranty claims can be easily made. In the following sections, we present two models for estimating the number and cost of warranty claims. The first model operates under the assumption that the probabilities of the lag time, l, for entering (or reporting) claims into the database are known. The second model considers the case when claims are entered as groups into the database.

9.5.1 Warranty Claims with Lag Times

We assume that the probability of entering a warranty claim into the database after a time lag l since the claim took place, f_l, is known. Let $F_1 = f_0 + f_1 + \ldots + f_l$. Moreover, the number of products (identical units of the same product) that are sold on day x, N_x, is known for $x = 0, 1, \ldots, T$, where T is the current date. Thus, the likelihood function for the claim frequency N_{xtl} is

$$L = \prod_{x+t+l\leq T}\prod\prod \frac{(N_x \lambda_t f_l)^{N_{xlt}} e^{-N_x \lambda_t f_l}}{N_{xlt}!}. \tag{9.71}$$

The maximum likelihood estimators obtained from Equation 9.71 are

$$\hat{\lambda}_t = \frac{N_e(t)}{R_{T-t}} \quad t = 0, 1, \ldots, T, \tag{9.72}$$

where

$$N_e(t) = \sum_{x+l\leq T-t}\sum N_{xlt} \tag{9.73}$$

is the total number of claims that have occurred at time (or age) t, and

$$R_{T-t} = \sum_{x=0}^{T-t} N_x F_{T-t-x} \tag{9.74}$$

is the adjusted count of the number of products at risk at time t. The number of products (units) sold on day x is adjusted by the probability that for a product in this group, a claim at age t would be reported by time T. In other words, to account for those claims that occurred before time T and would not be included in the analysis at T, we multiply N_x by a corresponding probability of reporting the claim before T. The average number of claims at time t for products put in service is

$$\hat{m}(t) = \hat{\lambda}_t \tag{9.75}$$

and

$$\hat{M}(t) = \sum_{u=0}^{t} \hat{\lambda}_u = \hat{\Lambda}_t. \tag{9.76}$$

It is important to note that if the time lag l is ignored or if the entering of the claims into the database is instantaneous, then the estimates of $\hat{m}(t)$ and $\hat{M}(t)$ are obtained with all of the F_l's (l = 0, 1, . . .) equal to 1. Moreover, R_{T-t} is, in effect, the total number of products sold that have an age of at least t at time T. Clearly, if there is a time lag l , and if it is purposely ignored in the analysis, then the estimates of λ_t are biased downward, resulting in serious errors in claim predictions.

It is also important to note that true age of a product at time t is greater than t, since products, in most cases, are temporarily stored in a warehouse as soon as they are produced until they are sold. Although the products are not in use while in the warehouse, their failure rates are affected. The longer the storage period, the higher the number of warranty claims during the warranty period since the warranty period starts from the time the product is sold regardless of the age of the product at that time. In this case, manufacturers may reduce such claims by either adjusting the production rate, such that the total inventory and claims cost are minimized, or by redesigning the product to significantly reduce its early failure rate.

The total cost of warranty claims can be estimated by multiplying the average number of claims at time t, $M(t)$, by the average cost of a claim. It can also be estimated by grouping the claims according to the cost as follows: suppose that claim costs are indexed by $c = 1, 2, \ldots, m$ and $k(c)$ is the cost of a claim in the cth group. Also, suppose that $\lambda_t^{(c)}$ is the expected number of claims of cost $k(c)$ for a product at age t, and that $N_{xlt}^{(c)}$ is Poission $(N_x \lambda_t^{(c)} f_l)$ independently for x, t, and l. Following the derivation of Equation 9.66, we obtain

$$\lambda_t^{(c)} = N_e^{(c)}(t) / R_{T-t}. \tag{9.77}$$

Similarly, $m^{(c)}(t)$ and $M^{(c)}(t)$ are natural extensions of Equations 9.75 and 9.76 representing the average number of claims of cost $k(c)$ at age t and up to age t for products sold over the period $0, 1, \ldots, \tau$. The average cost of all claims up to age t for all products sold in $t = 0, 1, \ldots, \tau$ is

$$K(t) = \sum_{c=1}^{m} k(c) M^{(c)}(t). \tag{9.78}$$

EXAMPLE 9.16

A manufacturer produces temperature and humidity chambers that are used for performing accelerated life testing. The chambers are introduced over a 60-day period with equal numbers of chambers being introduced every day. The warranty length of the chamber is 1 year. The true claim rate is 0.004 per chamber per day. Suppose that reporting lags of the claims are distributed over 0–59 days with probabilities $f_l = 1/80$ for $l = 0, 1, \ldots, 19$, and $40, 41, \ldots, 59$ days, and $f_l = 1/40$ for $l = 20, 21, \ldots, 39$ days. The average cost per claim is \$45. Determine the total warranty claims over a 2-month period.

SOLUTION

The estimate of the claim rate at time t is

$$\lambda_t = \frac{N_e(t)}{\sum_{x=0}^{T-t} N_x}.$$

The expected value of $N_e(t)$ is

$$E[N_e(t)] = \lambda_t R_{T-t},$$

where R_{T-t} is given by Equation 9.74.

Thus,

$$\hat{\lambda}_t = \frac{\lambda_t R_{T-t}}{\sum_{x=0}^{T-t} N_x} = \frac{\lambda_t \sum_{x=0}^{T-t} N_x F_{T-t-x}}{\sum_{x=0}^{T-t} N_x}.$$

Since $N_x = N$ for $x = 0, 1, \ldots$, we rewrite the above expression as

$$\hat{\lambda}_t = \frac{\lambda_t \sum_{x=0}^{T-t} F_{T-t-x}}{(T-t)}.$$

Substituting the values of λ_t and F_{T-t-x}, we obtain $\hat{\lambda}_t$ for $t = 0, 1, 2 \ldots, 59$ as shown in Table 9.2.

TABLE 9.2 $\hat{\lambda}_t$ **and** F_{T-t-x} **for Example 9.16**

t	F_{T-t-x}	$\sum_{x=0}^{T-t} F_{T-t-x}$	$\hat{\lambda}_t$
0	0.01250	30.48749	0.00207
1	0.01250	29.48749	0.00203
2	0.01250	28.49999	0.00200
3	0.01250	27.52499	0.00197
4	0.01250	26.56249	0.00193
5	0.01250	25.61250	0.00190
6	0.01250	24.67500	0.00186
7	0.01250	23.75000	0.00183
8	0.01250	22.83750	0.00179
9	0.01250	21.93750	0.00175
10	0.01250	21.05000	0.00172
11	0.01250	20.17500	0.00168
12	0.01250	19.31250	0.00164
13	0.01250	18.46250	0.00161
14	0.01250	17.62500	0.00157
15	0.01250	16.80000	0.00153
16	0.01250	15.98750	0.00149
17	0.01250	15.18750	0.00145
18	0.01250	14.40000	0.00140

TABLE 9.2 (*Continued*)

t	F_{T-t-x}	$\sum_{x=0}^{T-t} F_{T-t-x}$	$\hat{\lambda}_t$
19	0.02500	13.62500	0.00136
20	0.02500	12.86250	0.00132
21	0.02500	12.11250	0.00128
22	0.02500	11.38750	0.00123
23	0.02500	10.68750	0.00119
24	0.02500	10.01250	0.00114
25	0.02500	9.36250	0.00110
26	0.02500	8.73750	0.00106
27	0.02500	8.13750	0.00102
28	0.02500	7.56250	0.00098
29	0.02500	7.01250	0.00094
30	0.02500	6.48750	0.00089
31	0.02500	5.98750	0.00086
32	0.02500	5.51250	0.00082
33	0.02500	5.06250	0.00078
34	0.02500	4.63750	0.00074
35	0.02500	4.23750	0.00071
36	0.02500	3.86250	0.00067
37	0.02500	3.51250	0.00064
38	0.02500	3.18750	0.00061
39	0.01250	2.88750	0.00058
40	0.01250	2.61250	0.00055
41	0.01250	2.36250	0.00053
42	0.01250	2.12500	0.00050
43	0.01250	1.90000	0.00048
44	0.01250	1.68750	0.00045
45	0.01250	1.48750	0.00043
46	0.01250	1.30000	0.00040
47	0.01250	1.12500	0.00038
48	0.01250	0.96250	0.00035
49	0.01250	0.81250	0.00033
50	0.01250	0.67500	0.00030
51	0.01250	0.55000	0.00028
52	0.01250	0.43750	0.00025
53	0.01250	0.33750	0.00023
54	0.01250	0.25000	0.00020
55	0.01250	0.17500	0.00018
56	0.01250	0.11250	0.00015
57	0.01250	0.06250	0.00013
58	0.01250	0.02500	0.00010

■

The expected value for estimate $\hat{\Lambda}_{58} = \sum_{i=0}^{58} \lambda_i$, or $\hat{\Lambda}_{58} = 0.05928$ claims. Assume that 10 chambers are introduced every day. The expected warranty cost for the claims after 2 months is

$$\text{Claim cost} = 0.05928 \times 10 \times 58 \times 45$$

or

$$\text{Claim cost} = \$1,547.$$

9.5.2 Warranty Claims for Grouped Data

In this section we estimate the total warranty claims when the claims are grouped based on the age of the product. For example, we may know the total number of claims for all products whose ages fall between t_1 and t_2 days, $t_2 + 1$ and t_3, and so forth. All the claims for the products whose ages are within a time interval are reported as a group at a future time t—that is, all the products in the group have the same reporting lag. Again, we assume that the reporting lag distribution f_l is known.

Consider some age interval $t = [a, b]$, inclusive, the average number of claims per product for this age interval is

$$M(a,b) = \sum_{t=a}^{b} \lambda_t. \tag{9.79}$$

Using Equations 9.72 and 9.79, we estimate the average number of claims per product for the age interval $[a, b]$ as

$$\sum_{t=a}^{b} \hat{\lambda}_t = \sum_{t=a}^{b} \frac{N_e(t)}{R_{T-t}}. \tag{9.80}$$

If we only observe the total number of claims that have occurred during the interval $[a, b]$—that is, if we observe only $\sum_{t=a}^{b} N_e(t)$, then we approximate Equation 9.79 by

$$M(a,b) = \frac{\sum_{t=a}^{b} N_e(t)}{R(a,b)}, \tag{9.81}$$

where $R(a, b)$ is an estimate of the product days in service. An approximation of $R(a, b)$ is

$$R(a,b) = \frac{1}{2}(R_{T-a} + R_{T-b}) \tag{9.82}$$

or

$$R(a,b) = \frac{1}{b-a+1} \sum_{t=a}^{b} R_{T-t}. \tag{9.83}$$

The expected warranty claim cost, C_{total}, is

$$C_{total} = \bar{k}M(a, b)N_{a,b}, \tag{9.84}$$

where

$\bar{k}$ = is the average cost per claim

$N_{a,b}$ = is the number of products whose ages are between a and b, inclusive.

The approximation (9.84) becomes more accurate as the interval $[a, b]$ decreases.

PROBLEMS

9.1 A manufacturer of medical devices uses shape memory alloys (notably nickel-titanium) to manufacture novel devices. The alloys can be heated at one temperature then heated to recover their original shape. They can be elastically deformed 10–20% more than the conventional materials. The manufacturer produces a "micro vessel correction" device that, when inserted into blood vessels, is warmed by the blood and expands outward to maintain the desired vessel shape.

Experimental results show that the device experiences a constant failure rate of 0.008333 failures per month. The price of the device is $1250, and the yearly production of the device is 3000 devices. Assume that the manufacturer wishes to extend a 5-year warranty for this device. Determine the warranty reserve fund and the adjusted price of the device.

9.2 The producer of high precision instruments needs to extend a warranty for a new sensor that is capable of measuring temperature accurately in the range of 1500°F to 2000°F. Historical data show that the sensor's hazard rate can be expressed as

$$h(t) = kt,$$

where $k = 0.000085$. The cost of the sensor, not including warranty, is $80. Assume that the manufacturer is limiting the selling price to $90 after inclusion of the warranty cost. Determine the warranty length and the total warranty reserve fund for 3000 sensors.

9.3 Determine the lump-sum value to be paid to the customer when the product fails during the warranty period for Problem 9.2.

9.4 A manufacturer wishes to estimate the warranty cost for 3000 products. Assume that every time a repair is performed, it decreases the mean time to the next failure. The field data show that the failure-time distribution function is exponential with different means; that is, $F_i(t) = 1 - e^{-\lambda_i t}$ with $\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_w$, where λ_i is the failure rate of the last failure before the expiration of the warranty length, w. The manufacturer wishes to extend the warranty for 3 months. The failure rates of the first five failures are 0.5, 0.8, 1, 1.2, and 3 failures per year. The corresponding repair costs are 20, 19, 18, 18, 18. Determine the total warranty cost.

9.5 The failure distribution of a product consists of a mixture of two exponential densities that exhibit the desired failure-rate characteristics. This can be interpreted as having a mixture of two kinds of units—a proportion p of defectives with a high failure rate λ_1, and a proportion $1 - p$ of "normal" units with a lower failure rate λ_2. The p.d.f. of the model is

$$f(t) = p\lambda_1 e^{-\lambda_1 t} + (1-p)\lambda_2 e^{-\lambda_2 t}.$$

Assume $\lambda_1 = 4$ failures per year, $\lambda_2 = 3$ failures per year, and the cost per repair is \$80. Determine the warranty cost for a warranty length of 3 years. Also, assume $p = 0.3$.

9.6 Consider the following notation:

c = product price including warranty cost,

w = length of warranty,

m = MTTF of the product,

$C(t)$ = pro rata customer rebate at time t; $C(t) = C(1 - t/w)$, $0 < t < w$,

and

r = warranty reserve cost per unit.

Derive an expression for the total warranty reserve fund for L units of production assuming that the product exhibits the following hazard-rate function

$$h(t) = \frac{\beta}{\lambda^\beta} t^{\beta-1}.$$

9.7 Determine the proportion of the unit cost to be refunded as a lump-sum rebate that makes both the pro rata and lump-sum plan equivalent for Problem 9.6.

9.8 Develop the confidence interval for the expected cost of the pro rata warranty when the failure-time distribution is given by

$$f(t) = \lambda e^{-\lambda t} \quad \lambda > 0.$$

9.9 Develop the confidence interval for the expected cost of the full replacement warranty policy (FRW) when the failure time is given by

$$f(t) = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\gamma}\right] \quad t \geq 0, \gamma > 0, \theta > 0.$$

9.10 A car insurance company wishes to estimate the average warranty claims per year for a newly introduced car model. In collaboration with the manufacturer, the insurance company obtained the following failure data shown in Table 9.3 from the laboratory testing of different components of the car. The claims are

TABLE 9.3 Failure Times in Hours

Failure time	Number of failure units
1,000	10
2,100	12
3,400	9
4,400	11
5,800	10
7,000	14
8,200	10
10,000	8

approximately equal in value regardless of the type of failure. This implies that the failure data of all the components can be analyzed as if it came from one type of failure. The table shows data obtained from subjecting 84 cars to continuous testing. Assume that the failure time follows a Weibull distribution; the cost of a claim is \$85; the average miles per car is 15,000 per year; and 20,000 cars were introduced into the market. What is the total cost of claims per year for the next 5 years?

9.11 A manufacturer of portable telephones intends to extend a 36-month warranty on a new product. An accelerated test shows that the failure time of such products at normal operating conditions exhibits a Weibull distribution with a shape parameter of 2.2 and a scale parameter of 10,000. The price of a telephone unit is \$120 (not including warranty cost). Assuming a total production of 15,000, 20,000, and 25,000 in years 1, 2, and 3, respectively, determine the warranty reserve fund and the adjusted price of the telephone unit.

9.12 The manufacturer of the telephone units wishes to offer the customer a choice of one of the following warranties:

a. A full rebate policy for the duration of the warranty length.

b. A mixed policy which offers a full compensation if the product fails before time w_1, followed by a linearly prorated compensation up to the end of the warranty service of 36 months.

Design a mixed warranty policy whose reserve fund is equivalent to that of the full rebate policy.

9.13 Consider a good-as-new repair policy where the product is completely overhauled after a failure. The repair returns the product to its "new" condition. Assume that the failure-time distribution is

$$f(t) = \begin{cases} \dfrac{t^{\alpha_1 - 1}(1-t)^{\alpha_2 - 1}}{B(\alpha_1, \alpha_2)} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $B(\alpha_1, \alpha_2)$ is the beta function defined by

$$B(\alpha_1, \alpha_2) = \int_0^1 t^{\alpha_1 - 1}(1-t)^{\alpha_2 - 1}\,dt$$

for any real numbers $\alpha_1 > 0$ and $\alpha_2 > 0$. Also,

$$B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}.$$

The parameters of the failure distribution are$\alpha_1 = 1.5$ and $\alpha_2 = 3.0$. Assume $w = 1$ (1 year) and the repair cost $c_n = n$. What is the total warranty cost for a fixed production lot of 2000 units?

9.14 A warm standby system consists of two components in parallel. The cost of replacing a failed unit is \$20. The failure rates of the components are $\lambda_1 = 0.3$ and $\lambda_2 = 0.1$ failures per month. Assuming $W_\phi = 1$ and a warranty length of 6 months, what is the optimal replacement interval?

9.15 Design a new warranty policy for Problem 9.14. Under this policy, the manufacturer provides a replacement free of charge up to time w_1 from the time of the initial purchase. Replacement items in this time period assume the remaining warranty coverage of the original item. Failures in the interval w_1 to w ($>w_1$) are replaced at a pro rata cost. Replacement items in this interval are provided warranty coverage identical to that of the original item (Blischke and Murthy, 1994).

9.16 Burrs are considered a major problem in machining operations, punching, or casting processes. Many applications require that all burrs and sharp edges be removed to the extent that material fragments are not visible and sharpness cannot be felt. A manufacturer produces a cost-effective deburring tool that removes burrs and sharpness. The manufacturer intends to sell the tool (excluding warranty cost) for \$22 and provides a lump-sum warranty for a 6-month duration. The annual production is 6000 tools, and the manufacturer intends to invest the warranty fund at an interest rate of 4% per year and to increase the price by 3% after 6 months. Assume that the tools experience a constant failure rate of 0.006 failures per month. Determine the price of the tool including warranty cost, the proportion of the lump-sum rebate to be paid to the customer when the tool fails before 6 months, and the total warranty reserve fund.

9.17 A manufacturer wishes to change the current warranty policy on one of its products from being a pure pro rata rebate policy with a duration of 12 months to a full rebate policy. The full rebate consists of a lump-sum equivalent to the initial cost of the product if it fails before W_0 months from the date of purchase. The failure time of the product follows an Erlang distribution with three stages ($k = 3$). The parameter λ of the distribution is 0.005 failures per hour. The cost of the product is \$120, and its failure-time distribution function is

$$F(t) = 1 - e^{-\lambda t}\left\{\sum_{j=0}^{k-1} \frac{(\lambda t)^j}{j!}\right\}.$$

a. What is the expected number of failures during a 12-month period?

b. Determine the length of the full rebate that makes it equivalent, in cost, to the current warranty policy.

9.18 Oil well drilling requires 10" to 20" diameter drilling tools with a drill tip capable of drilling through rocks and similar material. A breakdown of the tip may result in significant losses. The manufacture of these tips provides a warranty of a period of 2 months (60 days) of continuous use (regardless of the terrain). The cost of a tip is \$100,000, and the losses due to failure are \$500,000. The tip experiences a Weibull failure rate with $\theta = 70$ (days) and $\gamma = 1.95$. Determine the warranty reserve fund assuming a linear pro rata policy.

9.19 Solve Problem 9.17 assuming that the failure time of the product follows a Weibull distribution with $\theta = 70$ h and $\gamma = 1.95$.

9.20 Assume that the manufacturer of a new printer wishes to provide a two-dimensional warranty with a warranty length of 12 months or 60,000 printed pages. The failure time due to age follows an exponential distribution with $\lambda_1 = 0.001$ failures per year and that the failure due to usage follows an exponential distribution with $\lambda_2 = 0.0005$ failures per page. The cost of the printer is \$1000, but it is replaced upon failure according to a linear pro rata replacement policy.

a. What is the expected number of failures during a 12-month period?

b. Determine the length of the full rebate that makes it equivalent, in cost, to the current warranty policy.

c. Determine the price of the printer that includes the warranty effect.

d. Determine the warranty reserve fund.

9.21 Assume the printer in Problem 9.20 has a failure time due to age that follows a Weibull distribution with $\theta = 5000$ h and the failures due to usage follows an exponential distribution with $\lambda = 0.0005$ failures per page.

a. What is the expected number of failures during a 12-month period?

b. Determine the length of the full rebate that makes it equivalent, in cost, to the current warranty policy.

c. Determine the price of the printer that includes the warranty effect.

d. Determine the warranty reserve fund.

9.22 A manufacturer introduces a new thermal printer over a 6-month period, and the number of printers sold at day x equals 5 when x is even and equals 10 when x is odd. Assume that the claims are entered into the database with time lags distributed over 0–29 days. The probabilities associated with the time lags are $f_l = 1/20$ for $l = 0, 1, \ldots, 9$, $f_l = 1/30$ for $l = 10, 11, \ldots, 19$, and $f_l = 1/60$ for $l = 20, 21, \ldots, 29$. Assume that the average claim rate is 0.001 per printer per day and that the average claim cost is $35. Determine the total warranty claims after 2 months of introducing the printers (the length of warranty is 6 months).

9.23 Assume that the manufacturer in Problem 9.22 decides to group the claims of the printers based on their age. In doing so, the manufacturer groups all the claims for printers whose ages fall within the same 15-day interval. After 2 months, the following claims were accumulated.

Age group	0–14 days	15–30 days	31–45 days	46–60 days
Number of claims	12	10	9	5

The probability distribution of the time lag for reporting the claims of a group is $f_l = 1/20$ for $l = 0, 1, \ldots, 9$, $f_l = 1/30$ for $l = 10, 11, \ldots, 19$, and $f_l = 1/60$ for $l = 20, 21, \ldots, 29$.

Determine the total cost of warranty claims over a 2-month period.

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CHAPTER 10

CASE STUDIES

10.1 CASE 1: A CRANE SPREADER SUBSYSTEM*

10.1.1 Introduction

Cranes are considered the primary method of transferring loads in ports. The operating conditions of a crane depend on the type and weight of the loads to be transferred, the coordinates of the points of pickup and discharge, the intensity of the flow of the loads, the location of the crane, and the effects of the environment (such as temperature, wind, snow, humidity, and dust content).

The coordinates of the points of pickup and discharge as well as the overall dimensions of the loads determine the principal dimensions of the crane (Kogan, 1976). The coordinates are given with known tolerances that determine the accuracy with which the loads are transferred. The tolerances affect the drive mechanisms and their operation.

Ports are usually equipped with several container handling gantry type cranes. Figure 10.1 is a diagrammatic sketch of a typical gantry type crane. During the unloading of a cargo ship, the crane picks up one container at a time from the ship and places it on a chassis of a transporter, which is then moved away to a designated location in the port. Loading a ship with containers is performed when a transporter carrying the container arrives at a specific location within reach of the crane. The container is then lifted by the crane and placed in a proper position on the ship.

Container handling gantry cranes are usually equipped with automated spreaders in order to permit rapid, safe, and efficient loading and unloading of containers. A remotely controlled, telescoping spreader is used to lift loads safely and transmit them vertically through the cornerposts of the container. The spreader depends upon hydraulic pumps to provide the power required for most of its operations. These pumps activate the hydraulic cylinders of the telescoping system, flippers, and twist locks.

The telescoping system of the spreader is used to adjust the length of the spreader to accommodate containers of different lengths (20-ft, 35-ft, or 40-ft containers). The flippers (or "gather guides") are retractable corner guides that help the operator in lowering the spreader onto the container. When a container is lifted, four twist locks must be engaged (one twist lock at each corner casting of the container). If one of the twist locks does not operate or function properly, none of the other three twist locks can be engaged as a safety precaution. The twist locks are turned into locked or unlocked positions through a hydraulic power unit.

* Based on actual operation of a major shipping company.

Reliability Engineering, Second Edition. Elsayed A. Elsayed.

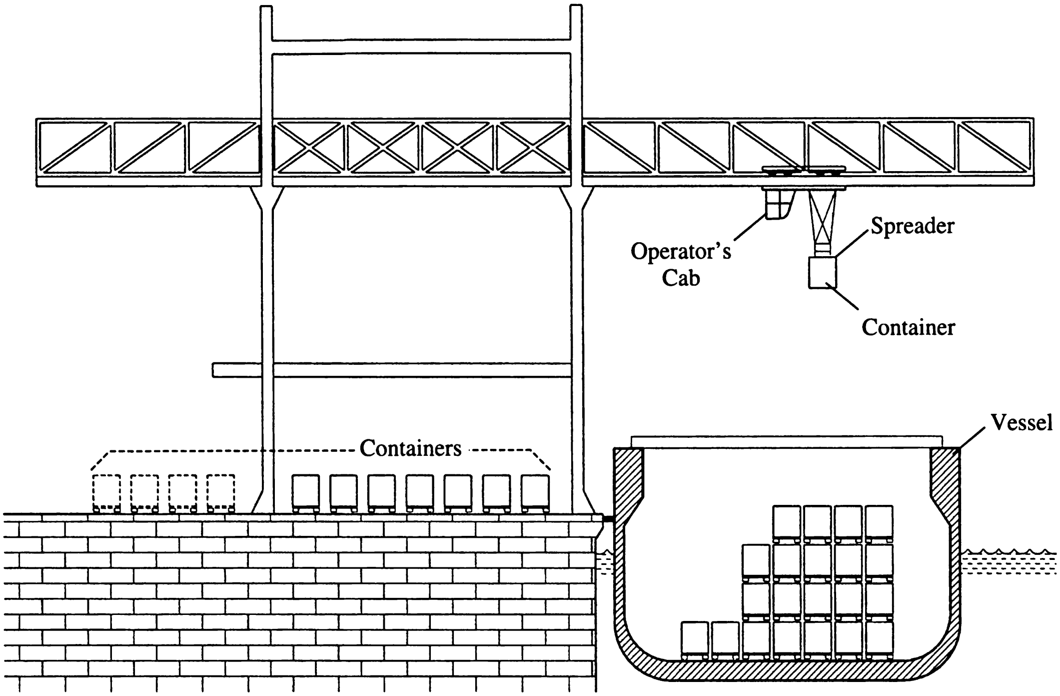

FIGURE 10.1 Diagrammatic sketch of a gantry type crane.

Another critical component of the spreader is the limit switch. The function of the limit switch is to alter the electrical circuit of a machine or piece of equipment so as to limit its motion. The twist lock limit switches are located on the hydraulic cylinder. There are two limit switches per twist lock to terminate the action of the hydraulic cylinder whenever the locked or unlocked positions are reached. Limit switches are also used by the telescoping system to terminate the expansion or contraction of the spreader whenever the desired length is reached. Figure 10.2 shows the plan view of a corner of the hydraulic system of a spreader subsystem.

The maintenance records for six cranes operated by a worldwide company for shipping and receiving containers show that failures involving the spreader and its components account for approximately 65% of all crane failures. The basic components of the spreader are shown in Table 10.1. The table also shows the number of failures, the failure-time distribution, and the parameter(s) of the distribution as estimated from the failure data.

10.1.2 Statement of the Problem

Cranes operate continuously for a period of 10 h/day including weekends. When a ship arrives at the port for loading or unloading, depending on the work load, one or more cranes immediately proceed with the task. Although spreaders are interchangeable between cranes, it is

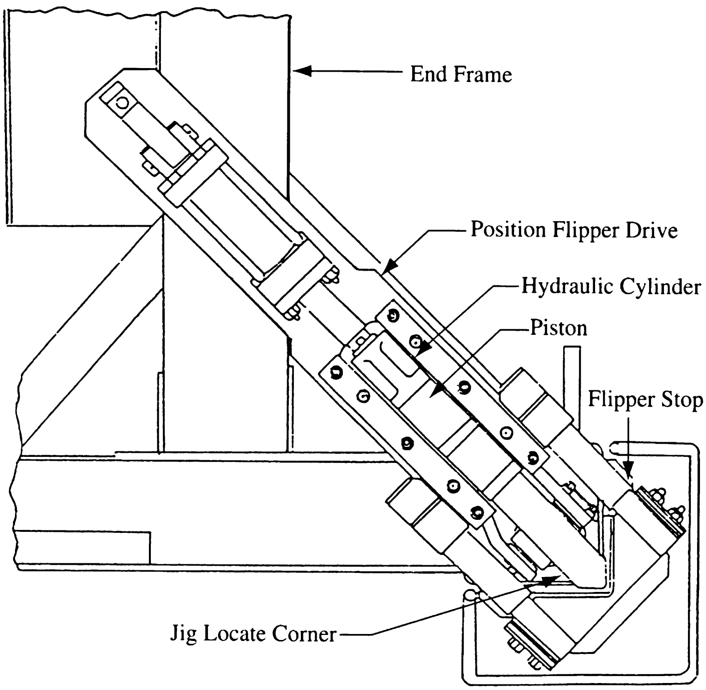

FIGURE 10.2 Hydraulic system of the spreader.

customary that only one spreader is assigned to each crane. The spreader requires an average repair time of 2 h when it fails, independent of the type of failure, that is, repair rate is constant. A failure of the spreader results in delaying the ship at a cost of $10,000/h. The cost of repairs is $200/h.

The management of the company is interested in choosing one of the following alternatives in order to minimize the total cost of the system.

1. Acquire additional spreaders at a cost of $100,000 per unit such that two spreaders are assigned to each crane. The expected life of a spreader is 5 years.
2. Increase the crew size in order to reduce the repair time to 1 h. In turn, this will increase the repair cost to $400/h.

We will investigate these two alternatives and make the proper recommendation.

10.1.3 Solution

The failure data of each critical component of the spreader are analyzed, and the failure-time distributions and their parameters are shown in Table 10.1. We construct the reliability block diagram of the spreader as shown in Figure 10.3. We use the notation *Xn* to refer to component

TABLE 10.1 Failure-Time Distributions of the Components

Type of failure	Components	Number of failures	Failure-time distribution	Parameter(s) of the distribution
A. Electrical	1. Loose connections	8[a]	—	—
	2. Short or open circuit	62	Exponential	7.14×10^{-4} failures/h
	3. Wires parted	10[a]	—	—
B. Flippers	1. Damaged (replaced)	129	Exponential	1.03×10^{-3} failures/h
	2. Hydraulic cylinder	29	Exponential	3.571×10^{-4} failures/h
	3. Flipper mechanism	44	Exponential	8.333×10^{-4} failures/h
C. Twist locks	1. Locks	85	Exponential[b]	6.666×10^{-4} failures/h
	2. Cylinder	55	Exponential[b]	4.640×10^{-4} failures/h
	3. Limit switches	178	Exponential[b]	5.319×10^{-4} failure/h
D. Telescoping system	1. Cylinders	12[a]	—	—
	2. Limit switches	112	Exponential	11.764×10^{-4} failures/h
E. Hydraulic system	1. Power unit	146	Exponential	12.5×10^{-4} failures/h
	2. Hydraulic piping	45	Exponential	8.33×10^{-4} failures/h
	3. Fittings (other than those on the power unit)	32	Exponential	9.756×10^{-4} failures/h
F. Frame (structural)	1. Main structure	5[a]	—	—
	2. Corners	7[a]	—	—
	3. Expanding trays	6[a]	—	—
	4. Corner trays	11[a]	—	—
G. Head block	1. Frame	6[a]	—	—
	2. Twist locks	10[a]	—	—
	3. Limit switches	18[a]	—	—

[a] There is insufficient data to determine the failure-time distribution (rare events during the 9-year period of the study).

[b] Both the Weibull and the exponential distributions appropriately fit the failure data. Comparisons of the sum of squares of errors show that the exponential distribution yields slightly lower values than the Weibull distribution.

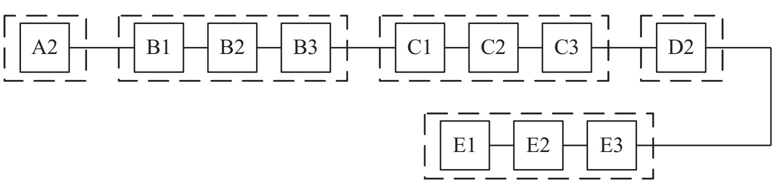

FIGURE 10.3 Block diagram of the spreader components.

n of failure type *X*. For example, *C3* refers to the limit switches of the twist locks (see Table 10.1). In constructing the block diagram, we drop those components that exhibit rare failures.

Using Equation 3.7 we develop an expression for the reliability of the spreader as

$$R_s(t) = \exp\left[-\left(\lambda_{A2} + \sum_{i=1}^{3} \lambda_{Bi} + \sum_{i=1}^{3} \lambda_{Ci} + \lambda_{D2} + \sum_{i=1}^{3} \lambda_{Ei}\right)t\right] \tag{10.1}$$

or

$$R_s(t) = \exp[-8.831\times 10^{-3} t]. \tag{10.2}$$

The hazard rate of the spreader subsystem is

$$h_s(t) = 8.831\times 10^{-3} \text{ failures per hour.}$$

Assuming that the preventive maintenance is performed at scheduled times that do not interrupt the normal operation of the crane, then the availability of the spreader is obtained by substituting $w^*(s) = \lambda/(s + \lambda)$ and $g^*(s) = \mu/(s + \mu)$ into Equation 3.41 or by using Equation 3.63 directly. Thus,

$$A(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

or

$$\begin{aligned} A(t) &= \frac{0.5}{8.831\times 10^{-3} + 0.5} + \frac{8.831\times 10^{-3}}{8.831\times 10^{-3}+0.5} e^{-(8.831\times 10^{-3}+0.5)t} \\ A(t) &= 0.9826 + 0.017355 e^{-0.50883t}. \end{aligned} \tag{10.3}$$

The average uptime availability of the crane during its 10 h of operation per day is obtained from Equation 3.69 as

$$\begin{aligned} A(10) &= \frac{1}{10}\int_0^{10} A(t)dt \\ A(10) &= \frac{1}{10}[0.9826t - 0.034107659 e^{-0.50833t}]\Big|_0^{10} \\ A(10) &= 0.9859897. \end{aligned}$$

If the average repair time is 2 h, the downtime cost of the current system in which one spreader is assigned to each crane can be calculated as

$$\text{Current downtime cost} = [1 - A(10)]\times 10{,}200 = \$142.905 \tag{10.4}$$

We now investigate alternatives 1 and 2. The repair rate of this alternative is 0.5 repairs per hour. The two spreaders will function as a cold standby system—that is, one spreader is used while the second spreader is not in operation, and its failure rate is zero. When the first spreader fails, it undergoes repairs, and the second spreader becomes the primary unit. Therefore, we use Equations 3.111 and 3.112 to obtain

$$\dot{P}_1(t) = -8.831\times10^{-3} P_1(t) + 0.5P_2(t) \tag{10.5}$$

$$\dot{P}_2(t) = -1.508831P_2(t) - 0.991169P_1(t) + 1. \tag{10.6}$$

Solving Equations 10.5 and 10.6 results in

$$P_1(t) = 0.982636 + 0.00868e^{-0.50015t}.$$
$$P_2(t) = 0.016329 - 0.016037e^{-0.50015t}.$$

The availability of the spreader is

$$A(t) = P_1(t) + P_2(t) = 0.998965 - 0.007357e^{-0.50015t}. \tag{10.7}$$

The average up-time availability of the crane is

$$A(10) = \frac{1}{10}\int_0^{10} A(t)dt = \frac{1}{10}[0.998965t + 0.014709e^{-0.50015t}]\Big|_0^{10}$$
$$A(10) = 0.997509.$$

Therefore, the loss due to downtime of the crane during the 10 h of operation is

$$\text{Downtime cost} = [1 - A(10)]10,200 = \$25.45.$$

The cost due to the acquisition of the crane is

$$\text{Cost of acquisition per hour} = \frac{100,000}{365\times10\times5} = \$5.47.$$

Therefore, the total cost of alternative 1 is

$$\text{Total Cost} = 25.45 + 5.47\times10 = \$80.15. \tag{10.8}$$

Alternative 2 increases the repair rate to one repair per hour. Substituting into Equation 3.41, we obtain

$$A(t) = \frac{1}{8.831\times10^{-3}+1} + \frac{8.831\times10^{-3}}{8.831\times10^{-3}+1}e^{-(8.831\times10^{-3}+1.0)t}$$
$$A(t) = 0.9912463 + 0.0087536e^{-1.008831t}.$$

The average uptime availability for alternative 2 is

$$A(10) = \frac{1}{10}\int_0^{10} [0.9912463 + 0.0087536e^{-1.008831t}]dt$$
$$A(10) = 0.9921140.$$

The downtime cost for alternative 2 is

$$\text{Downtime cost} = [1 - A(10)] \times 10,400 = \$82. \quad (10.9)$$

A comparison of Equations 10.4, 10.8, and 10.9 shows that alternative 1 is the preferred alternative since it results in a smaller downtime cost.

10.2 CASE 2: DESIGN OF A PRODUCTION LINE*

10.2.1 Introduction

A food processing line that is used to fill ingredients into packages and to seal and label those packages is shown in Figure 10.4. The food product in this case study is beef stew. The operation of the system is summarized below.

Raw material is manually moved into the production area, and a paper record is made of the lot number and the time and date when the material is placed into the product feeders. This is shown at the upper left of Figure 10.4. Federal government regulations for the food industry require that the material lots be traceable to the production lots in which they are produced. This is required in the event of product recall. The materials of the beef stew product are beef and mixed vegetables. The beef product feeder, which is a hopper with a conveyor, feeds the volumetric filler, shown at the top of Figure 10.4, which fills cups volumetrically. The transport system from the beef filler includes an in-line checkweigher that weighs the contents of the cup and recycles cups back to the filler if they are outside the weight specification (Boucher et al., 1996).

Acceptable cups move onto the vegetable filler where they receive a vegetable fill. When they arrive at the filling station of the packaging machine, they are moved into a cup dumper that overturns the cups into packages. Gravy is added to the package separately by the gravy filler, as shown at the bottom left of Figure 10.4. Through this series of events, an automatic fill is achieved.

The package is a polymer pouch, which is formed from roll stock on the packaging machine at the forming station, which is just prior to the cup dumper (filling) station. The packaging machine, which runs horizontally along the bottom of Figure 10.4, is a horizontal form-fill-seal (F/F/S) machine. The following steps are performed on that machine: after the materials are dumped into the package and gravy is automatically dispensed, the package is

* Based on an actual production line initiated by the Defense Logistics Agency and the combat Rations Advanced Manufacturing Technology Demonstration, Piscataway, New Jersey. The author acknowledges Thomas Boucher of Rutgers University for providing a summary of this case.

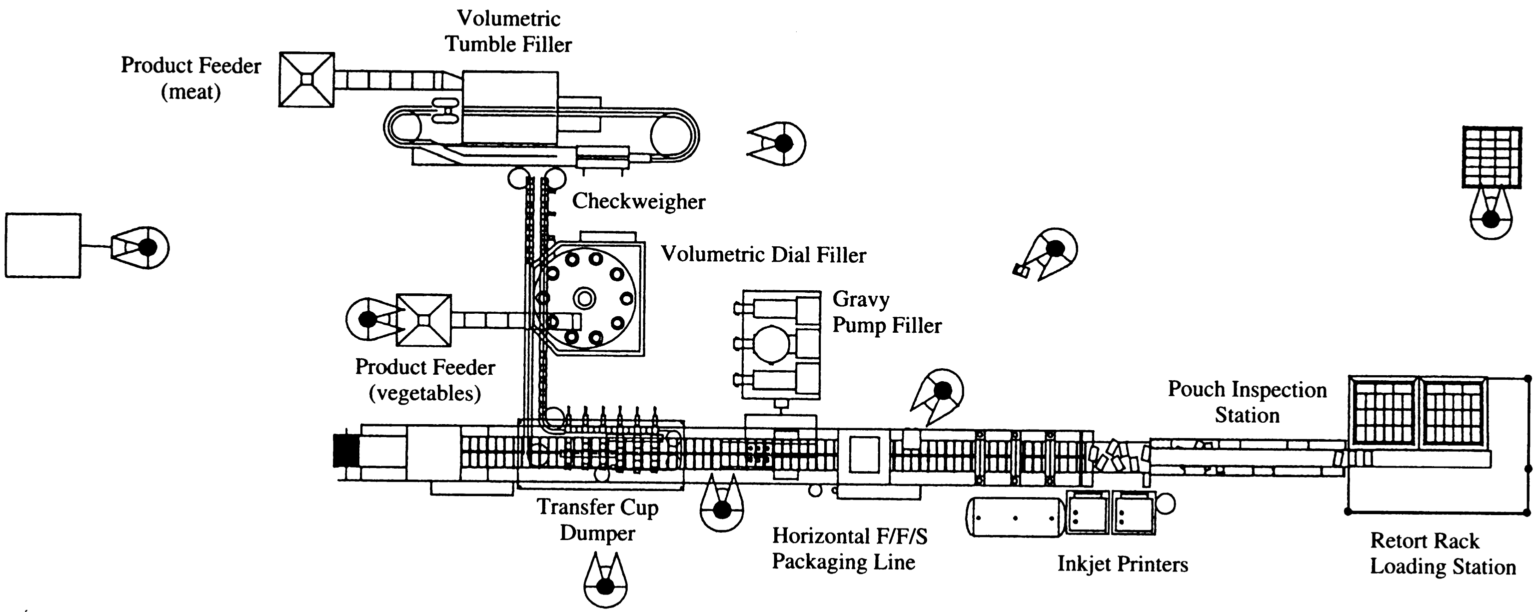

FIGURE 10.4 Layout of food processing facility.

indexed forward through a sealing station, where a top layer of polymer film is heat-sealed to the package. At the next forward index, a Videojet printer labels the package for product name and time of production, and a slitter cuts the roll stock into individual packages. Finally, pouches are inspected and loaded into racks to be taken to the retort station where they are subjected to a temperature of 250 F for 30 min.

The coordination of the cup filling and cup transport system with the packaging line is achieved using the F/F/S packaging machine controller. Each index of the F/F/S presents six pouches for filling. Upon completion of the index, the F/F/S controller signals the product transfer system controller, which in turn signals the cup dumper to fill the six pouches. A return signal from the product transfer system controller to the F/F/S controller acknowledges that the fill is complete and an index of the F/F/S can begin. Start signals to the Videojet printers and the gravy filler are also sent from the F/F/S controller, and a return signal is provided from the gravy filler when the six pouches are filled with gravy.

In this system, all the equipment along the filling and packaging line are controlled as unit operations. Fillers, product feeders, cup transfer system, and the F/F/S have their own controllers. All start and stop operations are accessible at the control panel of the individual operation. Line stoppages that can occur within any of the subsystems are reported to the relevant subsystem controller. A subsystem stoppage causes the line to stop, as the appropriate handshake is not exchanged to cause the F/F/S to continue with another index cycle.

The checkweigher at the exit of the beef filler provides a digital display of the most recent package weights. However, these data are not collected and permanently logged in the factory database. The only automatic data logging occurs on the F/F/S, which keeps a permanent record of certain events occurring during the sealing operation, such as seal temperature and pressure.

10.2.2 Statement of the Problem

The alternative to the current operation is the addition of a production line controller and a centralized control panel incorporating all of the unit operations along the line and providing additional data logging capability.

In this new system, all the unit operation controllers report to a production line controller. It is possible to operate the line centrally from the production line controller as well as locally using controllers for each subsystem. Subsystem status information will be reported to the central controller by each subsystem controller. Information displays and readouts on the central control panel provide status of all subsystem operations including fault conditions. Fault conditions and their downtime will be kept as a permanent record, and the data will be analyzed to identify recurring conditions that should be corrected. The central controller will be able to download information to the checkweigher, beef and vegetable fillers, F/F/S machine, and Videojet printer.

The failure rates of the individual machines and controllers are given in Table 10.2. The repair rate is 0.08 repairs per hour. Investigate the effect of the proposed alternative on the overall production line availability. Recommend changes in the design of the production line that will ensure a minimum production rate of 47,000 pouches per day.

TABLE 10.2 Failure-Rate Data for the Food Processing Line

Equipment	Number of units	Failure rate (constant)
Beef feeder	1	7.5×10^{-4} failures/h
Tumble filler	1	8.9×10^{-4} failures/h
Checkweigher	1	9.5×10^{-5} failures/h
Dial filler	1	5×10^{-4} failures/h
Vegetable feeder	1	4×10^{-4} failures/h
Transfer cup dumper	1	12×10^{-5} failures/h
Gravy pump filler	3	12×10^{-5} failures/h
Horizontal F/F/S	1	20×10^{-5} failures/h
Inkjet printers	2	8×10^{-6} failures/h
Controller for each unit	9	5×10^{-6} failures/h
Central controller	1	6×10^{-6} failures/h
Central control panel	1	6×10^{-7} failures/h

10.2.3 Solution

10.2.3.1 The Current Production Line The equipment of the current production line along with their controllers are considered a series system since the failure of any unit causes stoppage of the line. We construct the block reliability diagram as shown in Figure 10.5.

We estimate the reliability of each block (1 through 9) as follows:

$$R_1(t) = e^{-0.000755t}$$

$$R_2(t) = e^{-0.000895t}$$

$$R_3(t) = e^{-0.0001t}$$

$$R_4(t) = e^{-0.000505t}$$

$$R_5(t) = e^{-0.000405t}$$

$$R_6(t) = e^{-0.000125t}$$

$$R_7(t) = [3e^{-1.5\times10^{-4}t} - 3e^{-3\times10^{-4}t} + e^{-4.5\times10^{-4}t}]e^{-5\times10^{-6}t}$$

or

$$R_7(t) = 3e^{-1.55\times10^{-4}t} - 3e^{-3.05\times10^{-4}t} + e^{-4.55\times10^{-4}t}$$

$$R_8(t) = e^{-0.000205t}$$

$$R_9(t) = (2e^{-8\times10^{-6}t} - e^{-16\times10^{-6}t})e^{-5\times10^{-6}t}$$

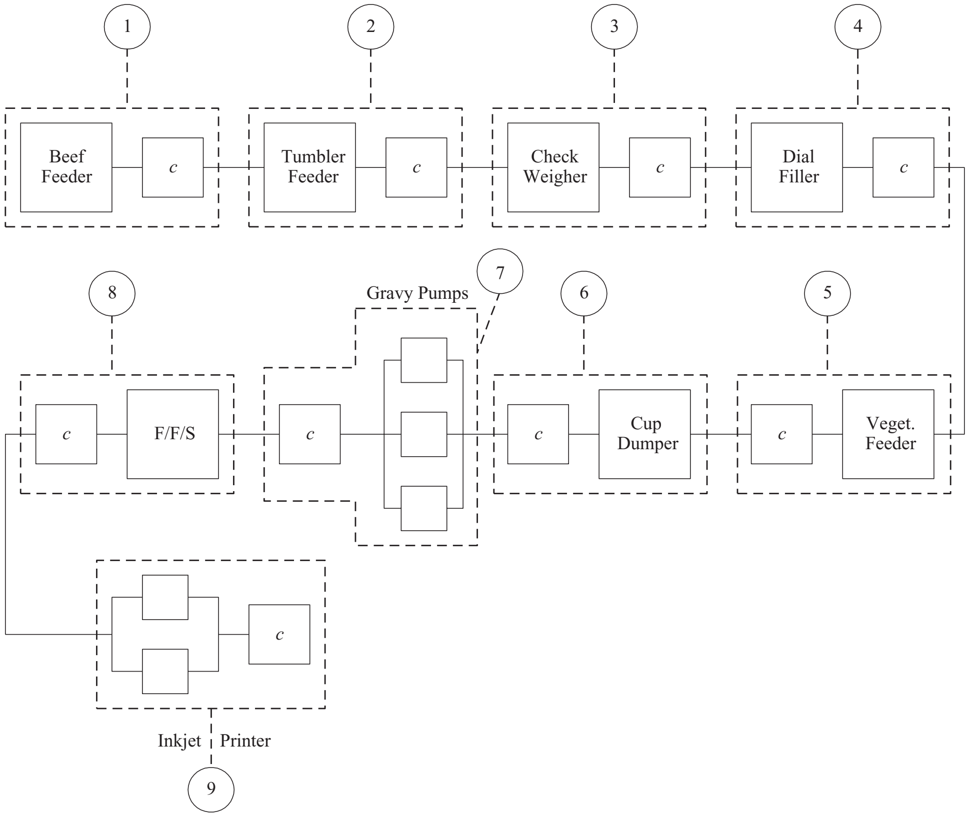

FIGURE 10.5 Block diagram of the current production line. *c* indicates local controller.

or

$$R_9(t) = 2e^{-13\times10^{-6}t} - e^{-21\times10^{-6}t}$$

The reliability of the current line is

$$R_{\text{current}}(t) = e^{-0.00299t}(3e^{-1.55\times10^{-4}t} - 3e^{-3.05\times10^{-4}t} + e^{-4.55\times10^{-4}t})(2e^{-13\times10^{-6}t} - e^{-21\times10^{-6}t})$$

or

$$R_{\text{current}}(t) = 6e^{-3.158\times10^{-3}t} - 6e^{-3.308\times10^{-3}t} + 2e^{-3.458\times10^{-3}t} - 3e^{-3.166\times10^{-3}t} + 3e^{-3.316\times10^{-3}t} - e^{-3.466\times10^{-3}t}. \tag{10.10}$$

Figure 10.6 shows the reliability of the line over an 8-h shift.

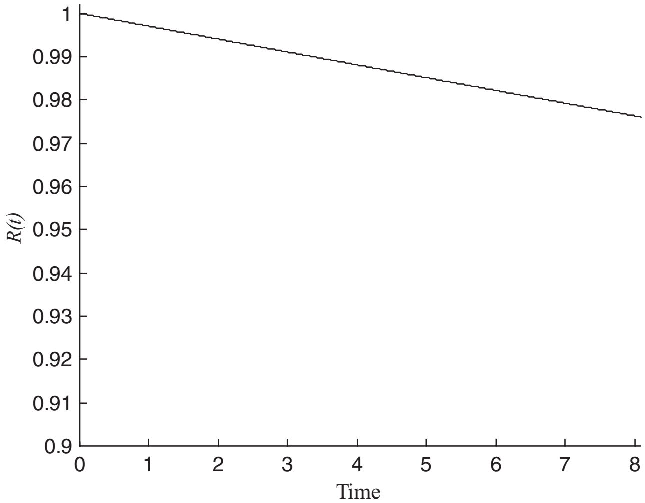

FIGURE 10.6 *R*(*t*) of the production line.

The p.d.f. of the current line is

$$f_{\text{current}}(t) = \frac{-dR_{\text{current}}(t)}{dt}$$

$$\begin{aligned} f_{\text{current}}(t) = {} & 18.948\times10^{-3}e^{-3.158\times10^{-3}t} - 19.848\times10^{-3}e^{-3.308\times10^{-3}t} + 6.916\times10^{-3}e^{-3.458\times10^{-3}} \\ & - 9.498\times10^{-3}e^{-3.166\times10^{-3}t} + 9.948\times10^{-3}e^{-3.316\times10^{-3}} - 3.466\times10^{-3}e^{-3.466\times10^{-3}t}. \end{aligned} \tag{10.11}$$

The effective failure rate of the system is obtained by dividing Equation 10.11 by Equation 10.10:

$$h_{\text{current}}(t) = \frac{f_{\text{current}}(t)}{R_{\text{current}}(t)}. \tag{10.12}$$

The instantaneous availability of the production line, $A_{\text{current}}(t)$, is estimated by using Equation 3.47:

$$A_{\text{current}}(t) = 1 - \frac{h_{\text{current}}(t)}{h_{\text{current}}(t) + \mu}. \tag{10.13}$$

The average uptime availability of the current production line is

$$A_{\text{current}}(8) = \frac{1}{8}\int_0^8 A(t)dt. \tag{10.14}$$

It should be noted that the failure rate of the line is constant, and its value is 3.00780×10^{-3} failures per hour. Thus,

$$A_{\text{current}}(8) = 0.96385545.$$

The number of pouches produced during an 8-h shift is $0.96385545 \times 480 \times 100 = 46{,}265$ pouches.

This quantity is less than the required minimum of 47,000 pouches. Doubling the repair rate to 0.16 repairs per hour results in

$$A(8) = 0.98159510.$$

The corresponding production rate is 47,116 pouches.

10.2.3.2 The Proposed Alternative The central controller and its panel can be considered as a redundant unit for the controller of each piece of equipment. For example, if the local controller of the checkweigher fails, the central controller will perform the functions of the local controller, and there will be no interruption of the production line. Similarly, if the central controller fails and the local controller is operating properly, the production line will not be interrupted. The reliability of the proposed system can be estimated by considering that the local controller ($\lambda = 5 \times 10^{-6}$) is, in effect, connected in parallel with the central controller ($\lambda = 6 \times 10^{-6}$) and its panel ($\lambda = 6 \times 10^{-7}$). Thus, the reliability of the controller system is

$$R_c(t) = (1 - e^{-6.6\times10^{-6}t})(1 - e^{-5\times10^{-6}t})$$

or

$$R_c(t) = e^{-6.6\times10^{-6}t} + e^{-5\times10^{-6}t} - e^{-11.6\times10^{-6}t}.$$

This "effective" controller is connected in series with each piece of equipment. Thus, the reliability of each block becomes

$$
\begin{aligned}
R_1(t) &= e^{-7.566\times10^{-4}t} + e^{-7.55\times10^{-4}t} - e^{7.616\times10^{-6}t}\\
R_2(t) &= e^{-8.966\times10^{-4}t} + e^{-8.95\times10^{-4}t} - e^{-9.016\times10^{-4}t}\\
R_3(t) &= e^{-1.016\times10^{-4}t} + e^{-1\times10^{-4}t} - e^{-1.066\times10^{-4}}\\
R_4(t) &= e^{-5.066\times10^{-4}t} + e^{-5.05\times10^{-4}t} - e^{-5.116\times10^{-6}t}\\
R_5(t) &= e^{-4.066\times10^{-4}t} + e^{-4.05\times10^{-4}t} - e^{-4.116\times10^{-4}t}\\
R_6(t) &= e^{-1.266\times10^{-4}t} + e^{-1.25\times10^{-4}t} - e^{-1.316\times10^{-4}t}\\
R_7(t) &= (3e^{-1.5\times10^{-4}t} + 3e^{-3.0\times10^{-4}t} + e^{-4.5\times10^{-4}t})R_c(t)\\
R_8(t) &= e^{-2.066\times10^{-4}t} + e^{-2.05\times10^{-4}t} - e^{-2.116\times10^{-4}t}\\
R_9(t) &= (2e^{-8\times10^{-6}t} - e^{-16\times10^{-6}t})R_c(t).
\end{aligned}
$$

TABLE 10.3 A Partial Listing of Reliability Values

Time	Reliability
0.017	0.99995035
0.033	0.99990165
0.050	0.99985248
0.067	0.99980289
0.083	0.99975342
0.100	0.99970460
—	—
—	—
7.917	0.97688252
7.933	0.97683388
7.950	0.97678584
7.967	0.97673810
7.983	0.97668970
8.000	0.97664177

Thus,

$$R_{\text{proposed}}(t) = \prod_{i=1}^{9} R_i(t). \tag{10.15}$$

Since direct estimation of $f(t) = -dR(t)/dt$ is difficult to obtain, we calculate $R_{\text{proposed}}(t)$ numerically for the 8-h shift. A sample of the results is shown in Table 10.3.

Examination of the results show that the failure rate of the system is 29.54×10^{-4}. The availability of the system is

$$A(8) = 0.964384861.$$

The number of pouches produced during an 8-h shift is $0.964384861 \times 480 \times 100 = 46{,}290$ pouches. Clearly, the proposed alternative has little effect on the system availability. Moreover, the daily production falls short, as in the current system, of the minimum required quantity of 47,000 pouches.

If the repair rate is doubled, the availability of the proposed system becomes

$$A(8) = 0.981869571.$$

The corresponding number of pouches per day is 47,129.

The availability of the system can further be improved by replacing the dial filler and the vegetable feeder by other equipment that exhibits reduced failure rates.

10.3 CASE 3: AN EXPLOSIVE DETECTION SYSTEM*

10.3.1 Introduction

Explosive detection is a major concern for law enforcement officers. Small size, but powerful, explosive devices and material can be easily concealed in handbags, briefcases, and baggage. Methods for explosive detection have been developed over the years. This has also been paralleled with similar developments in the concealments and the mixes of the explosive material. The result is that the current methods fall short of detecting such a variety of explosives.

In an effort to detect explosives in passenger's baggage in the airport, a manufacturer of scanning systems proposes to design an X-ray system capable of classifying material in baggage as explosive or non-explosive. The system is based on exposing the baggage to be inspected to X-rays generated at an excitation potential between 150 and 500 KeV. The X-ray beams scattered by the object being interrogated are collected and directed to the appropriate 13 detector elements. The X-ray spectra collected by the detector elements every 30 ms are then transferred, without interfering with the spectra acquisition, to 13 digital signal processors (DSP) that perform, using a neural network program, the discrimination between benign and suspicious spectra in real time. A summary of the technical data of this X-ray system is given in Table 10.4. The system consists of six major components as shown in Figure 10.7.

10.3.1.1 The Tower The tower is manufactured from steel tubing and welded to insure minimum flexing and movement during use. As shown in the figure, there is additional cross bracing to further stiffen the structure. The baggage handling system is not connected in any way to the tower, so that any vibrations generated in that subsystem will not be transmitted to the tower and any of the other subsystems.

10.3.1.2 The X-ray Generator The chief demands on the X-ray system are the generation of X-rays at an excitation potential between 150 and 200 KeV, at the highest possible tube current, with the effective linear dimension of the X-ray source (the focal spot) not exceeding 0.8 mm.

The X-ray tube is a closed, high vacuum rotating anode type tube, capable of generating X-rays continuously at 160 KeV. The anode is a circular tungsten strip mounted on appropriately profiled molybdenum disc of 200 mm radius, rotating at 10,000 rpm on a nearly frictionless, heat-conducting liquid metal bearing. Of course, continuous cooling must take place whenever the tube is operated. In addition, active cooling must be continued for 30 min after X-ray generation is switched off. Because of this requirement, an uninterruptible power supply is needed to drive the circulating pumps in the event of a power failure.

10.3.1.3 The Incident Beam Collimators The incident beam collimator is attached at its top to the tube shield to ensure continuous radiation shielding. At the bottom of the collimator is a complex set of movable slits to define the beam shape. The housing encasing these

* This system was developed by SCAN-TECH Security L.P., Northvale, New Jersey.

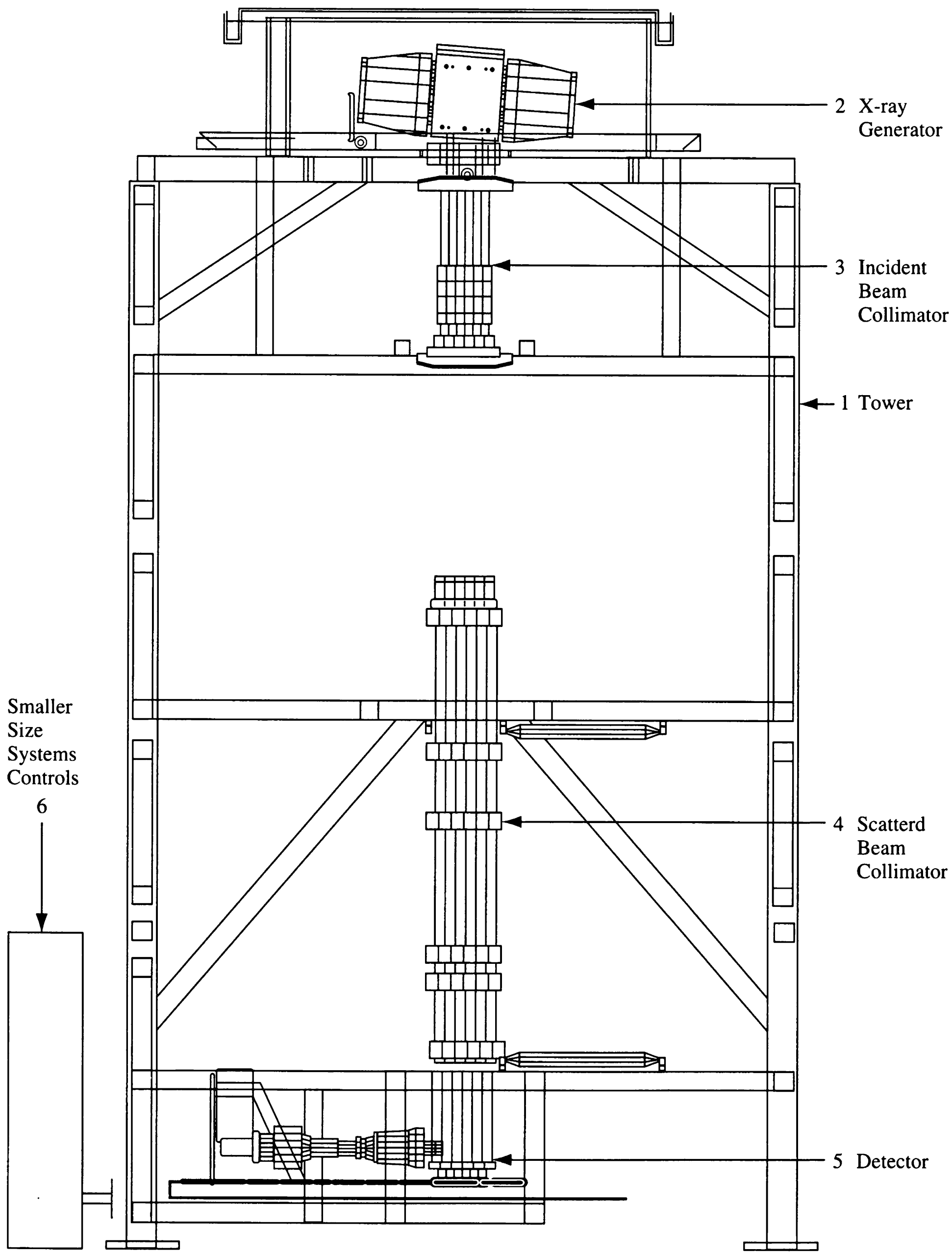

FIGURE 10.7 Schematic drawing of the explosive detection system.

TABLE 10.4 Technical Data for the Explosive Detection System

Inspection principle	Coherent X-ray scattering (CXRS) spectroscopy
Classification principle	Neural net run on digital signal processor
Classification result	Benign versus explosive, masked, or obscured
Detection probability	>99%
False alarm rate	<1%
Maximum bag size	L = 900 mm, W = 700 mm, H = 500 mm
Special resolution (voxel size)	L = 50 mm, W = 60 mm to 90 mm, H = 50 mm
Minimum detectable explosive	100 g (estimated)
Bag throughput	600 bags/h
X-ray high tension	160 kV
X-ray energy range	20 to 160 KeV
X-ray power	4.2 kW continuous, 9.6 kW pulsed (60 s)
Detector energy range	20 to 120 KeV
Detector energy resolution	<1.6 KeV at 60 KeV
Detector count rate	<40,000/s
Spectrum acquisition time	30 ms (for 10 bags/min)
Power for X-ray subsystem	3 phase 480 V/30 A 50/60 Hz
Power for detector subsystem	1 phase 220 V/20 A 50/60 Hz
Power for computer subsystem	1 phase 220 V/10 A 50/60 Hz
Operating temperature	+10°C…+40°C
Storage temperature	–20°C…+70°C

collimators (see Figure 10.8) is of a sliding telescope design that permits vertical movement of the components during alignment. Most of the components in the housing are made from steel. But, any component directly in contact with the X-ray beam is manufactured from a tungsten-10% copper alloy. The primary reason for choosing tungsten is that any characteristic radiation that is produced will superimpose with the lines from the X-ray tube itself. The tungsten is also an excellent X-ray absorber, and the slits need only be 5–10 mm thick for effective shielding.

10.3.1.4 The Scattered Beam Collimators The purpose of the scattered beam collimators is to collect and direct the X-ray beam scattered by the object being interrogated to the appropriate detector element. In order to have a high resolving power, the collimator must control the horizontal and tangential divergence of the scattered beam. This is achieved through the four radial and twelve star collimators, respectively. The radial collimators limit the angular divergence as measured in the plane containing the system axis and the diffracting point. The star collimators, on the other hand, control the divergence perpendicular to this plane. If the beam diverts in either plane, the resolving power will be reduced.

10.3.1.5 The Detector System The main component of the detector system is a cryogenically cooled, single Ge crystal X-ray detector. When an X-ray is absorbed in this crystal

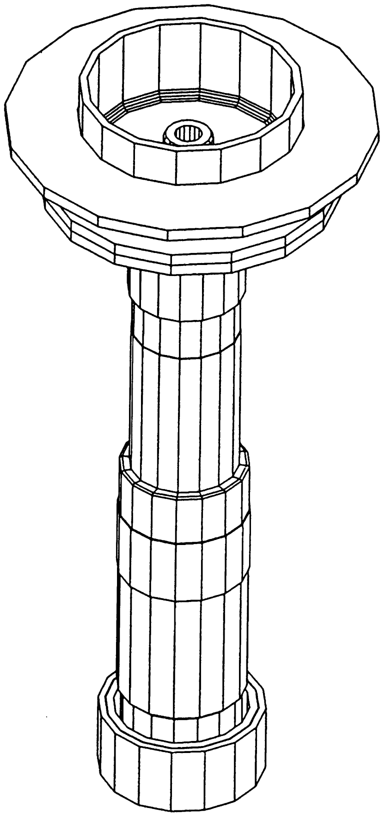

FIGURE 10.8 Telescoping primary beam collimator housing.

it creates a shower of electron-hole pairs, the number of which is proportional to the X-ray energy (E). A large bias voltage (1000 V) across the crystal sweeps the photo-induced charge, to the electrodes on either side of the crystal, creating a current pulse. The integrated area of this pulse is proportional to the photo induced charge and hence to the energy of the absorbed X-ray. The external electronics first amplify and shape this current pulse. A multichannel analyzer then sorts the pulses according to their net charge (X-ray energy) and increments the photon count in the appropriate energy bin.

10.3.1.6 System Controls and Electronics The system is controlled by a computer that acquires X-ray spectra from each of the 13 detector segments every 30 ms. This process consists of reading the energy data from the detector and forming energy spectra in the computer RAM by classifying these energy data. These spectra must then be transferred, without interfering with the spectra acquisition, to the 13 DSPs that perform the discrimination between benign and suspicious spectra in real time.

TABLE 10.5 Failure Times at 200 KeV

12.86	32.62	34.29	34.44	75.17	80.88	92.53	96.44	118.27	142.99
150.87	152.68	158.37	177.80	178.89	198.48	237.67	241.26	317.85	364.38
390.61	470.03	470.58	472.80	476.14	768.47[a]	768.47[a]	768.47[a]	768.47[a]	768.47[a]

[a] Indicates censoring.

TABLE 10.6 Failure Data of the Subsystems

Subsystem	Constant failure rate
X-ray generator	8.5×10^{-5} failures/h
Incident beam optics	7.2×10^{-6} failures/h
Scattered beam optics	10.2×10^{-6} failures/h
Control	7.35×10^{-5} failures/h

10.3.2 Statement of the Problem

The explosive detection system has more than 400 components. The high-voltage excursions in the X-ray generator have a direct effect on the failure times of the system's components. In order to provide availability measures of the system, the management conducts an accelerated life test on the most critical subsystem that is closest to the X-ray generator. This subsystem is identified as the detector. Thirty detector elements are subjected to an environment of 200 KeV and their failure times, in hours, are recorded as shown in Table 10.5. The system normally operates at 160 KeV (acceleration factor is 10).

The failure times of the remaining subsystems are observed over a 2-year period of operation. The failure rates of these subsystems are shown in Table 10.6. The explosive detection system is required to inspect baggage at a rate of one bag every 6 s or 600 bags per hour. When the system fails, it requires 30 min to cool down before repairs begin. The average time of the actual repair is 11 h, which is then followed by a warm-up period of 30 min.

A major airport receives in excess of 40,000 pieces of baggage per day for inspection. (The busy period of the airport is 12 h/day). The management of the airport is interested in determining the number and the configuration of several explosive detection systems that are capable of inspecting 40,000 pieces of baggage per day.

10.3.3 Solution

We use the failure-time data of the detectors to estimate the failure rate at normal operating conditions. Using Equation 5.8, we obtain

$$\lambda_s = \frac{r}{\sum_{i=1}^{r} t_i + \sum_{i=1}^{n-r} t_i^+},$$

where

λ_s = is the failure rate at stress level s;

r = is the number of non-censored failure data;

t_i = is the ith failure time; and

t_i^+ = is the ith censored time.

$$\lambda_s = \frac{25}{5{,}178.9 + 3{,}842.35} = 27.71 \times 10^{-4} \text{ failures per hour.}$$

The failure rate of a detector element at the normal operating conditions is

$$\lambda_{\text{element}} = \frac{\lambda_s}{A_F} = 2.771 \times 10^{-4} \text{ failures per hour.}$$

The detector system is composed of 13 detector elements connected in series. Thus, the failure rate of the detector subsystem is

$$\lambda_{\text{detector}} = 36.02 \times 10^{-4} \text{ failures per hour.}$$

All the subsystems of the explosive detection unit must operate properly for the system to function. Therefore, the subsystems are considered a series configuration with a failure rate of 0.00377 failures per hour (sum of all failure rates of the subsystems, including the detector).

If we assume that the availability of an explosive detection system is 1.0, then the number of systems needed to meet the inspection requirements is

$$\text{Number of systems} = \frac{40{,}000}{(600 \text{ bags per hour} \times 12)} \approx 6.$$

However, the failure and repair rates of the systems cause its availability to be less than 1.0. In order to estimate the availability of the six systems during the 12 h of the airport operation, we develop the state-transition probability as shown below.

Let $P_i(t)$ be the probability that there are i systems failed at time t ($i = 0, 1, \ldots, 6$). Following Equation 3.99 through Equation 3.103, we write

$$\dot{P}_0(t) = -\lambda P_0(t) + \mu P_1(t) \tag{10.16}$$

$$\dot{P}_i(t) = -(\lambda + \mu)P_i(t) + \lambda P_{i-1}(t) + \mu P_{i+1}(t) \quad (i = 1, 2, 3, 4, 5) \tag{10.17}$$

$$\dot{P}_6(t) = -\mu P_6(t) + \lambda P_5(t), \tag{10.18}$$

where μ is the repair rate of the system. Substituting $\lambda = 37.7 \times 10^{-4}$ failures per hour and $\mu = 0.333$ repairs per hour into Equation 10.16 through Equation 10.18 and solving numerically

TABLE 10.7 Partial Listing of the Solution

Time (s)	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	$P_5(t)$	$P_6(t)$
1	0.99981	0.00019	0.00000	0.00000	0.00000	0.00000	0.00000
2	0.99962	0.00038	0.00000	0.00000	0.00000	0.00000	0.00000
3	0.99944	0.00056	0.00000	0.00000	0.00000	0.00000	0.00000
4	0.99925	0.00075	0.00000	0.00000	0.00000	0.00000	0.00000
…..	…..	…..	…..	…..	…..	…..	…..
…..	…..	…..	…..	…..	…..	…..	…..
43,197	0.95476	0.04319	0.00195	0.00009	0.00000	0.00000	0.00000
43,198	0.95476	0.04319	0.00195	0.00009	0.00000	0.00000	0.00000
43,199	0.95476	0.04319	0.00195	0.00009	0.00000	0.00000	0.00000
43,200	0.95476	0.04319	0.00195	0.00009	0.00000	0.00000	0.00000

under the condition $\sum_{i=0}^{6} P_i(t) = 1$, we obtain the values of $P_i(t)$. A partial listing of the results is shown in Table 10.7.

If we define the availability as $P_0(t)$, that is, no failures of the explosive detection systems during the 12-h period, then

$$A(T = 12 \text{ h}) = \frac{1}{43{,}200} \sum_{t=1}^{43{,}200} P_0(t) = \frac{41{,}238.67188}{43{,}200} = 0.954598,$$

and the number of baggage inspected per 12 h is 41,238. This meets the minimum required baggage to be inspected.

10.4 CASE 4: RELIABILITY OF FURNACE TUBES*

10.4.1 Introduction

A major oil company produces 100 million barrels of a Sweet Oil Blend (SOB) per year. The production of the SOB requires hydrogen, which is supplied by five hydrogen plants. The production rate is proportional to the amount of hydrogen supplied—that is, more hydrogen production results in more production of oil until the maximum capacity of the plant is reached. Therefore, it is important that hydrogen plants operate without interruption or equipment failure.

Every hydrogen-producing plant operates a methane reformer furnace (MRF). Each furnace has hundreds of tubes that are filled with catalyst. Methane and steam pass through these tubes at high temperature where hydrogen is produced. The tubes are fabricated from a

* A partial description of this case was reprinted with permission. © 1995. Syncrude, Edmonton, Canada. With contributions by Ming J. Zuo, University of Alberta, Canada.

centrifugally cast alloy steel (chrome, nickel, carbon) in order to minimize corrosion and sustain the creep stress resulting from the high temperatures and pressures within the tubes. The cost of the tubes ranges between \$10 million and \$20 million and represents a high proportion of the total cost of the furnace.

The life of the furnace tubes is dependent on the operating conditions, namely, temperature and pressure. As mentioned earlier, increasing the hydrogen production increases the SOB production. However, increasing the hydrogen production decreases the tube's life and increases the risk of online tube failures.

The cost of the furnace tubes represents a high proportion of the total cost of the furnace. Therefore, the remaining life of the tubes should be accurately estimated so that the tubes are not replaced prematurely. Moreover, the tubes should be periodically inspected for possible crack propagations.

10.4.2 Statement of the Problem

The tubes are placed vertically in the furnace. The tubes have an internal diameter of 5.00 in., a wall thickness of 0.4 in., and a length of 45 ft. The design temperature of the tubes is 1710°F, and the design internal pressure is 400 psi (pounds per square inch). The tubes are flanged at the top end with a reduced diameter at the bottom end that leads into a smaller tube (common to a set of 15 tubes), which in turn feeds into an outlet collection header.

The expected design life of the tubes when the furnace operates at the normal operating conditions is 100,000 h. This design life is calculated based on the Larson–Miller design formula, which relates the properties of the tube material to the operating temperature and pressure. The formula is empirically developed.

Increasing the oil production requires an increase in the hydrogen production, which in turn increases the furnace burner rate. As a result, the temperature in the furnace tends to increase which causes a significant reduction in the remaining life of the tubes. Analysis of failure data collected over 8 years of operation shows that operating a tube at 25°F above the design temperature of 1710°F results in a loss of one-half of the tube remaining life. Temperature readings taken by an optical pyrometer show that approximately five out of fifteen tubes within the same set operate at temperatures of 1724°F.

The furnace fails when four out of fifteen tubes fail or when two consecutive tubes fail. The engineers of the oil company are interested in estimating the reliability of the furnace and the remaining life of each set of tubes. The furnace has 10 sets of tubes, and all are required for the proper function of the furnace. The engineers are also interested in determining the optimal preventive maintenance schedule that minimizes the downtime of the furnace.

The manufacturer of the tubes performs an accelerated life testing at 1835°F on 15 tubes and records the following failure times:

1958, 1013, 12,416, 755, 2901, 7225, 511, 2044, 191, 8034, 6038, 886, 1441, 11,479,
734, 327, 1986, 6701, 12,822, 3090, 3521, 1292, 1245, 8106, 8163.

A 25°F increase in the operating temperature of the furnace results in a 10% increase in the number of oil barrels produced. The net profit per barrel is \$20, and the cost of replacing all the tubes is \$15 million and requires 1 year. What is the operating temperature, above the design temperature that maximizes the profit? It should be noted that the furnace cannot operate beyond 1810°F.

10.4.3 Solution

We first test the validity of using a constant failure-rate model by calculating the Bartlett value, B_r, as follows:

$$\sum_{i=1}^{25} \ln t_i = 194.38$$

$$T = \sum_{i=1}^{n} t_i = 104{,}879.$$

Using Equation 5.2, we obtain

$$B_r = \frac{2\times 25\left[\ln\left(\frac{104{,}879}{25}\right) - \frac{1}{25}\times 194.38\right]}{1+(26)/(6\times 25)} = 24.14.$$

The critical values for a two-tailed test with $\alpha = 0.10$ are

$$\chi^2_{0.95,24} = 13.8484 \quad \text{and} \quad \chi^2_{0.05,24} = 36.4151.$$

Therefore, B_{25} does not contradict the hypothesis that the failure times can be modeled by an exponential distribution.

The failure rate at the stress level of 1835°F is

$$\lambda_{1835°\text{F}} = \frac{25}{104{,}879} = 2.3837\times 10^{-4} \text{ failures per hour.}$$

Since operating the furnace at 25°F above the design temperature results in a loss of one-half the remaining life of the tubes, the acceleration factor between the design temperature (1710°F) and the accelerated test temperature (1835°F) is 25, and

$$\lambda_{1710°\text{F}} = 9.53\times 10^{-6} \text{ failures per hour.}$$

The reliability of a single tube is

$$R(t) = e^{-9.53\times 10^{-6}t}. \tag{10.19}$$

The p.d.f. of the failure time of a single tube is

$$f(t) = \frac{-dR(t)}{dt} = 9.53 \times 10^{-6} e^{-9.53\times10^{-6}t}. \tag{10.20}$$

The mean life at the normal conditions is $1/\lambda = 104{,}931$ h.

If we assume that the tubes have been operating for 5 years (50,000 h), then the residual life of a tube is obtained using Equation 1.113

$$L(t) = \frac{1}{R(t)} \int_t^\infty \tau f(\tau) d\tau - t$$

$$L(50{,}000) = \frac{1}{R(50{,}000)} \int_{50{,}00}^\infty 9.53 \times 10^{-6} te^{-9.53\times10^{-6}t} dt - 50{,}000$$

or

$$L(50{,}000) = \frac{1}{\lambda} = 104{,}931 \text{ h.}$$

Since the exponential distribution has a memoryless property, then the remaining life at any time t is always $1/\lambda$.

The reliability of a set of 15 tubes is obtained by examining the two possible failure modes: (1) four out of fifteen tubes fail or (2) consecutive-2-out-of-15 F: system.

Reliability of the 4-out-of-15 system is

$$R_a(t) = \sum_{r=11}^{15} \binom{15}{r} (e^{-\lambda t})^r (1 - e^{-\lambda t})^{15-r}$$

or

$$R_a(t) = 1365(e^{-9.53\times10^{-6}t})^{11}(1 - e^{-9.53\times10^{-6}t})^4 + 455(e^{-9.53\times10^{-6}t})^{12}(1 - e^{-9.53\times10^{-6}t})^3$$
$$+105(e^{-9.53\times10^{-6}t})^{13}(1 - e^{-9.53\times10^{-6}t})^2 + 15(e^{-9.53\times10^{-6}t})^{14}(1 - e^{-9.53\times10^{-6}t}) + \left(e^{-9.53\times10^{-6}}\right)^{15}$$

or

$$R_a(t) = 1365e^{-10.483\times10^{-5}t} - 5005e^{-11.436\times10^{-5}t} + 6930e^{-12.389\times10^{-5}t} - 4290e^{-13.342\times10^{5}t} + 1001e^{-14.295\times10^{-5}t}. \tag{10.21}$$

Reliability of a consecutive-2-out-of-15 F: system is obtained using Equation 2.15 as

$$R_b(p, 2, n) = \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-j+1}{j} (1-p)^j \, p^{n-j}.$$

Thus,

$$R_b(p,2,15)=\sum_{j=0}^{8}\binom{15-j+1}{j}(1-p)^j\,p^{n-j}$$

$$\begin{aligned}R_b\left(e^{-\lambda t},2,15\right)=&\;e^{-6.671\times10^{-5}t}+28e^{-7.624\times10^{-5}t}-14e^{-8.577\times10^{-5}t}-98e^{-9.53\times10^{-5}t}+145e^{-10.483\times10^{-15}t}\\&-70e^{-11.436\times10^{-5}t}+5e^{-12.389\times10^{-5}t}+5e^{-13.342\times10^{-5}t}-e^{-14.295\times10^{-5}}.\end{aligned} \tag{10.22}$$

If we assume that the probability of a failure due to consecutive-2-out-of-15 equals the probability of a failure due to the 4-out-of-15 system, then the reliability of a set of tubes is

$$\begin{aligned}R_{\text{set}}(t)&=0.5R_a(t)+0.5R_b(t)\\&=0.5e^{-6.671\times10^{-5}t}+14e^{-7.624\times10^{-5}t}-7e^{-8.577\times10^{-5}t}-49e^{-9.53\times10^{-5}t}+755e^{-10.483\times10^{-5}t}\\&\quad-2537.5e^{-11.436\times10^{-5}t}+3467.5e^{-12.389\times10^{-5}t}-2142.5e^{-13.342\times10^{-5t}}+500e^{-14.295\times10^{-5}t}.\end{aligned} \tag{10.23}$$

The system consists of 10 sets of furnace tubes connected in series. Therefore, the reliability of the tubing system is

$$R_{\text{system}}=\prod_{l=1}^{10}R_{\text{set}}(t).$$

A plot of $R_{\text{system}}(t)$ versus time is shown in Figure 10.9.

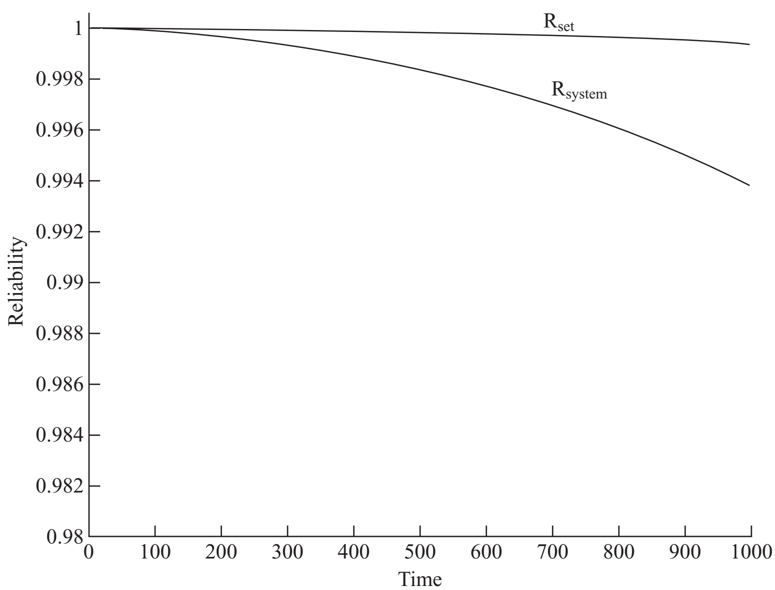

FIGURE 10.9 Reliability of the furnace tubes.

10.4.3.1 Maximization of Profit Operating the furnace at its design temperature of 1710°F results in a cycle of 10 years of operation and 1 year interruption for repair with repair cost $15 million. So the average profit per year in this 11-year cycle is 200 × 10 − 15/11 = 180.455 million per year.

Operating the furnace at its maximum temperature of 1810°F results in a 40% increase in the oil production and an increase of the tubes' failure rate to

$$\lambda_{1,810^\circ\mathrm{F}} = \frac{2.3837\times 10^{-4}}{2} = 1.1918\times 10^{-4} \text{ failures per hour.}$$

The mean life of the tubes is 8390 h or 1 year of operation. The profit resulting from increasing the oil production by 40% per year is 40×10^6 barrels × 20 = $800 million. After 1 year, the tubes are replaced at a cost of $15 million, and the plant is interrupted for 1 year with a loss of a profit of $200 million. So the average profit per year in a 2-year cycle is 200 + 800 − 15/2 = 497.5 million per year. Thus, it is economical to increase the temperature to 1810°F.

However, increasing the temperature to 1735°F results in a 10% increase per year in the oil production and an increase of the tubes' failure rate to

$$\lambda_{1735^\circ\mathrm{F}} = 1.9\times 10^{-5} \text{ failures per hour.}$$

The mean life of the tubes at 1735°F is 52,465 h or 5 years. The profit resulting from increasing the oil production by 10% per year is $50 \times 10^6 \times 20$ = $1000 million, and the cost of replacing the tubes after 5 years is $15 million + cost of plant interruption for 1 year. The average profit per year in a 6-year cycle is (200 + 20) × 5 − 15/6 = 180.833 million per year. Thus, it is economically justified to operate at temperature of 1735°F.

10.4.3.2 Preventive Maintenance Let c_1 be the cost per inspection for a tube and c_2 the cost of an undetected failure, then the optimum inspection interval, T, that minimizes the total expected cost per unit time is the value of T^* which minimizes Equation 8.92 or

$$c(t) = \frac{c_1 + c_2 T}{1 - e^{-\lambda T}} - \frac{c_2}{\lambda}.$$

From Figure 10.9, the failure rate of a set of tubes is $\lambda = 8.45 \times 10^{-5}$ failures per hour.

Assume c_2 = $900 and c_1 = $1,200. Then,

$$c(T^*) = \frac{1200 + 900T^*}{1 - e^{-8.45\times 10^{-5}}} - \frac{900}{8.45\times 10^{-5}}$$

and

$$T^* = 178 \text{ h.}$$

In other words, every set of tubes should be inspected after 178 h of operation.

10.5 CASE 5: RELIABILITY OF SMART CARDS*

10.5.1 Introduction

During the 1990s when smart cards were first introduced, one of the major challenges was to demonstrate its reliability to its early adopters. One of the largest organizations in an island state was keen to adopt smart cards as its employees' ID cards (see Figure 10.10). The organization has about 300,000 employees and operates from a few hundred sites that require security access. With smart card serving as ID for every employee, vital employees' data and passwords can be stored in the smart card. Smart card readers will then be installed at the entrances of various facilities and secured workstation so as to control, monitor, and record access centrally and remotely through computer network. It is thus important that smart cards and the readers must be highly reliable to ensure smooth work flow without compromising the desired level of safety and security.

Physically, the size and make of a smart card are similar to those of a credit card except that a memory chip with 24 memory cells is embedded in it. In fact, today, almost all credit cards are smart cards. The early generation of smart cards is slightly thicker than a typical credit card so as to ensure that the presence of the memory chip will not compromise its structural integrity and durability. With the structural durability out of sight, the key concern was the reliability in writing and reading data from the memory chip.

FIGURE 10.10 Smart cards.

* This case is contributed by Loon Ching Tang of The National University of Singapore.

Based on previous similar applications of memory chip designs, the smart card supplier claimed that the smart card can last for at least 5 years without the need for replacement. Before the adoption of smart card as the employee ID, the organization requested the smart card supplier to provide a third-party assessment of the reliability of the smart card.

10.5.2 Statement of the Problem

The first task is to determine the "real" design life of the smart cards as it might prove unworthy to adopt 5 years as the expected design life. For example, in the event that the failure-time distribution is exponential with mean time to failure (MTTF) = 5 years, 63.2% of the 300,000 cards would have failed by the end of 5 years. The disruptions to work flow and the associated logistics support to rectify these failures will be extensive.

The key consideration is thus to ensure that it will not be seen as a major problem in adopting the technology. Once the design life is determined, the primary objective is to design a test plan to demonstrate that smart card operations meet the reliability target.

Another issue is to relate the number of transactions to calendar time. It is decided to adopt the 80 : 20 principle by assuming that 80% of the users are "light" users with an average of 10 daily transactions while 20% are "heavy" users with an average of 50. It is also assumed that there are 300 working days per year.

10.5.3 Solution

We first provide a justification for translating the statement "it should last for five years" to a statistical reliability requirement.

Since failures may be inevitable within 5 years, from an organization view point, the administrator should not be burdened by too many complaints arising from failures. After discussions, it was decided that, on the average, no more than two complaints should be received daily. Based on 300 working days a year, this translates into maximum of 600 failures per year or 3000 failures over a 5-year period.

Since the total number of cards in use is 300,000, the statistical reliability requirement is thus "no more than 1% failure at the end of five years"; that is, the 1% percentile of the time to failure should be no less than 5 years.

The target MTTF can then be computed by assuming a constant failure rate:

$$\text{Target MTTF} = -TTF_p / \ln(1-p) = \frac{5}{[-\ln(0.99)]} = 497.5 \text{ years.}$$

In terms of number of transactions, we use this target and multiply it by the average number of transactions in a year:

$$\begin{aligned}\text{Target MTTF} &= 497.5 \times [(0.2 \times 50 + 0.8 \times 10) \times 300] \\ &= 2{,}686{,}477 \text{ transactions.}\end{aligned}$$

Alternatively, based on the total circulation of 300,000 cards, the total number of transactions over 5 years is

$$\begin{aligned}TTT &= 300,000 \times (0.2 \times 50 + 0.8 \times 10) \times 300 \times 5\\ &= 8,100,000,000.\end{aligned}$$

As the requirement is no more than 1% failure, the maximum number of failure, r, is 3000 which then results in a target MTTF of

$$\begin{aligned}\text{Target MTTF} &= \frac{TTR}{r} = 8,100,000,000/3000\\ &= 2,700,000 \text{ transactions.}\end{aligned} \tag{10.24}$$

Note that the two target MTTFs should be quite close as, $-\ln(1-p) \approx p$ for small p.

Next, we need to estimate the test time for each transaction. Each smart card is placed inside a reader, and a test message is generated via some random number generator to occupy all the 24 memory cells. A reader will read back and authenticate the written messages. This is because while in a typical application only one out of the 24 cells will be used, it could be any of the cells. Each cycle is treated as one transaction. As there are a few different types of smart card readers and each model takes different time to complete the entire cycle, the maximum duration of 2 min is used as the time taken to complete one transaction.

If no failure is allowed, the lower confidence limit of the MTTF is given by

$$\frac{2 \times TTT}{\chi^2_{2,\alpha}} = \frac{TTT}{-\ln(\alpha)} = \frac{\text{Sample size} \times \text{Test duration}}{-\ln(\alpha)}.$$

Equating this to the target MTTF in Equation 10.24 while setting $\alpha = 0.1$, we have

$$\text{Test duration} = -\ln(0.1) \times \text{Target MTTF/(Number of cards on test)}$$

Suppose 100 readers are used and the number of transactions needed to demonstrate the target MTTF is given by

$$\text{Test transactions} = -\ln(0.1) \times 2,700,000,000/100 = 62,170.$$

Since it can be performed continuously and each transaction takes at most 2 min, that is, 30 transactions per hour, the test duration is given by

$$\text{Test duration} = 62,170/(30 \times 24) = 87 \text{ days.}$$

Logistical support for the test is thus planned for 90 days.

On the other hand, the time line for implementation may present a natural constraint. Suppose that the test report must be completed in 2 months. Setting some allowance for report writing and other unforeseeable issues, the test duration should be no more than 52 days. The sample size needed is given by

$$\begin{aligned}\text{Number of cards needed} &= -\ln(0.1) \times \text{Target MTTF/(Test duration)}\\ &= -\ln(0.1) \times 2{,}700{,}000{,}000/(30 \times 24 \times 52)\\ &= 166 \text{ cards.}\end{aligned}$$

10.6 CASE 6: LIFE DISTRIBUTION OF SURVIVORS OF QUALIFICATION AND CERTIFICATION*

10.6.1 Introduction

System reliability modeling is performed using the life distributions of components comprising the system. For undersea system products and certain other high-consequence systems, components may undergo qualification and certification programs that are intended to increase the reliability of components used in the system. These procedures change the nature of the initial population. In particular, components that survive qualification and certification have a different life distribution than that of the initial population (i.e., the population before qualification and certification are performed). For reliability models for such systems, it is important to use the correct life distribution so that the results of the modeling will faithfully reflect what to expect in service. The purpose of this case is to describe how the life distribution of a population of components is changed by typical qualification and certification procedures.

The material presented here is not specific to any type of product. It applies equally to electronic, optical, or mechanical components.

10.6.2 Background

Most high-consequence systems make some effort to improve reliability by managing the population of components used in the system. Often, active measures are taken to remove from the population of components to be used those components that are judged to have short lifetimes. These measures sometimes are encapsulated in qualification and certification programs whose details may vary from instance to instance but whose purpose is always to acquire a population of components that is longer-lived than the population received from a supplier. The material contained in this case is drawn from the author's experience with reliability engineering for an intercontinental fiber-optic cable telecommunications system (Runge, 1992) developed at AT&T Bell Laboratories in the early 1980s. As such, particular details are cited that may not be shared by other component reliability management programs but, as the primary purpose of this case is to study such programs as decision processes, we expect that the results can be adapted to have broad applicability nonetheless.

* This case is contributed by Michael Tortorella, currently with Rutgers University, Piscataway, New Jersey.

10.6.2.1 Component Failure Modes Component reliability management begins with an examination of potential failure modes. Traditional analysis of component failure modes uses three categories:

1. Early life failures due to manufacturing defects caused by the component manufacturer;
2. Wear-out failures due to the changes in physical and chemical processes of the component with age and usage; and
3. So-called random failures due to the appearance at unpredictable times of environmental shocks that impose a stress exceeding the component's strength.

This model, while perhaps not universally applicable, has proven useful in many situations. It is primarily developed using the hazard rate of the component life distribution, where the so-called bathtub-shaped hazard-rate model corresponds to these three categories by type 1 failures in early life, type 2 failures at end of life, and type 3 failures in middle life (Billinton and Allan, 1992).

Components used in high-reliability, low-volume products like satellites, nuclear weapons, undersea cable telecommunications systems, and other so-called high-consequence systems, are not normally used as received from the supplier. Before use, they are subjected to various active measures whose purpose is to eliminate from the population of components to be used any whose lifetime is suspected to be less than the required service life for the system. These measures are summarized, for purposes of this case, in programs called qualification and certification. As the purpose of these programs is to alter the population of components by screening out those whose life length is suspected to be less than the required system life, the life distribution of the population of survivors of these programs is also altered.

Qualification Qualification is the process of ascertaining whether a population of components can be provided that is economically viable after certification is carried out. That is, qualification is intended to determine whether the proportion of components satisfying the certification criteria is sufficiently large that enough components pass the certification testing so that the overall cost of acquiring a certified population of components is reasonable. In the context of an undersea telecommunications cable system in which the reliability requirement is for a 25-year system life, we want to determine whether there are in the population enough components whose lifetimes are greater than 25 years so that a sensible certification can be implemented without compromising the system profit (clearly there is nothing essential about 25 years; replace it throughout by T if the requirement is T years). If F is the life distribution of the population to be used, then qualification attempts to determine the value of $F(25)$ with the hope that it is close to 0. In particular, let us suppose that there is some number θ, $0 < \theta < 1$, for which a sufficient condition for economical system deployment is $F(25) \le \theta$. That is, the definition of a population being qualified is that its life distribution satisfies $F(25) \le \theta$. For purposes of this case, it is not important how θ is determined; suffice it to say that cost and technology trade-offs are certainly part of this process. The choice of qualification criterion in this form reflects the notion that for a population to be qualified means that a large enough proportion of it has sufficiently long lifetimes that the cost of qualification and certification does not unduly impact the overall economics of the system.

Certification Certification is the selection, from the population judged to be qualified, of individual components for long life. That is, on the basis of additional data collected during a certification test on each individual component from the population judged to be qualified, a decision is made whether to use or to not use the component in assembly of the system. Certification is usually accomplished by some sort of degradation data testing, and again, as in qualification, components undergoing certification accumulate age during the test(s). We denote the accumulated age during qualification by τ_Q and the accumulated age during certification by τ_C, and we let $\tau = \tau_Q + \tau_C$ (allowing that τ_Q and/or τ_C might be zero; in particular, τ_Q will be zero if, as is frequently the case, components tested during qualification are not sent on to the certification process).

The key point now is that certification makes a use/do not use decision on each component individually, based on a judgment formed using data collected during certification testing about whether the component has more or less than a 25-year lifetime. As such, it is possible that the decision could be incorrect. A major objective of this case is to show how the quality of this decision influences the life distribution of the survivors of certification.

We formulate qualification and certification as decision problems and study how the Type I and Type II errors in these decisions influence the life distribution of the population of components that is chosen for use in assembling the system.

10.6.3 Qualification as a Decision Process

Qualification may be construed as a decision process: it is the gathering of information to support a judgment about whether $F(25) \le \theta$ or $F(25) > \theta$. As such, the decision is subject to Type I and Type II errors. The magnitude of these errors has an effect on the life distribution of the survivors of qualification and certification. This section explores the role of Type I and Type II errors in the qualification decision.

Qualification proceeds by a sequence of alternating reliability tests and product redesigns. After each reliability test, a judgment is made as to whether the population is qualified. Let us represent by Q and N the states of a population's being qualified and not qualified, respectively, using these letters both for the state of nature (the "actual," unknowable state of the population), and for the judgments made following reliability testing. Define also S_k to be the state of nature after the $(k-1)$st product redesign and J_k to be the judgment rendered after the kth reliability test, $k = 1, 2, \ldots$. S_k and J_k then take on the values N or Q. Finally, define F_k to be the life distribution of the population after the $(k-1)$st product redesign, with $F_1 = F$ being the original population distribution (before any testing or redesign is performed).

Necessarily, the sequence of judgments comprises some number of N values followed by a Q because once the population is deemed qualified, the sequence of product redesigns and reliability tests halts. Any judgment may individually be mistaken, including the last one, and so we need to examine not only the individual Type I and Type II errors at each step, but the aggregate or overall Type I and Type II errors in the final qualification decision.

TABLE 10.8 Qualification Decision Errors

	F(25) ≤ θ	**F(25) > θ**
Qualification accepts population	Correct decision	Type II error
Qualification rejects population	Type I error	Correct decision

For the qualification decision (after, say, k steps), the *overall* Type I error is rejecting the conclusion that $F_k(25) \le \theta$ when it is in fact true, and the *overall* Type II error is accepting the conclusion that $F_k(25) \le \theta$ when it is in fact false. Let us define $\alpha_Q(k)$ to be the probability of overall Type I error and $\beta_Q(k)$ be the probability of overall Type II error. Then $\alpha_Q(k) = P\{\text{Decide } F_k(25) > \theta \mid F_k(25) \le \theta\}$ and $\beta_Q(k) = P\{\text{Decide } F_k(25) \le \theta \mid F_k(25) > \theta\}$. For purposes of this case, which focuses on the magnitudes of the possible decision errors that can be made, the details of the qualification testing and the use of the information developed thereby to make the decision are not relevant. Obviously, in practice, we would like $\alpha_Q(k)$ and $\beta_Q(k)$ to be as small as possible, subject to whatever time and resource constraints may apply to the qualification undertaking. The present case concerns only how the values of $\alpha_Q(k)$ and $\beta_Q(k)$ influence the life distribution of the final survivors of certification that is performed on (what was decided to be) a qualified population. Table 10.8 provides the definitions of the overall Type I and Type II errors in this context.

Qualification proceeds through a sequential process of testing, decision, and modification of the product if the population is judged to be not qualified, until a decision is reached that the (suitably modified) population is qualified. Let us assume that there are $\nu - 1$ "unqualified" decisions followed by a "qualified" decision, at which point the modification process stops. While the distribution of ν is unlikely to be geometric, because the decisions are likely to be stochastically dependent, we assume that the Type I and Type II errors are independent from one trial to the next and that their probabilities (α and β, respectively) remain the same throughout. This is reasonable provided the type of testing that is done to qualify the population is substantially the same after each modification, the same personnel are involved in each decision, etc. Define $\sigma_k(N)$ (resp., $\sigma_k(Q)$) to be the number of N (resp., Q) states in $\{S_1, \ldots, S_k\}$; then $\sigma_k(N) + \sigma_k(Q) = k$. Recall that if $J_k = \mathrm{Q}$, then $J_1 = \ldots = J_{k-1} = N$. Then we have

$$P\{J_k = \mathrm{Q} \mid S_k = \mathrm{Q}\} = \sum_{n=1}^{k} (1-\alpha-\beta)^n \alpha^{k-n} P\{\sigma_{k-1}(\mathrm{N}) = n-1, \sigma_{k-1}(\mathrm{Q}) = k-n\} \tag{10.25}$$

and

$$P\{J_k = \mathrm{Q} \mid S_k = \mathrm{N}\} = \sum_{n=1}^{k} (1-\alpha-\beta)^{n-1} \alpha^{k+1} \beta P\{\sigma_{k-1}(\mathrm{N}) = n-1, \sigma_{k-1}(\mathrm{Q}) = k-n\}. \tag{10.26}$$

Then the overall Type I and Type II error probabilities at the kth step obtained by using Equations 10.25 and 10.26, respectively, are

$$\alpha_Q(k) = P\{J_k = \mathrm{N} \mid S_k = \mathrm{Q}\} = 1 - P\{J_k = \mathrm{Q} \mid S_k = \mathrm{Q}\}$$

and

$$\beta_Q(k) = P\{J_k = \mathrm{Q} \mid S_k = \mathrm{N}\}.$$

10.6.3.1 Type I and Type II Errors and the Qualified Life Distribution If $J_k = Q$, then we refer to the life distribution of the population judged qualified at step k as the "final" life distribution F_k. To say that $J_k = Q$ means that, as far as we know, $F_k(25) \le \theta$ (which is equivalent to $S_k = Q$). In this section, we study the quality of our knowledge about $F_k(25)$ based on the overall Type I and Type II errors in qualification. Accordingly, we wish to examine $P\{F_k(25) \le \theta \mid J_k = Q\}$. We have

$$\begin{aligned} P\{F_k(25) \le \theta \mid J_k = \mathrm{Q}\} &= P\{S_k = \mathrm{Q} \mid J_k = \mathrm{Q}\} \\ &= \frac{P\{J_k = \mathrm{Q} \mid S_k = \mathrm{Q}\} P\{S_k = \mathrm{Q}\}}{P\{J_k = \mathrm{Q}\}} \\ &= \frac{P\{J_k = \mathrm{Q} \mid S_k = \mathrm{Q}\} P\{S_k = \mathrm{Q}\}}{P\{J_k = \mathrm{Q} \mid S_k = \mathrm{Q}\} P\{S_k = \mathrm{Q}\} + P\{J_k = \mathrm{Q} \mid S_k = \mathrm{N}\} P\{S_k = \mathrm{N}\}} \qquad (10.27) \\ &= \frac{[1 - \alpha_Q(k)] P\{S_k = \mathrm{Q}\}}{[1 - \alpha_Q(k)] P\{S_k = \mathrm{Q}\} + \beta_Q(k) P\{S_k = \mathrm{N}\}}. \end{aligned}$$

Equation 10.27 represents a "degree of belief" in whether $F_k(25)$ is greater than or less than θ when a judgment is made at the kth step that it is. If both $\alpha_Q(k)$ and $\beta_Q(k)$ are equal to zero, then $P\{F_k(25) \le \theta \mid J_k = Q\} = 1$, and our judgment accurately reflects the state of nature. If either $\alpha_Q(k)$ or $\beta_Q(k)$ are positive, then our judgment is flawed, and the larger they are, the less accurate is our judgment.

To complete a computation with Equation 10.27, a model for the distribution of $\{S_1, \ldots, S_k\}$ is needed. A very accurate model would require detailed knowledge of the particular processes of testing and redesign in question, so the following remarks should be taken as illustrative only. A simple model for $\{S_1, \ldots, S_k\}$ is a two-state Markov chain having $p_{QQ} = P\{S_{j+1} = Q \mid S_j = Q\} = 1 - p_{NQ} =$ "large" (close to 1) and $p_{QN} = 1 - p_{NN} =$ "medium." A model like this would allow computation of the terms involving probabilities of events in the σ-field determined by $\{S_1, \ldots, S_k\}$ and so allow completion of computations in Equations 10.25–10.27. In practice, k is usually rather small, on the order of 2 or 3, so computations in this model would not be too onerous.

10.6.3.2 Survivors of Qualification Testing Qualification is usually accomplished through some accelerated life test(s). This means that components accumulate a certain amount of age during qualification, and this is reflected in the life distribution model by postulating a time τ_Q that represents the age consumed during qualification. Usually, the survivors of qualification testing are not used in downstream production; only the untested portion of the population (that is judged to be) qualified is used. However, in cases where the survivors are used in downstream production, if $J_k = Q$, then the life distribution of the survivors of qualification is given by

$$F_Q(t) = \frac{F_k(t+\tau_Q) - F_k(\tau_Q)}{1 - F_k(\tau_Q)}$$

for $t \geq 0$ (we use a new time origin for the population of survivors to be consistent with the actions taken in practice, where the survivors are considered a new, distinct population having a new life distribution that is zero at the time origin).

10.6.4 Certification as a Decision Process

Because certification makes a separate decision for each component regarding whether or not its lifetime exceeds 25 years, it is important to consider the possibility that the decision may be made incorrectly in particular cases. Let L denote the (random) lifetime of a given component from the population judged to be qualified that has survived the certification tests. That is, $L(\omega)$ is the lifetime of component ω, an element of the sample space that describes the population of components entering certification that survive the certification tests. Further, we divide this population into two parts: $A = \{L > 25\}$ and $U = \{L \leq 25\}$. These names are meant to call to mind that lifetimes exceeding 25 years are Acceptable and those not exceeding 25 years are Unacceptable. For each ω, let $C(\omega) = A$ (resp., U) if certification places component ω in A (resp., U). That is, $C(\omega)$ is the result of the certification decision on component ω. This slight abuse of notation should not cause confusion.

If $\omega \in A$ and $C(\omega) = A$, or if $\omega \in U$ and $C(\omega) = U$, then the certification decision is correct for ω. If, on the other hand, $\omega \in A$ and $C(\omega) = U$, or if $\omega \in U$ and $C(\omega) = A$, then the certification decision is incorrect for ω. In the first case, we have an example of *producer's risk*, or Type I error, in which an acceptable component is incorrectly discarded. In the second case, we have an example of *consumer's risk*, or Type II error, in which an unacceptable component is incorrectly retained. For purposes of most high-consequence systems, Type II error is much more significant because a component on which a Type II error is committed is one that is used in the system assembly and that will likely fail before 25 years. The probability of a Type I error is $P\{C = U \mid A\} = \alpha_C$, and the probability of a Type II error is $P\{C = A \mid U\} = \beta_C$. Table 10.9 shows the certification decision errors.

TABLE 10.9 Certification Decision Errors

	Component is in *A*	Component is in *U*
Certification marks component in *A*	Correct decision	Type II error
Certification marks component in *U*	Type I error	Correct decision

10.6.4.1 Life Distributions After the qualification sequence is complete, the life distribution of the population that is judged qualified is given by

$$F_Q(t) = \frac{F_k(t+\tau_Q) - F_k(\tau_Q)}{1 - F_k(\tau_Q)}$$

for $t \geq 0$ (allowing that τ_Q might be zero, which will be the case when only the untested members of the qualified population are sent to certification). In addition, we know that F_k satisfies Equation 10.27.

What is required, now, is the life distribution of the components that have been selected for use by the certification procedure, $P\{L \leq t \mid C = A\}$. However, it is easier to work with the survivor function, so we have

$$\begin{aligned} P\{L > t \mid C = A\} &= \frac{1}{P\{C = A\}} P\{L > t, C = A\} \\ &= \frac{1}{P\{C = A\}} [P\{L > t, C = A, A\} + P\{L > t, C = A, U\}] \end{aligned} \tag{10.28}$$

where $P\{C = A\}$ is given by

$$\begin{aligned} P\{C = A\} &= P\{C = A \mid A\} P(A) + P\{C = A \mid U\} P(U) \\ &= (1 - \alpha_C) P\{L > 25\} + \beta_C P\{L \leq 25\} \\ &= (1 - \alpha_C)[1 - F_Q(25)] + \beta_C F_Q(25). \end{aligned} \tag{10.29}$$

We now assume that, given A (or given U), the lifetime and the certification decision are conditionally independent. This reflects the idea that the decision maker does not know the lifetime of the device exactly. This is, of course, only an approximation because the certification decision is made based on some testing which may lead to an estimate of the device's lifetime, but this would result in a more complicated model that is beyond the scope of this case. Working now with the first term on the right-hand side of Equation 10.28, we obtain

$$\begin{aligned} P\{L > t, C = A, A\} &= P\{L > t, C = A \mid A\} P(A) \\ &= P\{L > t \mid A\} P\{C = A \mid A\} P(A) \\ &= (1 - \alpha_C) P\{L > t, L > 25\} \\ &= (1 - \alpha_C)[1 - F_Q(t \wedge 25)] \\ &= \begin{cases} (1 - \alpha_C)[1 - F_Q(25)], & 0 \leq t \leq 25 \\ (1 - \alpha_C)[1 - F_Q(t)], & t > 25 \end{cases}, \end{aligned} \tag{10.30}$$

where the conditional independence is used at the second step. Similarly, the second term on the right-hand side of Equation 10.28 yields

$$
\begin{aligned}
P\{L>t, C=A, U\} &= P\{L>t, C=A \mid U\}P(U) \\
&= P\{L>t \mid U\}P\{C=A \mid U\}P(U) \\
&= \beta_C P\{L \le t, L>25\} \\
&= \begin{cases} 0, & 0 \le t \le 25 \\ \beta_C\left[F_Q(t)-F_Q(25)\right], & t>25 \end{cases}.
\end{aligned}
$$

Altogether, we obtain

$$
P\{L>t \mid C=A\} = \begin{cases} \dfrac{(1-\alpha_C)[1-F_Q(25)]}{(1-\alpha_C)[1-F_Q(25)]+\beta_C F_Q(25)}, & 0 \le t \le 25 \\ \dfrac{(1-\alpha_C)[1-F_Q(t)]+\beta_C\left[F_Q(t)-F_Q(25)\right]}{(1-\alpha_C)[1-F_Q(25)]+\beta_C F_Q(25)}, & t>25 \end{cases}.
$$

This is the desired survivor function of the population of components that pass the certification screen.

Note that when $\alpha_C = \beta_C = 0$, that is, the certification decision is always correct, then $\{C = A\} = \{L > 25\}$ and we obtain

$$
P\{L>t \mid C=A\} = \begin{cases} 1, & 0 \le t \le 25 \\ \dfrac{1-F_Q(t)}{1-F_Q(25)}, & t>25 \end{cases}
$$

which is the same as $P\{L > t \mid L > 25\}$ as it should be.

In any case, as α and β increase, $P\{L > t \mid C = A\}$ decreases for each fixed t, indicating that the consequences of incorrect certification decisions become more costly as the probability of incorrect decision increases. In effect, what incorrect certification decisions do is increase the number of sub-25-year lifetime components in the population of components that survive qualification and certification, with the consequence that more failures will occur in service due to these components.

10.7 CASE 7: RELIABILITY MODELING OF TELECOMMUNICATION NETWORKS FOR THE AIR TRAFFIC CONTROL SYSTEM*

10.7.1 Introduction

Aircraft operating outside surveillance radar coverage areas, such as oceanic airspace, rely on high frequency (HF) radio for reporting position information (latitude, longitude, altitude, etc.) to the air traffic control system. Figure 10.11 shows the hardware subsystems used in the current oceanic operating environment. The HF radio link suffers from congestion, electrostatic, and

* This case was developed in collaboration with the FAA Technical Center, Atlantic City Airport, New Jersey.

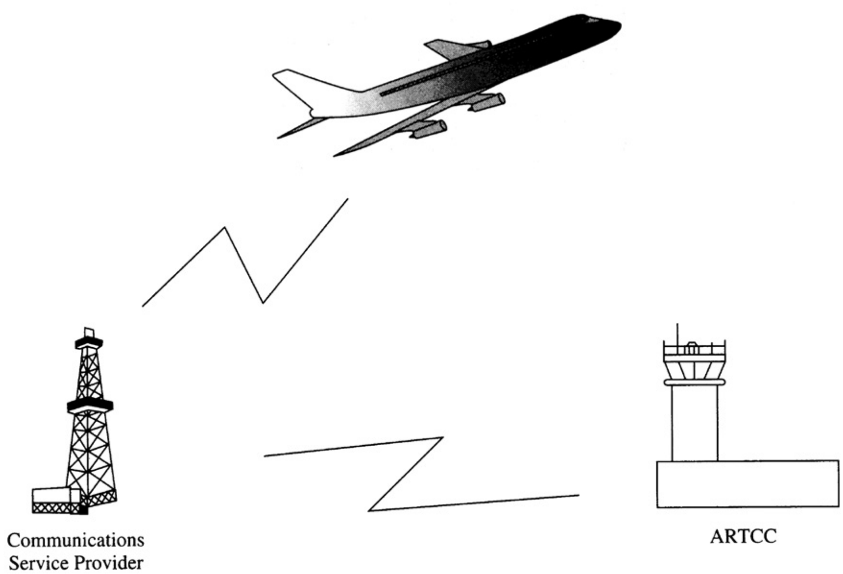

FIGURE 10.11 Current oceanic operating environment (HF radio).

sunspot interference, which cause frequent losses of contact between aircraft and the air traffic controller. This has necessitated a relatively large longitudinal separation of 60 mi between aircraft.

As part of the effort to improve the current air communication, navigation, and surveillance systems in order to meet the demand created by future increases in airspace traffic, the International Civil Aviation Organization (ICAO) defines the Automatic Dependent Surveillance Function (ADSF) as: "A function for use by air traffic services (ATSs) in which aircraft automatically transmit via data link, at intervals established by the ground ATS system, data derived from on-board navigation systems. As a minimum, the data include aircraft identification and three dimensional positions, additional data may be provided as appropriate" (International Civil Aviation Organization, Special Committee on Future Air Navigation and Surveillance, 1988).

Figure 10.11 shows one of several potential hardware subsystem configurations being considered by the Federal Aviation Administration (FAA) that could be used to carry out the ADSF in an oceanic operating environment. An ADSF-equipped aircraft, or Aeronautical Earth Station (AES), will generate position data from on-board navigation systems and automatically, that is, without pilot involvement, transmit the information to communication satellites, such as those of the International Maritime Satellite (INMARSAT) system. In turn, the message is sent to a ground earth station (GES), such as the Communication Satellite Corporation (COMSAT) facility in Southbury, Connecticut. The message is then received by a ground communication network service, similar to the network provided by Aeronautical Radio, Inc. (ARINC), which transfers the message to its intended destination, an en-route

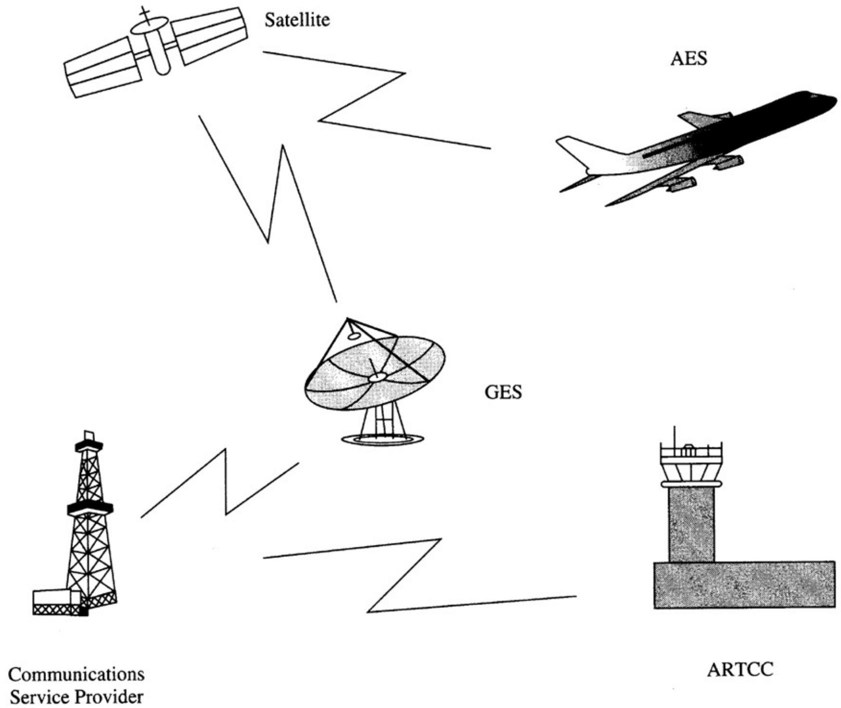

FIGURE 10.12 Proposed oceanic operating environment (ADS).

or oceanic controller's terminal at an Air Route Traffic Control Center (ARTCC), for control actions.

In addition to the main components shown in Figure 10.12, the ADSF can be considered as a collection of hardware, communication, and procedural systems.

The *hardware system* ranges in complexity from the orbital control components of a satellite to the simple telephone lines used to connect the ground communications network with the air traffic control center. The *communication system* allows the exchange of information between the various hardware subsystems. The hardware components (HC blocks) communicate with one another via these communication components (lines). That is, a hardware component is a physical piece of equipment or transmission medium that must be operational or accessible by the ADSF to be operational. A communication component is any protocol, channel, or software code that ensures that these hardware components remain accessible and connected with one another.

The *procedural system* is necessary to coordinate the use of the hardware and communication systems under different operating conditions. For example, in emergency or catastrophic failure situations, it may be necessary to establish a link with a satellite or GES that has a higher gain (signal transmission rate) or a higher level of reliability in order to ensure that messages are received by the ARTCC controller in the required amount of time.

10.7.2 Statement of the Problem

The ADS system is currently under development. The FAA is interested in analyzing the performance of different configurations of the system and in specifying reliability and availability values for the manufacturers of the system's equipment. Typical availability values for critical subsystems or equipment used in the air traffic control system are 0.99999 or higher.

More importantly, the critical components of the system should be identified. This will enable the FAA to recommend design changes of such components to ensure that the reliability objectives of the overall system are realized.

We now describe, in detail, the hardware components of the ADS system (refer to Figure 10.12).

10.7.2.1 Aeronautical Earth Station (AES) Any fixed or rotary wing aircraft is considered an AES. The minimum equipment required for an aircraft to be capable of operating in an ADS environment that utilizes a satellite data link is

- *Navigation systems*: These systems are responsible for generating information describing the location of the aircraft, such as latitude, longitude, and altitude. Many oceanic aircraft use what is known as an inertial reference system (IRS).
- *Automatic Dependent Surveillance Unit (ADSU)*: This can exist as a stand-alone single rack-mounted unit or can be software implementable in a Communications Management Unit, Line Replacement Unit (LRU), or a Flight Management Computer (FMC), as is the case in all Boeing 747-400's. It is the primary unit responsible for executing the ADS function onboard the AES.
- *Communications Management Unit (CMU)*: This acts as a "switcher," routing and forwarding messages to the desired air-ground link.
- *Satellite Data Unit (SDU)*: This unit determines modulation and demodulation, error correction, coding, data rates, and other signal parameters.
- *Radio Frequency Unit (RFU)*: The RFU, operating in full duplex mode, consists of low power amplifiers and frequency conversion electronics.
- *Antenna Subsystem*: This consists of splitters, combiners, high-power amplifiers, low-noise amplifiers, low-gain antennas (LGAs) and high-gain antennas (HGAs), and other radio frequency (RF) distribution units; the combination thereof depends on the level of service required by the AES.

A possible configuration of an AES avionics system is shown in Figure 10.13.

10.7.2.2 Satellite Communications The ADSF data link service will be supported by the satellites that occupy the INMARSAT constellation. The present constellation consists of four primary and seven backup satellites. There are 13 GESs available that receive the satellite signals directly.

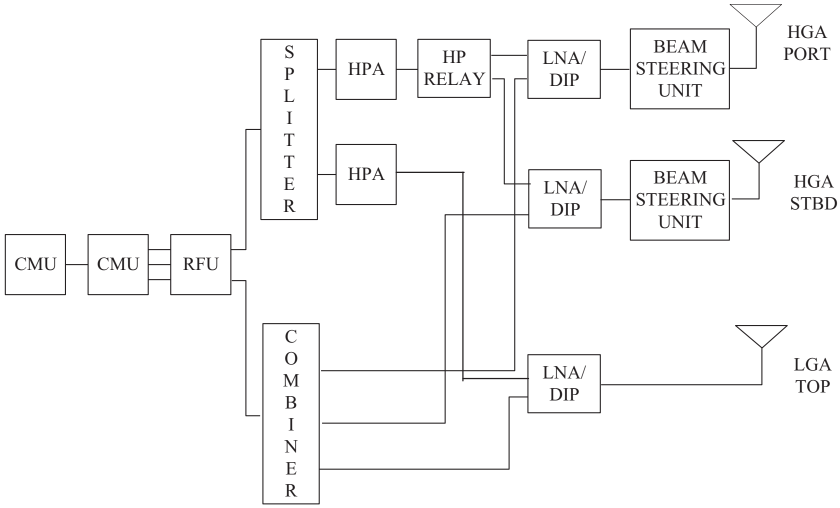

FIGURE 10.13 Possible AES avionics configuration.

10.7.2.3 Terrestrial Subnetwork-Communications Service Provider The terrestrial subnetwork connects the GES with the ARTCC. This network has a primary link and backup or secondary link. The components of each link include modems, an air–ground interface system, and a data network service.

10.7.2.4 Air Route Traffic Control Center The ARTCC provides the control service such as the assignments of airplanes to tracks and ensures that separation standards between the airplanes are maintained. The major components of the ARTCC are the National Airspace Data Interchange Network, a FAA router, and modems.

There are three components for the entire telecommunications networks: hardware, communication, and procedural systems. Their interactions are complex and difficult to model within the scope of this case study. We limit this case study to the modeling of the hardware components.

10.7.2.5 Reliability Data The failure rates of all components are constant. Since most of the equipment is under development, we utilize the reliability data projected by the manufacturer. They are shown in Table 10.10.

10.7.3 Solution

In order to analyze the reliability of the telecommunication networks for the air traffic control system, we first estimate the reliability of each major component separately as follows.

10.7.3.1 Aeronautical Earth Station (AES) One of the proposed avionic configurations of the AES is shown in Figure 10.13, which illustrates how the components of the AES

TABLE 10.10 Failure Data of the System's Components

Component/subsystem	Failure rate (failures/h)
Satellite Data Units (SDUs)	2.5×10^{-6}
Communications Management Unit (CMU)	1.42×10^{-6}
Radio Frequency Unit (RFU)	0.8×10^{-6}
Aeronautical Telecommunications Network (ATN)	1.75×10^{-4}
Air traffic services (ATSs)	2.85×10^{-4}
Automatic Dependent Surveillance Unit (ADSU)	5×10^{-4}
Splitter	3×10^{-6}
Combiner	5×10^{-6}
High-power antenna (HPA)	6×10^{-5}
High-power relay (HPR)	4×10^{-6}
High-gain antenna (HGA)	4×10^{-5}
Low-gain antenna (LGA)	3.5×10^{-5}
Low-noise antenna (LNA)	2×10^{-5}
Beam steering unit (BSU)	8.7×10^{-6}

are *physically* connected to each other. It does not directly show, however, the relationship between the components in terms of the reliability of the avionics subsystem. We assume that the dual HGA performs the same function (in terms of reliability) as the top-mounted LGA. Therefore, the components of the two antennae are connected in parallel to indicate redundancy as shown in Figure 10.14. Moreover, the AES is considered operational only if it is able to send *and* receive information. This implies that the Splitter and the Combiner are connected in series, not in parallel, as the physical diagram suggest.

There are two beam steering units connected in parallel, therefore the reliability of the beam steering is

$$R_{BSU}(t) = 1-(1-e^{-8.7\times10^{-6}t})(1-e^{-8.7\times10^{-6}t})$$
$$R_{BSU}(t) = 2e^{-8.7\times10^{-6}t} - e^{-17.4\times10^{-6}t}.$$

Similarly, the reliability of the low-noise antenna (LNA)/diplexer (DIP) is

$$R_{LNA}(t) = 2e^{-3.5\times10^{-5}t} - e^{-7\times10^{-5}t}.$$

The reliability of the upper path of the parallel configuration is

$$R_{upper}(t) = e^{-14.4\times10^{-5}t}(2e^{-8.7\times10^{-6}t} - e^{-17.4\times10^{-6}t})(2e^{-3.5\times10{-5t}} - e^{-7\times10^{-5}t})$$
$$R_{upper}(t) = 4e^{-18.77\times10^{-5}t} - 2e^{-19.64\times10^{-5}t} - 2e^{-22.27\times10^{-5}t} + e^{-23.14\times10^{-5}t}.$$

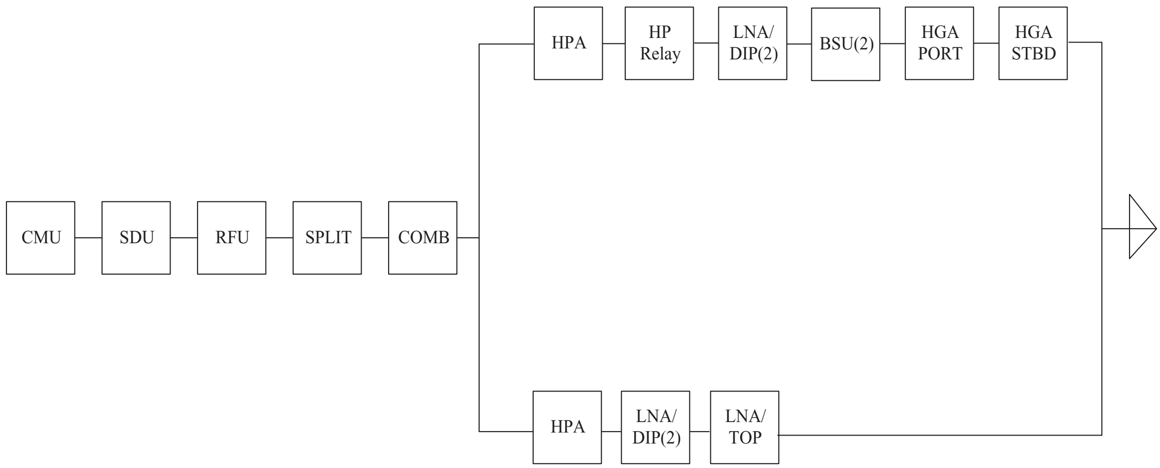

FIGURE 10.14 Reliability block diagram for the AES avionics.

The reliability of the lower path is

$$
\begin{aligned}
R_{\text{lower}}(t) &= e^{-11.5\times10^{-5}t} \\
R_{\text{parallel}}(t) &= 1-(1-R_{\text{upper}}(t))(1-R_{\text{lower}}(t)) \\
&= 4e^{-18.77\times10^{-5}t} - 2e^{-19.64\times10^{-5}t} - 2e^{-22.27\times10^{-5}t} + e^{-23.14\times10^{-5}} + e^{-11.5\times10^{-5}t} - 4e^{-30.27\times10^{-5}t} \\
&\quad + 2e^{-31.14\times10^{-5}t} + 2e^{-33.77\times10^{-5}t} - e^{-34.64\times10^{-5}t}.
\end{aligned} \tag{10.31}
$$

The reliability of an AES is

$$
\begin{aligned}
R_{\text{AES}}(t) &= e^{-12.72\times10^{-6}t}R_{\text{parallel}}(t) \\
R_{\text{AES}}(t) &= 4e^{-2.004\times10^{-4}t} - 2e^{-2.091\times10^{-4}t} - 2e^{-2.354\times10^{-4}t} + e^{-2.441\times10^{-4}t} + e^{-1.277\times10^{-4}t} - 4e^{-3.154\times10^{-4}t} \\
&\quad + 2e^{-3.214\times10^{-4}t} + 2e^{-3.5042\times10^{-4}t} - e^{-3.591\times10^{-4}t}.
\end{aligned} \tag{10.32}
$$

10.7.3.2 Satellite Communications The current satellite communications system includes four primary satellites and seven backup satellites. The failure rates of the primary satellites equal those of the backup satellites. We assume that the satellites are nonrepairable. The satellite subsystem can be modeled as a k-out-of-n or (4-out-of-11) system. Thus,

$$R_{\text{satellite}}(t) = \sum_{r=4}^{11}\binom{11}{r}(e^{-\lambda_s t})^r(1-e^{-\lambda_s t})^{11-r}$$

or

$$R_{\text{satellite}}(t) = 1 - \sum_{r=0}^{3} \binom{11}{r} (e^{-\lambda_s t})^r (1 - e^{-\lambda_s t})^{11-r},$$

where λ_s is the satellite failure rate

$$R_{\text{satellite}}(t) = 1 - \left[(1 - e^{-\lambda_s t})^{11} + 11e^{-\lambda_s t}(1 - e^{-\lambda_s t})^{10} + 55(e^{-\lambda_s t})^2(1 - e^{-\lambda_s t})^9 + 165(e^{-\lambda_s t})^3(1 - e^{-\lambda_s t})^8\right]. \tag{10.33}$$

The projected failure rate, λ_s, is 9.5×10^{-5} failures per hour.

10.7.3.3 Terrestrial Subnetwork This subnetwork has two identical links in parallel. Each link consists of modems, an air ground interface system, and a data network service. The reliability of the subnetwork is

$$R_{\text{subnetwork}}(t) = 2e^{-\lambda_{\text{net}} t} - e^{-2\lambda_{\text{net}} t}, \tag{10.34}$$

where λ_{net} is the failure rate of the subnetwork. Its projected value is 3×10^{-6} failures per hour.

10.7.3.4 Air Route Traffic Control Center (ARTCC) The major components of the ARTCC are the National Airspace Data Interchange Network, FAA router, and modems, all connected in series. Therefore, the reliability of the ARTCC is

$$R_{\text{ARTCC}}(t) = e^{-\lambda_{cc} t}, \tag{10.35}$$

where λ_{cc} is the sum of the failure rates of the individual components of the center. Again, the projected failure rate of the ARTCC is 7.2×10^{-6} failures per hour.

10.7.3.5 The Ground Earth Stations (GES) There are 13 GES available to receive the satellites' signals. Successful communication between the satellites and the GES requires a minimum of nine stations operating at any time. Thus, the reliability of the GES is

$$R_{\text{GES}}(t) = \sum_{r=9}^{13} \binom{13}{r} (e^{-\lambda_{\text{GES}} t})^r (1 - e^{-\lambda_{\text{GES}} t})^{13-r}$$

or

$$\begin{aligned} R_{\text{GES}}(t) = e^{-9\lambda_{\text{GES}} t}[&715(1 - e^{-\lambda_{\text{GES}} t})^4 + 286e^{-\lambda_{\text{GES}} t}(1 - e^{-\lambda_{\text{GES}} t})^3 \\ &+ 78e^{-2\lambda_{\text{GES}} t}(1 - e^{-\lambda_{\text{GES}} t})^2 + 13e^{-3\lambda_{\text{GES}} t}(1 - e^{-\lambda_{\text{GES}} t}) + e^{-4\lambda_{\text{GES}} t}], \end{aligned} \tag{10.36}$$

where λ_{GES} is the failure rate of the GES and its projected value is 3.75×10^{-6} failures per hour.

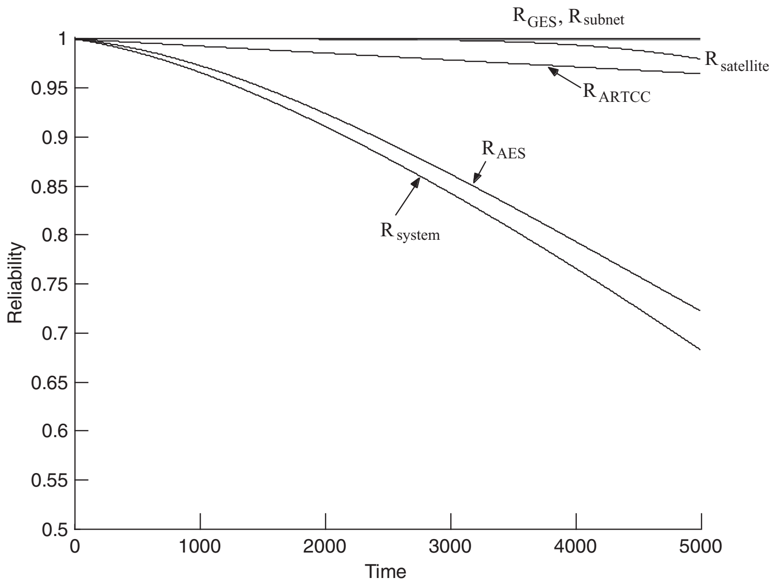

FIGURE 10.15 Reliability of the system and its subsystems.

The reliability of the telecommunication network for the air traffic control system is obtained by considering its five major hardware components as a series system. Thus,

$$R_{\text{system}}(t) = R_{\text{AES}}(t) \bullet R_{\text{satellite}}(t) \bullet R_{\text{subnetwork}}(t) \bullet R_{\text{ARTCC}}(t) \bullet R_{\text{GES}}(t). \tag{10.37}$$

The reliability of the individual components and that of the entire system are shown in Figure 10.15.

10.7.3.6 Importance of the Components Using Birnbaum's importance measure, as given by Equation 2.71, we obtain

$$G(\mathbf{q}(t)) = 1 - 1 - (q_{\text{AES}}(t))(1 - q_{\text{satellite}}(t))(1 - q_{\text{subnetwork}}(t))(1 - q_{\text{ARTCC}}(t)) \times (1 - q_{\text{GES}}(t))$$

At t = 1000 h

$$\begin{aligned}
I_B^{\text{AES}}(1000) &= (1 - q_{\text{satellite}}(1000))(1 - q_{\text{subnetwork}}(1000)) \times (1 - q_{\text{ARTCC}}(1000))(1 - q_{\text{GES}}(1000)) \\
I_B^{\text{AES}}(1000) &= 0.992816 \\
I_B^{\text{satellite}}(1000) &= 0.966502 \\
I_B^{\text{subnetwork}}(1000) &= 0.966510 \\
I_B^{\text{ARTCC}}(1000) &= 0.973387 \\
I_B^{\text{GES}}(1000) &= 0.966502.
\end{aligned}$$

The AES has the highest importance measure. Accordingly, it has the most impact on the overall system reliability.

Most of the AES components need to be redesigned in order to effectively reduce the failure rate of the AES. Similarly, the components of the ARTCC require design changes or redundancy for some of the components and links. The reliability of the GES exceeds the minimum requirement of the system. Their numbers are large enough to provide "inherent" redundancy in the system.

This analysis, though simple, shows that reliability techniques and modeling can be an effective design tool for complex configurations.

10.8 CASE 8: SYSTEM DESIGN USING RELIABILITY OBJECTIVES*

10.8.1 Introduction

Telecommunications service continuity is controlled by availability design objectives applicable to networks, network segments, and network elements. In general, these objectives are intended to control the amount of time that networks or portions of networks are unable to perform their required function. Availability objectives are typically stated as a single number equal to the long-term percentage of time that a system is expected to provide service. (*System* is a generic term used to describe an entity to which availability objectives apply. This could be a network, a network segment, or a network element.) As such, these objectives can significantly influence end-user perception of service quality.

This case discusses some of the implications of using traditional availability design objectives to control performance. This is done by first considering how current availability objectives apply to the *design of a single system*, and then by examining the resulting *availability service performance across a population of systems, each of which is designed to meet the same availability design objective*. New methods for describing end-user availability performance are discussed, and how they can be used to (1) evaluate current network and service objectives, (2) aid in the development of a top-down approach to setting availability objectives for new services, and (3) provide better network performance quality control.

The topics discussed in this case are addressed because there continue to be questions about the interpretation of availability design objectives for new services and supporting technologies. Two such questions are

- An objective for total downtime of a common control Bell System switching system was first established in the late 1950s, and evolved during the 1960s. The resulting objective required that there be "no more than two hours total downtime in 40 years." The same objective is used today and is stated as a downtime objective of 3 min per year for current switching systems. If a switching system is down for more than 3 min in a year, has it failed to meet its design objective?

* This case is contributed by the late Norman A. Marlow and Michael Tortorella of AT&T Bell Laboratories. It is modified by the author in its present form.

- If, after a period of several years, there are switching systems in a given population that have experienced (1) no downtime, and/or (2) downtime less than the objective, and/or (3) downtime greater than the objective, has the system design objective been met?

As we shall discuss, the answer to the first question is “not necessarily.” The answer to the second question is that we interpret the *design objective* as having been met if the *average of the population downtime distribution* does not exceed the objective value.

We first discuss traditional availability objectives, their relation to equipment reliability design objectives, and how they can be interpreted as a measure of long-term average service performance. Next we consider a population of identical systems, each designed to meet the same availability objective, and describe how downtime performance can vary across the population of system end users. Also discussed are differences in cumulative downtime performance that could be expected from simplex and duplex systems having the same availability objective and average unit restoration time. Extreme performance is considered by showing how the “longest restoration time” experienced over a given time period could vary across a population of systems. We conclude by discussing how these measures of population downtime performance can be used to assess the effects of system availability design objectives on end-user service performance and aid in the development of top-down availability objectives.

10.8.2 Availability Design Objectives

A typical availability objective might state that a system or service “be available at least 99.8 percent of the time.” Letting *uptime* denote the time during which a system or service is performing its required function, a consistent interpretation of this objective (and an interpretation that is in common use) is that, *when measured over a sufficiently long time interval*,

$$\frac{\text{Cumulative system uptime}}{\text{Cumulative observed time}} \geq 0.998. \tag{10.38}$$

“Cumulative observed time” in Equation 10.38 is cumulative system uptime, plus time when the system is not providing service but is supposed to be. The latter includes downtime in general, and consists of cumulative restoration times following system failures, planned maintenance downtimes, and other “out-of-service” times.

The definition of *instantaneous availability* given in Section 3.4 is consistent with Equation 10.38 and, as noted above, can apply both to systems and to the services they support. In the above example, the unavailability is 0.002, or 0.2%. This corresponds to 1051.2 min, or about 17.5 h *expected downtime* per year.

Similarly, the steady-state availability, A, of a system is defined as

$$A = \lim_{t \to \infty} P\{\text{system is operating at time } t\}. \tag{10.39}$$

From Equations 10.38 and 10.39, we obtain

$$E\{\text{cumulative system uptime in a steady-state period of length } T\} = A \times T, \quad (10.40)$$

where E denotes expected value. From Chapter 3, we rewrite the steady-state availability

$$A = \frac{MTTF}{MTTF + MTTR}, \quad (10.41)$$

where MTTF and MTTR are the mean time to failure and the mean time to repair, respectively.

The corresponding steady-state unavailability $\bar{A}$ in this example is then

$$\begin{aligned} \bar{A} &= 1 - A \\ \bar{A} &= \frac{MTTR}{MTTF + MTTR}. \end{aligned} \quad (10.42)$$

Letting $\lambda = 1/MTTF$ and $\mu = 1/MTTR$ (λ is the system failure rate and μ is the system restoration rate when the time to failure and time to repair distributions are exponential, and is henceforth assumed in this case), it follows from Equations 10.40 and 10.42 that the expected cumulative steady-state system downtime can be expressed as

$$E\{\text{cumulative downtime in a steady-state period of length } T\} = \frac{\lambda T}{\lambda + \mu}. \quad (10.43)$$

The performance measures given by the steady-state availability in Equation 10.41, the steady-state unavailability in Equation 10.42, or the expected downtime in Equation 10.43, are determined by the system MTTF and MTTR. MTTF is a basic design characteristic, while the MTTR is characteristic of a particular maintenance or operations policy. The MTTR may also depend on design features such as system modularity, self-diagnostic capability, or other factors.

Systems are designed to meet availability objectives by adjusting their MTTF and MTTR values within a model like Equation 10.43. For example, if the availability objective for a single simplex unit is 99.8%, then the corresponding *downtime objective* is 1051.2 min *expected downtime* per year. Using 525,600 min/year as a base, Equation 10.43 can be used to write this objective in the form

$$\frac{\lambda}{\lambda + \mu} \times 525{,}600 \le 1051.2 \text{ min/year}. \quad (10.44)$$

Assuming an average restoration time of at most 4 h $(1/\lambda) \le 240$ min, it follows from Equation 10.44 that the availability objective will be met if the unit failure rate λ satisfies

$$\lambda \le 4.38 \text{ failures per year}. \quad (10.45)$$

This is equivalent to an MTTF ($1/\lambda$) of at least 1996 h and is a system reliability design objective.

The same principles apply to more complex systems. For example, two identical simplex units operating *independently* in a load sharing parallel mode will have a system unavailability given by

$$\bar{A} = [\lambda/(\lambda + \mu))]^2. \tag{10.46}$$

In Equation 10.46, λ and μ are, respectively, the failure and restoration rates for each simplex unit. For example, if the parallel system downtime objective is 2 min/year (2 min/year is the downtime objective for parallel A-link access to the SS7 Common Channel Signaling network) (Bellcore, 1993), Equation 10.46 can be used to write the objective in the form

$$[\lambda/(\lambda + \mu)]^2 \times 525{,}600 \leq 2 \text{ min/year} \tag{10.47}$$

To meet the objective specified by Equation 10.47, each *simplex unit* in the parallel system must satisfy

$$\frac{\lambda}{\lambda + \mu} \times 525{,}600 \leq 1025.28 \text{ min (17 h)/year.} \tag{10.48}$$

Assuming an average restoration time of at most 4 h, it follows from Equation 10.48 that the parallel system downtime objective would be met if the failure rate λ of each simplex unit satisfies

$$\lambda \leq 4.2 \text{ failures per year.} \tag{10.49}$$

This corresponds to an MTTF of at least 2047 h for each simplex unit in the duplex system.

In the above example, the same downtime objective of 2 min/year could be met by using one simplex unit instead of two in parallel. However, with a mean restoration time of 4 h, the MTTF of the simplex unit would have to be at least 1,051,196 h or about 119 years.

In general, the unavailability of a complex system will be some function of its simplex network element failure and restoration rates

$$\bar{A} = \phi(\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots)$$

The form of this function generally depends on the system architecture, configuration of the components, and operating procedures (Birolini, 2010) as discussed in Chapters 2 and 3. As in the above examples, system downtime objectives can be used to select design values for network element failure rates and restoration rates by applying the relation

$$\phi(\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots) \times 525{,}600 \leq \text{Objective downtime (minutes) per year.} \tag{10.50}$$

In principle, Equation 10.50 represents a performance constraint subject to which system cost could be minimized.

With this as background, in this case we explore some ideas related to the following question: Suppose a system is designed to meet an availability objective using the procedure outlined above. What then are the properties of a performance indicator, or statistic, like Equation 10.38, when a large number of systems is put into service?

10.8.3 Availability Service Performance

Downtime objectives are intended to control the amount of time that networks, network segments, and network elements are unable to provide service. As discussed, these objectives are typically stated as a single number equal to the maximum expected downtime per year. As also discussed, the *same downtime objective can be met using different architectures*, and it is important to understand possible performance differences resulting from different implementations. Of course, by definition, the probability that a system is unable to perform its required function is equal to the system unavailability. The "design to availability objectives" procedure outlined above leads to the following interpretation of a downtime objective:

$$P\begin{Bmatrix} \textit{system cannot perform} \\ \textit{its required function} \end{Bmatrix} \le \frac{\text{Yearly downtime objective in minutes}}{525{,}600 \text{ min/year}}.$$

While this is an important *system performance measure*, analogous to a system "ineffective attempt rate," it does not adequately describe the full range of performance that can be expected in a population of such systems. In particular, if each system in a population meets the objective, all that can be said is that each system has an MTTF and MTTR meeting the specified objectives. However, individual times to system failure and corresponding restoration times will vary randomly, and will be different from the average or expected values to which they have been designed. The result is that observed yearly downtimes across a population will differ from the objective value. As such, variations in cumulative downtime over a given year will occur *across a population of systems*. If these systems are designed to meet a specified objective, then the mean or *average* of the population downtime distribution in 1 year should not exceed the objective value.

In assessing compliance with reliability objectives, one could simply stop here by ascertaining whether the sample mean of the population annual downtime exceeds the objective value. A simple hypothesis test would provide the understanding of statistical significance needed here. However, we maintain that by using appropriately the additional information contained in the model that underlies the "design to availability objectives" procedure outlined above, additional insight into the quality control of this key service satisfaction parameter can be obtained. We explore this idea later following further discussion of distributions of downtime in the population.

10.8.3.1 Cumulative Downtime Distributions To illustrate how yearly downtime can vary across a population, consider the single-unit system. Assume again that the cumulative downtime objective is 17.5 h (1051.2 min)/year and that the average restoration time is 4 h. Then, using Equation 10.44, the parameters MTTF = $1/\lambda$ and MTTR = $1/\mu$ are

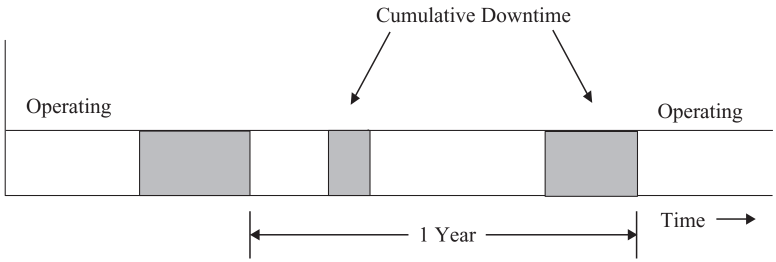

FIGURE 10.16 Cumulative yearly downtime.

$$\text{MTTF} = 1996 \text{ h}$$
$$\text{MTTR} = 4.0 \text{ h}.$$

Figure 10.16 shows how the cumulative downtime over a 1-year period might appear for a single unit.

The cumulative downtime during a given time period depends on both the times to failure and the corresponding restoration times. As discussed above, these times will vary from failure to failure and from system to system. This implies that the downtime in a given year will have some distribution across a population of units. In particular, using a Markov model, it is possible to obtain the predicted yearly cumulative downtime distribution (Takacs, 1957; Brownlee, 1960; Barlow and Proschan, 1965; Puri, 1971) as discussed in Chapter 3. This distribution is a function, $D(T)$, that depends on the simplex unit MTTF and MTTR, and is defined by

$$D(T) = P\{\text{cumulative system downtime in 1 year is} \leq T\}.$$

Using 1 year as a base, Figure 10.17 shows the complementary steady-state cumulative downtime distribution $1 - D(T)$ when the downtime objective is an average of 17.5 h (1051.2 min)/year and the average restoration time is 4 h.

Figure 10.17 illustrates important service consequences of using a design objective of 17.5 h average downtime per year and an average restoration time of 4 h. In particular, note that during one year, 43.2% of the units *in the population* are expected to have cumulative downtime *exceeding the objective*. For example, a particular unit in a population meeting an average cumulative downtime objective of 17.5 h has about a 20% chance of being down for 25 h or more in 1 year. Two other important service indicators are the probability of "zero downtime" in a given year, and the probability of exceeding a given threshold during a year. As Figure 10.17 also shows, about 1.2% of the unit population is expected to experience zero downtime during 1 year. In addition, about 5% of the population is expected to have cumulative downtime greater than 40 h.

Several important conclusions for a system designed to meet an unavailability objective interpreted as a long-term average follow:

- The objective value will be approached *as a time average* over a sufficiently long measurement period.
- In a large population of such systems, *the population average downtime* in a year should correspond to the objective.
- A single system may have downtime in a year exceeding objective value *while the population design objective is still met*.

In particular, it is not currently accepted practice to interpret the design objective as a maximum value that no system may exceed in service.

If a population of systems meets an average cumulative downtime objective of 17.5 h/year, then normal operating conditions will result in some large cumulative yearly downtimes as Figure 10.17 shows. On the other hand, if some systems in a population have cumulative yearly downtimes exceeding the limits one deduces from Figure 10.17, then factors other than those caused by nominal statistical variation may be contributing, and special corrective action may be needed. As a measure of typical performance, distributions of cumulative yearly downtimes could be used, together with "statistical control chart" techniques, to monitor and maintain objective levels of performance.

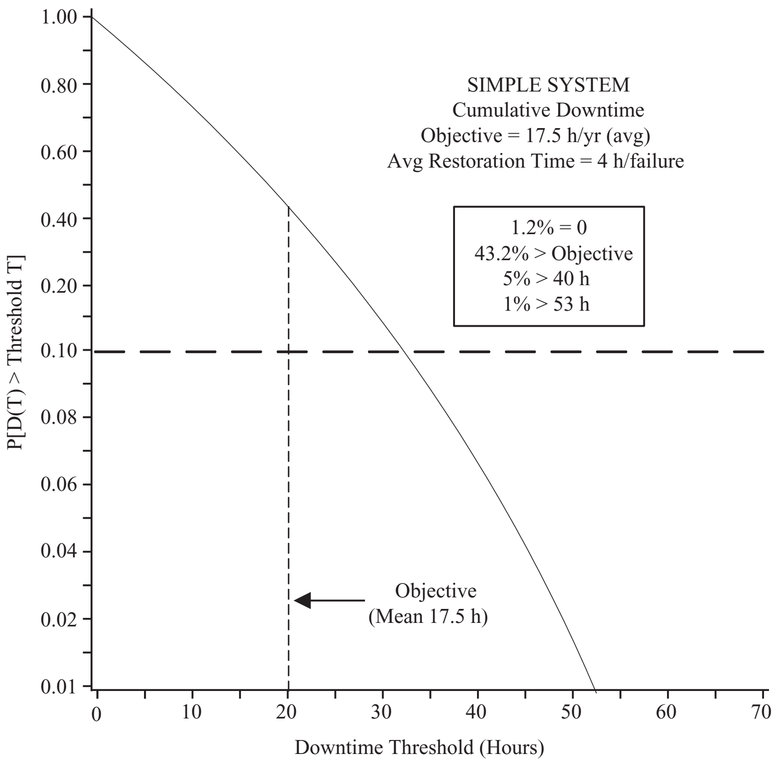

FIGURE 10.17 Simple unit cumulative downtime distribution.

10.8.3.2 Distribution of the Largest Restoration Time Each system in a population meeting the same average yearly cumulative downtime objective will experience variable restoration times and will be subject to a “largest” or maximum restoration time during a particular year. Across a population, there will be a corresponding distribution of largest restoration times. In contrast to the population distribution of cumulative yearly downtime, the distribution of the largest restoration time generally highlights poor performance associated with a single failure and provides an important measure of service quality resulting from specified design objectives. Using the previous example of a simplex system Markov model in which the average cumulative downtime objective is 17.5 h/year, and the average restoration time is 4 h, Figure 10.18 shows the predicted distribution of “largest restoration times” over 1 year (Marlow, n.d.).

This figure shows that while an MTTR of 4 h is used, the largest experienced restoration time is likely to be larger than this average value. In general, as more failures occur, the potential for larger restoration times increases.

Because the cumulative downtime experienced during a year will be no smaller than the largest single restoration time, Figure 10.18 also illustrates that the cumulative system

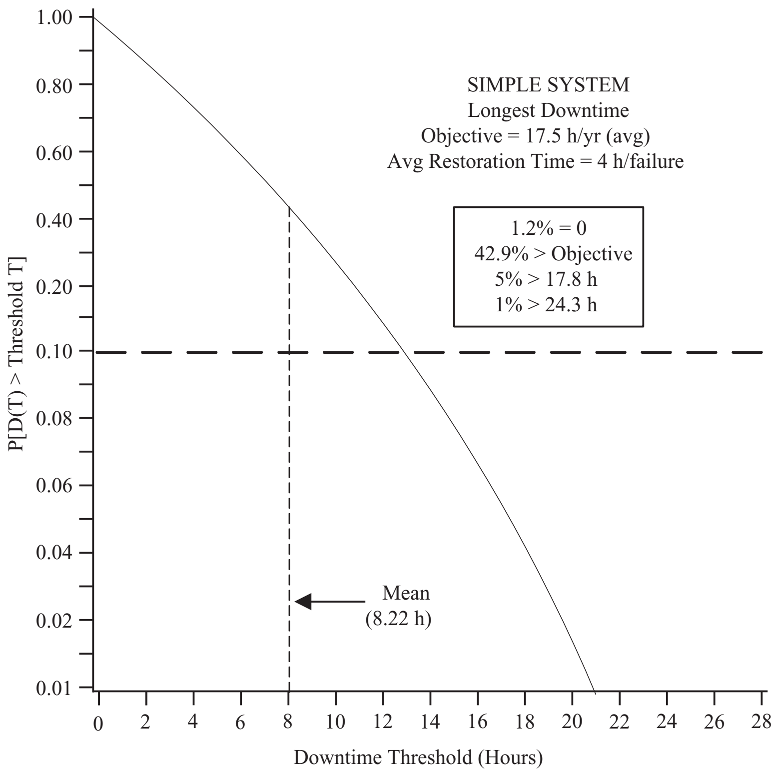

FIGURE 10.18 Distribution of the largest restoration time for a simplex system.

downtime experienced during a year may be larger than the average design value. In particular, this example shows that 5.3% of the systems are expected to have a “largest” single downtime exceeding the design value of 17.5 h *cumulative downtime* per year.

If a population of systems meets an average cumulative downtime objective of 17.5 h/year, then nominal operating conditions will result in some long individual restoration times as shown in Figure 10.18. On the other hand, if some systems in a population have restoration times exceeding these expected limits, then factors other than those caused by nominal statistical variation may be contributing to the increase in the restoration times, and special corrective maintenance action may be needed. As a measure of extreme performance, distributions of longest restoration times could be used, together with “statistical control chart” techniques, to monitor and maintain objective levels of performance.

10.8.4 Evaluation of Availability Design Objectives

To provide service continuity consistent with end-user expectations and network capabilities, availability design objectives should be established to control the full range of performance resulting from the objectives. In principle, this should follow a top-down approach in which service-level objectives are specified and networks are designed to meet the objectives. The example mentioned in the introduction highlights this approach: the goal is to provide a network service capability that is available 99.8% of the time. As discussed, this can be interpreted as an average service downtime of 1051.2 min/year and can be achieved by selecting architectures and network elements whose combined average downtime performance meets this objective. However, the resulting population downtime distribution shows that the *range* of service performance expected across a group of end users is not fully described by a single downtime objective for either the service or network design. In the following sections, we discuss how the selection of service and network design availability objectives can be better evaluated when the full range of downtime performance has been described using information provided by the population downtime distribution.

10.8.4.1 Evaluation of Current Network and Service Objectives Contemporary telecommunication network standards include downtime objectives for many network elements and for corresponding services. As an example, end office switching system access to the SS7 Common Channel Signaling network is provided by A-links connected to a mated pair of signaling transfer points (Bellcore, 1993). The average Common Channel Signaling (CCS) network access downtime objective is 2 min/year. To evaluate the full range of performance that can be expected from this objective, a parallel system Markov model was used to predict the cumulative yearly downtime distribution (Hamilton and Marlow, 1991). Figure 10.19 shows the complementary downtime distribution for an objective of 2 min/year and compares it to distributions that would result from objectives of 1 and 8 min/year, respectively. Each distribution is based on an average link restoration time of 4 h per failure. Figure 10.19 highlights important consequences of using a particular average downtime objective for signaling network access. In particular, with an objective of 2 min/year, about 1% of the end offices each year are expected to be unable to access CCS services for at least 1 h. With a tighter objective of

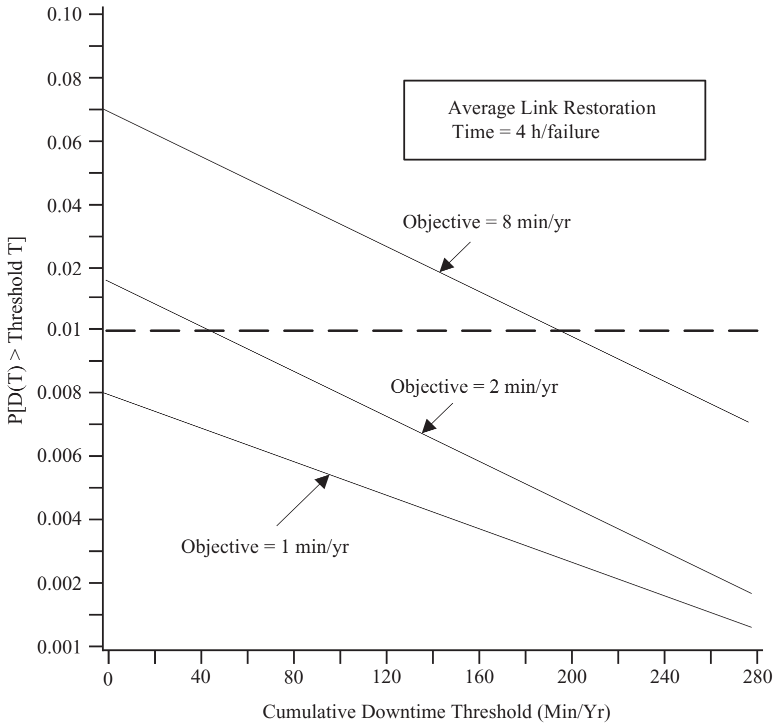

FIGURE 10.19 CCS network access downtime distributions over 1 year.

1 min, the percentage decreases to 0.5%, while it increases to about 4% with an objective of 8 min/year.

Each downtime objective also corresponds to an ineffective call attempt rate caused by access signaling failures and resulting downtime. For example, the 2-min objective is equivalent to an ineffective call attempt rate of $2/525{,}600 = 3.8 \times 10^{-6}$. Table 10.11 summarizes these performance characteristics.

The downtime objective for CCS network access was first published as a CCITT Recommendation in 1984 (CCITT, 1984a, 1984b). At that time, the objective was believed to be achievable, and was considered adequate for call setup performance expressed as an ineffective attempt rate. As the signaling network evolves, however, more services will be supported by this network and critical reviews of all downtime objectives will be needed. Using information such as that in Table 10.11, these reviews can be made more service oriented and should result in improved guidelines for future CCS network performance planning.

10.8.4.2 Establishing Top-Down Availability Objectives Top-down performance planning for a new service begins with a comprehensive understanding of customer needs, expectations, and perceptions of performance quality. End-user performance objectives are then

TABLE 10.11 SS7 Network Access Performance

Access downtime objective (min/year)	Ineffective attempt rate	Offices with access downtime >1 h/year
1	1.9×10^{-6}	0.5%
2	3.8×10^{-6}	1.0%
8	15×10^{-6}	4.0%

specified to meet these expectations and are then allocated so that network capabilities can be designed and implemented to meet these objectives.

While the above approach is desirable, it is sometimes difficult to achieve, because in practice, existing network capabilities, together with possible adjuncts, are used to support a given service. Then service planners must determine end-user service performance from that of the supporting network. Service availability objectives are often obtained in this way, resulting in a single number such as 99.8% for Public Switched Digital Services (PSDS) (Bellcore, 1985). This single number could be interpreted as an average service ineffective attempt rate or as an average yearly cumulative downtime across a population of end users. However, as the previous examples have shown, a single average does not adequately describe the range of performance that is likely to be experienced. Accordingly, in developing end-user service objectives, it would be appropriate in many instances to use network availability models to predict the population cumulative downtime distribution and ask if the expected range is acceptable for the type of service being planned. With this as a guide, possible changes in downtime design objectives for critical portions of the network could be made to better control service performance.

10.9 CASE 9: RELIABILITY MODELING OF HYDRAULIC FRACTURE PUMPS*

10.9.1 Introduction

Oil and gas extraction is an extensive process that begins after careful identification of their presence. The process begins by drilling a well which uses a steel casing to prevent the contamination of the water aquifer (200–500 ft underground) as shown in Figure 10.20. The drilling of the well continues vertically for about 5000–8000 ft depending on the presence of the shale which contains oil and/or gas. At the point of the end of the vertical part of the well, referred to as the kickoff point, the curved section of the wellbore begins (it is about 400–500 ft above the shale and the intended horizontal section of the well). Drilling continues until the desired horizontal length of the well is achieved. The next step involves the use of

* This case is a partial adaptation of Shaun Wolski's Master's project, Department of Industrial and Systems Engineering, Rutgers University, Spring 2011.

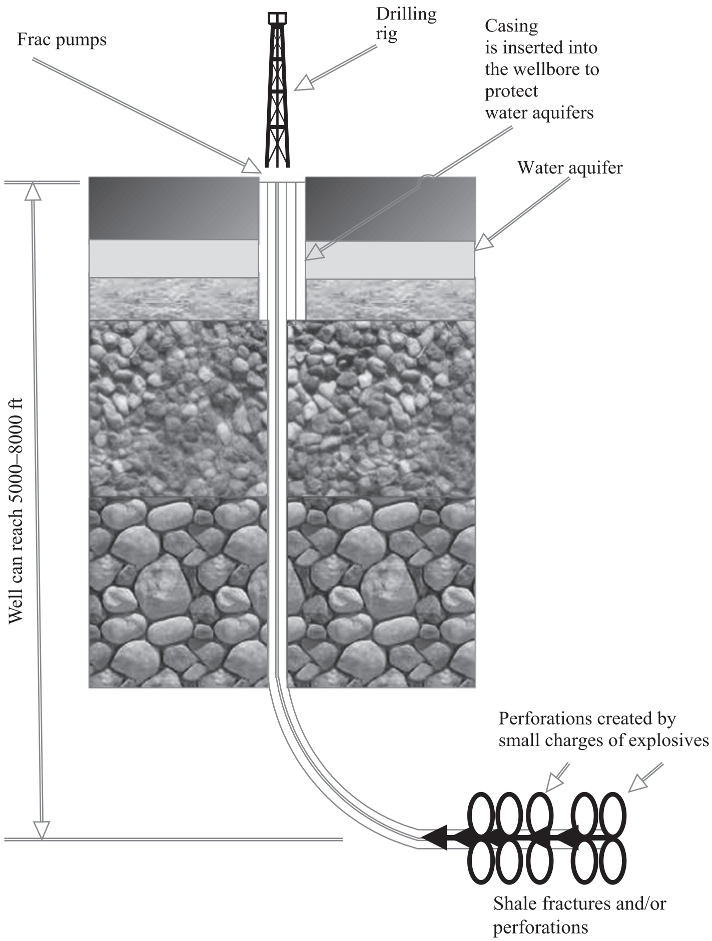

FIGURE 10.20 Well cross section and fracturing pumps.

shape charges (explosives) to develop perforation through the casing at the desired locations in the horizontal portion of the well. These will provide passages for oil and gas to enter the wellbore. These passages need to stay open during the entire life of the well. This is achieved as follows.

The well rig is removed and a procedure known as hydraulic fracturing or “fracking” which essentially involves pumping large amounts of a fluid (mostly water) downhole (wellbore) to break the rock open. Once the rock has been broken down, it is followed by a proppant (often sand or stronger materials depending on the pressures) which will remain in place and create a man-made permeable path. The primary purpose of this permeable path is to allow the hydrocarbons (oil and gas) to go through the formation to the drilled wellbore and eventually to the surface. The pumping of fluids is achieved by using several pumps each having 2250 brake horsepower (BHP). A typical pump is composed of (1) large engine (2) transmission (3) power end which converts the energy being delivered from the transmission into mechanical energy which drives the pistons that in turn drive the fluid end, and (4) the fluid end which pumps mixtures and fluids into the wellbore. The engine is the most critical component of the pump as it is a complex unit and is subject to frequent failures.

10.9.2 Pump Engine

A typical oil/gas well requires several fracturing pumps operate simultanously, and their number ranges from 5 to 35 pumps on the site. A minimum number of pumps is required to ensure the opening of the passage. However, pressure in the range of 15–20 kspi is required to force the oil/gas to the surface of the well. Consequently, the throughput of the well is dependent on the number of pumps in the system. The engine of the pump is considered the most critical component as it is quite complex and is subject to frequent failures. The reliability diagram of a typical engine is shown in Figure 10.21.

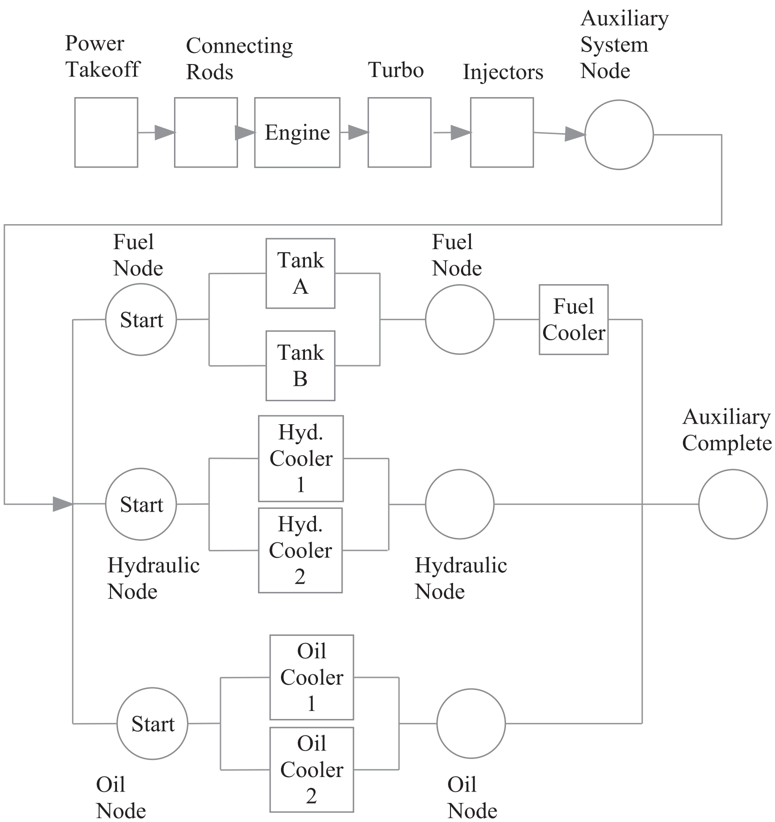

FIGURE 10.21 Reliability block diagram of the pump engine.

The reliability block diagram begins with a power takeoff which essentially is a device that transfers power from the engine on the tractor (pump is mounted on a tractor) to that of the pump engine. Other engine components include a turbo, injectors, and all the actual components and an auxiliary system which has three main parts (each with an active redundancy); all are required for operation of the pump, and if any part fails, the engine could overheat. The first part is the fuel system. It consists of two 150 gal fuel tanks labeled “A” and “B” which are located on the driver’s side and the passenger side, respectively. These are not to be confused with the fuel tanks on the tractor as those are used solely for the pump engine. The fuel cooler is placed in series with the fuel tanks. A fuel cooler is used to cool down the fuel and enhance the performance of fuel.

The second auxiliary system is the hydraulic system which controls numerous components on the system. Components run by the hydraulic system include the transmission, clutch, brake, and lubricating units. The hydraulic system has two cooling units in parallel. When both of coolers fail, then the entire system shuts down.

The third part is the oil system whose function is critical as it cools down the oil to a temperature that allows the oil to maintain its viscosity. This part is similar to the other two in that it has two oil coolers in parallel and requires only one to run. Note that the three parts (fuel, hydraulic, and oil nodes) are considered as a series system but depicted as parallel in Figure 10.21 for simplification.

10.9.3 Statement of the Problem

Extensive failure data of the pump engine components are collected, analyzed, and modeled as shown in Chapter 1. An example of the data collected for 100 failure times of the piston rods is shown in Table 10.12, and the corresponding failure-time distribution is shown in Figure 10.22.

Appropriate failure-time distributions are obtained for all the components as shown below.

TABLE 10.12 Failure-Time Data of the Pistons Connecting Rods

597	696	242	569	647	611	1159	313	598	1002
720	1086	543	540	563	472	306	424	1367	277
15	37	540	518	337	633	378	1030	451	595
709	1088	622	203	598	523	896	930	85	719
348	400	408	243	357	353	946	857	665	163
297	249	732	651	847	784	39	256	1126	214
941	887	411	796	471	196	565	954	694	888
614	924	205	405	1250	282	630	448	93	360
886	259	940	507	545	582	1000	465	1251	243
869	225	133	1247	316	459	592	776	694	808

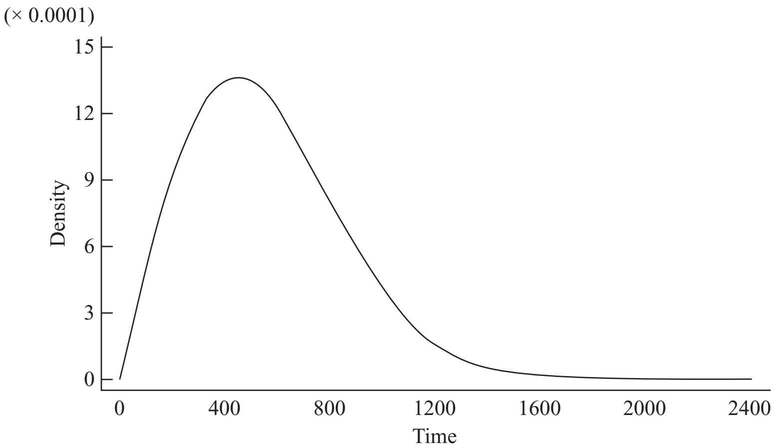

FIGURE 10.22 Failure-time distribution of the piston rods.

$$R_{power\ take\ off}(t) = e^{-0.00005t}$$
$$R_{rods}(t) = e^{-2.5118\times10^{-6}t^2}$$
$$R_{engine}(t) = e^{-2.8\times10^{-8}t^2}$$
$$R_{turbo}(t) = e^{-3.0\times10^{-5}t^{1.6}}$$
$$R_{injectors}(t) = e^{-0.000008t}$$
$$R_{fuel\ node}(t) = e^{-0.35\times10^{-4}t^{1.8}}$$
$$R_{hydraulic\ node}(t) = e^{-2.9\times10^{-8}t^2}$$
$$R_{oil\ node}(t) = e^{-0.000008t}.$$

A minimum of four pumps is needed to produce 100,000 barrels of oil per day. A typical cost of a pump is $1,000,000. The wellhead area can accommodate up to 11 pumps. Each additional pump increases the throughput of the well by 5000 barrels. Determine the optimum configuration of the pumps that results in maximum profit given that the profit per barrel is $10. The limiting time is the MTTF of the system.

10.9.4 Solution

Since all components (and parts) are connected in series then the reliability of a pump is the product of the reliability expressions given above. This results in

$$R_{pump}(t) = e^{-(6.6\times10^{-5}t+2.567\times10^{-6}t^2+3.0\times10^{-5}t^{1.6}+3.5\times10^{-5}t^{1.8})}. \tag{10.51}$$

We calculate the reliability of a pump system with a total of 11 pumps with four pumps operating, five pumps operating, and so on. We also calculate the corresponding MTTFs as follows:

TABLE 10.13 MTTF for Different Configurations

MTTF							
Config. 1	**Config. 2**	**Config. 3**	**Config. 4**	**Config. 5**	**Config. 6**	**Config. 7**	**Config. 8**
1-out-of-11	*2-out-of-11*	*3-out-of-11*	*4-out-of-11*	*5-out-of-11*	*6-out-of-11*	*7-out-of-11*	*8-out-of-11*
170.40	88.91	138.32	254.88	222.69	193.95	167.10	141.02

$$R_{system}(t) = \sum_{i=4}^{n} \frac{n!}{i!(n-i)!}(R_{pump}(t))^i (1 - R_{pump}(t))^{n-i}. \tag{10.52}$$

The MTTF for any system configuration is obtained as

$$MTTF_{system} = \int_0^\infty R_{system}(t)dt \cong \sum_{t=0}^{\infty} R_{system}(t). \tag{10.53}$$

The MTTF for different *k-out-of-n* with $k = 4, 5, 6, \ldots, 11$ and $n = 11$ are shown in Table 10.13.

The total profits after subtracting the cost of the pumps for Configurations 1 through 8 are

$$\begin{aligned}
\text{Config. } 8 &= 100{,}000 \times 10 \times 141.02 - 4{,}000{,}000 = \$137{,}020{,}000 \\
\text{Config. } 7 &= 105{,}000 \times 10 \times 167.10 - 5{,}000{,}000 = \$170{,}455{,}000 \\
\text{Config. } 6 &= 110{,}000 \times 10 \times 193.95 - 6{,}000{,}000 = \$207{,}345{,}000 \\
\text{Config. } 5 &= 115{,}000 \times 10 \times 222.69 - 7{,}000{,}000 = \$249{,}093{,}500 \\
\text{Config. } 4 &= 120{,}000 \times 10 \times 254.88 - 8{,}000{,}000 = \$297{,}856{,}000 \\
\text{Config. } 3 &= 125{,}000 \times 10 \times 138.32 - 9{,}000{,}000 = \$163{,}900{,}000.
\end{aligned}$$

As shown, the maximum return occurs when Configuration 4 is used. Clearly, other subsystems of the pump need to be considered in the analysis. These include the transmission, fluid end, power end, and others.

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APPENDICES

APPENDIX A

GAMMA TABLE

Gamma Function

n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
0.0100	99.4327	0.2600	3.4785	0.5100	1.7384	0.7600	1.2123
0.0200	49.4423	0.2700	3.3426	0.5200	1.7058	0.7700	1.1997
0.0300	32.7850	0.2800	3.2169	0.5300	1.6747	0.7800	1.1875
0.0400	24.4610	0.2900	3.1001	0.5400	1.6448	0.7900	1.1757
0.0500	19.4701	0.3000	2.9916	0.5500	1.6161	0.8000	1.1642
0.0600	16.1457	0.3100	2.8903	0.5600	1.5886	0.8100	1.1532
0.0700	13.7736	0.3200	2.7958	0.5700	1.5623	0.8200	1.1425
0.0800	11.9966	0.3300	2.7072	0.5800	1.5369	0.8300	1.1322
0.0900	10.6162	0.3400	2.6242	0.5900	1.5126	0.8400	1.1222
0.1000	9.5135	0.3500	2.5461	0.6000	1.4892	0.8500	1.1125
0.1100	8.6127	0.3600	2.4727	0.6100	1.4667	0.8600	1.1031
0.1200	7.8632	0.3700	2.4036	0.6200	1.4450	0.8700	1.0941
0.1300	7.2302	0.3800	2.3383	0.6300	1.4242	0.8800	1.0853
0.1400	6.6887	0.3900	2.2765	0.6400	1.4041	0.8900	1.0768
0.1500	6.2203	0.4000	2.2182	0.6500	1.3848	0.9000	1.0686
0.1600	5.8113	0.4100	2.1628	0.6600	1.3662	0.9100	1.0607
0.1700	5.4512	0.4200	2.1104	0.6700	1.3482	0.9200	1.0530
0.1800	5.1318	0.4300	2.0605	0.6800	1.3309	0.9300	1.0456
0.1900	4.8468	0.4400	2.0132	0.6900	1.3142	0.9400	1.0384
0.2000	4.5908	0.4500	1.9681	0.7000	1.2981	0.9500	1.0315
0.2100	4.3599	0.4600	1.9252	0.7100	1.2825	0.9600	1.0247
0.2200	4.1505	0.4700	1.8843	0.7200	1.2675	0.9700	1.0182
0.2300	3.9598	0.4800	1.8453	0.7300	1.2530	0.9800	1.0119
0.2400	3.7855	0.4900	1.8080	0.7400	1.2390	0.9900	1.0059
0.2500	3.6256	0.5000	1.7725	0.7500	1.2254	1.0000	1.0000

(*Continued*)

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n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
1.0100	0.9943	1.4300	0.8860	1.8500	0.9456	2.2700	1.1462
1.0200	0.9888	1.4400	0.8858	1.8600	0.9487	2.2800	1.1529
1.0300	0.9836	1.4500	0.8857	1.8700	0.9518	2.2900	1.1598
1.0400	0.9784	1.4600	0.8856	1.8800	0.9551	2.3000	1.1667
1.0500	0.9735	1.4700	0.8856	1.8900	0.9584	2.3100	1.1738
1.0600	0.9687	1.4800	0.8857	1.9000	0.9618	2.3200	1.1809
1.0700	0.9642	1.4900	0.8859	1.9100	0.9652	2.3300	1.1882
1.0800	0.9597	1.5000	0.8862	1.9200	0.9688	2.3400	1.1956
1.0900	0.9555	1.5100	0.8866	1.9300	0.9724	2.3500	1.2031
1.1000	0.9513	1.5200	0.8870	1.9400	0.9761	2.3600	1.2107
1.1100	0.9474	1.5300	0.8876	1.9500	0.9799	2.3700	1.2184
1.1200	0.9436	1.5400	0.8882	1.9600	0.9837	2.3800	1.2262
1.1300	0.9399	1.5500	0.8889	1.9700	0.9877	2.3900	1.2341
1.1400	0.9364	1.5600	0.8896	1.9800	0.9917	2.4000	1.2422
1.1500	0.9330	1.5700	0.8905	1.9900	0.9958	2.4100	1.2503
1.1600	0.9298	1.5800	0.8914	2.0000	1.0000	2.4200	1.2586
1.1700	0.9267	1.5900	0.8924	2.0100	1.0043	2.4300	1.2670
1.1800	0.9237	1.6000	0.8935	2.0200	1.0086	2.4400	1.2756
1.1900	0.9209	1.6100	0.8947	2.0300	1.0131	2.4500	1.2842
1.2000	0.9182	1.6200	0.8959	2.0400	1.0176	2.4600	1.2930
1.2100	0.9156	1.6300	0.8972	2.0500	1.0222	2.4700	1.3019
1.2200	0.9131	1.6400	0.8986	2.0600	1.0269	2.4800	1.3109
1.2300	0.9108	1.6500	0.9001	2.0700	1.0316	2.4900	1.3201
1.2400	0.9085	1.6600	0.9017	2.0800	1.0365	2.5000	1.3293
1.2500	0.9064	1.6700	0.9033	2.0900	1.0415	2.5100	1.3388
1.2600	0.9044	1.6800	0.9050	2.1000	1.0465	2.5200	1.3483
1.2700	0.9025	1.6900	0.9068	2.1100	1.0516	2.5300	1.3580
1.2800	0.9007	1.7000	0.9086	2.1200	1.0568	2.5400	1.3678
1.2900	0.8990	1.7100	0.9106	2.1300	1.0621	2.5500	1.3777
1.3000	0.8975	1.7200	0.9126	2.1400	1.0675	2.5600	1.3878
1.3100	0.8960	1.7300	0.9147	2.1500	1.0730	2.5700	1.3981
1.3200	0.8946	1.7400	0.9168	2.1600	1.0786	2.5800	1.4084
1.3300	0.8934	1.7500	0.9191	2.1700	1.0842	2.5900	1.4190
1.3400	0.8922	1.7600	0.9214	2.1800	1.0900	2.6000	1.4296
1.3500	0.8912	1.7700	0.9238	2.1900	1.0959	2.6100	1.4404
1.3600	0.8902	1.7800	0.9262	2.2000	1.1018	2.6200	1.4514
1.3700	0.8893	1.7900	0.9288	2.2100	1.1078	2.6300	1.4625
1.3800	0.8885	1.8000	0.9314	2.2200	1.1140	2.6400	1.4738
1.3900	0.8879	1.8100	0.9341	2.2300	1.1202	2.6500	1.4852
1.4000	0.8873	1.8200	0.9368	2.2400	1.1266	2.6600	1.4968
1.4100	0.8868	1.8300	0.9397	2.2500	1.1330	2.6700	1.5085
1.4200	0.8864	1.8400	0.9426	2.2600	1.1395	2.6800	1.5204

n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
2.6900	1.5325	3.1000	2.1976	3.5100	3.3603	3.9200	5.4313
2.7000	1.5447	3.1100	2.2189	3.5200	3.3977	3.9300	5.4988
2.7100	1.5571	3.1200	2.2405	3.5300	3.4357	3.9400	5.5673
2.7200	1.5696	3.1300	2.2623	3.5400	3.4742	3.9500	5.6368
2.7300	1.5824	3.1400	2.2845	3.5500	3.5132	3.9600	5.7073
2.7400	1.5953	3.1500	2.3069	3.5600	3.5529	3.9700	5.7789
2.7500	1.6084	3.1600	2.3297	3.5700	3.5930	3.9800	5.8515
2.7600	1.6216	3.1700	2.3528	3.5800	3.6338	3.9900	5.9252
2.7700	1.6351	3.1800	2.3762	3.5900	3.6751	4.0000	6.0000
2.7800	1.6487	3.1900	2.3999	3.6000	3.7170	4.0100	6.0759
2.7900	1.6625	3.2000	2.4240	3.6100	3.7595	4.0200	6.1530
2.8000	1.6765	3.2100	2.4483	3.6200	3.8027	4.0300	6.2312
2.8100	1.6907	3.2200	2.4731	3.6300	3.8464	4.0400	6.3106
2.8200	1.7051	3.2300	2.4981	3.6400	3.8908	4.0500	6.3912
2.8300	1.7196	3.2400	2.5235	3.6500	3.9358	4.0600	6.4730
2.8400	1.7344	3.2500	2.5493	3.6600	3.9814	4.0700	6.5560
2.8500	1.7494	3.2600	2.5754	3.6700	4.0277	4.0800	6.6403
2.8600	1.7646	3.2700	2.6018	3.6800	4.0747	4.0900	6.7258
2.8700	1.7799	3.2800	2.6287	3.6900	4.1223	4.1000	6.8126
2.8800	1.7955	3.2900	2.6559	3.7000	4.1707	4.1100	6.9008
2.8900	1.8113	3.3000	2.6834	3.7100	4.2197	4.1200	6.9902
2.9000	1.8274	3.3100	2.7114	3.7200	4.2694	4.1300	7.0811
2.9100	1.8436	3.3200	2.7398	3.7300	4.3199	4.1400	7.1733
2.9200	1.8600	3.3300	2.7685	3.7400	4.3711	4.1500	7.2669
2.9300	1.8767	3.3400	2.7976	3.7500	4.4230	4.1600	7.3619
2.9400	1.8936	3.3500	2.8272	3.7600	4.4757	4.1700	7.4584
2.9500	1.9108	3.3600	2.8571	3.7700	4.5291	4.1800	7.5563
2.9600	1.9281	3.3700	2.8875	3.7800	4.5833	4.1900	7.6557
2.9700	1.9457	3.3800	2.9183	3.7900	4.6384	4.2000	7.7567
2.9800	1.9636	3.3900	2.9495	3.8000	4.6942	4.2100	7.8592
2.9900	1.9817	3.4000	2.9812	3.8100	4.7508	4.2200	7.9632
3.0000	2.0000	3.4100	3.0133	3.8200	4.8083	4.2300	8.0689
3.0100	2.0186	3.4200	3.0459	3.8300	4.8666	4.2400	8.1762
3.0200	2.0374	3.4300	3.0789	3.8400	4.9257	4.2500	8.2851
3.0300	2.0565	3.4400	3.1124	3.8500	4.9857	4.2600	8.3957
3.0400	2.0759	3.4500	3.1463	3.8600	5.0466	4.2700	8.5080
3.0500	2.0955	3.4600	3.1807	3.8700	5.1084	4.2800	8.6220
3.0600	2.1153	3.4700	3.2156	3.8800	5.1711	4.2900	8.7378
3.0700	2.1355	3.4800	3.2510	3.8900	5.2348	4.3000	8.8554
3.0800	2.1559	3.4900	3.2869	3.9000	5.2993	4.3100	8.9747
3.0900	2.1766	3.5000	3.3233	3.9100	5.3648	4.3200	9.0960

(*Continued*)

n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
4.3300	9.2191	4.7500	16.5862	5.1700	31.1014	5.5900	60.5588
4.3400	9.3441	4.7600	16.8285	5.1800	31.5853	5.6000	61.5539
4.3500	9.4711	4.7700	17.0748	5.1900	32.0775	5.6100	62.5666
4.3600	9.6000	4.7800	17.3250	5.2000	32.5781	5.6200	63.5972
4.3700	9.7309	4.7900	17.5794	5.2100	33.0872	5.6300	64.6460
4.3800	9.8639	4.8000	17.8378	5.2200	33.6049	5.6400	65.7135
4.3900	9.9989	4.8100	18.1005	5.2300	34.1314	5.6500	66.7998
4.4000	10.1361	4.8200	18.3675	5.2400	34.6670	5.6600	67.9054
4.4100	10.2754	4.8300	18.6389	5.2500	35.2117	5.6700	69.0306
4.4200	10.4169	4.8400	18.9147	5.2600	35.7656	5.6800	70.1758
4.4300	10.5606	4.8500	19.1950	5.2700	36.3291	5.6900	71.3414
4.4400	10.7065	4.8600	19.4800	5.2800	36.9022	5.7000	72.5277
4.4500	10.8548	4.8700	19.7696	5.2900	37.4851	5.7100	73.7352
4.4600	11.0053	4.8800	20.0640	5.3000	38.0780	5.7200	74.9642
4.4700	11.1583	4.8900	20.3632	5.3100	38.6811	5.7300	76.2152
4.4800	11.3136	4.9000	20.6674	5.3200	39.2946	5.7400	77.4884
4.4900	11.4714	4.9100	20.9765	5.3300	39.9186	5.7500	78.7845
4.5000	11.6317	4.9200	21.2908	5.3400	40.5534	5.7600	80.1038
4.5100	11.7945	4.9300	21.6103	5.3500	41.1991	5.7700	81.4467
4.5200	11.9599	4.9400	21.9351	5.3600	41.8559	5.7800	82.8136
4.5300	12.1280	4.9500	22.2652	5.3700	42.5241	5.7900	84.2052
4.5400	12.2986	4.9600	22.6009	5.3800	43.2039	5.8000	85.6216
4.5500	12.4720	4.9700	22.9420	5.3900	43.8953	5.8100	87.0636
4.5600	12.6482	4.9800	23.2889	5.4000	44.5988	5.8200	88.5315
4.5700	12.8271	4.9900	23.6415	5.4100	45.3145	5.8300	90.0259
4.5800	13.0089	5.0000	24.0000	5.4200	46.0426	5.8400	91.5472
4.5900	13.1936	5.0100	24.3645	5.4300	46.7833	5.8500	93.0960
4.6000	13.3813	5.0200	24.7351	5.4400	47.5370	5.8600	94.6727
4.6100	13.5719	5.0300	25.1118	5.4500	48.3037	5.8700	96.2780
4.6200	13.7656	5.0400	25.4948	5.4600	49.0838	5.8800	97.9122
4.6300	13.9624	5.0500	25.8843	5.4700	49.8775	5.8900	99.5761
4.6400	14.1624	5.0600	26.2803	5.4800	50.6850	5.9000	101.2701
4.6500	14.3655	5.0700	26.6829	5.4900	51.5067	5.9100	102.9949
4.6600	14.5720	5.0800	27.0922	5.5000	52.3427	5.9200	104.7509
4.6700	14.7817	5.0900	27.5085	5.5100	53.1933	5.9300	106.5389
4.6800	14.9948	5.1000	27.9317	5.5200	54.0589	5.9400	108.3594
4.6900	15.2114	5.1100	28.3621	5.5300	54.9396	5.9500	110.2129
4.7000	15.4314	5.1200	28.7997	5.5400	55.8358	5.9600	112.1003
4.7100	15.6550	5.1300	29.2448	5.5500	56.7477	5.9700	114.0219
4.7200	15.8822	5.1400	29.6973	5.5600	57.6757	5.9800	115.9787
4.7300	16.1131	5.1500	30.1575	5.5700	58.6200	5.9900	117.9711
4.7400	16.3478	5.1600	30.6255	5.5800	59.5809	6.0000	120.0000

n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
6.0100	122.0661	6.4200	249.5509	6.8300	524.8511	7.2400	1133.5264
6.0200	124.1700	6.4300	254.0334	6.8400	534.6356	7.2500	1155.3823
6.0300	126.3123	6.4400	258.6011	6.8500	544.6116	7.2600	1177.6760
6.0400	128.4940	6.4500	263.2550	6.8600	554.7819	7.2700	1200.4185
6.0500	130.7156	6.4600	267.9975	6.8700	565.1516	7.2800	1223.6171
6.0600	132.9781	6.4700	272.8297	6.8800	575.7236	7.2900	1247.2832
6.0700	135.2820	6.4800	277.7539	6.8900	586.5032	7.3000	1271.4244
6.0800	137.6285	6.4900	282.7716	6.9000	597.4933	7.3100	1296.0537
6.0900	140.0181	6.5000	287.8849	6.9100	608.6996	7.3200	1321.1777
6.1000	142.4518	6.5100	293.0953	6.9200	620.1256	7.3300	1346.8097
6.1100	144.9303	6.5200	298.4052	6.9300	631.7754	7.3400	1372.9580
6.1200	147.4546	6.5300	303.8161	6.9400	643.6547	7.3500	1399.6354
6.1300	150.0255	6.5400	309.3305	6.9500	655.7667	7.3600	1426.8507
6.1400	152.6441	6.5500	314.9500	6.9600	668.1176	7.3700	1454.6176
6.1500	155.3111	6.5600	320.6770	6.9700	680.7109	7.3800	1482.9451
6.1600	158.0274	6.5700	326.5134	6.9800	693.5528	7.3900	1511.8477
6.1700	160.7941	6.5800	332.4616	6.9900	706.6470	7.4000	1541.3342
6.1800	163.6120	6.5900	338.5236	7.0000	720.0000	7.4100	1571.4203
6.1900	166.4825	6.6000	344.7020	7.0100	733.6171	7.4200	1602.1152
6.2000	169.4060	6.6100	350.9986	7.0200	747.5034	7.4300	1633.4349
6.2100	172.3841	6.6200	357.4164	7.0300	761.6632	7.4400	1665.3906
6.2200	175.4175	6.6300	363.9571	7.0400	776.1037	7.4500	1697.9950
6.2300	178.5075	6.6400	370.6239	7.0500	790.8292	7.4600	1731.2637
6.2400	181.6549	6.6500	377.4185	7.0600	805.8471	7.4700	1765.2081
6.2500	184.8612	6.6600	384.3443	7.0700	821.1620	7.4800	1799.8456
6.2600	188.1272	6.6700	391.4035	7.0800	836.7813	7.4900	1835.1874
6.2700	191.4543	6.6800	398.5985	7.0900	852.7099	7.5000	1871.2517
6.2800	194.8435	6.6900	405.9326	7.1000	868.9559	7.5100	1908.0504
6.2900	198.2962	6.7000	413.4079	7.1100	885.5239	7.5200	1945.6019
6.3000	201.8134	6.7100	421.0280	7.1200	902.4222	7.5300	1983.9192
6.3100	205.3968	6.7200	428.7951	7.1300	919.6564	7.5400	2023.0216
6.3200	209.0471	6.7300	436.7129	7.1400	937.2346	7.5500	2062.9221
6.3300	212.7661	6.7400	444.7835	7.1500	955.1622	7.5600	2103.6414
6.3400	216.5549	6.7500	453.0110	7.1600	973.4484	7.5700	2145.1926
6.3500	220.4150	6.7600	461.3976	7.1700	992.0996	7.5800	2187.5977
6.3600	224.3476	6.7700	469.9473	7.1800	1011.1224	7.5900	2230.8706
6.3700	228.3544	6.7800	478.6627	7.1900	1030.5265	7.6000	2275.0332
6.3800	232.4366	6.7900	487.5479	7.2000	1050.3174	7.6100	2320.1006
6.3900	236.5959	6.8000	496.6054	7.2100	1070.5054	7.6200	2366.0967
6.4000	240.8335	6.8100	505.8398	7.2200	1091.0967	7.6300	2413.0356
6.4100	245.1514	6.8200	515.2535	7.2300	1112.1016	7.6400	2460.9426

(*Continued*)

n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
7.6500	2509.8330	8.0700	5805.6143	8.4900	13,745.5537	8.9100	33,270.3555
7.6600	2559.7332	8.0800	5924.4116	8.5000	14,034.3877	8.9200	33,986.8477
7.6700	2610.6589	8.0900	6045.7188	8.5100	14,329.4697	8.9300	34,719.1172
7.6800	2662.6379	8.1000	6169.5806	8.5200	14,630.9121	8.9400	35,467.6445
7.6900	2715.6887	8.1100	6296.0747	8.5300	14,938.9092	8.9500	36,232.7461
7.7000	2769.8330	8.1200	6425.2466	8.5400	15,253.5820	8.9600	37,014.7852
7.7100	2825.0981	8.1300	6557.1558	8.5500	15,575.0781	8.9700	37,814.1406
7.7200	2881.5032	8.1400	6691.8481	8.5600	15,903.5146	8.9800	38,631.1328
7.7300	2939.0776	8.1500	6829.4102	8.5700	16,239.1074	8.9900	39,466.3086
7.7400	2997.8406	8.1600	6969.8911	8.5800	16,581.9902	9.0000	40,320.0000
7.7500	3057.8242	8.1700	7113.3540	8.5900	16,932.3242	9.0100	41,192.7070
7.7600	3119.0474	8.1800	7259.8521	8.6000	17,290.2344	9.0200	42,084.6953
7.7700	3181.5435	8.1900	7409.4775	8.6100	17,655.9668	9.0300	42,996.5742
7.7800	3245.3328	8.2000	7562.2842	8.6200	18,029.6543	9.0400	43,928.7188
7.7900	3310.4497	8.2100	7718.3447	8.6300	18,411.4805	9.0500	44,881.5820
7.8000	3376.9170	8.2200	7877.7256	8.6400	18,801.5801	9.0600	45,855.5547
7.8100	3444.7686	8.2300	8040.4858	8.6500	19,200.2207	9.0700	46,851.3047
7.8200	3514.0283	8.2400	8206.7314	8.6600	19,607.5547	9.0800	47,869.2461
7.8300	3584.7332	8.2500	8376.5215	8.6700	20,023.7734	9.0900	48,909.8711
7.8400	3656.9072	8.2600	8549.9346	8.6800	20,449.0371	9.1000	49,973.5977
7.8500	3730.5891	8.2700	8727.0332	8.6900	20,883.6270	9.1100	51,061.1602
7.8600	3805.8037	8.2800	8907.9307	8.7000	21,327.7109	9.1200	52,173.0078
7.8700	3882.5908	8.2900	9092.6934	8.7100	21,781.5059	9.1300	53,309.6797
7.8800	3960.9780	8.3000	9281.4092	8.7200	22,245.2266	9.1400	54,471.6328
7.8900	4041.0063	8.3100	9474.1416	8.7300	22,719.0449	9.1500	55,659.6914
7.9000	4122.7036	8.3200	9671.0205	8.7400	23,203.2871	9.1600	56,874.3047
7.9100	4206.1133	8.3300	9872.1152	8.7500	23,698.1367	9.1700	58,116.1055
7.9200	4291.2651	8.3400	10,077.5205	8.7600	24,203.8340	9.1800	59,385.5820
7.9300	4378.2036	8.3500	10,287.3096	8.7700	24,720.5664	9.1900	60,683.6133
7.9400	4466.9629	8.3600	10,501.6201	8.7800	25,248.6895	9.2000	62,010.7266
7.9500	4557.5786	8.3700	10,720.5322	8.7900	25,788.4023	9.2100	63,367.6016
7.9600	4650.0986	8.3800	10,944.1436	8.8000	26,339.9766	9.2200	64,754.9023
7.9700	4744.5557	8.3900	11,172.5430	8.8100	26,903.6133	9.2300	66,173.1953
7.9800	4840.9985	8.4000	11,405.8721	8.8200	27,479.7012	9.2400	67,623.4688
7.9900	4939.4629	8.4100	11,644.2227	8.8300	28,068.4609	9.2500	69,106.3047
8.0000	5040.0000	8.4200	11,887.7080	8.8400	28,670.1797	9.2600	70,622.4609
8.0100	5142.6606	8.4300	12,136.4102	8.8500	29,285.0918	9.2700	72,172.5547
8.0200	5247.4688	8.4400	12,390.4961	8.8600	29,913.6172	9.2800	73,757.6641
8.0300	5354.4927	8.4500	12,650.0625	8.8700	30,555.9902	9.2900	75,378.4297
8.0400	5463.7700	8.4600	12,915.2285	8.8800	31,212.5391	9.3000	77,035.6953
8.0500	5575.3521	8.4700	13,186.1191	8.8900	31,883.5117	9.3100	78,730.1172
8.0600	5689.2749	8.4800	13,462.8301	8.9000	32,569.3535	9.3200	80,462.8984

n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
9.3300	82,234.7188	9.5000	119,292.2969	9.6700	173,606.1250	9.8400	253,444.3906
9.3400	84,046.5312	9.5100	121,943.7969	9.6800	177,497.6250	9.8500	259,173.0625
9.3500	85,899.0312	9.5200	124,655.3359	9.6900	181,478.7188	9.8600	265,034.6250
9.3600	87,793.5469	9.5300	127,428.8906	9.7000	185,551.0938	9.8700	271,031.6250
9.3700	89,730.8516	9.5400	130,265.6094	9.7100	189,716.9219	9.8800	277,167.3125
9.3800	91,711.9297	9.5500	133,166.9062	9.7200	193,977.9688	9.8900	283,444.3438
9.3900	93,737.6328	9.5600	136,134.0625	9.7300	198,337.2812	9.9000	289,867.2188
9.4000	95,809.3203	9.5700	139,169.1562	9.7400	202,796.7500	9.9100	296,438.8438
9.4100	97,927.9297	9.5800	142,273.4688	9.7500	207,358.7031	9.9200	303,162.7500
9.4200	100,094.4922	9.5900	145,448.6562	9.7600	212,025.5625	9.9300	310,041.6562
9.4300	102,309.9219	9.6000	148,696.0156	9.7700	216,799.3438	9.9400	317,080.7500
9.4400	104,575.7812	9.6100	152,017.8594	9.7800	221,683.4844	9.9500	324,283.0938
9.4500	106,893.0078	9.6200	155,415.6406	9.7900	226,680.0938	9.9600	331,652.4688
9.4600	109,262.8281	9.6300	158,891.0625	9.8000	231,791.7969	9.9700	339,192.1875
9.4700	111,686.1875	9.6400	162,445.6406	9.8100	237,020.8281	9.9800	346,907.5625
9.4800	114,164.8047	9.6500	166,081.8906	9.8200	242,370.9844	9.9900	354,802.0625
9.4900	116,699.7344	9.6600	169,801.4062	9.8300	247,844.5156	10.0000	362,880.0000

APPENDIX **B**

COMPUTER PROGRAM TO CALCULATE THE RELIABILITY OF A CONSECUTIVE-K-OUT-OF-N:F SYSTEM

```
      DIMENSION Q(20),P(20),F(0:20)
10
      PRINT*,"Enter Total Number of Units n"
      READ*,N
      PRINT*,"Enter Total Number of Consecutive Failed Units k"
      READ*,K
      IF (K.LE. 1 .OR. K .GE. N) GOTO 10
20
      DO 100 I =1,N
      PRINT*,"Enter the reliability of component", I
      READ*,P(I)
      IF (P(I) .LE. 0 .OR. P(I) .GT. 1) GOTO 20
      Q(I)=1-P(I)
100
      CONTINUE
C
C

      DO 200 J=1,K
      IJ=J-1
      F(J)=0.0
200
      CONTINUE
```

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```
C
      QP=1
      DO 300 II=1,K
      QP=QP*Q(II)
300   CONTINUE
C
      F(K)=QP
      DO 400 IK=K+1, N
      QP=QP*(Q(IK)/Q(IK-K))
      F(IK)=F(IK-1)+ (1-F(IK-K-1))*P(IK-K)*QP
400   CONTINUE
      RS=1-F(N)
      PRINT*,“Reliability of the system is”, RS
C
      STOP
      END
```

APPENDIX C

OPTIMUM ARRANGEMENT OF COMPONENTS IN CONSECUTIVE-2-OUT-OF-N:F SYSTEMS

```
c
c
      integer n,nmax,i,j,permutation,temp,k
      parameter(nmax=20)
c     Components are stored in component(nmax)
      integer component(nmax),pre_order(nmax)
c     Probabilities are stored in q(nmax)
      double precision q(nmax),previous,swap,seed,reliability
      real rand
      logical flag,disregard
      common /block/ q,previous,pre_order,n,reliability
      print*, "Please input the number of components : "
      read*,n
      do i=1,n
      print*, "Enter the unreliability of component", i
         read*,q(i)
       enddo
c      Sort them in descending order
          do i=1,n
             do j=i+1,n
                =if (q(i).lt.q(j)) then
                   swap=q(i)
                   q(i)=q(j)
                   q(j)=swap
             endif
          enddo
       enddo
```

```
c   Maximum reliability and the corresponding order of components
c   will be stored in "previous" and "pre_order" respectively .
      previous=0.0
      do i=1,n
        pre_order(i)=i
      enddo
c   Now start enumarating the components.(There will be n!
c   of them.If the ones in reversed sequence are eliminated,
c   then there will be n!/2 sequences with distinct reliabilities. )
c   Initialize component(i) to 1 2 3 . . . .n
      do i=1,n
        component(i)=i
      enddo
c   Now calculate the reliability of the first sequence
      call calculate_reliability(component)
      permutation = 1
c   Swap the last two elements
5   temp=component(n-1)
      component(n-1)=component(n)
      component(n)=temp
c   Check whether this sequence appeared in reversed order before.
      disregard=.false.
      do i=1,component(1)-1
        if (component(n).eq.i) disregard=.true.
      enddo
c   Calculate the reliability of the next sequence (if it didn't
c   appear in reversed order before ) which is obtained
c   from the previous one by swapping the last two elements.
      if (.not.disregard) then
        permutation=permutation+1
        call calculate_reliability(component,permutation)
      endif
c   Now in so-called dynamical do-loops the next sequence is generated.
c   First (n-2)nd element is increased by one ,the position of its
c   new entry in the old sequence is found and replaced by the old
c   entry . If (n-2)nd element cannot be increased any further , then
c   the (n-3)rd entry is checked upon , and so on . If a new sequence
c   is obtained by increasing the entry in the kth position , then the
c   entries to the right of k (k+1,k+2,..,n) are sorted in ascending order.
c   That way it is possible to generate n! sequences .
      k=n-2
10  continue
c   If k is zero , this signals that all n!/2 sequences have been
c   generated.
```

```
      if (k.gt.0) then
         temp = component(k)
20          component(k)=component(k)+1
         if (component(k).eq.(n+1)) then
            component(k)= temp
            k=k-1
            goto 10
         endif
         flag=.false.
         do i=1,k-1
            if (component(i).eq.component(k)) flag=.true.
         enddo
         if (flag) goto 20
         component(k)=component(k)
         do i=k+1,n
            if (component(k).eq.component(i)) then
               component(i)=temp
            endif
         enddo
c     Now sort the elements to the right of k. (Starting from
c     (k+1) until n.)
         do i=k+1,n
            do j=i+1,n
               if (component(i).gt.component(j)) then
                  temp = component(i)
                  component(i)=component(j)
                  component(j)=temp
               endif
            enddo
         enddo
c     Check whether this sequence appeared in reversed order before.
         disregard=.false.
         do i=1,component(1)-1
            if (component(n).eq.i) disregard=.true.
         enddo
c     If a new sequence is found , then its reliability will be calculated.
         if (.not.disregard) then
            permutation=permutation+1
            call calculate_reliability(component)
         endif
         goto 5
      endif
c
```

```
c   Now the result will be printed
c
      print*, "The number of components in this run is",n
      print*," "
      print *,"The following sequence of components has the maximum",
      1   " reliability : "
      write(6,*)(pre_order(i),i=1,n)
      print*," "
      print*,"Its Reliability is ",previous
      print*," "
      do i=1,n
      print*,pre_order(i)," has the probability of failure ",
      1   q(pre_order(i))
      enddo
      end
c
c   The following subroutine calculates the reliability
c   of a given sequence of components.
c
      subroutine calculate_reliability (component)
      integer n,nmax,i,k,step,j1,j2,j,temp
      parameter(nmax=20)
c   counter(nmax) will index the dynamical do-loops.
c   neighbour(nmax,2) keeps the two consecutive components.
c   Note that there are (n-1) of them in an n-component system.
        integer counter(nmax),component(nmax),pre_order(nmax)
        1   ,neighbour(nmax,2)
        double precision q(nmax),probability,coef,previous,product
        double precision reliability
        common /block/ q,previous,pre_order,n,reliability
c   First set the register "neighbour".
        do i=1,n-1
          neighbour(i,1)=component(i)
          neighbour(i,2)=component(i+1)
        enddo
c   Calculate the sum of the products of failure probabilities
c   for two consecutive components .
        probability=0.0
        do i=1,n-1
          probability=probability+q(neighbour(i,1))*q(neighbour(i,2))
        enddo
c   Now there will be "step" many dynamical "do-loops" , which
c   will be indexed by counter(1..step) . This way   the rest of
```

```
c   the terms in the equation will be calculated.
      do step=2,n-1
         coef = 1.0d0
      if ((step-2*(step/2)).eq.0) coef=-1.0d0
c   Initialize the counters for the dynamical do-loops.
      do i=1,step
         counter(i)=i
      enddo
5        product=1.0
c   For the enumarated sequence the probability is calculated.
      do i=1,step
         j=counter(i)
         product=product*q(neighbour(j,1))*q(neighbour(j,2))
      enddo
      do i=1,step-1
         j1=counter(i)
         j2=counter(i+1)
         if(neighbour(j1,2).eq.neighbour(j2,1)) then
            product = product/q(neighbour(j1,2))
         endif
      enddo
      probability = probability + coef*product
c   Now increment the counter of the innermost do-loop. If the
c   upper limit is reached, then the counter of the next innermost
c   do-loop is incremented, and so on.
      k=step
10    temp=counter(k)
      counter(k)=counter(k)+1
      if(counter(k).eq.(n-step+k)) then
         counter(k)=temp
         k=k-1
         if (k.gt.0) goto 10
      else
         do i=k+1,step
            counter(i)=counter(i-1)+1
         enddo
         if (counter(step).lt.n) goto 5
      endif
    enddo
c
c   At this point the probability of failure has been calculated
c   and now the reliability will be obtained .
c
    reliability = 1.0d0-probability
```

```
c
c   If the calculated reliability is greater than the largest
c   of the previous sequences , then store the present reliability
c   as the largest one.
c
      if (reliability.gt.previous) then
         previous=reliability
         do i=1,n
            pre_order(i)=component(i)
         enddo
      endif
   end
```

APPENDIX D

COMPUTER PROGRAM FOR SOLVING THE TIME-DEPENDENT EQUATIONS USING RUNGE-KUTTA'S METHOD

```
      EXTERNAL VECTOR
      DIMENSION TT(1000)
      REAL*8 X (10),XDOT(10)
      COMMON ALMDA1, ALMDA2, ALMDA3, AMU
      ALMDA1=0.00001
      ALMDA2=ALMDA1/10.0
      ALMDA3=ALMDA2/10.0
      AMU=0.0001
      N=5
      H=0.05
      T=0.0
      DO 100 I=1,5
      X(I)=0.0
100   CONTINUE
      X(I)=1.0
      DO 200 I=1,1000
      TT(I)=T
      CALL RKINT(T,X,N,H,VECTOR)
      WRITE(6,44) T, X(1) , X(2),X(3),X(4)
44    FORMAT
      (1X,'T=',F15.3,3X,F10.7,3X,F10.7,3X,F10.7)
200   CONTINUE
      STOP
      END
```

```
C
C
      SUBROUTINE RKINT(T,X,N,H,VECTOR)
      EXTERNAL VECTOR
      REAL*8 X(10),XDOT(10),K1(10),K2(10),K3(10),K4(10),SAVEX(10)
      DO 10 J=1,N
      SAVEX(J)=X(J)
10    CONTINUE
      T=T+1
      CALL VECTOR(T,X,XDOT,N)
      DO 11 J=1,N
      K1(J)=XDOT(J)
11    X(J)=SAVEX(J)+0.5*H*K1(J)
      T=T+0.5*H
      CALL VECTOR(T,X,XDOT,N)
      DO 12 J=1,N
      K2(J)=XDOT(J)
12    X(J)=SAVEX(J)+0.5*H*K2(J)
      T=T+0.5*H
      CALL VECTOR(T,X,XDOT,N)
      DO 13 J=1,N
      K3(J)=XDOT(J)
13    X(J)=SAVEX(J)+0.5*H*K3(J)
      T=T+0.5*H
      CALL VECTOR(T,X,XDOT,N)
      DO 14J=1,N
      K4(J)=XDOT(J)
14    X(J)=SAVEX(J)+(H/6.)*(K1(J)+2*K2(J)+2*K3(J)+K4(J))
      RETURN
      END
C
      SUBROUTINE VECTOR (T,X,XDOT,N)
      REAL*8 XDOT(10),X(10)
      COMMON ALMDA, ALMDBR,AMU
      XDOT(1)=-(ALMDA1+ALMDA2+ALMDA3)*X(1)+AMU*X(2)
      &+AMU*X(3)+AMU*X(4)
      XDOT(2)=-AMU*X(2)+ALMDA1*X(1)
      XDOT(3)=-AMU*X(3)+ALMDA2*X(2)
      XDOT(4)=-AMU*X(4)+ALMDA3*X(3)
      RETURN
      END
```

APPENDIX **E**

THE NEWTON–RAPHSON METHOD

This method is used for solving nonlinear equations iteratively. We consider first a single variable equation. Let x_0 be a point which is not a root of the function $f(x)$ but a close estimate of the root. So the function $f(x)$ can be expanded using Taylor series about x_0 as

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \ldots$$

If $f(x) = 0$, then x must be a root, and the right-hand side of the above equation constitutes an equation for the root x. The equation is a polynomial of degree infinity and an approximate value of x can be obtained by setting $f(x)$ to zero and taking the first two terms of the right-hand side to yield

$$0 = f(x_0) + (x - x_0)f'(x_0).$$

Solving for x gives

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Now x represents an improved estimate of the root and can be used in lieu of x_0 in the above equation to obtain a better estimate of the root. The process is repeated until the difference between two consecutive estimates of the root is acceptable.

These steps can be summarized as follows:

1. Determine an initial estimate of x—say, $\hat{x}_0$ —such that $f(\hat{x}_0) \simeq 0$.
2. $\hat{x}_1 = \hat{x}_0 - (f(\hat{x}_0) / f'(\hat{x}_0))$. $f'(\hat{x}_0)$ first derivative of $f(x)$ at $x = \hat{x}_0$.
3. $\hat{x}_{k+1} = \hat{x} - (f(\hat{x}_k) / f'(\hat{x}_k))$.
4. Stop when $|d| = \hat{x}_k - \hat{x}_{k+1}$ is less than or equal to ε.

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EXAMPLE E.1

Find the value of x which results in the following function $f(x) = 0$:

$$f(x) = x^3 - 2x^2 + 5.$$

$$f'(x) = 3x^2 - 4x$$

SOLUTION

Let

$$\hat{x}_0 = -1$$

$$f(\hat{x}_0) = 2 \quad f'(\hat{x}_0) = 7$$

$$\hat{x}_1 = -1 - \frac{2}{7} = -1.285714$$

$$f(\hat{x}_1) = 0.431484$$

$$f_1' = (\hat{x}_1) = 10.102037$$

- $\hat{x}_2 = -1.285714 - \dfrac{0.431484}{10.102037} = -1.243001$

  $f(\hat{x}_2) = -0.010607 \quad f'(\hat{x}_2) = 9.607163$

- $\hat{x}_3 = -1.243001 + \dfrac{0.010607}{9.607163} = -1.241897$

  $f(\hat{x}_3) = -4.0673 \times 10^{-6} \quad f'(\hat{x}_3) = 9.594511$

- $\hat{x}_4 = -1.241897 + \dfrac{4.0673 \times 10^{-6}}{9.594511} = -1.241896.$

The value of x that minimizes the function $f(x)$ is −1.241896. ■

This method can be extended to solve a system of equations with more than one unknown. For example, determine $x_1, x_2, \ldots, x_p$ such that

$$f_1(x_1, x_2, \ldots, x_p) = 0$$

$$f_2(x_1, x_2, \ldots, x_p) = 0$$

$$f_p(x_1, x_2, \ldots, x_p) = 0.$$

Let a_{ij} be the partial derivative of f_i w.r.t. x_j and $a_{ij} = \partial f_i/\partial x_j$. Construct the Jacobian Matrix J as

$$J = \begin{bmatrix} a_{11} & \dots & a_{1p} \\ a_{21} & \dots & a_{2p} \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ a_{p1} & \dots & a_{pp} \end{bmatrix}.$$

Let $x_1^k, x_2^k, \dots x_p^k$ be the approximate roots at the kth iteration. Let $f_1^k, \dots, f_p^k$ be the corresponding values of the functions $f_1, \dots, f_p$—that is,

$$\begin{aligned} f_1^k &= f_1(x_1^k, \dots, x_p^k) \\ f_2^k &= f_2(x_1^k, \dots, x_p^k) \\ f_3^k &= f_3(x_1^k, \dots, x_p^k). \end{aligned}$$

Let b_{ij}^k be the ijth element of J^{-1} evaluated at $x_1^k, x_2^k, \dots x_p^k$. The net approximation is given by

$$\begin{aligned} x_1^{k+1} &= x_1^k - (b_{11}^k f_1^k + b_{12}^k f_2^k + \dots + f_{1p}^k f_p^k) \\ x_2^{k+1} &= x_2^k - (b_{21}^k f_1^k + b_{22}^k f_2^k + \dots + f_{2p}^k f_p^k) \\ x_p^{k+1} &= x_p^k - (b_{p1}^k f_1^k + b_{p2}^k f_2^k + \dots + f_{pp}^k f_p^k). \end{aligned}$$

Let $x_1^0, x_2^0, \dots, x_p^0$ be the initial values of x_i. The above iteration steps are continued until either $f_1, f_2, \dots, f_p$ are close enough to zero or when the differences in the x values between two consecutive iterations are less than a specified amount ε.

EXAMPLE E.2

Find the values of x_1 and such x_2 that

$$\begin{aligned} x_1^2 - x_1 x_2 + 2x_2 - 4 &= 0 \\ x_2^2 + x_1 x_2 - 4x_1 &= 0. \end{aligned}$$

SOLUTION

$$\begin{aligned} p &= 2 \\ f_1 &= x_1^2 - x_1 x_2 + 2x_2 - 4 \\ f_2 &= x_2^2 + x_1 x_2 - 4x_1. \end{aligned}$$

We obtain the partial derivatives as

$$\frac{\partial f_1}{\partial x_1} = 2x_1 - x_2$$
$$\frac{\partial f_1}{\partial x_2} = -x_1 + 2$$
$$\frac{\partial f_2}{\partial x_1} = x_2 - 4$$
$$\frac{\partial f_2}{\partial x_2} = 2x_2 + x_1.$$

The Jacobian Matrix is

$$J = \begin{bmatrix} 2x_1 - x_2 & -x_1 + 2 \\ x_2 - 4 & 2x_2 + x_1 \end{bmatrix}.$$

Let the initial estimates of $x_1^0 = 1,\ x_2^0 = 2,\ f_1^0 = -1,\ f_2^0 = -2$

$$J = \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix}$$
$$J^{-1} = \begin{bmatrix} 2.5 & -0.5 \\ 1 & 0 \end{bmatrix}.$$

Iteration 1

$$x_1^1 = 1 - [(-1)(2.5) + (-2)(-0.5)] = 2.5$$
$$x_2^1 = 2 - [(-1)(1) + (0)(2)] = 3$$
$$f_1^1 = 0.75, \quad f_2^1 = 6.5$$
$$J = \begin{bmatrix} 2 & -0.5 \\ -1 & 8.5 \end{bmatrix}$$
$$J^{-1} = \begin{bmatrix} 0.5152 & 0.0303 \\ 0.0606 & 0.1212 \end{bmatrix}.$$

Iteration 2

$$x_1^2 = 2.5 - [(0.5152)(0.75) + (0.0303)(6.5)] = 1.91665$$
$$x_2^2 = 3 - [(0.0606)(0.75) + (0.1212)(6.5)] = 2.16675$$

Substituting in f_1 and f_2 to obtain

$$f_1^2 = -0.14585, \quad f_2^2 = 1.1811$$

$$J = \begin{bmatrix} 1.666 & 0.0833 \\ -1.8333 & 6.250 \end{bmatrix}$$

$$J^{-1} = \begin{bmatrix} 0.5916 & 0.1735 \\ -0.0079 & 0.1577 \end{bmatrix}.$$

Iteration 3

$$x_1^3 = 1.9166 - [(1.5916)(-0.1458) + (-0.0079)(1.1811)] = 2.0017$$

$$x_2^3 = 2.16675 - [(0.1735)(-0.1458) + (0.1577)(1.1811)] = 2.00579$$

Substituting in f_1 and f_2 to obtain

$$f_1^3 = 0.003558, \quad f_2^3 = 0.03125$$

$$J = \begin{bmatrix} 1.9977 & -0.0017 \\ -1.9942 & 0.0133 \end{bmatrix}$$

$$J^{-1} = \begin{bmatrix} 0.5007 & 0.0001 \\ 0.1661 & 0.1663 \end{bmatrix}.$$

Iteration 4

$$x_1^4 = 2.0017 - [(0.5007)(0.003558) + (0.0001)(0.03125)] = 1.9999$$

$$x_2^4 = 2.0057 - [(0.1661)(0.003558) + (0.1663)(0.03125)] = 2.000007.$$

Substituting in f_1 and f_2 to obtain

$$f_1^4 = -0.00002, \quad f_2^4 = 0.00006.$$

Since the values of the functions are very close to zero, the iteration is terminated and the solution of the equations is

$$x_1 = 2, \quad x_2 = 2. \qquad \blacksquare$$

APPENDIX F

COEFFICIENTS OF b_i's FOR $i = 1, \ldots, n$*

The best linear unbiased estimator $\theta_2^* = \sum b_i y_{n(i)}$ when the threshold parameter θ_1 is known on k order statistics with optimum ranks from the Rayleigh distribution for sample sizes $n = 5(1)25(5)45$ with censoring from the right for $r = 0(1)n - 2$.

* From Hassanein, K. M., Saleh, A. K., and Brown, E. (1995), "Best linear unbiased estimate and confidence interval for Rayleigh's scale parameter when the threshold parameter is known for data with censored observations from the right." Report from the University of Kansas Medical School. Reproduced by permission from the authors.

i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i
$n = 5$	$r = 0$	$n = 7$	$r = 0$	5	0.09412	$n = 9$	$r = 2$	3	0.04479	2	1.57120	$n = 11$	$r = 5$	11	0.19806
				6	0.11041			4	0.05373						
1	0.06038	1	0.03655	7	0.28344	1	0.03229	5	0.06258	$n = 11$	$r = 0$	1	0.03405	$n = 12$	$r = 2$
2	0.09279	2	0.05485			2	0.04781	6	0.07176			2	0.04994		
3	0.12404	3	0.07110	$n = 8$	$r = 2$	3	0.06103	7	0.08174	1	0.01865	3	0.06312	1	0.01966
4	0.16048	4	0.08739			4	0.07351	8	0.09325	2	0.02742	4	0.07513	2	0.02881
5	0.23520	5	0.10537	1	0.03988	5	0.08604	9	0.23258	3	0.03476	5	0.08662	3	0.03640
		6	0.12799	2	0.05935	6	0.09924			4	0.04153	6	0.63451	4	0.04332
$n = 5$	$r = 1$	7	0.17785	3	0.07615	7	0.36983	$n = 10$	$r = 2$	5	0.04815			5	0.04999
				4	0.09228					6	0.05488	$n = 11$	$r = 6$	6	0.05665
1	0.07530	$n = 7$	$r = 1$	5	0.10882	$n = 9$	$r = 3$	1	0.02685	7	0.06202			7	0.06351
2	0.11534			6	0.41420			2	0.03959	8	0.06994	1	0.04069	8	0.07080
3	0.15324	1	0.04261			1	0.03762	3	0.05033	9	0.07932	2	0.05962	9	0.07882
4	0.43358	2	0.06390	$n = 8$	$r = 3$	2	0.05565	4	0.06033	10	0.09177	3	0.07522	10	0.28269
		3	0.08272			3	0.07094	5	0.07019	11	0.12119	4	0.08934		
$n = 5$	$r = 2$	4	0.10145	1	0.04773	4	0.08529	6	0.08036			5	0.78208	$n = 12$	$r = 3$
		5	0.12177	2	0.07092	5	0.09952	7	0.09127	$n = 11$	$r = 1$				
1	0.09957	6	0.31934	3	0.09079	6	0.49590	8	0.33479			$n = 11$	$r = 7$	1	0.02184
2	0.15151			4	0.10964					1	0.02051			2	0.03199
3	0.67477	$n = 7$	$r = 2$	5	0.56289	$n = 9$	$r = 4$	$n = 10$	$r = 3$	2	0.03015	1	0.05052	3	0.04040
										3	0.03821	2	0.07386	4	0.04807
$n = 5$	$r = 3$	1	0.05103	$n = 8$	$r = 4$	1	0.04500	1	0.03066	4	0.04565	3	0.09296	5	0.05543
		2	0.07640			2	0.06647	2	0.04518	5	0.05290	4	0.97012	6	0.06276
1	0.14600	3	0.09867	1	0.05933	3	0.08456	3	0.05739	6	0.06027			7	0.07027
2	1.03016	4	0.12049	2	0.08793	4	0.10137	4	0.06872	7	0.06805	$n = 11$	$r = 8$	8	0.07817
		5	0.47254	3	0.11216	5	0.64294	5	0.07982	8	0.07663			9	0.37083
$n = 6$	$r = 0$			4	0.74509			6	0.09114	9	0.08662	1	0.06650		
		$n = 7$	$r = 3$			$n = 9$	$r = 5$	7	0.44466	10	0.21383	2	0.09693	$n = 12$	$r = 4$
1	0.04599			$n = 8$	$r = 5$							3	1.23222		
2	0.06969	1	0.06348			1	0.05590	$n = 10$	$r = 4$	$n = 11$	$r = 2$			1	0.02455

3	0.09142
4	0.11434
5	0.14236
6	0.20229

$n = 6$ $r = 1$

1	0.05512
2	0.08341
3	0.10912
4	0.13579
5	0.36696

$n = 6$ $r = 2$

1	0.06864
2	0.10354
3	0.13477
4	0.55338

$n = 6$ $r = 3$

1	0.09063
2	0.13594
3	0.79308

$n = 6$ $r = 4$

1	0.13274
2	1.15833

2	0.09478
3	0.12187
4	0.65522

$n = 7$ $r = 4$

1	0.08373
2	0.12440
3	0.89716

$n = 7$ $r = 5$

1	0.12255
2	1.27394

$n = 8$ $r = 0$

1	0.02996
2	0.04464
3	0.05739
4	0.06977
5	0.08274
6	0.09746
7	0.11634
8	0.15894

$n = 8$ $r = 1$

1	0.03422
2	0.05097
3	0.06549
4	0.07953

1	0.07820
2	0.11540
3	0.99106

$n = 8$ $r = 6$

1	0.11440
2	1.38007

$n = 9$ $r = 0$

1	0.02514
2	0.03725
3	0.04761
4	0.05745
5	0.06743
6	0.07816
7	0.09057
8	0.10672
9	0.14384

$n = 9$ $r = 1$

1	0.02827
2	0.04189
3	0.05351
4	0.06453
5	0.07566
6	0.08755
7	0.10106
8	0.25529

2	0.08238
3	0.10448
4	0.82629

$n = 9$ $r = 6$

1	0.07364
2	0.10812
3	1.07720

$n = 9$ $r = 7$

1	0.10768
2	1.47869

$n = 10$ $r = 0$

1	0.02149
2	0.03171
3	0.04034
4	0.04841
5	0.05641
6	0.06475
7	0.07388
8	0.08457
9	0.09865
10	0.13149

$n = 10$ $r = 1$

1	0.02387
2	0.03522

1	0.03570
2	0.05256
3	0.06668
4	0.07971
5	0.09237
6	0.56855

$n = 10$ $r = 5$

1	0.04268
2	0.06276
3	0.07948
4	0.09477
5	0.71543

$n = 10$ $r = 6$

1	0.05301
2	0.07777
3	0.09821
4	0.90085

$n = 10$ $r = 7$

1	0.06980
2	0.10207
3	1.15721

$n = 10$ $r = 8$

1	0.10203

1	0.02279
2	0.03349
3	0.04243
4	0.05066
5	0.05867
6	0.06678
7	0.07529
8	0.08456
9	0.30633

$n = 11$ $r = 3$

1	0.02562
2	0.03764
3	0.04766
4	0.05687
5	0.06580
6	0.07479
7	0.08413
8	0.40400

$n = 11$ $r = 4$

1	0.02925
2	0.04294
3	0.05433
4	0.06476
5	0.07482
6	0.08487
7	0.51146

$n = 11$ $r = 9$

1	0.09718
2	1.65860

$n = 12$ $r = 0$

1	0.01639
2	0.02402
3	0.03036
4	0.03616
5	0.04175
6	0.04735
7	0.05315
8	0.05938
9	0.06636
10	0.07469
11	0.08583
12	0.11246

$n = 12$ $r = 1$

1	0.01788
2	0.02620
3	0.03310
4	0.03942
5	0.04550
6	0.05159
7	0.05789
8	0.06462
9	0.07212
10	0.08092

2	0.03595
3	0.04537
4	0.05395
5	0.06216
6	0.07029
7	0.07855
8	0.46601

$n = 12$ $r = 5$

1	0.02801
2	0.04100
3	0.05171
4	0.06143
5	0.07069
6	0.07979
7	0.57226

$n = 12$ $r = 6$

1	0.03260
2	0.04768
3	0.06007
4	0.07126
5	0.08186
6	0.69526

$n = 12$ $r = 7$

1	0.03896
2	0.05690
3	0.07159

i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i
4	0.08476	3	0.03481	3	1.37029	$n = 14$	$r = 4$	$n = 14$	$r = 11$	7	0.04529	2	0.04238	$n = 16$	$r = 2$
5	0.84405	4	0.04132							8	0.04980	3	0.05305		
		5	0.04752	$n = 13$	$r = 11$	1	0.01823	1	0.05884	9	0.05447	4	0.06252	1	0.01221
$n = 12$	$r = 8$	6	0.05363			2	0.02659	2	0.08522	10	0.05941	5	0.07129	2	0.01775
		7	0.05981	1	0.08925	3	0.03342	3	1.43445	11	0.06474	6	0.85505	3	0.02225
1	0.04835	8	0.06623	2	1.82095	4	0.03958			12	0.29953			4	0.02627
2	0.07049	9	0.07306			5	0.04539	$n = 14$	$r = 12$			$n = 15$	$r = 10$	5	0.03004
3	0.08848	10	0.34317	$n = 14$	$r = 0$	6	0.05106			$n = 15$	$r = 4$			6	0.03367
4	1.03506					7	0.05673	1	0.08596			1	0.03484	7	0.03726
		$n = 13$	$r = 4$	1	0.01303	8	0.06251	2	1.89695	1	0.01603	2	0.05056	8	0.04088
$n = 12$	$r = 9$			2	0.01901	9	0.06854			2	0.02334	3	0.06322	9	0.04460
		1	0.02100	3	0.02392	10	0.39769	$n = 15$	$r = 0$	3	0.02929	4	0.07439	10	0.04849
1	0.06362	2	0.03068	4	0.02834					4	0.03462	5	1.00921	11	0.05264
2	0.09251	3	0.03864	5	0.03254	$n = 14$	$r = 5$	1	0.01176	5	0.03963			12	0.05719
3	1.30303	4	0.04584	6	0.03665			2	0.01713	6	0.04449	$n = 15$	$r = 11$	13	0.06232
		5	0.05269	7	0.04079	1	0.02024	3	0.02151	7	0.04931			14	0.21767
$n = 12$	$r = 10$	6	0.05941	8	0.04506	2	0.02951	4	0.02544	8	0.05419	1	0.04321		
		7	0.06619	9	0.04959	3	0.03709	5	0.02914	9	0.05922	2	0.06261	$n = 16$	$r = 3$
1	0.09297	8	0.07317	10	0.05453	4	0.04390	6	0.03274	10	0.06449	3	0.07816		
2	1.74166	9	0.42880	11	0.06015	5	0.05032	7	0.03633	11	0.37123	4	1.21011	1	0.01315
				12	0.06693	6	0.05656	8	0.03999					2	0.01911
$n = 13$	$r = 0$	$n = 13$	$r = 5$	13	0.07611	7	0.06278	9	0.04381	$n = 15$	$r = 5$	$n = 15$	$r = 12$	3	0.02396
				14	0.09844	8	0.06909	10	0.04788					4	0.02828
1	0.01455	1	0.02360			9	0.48182	11	0.05235	1	0.01763	1	0.05683	5	0.03233
2	0.02127	2	0.03447	$n = 14$	$r = 1$			12	0.05745	2	0.02565	2	0.08216	6	0.03623
3	0.02682	3	0.04339			$n = 14$	$r = 6$	13	0.06365	3	0.03219	3	1.49592	7	0.04009
4	0.03185	4	0.05144	1	0.01403			14	0.07208	4	0.03804			8	0.04397
5	0.03666	5	0.05908	2	0.02047	1	0.02275	15	0.09273	5	0.04353	$n = 15$	$r = 13$	9	0.04795
6	0.04142	6	0.06655	3	0.02575	2	0.03315			6	0.04884			10	0.05210
7	0.04627	7	0.07402	4	0.03051	3	0.04165	$n = 15$	$r = 1$	7	0.05409	1	0.08300	11	0.05651
8	0.05136	8	0.52258	5	0.03502	4	0.04926			8	0.05939	2	1.97003	12	0.06130
9	0.05688			6	0.03944	5	0.05642	1	0.01260	9	0.06482			13	0.28196
10	0.06311	$n = 13$	$r = 6$	7	0.04389	6	0.06336	2	0.01835	10	0.44766	$n = 16$	$r = 0$		
11	0.07059			8	0.04848	7	0.07023	3	0.02304					$n = 16$	$r = 4$

12	0.08066
13	0.10496

$n = 13 \quad r = 1$

1	0.01576
2	0.02304
3	0.02904
4	0.03449
5	0.03969
6	0.04483
7	0.05007
8	0.05555
9	0.06147
10	0.06811
11	0.07597
12	0.18460

$n = 13 \quad r = 2$

1	0.01719
2	0.02513
3	0.03167
4	0.03760
5	0.04326
6	0.04885
7	0.05453
8	0.06045
9	0.06681
10	0.07386
11	0.26271

$n = 13 \quad r = 3$

1	0.01891
2	0.02763

1	0.02692
2	0.03930
3	0.04945
4	0.05857
5	0.06719
6	0.07557
7	0.62836

$n = 13 \quad r = 7$

1	0.03133
2	0.04570
3	0.05743
4	0.06795
5	0.07782
6	0.75181

$n = 13 \quad r = 8$

1	0.03743
2	0.05453
3	0.06844
4	0.08083
5	0.90218

$n = 13 \quad r = 9$

1	0.04643
2	0.06754
3	0.08460
4	1.09637

$n = 13 \quad r = 10$

1	0.06109
2	0.08864

9	0.05332
10	0.05859
11	0.06455
12	0.07163
13	0.17297

$n = 14 \quad r = 2$

1	0.01520
2	0.02217
3	0.02789
4	0.03304
5	0.03792
6	0.04269
7	0.04749
8	0.05243
9	0.05762
10	0.06325
11	0.06953
12	0.24558

$n = 14 \quad r = 3$

1	0.01658
2	0.02418
3	0.03041
4	0.03602
5	0.04133
6	0.04651
7	0.05171
8	0.05704
9	0.06263
10	0.06863
11	0.31971

8	0.57487

$n = 14 \quad r = 7$

1	0.02595
2	0.03780
3	0.04745
4	0.05608
5	0.06417
6	0.07197
7	0.68065

$n = 14 \quad r = 8$

1	0.03019
2	0.04394
3	0.05511
4	0.06506
5	0.07434
6	0.80489

$n = 14 \quad r = 9$

1	0.03606
2	0.05243
3	0.06567
4	0.07741
5	0.95708

$n = 14 \quad r = 10$

1	0.04473
2	0.06494
3	0.08118
4	1.15458

4	0.02725
5	0.03121
6	0.03506
7	0.03890
8	0.04281
9	0.04688
10	0.05122
11	0.05596
12	0.06135
13	0.06779
14	0.16280

$n = 15 \quad r = 2$

1	0.01357
2	0.01976
3	0.02481
4	0.02933
5	0.03359
6	0.03774
7	0.04186
8	0.04605
9	0.05041
10	0.05503
11	0.06006
12	0.06571
13	0.23071

$n = 15 \quad r = 3$

1	0.01470
2	0.02140
3	0.02686
4	0.03176
5	0.03637
6	0.04084

$n = 15 \quad r = 6$

1	0.01957
2	0.02847
3	0.03572
4	0.04219
5	0.04825
6	0.05410
7	0.05987
8	0.06566
9	0.53091

$n = 15 \quad r = 7$

1	0.02199
2	0.03198
3	0.04010
4	0.04734
5	0.05410
6	0.06061
7	0.06699
8	0.62367

$n = 15 \quad r = 8$

1	0.02508
2	0.03646
3	0.04568
4	0.05389
5	0.06153
6	0.06885
7	0.72977

$n = 15 \quad r = 9$

1	0.02917

1	0.01069
2	0.01554
3	0.01948
4	0.02300
5	0.02630
6	0.02949
7	0.03264
8	0.03583
9	0.03910
10	0.04254
11	0.04624
12	0.05031
13	0.05499
14	0.06069
15	0.06847
16	0.08767

$n = 16 \quad r = 1$

1	0.01140
2	0.01657
3	0.02077
4	0.02453
5	0.02804
6	0.03144
7	0.03480
8	0.03819
9	0.04168
10	0.04533
11	0.04925
12	0.05355
13	0.05847
14	0.06436
15	0.15384

1	0.01424
2	0.02070
3	0.02594
4	0.03062
5	0.03500
6	0.03921
7	0.04337
8	0.04755
9	0.05182
10	0.05626
11	0.06094
12	0.34841

$n = 16 \quad r = 5$

1	0.01553
2	0.02257
3	0.02828
4	0.03337
5	0.03813
6	0.04271
7	0.04722
8	0.05174
9	0.05634
10	0.06110
11	0.41855

$n = 16 \quad r = 6$

1	0.01708
2	0.02481
3	0.03108
4	0.03666
5	0.04188
6	0.04688

i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i
7	0.05180	$n = 16$	$r = 14$	7	0.03582	2	0.02668	4	0.01915	14	0.05548	4	0.03432	$n = 18$	$r = 16$
8	0.05672			8	0.03920	3	0.03337	5	0.02184	15	0.25279	5	0.03908		
9	0.06170	1	0.08032	9	0.04264	4	0.03929	6	0.02441			6	0.04361	1	0.07567
10	0.49401	2	2.04050	10	0.04619	5	0.04478	7	0.02692	$n = 18$	$r = 4$	7	0.04800	2	2.17464
				11	0.04991	6	0.05002	8	0.02942			8	0.05234		
$n = 16$	$r = 7$	$n = 17$	$r = 0$	12	0.05389	7	0.05512	9	0.03194	1	0.01153	9	0.05666	$n = 19$	$r = 0$
				13	0.05823	8	0.06016	10	0.03453	2	0.01671	10	0.57825		
1	0.01896	1	0.00977	14	0.26650	9	0.61998	11	0.03724	3	0.02090			1	0.00828
2	0.02753	2	0.01418					12	0.04010	4	0.02460	$n = 18$	$r = 9$	2	0.01199
3	0.03448	3	0.01775	$n = 17$	$r = 4$	$n = 17$	$r = 9$	13	0.04321	5	0.02804			3	0.01497
4	0.04066	4	0.02093					14	0.04667	6	0.03133	1	0.01789	4	0.01761
5	0.04642	5	0.02389	1	0.01277	1	0.02067	15	0.05067	7	0.03454	2	0.02591	5	0.02006
6	0.05194	6	0.02674	2	0.01853	2	0.02996	16	0.05557	8	0.03772	3	0.03236	6	0.02239
7	0.05734	7	0.02955	3	0.02319	3	0.03746	17	0.06230	9	0.04093	4	0.03805	7	0.02466
8	0.06272	8	0.03235	4	0.02734	4	0.04408	18	0.07911	10	0.04421	5	0.04331	8	0.02690
9	0.57679	9	0.03521	5	0.03120	5	0.05021			11	0.04761	6	0.04831	9	0.02916
		10	0.03818	6	0.03490	6	0.05605	$n = 18$	$r = 1$	12	0.05117	7	0.05315	10	0.03145
$n = 16$	$r = 8$	11	0.04131	7	0.03854	7	0.06170			13	0.05498	8	0.05790	11	0.03382
		12	0.04468	8	0.04216	8	0.71300	1	0.00950	14	0.31094	9	0.66088	12	0.03631
1	0.02130	13	0.04843	9	0.04584			2	0.01377					13	0.03895
2	0.03092	14	0.05274	10	0.04963	$n = 17$	$r = 10$	3	0.01722	$n = 18$	$r = 5$	$n = 18$	$r = 10$	14	0.04182
3	0.03871	15	0.05801	11	0.05359			4	0.02027					15	0.04504
4	0.04562	16	0.06523	12	0.05779	1	0.02357	5	0.02311	1	0.01241	1	0.02009	16	0.04876
5	0.05205	17	0.08316	13	0.32850	2	0.03415	6	0.02583	2	0.01799	2	0.02909	17	0.05333
6	0.05819					3	0.04267	7	0.02849	3	0.02249	3	0.03632	18	0.05964
7	0.06418	$n = 17$	$r = 1$	$n = 17$	$r = 5$	4	0.05018	8	0.03113	4	0.02648	4	0.04269	19	0.07545
8	0.66957					5	0.05712	9	0.03380	5	0.03018	5	0.04856		
		1	0.01037	1	0.01383	6	0.06369	10	0.03654	6	0.03371	6	0.05413	$n = 19$	$r = 1$
$n = 16$	$r = 9$	2	0.01506	2	0.02006	7	0.82036	11	0.03940	7	0.03715	7	0.05950		
		3	0.01886	3	0.02511			12	0.04242	8	0.04057	8	0.75430	1	0.00874
1	0.02429	4	0.02223	4	0.02960	$n = 17$	$r = 11$	13	0.04569	9	0.04400			2	0.01265
2	0.03525	5	0.02538	5	0.03377			14	0.04932	10	0.04750	$n = 18$	$r = 11$	3	0.01580
3	0.04410	6	0.02841	6	0.03777	1	0.02741	15	0.05348	11	0.05112			4	0.01859
4	0.05194	7	0.03138	7	0.04169	2	0.03969	16	0.05852	12	0.05490	1	0.02291	5	0.02117

5	0.05920
6	0.06612
7	0.77622

$n = 16$ $r = 10$

1	0.02825
2	0.04097
3	0.05121
4	0.06025
5	0.06860
6	0.90270

$n = 16$ $r = 11$

1	0.03373
2	0.04887
3	0.06102
4	0.07170
5	1.05894

$n = 16$ $r = 12$

1	0.04182
2	0.06052
3	0.07545
4	1.26329

$n = 16$ $r = 13$

1	0.05500
2	0.07942
3	1.55500

8	0.03436
9	0.03739
10	0.04053
11	0.04384
12	0.04741
13	0.05135
14	0.05586
15	0.06129
16	0.14587

$n = 17$ $r = 2$

1	0.01107
2	0.01606
3	0.02011
4	0.02371
5	0.02706
6	0.03029
7	0.03346
8	0.03662
9	0.03985
10	0.04318
11	0.04669
12	0.05046
13	0.05459
14	0.05928
15	0.20613

$n = 17$ $r = 3$

1	0.01186
2	0.01721
3	0.02154
4	0.02539
5	0.02899
6	0.03243
8	0.04559
9	0.04954
10	0.05360
11	0.05781
12	0.39340

$n = 17$ $r = 6$

1	0.01508
2	0.02188
3	0.02737
4	0.03225
5	0.03679
6	0.04114
7	0.04539
8	0.04961
9	0.05388
10	0.05823
11	0.46251

$n = 17$ $r = 7$

1	0.01657
2	0.02404
3	0.03008
4	0.03543
5	0.04040
6	0.04515
7	0.04979
8	0.05439
9	0.05901
10	0.53738

$n = 17$ $r = 8$

1	0.01840
3	0.04955
4	0.05822
5	0.06620
6	0.94816

$n = 17$ $r = 12$

1	0.03272
2	0.04734
3	0.05904
4	0.06929
5	1.10656

$n = 17$ $r = 13$

1	0.04057
2	0.05862
3	0.07300
4	1.31439

$n = 17$ $r = 14$

1	0.05334
2	0.07693
3	1.61195

$n = 17$ $r = 15$

1	0.07790
2	2.10863

$n = 18$ $r = 0$

1	0.00897
2	0.01301
3	0.01626
17	0.13873

$n = 18$ $r = 2$

1	0.01009
2	0.01463
3	0.01829
4	0.02154
5	0.02455
6	0.02744
7	0.03026
8	0.03306
9	0.03589
10	0.03879
11	0.04181
12	0.04500
13	0.04844
14	0.05224
15	0.05655
16	0.19584

$n = 18$ $r = 3$

1	0.01076
2	0.01560
3	0.01951
4	0.02297
5	0.02618
6	0.02926
7	0.03226
8	0.03524
9	0.03825
10	0.04133
11	0.04453
12	0.04790
13	0.05152
13	0.37140

$n = 18$ $r = 6$

1	0.01344
2	0.01949
3	0.02436
4	0.02867
5	0.03266
6	0.03648
7	0.04019
8	0.04387
9	0.04756
10	0.05131
11	0.05517
12	0.43525

$n = 18$ $r = 7$

1	0.01466
2	0.02124
3	0.02655
4	0.03124
5	0.03559
6	0.03973
7	0.04376
8	0.04774
9	0.05172
10	0.05576
11	0.50370

$n = 18$ $r = 8$

1	0.01611
2	0.02335
3	0.02917
2	0.03315
3	0.04137
4	0.04860
5	0.05524
6	0.06152
7	0.86250

$n = 18$ $r = 12$

1	0.02663
2	0.03852
3	0.04804
4	0.05638
5	0.06403
6	0.99171

$n = 18$ $r = 13$

1	0.03180
2	0.04595
3	0.05724
4	0.06711
5	1.15232

$n = 18$ $r = 14$

1	0.03942
2	0.05690
3	0.07078
4	1.36364

$n = 18$ $r = 15$

1	0.05183
2	0.07466
3	1.66698
6	0.02363
7	0.02602
8	0.02839
9	0.03076
10	0.03318
11	0.03568
12	0.03829
13	0.04107
14	0.04409
15	0.04744
16	0.05131
17	0.05600
18	0.13231

$n = 19$ $r = 2$

1	0.00925
2	0.01339
3	0.01673
4	0.01968
5	0.02241
6	0.02501
7	0.02754
8	0.03005
9	0.03256
10	0.03511
11	0.03775
12	0.04050
13	0.04342
14	0.04659
15	0.05009
16	0.05408
17	0.18660

i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i
$n = 19$	$r = 3$	$n = 19$	$r = 7$	4	0.04715	6	0.02172	13	0.04401	$n = 20$	$r = 9$	$n = 20$	$r = 15$	19	0.05160
				5	0.05354	7	0.02389	14	0.04697					20	0.12118
1	0.00983	1	0.01309	6	0.05956	8	0.02603	15	0.05016	1	0.01392	1	0.03016		
2	0.01423	2	0.01895	7	0.90288	9	0.02816	16	0.28137	2	0.02012	2	0.04350	$n = 21$	$r = 2$
3	0.01777	3	0.02366			10	0.03032			3	0.02510	3	0.05409		
4	0.02091	4	0.02782	$n = 19$	$r = 13$	11	0.03253	$n = 20$	$r = 5$	4	0.02947	4	0.06330	1	0.00788
5	0.02380	5	0.03166			12	0.03482			5	0.03349	5	1.23900	2	0.01139
6	0.02657	6	0.03531	1	0.02593	13	0.03723	1	0.01022	6	0.03730			3	0.01420
7	0.02925	7	0.03885	2	0.03745	14	0.03980	2	0.01479	7	0.04097	$n = 20$	$r = 16$	4	0.01668
8	0.03191	8	0.04233	3	0.04666	15	0.04259	3	0.01845	8	0.04456			5	0.01895
9	0.03457	9	0.04580	4	0.05471	16	0.04571	4	0.02168	9	0.04812	1	0.03738	6	0.02111
10	0.03727	10	0.04931	5	0.06207	17	0.04932	5	0.02465	10	0.05167	2	0.05385	7	0.02319
11	0.04006	11	0.05288	6	1.03355	18	0.05370	6	0.02748	11	0.57941	3	0.06688	8	0.02523
12	0.04297	12	0.47451			19	0.12648	7	0.03021			4	1.45726	9	0.02727
13	0.04604			$n = 19$	$r = 14$			8	0.03290	$n = 20$	$r = 10$			10	0.02931
14	0.04936	$n = 19$	$r = 8$			$n = 20$	$r = 2$	9	0.03558			$n = 20$	$r = 17$	11	0.03138
15	0.05300			1	0.03095			10	0.03828	1	0.01530			12	0.03351
16	0.24053	1	0.01428	2	0.04467	1	0.00852	11	0.04103	2	0.02211	1	0.04914	13	0.03573
		2	0.02066	3	0.05560	2	0.01233	12	0.04386	3	0.02757	2	0.07067	14	0.03807
$n = 19$	$r = 4$	3	0.02579	4	0.06512	3	0.01538	13	0.04682	4	0.03237	3	1.77197	15	0.04056
		4	0.03032	5	1.19641	4	0.01807	14	0.04993	5	0.03678			16	0.04328
1	0.01048	5	0.03449			5	0.02056	15	0.33470	6	0.04094	$n = 20$	$r = 18$	17	0.04631
2	0.01518	6	0.03846	$n = 19$	$r = 15$	6	0.02292			7	0.04495			18	0.04977
3	0.01896	7	0.04229			7	0.02521	$n = 20$	$r = 6$	8	0.04886	1	0.07175	19	0.17067
4	0.02229	8	0.04606	1	0.03836	8	0.02746			9	0.05273	2	2.30099		
5	0.02538	9	0.04982	2	0.05531	9	0.02971	1	0.01095	10	0.65388			$n = 21$	$r = 3$
6	0.02832	10	0.05359	3	0.06875	10	0.03199	2	0.01584			$n = 21$	$r = 0$		
7	0.03118	11	0.54255	4	1.41121	11	0.03431	3	0.01976	$n = 20$	$r = 11$			1	0.00832
8	0.03401					12	0.03672	4	0.02321			1	0.00713	2	0.01202
9	0.03684	$n = 19$	$r = 9$	$n = 19$	$r = 16$	13	0.03925	5	0.02640	1	0.01698	2	0.01031	3	0.01499
10	0.03971					14	0.04195	6	0.02942	2	0.02454	3	0.01285	4	0.01760
11	0.04266	1	0.01569	1	0.05043	15	0.04487	7	0.03234	3	0.03058	4	0.01509	5	0.02000
12	0.04573	2	0.02271	2	0.07259	16	0.04812	8	0.03521	4	0.03589	5	0.01715	6	0.02228
13	0.04898	3	0.02834	3	1.72027	17	0.05183	9	0.03807	5	0.04076	6	0.01911	7	0.02447

14	0.05245
15	0.29534

$n = 19$	$r = 5$
1	0.01123
2	0.01626
3	0.02030
4	0.02388
5	0.02718
6	0.03033
7	0.03338
8	0.03640
9	0.03942
10	0.04248
11	0.04561
12	0.04887
13	0.05228
14	0.35199

$n = 19$	$r = 6$
1	0.01209
2	0.01750
3	0.02186
4	0.02570
5	0.02925
6	0.03263
7	0.03591
8	0.03915
9	0.04238
10	0.04564
11	0.04899
12	0.05244
13	0.41138

4	0.03330
5	0.03787
6	0.04221
7	0.04640
8	0.05051
9	0.05459
10	0.61699

$n = 19$	$r = 10$
1	0.01741
2	0.02519
3	0.03143
4	0.03692
5	0.04198
6	0.04676
7	0.05137
8	0.05588
9	0.69980

$n = 19$	$r = 11$
1	0.01956
2	0.02829
3	0.03528
4	0.04142
5	0.04707
6	0.05240
7	0.05752
8	0.79374

$n = 19$	$r = 12$
1	0.02230
2	0.03224
3	0.04018

$n = 19$	$r = 17$
1	0.07363
2	2.23870

$n = 20$	$r = 0$
1	0.00767
2	0.01110
3	0.01385
4	0.01627
5	0.01851
6	0.02064
7	0.02270
8	0.02473
9	0.02676
10	0.02882
11	0.03092
12	0.03310
13	0.03539
14	0.03784
15	0.04052
16	0.04351
17	0.04699
18	0.05128
19	0.05721
20	0.07214

$n = 20$	$r = 1$
1	0.00807
2	0.01168
3	0.01457
4	0.01713
5	0.01948

18	0.17825

$n = 20$	$r = 3$
1	0.00902
2	0.01305
3	0.01628
4	0.01914
5	0.02177
6	0.02426
7	0.02669
8	0.02907
9	0.03145
10	0.03385
11	0.03630
12	0.03884
13	0.04150
14	0.04433
15	0.04738
16	0.05075
17	0.22950

$n = 20$	$r = 4$
1	0.00958
2	0.01386
3	0.01730
4	0.02033
5	0.02312
6	0.02577
7	0.02834
8	0.03087
9	0.03339
10	0.03593
11	0.03852
12	0.04120

10	0.04094
11	0.04387
12	0.04687
13	0.04999
14	0.39028

$n = 20$	$r = 7$
1	0.01179
2	0.01705
3	0.02127
4	0.02499
5	0.02841
6	0.03165
7	0.03479
8	0.03787
9	0.04093
10	0.04400
11	0.04712
12	0.05031
13	0.44893

$n = 20$	$r = 8$
1	0.01277
2	0.01846
3	0.02303
4	0.02704
5	0.03074
6	0.03425
7	0.03763
8	0.04095
9	0.04424
10	0.04753
11	0.05087
12	0.51159

6	0.04536
7	0.04977
8	0.05407
9	0.73699

$n = 20$	$r = 12$
1	0.01907
2	0.02755
3	0.03432
4	0.04026
5	0.04570
6	0.05083
7	0.05573
8	0.83154

$n = 20$	$r = 13$
1	0.02174
2	0.03139
3	0.03909
4	0.04583
5	0.05199
6	0.05778
7	0.94168

$n = 20$	$r = 14$
1	0.02527
2	0.03647
3	0.04539
4	0.05318
5	0.06027
6	1.07387

7	0.02099
8	0.02284
9	0.02468
10	0.02654
11	0.02842
12	0.03036
13	0.03237
14	0.03450
15	0.03679
16	0.03929
17	0.04209
18	0.04536
19	0.04939
20	0.05498
21	0.06911

$n = 21$	$r = 1$
1	0.00749
2	0.01082
3	0.01349
4	0.01584
5	0.01801
6	0.02006
7	0.02204
8	0.02398
9	0.02591
10	0.02785
11	0.02983
12	0.03186
13	0.03397
14	0.03620
15	0.03859
16	0.04120
17	0.04411
18	0.04749

8	0.02663
9	0.02877
10	0.03092
11	0.03310
12	0.03535
13	0.03768
14	0.04012
15	0.04274
16	0.04557
17	0.04869
18	0.21952

$n = 21$	$r = 4$
1	0.00881
2	0.01273
3	0.01587
4	0.01863
5	0.02118
6	0.02358
7	0.02590
8	0.02818
9	0.03044
10	0.03271
11	0.03502
12	0.03738
13	0.03983
14	0.04240
15	0.04514
16	0.04808
17	0.26877

$n = 21$	$r = 5$
1	0.00936
2	0.01352

i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i
3	0.01686	3	0.02244	$n = 21$	$r = 15$	9	0.02395	10	0.02995	2	0.01509	10	0.72296	$n = 23$	$r = 0$
4	0.01979	4	0.02633			10	0.02571	11	0.03202	3	0.01879				
5	0.02249	5	0.02990	1	0.02466	11	0.02749	12	0.03412	4	0.02204	$n = 22$	$r = 13$	1	0.00623
6	0.02505	6	0.03328	2	0.03556	12	0.02931	13	0.03629	5	0.02502			2	0.00899
7	0.02751	7	0.03652	3	0.04422	13	0.03118	14	0.03855	6	0.02782	1	0.01620	3	0.01119
8	0.02992	8	0.03969	4	0.05177	14	0.03315	15	0.04092	7	0.03052	2	0.02336	4	0.01312
9	0.03232	9	0.04283	5	0.05863	15	0.03522	16	0.04345	8	0.03315	3	0.02907	5	0.01489
10	0.03472	10	0.04595	6	1.11282	16	0.03745	17	0.04618	9	0.03575	4	0.03406	6	0.01656
11	0.03716	11	0.04909			17	0.03989	18	0.25735	10	0.03833	5	0.03862	7	0.01816
12	0.03966	12	0.54678	$n = 21$	$r = 16$	18	0.04262			11	0.04093	6	0.04289	8	0.01972
13	0.04224					19	0.04579	$n = 22$	$r = 5$	12	0.04357	7	0.04698	9	0.02126
14	0.04494	$n = 21$	$r = 10$	1	0.02943	20	0.04967			13	0.04626	8	0.05093	10	0.02280
15	0.04780			2	0.04241	21	0.11633	1	0.00861	14	0.46035	9	0.80693	11	0.02435
16	0.31920	1	0.01359	3	0.05269			2	0.01243					12	0.02592
		2	0.01963	4	0.06162	$n = 22$	$r = 2$	3	0.01549	$n = 22$	$r = 9$	$n = 22$	$r = 14$	13	0.02753
$n = 21$	$r = 6$	3	0.02446	5	1.28023			4	0.01817					14	0.02921
		4	0.02869			1	0.00732	5	0.02063	1	0.01125	1	0.01819	15	0.03096
1	0.00998	5	0.03258	$n = 21$	$r = 17$	2	0.01057	6	0.02295	2	0.01624	2	0.02623	16	0.03282
2	0.01442	6	0.03624			3	0.01317	7	0.02519	3	0.02023	3	0.03262	17	0.03483
3	0.01798	7	0.03977	1	0.03648	4	0.01545	8	0.02737	4	0.02372	4	0.03821	18	0.03703
4	0.02111	8	0.04320	2	0.05251	5	0.01755	9	0.02953	5	0.02692	5	0.04330	19	0.03951
5	0.02398	9	0.04659	3	0.06516	6	0.01952	10	0.03169	6	0.02993	6	0.04807	20	0.04242
6	0.02670	10	0.04996	4	1.50194	7	0.02143	11	0.03387	7	0.03283	7	0.05261	21	0.04602
7	0.02932	11	0.61453			8	0.02329	12	0.03609	8	0.03565	8	0.90291	22	0.05103
8	0.03189			$n = 21$	$r = 18$	9	0.02514	13	0.03837	9	0.03843			23	0.06380
9	0.03444	$n = 21$	$r = 11$			10	0.02698	14	0.04074	10	0.04120	$n = 22$	$r = 15$		
10	0.03699			1	0.04795	11	0.02885	15	0.04322	11	0.04397			$n = 23$	$r = 1$
11	0.03958	1	0.01493	2	0.06891	12	0.03076	16	0.04586	12	0.04678	1	0.02073		
12	0.04222	2	0.02157	3	1.82222	13	0.03272	17	0.30520	13	0.51813	2	0.02988	1	0.00651
13	0.04495	3	0.02687			14	0.03478					3	0.03716	2	0.00940
14	0.04779	4	0.03151	$n = 21$	$r = 19$	15	0.03695	$n = 22$	$r = 6$	$n = 22$	$r = 10$	4	0.04349	3	0.01170
15	0.37147	5	0.03577			16	0.03927					5	0.04926	4	0.01372
		6	0.03978	1	0.07000	17	0.04181	1	0.00915	1	0.01218	6	0.05465	5	0.01556
$n = 21$	$r = 7$	7	0.04363	2	2.36163	18	0.04463	2	0.01321	2	0.01758	7	1.01520	6	0.01731

1	0.01069
2	0.01545
3	0.01926
4	0.02260
5	0.02568
6	0.02859
7	0.03139
8	0.03413
9	0.03685
10	0.03957
11	0.04232
12	0.04512
13	0.04801
14	0.42629

$n = 21$ $r = 8$

1	0.01151
2	0.01663
3	0.02073
4	0.02433
5	0.02763
6	0.03076
7	0.03377
8	0.03671
9	0.03962
10	0.04253
11	0.04546
12	0.04844
13	0.48442

$n = 21$ $r = 9$

1	0.01247
2	0.01801
8	0.04737
9	0.05106
10	0.68914

$n = 21$ $r = 12$

1	0.01657
2	0.02393
3	0.02980
4	0.03494
5	0.03964
6	0.04407
7	0.04831
8	0.05243
9	0.77264

$n = 21$ $r = 13$

1	0.01861
2	0.02686
3	0.03344
4	0.03919
5	0.04445
6	0.04939
7	0.05411
8	0.86788

$n = 21$ $r = 14$

1	0.02122
2	0.03061
3	0.03809
4	0.04462
5	0.05057
6	0.05615
7	0.97907

$n = 22$ $r = 0$

1	0.00666
2	0.00961
3	0.01198
4	0.01405
5	0.01596
6	0.01776
7	0.01949
8	0.02119
9	0.02287
10	0.02455
11	0.02625
12	0.02799
13	0.02978
14	0.03166
15	0.03365
16	0.03578
17	0.03812
18	0.04076
19	0.04383
20	0.04764
21	0.05292
22	0.06634

$n = 22$ $r = 1$

1	0.00697
2	0.01007
3	0.01254
4	0.01472
5	0.01671
6	0.01860
7	0.02041
8	0.02219
19	0.04788
20	0.16375

$n = 22$ $r = 3$

1	0.00771
2	0.01113
3	0.01386
4	0.01626
5	0.01847
6	0.02055
7	0.02255
8	0.02451
9	0.02645
10	0.02839
11	0.03035
12	0.03235
13	0.03442
14	0.03657
15	0.03884
16	0.04126
17	0.04389
18	0.04681
19	0.21043

$n = 22$ $r = 4$

1	0.00813
2	0.01174
3	0.01463
4	0.01716
5	0.01949
6	0.02169
7	0.02380
8	0.02586
9	0.02791
3	0.01645
4	0.01930
5	0.02191
6	0.02438
7	0.02675
8	0.02907
9	0.03135
10	0.03364
11	0.03594
12	0.03829
13	0.04069
14	0.04318
15	0.04579
16	0.35458

$n = 22$ $r = 7$

1	0.00976
2	0.01409
3	0.01755
4	0.02058
5	0.02336
6	0.02599
7	0.02851
8	0.03098
9	0.03341
10	0.03584
11	0.03828
12	0.04076
13	0.04330
14	0.04593
15	0.40608

$n = 22$ $r = 8$

1	0.01045
3	0.02190
4	0.02567
5	0.02913
6	0.03239
7	0.03551
8	0.03855
9	0.04154
10	0.04452
11	0.04749
12	0.58034

$n = 22$ $r = 11$

1	0.01328
2	0.01917
3	0.02386
4	0.02797
5	0.03173
6	0.03527
7	0.03866
8	0.04196
9	0.04520
10	0.04840
11	0.64812

$n = 22$ $r = 12$

1	0.01460
2	0.02106
3	0.02621
4	0.03072
5	0.03484
6	0.03871
7	0.04242
8	0.04602
9	0.04954

$n = 22$ $r = 16$

1	0.02410
2	0.03471
3	0.04314
4	0.05046
5	0.05711
6	1.15052

$n = 22$ $r = 17$

1	0.02876
2	0.04140
3	0.05140
4	0.06008
5	1.32022

$n = 22$ $r = 18$

1	0.03564
2	0.05125
3	0.06357
4	1.54534

$n = 22$ $r = 19$

1	0.04684
2	0.06726
3	1.87113

$n = 22$ $r = 20$

1	0.06837
2	2.42076
7	0.01898
8	0.02061
9	0.02222
10	0.02383
11	0.02544
12	0.02709
13	0.02877
14	0.03052
15	0.03235
16	0.03429
17	0.03637
18	0.03866
19	0.04124
20	0.04422
21	0.04788
22	0.11187

$n = 23$ $r = 2$

1	0.00682
2	0.00984
3	0.01226
4	0.01437
5	0.01630
6	0.01813
7	0.01988
8	0.02159
9	0.02327
10	0.02495
11	0.02665
12	0.02836
13	0.03013
14	0.03195
15	0.03386
16	0.03588
17	0.03806

i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i
18	0.04043	$n = 23$	$r = 6$	12	0.04217	7	0.04575	9	0.01984	$n = 24$	$r = 3$	$n = 24$	$r = 6$	9	0.03160
19	0.04309			13	0.04471	8	0.04955	10	0.02125					10	0.03382
20	0.04614	1	0.00843	14	0.49274	9	0.83999	11	0.02267	1	0.00668	1	0.00780	11	0.03604
21	0.15740	2	0.01216					12	0.02410	2	0.00964	2	0.01124	12	0.03827
		3	0.01513	$n = 23$	$r = 10$	$n = 23$	$r = 15$	13	0.02556	3	0.01199	3	0.01398	13	0.04053
$n = 23$	$r = 3$	4	0.01774					14	0.02707	4	0.01404	4	0.01638	14	0.04283
		5	0.02013	1	0.01101	1	0.01779	15	0.02863	5	0.01593	5	0.01857	15	0.47003
1	0.00716	6	0.02237	2	0.01588	2	0.02563	16	0.03028	6	0.01769	6	0.02063		
2	0.01034	7	0.02453	3	0.01976	3	0.03186	17	0.03203	7	0.01939	7	0.02260	$n = 24$	$r = 10$
3	0.01287	8	0.02663	4	0.02315	4	0.03729	18	0.03392	8	0.02104	8	0.02452		
4	0.01509	9	0.02870	5	0.02626	5	0.04223	19	0.03599	9	0.02266	9	0.02640	1	0.01002
5	0.01712	10	0.03076	6	0.02917	6	0.04685	20	0.03834	10	0.02427	10	0.02827	2	0.01444
6	0.01903	11	0.03283	7	0.03197	7	0.05124	21	0.04109	11	0.02588	11	0.03014	3	0.01795
7	0.02087	12	0.03493	8	0.03468	8	0.93675	22	0.04451	12	0.02751	12	0.03203	4	0.02103
8	0.02266	13	0.03707	9	0.03735			23	0.04927	13	0.02918	13	0.03395	5	0.02383
9	0.02443	14	0.03927	10	0.03999	$n = 23$	$r = 16$	24	0.06145	14	0.03089	14	0.03592	6	0.02646
10	0.02619	15	0.04156	11	0.04263					15	0.03266	15	0.03795	7	0.02898
11	0.02797	16	0.04396	12	0.04529	1	0.02028	$n = 24$	$r = 1$	16	0.03452	16	0.04007	8	0.03142
12	0.02977	17	0.33931	13	0.55030	2	0.02921			17	0.03649	17	0.04229	9	0.03382
13	0.03161					3	0.03629	1	0.00610	18	0.03861	18	0.32544	10	0.03618
14	0.03352	$n = 23$	$r = 7$	$n = 23$	$r = 11$	4	0.04245	2	0.00880	19	0.04092			11	0.03854
15	0.03552					5	0.04805	3	0.01095	20	0.04348	$n = 24$	$r = 7$	12	0.04091
16	0.03763	1	0.00895	1	0.01192	6	0.05327	4	0.01282	21	0.19451			13	0.04331
17	0.03989	2	0.01291	2	0.01719	7	1.05017	5	0.01454			1	0.00825	14	0.52366
18	0.04235	3	0.01608	3	0.02139			6	0.01616	$n = 24$	$r = 4$	2	0.01190		
19	0.04508	4	0.01884	4	0.02506	$n = 23$	$r = 17$	7	0.01771			3	0.01480	$n = 24$	$r = 11$
20	0.20213	5	0.02138	5	0.02841			8	0.01921	1	0.00702	4	0.01734		
		6	0.02376	6	0.03156	1	0.02357	9	0.02070	2	0.01012	5	0.01966	1	0.01078
$n = 23$	$r = 4$	7	0.02605	7	0.03458	2	0.03393	10	0.02217	3	0.01259	6	0.02184	2	0.01554
		8	0.02828	8	0.03751	3	0.04213	11	0.02365	4	0.01475	7	0.02392	3	0.01932
1	0.00754	9	0.03047	9	0.04038	4	0.04925	12	0.02514	5	0.01672	8	0.02595	4	0.02263
2	0.01088	10	0.03265	10	0.04322	5	0.05571	13	0.02666	6	0.01858	9	0.02794	5	0.02564
3	0.01354	11	0.03484	11	0.04605	6	1.18709	14	0.02823	7	0.02036	10	0.02991	6	0.02847
4	0.01588	12	0.03705	12	0.61246			15	0.02986	8	0.02208	11	0.03188	7	0.03117

5	0.01802
6	0.02003
7	0.02196
8	0.02385
9	0.02571
10	0.02756
11	0.02942
12	0.03131
13	0.03325
14	0.03525
15	0.03734
16	0.03954
17	0.04189
18	0.04444
19	0.24695

$n = 23$ $r = 5$

1	0.00796
2	0.01148
3	0.01430
4	0.01676
5	0.01901
6	0.02114
7	0.02318
8	0.02516
9	0.02712
10	0.02907
11	0.03103
12	0.03302
13	0.03506
14	0.03715
15	0.03934
16	0.04164
17	0.04409
18	0.29250

13	0.03931
14	0.04163
15	0.04404
16	0.38791

$n = 23$ $r = 8$

1	0.00955
2	0.01377
3	0.01714
4	0.02009
5	0.02279
6	0.02533
7	0.02777
8	0.03014
9	0.03247
10	0.03478
11	0.03711
12	0.03945
13	0.04184
14	0.04429
15	0.43885

$n = 23$ $r = 9$

1	0.01023
2	0.01475
3	0.01836
4	0.02151
5	0.02440
6	0.02712
7	0.02972
8	0.03225
9	0.03474
10	0.03721
11	0.03968

$n = 23$ $r = 12$

1	0.01299
2	0.01874
3	0.02331
4	0.02730
5	0.03095
6	0.03438
7	0.03765
8	0.04082
9	0.04393
10	0.04700
11	0.68035

$n = 23$ $r = 13$

1	0.01428
2	0.02058
3	0.02560
4	0.02998
5	0.03398
6	0.03773
7	0.04131
8	0.04477
9	0.04815
10	0.75550

$n = 23$ $r = 14$

1	0.01584
2	0.02283
3	0.02839
4	0.03324
5	0.03766
6	0.04180

$n = 23$ $r = 18$

1	0.02812
2	0.04046
3	0.05020
4	0.05864
5	1.35906

$n = 23$ $r = 19$

1	0.03485
2	0.05009
3	0.06209
4	1.58759

$n = 23$ $r = 20$

1	0.04580
2	0.06573
3	1.91881

$n = 23$ $r = 21$

1	0.06686
2	2.47849

$n = 24$ $r = 0$

1	0.00585
2	0.00843
3	0.01049
4	0.01229
5	0.01394
6	0.01549
7	0.01697
8	0.01842

16	0.03157
17	0.03339
18	0.03536
19	0.03751
20	0.03994
21	0.04277
22	0.04623
23	0.10777

$n = 24$ $r = 2$

1	0.00638
2	0.00920
3	0.01144
4	0.01341
5	0.01520
6	0.01689
7	0.01851
8	0.02008
9	0.02163
10	0.02317
11	0.02471
12	0.02627
13	0.02786
14	0.02950
15	0.03120
16	0.03298
17	0.03488
18	0.03692
19	0.03915
20	0.04165
21	0.04453
22	0.15157

9	0.02378
10	0.02547
11	0.02716
12	0.02887
13	0.03061
14	0.03240
15	0.03426
16	0.03620
17	0.03825
18	0.04045
19	0.04283
20	0.23742

$n = 24$ $r = 5$

1	0.00739
2	0.01065
3	0.01325
4	0.01552
5	0.01760
6	0.01955
7	0.02142
8	0.02324
9	0.02503
10	0.02680
11	0.02858
12	0.03037
13	0.03220
14	0.03407
15	0.03601
16	0.03804
17	0.04018
18	0.04245
19	0.28091

12	0.03387
13	0.03590
14	0.03796
15	0.04010
16	0.04231
17	0.37148

$n = 24$ $r = 8$

1	0.00877
2	0.01264
3	0.01572
4	0.01842
5	0.02088
6	0.02319
7	0.02540
8	0.02755
9	0.02966
10	0.03175
11	0.03384
12	0.03594
13	0.03807
14	0.04025
15	0.04249
16	0.41951

$n = 24$ $r = 9$

1	0.00935
2	0.01348
3	0.01676
4	0.01963
5	0.02226
6	0.02472
7	0.02708
8	0.02936

8	0.03379
9	0.03636
10	0.03889
11	0.04141
12	0.04394
13	0.58111

$n = 24$ $r = 12$

1	0.01167
2	0.01682
3	0.02091
4	0.02449
5	0.02775
6	0.03080
7	0.03372
8	0.03654
9	0.03930
10	0.04203
11	0.04473
12	0.64329

$n = 24$ $r = 13$

1	0.01272
2	0.01833
3	0.02279
4	0.02668
5	0.03022
6	0.03355
7	0.03671
8	0.03977
9	0.04276
10	0.04571
11	0.71137

i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i
$n = 24$	$r = 14$	$n = 24$	$r = 21$	3	0.01072	$n = 25$	$r = 5$	3	0.01449	$n = 25$	$r = 12$	$n = 25$	$r = 17$	13	0.01751
				4	0.01255			4	0.01696					14	0.01843
1	0.01398	1	0.04483	5	0.01422	1	0.00688	5	0.01922	1	0.01057	1	0.01707	15	0.01935
2	0.02014	2	0.06430	6	0.01579	2	0.00991	6	0.02134	2	0.01522	2	0.02456	16	0.02029
3	0.02503	3	1.96534	7	0.01729	3	0.01232	7	0.02336	3	0.01891	3	0.03049	17	0.02124
4	0.02930			8	0.01875	4	0.01443	8	0.02532	4	0.02213	4	0.03564	18	0.02223
5	0.03318	$n = 24$	$r = 22$	9	0.02018	5	0.01635	9	0.02723	5	0.02507	5	0.04031	19	0.02325
6	0.03682			10	0.02159	6	0.01815	10	0.02913	6	0.02782	6	0.04467	20	0.02432
7	0.04028	1	0.06544	11	0.02301	7	0.01988	11	0.03102	7	0.03043	7	0.04879	21	0.02544
8	0.04362	2	2.53490	12	0.02443	8	0.02155	12	0.03292	8	0.03297	8	1.00130	22	0.02663
9	0.04688			13	0.02588	9	0.02319	13	0.03484	9	0.03544			23	0.02791
10	0.78688	$n = 25$	$r = 0$	14	0.02736	10	0.02481	14	0.03679	10	0.03787	$n = 25$	$r = 18$	24	0.02930
				15	0.02888	11	0.02643	15	0.03879	11	0.04029			25	0.03084
$n = 24$	$r = 15$	1	0.00550	16	0.03047	12	0.02806	16	0.04086	12	0.04271	1	0.01945	26	0.03260
		2	0.00793	17	0.03215	13	0.02971	17	0.40200	13	0.61070	2	0.02798	27	0.03468
1	0.01551	3	0.00986	18	0.03393	14	0.03140					3	0.03472	28	0.03728
2	0.02234	4	0.01155	19	0.03584	15	0.03313	$n = 25$	$r = 9$	$n = 25$	$r = 13$	4	0.04057	29	0.04093
3	0.02776	5	0.01309	20	0.03795	16	0.03494					5	0.04587	30	0.05044
4	0.03248	6	0.01453	21	0.04031	17	0.03682	1	0.00859	1	0.01144	6	0.05079		
5	0.03678	7	0.01591	22	0.04304	18	0.03882	2	0.01238	2	0.01647	7	1.11703	$n = 30$	$r = 1$
6	0.04079	8	0.01725	23	0.14618	19	0.04095	3	0.01539	3	0.02047				
7	0.04461	9	0.01857			20	0.27028	4	0.01802	4	0.02395	$n = 25$	$r = 19$	1	0.00434
8	0.04829	10	0.01987	$n = 25$	$r = 3$			5	0.02041	5	0.02713			2	0.00624
9	0.87194	11	0.02118			$n = 25$	$r = 6$	6	0.02266	6	0.03009	1	0.02261	3	0.00774
		12	0.02249	1	0.00625			7	0.02480	7	0.03292	2	0.03250	4	0.00904
$n = 24$	$r = 16$	13	0.02382	2	0.00901	1	0.00724	8	0.02688	8	0.03565	3	0.04031	5	0.01022
		14	0.02519	3	0.01120	2	0.01043	9	0.02891	9	0.03831	4	0.04708	6	0.01132
1	0.01742	15	0.02660	4	0.01312	3	0.01297	10	0.03092	10	0.04093	5	0.05319	7	0.01236
2	0.02508	16	0.02807	5	0.01487	4	0.01518	11	0.03292	11	0.04353	6	1.25717	8	0.01337
3	0.03115	17	0.02961	6	0.01651	5	0.01721	12	0.03493	12	0.67298			9	0.01434
4	0.03644	18	0.03126	7	0.01808	6	0.01910	13	0.03695			$n = 25$	$r = 20$	10	0.01530
5	0.04124	19	0.03305	8	0.01960	7	0.02092	14	0.03901	$n = 25$	$r = 14$			11	0.01624
6	0.04572	20	0.03502	9	0.02109	8	0.02267	15	0.04112			1	0.02697	12	0.01718
7	0.04997	21	0.03724	10	0.02257	9	0.02440	16	0.44959	1	0.01247	2	0.03875	13	0.01812

8	0.96952

$n = 24$ $r = 17$

1	0.01985
2	0.02857
3	0.03548
4	0.04148
5	0.04692
6	0.05199
7	1.08408

$n = 24$ $r = 18$

1	0.02307
2	0.03319
3	0.04119
4	0.04813
5	0.05441
6	1.22261

$n = 24$ $r = 19$

1	0.02753
2	0.03958
3	0.04908
4	0.05730
5	1.39686

$n = 24$ $r = 20$

1	0.03411
2	0.04900
3	0.06071
4	1.62875

22	0.03985
23	0.04310
24	0.04763
25	0.05927

$n = 25$ $r = 1$

1	0.00573
2	0.00826
3	0.01027
4	0.01203
5	0.01363
6	0.01513
7	0.01657
8	0.01797
9	0.01934
10	0.02070
11	0.02205
12	0.02342
13	0.02481
14	0.02623
15	0.02769
16	0.02922
17	0.03083
18	0.03254
19	0.03439
20	0.03643
21	0.03873
22	0.04141
23	0.04469
24	0.10397

$n = 25$ $r = 2$

1	0.00598
2	0.00862

11	0.02404
12	0.02553
13	0.02704
14	0.02859
15	0.03018
16	0.03183
17	0.03357
18	0.03542
19	0.03741
20	0.03958
21	0.04201
22	0.18748

$n = 25$ $r = 4$

1	0.00655
2	0.00944
3	0.01174
4	0.01374
5	0.01557
6	0.01729
7	0.01893
8	0.02053
9	0.02209
10	0.02364
11	0.02518
12	0.02674
13	0.02831
14	0.02993
15	0.03159
16	0.03332
17	0.03513
18	0.03705
19	0.03911
20	0.04135
21	0.22867

10	0.02610
11	0.02780
12	0.02951
13	0.03125
14	0.03301
15	0.03483
16	0.03672
17	0.03869
18	0.04076
19	0.31277

$n = 25$ $r = 7$

1	0.00764
2	0.01101
3	0.01369
4	0.01602
5	0.01816
6	0.02016
7	0.02207
8	0.02392
9	0.02574
10	0.02753
11	0.02932
12	0.03112
13	0.03295
14	0.03480
15	0.03671
16	0.03868
17	0.04073
18	0.35654

$n = 25$ $r = 8$

1	0.00809
2	0.01166

$n = 25$ $r = 10$

1	0.00917
2	0.01320
3	0.01641
4	0.01921
5	0.02176
6	0.02415
7	0.02644
8	0.02864
9	0.03081
10	0.03294
11	0.03506
12	0.03719
13	0.03933
14	0.04151
15	0.49982

$n = 25$ $r = 11$

1	0.00982
2	0.01414
3	0.01757
4	0.02057
5	0.02330
6	0.02586
7	0.02830
8	0.03066
9	0.03296
10	0.03524
11	0.03750
12	0.03976
13	0.04204
14	0.55328

2	0.01795
3	0.02231
4	0.02610
5	0.02955
6	0.03277
7	0.03584
8	0.03880
9	0.04169
10	0.04452
11	0.74130

$n = 25$ $r = 15$

1	0.01370
2	0.01972
3	0.02450
4	0.02866
5	0.03244
6	0.03597
7	0.03933
8	0.04256
9	0.04570
10	0.81722

$n = 25$ $r = 16$

1	0.01520
2	0.02188
3	0.02717
4	0.03177
5	0.03595
6	0.03985
7	0.04356
8	0.04711
9	0.90287

3	0.04804
4	0.05605
5	1.43370

$n = 25$ $r = 21$

1	0.03342
2	0.04798
3	0.05941
4	1.66892

$n = 25$ $r = 22$

1	0.04392
2	0.06296
3	2.01080

$n = 25$ $r = 23$

1	0.06410
2	2.59008

$n = 30$ $r = 0$

1	0.00420
2	0.00603
3	0.00748
4	0.00874
5	0.00988
6	0.01094
7	0.01195
8	0.01292
9	0.01387
10	0.01479
11	0.01570
12	0.01661

14	0.01906
15	0.02001
16	0.02098
17	0.02197
18	0.02299
19	0.02405
20	0.02515
21	0.02630
22	0.02753
23	0.02885
24	0.03028
25	0.03187
26	0.03367
27	0.03579
28	0.03840
29	0.08856

$n = 30$ $r = 2$

1	0.00450
2	0.00646
3	0.00801
4	0.00936
5	0.01058
6	0.01172
7	0.01280
8	0.01384
9	0.01485
10	0.01584
11	0.01682
12	0.01779
13	0.01876
14	0.01973
15	0.02072
16	0.02172
17	0.02275

i	b_i	*i*	b_i	*i*	b_i	*i*	b_i	*i*	b_i	*i*	b_i	*i*	b_i	*i*	b_i
18	0.02380	$n = 30$	$r = 5$	14	0.02399	9	0.02076	13	0.03070	9	0.03172	$n = 30$	$r = 23$	24	0.02144
19	0.02489			15	0.02518	10	0.02213	14	0.03227	10	0.03380			25	0.02229
20	0.02603	1	0.00504	16	0.02639	11	0.02349	15	0.03384	11	0.03583	1	0.01776	26	0.02319
21	0.02723	2	0.00723	17	0.02762	12	0.02484	16	0.03543	12	0.03784	2	0.02548	27	0.02415
22	0.02849	3	0.00897	18	0.02889	13	0.02618	17	0.53602	13	0.74416	3	0.03155	28	0.02518
23	0.02985	4	0.01048	19	0.03019	14	0.02753					4	0.03679	29	0.02632
24	0.03133	5	0.01185	20	0.03155	15	0.02888	$n = 30$	$r = 14$	$n = 30$	$r = 18$	5	0.04151	30	0.02759
25	0.03296	6	0.01313	21	0.03296	16	0.03026					6	0.04587	31	0.02903
26	0.03480	7	0.01434	22	0.03444	17	0.03166	1	0.00786	1	0.01046	7	1.26963	32	0.03075
27	0.03694	8	0.01550	23	0.29813	18	0.03309	2	0.01129	2	0.01502			33	0.03291
28	0.12439	9	0.01663			19	0.03456	3	0.01400	3	0.01861	$n = 30$	$r = 24$	34	0.03595
		10	0.01774	$n = 30$	$r = 8$	20	0.40994	4	0.01635	4	0.02173			35	0.04398
$n = 30$	$r = 3$	11	0.01883					5	0.01848	5	0.02455	1	0.02064		
		12	0.01991	1	0.00572	$n = 30$	$r = 11$	6	0.02046	6	0.02716	2	0.02960	$n = 35$	$r = 1$
1	0.00466	13	0.02100	2	0.00822			7	0.02233	7	0.02964	3	0.03663		
2	0.00670	14	0.02209	3	0.01020	1	0.00662	8	0.02413	8	0.03201	4	0.04269	1	0.00343
3	0.00831	15	0.02319	4	0.01191	2	0.00952	9	0.02588	9	0.03430	5	0.04814	2	0.00493
4	0.00971	16	0.02431	5	0.01346	3	0.01180	10	0.02758	10	0.03653	6	1.41782	3	0.00610
5	0.01097	17	0.02545	6	0.01491	4	0.01378	11	0.02926	11	0.03872			4	0.00711
6	0.01216	18	0.02662	7	0.01628	5	0.01558	12	0.03092	12	0.80758	$n = 30$	$r = 25$	5	0.00803
7	0.01328	19	0.02783	8	0.01760	6	0.01725	13	0.03257					6	0.00888
8	0.01435	20	0.02909	9	0.01888	7	0.01884	14	0.03423	$n = 30$	$r = 19$	1	0.02462	7	0.00968
9	0.01540	21	0.03041	10	0.02014	8	0.02036	15	0.03589			2	0.03529	8	0.01044
10	0.01643	22	0.03181	11	0.02138	9	0.02184	16	0.58276	1	0.01140	3	0.04365	9	0.01118
11	0.01744	23	0.03329	12	0.02260	10	0.02329			2	0.01636	4	0.05084	10	0.01190
12	0.01844	24	0.03489	13	0.02383	11	0.02471	$n = 30$	$r = 15$	3	0.02028	5	1.60550	11	0.01261
13	0.01945	25	0.22806	14	0.02506	12	0.02612			4	0.02367			12	0.01331
14	0.02046			15	0.02631	13	0.02753	1	0.00838	5	0.02674	$n = 30$	$r = 26$	13	0.01400
15	0.02148	$n = 30$	$r = 6$	16	0.02757	14	0.02895	2	0.01204	6	0.02958			14	0.01468
16	0.02252			17	0.02885	15	0.03037	3	0.01493	7	0.03227	1	0.03050	15	0.01537
17	0.02358	1	0.00524	18	0.03017	16	0.03181	4	0.01743	8	0.03484	2	0.04368	16	0.01606
18	0.02468	2	0.00754	19	0.03152	17	0.03327	5	0.01970	9	0.03733	3	0.05399	17	0.01675
19	0.02580	3	0.00935	20	0.03293	18	0.03477	6	0.02181	10	0.03974	4	1.85693	18	0.01746
20	0.02698	4	0.01092	21	0.03439	19	0.44998	7	0.02380	11	0.87764			19	0.01817

21	0.02822
22	0.02952
23	0.03092
24	0.03244
25	0.03411
26	0.03598
27	0.15920

$n = 30$ $r = 4$

1	0.00484
2	0.00696
3	0.00863
4	0.01008
5	0.01140
6	0.01262
7	0.01379
8	0.01490
9	0.01599
10	0.01706
11	0.01811
12	0.01915
13	0.02019
14	0.02124
15	0.02230
16	0.02338
17	0.02448
18	0.02561
19	0.02678
20	0.02800
21	0.02927
22	0.03062
23	0.03207
24	0.03362
25	0.03533
26	0.19362

5	0.01234
6	0.01367
7	0.01493
8	0.01614
9	0.01732
10	0.01847
11	0.01961
12	0.02074
13	0.02186
14	0.02300
15	0.02414
16	0.02530
17	0.02649
18	0.02771
19	0.02897
20	0.03027
21	0.03164
22	0.03308
23	0.03460
24	0.26281

$n = 30$ $r = 7$

1	0.00547
2	0.00786
3	0.00975
4	0.01139
5	0.01288
6	0.01427
7	0.01558
8	0.01684
9	0.01807
10	0.01927
11	0.02045
12	0.02163
13	0.02281

22	0.33427

$n = 30$ $r = 9$

1	0.00599
2	0.00861
3	0.01068
4	0.01248
5	0.01410
6	0.01562
7	0.01706
8	0.01844
9	0.01978
10	0.02109
11	0.02238
12	0.02367
13	0.02495
14	0.02624
15	0.02754
16	0.02885
17	0.03019
18	0.03156
19	0.03297
20	0.03443
21	0.37145

$n = 30$ $r = 10$

1	0.00629
2	0.00904
3	0.01121
4	0.01310
5	0.01480
6	0.01640
7	0.01790
8	0.01935

$n = 30$ $r = 12$

1	0.00699
2	0.01004
3	0.01245
4	0.01454
5	0.01644
6	0.01820
7	0.01988
8	0.02148
9	0.02304
10	0.02456
11	0.02606
12	0.02755
13	0.02903
14	0.03052
15	0.03201
16	0.03352
17	0.03506
18	0.49190

$n = 30$ $r = 13$

1	0.00740
2	0.01063
3	0.01318
4	0.01539
5	0.01740
6	0.01927
7	0.02103
8	0.02273
9	0.02438
10	0.02599
11	0.02757
12	0.02914

8	0.02572
9	0.02757
10	0.02939
11	0.03117
12	0.03293
13	0.03468
14	0.03643
15	0.63261

$n = 30$ $r = 16$

1	0.00897
2	0.01289
3	0.01598
4	0.01866
5	0.02109
6	0.02334
7	0.02548
8	0.02752
9	0.02950
10	0.03144
11	0.03334
12	0.03522
13	0.03708
14	0.68616

$n = 30$ $r = 17$

1	0.00966
2	0.01387
3	0.01720
4	0.02008
5	0.02269
6	0.02511
7	0.02740
8	0.02960

$n = 30$ $r = 20$

1	0.01252
2	0.01797
3	0.02227
4	0.02599
5	0.02935
6	0.03247
7	0.03541
8	0.03822
9	0.04093
10	0.95603

$n = 30$ $r = 21$

1	0.01389
2	0.01993
3	0.02470
4	0.02881
5	0.03253
6	0.03598
7	0.03922
8	0.04232
9	1.04503

$n = 30$ $r = 22$

1	0.01559
2	0.02237
3	0.02771
4	0.03232
5	0.03648
6	0.04033
7	0.04395
8	1.14793

$n = 30$ $r = 27$

1	0.04007
2	0.05733
3	2.22428

$n = 30$ $r = 28$

1	0.05848
2	2.85005

$n = 35$ $r = 0$

1	0.00334
2	0.00479
3	0.00593
4	0.00691
5	0.00780
6	0.00863
7	0.00940
8	0.01015
9	0.01087
10	0.01157
11	0.01225
12	0.01293
13	0.01360
14	0.01427
15	0.01493
16	0.01560
17	0.01628
18	0.01696
19	0.01766
20	0.01837
21	0.01910
22	0.01985
23	0.02062
20	0.01890
21	0.01965
22	0.02042
23	0.02122
24	0.02206
25	0.02293
26	0.02386
27	0.02485
28	0.02591
29	0.02707
30	0.02837
31	0.02984
32	0.03158
33	0.03375
34	0.07730

$n = 35$ $r = 2$

1	0.00354
2	0.00507
3	0.00628
4	0.00733
5	0.00827
6	0.00915
7	0.00997
8	0.01076
9	0.01152
10	0.01226
11	0.01299
12	0.01371
13	0.01442
14	0.01513
15	0.01583
16	0.01654
17	0.01726
18	0.01798

i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i
19	0.01872	18	0.01914	21	0.02300	$n = 35$	$r = 9$	12	0.01882	4	0.01151	5	0.01513	15	0.74620
20	0.01947	19	0.01992	22	0.02390			13	0.01979	5	0.01298	6	0.01673		
21	0.02024	20	0.02072	23	0.02483	1	0.00449	14	0.02075	6	0.01435	7	0.01823	$n = 35$	$r = 21$
22	0.02104	21	0.02154	24	0.02579	2	0.00644	15	0.02172	7	0.01565	8	0.01967		
23	0.02186	22	0.02238	25	0.02680	3	0.00797	16	0.02268	8	0.01688	9	0.02105	1	0.00832
24	0.02272	23	0.02325	26	0.02786	4	0.00930	17	0.02366	9	0.01807	10	0.02239	2	0.01192
25	0.02362	24	0.02416	27	0.02898	5	0.01050	18	0.02464	10	0.01923	11	0.02371	3	0.01476
26	0.02457	25	0.02511	28	0.03018	6	0.01160	19	0.02563	11	0.02036	12	0.02500	4	0.01720
27	0.02558	26	0.02612	29	0.22760	7	0.01265	20	0.02665	12	0.02148	13	0.02628	5	0.01941
28	0.02667	27	0.02718			8	0.01365	21	0.02768	13	0.02258	14	0.02755	6	0.02144
29	0.02786	28	0.02833	$n = 35$	$r = 7$	9	0.01462	22	0.02874	14	0.02368	15	0.02881	7	0.02336
30	0.02919	29	0.02957			10	0.01556	23	0.02983	15	0.02477	16	0.03007	8	0.02519
31	0.03069	30	0.03094	1	0.00417	11	0.01648	24	0.38230	16	0.02587	17	0.03134	9	0.02696
32	0.03244	31	0.16847	2	0.00598	12	0.01738			17	0.02697	18	0.60476	10	0.02866
33	0.10854			3	0.00740	13	0.01828	$n = 35$	$r = 12$	18	0.02808			11	0.03033
		$n = 35$	$r = 5$	4	0.00864	14	0.01917			19	0.02920	$n = 35$	$r = 18$	12	0.03197
$n = 35$	$r = 3$			5	0.00975	15	0.02007	1	0.00508	20	0.03034			13	0.03358
		1	0.00389	6	0.01078	16	0.02096	2	0.00728	21	0.48620	1	0.00686	14	0.80064
1	0.00365	2	0.00558	7	0.01175	17	0.02186	3	0.00901			2	0.00983		
2	0.00523	3	0.00691	8	0.01268	18	0.02277	4	0.01051	$n = 35$	$r = 15$	3	0.01217	$n = 35$	$r = 22$
3	0.00648	4	0.00806	9	0.01358	19	0.02370	5	0.01186			4	0.01419		
4	0.00756	5	0.00910	10	0.01445	20	0.02464	6	0.01311	1	0.00583	5	0.01602	1	0.00895
5	0.00853	6	0.01006	11	0.01530	21	0.02560	7	0.01429	2	0.00837	6	0.01770	2	0.01283
6	0.00943	7	0.01097	12	0.01615	22	0.02659	8	0.01542	3	0.01036	7	0.01929	3	0.01588
7	0.01028	8	0.01183	13	0.01698	23	0.02761	9	0.01651	4	0.01208	8	0.02081	4	0.01851
8	0.01110	9	0.01267	14	0.01781	24	0.02867	10	0.01757	5	0.01363	9	0.02227	5	0.02088
9	0.01188	10	0.01349	15	0.01864	25	0.02977	11	0.01861	6	0.01507	10	0.02369	6	0.02307
10	0.01265	11	0.01429	16	0.01948	26	0.31845	12	0.01963	7	0.01642	11	0.02508	7	0.02513
11	0.01340	12	0.01508	17	0.02032			13	0.02064	8	0.01772	12	0.02645	8	0.02709
12	0.01414	13	0.01586	18	0.02117	$n = 35$	$r = 10$	14	0.02165	9	0.01897	13	0.02779	9	0.02898
13	0.01487	14	0.01663	19	0.02203			15	0.02265	10	0.02018	14	0.02913	10	0.03082
14	0.01560	15	0.01741	20	0.02291	1	0.00467	16	0.02366	11	0.02137	15	0.03046	11	0.03261
15	0.01633	16	0.01819	21	0.02381	2	0.00670	17	0.02467	12	0.02254	16	0.03179	12	0.03436
16	0.01706	17	0.01897	22	0.02474	3	0.00829	18	0.02569	13	0.02369	17	0.64882	13	0.85991

17	0.01779	18	0.01977	23	0.02569	4	0.00967	19	0.02672	14	0.02484				
18	0.01854	19	0.02058	24	0.02669	5	0.01091	20	0.02778	15	0.02599	$n = 35$	$r = 19$	$n = 35$	$r = 23$
19	0.01930	20	0.02140	25	0.02772	6	0.01207	21	0.02885	16	0.02713				
20	0.02008	21	0.02225	26	0.02881	7	0.01315	22	0.02995	17	0.02828	1	0.00728	1	0.00969
21	0.02087	22	0.02311	27	0.02996	8	0.01419	23	0.41567	18	0.02944	2	0.01044	2	0.01389
22	0.02169	23	0.02401	28	0.25742	9	0.01520			19	0.03061	3	0.01293	3	0.01718
23	0.02254	24	0.02495			10	0.01617	$n = 35$	$r = 13$	20	0.52376	4	0.01507	4	0.02003
24	0.02342	25	0.02593	$n = 35$	$r = 8$	11	0.01713					5	0.01701	5	0.02259
25	0.02434	26	0.02696			12	0.01807	1	0.00531	$n = 35$	$r = 16$	6	0.01880	6	0.02495
26	0.02532	27	0.02805	1	0.00432	13	0.01901	2	0.00761			7	0.02048	7	0.02718
27	0.02636	28	0.02923	2	0.00620	14	0.01993	3	0.00942	1	0.00614	8	0.02209	8	0.02930
28	0.02748	29	0.03050	3	0.00768	15	0.02086	4	0.01098	2	0.00880	9	0.02364	9	0.03134
29	0.02870	30	0.19801	4	0.00896	16	0.02179	5	0.01240	3	0.01090	10	0.02515	10	0.03332
30	0.03004			5	0.01011	17	0.02272	6	0.01371	4	0.01271	11	0.02662	11	0.03524
31	0.03156	$n = 35$	$r = 6$	6	0.01118	18	0.02367	7	0.01494	5	0.01434	12	0.02806	12	0.92503
32	0.13876			7	0.01218	19	0.02463	8	0.01612	6	0.01585	13	0.02949		
		1	0.00403	8	0.01315	20	0.02561	9	0.01726	7	0.01728	14	0.03090	$n = 35$	$r = 24$
$n = 35$	$r = 4$	2	0.00577	9	0.01408	21	0.02660	10	0.01836	8	0.01864	15	0.03231		
		3	0.00715	10	0.01498	22	0.02763	11	0.01945	9	0.01995	16	0.69580	1	0.01056
1	0.00377	4	0.00834	11	0.01587	23	0.02868	12	0.02051	10	0.02123			2	0.01513
2	0.00540	5	0.00941	12	0.01674	24	0.02977	13	0.02157	11	0.02248	$n = 35$	$r = 20$	3	0.01872
3	0.00669	6	0.01041	13	0.01761	25	0.34994	14	0.02262	12	0.02371			4	0.02182
4	0.00780	7	0.01134	14	0.01847			15	0.02366	13	0.02492	1	0.00777	5	0.02460
5	0.00880	8	0.01224	15	0.01933	$n = 35$	$r = 11$	16	0.02471	14	0.02613	2	0.01113	6	0.02717
6	0.00974	9	0.01311	16	0.02019			17	0.02577	15	0.02733	3	0.01378	7	0.02959
7	0.01061	10	0.01395	17	0.02106	1	0.00486	18	0.02683	16	0.02853	4	0.01607	8	0.03190
8	0.01145	11	0.01478	18	0.02194	2	0.00698	19	0.02791	17	0.02973	5	0.01813	9	0.03411
9	0.01226	12	0.01559	19	0.02283	3	0.00864	20	0.02900	18	0.03095	6	0.02003	10	0.03625
10	0.01305	13	0.01640	20	0.02374	4	0.01007	21	0.03012	19	0.56318	7	0.02183	11	0.99731
11	0.01383	14	0.01720	21	0.02467	5	0.01137	22	0.45024			8	0.02354		
12	0.01459	15	0.01801	22	0.02563	6	0.01257			$n = 35$	$r = 17$	9	0.02519	$n = 35$	$r = 25$
13	0.01535	16	0.01881	23	0.02662	7	0.01370	$n = 35$	$r = 14$			10	0.02679		
14	0.01610	17	0.01962	24	0.02764	8	0.01478			1	0.00648	11	0.02836	1	0.01160
15	0.01685	18	0.02044	25	0.02871	9	0.01583	1	0.00556	2	0.00929	12	0.02989	2	0.01662
16	0.01760	19	0.02128	26	0.02983	10	0.01684	2	0.00797	3	0.01150	13	0.03140	3	0.02056
17	0.01837	20	0.02213	27	0.28766	11	0.01784	3	0.00987	4	0.01341	14	0.03290	4	0.02395

i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i
5	0.02701	$n = 35$	$r = 33$	21	0.01548	8	0.00891	$n = 40$	$r = 5$	31	0.02493	28	0.02385	29	0.02625
6	0.02983			22	0.01603	9	0.00953			32	0.02585	29	0.02467	30	0.30662
7	0.03248	1	0.05411	23	0.01659	10	0.01013	1	0.00313	33	0.02683	30	0.02552		
8	0.03500	2	3.08823	24	0.01716	11	0.01071	2	0.00448	34	0.20130	31	0.02642	$n = 40$	$r = 11$
9	0.03742			25	0.01775	12	0.01128	3	0.00553			32	0.25336		
10	1.07854	$n = 40$	$r = 0$	26	0.01836	13	0.01185	4	0.00645	$n = 40$	$r = 7$			1	0.00377
				27	0.01899	14	0.01240	5	0.00727			$n = 40$	$r = 9$	2	0.00540
$n = 35$	$r = 26$	1	0.00274	28	0.01964	15	0.01296	6	0.00803	1	0.00331			3	0.00668
		2	0.00392	29	0.02033	16	0.01351	7	0.00874	2	0.00475	1	0.00353	4	0.00778
1	0.01286	3	0.00484	30	0.02105	17	0.01406	8	0.00942	3	0.00587	2	0.00505	5	0.00877
2	0.01843	4	0.00564	31	0.02182	18	0.01462	9	0.01007	4	0.00684	3	0.00625	6	0.00968
3	0.02279	5	0.00636	32	0.02264	19	0.01517	10	0.01070	5	0.00771	4	0.00728	7	0.01054
4	0.02655	6	0.00702	33	0.02353	20	0.01574	11	0.01132	6	0.00851	5	0.00820	8	0.01136
5	0.02993	7	0.00765	34	0.02450	21	0.01631	12	0.01193	7	0.00927	6	0.00906	9	0.01215
6	0.03305	8	0.00824	35	0.02559	22	0.01689	13	0.01252	8	0.00999	7	0.00987	10	0.01291
7	0.03598	9	0.00882	36	0.02684	23	0.01748	14	0.01311	9	0.01068	8	0.01063	11	0.01366
8	0.03876	10	0.00937	37	0.02831	24	0.01808	15	0.01370	10	0.01135	9	0.01137	12	0.01438
9	1.17117	11	0.00991	38	0.03015	25	0.01870	16	0.01428	11	0.01201	10	0.01208	13	0.01510
		12	0.01044	39	0.06870	26	0.01934	17	0.01486	12	0.01265	11	0.01278	14	0.01581
$n = 35$	$r = 27$	13	0.01096			27	0.02000	18	0.01545	13	0.01328	12	0.01346	15	0.01651
		14	0.01148	$n = 40$	$r = 2$	28	0.02069	19	0.01604	14	0.01390	13	0.01413	16	0.01722
1	0.01444	15	0.01199			29	0.02141	20	0.01663	15	0.01452	14	0.01480	17	0.01792
2	0.02068	16	0.01250	1	0.00288	30	0.02217	21	0.01723	16	0.01514	15	0.01546	18	0.01862
3	0.02557	17	0.01301	2	0.00412	31	0.02297	22	0.01785	17	0.01576	16	0.01611	19	0.01933
4	0.02978	18	0.01353	3	0.00510	32	0.02383	23	0.01847	18	0.01638	17	0.01677	20	0.02004
5	0.03356	19	0.01404	4	0.00594	33	0.02476	24	0.01911	19	0.01700	18	0.01743	21	0.02076
6	0.03705	20	0.01456	5	0.00669	34	0.02577	25	0.01976	20	0.01763	19	0.01809	22	0.02149
7	0.04032	21	0.01509	6	0.00739	35	0.02690	26	0.02043	21	0.01827	20	0.01876	23	0.02223
8	1.27871	22	0.01563	7	0.00805	36	0.02817	27	0.02113	22	0.01892	21	0.01944	24	0.02299
		23	0.01618	8	0.00868	37	0.12323	28	0.02185	23	0.01958	22	0.02012	25	0.02377
$n = 35$	$r = 28$	24	0.01674	9	0.00928			29	0.02261	24	0.02025	23	0.02082	26	0.02456
		25	0.01731	10	0.00986	$n = 40$	$r = 4$	30	0.02341	25	0.02094	24	0.02153	27	0.02538
1	0.01645	26	0.01790	11	0.01043			31	0.02425	26	0.02165	25	0.02227	28	0.02623
2	0.02355	27	0.01852	12	0.01099	1	0.00304	32	0.02514	27	0.02239	26	0.02302	29	0.33395

3	0.02912
4	0.03390
5	0.03819
6	0.04215
7	1.40639

$n = 35$ $r = 29$

1	0.01911
2	0.02735
3	0.03381
4	0.03934
5	0.04430
6	1.56247

$n = 35$ $r = 30$

1	0.02279
2	0.03261
3	0.04028
4	0.04685
5	1.76091

$n = 35$ $r = 31$

1	0.02823
2	0.04037
3	0.04983
4	2.02774

$n = 35$ $r = 32$

1	0.03708
2	0.05298
3	2.41907

28	0.01916
29	0.01983
30	0.02053
31	0.02128
32	0.02208
33	0.02295
34	0.02391
35	0.02498
36	0.02620
37	0.02766
38	0.02950
39	0.03211
40	0.03903

$n = 40$ $r = 1$

1	0.00280
2	0.00402
3	0.00497
4	0.00579
5	0.00652
6	0.00720
7	0.00784
8	0.00845
9	0.00904
10	0.00961
11	0.01016
12	0.01071
13	0.01124
14	0.01177
15	0.01230
16	0.01282
17	0.01334
18	0.01387
19	0.01440
20	0.01493

13	0.01154
14	0.01208
15	0.01262
16	0.01316
17	0.01369
18	0.01423
19	0.01478
20	0.01533
21	0.01588
22	0.01645
23	0.01702
24	0.01761
25	0.01821
26	0.01884
27	0.01948
28	0.02015
29	0.02086
30	0.02160
31	0.02238
32	0.02322
33	0.02413
34	0.02512
35	0.02623
36	0.02749
37	0.02897
38	0.09646

$n = 40$ $r = 3$

1	0.00296
2	0.00423
3	0.00524
4	0.00610
5	0.00688
6	0.00759
7	0.00827

2	0.00435
3	0.00538
4	0.00627
5	0.00707
6	0.00780
7	0.00850
8	0.00916
9	0.00979
10	0.01041
11	0.01101
12	0.01160
13	0.01217
14	0.01275
15	0.01332
16	0.01388
17	0.01445
18	0.01502
19	0.01559
20	0.01617
21	0.01676
22	0.01735
23	0.01796
24	0.01858
25	0.01922
26	0.01987
27	0.02055
28	0.02126
29	0.02200
30	0.02277
31	0.02359
32	0.02447
33	0.02542
34	0.02645
35	0.02759
36	0.14946

33	0.02611
34	0.02715
35	0.17542

$n = 40$ $r = 6$

1	0.00322
2	0.00461
3	0.00570
4	0.00664
5	0.00748
6	0.00826
7	0.00900
8	0.00970
9	0.01037
10	0.01102
11	0.01165
12	0.01228
13	0.01289
14	0.01350
15	0.01410
16	0.01470
17	0.01530
18	0.01590
19	0.01651
20	0.01712
21	0.01774
22	0.01837
23	0.01901
24	0.01966
25	0.02033
26	0.02102
27	0.02174
28	0.02248
29	0.02326
30	0.02407

28	0.02315
29	0.02394
30	0.02478
31	0.02566
32	0.02659
33	0.22724

$n = 40$ $r = 8$

1	0.00342
2	0.00489
3	0.00605
4	0.00705
5	0.00795
6	0.00878
7	0.00956
8	0.01030
9	0.01101
10	0.01171
11	0.01238
12	0.01304
13	0.01369
14	0.01434
15	0.01498
16	0.01561
17	0.01625
18	0.01689
19	0.01753
20	0.01818
21	0.01883
22	0.01950
23	0.02018
24	0.02087
25	0.02158
26	0.02231
27	0.02307

27	0.02379
28	0.02460
29	0.02544
30	0.02631
31	0.27979

$n = 40$ $r = 10$

1	0.00365
2	0.00522
3	0.00646
4	0.00752
5	0.00848
6	0.00936
7	0.01019
8	0.01099
9	0.01175
10	0.01248
11	0.01320
12	0.01391
13	0.01460
14	0.01529
15	0.01597
16	0.01665
17	0.01732
18	0.01800
19	0.01869
20	0.01938
21	0.02008
22	0.02078
23	0.02150
24	0.02224
25	0.02299
26	0.02377
27	0.02456
28	0.02539

$n = 40$ $r = 12$

1	0.00391
2	0.00559
3	0.00692
4	0.00806
5	0.00908
6	0.01003
7	0.01092
8	0.01177
9	0.01258
10	0.01337
11	0.01414
12	0.01489
13	0.01564
14	0.01637
15	0.01710
16	0.01783
17	0.01855
18	0.01928
19	0.02001
20	0.02074
21	0.02149
22	0.02224
23	0.02301
24	0.02379
25	0.02459
26	0.02541
27	0.02626
28	0.36189

$n = 40$ $r = 13$

1	0.00405
2	0.00580

i	b_i	*i*	b_i	*i*	b_i	*i*	b_i	*i*	b_i	*i*	b_i	*i*	b_i	*i*	b_i
3	0.00717	14	0.01832	3	0.00880	$n = 40$	$r = 21$	6	0.01748	6	0.02321	$n = 40$	$r = 34$	34	0.01842
4	0.00835	15	0.01913	4	0.01025			7	0.01903	7	0.02525			35	0.01904
5	0.00942	16	0.01994	5	0.01155	1	0.00575	8	0.02050	8	0.02719	1	0.01787	36	0.01965
6	0.01040	17	0.02075	6	0.01275	2	0.00823	9	0.02190	9	0.02904	2	0.02555	37	0.02035
7	0.01132	18	0.02156	7	0.01388	3	0.01018	10	0.02327	10	0.03083	3	0.03155	38	0.02111
8	0.01220	19	0.02237	8	0.01496	4	0.01185	11	0.02459	11	0.03257	4	0.03667	39	0.02191
9	0.01305	20	0.02319	9	0.01599	5	0.01336	12	0.02589	12	1.03038	5	0.04126	40	0.02284
10	0.01386	21	0.02402	10	0.01699	6	0.01475	13	0.02716			6	1.69507	41	0.02390
11	0.01466	22	0.02485	11	0.01797	7	0.01605	14	0.02841	$n = 40$	$r = 29$			42	0.02516
12	0.01544	23	0.02570	12	0.01892	8	0.01730	15	0.02965			$n = 40$	$r = 35$	43	0.02676
13	0.01621	24	0.02657	13	0.01986	9	0.01849	16	0.79541	1	0.00988			44	0.02904
14	0.01697	25	0.45042	14	0.02078	10	0.01964			2	0.01414	1	0.02132	45	0.03513
15	0.01773			15	0.02170	11	0.02077	$n = 40$	$r = 25$	3	0.01747	2	0.03046		
16	0.01848	$n = 40$	$r = 16$	16	0.02262	12	0.02186			4	0.02034	3	0.03759	$n = 45$	$r = 1$
17	0.01923			17	0.02353	13	0.02294	1	0.00727	5	0.02291	4	0.04368		
18	0.01998	1	0.00456	18	0.02444	14	0.02401	2	0.01041	6	0.02527	5	1.90384	1	0.00235
19	0.02074	2	0.00652	19	0.02536	15	0.02507	3	0.01287	7	0.02749			2	0.00336
20	0.02150	3	0.00807	20	0.02628	16	0.02611	4	0.01498	8	0.02960	$n = 40$	$r = 36$	3	0.00415
21	0.02227	4	0.00940	21	0.02720	17	0.02716	5	0.01688	9	0.03161			4	0.00483
22	0.02305	5	0.01059	22	0.54870	18	0.02820	6	0.01863	10	0.03355	1	0.02640	5	0.00544
23	0.02384	6	0.01169			19	0.66131	7	0.02028	11	1.10508	2	0.03771	6	0.00600
24	0.02465	7	0.01273	$n = 40$	$r = 19$			8	0.02184			3	0.04650	7	0.00652
25	0.02548	8	0.01372			$n = 40$	$r = 22$	9	0.02334	$n = 40$	$r = 30$	4	2.18534	8	0.00702
26	0.02632	9	0.01467	1	0.00520			10	0.02479					9	0.00751
27	0.39054	10	0.01559	2	0.00745	1	0.00607	11	0.02620	1	0.01085	$n = 40$	$r = 37$	10	0.00796
		11	0.01648	3	0.00921	2	0.00869	12	0.02757	2	0.01553			11	0.00842
$n = 40$	$r = 14$	12	0.01736	4	0.01073	3	0.01074	13	0.02892	3	0.01919	1	0.03468	12	0.00885
		13	0.01822	5	0.01210	4	0.01251	14	0.03025	4	0.02233	2	0.04949	13	0.00929
1	0.00421	14	0.01907	6	0.01335	5	0.01410	15	0.84677	5	0.02515	3	2.59935	14	0.00972
2	0.00602	15	0.01992	7	0.01454	6	0.01556			6	0.02774			15	0.01010
3	0.00745	16	0.02076	8	0.01566	7	0.01694	$n = 40$	$r = 26$	7	0.03017	$n = 40$	$r = 38$	16	0.01057
4	0.00868	17	0.02160	9	0.01675	8	0.01825			8	0.03247			17	0.01094
5	0.00978	18	0.02244	10	0.01779	9	0.01950	1	0.00779	9	0.03468	1	0.05060	18	0.01147
6	0.01080	19	0.02329	11	0.01881	10	0.02072	2	0.01114	10	1.18929	2	3.30932	19	0.01167

7	0.01176
8	0.01267
9	0.01355
10	0.01439
11	0.01522
12	0.01603
13	0.01683
14	0.01762
15	0.01840
16	0.01918
17	0.01996
18	0.02074
19	0.02152
20	0.02231
21	0.02311
22	0.02392
23	0.02474
24	0.02557
25	0.02643
26	0.42001

$n = 40$	$r = 15$
1	0.00437
2	0.00626
3	0.00775
4	0.00902
5	0.01017
6	0.01123
7	0.01222
8	0.01317
9	0.01408
10	0.01497
11	0.01583
12	0.01667
13	0.01750

20	0.02413
21	0.02499
22	0.02586
23	0.02674
24	0.48190

$n = 40$	$r = 17$
1	0.00475
2	0.00681
3	0.00842
4	0.00980
5	0.01105
6	0.01220
7	0.01328
8	0.01431
9	0.01530
10	0.01626
11	0.01719
12	0.01811
13	0.01900
14	0.01989
15	0.02077
16	0.02165
17	0.02252
18	0.02340
19	0.02428
20	0.02516
21	0.02605
22	0.02695
23	0.51461

$n = 40$	$r = 18$
1	0.00497
2	0.00711

12	0.01981
13	0.02079
14	0.02176
15	0.02272
16	0.02368
17	0.02463
18	0.02558
19	0.02653
20	0.02749
21	0.58436

$n = 40$	$r = 20$
1	0.00546
2	0.00782
3	0.00967
4	0.01126
5	0.01270
6	0.01402
7	0.01526
8	0.01644
9	0.01757
10	0.01867
11	0.01974
12	0.02079
13	0.02182
14	0.02283
15	0.02384
16	0.02484
17	0.02583
18	0.02683
19	0.02783
20	0.62181

11	0.02190
12	0.02306
13	0.02420
14	0.02532
15	0.02643
16	0.02753
17	0.02863
18	0.70316

$n = 40$	$r = 23$
1	0.00642
2	0.00919
3	0.01137
4	0.01324
5	0.01492
6	0.01647
7	0.01792
8	0.01931
9	0.02063
10	0.02192
11	0.02317
12	0.02439
13	0.02559
14	0.02678
15	0.02795
16	0.02911
17	0.74772

$n = 40$	$r = 24$
1	0.00682
2	0.00976
3	0.01207
4	0.01406
5	0.01584

3	0.01378
4	0.01604
5	0.01807
6	0.01995
7	0.02170
8	0.02337
9	0.02498
10	0.02652
11	0.02803
12	0.02949
13	0.03093
14	0.90246

$n = 40$	$r = 27$
1	0.00838
2	0.01199
3	0.01482
4	0.01726
5	0.01944
6	0.02145
7	0.02334
8	0.02514
9	0.02686
10	0.02852
11	0.03013
12	0.03170
13	0.96330

$n = 40$	$r = 28$
1	0.00907
2	0.01298
3	0.01604
4	0.01867
5	0.02103

$n = 40$	$r = 31$
1	0.01204
2	0.01722
3	0.02127
4	0.02475
5	0.02787
6	0.03074
7	0.03343
8	0.03597
9	1.28561

$n = 40$	$r = 32$
1	0.01351
2	0.01932
3	0.02387
4	0.02776
5	0.03125
6	0.03446
7	0.03746
8	1.39777

$n = 40$	$r = 33$
1	0.01539
2	0.02201
3	0.02718
4	0.03160
5	0.03557
6	0.03921
7	1.53133

$n = 45$	$r = 0$
1	0.00230
2	0.00328
3	0.00406
4	0.00472
5	0.00532
6	0.00587
7	0.00638
8	0.00687
9	0.00734
10	0.00779
11	0.00823
12	0.00865
13	0.00909
14	0.00951
15	0.00988
16	0.01034
17	0.01070
18	0.01122
19	0.01142
20	0.01197
21	0.01236
22	0.01273
23	0.01318
24	0.01359
25	0.01401
26	0.01445
27	0.01490
28	0.01535
29	0.01582
30	0.01630
31	0.01679
32	0.01732
33	0.01785
20	0.01224
21	0.01264
22	0.01302
23	0.01348
24	0.01390
25	0.01433
26	0.01477
27	0.01524
28	0.01569
29	0.01617
30	0.01666
31	0.01717
32	0.01771
33	0.01825
34	0.01883
35	0.01947
36	0.02009
37	0.02080
38	0.02158
39	0.02239
40	0.02334
41	0.02441
42	0.02569
43	0.02728
44	0.06190

$n = 45$	$r = 2$
1	0.00240
2	0.00343
3	0.00424
4	0.00494
5	0.00556
6	0.00614
7	0.00668
8	0.00719

i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i
9	0.00768	33	0.01911	14	0.01071	$n = 45$	$r = 7$	28	0.01864	19	0.01413	14	0.01334	13	0.01305
10	0.00815	34	0.01971	15	0.01107			29	0.01920	20	0.01610	15	0.01286	14	0.01401
11	0.00862	35	0.02038	16	0.01169	1	0.00272	30	0.01977	21	0.01518	16	0.01484	15	0.01406
12	0.00905	36	0.02102	17	0.01193	2	0.00389	31	0.02037	22	0.01679	17	0.01378	16	0.01527
13	0.00951	37	0.02176	18	0.01273	3	0.00480	32	0.02100	23	0.01677	18	0.01592	17	0.01525
14	0.00995	38	0.02257	19	0.01270	4	0.00559	33	0.02164	24	0.01742	19	0.01507	18	0.01630
15	0.01034	39	0.02341	20	0.01370	5	0.00629	34	0.02230	25	0.01801	20	0.01676	19	0.01659
16	0.01081	40	0.02439	21	0.01360	6	0.00694	35	0.02304	26	0.01853	21	0.01641	20	0.01737
17	0.01120	41	0.02547	22	0.01457	7	0.00755	36	0.02375	27	0.01911	22	0.01757	21	0.01782
18	0.01173	42	0.11101	23	0.01465	8	0.00813	37	0.22695	28	0.01968	23	0.01782	22	0.01850
19	0.01195			24	0.01535	9	0.00869			29	0.02027	24	0.01853	23	0.01904
20	0.01252	$n = 45$	$r = 4$	25	0.01574	10	0.00922	$n = 45$	$r = 9$	30	0.02088	25	0.01905	24	0.01967
21	0.01294			26	0.01624	11	0.00976			31	0.02150	26	0.01965	25	0.02027
22	0.01332	1	0.00252	27	0.01675	12	0.01024	1	0.00287	32	0.02216	27	0.02024	26	0.02089
23	0.01379	2	0.00360	28	0.01725	13	0.01074	2	0.00410	33	0.02284	28	0.02085	27	0.02152
24	0.01422	3	0.00445	29	0.01777	14	0.01128	3	0.00507	34	0.02352	29	0.02147	28	0.02216
25	0.01465	4	0.00518	30	0.01831	15	0.01165	4	0.00590	35	0.27367	30	0.02211	29	0.02282
26	0.01512	5	0.00583	31	0.01887	16	0.01214	5	0.00664			31	0.02277	30	0.02350
27	0.01558	6	0.00644	32	0.01946	17	0.01289	6	0.00733	$n = 45$	$r = 11$	32	0.02346	31	0.37131
28	0.01606	7	0.00700	33	0.02005	18	0.01333	7	0.00797			33	0.32159		
29	0.01654	8	0.00754	34	0.02068	19	0.01316	8	0.00858	1	0.00304			$n = 45$	$r = 15$
30	0.01705	9	0.00805	35	0.02137	20	0.01440	9	0.00917	2	0.00434	$n = 45$	$r = 13$		
31	0.01756	10	0.00855	36	0.02204	21	0.01454	10	0.00972	3	0.00537			1	0.00344
32	0.01812	11	0.00904	37	0.02281	22	0.01527	11	0.01033	4	0.00625	1	0.00323	2	0.00492
33	0.01867	12	0.00949	38	0.02365	23	0.01534	12	0.01070	5	0.00703	2	0.00462	3	0.00608
34	0.01926	13	0.00997	39	0.02451	24	0.01629	13	0.01155	6	0.00776	3	0.00570	4	0.00708
35	0.01991	14	0.01042	40	0.15777	25	0.01645	14	0.01158	7	0.00844	4	0.00663	5	0.00797
36	0.02055	15	0.01086			26	0.01712	15	0.01279	8	0.00909	5	0.00747	6	0.00879
37	0.02127	16	0.01130	$n = 45$	$r = 6$	27	0.01761	16	0.01255	9	0.00971	6	0.00824	7	0.00956
38	0.02207	17	0.01180			28	0.01815	17	0.01372	10	0.01029	7	0.00897	8	0.01030
39	0.02289	18	0.01225	1	0.00265	29	0.01870	18	0.01356	11	0.01094	8	0.00966	9	0.01099
40	0.02386	19	0.01259	2	0.00379	30	0.01926	19	0.01452	12	0.01134	9	0.01031	10	0.01169
41	0.02494	20	0.01305	3	0.00468	31	0.01984	20	0.01467	13	0.01221	10	0.01095	11	0.01231
42	0.02621	21	0.01367	4	0.00544	32	0.02046	21	0.01576	14	0.01224	11	0.01158	12	0.01297

43	0.08692

$n = 45$ $r = 3$

1	0.00246
2	0.00352
3	0.00434
4	0.00506
5	0.00569
6	0.00628
7	0.00683
8	0.00736
9	0.00786
10	0.00834
11	0.00882
12	0.00927
13	0.00973
14	0.01019
15	0.01058
16	0.01108
17	0.01144
18	0.01204
19	0.01219
20	0.01287
21	0.01319
22	0.01367
23	0.01410
24	0.01457
25	0.01501
26	0.01547
27	0.01596
28	0.01643
29	0.01694
30	0.01745
31	0.01798
32	0.01854

22	0.01385
23	0.01457
24	0.01486
25	0.01538
26	0.01584
27	0.01635
28	0.01683
29	0.01735
30	0.01787
31	0.01841
32	0.01899
33	0.01957
34	0.02018
35	0.02086
36	0.02152
37	0.02227
38	0.02310
39	0.02395
40	0.02493
41	0.13454

$n = 45$ $r = 5$

1	0.00258
2	0.00369
3	0.00456
4	0.00531
5	0.00598
6	0.00660
7	0.00718
8	0.00773
9	0.00825
10	0.00876
11	0.00926
12	0.00973
13	0.01022

5	0.00613
6	0.00677
7	0.00736
8	0.00792
9	0.00847
10	0.00898
11	0.00950
12	0.00998
13	0.01050
14	0.01094
15	0.01145
16	0.01179
17	0.01244
18	0.01290
19	0.01333
20	0.01344
21	0.01476
22	0.01419
23	0.01548
24	0.01558
25	0.01617
26	0.01664
27	0.01717
28	0.01769
29	0.01823
30	0.01878
31	0.01934
32	0.01995
33	0.02055
34	0.02120
35	0.02190
36	0.02259
37	0.02336
38	0.02422
39	0.18082

33	0.02108
34	0.02173
35	0.02246
36	0.02315
37	0.02395
38	0.20386

$n = 45$ $r = 8$

1	0.00279
2	0.00399
3	0.00493
4	0.00574
5	0.00646
6	0.00713
7	0.00776
8	0.00835
9	0.00892
10	0.00947
11	0.00999
12	0.01056
13	0.01096
14	0.01173
15	0.01182
16	0.01281
17	0.01245
18	0.01449
19	0.01323
20	0.01471
21	0.01497
22	0.01548
23	0.01620
24	0.01627
25	0.01716
26	0.01751
27	0.01808

22	0.01581
23	0.01642
24	0.01702
25	0.01743
26	0.01805
27	0.01858
28	0.01914
29	0.01972
30	0.02031
31	0.02092
32	0.02157
33	0.02221
34	0.02290
35	0.02365
36	0.25018

$n = 45$ $r = 10$

1	0.00295
2	0.00422
3	0.00521
4	0.00607
5	0.00683
6	0.00754
7	0.00820
8	0.00882
9	0.00944
10	0.01001
11	0.01055
12	0.01120
13	0.01152
14	0.01261
15	0.01187
16	0.01432
17	0.01296
18	0.01490

15	0.01358
16	0.01306
17	0.01484
18	0.01384
19	0.01621
20	0.01502
21	0.01677
22	0.01657
23	0.01755
24	0.01789
25	0.01852
26	0.01907
27	0.01966
28	0.02025
29	0.02086
30	0.02148
31	0.02211
32	0.02280
33	0.02347
34	0.29744

$n = 45$ $r = 12$

1	0.00313
2	0.00448
3	0.00553
4	0.00643
5	0.00725
6	0.00799
7	0.00870
8	0.00936
9	0.01001
10	0.01061
11	0.01121
12	0.01186
13	0.01218

12	0.01206
13	0.01302
14	0.01291
15	0.01449
16	0.01384
17	0.01570
18	0.01507
19	0.01660
20	0.01647
21	0.01749
22	0.01779
23	0.01852
24	0.01904
25	0.01965
26	0.02025
27	0.02086
28	0.02149
29	0.02213
30	0.02279
31	0.02345
32	0.34620

$n = 45$ $r = 14$

1	0.00333
2	0.00476
3	0.00589
4	0.00685
5	0.00771
6	0.00851
7	0.00926
8	0.00996
9	0.01066
10	0.01127
11	0.01195
12	0.01259

13	0.01367
14	0.01411
15	0.01498
16	0.01528
17	0.01621
18	0.01655
19	0.01731
20	0.01783
21	0.01848
22	0.01906
23	0.01969
24	0.02030
25	0.02093
26	0.02157
27	0.02222
28	0.02288
29	0.02356
30	0.39702

$n = 45$ $r = 16$

1	0.00356
2	0.00509
3	0.00629
4	0.00732
5	0.00824
6	0.00909
7	0.00989
8	0.01065
9	0.01138
10	0.01206
11	0.01277
12	0.01341
13	0.01406
14	0.01476
15	0.01528

i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i	i	b_i
16	0.01604	16	0.01715	20	0.02137	2	0.00671	16	0.02305	13	0.02385	2	0.01130	7	0.03135
17	0.01655	17	0.01781	21	0.02209	3	0.00828	17	0.02394	14	0.02492	3	0.01395	8	0.03371
18	0.01726	18	0.01849	22	0.02281	4	0.00964	18	0.02482	15	0.02598	4	0.01623	9	1.39103
19	0.01782	19	0.01915	23	0.02354	5	0.01085	19	0.02570	16	0.02702	5	0.01827		
20	0.01847	20	0.01981	24	0.02427	6	0.01197	20	0.70897	17	0.83664	6	0.02014	$n = 45$	$r = 37$
21	0.01909	21	0.02048	25	0.53753	7	0.01302					7	0.02189		
22	0.01972	22	0.02115			8	0.01402	$n = 45$	$r = 26$	$n = 45$	$r = 29$	8	0.02355	1	0.01274
23	0.02035	23	0.02183	$n = 45$	$r = 21$	9	0.01497					9	0.02514	2	0.01820
24	0.02099	24	0.02251			10	0.01589	1	0.00543	1	0.00643	10	0.02667	3	0.02246
25	0.02164	25	0.02320	1	0.00430	11	0.01678	2	0.00776	2	0.00920	11	0.02815	4	0.02611
26	0.02230	26	0.02391	2	0.00615	12	0.01765	3	0.00958	3	0.01136	12	0.02958	5	0.02936
27	0.02297	27	0.47855	3	0.00760	13	0.01850	4	0.01115	4	0.01322	13	1.05748	6	0.03235
28	0.02365			4	0.00884	14	0.01934	5	0.01255	5	0.01488			7	0.03514
29	0.42340	$n = 45$	$r = 19$	5	0.00995	15	0.02017	6	0.01385	6	0.01641	$n = 45$	$r = 33$	8	1.50774
				6	0.01098	16	0.02099	7	0.01506	7	0.01785				
$n = 45$	$r = 17$	1	0.00397	7	0.01194	17	0.02180	8	0.01621	8	0.01920	1	0.00855	$n = 45$	$r = 38$
		2	0.00568	8	0.01286	18	0.02261	9	0.01731	9	0.02050	2	0.01223		
1	0.00369	3	0.00701	9	0.01373	19	0.02342	10	0.01837	10	0.02176	3	0.01510	1	0.01451
2	0.00527	4	0.00816	10	0.01457	20	0.02422	11	0.01940	11	0.02297	4	0.01756	2	0.02073
3	0.00651	5	0.00919	11	0.01539	21	0.02503	12	0.02040	12	0.02415	5	0.01976	3	0.02557
4	0.00758	6	0.01014	12	0.01619	22	0.63544	13	0.02138	13	0.02531	6	0.02179	4	0.02972
5	0.00854	7	0.01103	13	0.01698			14	0.02235	14	0.02644	7	0.02368	5	0.03342
6	0.00942	8	0.01187	14	0.01775	$n = 45$	$r = 24$	15	0.02330	15	0.02756	8	0.02547	6	0.03681
7	0.01024	9	0.01268	15	0.01851			16	0.02424	16	0.88528	9	0.02719	7	1.64704
8	0.01103	10	0.01346	16	0.01926	1	0.00491	17	0.02517			10	0.02883		
9	0.01178	11	0.01422	17	0.02001	2	0.00702	18	0.02610	$n = 45$	$r = 30$	11	0.03043	$n = 45$	$r = 39$
10	0.01251	12	0.01496	18	0.02076	3	0.00868	19	0.74889			12	1.12662		
11	0.01320	13	0.01568	19	0.02150	4	0.01009			1	0.00686			1	0.01685
12	0.01391	14	0.01640	20	0.02225	5	0.01137	$n = 45$	$r = 27$	2	0.00981	$n = 45$	$r = 34$	2	0.02407
13	0.01457	15	0.01710	21	0.02299	6	0.01254			3	0.01211			3	0.02969
14	0.01523	16	0.01780	22	0.02374	7	0.01364	1	0.00573	4	0.01409	1	0.00932	4	0.03449

15	0.01590	17	0.01849	23	0.02449	8	0.01468	2	0.00819	5	0.01586	2	0.01332	5	0.03877
16	0.01652	18	0.01918	24	0.56877	9	0.01567	3	0.01011	6	0.01749	3	0.01645	6	1.81818
17	0.01721	19	0.01987			10	0.01664	4	0.01176	7	0.01902	4	0.01912		
18	0.01782	20	0.02056	*n* = 45	*r* = 22	11	0.01757	5	0.01325	8	0.02046	5	0.02152	*n* = 45	*r* = 40
19	0.01848	21	0.02126			12	0.01848	6	0.01461	9	0.02185	6	0.02373		
20	0.01911	22	0.02195	1	0.00449	13	0.01937	7	0.01589	10	0.02318	7	0.02578	1	0.02010
21	0.01976	23	0.02265	2	0.00642	14	0.02025	8	0.01710	11	0.02447	8	0.02773	2	0.02869
22	0.02041	24	0.02336	3	0.00793	15	0.02112	9	0.01826	12	0.02573	9	0.02959	3	0.03538
23	0.02107	25	0.02407	4	0.00922	16	0.02197	10	0.01937	13	0.02696	10	0.03138	4	0.04108
24	0.02173	26	0.50750	5	0.01038	17	0.02282	11	0.02046	14	0.02816	11	1.20381	5	2.03686
25	0.02240			6	0.01146	18	0.02367	12	0.02151	15	0.93781				
26	0.02307	*n* = 45	*r* = 20	7	0.01246	19	0.02451	13	0.02255			*n* = 45	*r* = 35	*n* = 45	*r* = 41
27	0.02376			8	0.01341	20	0.02535	14	0.02356	*n* = 45	*r* = 31				
28	0.45055	1	0.00413	9	0.01432	21	0.67124	15	0.02457			1	0.01024	1	0.02489
		2	0.00590	10	0.01520			16	0.02556	1	0.00734	2	0.01463	2	0.03551
n = 45	*r* = 18	3	0.00729	11	0.01606	*n* = 45	*r* = 25	17	0.02654	2	0.01050	3	0.01806	3	0.04376
		4	0.00849	12	0.01689			18	0.79132	3	0.01297	4	0.02100	4	2.33237
1	0.00382	5	0.00956	13	0.01771	1	0.00516			4	0.01508	5	0.02363		
2	0.00547	6	0.01054	14	0.01851	2	0.00737	*n* = 45	*r* = 28	5	0.01698	6	0.02604	*n* = 45	*r* = 42
3	0.00676	7	0.01147	15	0.01930	3	0.00911			6	0.01872	7	0.02830		
4	0.00786	8	0.01235	16	0.02009	4	0.01060	1	0.00606	7	0.02036	8	0.03043	1	0.03268
5	0.00885	9	0.01319	17	0.02087	5	0.01193	2	0.00866	8	0.02190	9	0.03247	2	0.04661
6	0.00977	10	0.01400	18	0.02165	6	0.01316	3	0.01070	9	0.02338	10	1.29103	3	2.76794
7	0.01062	11	0.01478	19	0.02242	7	0.01431	4	0.01245	10	0.02480				
8	0.01144	12	0.01555	20	0.02319	8	0.01540	5	0.01402	11	0.02618	*n* = 45	*r* = 36	*n* = 45	*r* = 43
9	0.01222	13	0.01631	21	0.02397	9	0.01645	6	0.01546	12	0.02752				
10	0.01296	14	0.01705	22	0.02475	10	0.01746	7	0.01681	13	0.02883	1	0.01135	1	0.04769
11	0.01370	15	0.01778	23	0.60134	11	0.01844	8	0.01809	14	0.99492	2	0.01622	2	3.51655
12	0.01440	16	0.01850			12	0.01939	9	0.01931			3	0.02002		
13	0.01511	17	0.01922	*n* = 45	*r* = 23	13	0.02033	10	0.02050	*n* = 45	*r* = 32	4	0.02328		
14	0.01580	18	0.01994			14	0.02125	11	0.02164			5	0.02619		
15	0.01647	19	0.02066	1	0.00469	15	0.02215	12	0.02276	1	0.00790	6	0.02886		

APPENDIX G

VARIANCE OF θ_2^*'s IN TERMS OF θ_2^2/n AND K_3/K_2^* *

The variance, K_3/K_2 and relative efficiency of the scale parameter θ_2^* when the threshold parameter θ_1 is known for sample sizes $n = 5(1)25(5)45$ from the Rayleigh distribution with censoring from the right for $r = 0(1)n - 2$.

* From Hassanein, K. M., Saleh, A. K., and Brown E. (1995), "Best linear unbiased estimate and confidence interval for Rayleigh's scale parameter when the threshold parameter is known for data with censored observations from the right." Report from the University of Kansas Medical School. Reproduced by permission from the authors.

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n	r	$Var(\theta_2^*)$	K_3/K_2	$Eff(\theta_2^*)$	n	r	$Var(\theta_2^*)$	K_3/K_2	$Eff(\theta_2^*)$	n	r	$Var(\theta_2^*)$	K_3/K_2	$Eff(\theta_2^*)$
5	0	0.05135	0.67288	1.00000	9	6	0.08652	1.25896	0.32624	12	7	0.05120	1.09627	0.41195
5	1	0.06443	0.77746	0.79699	9	7	0.13178	1.58637	0.21418	12	8	0.06434	1.24238	0.32783
5	2	0.08657	0.92585	0.59318	10	0	0.02537	0.65170	1.00000	12	9	0.08651	1.45916	0.24381
5	3	0.13181	1.17616	0.38959	10	1	0.02820	0.69952	0.89956	12	10	0.13178	1.83462	0.16006
6	0	0.04263	0.66609	1.00000	10	2	0.03176	0.75371	0.79868	13	0	0.01945	0.64641	1.00000
6	1	0.05127	0.75039	0.83139	10	3	0.03636	0.81757	0.69764	13	1	0.02108	0.68265	0.92294
6	2	0.06439	0.86033	0.66203	10	4	0.04252	0.89558	0.59651	13	2	0.02301	0.72206	0.84556
6	3	0.08655	1.01965	0.49256	10	5	0.05121	0.99513	0.49536	13	3	0.02533	0.76611	0.76805
6	4	0.13180	1.29107	0.32344	10	6	0.06434	1.12984	0.39423	13	4	0.02817	0.81642	0.69048
7	0	0.03643	0.66109	1.00000	10	7	0.08651	1.32908	0.29321	13	5	0.03174	0.87512	0.61286
7	1	0.04257	0.73179	0.85584	10	8	0.13178	1.67323	0.19249	13	6	0.03634	0.94536	0.53523
7	2	0.05124	0.81914	0.71102	11	0	0.02303	0.64964	1.00000	13	7	0.04251	1.03203	0.45758
7	3	0.06437	0.93534	0.56602	11	1	0.02534	0.69284	0.90879	13	8	0.05120	1.14341	0.37995
7	4	0.08653	1.10530	0.42105	11	2	0.02819	0.74099	0.81718	13	9	0.06434	1.29495	0.30236
7	5	0.13179	1.39649	0.27646	11	3	0.03175	0.79651	0.72542	13	10	0.08651	1.52002	0.22487
8	0	0.03181	0.65724	1.00000	11	4	0.03635	0.86243	0.63358	13	11	0.13177	1.91020	0.14762
8	1	0.03639	0.71817	0.87410	11	5	0.04252	0.94336	0.54171	14	0	0.01805	0.64512	1.00000
8	2	0.04255	0.79067	0.74764	11	6	0.05120	1.04695	0.44983	14	1	0.01944	0.67867	0.92849
8	3	0.05123	0.88197	0.62097	11	7	0.06434	1.18747	0.35798	14	2	0.02107	0.71481	0.85669
8	4	0.06436	1.00450	0.49426	11	8	0.08651	1.39565	0.26624	14	3	0.02300	0.75476	0.78477
8	5	0.08652	1.18466	0.36764	11	9	0.13178	1.75578	0.17479	14	4	0.02532	0.79974	0.71278
8	6	0.13178	1.49446	0.24138	12	0	0.02109	0.64790	1.00000	14	5	0.02817	0.85132	0.64076
9	0	0.02823	0.65419	1.00000	12	1	0.02301	0.68731	0.91646	14	6	0.03174	0.91169	0.56872
9	1	0.03178	0.70776	0.88826	12	2	0.02533	0.73064	0.83257	14	7	0.03634	0.98408	0.49666
9	2	0.03637	0.76976	0.77603	12	3	0.02818	0.77976	0.74853	14	8	0.04251	1.07354	0.42460
9	3	0.04253	0.84491	0.66361	12	4	0.03174	0.83683	0.66442	14	9	0.05120	1.18865	0.35256
9	4	0.05122	0.94034	0.55112	12	5	0.03635	0.90491	0.58027	14	10	0.06433	1.34544	0.28056
9	5	0.06435	1.06906	0.43863	12	6	0.04251	0.98873	0.49611	14	11	0.08651	1.57851	0.20865

(*Continued*)

n	r	$Var(\theta_2^*)$	K_3/K_2	$Eff(\theta_2^*)$	n	r	$Var(\theta_2^*)$	K_3/K_2	$Eff(\theta_2^*)$	n	r	$Var(\theta_2^*)$	K_3/K_2	$Eff(\theta_2^*)$
14	12	0.13177	1.98290	0.13697	16	14	0.13177	2.12083	0.11971	18	12	0.04250	1.22531	0.32956
15	0	0.01684	0.64399	1.00000	17	0	0.01484	0.64211	1.00000	18	13	0.05119	1.35441	0.27363
15	1	0.01804	0.67522	0.93330	17	1	0.01577	0.66957	0.94120	18	14	0.06433	1.53073	0.21775
15	2	0.01943	0.70862	0.86632	17	2	0.01682	0.69856	0.88216	18	15	0.08650	1.79347	0.16193
15	3	0.02106	0.74517	0.79923	17	3	0.01803	0.72982	0.82302	18	16	0.13177	2.25031	0.10630
15	4	0.02300	0.78584	0.73209	17	4	0.01943	0.76397	0.76384	19	0	0.01326	0.64060	1.00000
15	5	0.02532	0.83183	0.66491	17	5	0.02106	0.80177	0.70463	19	1	0.01400	0.66512	0.94742
15	6	0.02817	0.88473	0.59771	17	6	0.02299	0.84413	0.64539	19	2	0.01483	0.69075	0.89464
15	7	0.03174	0.94677	0.53049	17	7	0.02532	0.89226	0.58614	19	3	0.01576	0.71806	0.84177
15	8	0.03634	1.02127	0.46327	17	8	0.02816	0.94781	0.52688	19	4	0.01682	0.74752	0.78887
15	9	0.04251	1.11346	0.39605	17	9	0.03173	1.01313	0.46762	19	5	0.01802	0.77961	0.73593
15	10	0.05119	1.23221	0.32885	17	10	0.03634	1.09175	0.40835	19	6	0.01942	0.81492	0.68297
15	11	0.06433	1.39408	0.26169	17	11	0.04250	1.18922	0.34910	19	7	0.02106	0.85417	0.63000
15	12	0.08650	1.63491	0.19462	17	12	0.05119	1.31495	0.28986	19	8	0.02299	0.89831	0.57702
15	13	0.13177	2.05302	0.12776	17	13	0.06433	1.48658	0.23066	19	9	0.02531	0.94860	0.52404
16	0	0.01577	0.64299	1.00000	17	14	0.08650	1.74222	0.17154	19	10	0.02816	1.00676	0.47105
16	1	0.01683	0.67222	0.93749	17	15	0.13177	2.18653	0.11261	19	11	0.03173	1.07527	0.41806
16	2	0.01803	0.70325	0.87474	18	0	0.01401	0.64131	1.00000	19	12	0.03634	1.15785	0.36507
16	3	0.01943	0.73694	0.81188	18	1	0.01483	0.66722	0.94448	19	13	0.04250	1.26036	0.31209
16	4	0.02106	0.77407	0.74897	18	2	0.01576	0.69443	0.88875	19	14	0.05119	1.39275	0.25913
16	5	0.02299	0.81557	0.68602	18	3	0.01682	0.72357	0.83292	19	15	0.06433	1.57363	0.20620
16	6	0.02532	0.86262	0.62305	4	4	0.01803	0.75520	0.77705	19	16	0.08650	1.84328	0.15335
16	7	0.02816	0.91684	0.56007	5	5	0.01942	0.78992	0.72115	19	17	0.13177	2.31233	0.10067
16	8	0.03173	0.98053	0.49708	18	6	0.02106	0.82843	0.66523	20	0	0.01260	0.63995	1.00000
16	9	0.03634	1.05711	0.43408	18	7	0.02299	0.87168	0.60930	20	1	0.01326	0.66323	0.95007
16	10	0.04251	1.15197	0.37110	18	8	0.02531	0.92088	0.55335	20	2	0.01400	0.68746	0.89994
16	11	0.05119	1.27426	0.30813	18	9	0.02816	0.97774	0.49740	20	3	0.01482	0.71316	0.84973
16	12	0.06433	1.44108	0.24520	18	10	0.03173	1.04468	0.44145	20	4	0.01576	0.74072	0.79949
16	13	0.08650	1.68942	0.18235	18	11	0.03634	1.12530	0.38550	20	5	0.01681	0.77057	0.74922

20	6	0.01802	0.80316	0.69893	21	17	0.06433	1.65608	0.18643	23	6	0.01482	0.77552	0.73842
20	7	0.01942	0.83909	0.64862	21	18	0.08650	1.93907	0.13864	23	7	0.01575	0.80417	0.69472
20	8	0.02105	0.87910	0.59831	21	19	0.13177	2.43163	0.09101	23	8	0.01681	0.83536	0.65101
20	9	0.02299	0.92414	0.54799	22	0	0.01144	0.63883	1.00000	23	9	0.01802	0.86959	0.60729
20	10	0.02531	0.97549	0.49767	22	1	0.01199	0.65996	0.95463	23	10	0.01942	0.90745	0.56357
20	11	0.02816	1.03493	0.44734	22	2	0.01259	0.68182	0.90908	23	11	0.02105	0.94972	0.51984
20	12	0.03173	1.10500	0.39701	22	3	0.01325	0.70481	0.86347	23	12	0.02299	0.99741	0.47611
20	13	0.03633	1.18950	0.34669	22	4	0.01399	0.72925	0.81782	23	13	0.02531	1.05190	0.43238
20	14	0.04250	1.29445	0.29638	22	5	0.01482	0.75542	0.77215	23	14	0.02816	1.11507	0.38865
20	15	0.05119	1.43005	0.24608	22	6	0.01575	0.78368	0.72646	23	15	0.03173	1.18965	0.34492
20	16	0.06433	1.61538	0.19582	22	7	0.01681	0.81441	0.68076	23	16	0.03633	1.27970	0.30120
20	17	0.08650	1.89179	0.14563	22	8	0.01802	0.84808	0.63505	23	17	0.04250	1.39167	0.25749
20	18	0.13177	2.37273	0.09560	22	9	0.01942	0.88529	0.58933	23	18	0.05119	1.53649	0.21379
21	0	0.01199	0.63937	1.00000	22	10	0.02105	0.92681	0.54361	23	19	0.06433	1.73461	0.17012
21	1	0.01259	0.66152	0.95246	22	11	0.02299	0.97363	0.49788	23	20	0.08650	2.03034	0.12651
21	2	0.01326	0.68450	0.90473	22	12	0.02531	1.02708	0.45215	23	21	0.13177	2.54534	0.08305
21	3	0.01400	0.70877	0.85693	22	13	0.02816	1.08903	0.40643	24	0	0.01048	0.63788	1.00000
21	4	0.01482	0.73467	0.80909	22	14	0.03173	1.16213	0.36070	24	1	0.01094	0.65724	0.95843
21	5	0.01575	0.76256	0.76123	22	15	0.03633	1.25037	0.31498	24	2	0.01144	0.67715	0.91670
21	6	0.01681	0.79283	0.71335	22	16	0.04250	1.36005	0.26926	24	3	0.01198	0.69796	0.87491
21	7	0.01802	0.82596	0.66546	22	17	0.05119	1.50185	0.22357	24	4	0.01259	0.71991	0.83308
21	8	0.01942	0.86253	0.61756	22	18	0.06433	1.69580	0.17790	24	5	0.01325	0.74324	0.79124
21	9	0.02105	0.90329	0.56965	22	19	0.08650	1.98523	0.13230	24	6	0.01399	0.76818	0.74938
21	10	0.02299	0.94922	0.52174	22	20	0.13177	2.48914	0.08685	24	7	0.01482	0.79502	0.70751
21	11	0.02531	1.00163	0.47382	23	0	0.01094	0.63834	1.00000	24	8	0.01575	0.82408	0.66563
21	12	0.02816	1.06233	0.42590	23	1	0.01144	0.65854	0.95661	24	9	0.01681	0.85576	0.62375
21	13	0.03173	1.13393	0.37799	23	2	0.01199	0.67938	0.91306	24	10	0.01802	0.89054	0.58186
21	14	0.03633	1.22032	0.33007	23	3	0.01259	0.70123	0.86944	24	11	0.01942	0.92905	0.53996
21	15	0.04250	1.32766	0.28217	23	4	0.01325	0.72435	0.82579	24	12	0.02105	0.97206	0.49806
21	16	0.05119	1.46639	0.23429	23	5	0.01399	0.74902	0.78211	24	13	0.02298	1.02062	0.45616

(*Continued*)

n	r	$Var\,(\theta_2^*)$	K_3/K_2	$Eff\,(\theta_2^*)$	n	r	$Var\,(\theta_2^*)$	K_3/K_2	$Eff\,(\theta_2^*)$	n	r	$Var\,(\theta_2^*)$	K_3/K_2	$Eff\,(\theta_2^*)$
24	14	0.02531	1.07613	0.41426	25	20	0.05119	1.60351	0.19659	30	25	0.05119	1.75990	0.16367
24	15	0.02816	1.14051	0.37237	25	21	0.06433	1.80973	0.15643	30	26	0.06433	1.98510	0.13023
24	16	0.03173	1.21654	0.33047	25	22	0.08650	2.11768	0.11633	30	27	0.08650	2.32167	0.09685
24	17	0.03633	1.30837	0.28858	25	23	0.13177	2.65419	0.07637	30	28	0.13177	2.90852	0.06358
24	18	0.04250	1.42259	0.24669	30	0	0.00838	0.63577	1.00000	35	0	0.00718	0.63454	1.00000
24	19	0.05119	1.57036	0.20483	30	1	0.00867	0.65125	0.96677	35	1	0.00739	0.64782	0.97153
24	20	0.06433	1.77257	0.16299	30	2	0.00898	0.66699	0.93342	35	2	0.00761	0.66123	0.94296
24	21	0.08650	2.07447	0.12121	30	3	0.00931	0.68321	0.90002	35	3	0.00785	0.67495	0.91435
24	22	0.13177	2.60034	0.07957	30	4	0.00967	0.70006	0.86661	35	4	0.00810	0.68909	0.88573
25	0	0.01006	0.63746	1.00000	30	5	0.01005	0.71767	0.83317	35	5	0.00837	0.70373	0.85709
25	1	0.01048	0.65605	0.96010	30	6	0.01048	0.73614	0.79972	35	6	0.00866	0.71894	0.82844
25	2	0.01094	0.67511	0.92004	30	7	0.01093	0.75561	0.76627	35	7	0.00897	0.73480	0.79978
25	3	0.01144	0.69498	0.87993	30	8	0.01143	0.77620	0.73280	35	8	0.00931	0.75139	0.77112
25	4	0.01198	0.71587	0.83979	30	9	0.01198	0.79808	0.69934	35	9	0.00966	0.76878	0.74245
25	5	0.01258	0.73799	0.79963	30	10	0.01258	0.82141	0.66586	35	10	0.01005	0.78707	0.71378
25	6	0.01325	0.76156	0.75946	30	11	0.01325	0.84640	0.63239	35	11	0.01047	0.80638	0.68511
25	7	0.01399	0.78680	0.71927	30	12	0.01399	0.87329	0.59891	35	12	0.01093	0.82680	0.65644
25	8	0.01482	0.81399	0.67908	30	13	0.01482	0.90237	0.56543	35	13	0.01143	0.84849	0.62776
25	9	0.01575	0.84347	0.63888	30	14	0.01575	0.93399	0.53195	35	14	0.01198	0.87159	0.59908
25	10	0.01681	0.87564	0.59867	30	15	0.01681	0.96858	0.49846	35	15	0.01258	0.89629	0.57040
25	11	0.01802	0.91098	0.55846	30	16	0.01802	1.00667	0.46498	35	16	0.01325	0.92281	0.54171
25	12	0.01942	0.95013	0.51825	30	17	0.01942	1.04895	0.43149	35	17	0.01399	0.95139	0.51303
25	13	0.02105	0.99388	0.47804	30	18	0.02105	1.09629	0.39800	35	18	0.01481	0.98236	0.48435
25	14	0.02298	1.04330	0.43782	30	19	0.02298	1.14985	0.36451	35	19	0.01575	1.01608	0.45566
25	15	0.02531	1.09981	0.39760	30	20	0.02531	1.21117	0.33103	35	20	0.01681	1.05302	0.42697
25	16	0.02816	1.16538	0.35739	30	21	0.02816	1.28241	0.29755	35	21	0.01802	1.09375	0.39829
25	17	0.03173	1.24283	0.31718	30	22	0.03173	1.36667	0.26406	35	22	0.01941	1.13901	0.36960
25	18	0.03633	1.33642	0.27697	30	23	0.03633	1.46860	0.23059	35	23	0.02105	1.18973	0.34092
25	19	0.04250	1.45285	0.23677	30	24	0.04250	1.59551	0.19712	35	24	0.02298	1.24717	0.31223

35	25	0.02531	1.31299	0.28355	40	22	0.01399	1.02329	0.44868	45	14	0.00810	0.80214	0.68845
35	26	0.02815	1.38952	0.25486	40	23	0.01481	1.05608	0.42359	45	15	0.00837	0.81776	0.66616
35	27	0.03172	1.48011	0.22618	40	24	0.01575	1.09183	0.39850	45	16	0.00866	0.83412	0.64388
35	28	0.03633	1.58976	0.19751	40	25	0.01680	1.13101	0.37341	45	17	0.00897	0.85127	0.62159
35	29	0.04250	1.72638	0.16884	40	26	0.01802	1.17425	0.34832	45	18	0.00930	0.86929	0.59930
35	30	0.05119	1.90345	0.14019	40	27	0.01941	1.22233	0.32324	45	19	0.00966	0.88827	0.57701
35	31	0.06433	2.14617	0.11155	40	28	0.02105	1.27625	0.29815	45	20	0.01005	0.90831	0.55472
35	32	0.08650	2.50913	0.08296	40	29	0.02298	1.33735	0.27306	45	21	0.01047	0.92952	0.53243
35	33	0.13177	3.14234	0.05446	40	30	0.02531	1.40740	0.24797	45	22	0.01093	0.95204	0.51014
40	0	0.00628	0.63361	1.00000	40	31	0.02815	1.48890	0.22289	45	23	0.01143	0.97600	0.48784
40	1	0.00644	0.64524	0.97510	40	32	0.03172	1.58541	0.19781	45	24	0.01198	1.00158	0.46555
40	2	0.00660	0.65693	0.95011	40	33	0.03633	1.70228	0.17273	45	25	0.01258	1.02899	0.44326
40	3	0.00678	0.66883	0.92509	40	34	0.04250	1.84798	0.14766	45	26	0.01324	1.05848	0.42096
40	4	0.00697	0.68102	0.90005	40	35	0.05119	2.03689	0.12260	45	27	0.01399	1.09032	0.39867
40	5	0.00717	0.69356	0.87501	40	36	0.06433	2.29595	0.09755	45	28	0.01481	1.12486	0.37638
40	6	0.00738	0.70650	0.84995	40	37	0.08650	2.68351	0.07255	45	29	0.01575	1.16254	0.35408
40	7	0.00761	0.71989	0.82489	40	38	0.13177	3.35992	0.04762	45	30	0.01680	1.20387	0.33179
40	8	0.00785	0.73379	0.79982	45	0	0.00558	0.63288	1.00000	45	31	0.01802	1.24950	0.30950
40	9	0.00810	0.74823	0.77475	45	1	0.00570	0.64323	0.97787	45	32	0.01941	1.30025	0.28720
40	10	0.00837	0.76328	0.74968	45	2	0.00583	0.65360	0.95567	45	33	0.02105	1.35721	0.26491
40	11	0.00866	0.77901	0.72461	45	3	0.00597	0.66411	0.93343	45	34	0.02298	1.42176	0.24262
40	12	0.00897	0.79546	0.69953	45	4	0.00612	0.67483	0.91119	45	35	0.02531	1.49582	0.22033
40	13	0.00930	0.81273	0.67445	45	5	0.00627	0.68581	0.88893	45	36	0.02815	1.58201	0.19804
40	14	0.00966	0.83090	0.64937	45	6	0.00643	0.69707	0.86667	45	37	0.03172	1.68411	0.17576
40	15	0.01005	0.85005	0.62429	45	7	0.00660	0.70867	0.84440	45	38	0.03633	1.80779	0.15347
40	16	0.01047	0.87030	0.59920	45	8	0.00678	0.72063	0.82213	45	39	0.04250	1.96204	0.13120
40	17	0.01093	0.89177	0.57412	45	9	0.00697	0.73299	0.79985	45	40	0.05119	2.16210	0.10893
40	18	0.01143	0.91460	0.54903	45	10	0.00717	0.74578	0.77758	45	41	0.06433	2.43654	0.08668
40	19	0.01198	0.93895	0.52395	45	11	0.00738	0.75905	0.75530	45	42	0.08650	2.84723	0.06446
40	20	0.01258	0.96503	0.49886	45	12	0.00761	0.77283	0.73302	45	43	0.13177	3.56424	0.04231
40	21	0.01325	0.99305	0.47377	45	13	0.00784	0.78718	0.71073					

APPENDIX H

COMPUTER LISTING OF THE NEWTON–RAPHSON METHOD

```
      implicit real*8(a-h,o-z)
      dimension t(100),s(100)
      common teta
      print *, 'what is the number of observations?'
      read *,n
      print *, 'please enter failure time associated
     % with each observation,pressing CR after each one'
      do 10 i=1,n
      read *,t(i)
      print *,i,':',t(i)
10    continue
      gam=3.0
      print *, gam
11    gkap=gam
      print *,'gamma:', gkap
      gam=gkap-dif(gkap,n,t)/dp(gkap,n,t)
      ep=gkap-gam
      epp=abs(ep)
      print *,'ep',ep
      if(epp.le.0.000001) goto 20
      goto 11
20    print *,'estimated gamma',gam
      fundif=dif(gam,n,t)
      print *,'value of difference func.',fundif
      print *,'theta:',teta
      print *,' reliabilities at given times'
      do 22 i=1,n
      s(i)=exp(-t(i)**gam/teta)
      print *,'s',i,':',s(i)
```

Reliability Engineering, Second Edition. Elsayed A. Elsayed.

```
22    continue
      stop
      end
c
      function dif(gg,m,tt)
      implicit real*8(a-h,o-z)
      dimension tt(m)
      common teta
c     print *,'n is',m
      s1=0.0
      s2=0.0
      s3=0.0
      do 123 i=1,m
      s1=s1+tt(i)**gg*dlog(tt(i))
      s2=s2+tt(i)**gg
123   s3=s3+dlog(tt(i))
      print *,'sums',s1,s2,s3
      teta=s2/m
      dffif=s1/s2-s3/m-1/gg
      print *,'d func.',dffif
      dif=dffif
      return
      end
c
c
      function dp(ggg,mm,tt)
      implicit real*8(a-h,o-z)
      dimension tt(mm)
      sum1=0.0
      sum2=0.0
      sum3=0.0
      do 125 i=1,mm
      sum1=sum1+tt(i)**ggg*dlog(tt(i))**2
      sum2=sum2+tt(i)**ggg
125   sum3=sum3+tt(i)**ggg*dlog(tt(i))
      dpp=(sum1*sum2-sum3**2)/sum2**2+1/ggg**2
      dp=dpp
      return
      end
```

APPENDIX I

COEFFICIENTS (a_i AND b_i) OF THE BEST ESTIMATES OF THE MEAN (μ) AND STANDARD DEVIATION (σ) IN CENSORED SAMPLES UP TO $n = 20$ FROM A NORMAL POPULATION*

***n* = 2**			
n* – *r		$t_{(1)}$	$t_{(2)}$
0	μ	0.5000	0.5000
	σ	–0.8862	0.8862

***n* = 3**				
n* – *r		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$
0	μ	0.3333	0.3333	0.3333
	σ	–0.5908	0.0000	0.5908
1	μ	0.0000	1.0000	
	σ	–1.1816	1.1816	

* From A. E. Sarhan and B. G. Greenberg, "Estimation of location and scale parameters by order statistics from singly and doubly censored samples, parts I and II." *Annals of Mathematics Statistics* Part I, Vol. 27, No. 1 (1956), pp. 427–451. Reproduced by permission of *Annals of Mathematical Statistics*.

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$n = 4$					
$n - r$		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$
0	μ	0.2500	0.2500	0.2500	0.2500
	σ	–0.4539	–0.1102	0.1102	0.4539
1	μ	0.1161	0.2408	0.6431	
	σ	–0.6971	–0.1268	0.8239	
2	μ	–0.4506	1.4056		
	σ	–1.3654	1.3654		

$n = 5$						
$n - r$		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$
0	μ	0.2000	0.2000	0.2000	0.2000	0.2000
	σ	–0.3724	–0.1352	0.0000	0.1352	0.3724
1	μ	0.1252	0.1830	0.2147	0.4771	
	σ	–0.5117	–0.1668	0.0274	0.6511	
2	μ	–0.0638	0.1498	0.9139		
	σ	–0.7696	–0.2121	0.9817		
3	μ	–0.7411	1.7411			
	σ	–1.4971	1.4971			

$n = 6$							
$n - r$		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$
0	μ	0.1667	0.1667	0.1667	0.1667	0.1667	0.1667
	σ	–0.3175	–0.1386	–0.0432	0.0432	0.1386	0.3175
1	μ	0.1183	0.1510	0.1680	0.1828	0.3799	
	σ	–0.4097	–0.1685	0.0406	0.0740	0.5448	
2	μ	0.0185	0.1226	0.1761	0.6828		
	σ	–0.5528	–0.2091	0.0290	0.7909		
3	μ	–0.2159	0.0649	1.1511			
	σ	–0.8244	–0.2760	1.1004			
4	μ	–1.0261	2.0261				
	σ	–1.5988	1.5988				

n = 7								
n* – *r		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$	$t_{(7)}$
0	μ	0.1429	0.1429	0.1429	0.1429	0.1429	0.1429	0.1429
	σ	–0.2778	–0.1351	–0.0625	0.0000	0.0625	0.1351	0.2778
1	μ	0.1088	0.1295	0.1400	0.1487	0.1571	0.3159	
	σ	–0.3440	–0.1610	–0.0681	0.0114	0.0901	0.4716	
2	μ	0.0465	0.1072	0.1375	0.1626	0.5462		
	σ	–0.4370	–0.1943	–0.0718	0.0321	0.6709		
3	μ	–0.0738	0.0677	0.1375	0.8686			
	σ	–0.5848	–0.2428	–0.0717	0.8994			
4	μ	–0.3474	–0.0135	1.3609				
	σ	–0.8682	–0.3269	1.1951				
5	μ	–1.2733	2.2733					
	σ	–1.6812	1.6812					

n = 8									
n* – *r		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$	$t_{(7)}$	$t_{(8)}$
0	μ	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250
	σ	–0.2476	–0.1294	–0.0713	–0.0230	0.0230	0.0713	0.1294	0.2476
1	μ	0.0997	0.1139	0.1208	0.1265	0.1318	0.1370	0.2704	
	σ	–0.2978	–0.1515	–0.0796	–0.0200	0.0364	0.0951	0.4175	
2	μ	0.0569	0.0962	0.1153	0.1309	0.1451	0.4555		
	σ	–0.3638	–0.1788	–0.0881	–0.0132	0.0570	0.5868		
3	μ	–0.0167	0.0677	0.1084	0.1413	0.6993			
	σ	–0.4586	–0.2156	–0.0970	0.0002	0.7709			
4	μ	–0.1549	0.0176	0.1001	1.0372				
	σ	–0.6110	–0.2707	–0.1061	0.9878				
5	μ	–0.4632	–0.0855	1.5487					
	σ	–0.9045	–0.3690	1.2735					
6	μ	–1.4915	2.4915						
	σ	–1.7502	1.7502						

						***n* = 9**				
n* – *r		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$	$t_{(7)}$	$t_{(8)}$	$t_{(9)}$
0	μ	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111
	σ	–0.2237	–0.1233	–0.0751	–0.0360	0.0000	0.0360	0.0751	0.1233	0.2237
1	μ	0.0915	0.1018	0.1067	0.1106	0.1142	0.1177	0.1212	0.2365	
	σ	–0.2633	–0.1421	–0.0841	–0.0370	0.0062	0.0492	0.0954	0.3757	
2	μ	0.0602	0.0876	0.1006	0.1110	0.1204	0.1294	0.3909		
	σ	–0.3129	–0.1647	–0.0938	–0.0364	0.0160	0.0678	0.5239		
3	μ	0.0104	0.0660	0.0923	0.1133	0.1320	0.5860			
	σ	–0.3797	–0.1936	–0.1048	–0.0333	0.0317	0.6797			
4	μ	–0.0731	0.0316	0.0809	0.1199	0.8408				
	σ	–0.4766	–0.2335	–0.1181	–0.0256	0.8537				
5	μ	–0.2272	–0.0284	0.0644	1.1912					
	σ	–0.6330	–0.2944	–0.1348	1.0622					
6	μ	–0.5664	–0.1521	1.7185						
	σ	–0.9355	–0.4047	1.3402						
7	μ	–1.6868	2.6868							
	σ	–1.8092	1.8092							

						***n* = 10**					
n* – *r		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$	$t_{(7)}$	$t_{(8)}$	$t_{(9)}$	$t_{(10)}$
0	μ	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
	σ	–0.2044	–0.1172	–0.0763	–0.0436	–0.0142	0.0142	0.0436	0.0763	0.1172	0.2044
1	μ	0.0843	0.0921	0.0957	0.0986	0.1011	0.1036	0.1060	0.1085	0.2101	
	σ	–0.2364	–0.1334	–0.0851	–0.0465	–0.0119	0.0215	0.0559	0.0937	0.3423	
2	μ	0.0605	0.0804	0.0898	0.0972	0.1037	0.1099	0.1161	0.3424		
	σ	–0.2753	–0.1523	–0.0947	–0.0488	–0.0077	0.0319	0.0722	0.4746		
3	μ	0.0244	0.0636	0.0818	0.0962	0.1089	0.1207	0.5045			
	σ	–0.3252	–0.1758	–0.1058	–0.0502	–0.0006	0.0469	0.6107			
4	μ	–0.0316	0.0383	0.0707	0.0962	0.1185	0.7078				
	σ	–0.3930	–0.2063	–0.1192	–0.0501	0.0111	0.7576				
5	μ	–0.1240	–0.0016	0.0549	0.0990	0.9718					
	σ	–0.4919	–0.2491	–0.1362	–0.0472	0.9243					
6	μ	–0.2923	–0.0709	0.0305	1.3327						
	σ	–0.6520	–0.3150	–0.1593	1.1263						
7	μ	–0.6596	–0.2138	1.8734							
	σ	–0.9625	–0.4357	1.3981							
8	μ	–1.8634	2.8634								
	σ	–1.8608	1.8608								

$n-r$		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$	$t_{(7)}$	$t_{(8)}$	$t_{(9)}$	$t_{(10)}$	$t_{(11)}$
							***n* = 11**					
0	μ	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909
	σ	−0.1883	−0.1115	−0.0760	−0.0481	−0.0234	0.0000	0.0234	0.0481	0.0760	0.1115	0.1883
1	μ	0.0781	0.0841	0.0869	0.0891	0.0910	0.0928	0.0945	0.0963	0.0982	0.1891	
	σ	−0.2149	−0.1256	−0.0843	−0.0519	−0.0233	0.0038	0.0309	0.0593	0.0911	0.3149	
2	μ	0.0592	0.0744	0.0814	0.0869	0.0917	0.0962	0.1005	0.1049	0.3047		
	σ	−0.2463	−0.1417	−0.0934	−0.0555	−0.0220	0.0095	0.0409	0.0736	0.4349		
3	μ	0.0320	0.0609	0.0741	0.0845	0.0935	0.1020	0.1101	0.4430			
	σ	−0.2852	−0.1610	−0.1038	−0.0589	0.0194	0.0178	0.0545	0.5562			
4	μ	0.0082	0.0415	0.0642	0.0820	0.0974	0.1116	0.6116				
	σ	−0.3357	−0.1854	−0.1163	−0.0621	−0.0146	0.0299	0.6842				
5	μ	−0.0698	0.0128	0.0504	0.0797	0.1049	0.8220					
	σ	−0.4045	−0.2175	−0.1317	−0.0647	−0.0061	0.8246					
6	μ	−0.1702	−0.0323	0.0303	0.0786	1.0937						
	σ	−0.5053	−0.2627	−0.1519	−0.0657	0.9857						
7	μ	−0.3516	−0.1104	−0.0016	1.4636							
	σ	−0.6687	−0.3331	−0.1807	1.1825							
8	μ	−0.7445	−0.2712	2.0157								
	σ	−0.9862	−0.4630	1.4402								
9	μ	−2.0245	3.0245									
	σ	−1.9065	1.9065									

$n-r$		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$	$t_{(7)}$	$t_{(8)}$	$t_{(9)}$	$t_{(10)}$	$t_{(11)}$	$t_{(12)}$
							***n* = 12**						
0	μ	0.0833	0.0833	0.0833	0.0833	0.0833	0.0833	0.0833	0.0833	0.0833	0.0833	0.0833	0.0833
	σ	−0.1748	−0.1061	−0.0749	−0.0506	−0.0294	−0.0097	0.0097	0.0294	0.0506	0.0749	0.1061	0.1748
1	μ	0.0726	0.0775	0.0796	0.0813	0.0828	0.0842	0.0855	0.0868	0.0882	0.0896	0.1719	
	σ	−0.1972	−0.1185	−0.0827	−0.0548	−0.0305	−0.0079	0.0142	0.0367	0.0608	0.0881	0.2919	
2	μ	0.0574	0.0693	0.0747	0.0789	0.0825	0.0859	0.0891	0.0923	0.0956	0.2745		
	σ	−0.2232	−0.1324	−0.0911	−0.0590	−0.0310	−0.0050	0.0203	0.0461	0.0733	0.4020		
3	μ	0.0360	0.0581	0.0682	0.0759	0.0827	0.0888	0.0948	0.1006	0.3950			
	σ	−0.2545	−0.1487	−0.1007	−0.0633	−0.0308	−0.0007	0.0286	0.0582	0.5119			
4	μ	0.0057	0.0428	0.0595	0.0724	0.0836	0.0938	0.1036	0.5386				
	σ	−0.2937	−0.1686	−0.1119	−0.0678	−0.0296	0.0058	0.0400	0.6259				
5	μ	−0.0382	0.0210	0.0477	0.0684	0.0861	0.1022	0.7128					
	σ	−0.3448	−0.1939	−0.1255	−0.0726	−0.0267	0.0155	0.7479					
6	μ	−0.1048	−0.0109	0.0313	0.0637	0.0915	0.9292						
	σ	−0.4146	−0.2274	−0.1428	−0.0774	−0.0210	0.8833						
7	μ	−0.2125	−0.0609	0.0070	0.0589	1.2075							
	σ	−0.5171	−0.2749	−0.1659	−0.0820	1.0399							
8	μ	−0.4059	−0.1472	−0.0321	1.5852								
	σ	−0.6836	−0.3493	−0.1996	1.2324								
9	μ	−0.8225	−0.3249	2.1474									
	σ	−1.0075	−0.4874	1.4948									
10	μ	−2.1728	3.1728										
	σ	−1.9474	1.9474										

$n-r$		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$	$t_{(7)}$	$t_{(8)}$	$t_{(9)}$	$t_{(10)}$	$t_{(11)}$	$t_{(12)}$	$t_{(13)}$
							$n = 13$							
0	μ	0.0769	0.0769	0.0769	0.0769	0.0769	0.0769	0.0769	0.0769	0.0769	0.0769	0.0769	0.0769	0.0769
	σ	−0.1632	−0.1013	−0.0735	−0.0520	−0.0335	−0.0164	0.0000	0.0164	0.0335	0.0520	0.0735	0.1013	0.1632
1	μ	0.0679	0.0718	0.0735	0.0749	0.0761	0.0771	0.0781	0.0792	0.0802	0.0813	0.0824	0.1576	
	σ	−0.1824	−0.1122	−0.0806	−0.0563	−0.0353	−0.1060	0.0026	0.0212	0.0404	0.0612	0.0850	0.2724	
2	μ	0.0552	0.0648	0.0691	0.0724	0.0752	0.0778	0.0803	0.0827	0.0852	0.0877	0.2497		
	σ	−0.2043	−0.1243	−0.0884	−0.0607	−0.0368	−0.0148	0.0063	0.0273	0.0490	0.0723	0.3743		
3	μ	0.0380	0.0555	0.0633	0.0693	0.0745	0.0792	0.0836	0.0880	0.0924	0.3564			
	σ	−0.2301	−0.1382	−0.0970	−0.0653	−0.0379	−0.0128	0.0113	0.0352	0.0598	0.4750			
4	μ	0.0144	0.0430	0.0557	0.0655	0.0739	0.0816	0.0888	0.0958	0.4813				
	σ	−0.2616	−0.1549	−0.1071	−0.0703	−0.0386	−0.0095	0.0182	0.0456	0.5781				
5	μ	−0.0185	0.0259	0.0457	0.0610	0.0740	0.0857	0.0968	0.6294					
	σ	−0.3011	−0.1754	−0.1191	−0.0758	−0.0386	−0.0046	0.0278	0.6867					
6	μ	−0.0659	0.0020	0.0322	0.0553	0.0750	0.0928	0.8085						
	σ	−0.3528	−0.2015	−0.1339	−0.0819	−0.0374	0.0032	0.8042						
7	μ	−0.1371	−0.0330	0.0132	0.0484	0.0784	1.0301							
	σ	−0.4236	−0.2363	−0.1528	−0.0888	−0.0341	0.9355							
8	μ	−0.2516	−0.0876	−0.0151	0.0400	1.3143								
	σ	−0.5276	−0.2859	−0.1785	−0.0964	1.0884								
9	μ	−0.4561	−0.1817	−0.0610	1.6988									
	σ	−0.6969	−0.3638	−0.2165	1.2773									
10	μ	−0.8946	−0.3753	2.2699										
	σ	−1.0266	−0.5094	1.5360										
11	μ	−2.3101	3.3101											
	σ	−1.9845	1.9845											

							$n = 14$								
$n-r$		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$	$t_{(7)}$	$t_{(8)}$	$t_{(9)}$	$t_{(10)}$	$t_{(11)}$	$t_{(12)}$	$t_{(13)}$	$t_{(14)}$
0	μ	0.0714	0.0714	0.0714	0.0714	0.0714	0.0714	0.0714	0.0714	0.0714	0.0714	0.0714	0.0714	0.0714	0.0714
	σ	–0.1532	–0.0968	–0.0717	–0.0526	–0.0362	–0.0212	–0.0070	0.0070	0.0212	0.0362	0.0526	0.0717	0.0968	0.1532
1	μ	0.0637	0.0669	0.0683	0.0694	0.0704	0.0712	0.0721	0.0728	0.0736	0.0745	0.0753	0.0762	0.1455	
	σ	–0.1698	–0.1065	–0.0784	–0.0568	–0.0384	–0.0216	–0.0056	0.0100	0.0259	0.0426	0.0609	0.0820	0.2556	
2	μ	0.0530	0.0609	0.0643	0.0670	0.0692	0.0713	0.0732	0.0751	0.0770	0.0789	0.0809	0.2291		
	σ	–0.1885	–0.1171	–0.0854	–0.0612	–0.0404	–0.0215	–0.0036	0.0140	0.0319	0.0505	0.0707	0.3506		
3	μ	0.0388	0.0529	0.0592	0.0639	0.0680	0.0717	0.0752	0.0785	0.0819	0.0852	0.3247			
	σ	–0.2102	–0.1292	–0.0933	–0.0658	–0.0423	–0.0209	–0.0006	0.0192	0.0393	0.0601	0.4438			
4	μ	0.0199	0.0426	0.0526	0.0602	0.0667	0.0726	0.0782	0.0835	0.0887	0.4350				
	σ	–0.2361	–0.1434	–0.1023	–0.0709	–0.0440	–0.0196	0.0035	0.0260	0.0487	0.5382				
5	μ	–0.0057	0.0288	0.0440	0.0557	0.0655	0.0744	0.0828	0.0908	0.5637					
	σ	–0.2678	–0.1604	–0.1129	–0.0765	–0.0455	–0.0174	0.0092	0.0350	0.6363					
6	μ	–0.0411	0.0102	0.0328	0.0500	0.0646	0.0777	0.0899	0.7159						
	σ	–0.3077	–0.1815	–0.1256	–0.0829	–0.0466	–0.0137	0.0172	0.7407						
7	μ	–0.0915	–0.0158	0.0175	0.0429	0.0643	0.0835	0.8992							
	σ	–0.3599	–0.2084	–0.1414	–0.0903	–0.0469	–0.0077	0.8546							
8	μ	–0.1670	–0.0537	–0.0040	0.0338	0.0655	1.1255								
	σ	–0.4317	–0.2444	–0.1618	–0.0990	–0.0457	0.9825								
9	μ	–0.2879	–0.1127	–0.0360	0.0218	1.4148									
	σ	–0.5372	–0.2959	–0.1898	–0.1094	1.1322									
10	μ	–0.5027	–0.2142	–0.0886	1.8054										
	σ	–0.7091	–0.3771	–0.2318	1.3180										
11	μ	–0.9616	–0.4228	2.3843											
	σ	–1.0441	–0.5293	1.5734											
12	μ	–2.4378	3.4378												
	σ	–2.0182	2.0182												

								$n = 15$								
$n - r$		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$	$t_{(7)}$	$t_{(8)}$	$t_{(9)}$	$t_{(10)}$	$t_{(11)}$	$t_{(12)}$	$t_{(13)}$	$t_{(14)}$	$t_{(15)}$
0	μ	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667
	σ	–0.1444	–0.0927	–0.0699	–0.0526	0.0379	–0.0247	–0.0122	0.0000	0.0122	0.0247	0.0379	0.0526	0.0699	0.0927	0.1444
1	μ	0.0599	0.0627	0.0639	0.0648	0.0655	0.0662	0.0669	0.0675	0.0682	0.0688	0.0695	0.0702	0.0709	0.0709	
	σ	–0.1590	–0.1013	–0.0760	–0.0568	–0.0404	–0.0256	–0.0116	0.0019	0.0154	0.0293	0.0440	0.0602	0.0791	0.0791	
2	μ	0.0508	0.0574	0.0602	0.0624	0.0642	0.0659	0.0675	0.0690	0.0704	0.0719	0.0735	0.0751	0.2116		
	σ	–0.1752	–0.1108	–0.0825	–0.0610	–0.0427	–0.0262	–0.0106	0.0044	0.0195	0.0349	0.0512	0.0690	0.3300		
3	μ	0.0390	0.0506	0.0556	0.0595	0.0628	0.0657	0.0685	0.0711	0.0737	0.0763	0.0790	0.2982			
	σ	–0.1937	–0.1214	–0.0897	–0.0655	–0.0450	–0.0265	–0.0091	0.0078	0.0246	0.0417	0.0598	0.4169			
4	μ	0.0234	0.0418	0.0498	0.0560	0.0611	0.0658	0.0701	0.0743	0.0784	0.0824	0.0397				
	σ	–0.2154	–0.1336	–0.0977	–0.0705	–0.0473	–0.0264	–0.0068	0.0122	0.0310	0.0502	0.5042				
5	μ	0.0030	0.0305	0.0425	0.0516	0.0593	0.0663	0.0727	0.0789	0.0849	0.5104					
	σ	–0.2414	–0.1481	–0.1071	–0.0760	–0.0496	–0.0258	–0.0035	0.0180	0.0393	0.5940					
6	μ	–0.0244	0.0155	0.0330	0.0462	0.0574	0.0674	0.0767	0.0856	0.6425						
	σ	–0.2733	–0.1654	–0.1181	–0.0822	–0.0518	–0.0244	0.0012	0.0258	0.6882						
7	μ	–0.0621	–0.0046	0.0205	0.0395	0.0555	0.0698	0.0830	0.7983							
	σ	–0.3136	–0.1870	–0.1315	–0.0894	–0.0538	–0.0219	0.0079	0.7892							
8	μ	–0.1155	–0.0326	0.0036	0.0309	0.0539	0.0743	0.9854								
	σ	–0.3664	–0.2146	–0.1482	–0.0979	–0.0555	–0.0174	0.9001								
9	μ	–0.1950	–0.0732	–0.0203	0.0196	0.0531	1.2157									
	σ	–0.4390	–0.2518	–0.1700	–0.1082	–0.0562	1.0252									
10	μ	–0.3217	–0.1364	–0.0560	0.0043	1.5097										
	σ	–0.5459	–0.3050	–0.2002	–0.1211	1.1722										
11	μ	–0.5462	–0.2448	–0.1148	1.9058											
	σ	–0.7201	–0.3892	–0.2458	1.3552											
12	μ	–1.0242	–0.4676	2.4918												
	σ	–1.0601	–0.5477	1.6077												
13	μ	–2.5574	3.5574													
	σ	–2.0493	2.0493													

									***n* = 16**								
n* − *r		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$	$t_{(7)}$	$t_{(8)}$	$t_{(9)}$	$t_{(10)}$	$t_{(11)}$	$t_{(12)}$	$t_{(13)}$	$t_{(14)}$	$t_{(15)}$	$t_{(16)}$
0	μ	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625
	σ	−0.1366	−0.0889	−0.0681	−0.0524	−0.0391	−0.0272	−0.0160	−0.0053	0.0053	0.0160	0.0272	0.0391	0.0524	0.0681	0.0889	0.1366
1	μ	0.0566	0.0589	0.0599	0.0607	0.0613	0.0619	0.0625	0.0630	0.0635	0.0640	0.0645	0.0651	0.0657	0.0663	0.1261	
	σ	−0.1495	−0.0967	−0.0737	−0.0563	−0.0416	−0.0284	−0.0161	−0.0042	0.0075	0.0193	0.0316	0.0447	0.0593	0.0763	0.2279	
2	μ	0.0487	0.0543	0.0566	0.0585	0.0600	0.0614	0.0626	0.0638	0.0650	0.0662	0.0674	0.0687	0.0700	0.1967		
	σ	−0.1637	−0.1051	−0.0797	−0.0604	−0.0441	−0.0294	−0.0158	−0.0027	0.0103	0.0233	0.0369	0.0513	0.0671	0.3120		
3	μ	0.0386	0.0483	0.0525	0.0557	0.0584	0.0608	0.0630	0.0652	0.0673	0.0693	0.0714	0.0736	0.2757			
	σ	−0.1797	−0.1145	−0.0862	−0.0647	−0.0466	−0.0303	−0.0151	−0.0006	0.0138	0.0282	0.0432	0.0590	0.3935			
4	μ	0.0257	0.0408	0.0474	0.0524	0.0566	0.0604	0.0638	0.0671	0.0704	0.0736	0.0768	0.3649				
	σ	−0.1982	−0.1252	−0.0935	−0.0694	−0.0491	−0.0310	−0.0140	0.0022	0.0182	0.0343	0.0508	0.4748				
5	μ	0.0090	0.0313	0.0410	0.0484	0.0545	0.0601	0.0652	0.0700	0.0747	0.4664						
	σ	−0.2200	−0.1376	−0.1018	−0.0747	−0.0518	−0.0313	−0.0123	0.0060	0.0239	0.0419	0.5577					
6	μ	−0.0129	0.0191	0.0329	0.0434	0.0522	0.0601	0.0673	0.0742	0.0808	0.5829						
	σ	−0.2461	−0.1523	−0.1114	−0.0806	−0.0546	−0.0313	−0.0097	0.0110	0.0312	0.6439						
7	μ	−0.0420	0.0030	0.0225	0.0373	0.0496	0.0606	0.0707	0.0803	0.7180							
	σ	−0.2782	−0.1700	−0.1229	−0.0874	−0.0575	−0.0307	−0.0059	0.0177	0.7350							
8	μ	−0.0817	−0.0185	0.0089	0.0295	0.0467	0.0621	0.0762	0.8769								
	σ	−0.3189	−0.1920	−0.1369	−0.0954	−0.0604	−0.0292	−0.0004	0.8333								
9	μ	−0.1379	−0.0484	−0.0096	0.0194	0.0438	0.0653	1.0674									
	σ	−0.3723	−0.2204	−0.1545	−0.1049	−0.0632	−0.0262	0.9415									
10	μ	−0.2211	−0.0916	−0.0358	0.0061	0.0410	1.3015										
	σ	−0.4457	−0.2586	−0.1776	−0.1167	−0.0657	1.0642										
11	μ	−0.3534	−0.1587	−0.0750	−0.0125	1.5996											
	σ	−0.5538	−0.3134	−0.2097	−0.1319	1.2088											
12	μ	−0.5869	−0.2739	−0.1398	2.0006												
	σ	−0.7303	−0.4004	−0.2586	1.3894												
13	μ	−1.0829	−0.5101	2.5931													
	σ	−1.0748	−0.5645	1.6394													
14	μ	−2.6696	3.6696														
	σ	−2.0779	2.0779														

$n = 17$																		
$n - r$		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$	$t_{(7)}$	$t_{(8)}$	$t_{(9)}$	$t_{(10)}$	$t_{(11)}$	$t_{(12)}$	$t_{(13)}$	$t_{(14)}$	$t_{(15)}$	$t_{(16)}$	$t_{(17)}$
0	μ	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588	0.0588
	σ	–0.1297	–0.0854	–0.0663	–0.0519	–0.0398	–0.0290	–0.0189	–0.0094	0.0000	0.0094	0.0189	0.0290	0.0398	0.0519	0.0663	0.0854	0.1297
1	μ	0.0536	0.0596	0.0565	0.0571	0.0577	0.0582	0.0586	0.0590	0.0595	0.0599	0.0603	0.0607	0.0612	0.0617	0.0622	0.1183	
	σ	–0.1412	–0.0925	–0.0715	–0.0556	–0.0423	–0.0304	–0.0194	–0.0089	0.0014	0.0117	0.0222	0.0332	0.0450	0.0582	0.0737	0.2164	
2	μ	0.0468	0.0515	0.0535	0.0550	0.0563	0.0574	0.0585	0.0595	0.0605	0.0615	0.0624	0.0634	0.0645	0.0656	0.1837		
	σ	–0.1537	–0.1001	–0.0769	–0.0595	0.0448	–0.0317	–0.0196	–0.0080	0.0033	0.0146	0.0261	0.0381	0.0510	0.0653	0.296		
3	μ	0.0381	0.0463	0.0498	0.0525	0.0547	0.0567	0.0585	0.0603	0.0620	0.0636	0.0653	0.0670	0.0688	0.2564			
	σ	–0.1677	–0.1085	–0.0829	–0.0636	–0.0474	–0.0329	–0.0196	–0.0068	0.0057	0.0181	0.0308	0.0439	0.0580	0.3728			
4	μ	0.0271	0.0398	0.0453	0.0494	0.0528	0.0559	0.0588	0.0615	0.0641	0.0666	0.0692	0.0718	0.3378				
	σ	–0.1837	–0.1179	–0.0895	–0.0681	–0.0501	–0.0341	–0.0102	–0.0051	0.0087	0.0225	0.0364	0.0509	0.4491				
5	μ	0.0131	0.0317	0.0397	0.0457	0.0507	0.0552	0.0593	0.0632	0.0670	0.0707	0.0744	0.4294					
	σ	–0.2022	–0.1287	–0.0969	–0.0730	–0.0529	–0.0351	–0.0185	–0.0028	0.0126	0.0278	0.0433	0.5263					
6	μ	–0.0047	0.0214	0.0327	0.0411	0.0482	0.0545	0.0603	0.0658	0.0710	0.0762	0.5334						
	σ	–0.2241	–0.1412	–0.1055	–0.0786	–0.0560	–0.0359	–0.0173	0.0004	0.0176	0.0346	0.6059						
7	μ	–0.0278	0.0083	0.0239	0.0356	0.0454	0.0540	0.0620	0.0695	0.0767	0.6525							
	σ	–0.2504	–0.1561	–0.1154	–0.0849	–0.0592	–0.0364	–0.0154	0.0046	0.0240	0.6892							
8	μ	–0.0585	–0.0088	0.0126	0.0286	0.0421	0.0539	0.0648	0.0750	0.7903								
	σ	–0.2828	–0.1742	–0.1274	–0.0922	–0.0627	–0.0365	–0.0124	0.0105	0.7777								
9	μ	–0.1002	–0.0317	–0.0022	0.0199	0.0383	0.0545	0.0694	0.9521									
	σ	–0.3238	–0.1967	–0.1419	–0.1009	–0.0665	–0.0359	–0.0079	0.8736									
10	μ	–0.1589	–0.0633	–0.0223	0.0084	0.0340	0.0565	1.1456										
	σ	–0.3777	–0.2257	–0.1603	–0.1113	–0.0704	–0.0341	0.9796										
11	μ	–0.2457	–0.1091	–0.0506	–0.0070	0.0293	1.3831											
	σ	–0.4519	–0.2649	–0.1846	–0.1245	–0.0744	1.1002											
12	μ	–0.3832	–0.1799	–0.0932	–0.0287	1.6850												
	σ	–0.5612	–0.3212	–0.2185	–0.1418	1.2427												
13	μ	–0.6253	–0.3014	–0.1637	2.0904													
	σ	–0.7398	–0.4108	–0.2705	1.4211													
14	μ	–1.1383	–0.5506	2.6888														
	σ	–1.0885	–0.5802	1.6687														
15	μ	–2.7754	3.7754															
	σ	–2.1046	2.1046															

									$n = 18$										
$n - r$		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$	$t_{(7)}$	$t_{(8)}$	$t_{(9)}$	$t_{(10)}$	$t_{(11)}$	$t_{(12)}$	$t_{(13)}$	$t_{(14)}$	$t_{(15)}$	$t_{(16)}$	$t_{(17)}$	$t_{(18)}$
0	μ	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556
	σ	−0.1235	−0.0822	−0.0645	−0.0512	−0.0401	−0.0302	−0.0211	−0.0125	−0.0041	0.0041	0.0125	0.0211	0.0302	0.0401	0.0512	0.0645	0.0822	0.1235
1	μ	0.0509	0.0526	0.0534	0.0540	0.0544	0.0548	0.0552	0.0556	0.0559	0.0563	0.0566	0.0570	0.0574	0.0577	0.0582	0.0586	0.1113	
	σ	−0.1338	−0.0887	−0.0693	−0.0548	−0.0426	−0.0318	−0.0219	−0.0124	−0.0033	0.0058	0.0149	0.0243	0.0450	0.0570	0.0712	0.0712	0.2061	
2	μ	0.0449	0.0489	0.0507	0.0520	0.0531	0.0540	0.0549	0.0558	0.0566	0.0574	0.0582	0.0590	0.0598	0.0607	0.0616	0.1723		
	σ	−0.1449	−0.0955	−0.0743	−0.0584	−0.0451	−0.0333	−0.0224	−0.0121	−0.0021	0.0078	0.0178	0.0281	0.0389	0.0505	0.0634	0.2818		
3	μ	0.0373	0.0443	0.0474	0.0496	0.0515	0.0532	0.0547	0.0562	0.0576	0.0589	0.0603	0.0617	0.0631	0.0646	0.2396			
	σ	−0.1573	−0.1030	−0.0798	−0.0623	−0.0477	−0.0347	−0.0228	−0.0115	−0.0005	0.0103	0.0213	0.0325	0.0442	0.0568	0.3545			
4	μ	0.0279	0.0387	0.0433	0.0468	0.0497	0.0522	0.0546	0.0568	0.0589	0.0610	0.0631	0.0652	0.0674	0.3144				
	σ	−0.1713	−0.1114	−0.0858	−0.0665	−0.0504	−0.0361	−0.0230	−0.0105	0.0015	0.0135	0.0254	0.0377	0.0505	0.4264				
5	μ	0.0161	0.0317	0.0384	0.0434	0.0475	0.0512	0.0546	0.0578	0.0609	0.0639	0.0669	0.0698	0.3979					
	σ	−0.1872	−0.1209	−0.0925	−0.0711	−0.0533	−0.0375	−0.0230	−0.0092	0.0042	0.0173	0.0305	0.0440	0.4988					
6	μ	0.0013	0.0230	0.0323	0.0392	0.0450	0.0502	0.0549	0.0593	0.0636	0.0677	0.0718	0.4918						
	σ	−0.2058	−0.1318	−0.1001	−0.0763	−0.0564	−0.0388	−0.0226	−0.0073	0.0075	0.0221	0.0367	0.5728						
7	μ	−0.0176	0.0121	0.0247	0.0342	0.0421	0.0491	0.0555	0.0615	0.0673	0.0729	0.5980							
	σ	−0.2278	−0.1445	−0.1089	−0.0821	−0.0598	−0.0400	−0.0219	−0.0047	0.0119	0.0282	0.6497							
8	μ	−0.0149	−0.0019	0.0153	0.0281	0.0387	0.0482	0.0568	0.0648	0.0726	0.7194								
	σ	−0.2543	−0.1597	−0.1191	−0.0888	−0.0635	−0.0411	−0.0205	−0.0011	0.0176	0.7305								
9	μ	−0.0741	−0.0200	0.0031	0.0204	0.0348	0.0474	0.0590	0.0698	0.8597									
	σ	−0.2869	−0.1781	−0.1315	−0.0966	−0.0676	−0.0418	−0.0183	0.0039	0.8168									
10	μ	−0.1176	−0.0441	−0.0128	0.0106	0.0300	0.0471	0.0627	1.0241										
	σ	−0.3283	−0.2010	−0.1466	−0.1059	−0.0720	−0.0421	−0.0147	0.9107										
11	μ	−0.1788	−0.0775	−0.0343	−0.0022	0.0245	0.0479	1.2204											
	σ	−0.3827	−0.2307	−0.1657	−0.1173	−0.0770	−0.0414	1.0148											
12	μ	−0.2689	−0.1257	−0.0648	−0.0195	0.0180	1.4609												
	σ	−0.4576	−0.2707	−0.1911	−0.3317	−0.0824	1.1335												
13	μ	−0.4113	−0.2001	−0.1106	−0.0442	1.7662													
	σ	−0.5681	−0.3285	−0.2266	−0.1509	1.2741													
14	μ	−0.6615	−0.3276	−0.1867	2.1758														
	σ	−0.7486	−0.4205	−0.2815	1.4505														
15	μ	−1.1905	−0.5891	2.7796															
	σ	−1.1013	−0.5948	1.6960															
16	μ	−2.8756	3.8756																
	σ	−2.1294	2.1294																

										$n = 19$										
$n - r$		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$	$t_{(7)}$	$t_{(8)}$	$t_{(9)}$	$t_{(10)}$	$t_{(11)}$	$t_{(12)}$	$t_{(13)}$	$t_{(14)}$	$t_{(15)}$	$t_{(16)}$	$t_{(17)}$	$t_{(18)}$	$t_{(19)}$
0	μ	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526
	σ	–0.1178	–0.0792	–0.0628	–0.0505	–0.0402	–0.0312	–0.0228	–0.0150	–0.0074	0.0000	0.0074	0.0150	0.0228	0.0312	0.0402	0.0505	0.0628	0.0792	0.1178
1	μ	0.0485	0.0500	0.0506	0.0511	0.0519	0.0522	0.0525	0.0528	0.0531	0.0534	0.0537	0.0540	0.0543	0.0547	0.0550	0.0554	0.1052	0.1052	
	σ	–0.1272	–0.0851	–0.0672	–0.0538	–0.0427	–0.0328	–0.0238	–0.0152	–0.0070	0.0011	0.0092	0.0174	0.0259	0.0349	0.0447	0.0558	0.0689	0.1969	
2	μ	0.0431	0.0467	0.0482	0.0493	0.0502	0.0511	0.0518	0.0525	0.0532	0.0539	0.0546	0.0552	0.0559	0.0566	0.0574	0.0581	0.1623		
	σ	–0.1372	–0.0914	–0.0719	–0.0573	–0.0451	–0.0344	–0.0245	–0.0152	–0.0063	0.0025	0.0113	0.0202	0.0295	0.0392	0.0498	0.0616	0.2690		
3	μ	0.0365	0.0426	0.0451	0.0471	0.0487	0.0501	0.0514	0.0526	0.0538	0.0550	0.0561	0.0572	0.0584	0.0596	0.0608	0.2249			
	σ	–0.1482	–0.0982	–0.0769	–0.0609	–0.0477	–0.0359	–0.0252	–0.0150	–0.0053	0.0043	0.0139	0.0236	0.0336	0.0442	0.0556	0.3381			
4	μ	0.0283	0.0376	0.0415	0.0445	0.0469	0.0491	0.0510	0.0529	0.0547	0.0564	0.0582	0.0599	0.0616	0.0634	0.2940				
	σ	–0.1605	–0.1057	–0.0823	–0.0649	–0.0504	–0.0375	–0.0257	–0.0146	–0.0040	0.0065	0.0170	0.0276	0.0385	0.0499	0.4062				
5	μ	0.0182	0.0315	0.0371	0.0413	0.0448	0.0479	0.0507	0.0534	0.0559	0.0584	0.0609	0.0633	0.0658	0.3707					
	σ	–0.1744	–0.1141	–0.0884	–0.0692	–0.0532	–0.0391	–0.0262	–0.0140	–0.0022	0.0093	0.0207	0.0323	0.0442	0.4744					
6	μ	0.0057	0.0240	0.0318	0.0376	0.0424	0.0467	0.0506	0.0542	0.0577	0.0611	0.0644	0.0678	0.4561						
	σ	–0.1905	–0.1237	–0.0952	–0.0740	–0.0563	–0.0407	–0.0264	–0.0129	0.0000	0.0127	0.0253	0.0380	0.5437						
7	μ	–0.0100	0.0147	0.0253	0.0331	0.0396	0.0453	0.0506	0.0554	0.0601	0.0647	0.0691	0.5520							
	σ	–0.2091	–0.1347	–0.1030	–0.0794	–0.0597	–0.0423	–0.0264	–0.0114	0.0029	0.0170	0.0309	0.6152							
8	μ	–0.0298	–0.0032	0.0172	0.0276	0.0363	0.0439	0.0509	0.0574	0.0636	0.0696	0.6603								
	σ	–0.2312	–0.1476	–0.1120	–0.0855	–0.0634	–0.0439	–0.0261	–0.0093	0.0067	0.0224	0.6898								
9	μ	–0.0553	–0.0115	0.0070	0.0209	0.0324	0.0425	0.0516	0.0602	0.0684	0.7839									
	σ	–0.2578	–0.1630	–0.1226	–0.0925	–0.0675	–0.0454	–0.0252	0.0063	0.0118	0.7685									
10	μ	–0.0888	–0.0306	–0.0059	0.0125	0.0277	0.0411	0.0532	0.0646	0.9264										
	σ	–0.2907	–0.1817	–0.1353	–0.1007	–0.0720	–0.0468	–0.0237	–0.0021	0.8530										
11	μ	–0.1341	–0.0560	–0.0229	0.0017	0.0221	0.0399	0.0561	1.0931											
	σ	–0.3324	–0.2051	–0.1509	–0.1106	–0.0772	–0.0478	–0.0210	0.9450											
12	μ	–0.1976	–0.0910	–0.0459	–0.0124	0.0153	0.0395	1.2920												
	σ	–0.3873	–0.2353	–0.1708	–0.1228	–0.0831	–0.0482	1.0474												
13	μ	–0.2909	–0.1415	–0.0783	–0.0316	0.0070	1.5353													
	σ	–0.4629	–0.2762	–0.1971	–0.1384	–0.0898	1.1645													
14	μ	–0.4379	–0.2194	–0.1273	–0.0593	1.8438														
	σ	–0.5744	–0.3353	–0.2342	–0.1594	1.3034														
15	μ	–0.6958	–0.3527	–0.2086	2.2572															
	σ	–0.7568	–0.4295	–0.2918	1.4781															
16	μ	–1.2401	–0.6258	2.8659																
	σ	–1.1132	–0.6084	1.7216																
17	μ	–2.9705	3.9705																	
	σ	–2.1527	2.1527																	

											$n = 20$										
$n - r$		$t_{(1)}$	$t_{(2)}$	$t_{(3)}$	$t_{(4)}$	$t_{(5)}$	$t_{(6)}$	$t_{(7)}$	$t_{(8)}$	$t_{(9)}$	$t_{(10)}$	$t_{(11)}$	$t_{(12)}$	$t_{(13)}$	$t_{(14)}$	$t_{(15)}$	$t_{(16)}$	$t_{(17)}$	$t_{(18)}$	$t_{(19)}$	$t_{(20)}$
0	μ	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
	σ	–0.1128	–0.0765	–0.0611	–0.0497	–0.0402	–0.0318	–0.0241	–0.0169	–0.0101	–0.0033	0.0033	0.0101	0.0169	0.0241	0.0318	0.0402	0.0497	0.0611	0.0765	0.1128
1	μ	0.0462	0.0476	0.0482	0.0486	0.0489	0.0493	0.0495	0.0498	0.0501	0.0503	0.0506	0.0508	0.0511	0.0513	0.0516	0.0519	0.0522	0.0525	0.0996	
	σ	–0.1212	–0.0819	–0.0652	–0.0528	–0.0425	–0.0335	–0.0252	–0.0174	–0.0099	–0.0026	0.0046	0.0119	0.0193	0.0271	0.0354	0.0444	0.0546	0.0667	0.1184	
2	μ	0.0415	0.0446	0.0459	0.0469	0.0477	0.0484	0.0491	0.0497	0.0502	0.0508	0.0514	0.0519	0.0525	0.0531	0.0537	0.0544	0.0550	0.1533		
	σ	–0.1303	–0.0876	–0.0695	–0.0561	–0.0449	–0.0351	–0.0261	–0.0177	–0.0096	–0.0017	0.0061	0.0140	0.0221	0.0305	0.0394	0.0491	0.0599	0.2574		
3	μ	0.0356	0.0409	0.0431	0.0448	0.0462	0.0474	0.0485	0.0496	0.0506	0.0516	0.0525	0.0535	0.0544	0.0554	0.0564	0.0575	0.2119			
	σ	–0.1401	–0.0938	–0.0741	–0.0595	–0.0474	–0.0367	–0.0269	–0.0178	–0.0090	–0.0004	0.0080	0.0166	0.0253	0.0344	0.0440	0.0543	0.3233			
4	μ	0.0284	0.0365	0.0399	0.0424	0.0445	0.0463	0.0480	0.0496	0.0511	0.0526	0.0540	0.0554	0.0569	0.0584	0.0599	0.2762				
	σ	–0.1511	–0.1006	–0.0792	–0.0632	–0.0500	–0.0384	–0.0277	–0.0178	–0.0082	0.0011	0.0103	0.0196	0.0291	0.0389	0.0492	0.3880				
5	μ	0.0197	0.0311	0.0359	0.0395	0.0425	0.0451	0.0475	0.0497	0.0519	0.0539	0.0560	0.0580	0.0600	0.0621	0.3470					
	σ	–0.1634	–0.1081	–0.0847	–0.0673	–0.0528	–0.0401	–0.0285	–0.0176	–0.0071	0.0030	0.0131	0.0232	0.0335	0.0441	0.4526					
6	μ	0.0090	0.0246	0.0312	0.0361	0.0402	0.0438	0.0470	0.0501	0.0530	0.0558	0.0586	0.0613	0.0640	0.4254						
	σ	–0.1773	–0.1166	–0.0908	–0.0717	–0.0558	–0.0418	–0.0291	–0.0171	–0.0057	0.0055	0.0165	0.0275	0.0387	0.5179						
7	μ	–0.0042	0.0167	0.0255	0.0321	0.0375	0.0423	0.0466	0.0507	0.0545	0.0583	0.0619	0.0656	0.5126							
	σ	–0.1934	–0.1262	–0.0978	–0.0766	–0.0591	–0.0437	–0.0296	–0.0164	–0.0038	0.0085	0.0206	0.0327	0.5848							
8	μ	–0.0206	0.0069	0.0185	0.0272	0.0343	0.0406	0.0463	0.0517	0.0567	0.0616	0.0664	0.6103								
	σ	–0.2121	–0.1374	–0.1057	–0.0822	–0.0627	–0.0455	–0.0299	–0.0153	–0.0013	0.0123	0.0257	0.6541								
9	μ	–0.0414	–0.0053	0.0099	0.0213	0.0306	0.0388	0.0463	0.0532	0.0598	0.0662	0.7205									
	σ	–0.2343	–0.1504	–0.1149	–0.0886	–0.0667	–0.0475	–0.0300	–0.0136	0.0020	0.0172	0.7268									
10	μ	–0.0680	–0.0208	–0.0008	0.0140	0.0262	0.0369	0.0466	0.0557	0.0642	0.8460										
	σ	–0.2612	–0.1660	–0.1258	–0.0959	–0.0712	–0.0494	–0.0296	–0.0111	0.0065	0.8038										
11	μ	–0.1028	–0.0408	–0.0146	0.0048	0.0209	0.0349	0.0476	0.0594	0.9905											
	σ	–0.2943	–0.1850	–0.1389	–0.1046	–0.0762	–0.0513	–0.0287	–0.0076	0.8866											
12	μ	–0.1498	–0.0673	–0.0326	–0.0068	0.0144	0.0329	0.0497	1.1595												
	σ	–0.3363	–0.2088	–0.1550	–0.1151	–0.0820	–0.0531	–0.0268	0.9771												
13	μ	–0.2154	–0.1039	–0.0569	–0.0222	0.0064	0.0313	1.3607													
	σ	–0.3916	–0.2396	–0.1755	–0.1280	–0.0888	–0.0544	1.0779													
14	μ	–0.3117	–0.1565	–0.0914	–0.0433	–0.0037	1.6066														
	σ	–0.4679	–0.2813	–0.2028	–0.1447	–0.0967	1.1934														
15	μ	–0.4632	–0.2378	–0.1433	–0.0738	1.9180															
	σ	–0.5804	–0.3417	–0.2414	–0.1673	1.3307															
16	μ	–0.7284	–0.3766	–0.2298	2.3348																
	σ	–0.7645	–0.4380	–0.3014	1.5039																
17	μ	–1.2872	–0.6610	2.9482																	
	σ	–1.1244	–0.6212	1.7456																	
18	μ	–3.0609	4.0609																		
	σ	–2.1745	2.1745																		

APPENDIX J

BAKER'S ALGORITHM*

```
#include <stdio.h> /* for reading data file */
#include <math.h> /* for square root */
#include <stdlib.h>
#define MAXNUM 600 /* max. sample size */
#define MAXBIN 101 /* max array size for H array */
#define LEN 40 /* max. length of filename */
int compare(float *x, float *y)
{
    if(*x==*y)return(0);
    return *x>*y?1:-1;
}
int geth(float *,int ,float ,float *,float *,int *,
int *,float *,float *);
void main()
{
    FILE *fp;
    float a[MAXNUM], dt, h[MAXBIN],period,mu,sigsq;
    char filename[LEN]; /* array to hold filename */
    int flag, hsize=101,n,i,scanval,binsreq;
    for(i=0;i<MAXBIN;i++)h[i]=0.;
    puts("\nEnter input file name: ");
    scanf("%40s",filename); /* get file name (MAX 40 CHARS) */
    if ((fp = fopen(filename,"r")) == 0 ) /* open file */
    {
        puts ("Can't open input file"); /* unsuccessful */
        return;
    }
```

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```
    fscanf(fp,“%f”,&period);
    n=0;
    while(( scanval=fscanf(fp,“%f”,&a[n])) != EOF && scanval != NULL)
    {
        ++n;
        if(n>=MAXNUM)
        {
            printf(“sample size too large. Maximum is %d0,MAXNUM);
            return;
        }
    }
    fclose(fp);
    printf(“read in %d numbers0,n);
    flag=geth(a,n,period,&dt,h,&hsize,&binsreq,&mu,&sigsq);
    if(flag<0){
      printf(“%d array elements requested, which is too many.0,
      binsreq);
      return;
    }
    else   {
        printf(“%d array elements used0,binsreq);
        printf(“h array size %d, time step %f0,hsize,dt);
        for(i=1;i<hsize;i++)printf(“%f %f %f0,(float)(i*dt),h[i],
    (float)(i*dt)/mu+sigsq/(2.*mu*mu)-0.5);
    }
}
int geth(float *a,int n,float period,float *dt,float *h,int *hsize,
int *binsreq,float *mu,float *sigsq)
{
    float sigma=0., *perm_factor,scale, en,
    *cumsum,*total, *b;
    int i,k,n0,n0dash,s,bin,m, ndash,ntop, *x, *dope;
    *mu=0.;
    cumsum=(float *)calloc(n,sizeof(float)); /* allocate workspace */
    dope=(int *)calloc(n,sizeof(int));
    x=(int *)calloc(n,sizeof(int));
    total=(float *)calloc(n,sizeof(float));
    if(!cumsum || !dope || !x || !total )return (-1);
    /*return if cant allocate workspace */
    qsort(a,n,sizeof(float),compare); /* sort failure times */
    for(i=0;i<n;i++){
      *mu+=a[i]; /* find mean */
        sigma+=a[i]*a[i];
    }
```

```
*mu/=n;
sigma=(sigma-n*(*mu)*(*mu))/(n-1.);
*sigsq=sigma;
bin=*hsize;
*dt=period/(bin-1.);
scale=(bin-1.)/period;
for(i=0;i<n;i++) x[i]=(int)(scale*a[i]+0.5);
cumsum[0]=a[0]; /* find sums of order statistics */
for(i=1,m=1;i<n;i++){
   cumsum[i]=cumsum[i-1]+a[i];
   if(cumsum[i] < period)m=i+1;
}
if(m==n)printf("Maximum value of m, i.e. %d attained0,m);
*binsreq=(m+1)*(bin+1);
b=(float *)calloc(*binsreq,sizeof(float));
if(!b) return(-1.); /* not enough workspace */
for(i=0;i<m;i++)dope[i]=i*(bin+1); /* offsets to mimic 2 dim. array */
if(x[0]<bin)++b[dope[0]+x[0]]; /* start histogram off */
n0=x[0];
for(s=1;s<n;s++)
{
    n0dash=n0+x[s]; /* translate each level by x[s] */
    for(k=(s>(m-1))?(m-1):s;k>=1;k–) /* only build up histograms
            as far as mth level, */
    {
        ntop=(bin>n0dash)?n0dash:bin;
        for(ndash=x[s];ndash<=ntop;ndash++)
            b[dope[k]+ndash]+=b[dope[k-1]+ndash-x[s]];
    }
    if(x[s]<bin)++b[dope[0]+x[s]]; /* add new term to lowest level */
    n0=n0dash;
}
/* convert histograms to p.d.f.s, cumulate to dist. funs, and add. */
perm_factor=cumsum; /* re-use cumsum space for perm numbers */
en=(float)n;
perm_factor[0]=1./en;
for(i=1;i<m;i++)
    perm_factor[i]=(float)(perm_factor[i-1]*(i+1)/(en-i));
for(i=0;i<bin;i++){
    h[i]=0.;
    for(s=0;s<m;s++){
        total[s]+=b[dope[s]+i]*perm_factor[s];
        h[i]+=total[s];
```

```
        }
    }
    /* make continuity correction…only want half the mass at last point */
    for(i=bin-1;i>0;i–)
        h[i]=0.5*(h[i]+h[i-1]);
    free(b);
    free(cumsum);
    free(x);
    free(dope);
    free(total);
    return(0);
}
```

APPENDIX K

STANDARD NORMAL DISTRIBUTION*

Normal Distribution and Related Functions

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\, dt$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

x	$F(x)$	$1 - F(x)$	$f(x)$	x	$F(x)$	$1 - F(x)$	$f(x)$
0.00	0.5000	0.5000	0.3989	0.15	0.5596	0.4404	0.3945
0.01	0.5040	0.4960	0.3989	0.16	0.5636	0.4364	0.3939
0.02	0.5080	0.4920	0.3989	0.17	0.5675	0.4325	0.3932
0.03	0.5120	0.4880	0.3988	0.18	0.5714	0.4286	0.3925
0.04	0.5160	0.4840	0.3986	0.19	0.5753	0.4247	0.3918
0.05	0.5199	0.4801	0.3984	0.20	0.5793	0.4207	0.3910
0.06	0.5239	0.4761	0.3982	0.21	0.5832	0.4168	0.3902
0.07	0.5279	0.4721	0.3980	0.22	0.5871	0.4129	0.3894
0.08	0.5319	0.4681	0.3977	0.23	0.5910	0.4090	0.3885
0.09	0.5359	0.4641	0.3973	0.24	0.5948	0.4052	0.3876
0.10	0.5398	0.4602	0.3970	0.25	0.5987	0.4013	0.3867
0.11	0.5438	0.4562	0.3965	0.26	0.6026	0.3974	0.3857
0.12	0.5478	0.4522	0.3961	0.27	0.6064	0.3936	0.3847
0.13	0.5517	0.4483	0.3956	0.28	0.6103	0.3897	0.3836
0.14	0.5557	0.4443	0.3951	0.29	0.6141	0.3859	0.3825

(*Continued*)

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x	$F(x)$	$1 - F(x)$	$f(x)$	x	$F(x)$	$1 - F(x)$	$f(x)$
0.30	0.6179	0.3821	0.3814	0.72	0.7642	0.2358	0.3079
0.31	0.6217	0.3783	0.3802	0.73	0.7673	0.2327	0.3056
0.32	0.6255	0.3745	0.3790	0.74	0.7704	0.2296	0.3034
0.33	0.6293	0.3707	0.3778	0.75	0.7734	0.2266	0.3011
0.34	0.6331	0.3669	0.3765	0.76	0.7764	0.2236	0.2989
0.35	0.6368	0.3632	0.3752	0.77	0.7794	0.2206	0.2966
0.36	0.6406	0.3594	0.3739	0.78	0.7823	0.2177	0.2943
0.37	0.6443	0.3557	0.3725	0.79	0.7852	0.2148	0.2920
0.38	0.6480	0.3520	0.3712	0.80	0.7881	0.2119	0.2897
0.39	0.6517	0.3483	0.3697	0.81	0.7910	0.2090	0.2874
0.40	0.6554	0.3446	0.3683	0.82	0.7939	0.2061	0.2850
0.41	0.6591	0.3409	0.3668	0.83	0.7967	0.2033	0.2827
0.42	0.6628	0.3372	0.3653	0.84	0.7995	0.2005	0.2803
0.43	0.6664	0.3336	0.3637	0.85	0.8023	0.1977	0.2780
0.44	0.6700	0.3300	0.3621	0.86	0.8051	0.1949	0.2756
0.45	0.6736	0.3264	0.3605	0.87	0.8078	0.1922	0.2732
0.46	0.6772	0.3228	0.3589	0.88	0.8106	0.1894	0.2709
0.47	0.6808	0.3192	0.3572	0.89	0.8133	0.1867	0.2685
0.48	0.6844	0.3156	0.3555	0.90	0.8159	0.1841	0.2661
0.49	0.6879	0.3121	0.3538	0.91	0.8186	0.1814	0.2637
0.50	0.6915	0.3085	0.3521	0.92	0.8212	0.1788	0.2613
0.51	0.6950	0.3050	0.3503	0.93	0.8238	0.1762	0.2589
0.52	0.6985	0.3015	0.3485	0.94	0.8264	0.1736	0.2565
0.53	0.7019	0.2981	0.3467	0.95	0.8289	0.1711	0.2541
0.54	0.7054	0.2946	0.3448	0.96	0.8315	0.1685	0.2516
0.55	0.7088	0.2912	0.3429	0.97	0.8340	0.1660	0.2492
0.56	0.7123	0.2877	0.3410	0.98	0.8365	0.1635	0.2468
0.57	0.7157	0.2843	0.3391	0.99	0.8389	0.1611	0.2444
0.58	0.7190	0.2810	0.3372	1.00	0.8413	0.1587	0.2420
0.59	0.7224	0.2776	0.3352	1.01	0.8438	0.1562	0.2396
0.60	0.7257	0.2743	0.3332	1.02	0.8461	0.1539	0.2371
0.61	0.7291	0.2709	0.3312	1.03	0.8485	0.1515	0.2347
0.62	0.7324	0.2676	0.3292	1.04	0.8508	0.1492	0.2323
0.63	0.7357	0.2643	0.3271	1.05	0.8531	0.1469	0.2299
0.64	0.7389	0.2611	0.3251	1.06	0.8554	0.1446	0.2275
0.65	0.7422	0.2578	0.3230	1.07	0.8577	0.1423	0.2251
0.66	0.7454	0.2546	0.3209	1.08	0.8599	0.1401	0.2227
0.67	0.7486	0.2514	0.3187	1.09	0.8621	0.1379	0.2203
0.68	0.7517	0.2483	0.3166	1.10	0.8643	0.1357	0.2179
0.69	0.7549	0.2451	0.3144	1.11	0.8665	0.1335	0.2155
0.70	0.7580	0.2420	0.3123	1.12	0.8686	0.1314	0.2131
0.71	0.7611	0.2389	0.3101	1.13	0.8708	0.1292	0.2107

x	$F(x)$	$1 - F(x)$	$f(x)$	x	$F(x)$	$1 - F(x)$	$f(x)$
1.14	0.8729	0.1271	0.2083	1.55	0.9394	0.0606	0.1200
1.15	0.8749	0.1251	0.2059	1.56	0.9406	0.0594	0.1182
1.16	0.8770	0.1230	0.2036	1.57	0.9418	0.0582	0.1163
1.17	0.8790	0.1210	0.2012	1.58	0.9429	0.0571	0.1145
1.18	0.8810	0.1190	0.1989	1.59	0.9441	0.0559	0.1127
1.19	0.8830	0.1170	0.1965	1.60	0.9452	0.0548	0.1109
1.20	0.8849	0.1151	0.1942	1.61	0.9463	0.0537	0.1092
1.21	0.8869	0.1131	0.1919	1.62	0.9474	0.0526	0.1074
1.22	0.8888	0.1112	0.1895	1.63	0.9484	0.0516	0.1057
1.23	0.8907	0.1093	0.1872	1.64	0.9495	0.0505	0.1040
1.24	0.8925	0.1075	0.1849	1.65	0.9505	0.0495	0.1023
1.25	0.8944	0.1056	0.1826	1.66	0.9515	0.0485	0.1006
1.26	0.8962	0.1038	0.1804	1.67	0.9525	0.0475	0.0989
1.27	0.8980	0.1020	0.1781	1.68	0.9535	0.0465	0.0973
1.28	0.8997	0.1003	0.1758	1.69	0.9545	0.0455	0.0957
1.29	0.9015	0.0985	0.1736	1.70	0.9554	0.0446	0.0940
1.30	0.9032	0.0968	0.1714	1.71	0.9564	0.0436	0.0925
1.31	0.9049	0.0951	0.1691	1.72	0.9573	0.0427	0.0909
1.32	0.9066	0.0934	0.1669	1.73	0.9582	0.0418	0.0893
1.33	0.9082	0.0918	0.1647	1.74	0.9591	0.0409	0.0878
1.34	0.9099	0.0901	0.1626	1.75	0.9599	0.0401	0.0863
1.35	0.9115	0.0885	0.1604	1.76	0.9608	0.0392	0.0848
1.36	0.9131	0.0869	0.1582	1.77	0.9616	0.0384	0.0833
1.37	0.9147	0.0853	0.1561	1.78	0.9625	0.0375	0.0818
1.38	0.9162	0.0838	0.1539	1.79	0.9633	0.0367	0.0804
1.39	0.9177	0.0823	0.1518	1.80	0.9641	0.0359	0.0790
1.40	0.9192	0.0808	0.1497	1.81	0.9649	0.0351	0.0775
1.41	0.9207	0.0793	0.1476	1.82	0.9656	0.0344	0.0761
1.42	0.9222	0.0778	0.1456	1.83	0.9664	0.0336	0.0748
1.43	0.9236	0.0764	0.1435	1.84	0.9671	0.0329	0.0734
1.44	0.9251	0.0749	0.1415	1.85	0.9678	0.0322	0.0721
1.45	0.9265	0.0735	0.1394	1.86	0.9686	0.0314	0.0707
1.46	0.9279	0.0721	0.1374	1.87	0.9693	0.0307	0.0694
1.47	0.9292	0.0708	0.1354	1.88	0.9699	0.0301	0.0681
1.48	0.9306	0.0694	0.1334	1.89	0.9706	0.0294	0.0669
1.49	0.9319	0.0681	0.1315	1.90	0.9713	0.0287	0.0656
1.50	0.9332	0.0668	0.1295	1.91	0.9719	0.0281	0.0644
1.51	0.9345	0.0655	0.1276	1.92	0.9726	0.0274	0.0632
1.52	0.9357	0.0643	0.1257	1.93	0.9732	0.0268	0.0620
1.53	0.9370	0.0630	0.1238	1.94	0.9738	0.0262	0.0608
1.54	0.9382	0.0618	0.1219	1.95	0.9744	0.0256	0.0596

(*Continued*)

x	$F(x)$	$1 - F(x)$	$f(x)$	x	$F(x)$	$1 - F(x)$	$f(x)$
1.96	0.9750	0.0250	0.0584	2.38	0.9913	0.0087	0.0235
1.97	0.9756	0.0244	0.0573	2.39	0.9916	0.0084	0.0229
1.98	0.9761	0.0239	0.0562	2.40	0.9918	0.0082	0.0224
1.99	0.9767	0.0233	0.0551	2.41	0.9920	0.0080	0.0219
2.00	0.9772	0.0228	0.0540	2.42	0.9922	0.0078	0.0213
2.01	0.9778	0.0222	0.0529	2.43	0.9925	0.0075	0.0208
2.02	0.9783	0.0217	0.0519	2.44	0.9927	0.0073	0.0203
2.03	0.9788	0.0212	0.0508	2.45	0.9929	0.0071	0.0198
2.04	0.9793	0.0207	0.0498	2.46	0.9931	0.0069	0.0194
2.05	0.9798	0.0202	0.0488	2.47	0.9932	0.0068	0.0189
2.06	0.9803	0.0197	0.0478	2.48	0.9934	0.0066	0.0184
2.07	0.9808	0.0192	0.0468	2.49	0.9936	0.0064	0.0180
2.08	0.9812	0.0188	0.0459	2.50	0.9938	0.0062	0.0175
2.09	0.9817	0.0183	0.0449	2.51	0.9940	0.0060	0.0717
2.10	0.9821	0.0179	0.0440	2.52	0.9941	0.0059	0.0167
2.11	0.9826	0.0174	0.0431	2.53	0.9943	0.0057	0.0163
2.12	0.9830	0.0170	0.0422	2.54	0.9945	0.0055	0.0158
2.13	0.9834	0.0166	0.0413	2.55	0.9946	0.0054	0.0155
2.14	0.9838	0.0162	0.0404	2.56	0.9948	0.0052	0.0151
2.15	0.9842	0.0158	0.0396	2.57	0.9949	0.0051	0.0147
2.16	0.9846	0.0154	0.0387	2.58	0.9951	0.0049	0.0143
2.17	0.9850	0.0150	0.0379	2.59	0.9952	0.0048	0.0139
2.18	0.9854	0.0146	0.0371	2.60	0.9953	0.0047	0.0136
2.19	0.9857	0.0143	0.0363	2.61	0.9955	0.0045	0.0132
2.20	0.9861	0.0139	0.0355	2.62	0.9956	0.0044	0.0129
2.21	0.9864	0.0136	0.0347	2.63	0.9957	0.0043	0.0126
2.22	0.9868	0.0132	0.0339	2.64	0.9959	0.0041	0.0122
2.23	0.9871	0.0129	0.0332	2.65	0.9960	0.0040	0.0119
2.24	0.9875	0.0125	0.0325	2.66	0.9961	0.0039	0.0116
2.25	0.9878	0.0122	0.0317	2.67	0.9962	0.0038	0.0113
2.26	0.9881	0.0119	0.0310	2.68	0.9963	0.0037	0.0110
2.27	0.9884	0.0116	0.0303	2.69	0.9964	0.0036	0.0107
2.28	0.9887	0.0113	0.0297	2.70	0.9965	0.0035	0.0104
2.29	0.9890	0.0110	0.0290	2.71	0.9966	0.0034	0.0101
2.30	0.9893	0.0107	0.0283	2.72	0.9967	0.0033	0.0099
2.31	0.9896	0.0104	0.0277	2.73	0.9968	0.0032	0.0096
2.32	0.9898	0.0102	0.0270	2.74	0.9969	0.0031	0.0093
2.33	0.9901	0.0099	0.0264	2.75	0.9970	0.0030	0.0091
2.34	0.9904	0.0096	0.0258	2.76	0.9971	0.0029	0.0088
2.35	0.9906	0.0094	0.0252	2.77	0.9972	0.0028	0.0086
2.36	0.9909	0.0091	0.0246	2.78	0.9973	0.0027	0.0084
2.37	0.9911	0.0089	0.0241	2.79	0.9974	0.0026	0.0081

x	$F(x)$	$1 - F(x)$	$f(x)$	x	$F(x)$	$1 - F(x)$	$f(x)$
2.80	0.9974	0.0026	0.0079	3.21	0.9993	0.0007	0.0023
2.81	0.9975	0.0025	0.0077	3.22	0.9994	0.0006	0.0022
2.82	0.9976	0.0024	0.0075	3.23	0.9994	0.0006	0.0022
2.83	0.9977	0.0023	0.0073	3.24	0.9994	0.0006	0.0021
2.84	0.9977	0.0023	0.0071	3.25	0.9994	0.0006	0.0020
2.85	0.9978	0.0022	0.0069	3.26	0.9994	0.0006	0.0020
2.86	0.9979	0.0021	0.0067	3.27	0.9995	0.0005	0.0019
2.87	0.9979	0.0021	0.0065	3.28	0.9995	0.0005	0.0018
2.88	0.9980	0.0020	0.0063	3.29	0.9995	0.0005	0.0018
2.89	0.9981	0.0019	0.0061	3.30	0.9995	0.0005	0.0017
2.90	0.9981	0.0019	0.0060	3.31	0.9995	0.0005	0.0017
2.91	0.9982	0.0018	0.0058	3.32	0.9995	0.0005	0.0016
2.92	0.9982	0.0018	0.0056	3.33	0.9996	0.0004	0.0016
2.93	0.9983	0.0017	0.0055	3.34	0.9996	0.0004	0.0015
2.94	0.9984	0.0016	0.0053	3.35	0.9996	0.0004	0.0015
2.95	0.9984	0.0016	0.0051	3.36	0.9996	0.0004	0.0014
2.96	0.9985	0.0015	0.0050	3.37	0.9996	0.0004	0.0014
2.97	0.9985	0.0015	0.0048	3.38	0.9996	0.0004	0.0013
2.98	0.9986	0.0014	0.0047	3.39	0.9997	0.0003	0.0013
2.99	0.9986	0.0014	0.0046	3.40	0.9997	0.0003	0.0012
3.00	0.9987	0.0013	0.0044	3.41	0.9997	0.0003	0.0012
3.01	0.9987	0.0013	0.0043	3.42	0.9997	0.0003	0.0012
3.02	0.9987	0.0013	0.0042	3.43	0.9997	0.0003	0.0011
3.03	0.9988	0.0012	0.0040	3.44	0.9997	0.0003	0.0011
3.04	0.9988	0.0012	0.0039	3.45	0.9997	0.0003	0.0010
3.05	0.9989	0.0011	0.0038	3.46	0.9997	0.0003	0.0010
3.06	0.9989	0.0011	0.0037	3.47	0.9997	0.0003	0.0010
3.07	0.9989	0.0011	0.0036	3.48	0.9997	0.0003	0.0009
3.08	0.9990	0.0010	0.0035	3.49	0.9998	0.0002	0.0009
3.09	0.9990	0.0010	0.0034	3.50	0.9998	0.0002	0.0009
3.10	0.9990	0.0010	0.0033	3.51	0.9998	0.0002	0.0008
3.11	0.9991	0.0009	0.0032	3.52	0.9998	0.0002	0.0008
3.12	0.9991	0.0009	0.0031	3.53	0.9998	0.0002	0.0008
3.13	0.9991	0.0009	0.0030	3.54	0.9998	0.0002	0.0008
3.14	0.9992	0.0008	0.0029	3.55	0.9998	0.0002	0.0007
3.15	0.9992	0.0008	0.0028	3.56	0.9998	0.0002	0.0007
3.16	0.9992	0.0008	0.0027	3.57	0.9998	0.0002	0.0007
3.17	0.9992	0.0008	0.0026	3.58	0.9998	0.0002	0.0007
3.18	0.9993	0.0007	0.0025	3.59	0.9998	0.0002	0.0006
3.19	0.9993	0.0007	0.0025	3.60	0.9998	0.0002	0.0006
3.20	0.9993	0.0007	0.0024	3.61	0.9998	0.0002	0.0006

(*Continued*)

x	$F(x)$	$1 - F(x)$	$f(x)$	x	$F(x)$	$1 - F(x)$	$f(x)$
3.62	0.9999	0.0001	0.0006	3.82	0.9999	0.0001	0.0003
3.63	0.9999	0.0001	0.0005	3.83	0.9999	0.0001	0.0003
3.64	0.9999	0.0001	0.0005	3.84	0.9999	0.0001	0.0003
3.65	0.9999	0.0001	0.0005	3.85	0.9999	0.0001	0.0002
3.66	0.9999	0.0001	0.0005	3.86	0.9999	0.0001	0.0002
3.67	0.9999	0.0001	0.0005	3.87	0.9999	0.0001	0.0002
3.68	0.9999	0.0001	0.0005	3.88	0.9999	0.0001	0.0002
3.69	0.9999	0.0001	0.0004	3.89	1.0000	0.0000	0.0002
3.70	0.9999	0.0001	0.0004	3.90	1.0000	0.0000	0.0002
3.71	0.9999	0.0001	0.0004	3.91	1.0000	0.0000	0.0002
3.72	0.9999	0.0001	0.0004	3.92	1.0000	0.0000	0.0002
3.73	0.9999	0.0001	0.0004	3.93	1.0000	0.0000	0.0002
3.74	0.9999	0.0001	0.0004	3.94	1.0000	0.0000	0.0002
3.75	0.9999	0.0001	0.0004	3.95	1.0000	0.0000	0.0002
3.76	0.9999	0.0001	0.0003	3.96	1.0000	0.0000	0.0002
3.77	0.9999	0.0001	0.0003	3.97	1.0000	0.0000	0.0002
3.78	0.9999	0.0001	0.0003	3.98	1.0000	0.0000	0.0001
3.79	0.9999	0.0001	0.0003	3.99	1.0000	0.0000	0.0001
3.80	0.9999	0.0001	0.0003	4.00	1.0000	0.0000	0.0001
3.81	0.9999	0.0001	0.0003				

APPENDIX L

CRITICAL VALUES OF χ^2

Critical Values of χ^2

Degrees of freedom	$\chi^2_{0.995}$	$\chi^2_{0.990}$	$\chi^2_{0.975}$	$\chi^2_{0.950}$	$\chi^2_{0.900}$
1	0.0000393	0.0001571	0.0009821	0.0039321	0.0157908
2	0.0100251	0.0201007	0.0506356	0.102587	0.21072
3	0.0717212	0.114832	0.215795	0.351846	0.584375
4	0.20699	0.29711	0.484419	0.710721	1.063623
5	0.41174	0.5543	0.831211	1.145476	1.61031
6	0.675727	0.872085	1.237347	1.63539	2.20413
7	0.989265	1.239043	1.68987	2.16735	2.83311
8	1.344419	1.646482	2.17973	2.73264	3.48954
9	1.734926	2.087912	2.70039	3.32511	4.16816
10	2.15585	2.55821	3.24697	3.9403	4.86518
11	2.60321	3.05347	3.81575	4.57481	5.57779
12	3.07382	3.57056	4.40379	5.22603	6.3038
13	3.56503	4.10691	5.00874	5.89186	7.0415
14	4.07468	4.66043	5.62872	6.57063	7.78953
15	4.60094	5.22935	6.26214	7.26094	8.54675
16	5.14224	5.81221	6.90766	7.96164	9.31223
17	5.69724	6.40776	7.56418	8.67176	10.0852
18	6.26481	7.01491	8.23075	9.39046	10.8649
19	6.84398	7.63273	8.90655	10.117	11.6509
20	7.43386	8.2604	9.59083	10.8508	12.4426
21	8.03366	8.8972	10.28293	11.5913	13.2396
22	8.64272	9.54249	10.9823	12.338	14.0415
23	9.26042	10.19567	11.6885	13.0905	14.8479
24	9.88623	10.8564	12.4011	13.8484	15.6587

(*Continued*)

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Degrees of freedom	$\chi^2_{0.995}$	$\chi^2_{0.990}$	$\chi^2_{0.975}$	$\chi^2_{0.950}$	$\chi^2_{0.900}$
25	10.5197	11.524	13.1197	14.6114	16.4734
26	11.1603	12.1981	13.8439	15.3791	17.2919
27	11.8076	12.8786	14.5733	16.1513	18.1138
28	12.4613	13.5648	15.3079	16.9279	18.9392
29	13.1211	14.2565	16.0471	17.7083	19.7677
30	13.7867	14.9535	16.7908	18.4926	20.5992
40	20.7065	22.1643	24.4331	26.5093	29.0505
50	27.9907	29.7067	32.3574	34.7642	37.6886
60	35.5346	37.4848	40.4817	43.1879	46.4589
70	43.2752	45.4418	48.7576	51.7393	55.329
80	51.172	53.54	57.1532	60.3915	64.2778
90	59.1963	61.7541	65.6466	69.126	73.2912
100	67.3276	70.0648	74.2219	77.9295	82.3581

Critical Values of χ^2 (Continued)

Degrees of freedom	$\chi^2_{0.100}$	$\chi^2_{0.050}$	$\chi^2_{0.025}$	$\chi^2_{0.010}$	$\chi^2_{0.005}$
1	2.70554	3.84146	5.02389	6.6349	7.87944
2	4.60517	5.99147	7.37776	9.21034	10.5966
3	6.25139	7.81473	9.3484	11.3449	12.8381
4	7.77944	9.48773	11.1433	13.2767	14.8602
5	9.23635	11.0705	12.8325	15.0863	16.7496
6	10.6446	12.5916	14.4494	16.8119	18.5476
7	12.017	14.0671	16.0128	18.4753	20.2777
8	13.3616	15.5073	17.5346	20.0902	21.955
9	14.6837	16.919	19.0228	21.666	23.5893
10	15.9871	18.307	20.4831	23.2093	25.1882
11	17.275	19.6751	21.92	24.725	26.7569
12	18.5494	21.0261	23.3367	26.217	28.2995
13	19.8119	22.3621	24.7356	27.6883	29.8194
14	21.0642	23.6848	26.119	29.1413	31.3193
15	22.3072	24.9958	27.4884	30.5779	32.8013
16	23.5418	26.2962	28.8454	31.9999	34.2672
17	24.769	27.5871	30.191	33.4087	35.7185
18	25.9894	28.8693	31.5264	34.8053	37.1564
19	27.2036	30.1435	32.8523	36.1908	38.5822
20	28.412	31.4104	34.1696	37.5662	39.9968

Degrees of freedom	$\chi^2_{0.100}$	$\chi^2_{0.050}$	$\chi^2_{0.025}$	$\chi^2_{0.010}$	$\chi^2_{0.005}$
21	29.6151	32.6705	35.4789	38.9321	41.401
22	30.8133	33.9244	36.7807	40.2894	42.7956
23	32.0069	35.1725	38.0757	41.6384	44.1813
24	33.1963	36.4151	39.3641	42.9798	45.5585
25	34.3816	37.6525	40.6465	44.3141	46.9278
26	35.5631	38.8852	41.9232	45.6417	48.2899
27	36.7412	40.1133	43.1944	46.963	49.6449
28	37.9159	41.3372	44.4607	48.2782	50.9933
29	39.0875	42.5569	45.7222	49.5879	52.3356
30	40.256	43.7729	46.9792	50.8922	53.672
40	51.805	55.7585	59.3417	63.6907	66.7659
50	63.1671	67.5048	71.4202	76.1539	79.49
60	74.397	79.0819	83.2976	88.3794	91.9517
70	85.5271	90.5312	95.0231	100.425	104.215
80	96.5782	101.879	106.629	112.329	116.321
90	107.565	113.145	118.136	124.116	128.299
100	118.498	124.342	129.561	135.807	140.169

APPENDIX M

SOLUTIONS OF SELECTED PROBLEMS

CHAPTER 1

1.1 $\mu = \dfrac{a+b}{2}$

$$\sigma^2 = \frac{(b-a)^2}{12}$$

1.3 $\mathrm{var}(t) = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$

$P[t \le med] = 0.5$

then

$$0 = \frac{\ln(\text{med}) - \mu}{\sigma}$$

$\mu = \ln(\text{med})$

$\text{med} = e^{\mu}$

1.11 $R(t) = \exp[-(k\lambda t^c + (1-k)b(e^{\beta t^b} - 1))]$

If $c = 1$, then $h(t)$ = constant = λ.

If $c > 1$, $k = 1$, then $h(t)$ is an increasing function with t.

If $c = 1$, $0 \le k < 1$, $b > 1$, then $h(t)$ is a decreasing function with t.

1.12 $R(t) = e^{-t/290} \displaystyle\sum_{k=0}^{2} \frac{(t/290)^k}{k!}$

$R(100) = e^{-0.345}(1 + 0.3448 + 0.05945) = 0.9945$

$E(t) = \gamma\theta = 870$ h

Residual life = 770 h.

1.13 $h(t) = \dfrac{\frac{8}{7}e^{-t} - \frac{8}{7}e^{-8t}}{\frac{8}{7}e^{-t} - \frac{1}{7}e^{-8t}}$

$MTTF = \dfrac{63}{56}$

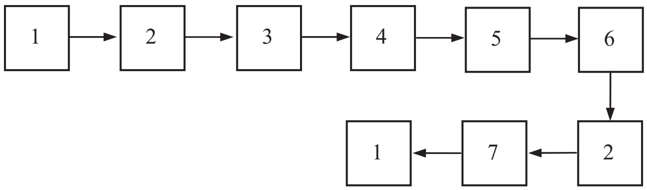
1.24 **a.** 0.768 failures

b. 10^4 h

c. 0.9824

CHAPTER 2

2.3

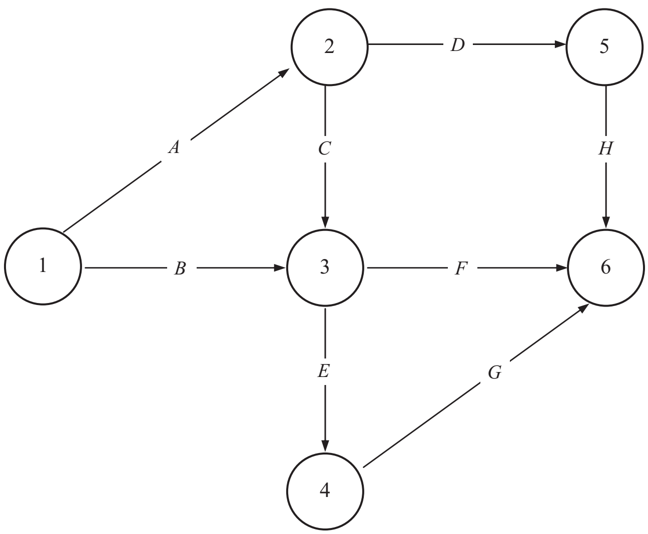


$R(t) = e^{-\sum \lambda_i t} = e^{-0.00285t}$

This is the reliability of one cassette. The reliability of recording from one cassette to another is

$R(t) = e^{-0.0057t}$

2.9



a. $R_{system} = 0.57632$

b. $R_A(t) = e^{-\int_0^t 0.2577 e^{0.0027t} dt}$

$R_A(t) = e^{-9.544(e^{0.0027t-1})}$

$R_B(t)$, $R_C(t)$, . . . , $R_h(t)$ can be determined as above. We then substitute the corresponding tie sets into the R_{system} equation to yield an expression that is a function of time. Integrating this expression from zero to infinity results in the MTTF. No closed-form expression will exist; therefore, use a numerical approach to estimate the MTTF.

c. To ensure a reliability of 0.98 after 2 years of service, the pipes connecting nodes 1 and 2 and those connecting 1 and 3, 2 and 3, and 3 and 6 must have redundant pipes in parallel with each of them.

2.13 **a.** $R = 0.90961$

b. $I_B^{10} = 0.00006$

$I_B^{13} = 0.02491$

2.18 $R(0.98, 2, 6) = 0.998032$

2.22 **a.** $R(t) = e^{-5\lambda t}$

b. $R(2; 5, p) = 10e^{-2\lambda t} - 20e^{-3\lambda t} + 15e^{-4\lambda t} - 4e^{-5\lambda t}$

c. $R(3; 5, p) = 10e^{-3\lambda t} - 15e^{-4\lambda t} + 6e^{-5\lambda t}$

d. $R = 5e^{-\lambda t} - 10e^{-2\lambda t} + 10e^{-3\lambda t} - 5e^{-4\lambda t} + e^{-5\lambda t}$

2.27 **a.** The cut-sets are

$C_1 = \overline{ab}\,\overline{ac}\,\overline{ad}$,

$C_2 = \overline{ab}\,\overline{db}\,\overline{cb}$,

$C_3 = \overline{ab}\,\overline{ad}\,\overline{cb}$, and

$C_4 = \overline{ab}\,\overline{dc}\,\overline{ac}\,\overline{db}$.

The tie-sets are

$T_1 = ab$.

$T_2 = ac\ cb$,

$T_3 = ad\ db$,

$T_4 = ad\ dc\ cb$, and

$T_5 = ad\ dc\ ca\ ab$.

b. $R = P + 2P^2 - P^3 - 4P^4 + 6P^5 - 2P^6$

2.30 **a.** $MTTF = \int_0^\infty R(t)dt = \dfrac{1}{\lambda_1 + \lambda_2 + \lambda_3}$

b. $MTTF = \sqrt{\dfrac{\pi}{2(\lambda_1 + \lambda_2 + \lambda_3)}}$

c. No closed-form expression

CHAPTER 3

3.1 $MTTF = \dfrac{1}{\lambda} \displaystyle\sum_{j=0}^{\left[\frac{n+1}{2}\right]} \binom{n-j-1}{j} \left[\frac{1}{j+1} - \binom{n-j-1}{1} \frac{1}{j+2} + \ldots + (-1)^{n-j-1} \frac{1}{n} \right]$

There is no closed form for this expression. For certain values of n and j, some integrals may be evaluated numerically.

3.4 $R_S(t) = 4e^{-4t^\gamma/\theta} - 6e^{-5t^\gamma/\theta} + 4e^{-6t^\gamma/\theta} - e^{-7t^\gamma/\theta}$

$$MTTF = \frac{\Gamma(1/\gamma)}{\gamma} \left(4\left(\frac{\theta}{4}\right)^{1/\gamma} - 6\left(\frac{\theta}{5}\right)^{1/\gamma} + 4\left(\frac{\theta}{6}\right)^{1/\gamma} + \left(\frac{\theta}{7}\right)^{1/\gamma} \right)$$

$$h_S(t) = \frac{f_s(t)}{R_s(t)} = \frac{\dfrac{\gamma}{\theta} t^{\gamma-1} \left[16 - 30e^{\frac{-t^\gamma}{\theta}} + 24e^{\frac{-2t^\gamma}{\theta}} - 7e^{\frac{-3t^\gamma}{\theta}} \right]}{\left[4 - 6e^{\frac{-t^\gamma}{\theta}} + 4e^{\frac{-2t^\gamma}{\theta}} - 7e^{\frac{-3t^\gamma}{\theta}} \right]}$$

3.11 **a.** $\lambda_1 = 5.70 \times 10^{-5}$, $\lambda_2 = 1.14 \times 10^{-4}$, $\lambda_3 = 3.42 \times 10^{-4}$ failures per hour

b. MTTF = 1949.32 h

c. 6.3072×10^{-4}

d. $\lambda_1 = 2.9236 \times 10^{-6}$

$\lambda_2 = 5.8473 \times 10^{-6}$

$\lambda_3 = 1.7542 \times 10^{-5}$

3.12 **a.** MTTF = 2λ

$$\sigma^2_{\text{MTTF}} = \int_0^\infty t^2 f(t)dt - (\text{MTTF})^2 = 6\lambda^2 - 4\lambda^2 = 2\lambda^2$$

b. $R(t) = 0.001497$

MTTF = 3666 h

3.13 $R_{sys} = e^{-2.17\times10^{-9} t^{2.3}} + e^{-2.5\times10^{-8} t^2 - 1.14\times10^{-6} t^{2.2}} + e^{-0.5\times10^{-7} t - 1.14\times10^{-6} t^{2.2}}$

$- e^{-0.5\times10^{-7} t - 2.5\times10^{-8} t^2 - 1.14\times10^{-6} t^{2.2}}$

$- e^{-2.5\times10^{-8} t^2 - 1.14\times10^{-6} t^{2.2} - 2.17\times10^{-9} t^{2.3}}$

$- e^{-0.5\times10^{-7} t - 1.14\times10^{-6} t^{2.2} - 2.17\times10^{-9} t^{2.3}}$

$+ e^{-0.5\times10^{-7} t - 2.5\times10^{-8} t^2 - 1.14\times10^{-6} t^{2.2} - 2.17\times10^{-9} t^{2.3}}$

$$R_{sys}(1{,}000) = 0.9830935$$

$$\text{MTTF} = \int_0^\infty R_{sys}(t) = \int_0^\infty e^{-2.17\times10^{-9}t^{2.3}}dt + \int_0^\infty e^{-2.5\times10^{-8}t^2+1.14\times10^{-6}t^{2.2}}dt$$
$$+\ldots+\int_0^\infty e^{-0.5\times10^{-7}t-2.5\times10^{-8}t^2-1.14\times10^{-6}t^{2.2}-2.17\times10^{-9}t^{2.3}}dt$$

There is no closed-form expression for the MTTF. An approximate value may be obtained through numerical analysis.

3.18 $R_s(t) = 4(R_{ps}(t))^3 - 3(R_{ps}(t))^4$

$$R_{ps}(t) = \exp\left[-0.00133t - 0.00144t^2 - \frac{t^{1.3}}{2.3\times10^{-3}} - 3333.33e^{0.3t}\right]$$

3.23 $n = 3$

$\text{MTTF} = 10{,}233\text{ h}$

3.24 One component

3.26 $R = 0.98597$

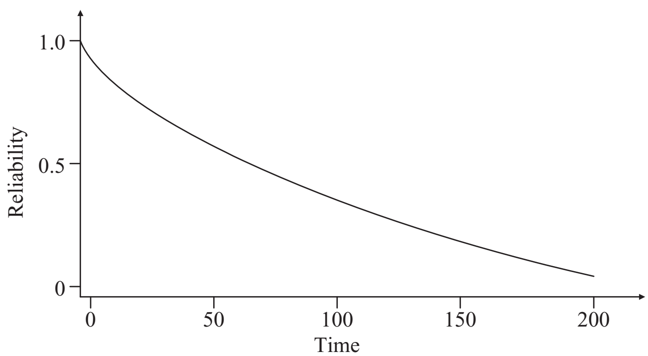
CHAPTER 4

4.1 **a.** $\hat{\lambda} = 0.010353$

b. $\lambda = \frac{1}{n}\sum x_i$

c. $R(49) = 0.6021$

d.



4.2 **a.** $\alpha = 3.83$

$\beta = 5.9766$

b. $R(t) = 1 - (400t^{4.830} - 178.6t^{10.806})$

$$h(t) = \frac{f(t)}{R(t)} = \frac{1{,}930.17t^{4.830} - 1{,}930t^{10.806}}{1 - 400t^{4.830} + 178.6t^{10.806}}$$

4.9 **a.** From the first moment:

$$\theta = \frac{\sum t_i(\lambda_1\lambda_2) - n\lambda_1}{n(\lambda_2 - \lambda_1)}.$$

From the second moment:

$$\theta = \frac{(\lambda_1^2\lambda_2^2)\left(\frac{1}{n}\sum t_i^2\right) - 2\lambda_1^2}{2(\lambda_2^2 - \lambda_1^2)}.$$

From the third moment:

$$\theta = \frac{(\lambda_1^3\lambda_2^3)\left(\frac{1}{n}\sum t_i^3\right) - 2\lambda_1^3}{6(\lambda_2^3 - \lambda_1^3)}.$$

Solve these equations to obtain λ_1, λ_2, θ.

4.16 **a.** $M_1 = v$

$M_2 = 2v + v^2$

or

$\hat{v} = -1 \pm \sqrt{1 + M_2}$

CHAPTER 5

5.1 **a.** It contradicts the hypothesis that the failure times can be modeled by an exponential distribution when $\alpha = 0.10$.

b. Yes

c. Yes

d. MTTF = 29, 038.5 h

5.3 **a.** When $B_{10} > \chi^2_{0.99,9}$, the hypothesis that the data can be modeled by an exponential distribution cannot be rejected. t_1 is not abnormally short. t_{last} is not abnormally long.

b. $\hat{\lambda} = 3.55 \times 10^{-5}$

c. $R(20{,}000) = 0.4916$

d. MTTF = 28, 169 h

5.7 **a.** $\hat{\gamma} = 1.125$

$\hat{\theta} = 130{,}979.19$

b. $R(50{,}000) = 0.228{,}50$

5.11 **a.** $\hat{\mu} = 2.8998$

$\hat{\sigma} = 0.7107$

$$2.5497 < \mu < 3.2498$$

$$0.3890 < \sigma < 2.2350$$

b. $\hat{\gamma} = 1.97, \hat{\theta}_1 = 22.8337$

$$0.9423 < \gamma < 2.7573$$

$$22.8337\exp[-U_{a/2} / 6.8243] \leq \theta_1 \leq 22.8337[-U_{1-a/2} / 6.8243]$$

5.17 $\hat{\lambda} = \dfrac{1}{2r}\left[\sum_{i=1}^{\gamma} t_i + \sum_{i=1}^{n-\gamma} t_i^+\right]$

5.24 $R_{\text{PLE}}(8.5 \times 10^9) = 0.088506$

$R_{\text{CHE}}(8.5 \times 10^9) = 0.022823$

$\text{MTTF}_{\text{PLE}} = 4.818 \times 10^9$ cycles

$\text{MTTF}_{\text{CHE}} = 4.705 \times 10^9$ cycles

5.25 $\hat{\lambda} = 10{,}656.12$

$\text{Var}(\hat{\lambda}) = 2.2710 \times 10^8$

$5985.872 < \lambda < 15{,}326.377$

$\text{MTTF} = 21{,}312.25$

CHAPTER 6

6.6 $\hat{\lambda}_o = 5.615 \times 10^{-5}$ failures per hour

$R(10^4) = 0.570{,}35$

6.7 $\hat{\theta}_o = 188{,}720$

$R(50{,}000) = 0.998{,}76$

The reliability requirements are met.

6.8 $f_o(t) = \dfrac{t}{A_F^2 \lambda_s^2} e^{-t/A_F\lambda_s}$

$$F_o(t) = 1 - \left(\frac{t}{A_F} + \lambda_s\right) / \lambda_s e^{-t/A_F\lambda_s}$$

$$R_o(t) = \left(\frac{t}{A_F} + \lambda_s\right) / \lambda_s e^{-t/A_F\lambda_s}$$

$$h_o(t) = \frac{t}{A_F\lambda_s(t + \lambda_s A_F)}$$

6.12 Life at 30°C and 5 V is 61,206 h.

6.14 **a.** Life at normal conditions is 32,764,059 h.

b. L_{OT} = 917.359 h

6.16 **a.** $R(10{,}000) = 0.573$

b. $A_F(\text{bet. 50 V and 5 V}) = 42.3$

$A_F(\text{bet. 80 V and 5 V}) = 162.1$

c. $L_o = 3.14 \times 10^{11}$ h

CHAPTER 7

7.1 0.5 failures

7.2 $M(10^4) = 2.25$ failures
$A = 0.9523$

7.4 $M(20)_{\text{asymptotic}} = 0.214$
$M(20)_{\text{asymptotic}} = 0.9285$
$M(20)_{\text{exponential}} = 0.7137$
$M(40)_{\text{exponential}} = 1.427$

7.8 $M(10^4)$brush motors = 2.039
$M(10^4)$BLDC motors = 0.861
$x = 1.178y$

7.10 **a.** $M_A(200) = 41{,}371.21$
$M_B(200) = 41{,}370.89$

b. $P_A(200) = 0.68$

7.14 **a.** $\lambda = 0.010{,}08$
$M(10^4) = 101.042$
$\text{Var}\,(N(M(10^4)) = 0.0$

b. Confidence interval
$101.042 \leq N(\text{t}) \leq 101.042$

7.16 **a.** $E[N(600) - N(200)] = 708.15$

b. $R(600) = 0.0$

CHAPTER 8

8.1 **a.** $t_p^* = 0.999$
$L = 297.97$

8.2 Replace at failure; no inventory

8.7 $N^* = 1$
$\phi^* = \$95.334$

8.11 Perform replacements on failure regardless of group size

CHAPTER 9

9.4 Warranty cost = \$252,530.

9.5 **a.** Warranty cost = \$779.53

9.11 c = \$160.86
R = \$2,451,506

9.17 **a.** $M(12) \cong 16.66$
b. $W_0 \cong 4$ months

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