Computational Imaging Course

Partial Differential Equations, Inverse problems & Computational Imaging

Ronan Fablet

Prof. IMT Atlantique, Lab-STICC

ronan.fablet@imt-atlantique.fr

Web : https://rfablet.github.io/

Objectives

Key notions

Varionatonal schemes **(ill-posed) Inverse problems**

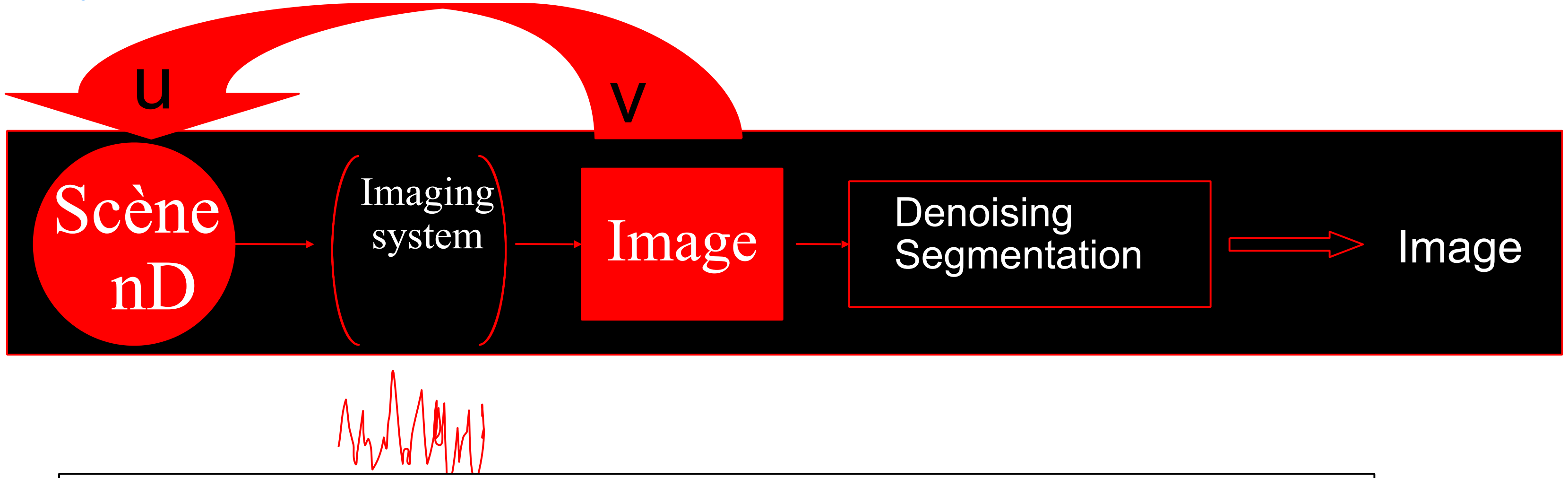
Regularisation **Minimisation**

Course content

- Inverse problems: the image denoising example
- Quadratic prior and the heat equation
- Geometrical interpretation of the diffusion process
- PDE & anisitropic diffusion

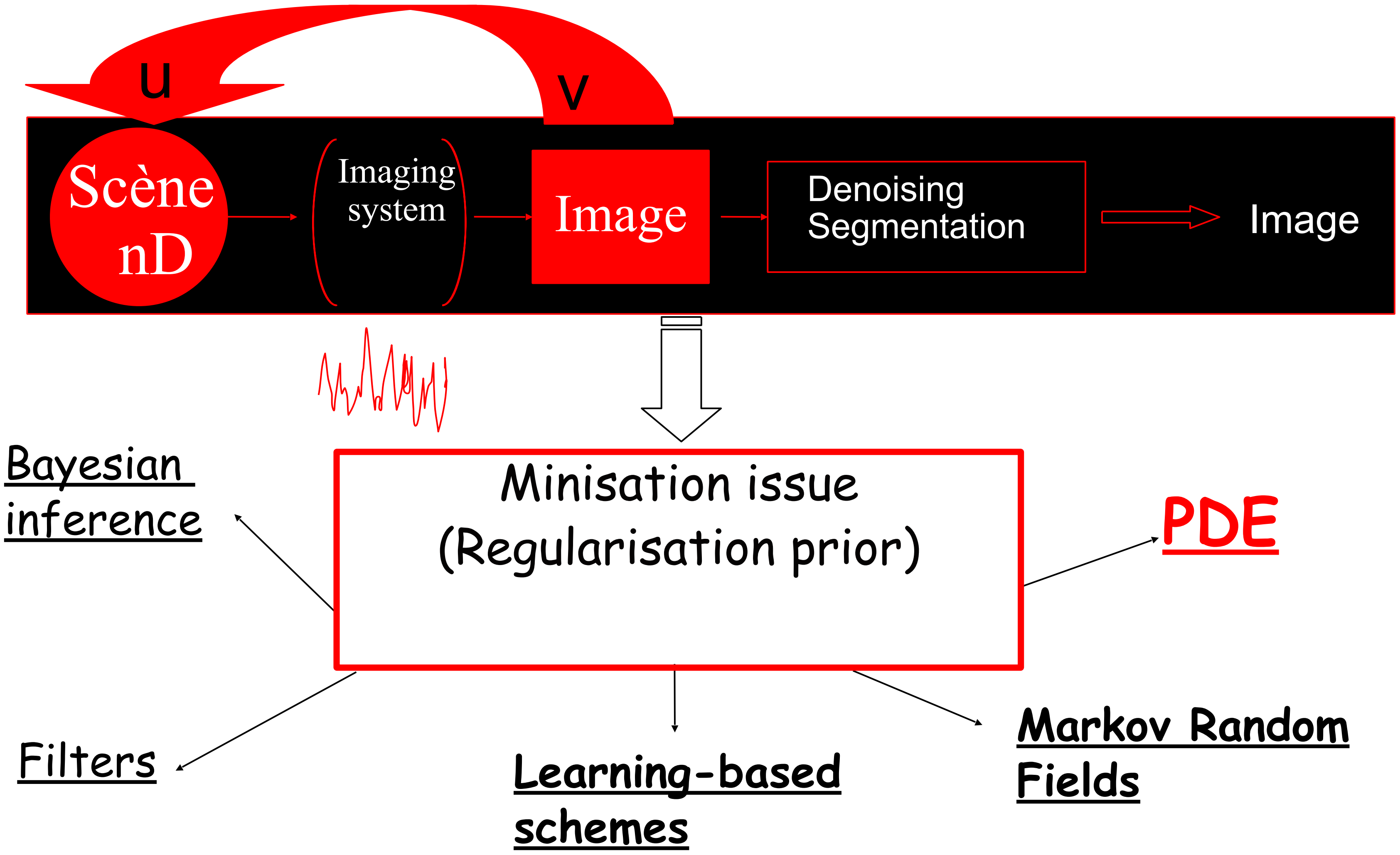
Inverse problems in computational imaging

Retrieving x knowing y given hypothesis on the noise process, the sensor and .....



- denoising
- Deconvolution
- Reconstruction
- Motion estimation
- Contour detection
- ...

Inverse problems in computational imaging



Inverse problem: image denoising as an example

Trade-off between the consistency to:

- **the observation**
- **the prior knowledge**

Minimization of a unique cost function

Solution stated as the solution of a minimization issue

$$\widehat{u} = \arg\min_{u \in \mathcal{U}} E_{obs}(u, v) + E_{reg}(u)$$

- **Consistency between the observations and the solution**
- **Agreement to the expected properties to be depicted by the solution (e.g., regularity prior)**
- **Illustration with a quadratic formulation**

$$\widehat{u} = \arg\min_{u \in \mathcal{U}} \|u - v\|^2 + \|\nabla u\|^2$$

Euler-Lagrange Equation & Heat Equation

Minimization issue: variational formulation

Solution stated as the solution of a minimization issue

$$\widehat{u} = \arg\min_{u \in \mathcal{U}} \|u - v\|^2 + \|\nabla u\|^2$$

- **Variables x,y regarded as a scalar function**
- **Defintion of the variational cost:**

$$E(u,v) = \int_{p \in \Omega} \|u(p) - v(p)\|^2 \, dp + \int_{p \in \Omega} \|\nabla u\|^2 \, dp$$

- **Gradient-based minimization**

Minimization issue: variational formulation

Solution stated as the solution of a minimization issue

$$\widehat{u} = \arg\min_{u \in \mathcal{U}} \|u - v\|^2 + \|\nabla u\|^2$$

- **Variables x,y regarded as a scalar function**
- **Defintion of the variational cost:**

$$E(u,v) = \int_{p \in \Omega} \|u(p) - v(p)\|^2 \, dp + \int_{p \in \Omega} \|\nabla u\|^2 \, dp$$

- **Gradient-based minimization**

$$u^{(k+1)} = u^{(k)} - \lambda \cdot \nabla_u E(u^{(k)}, v)$$

Minimization issue: variational formulation

Solution stated as the solution of a minimization issue

$$\widehat{u} = \arg\min_{u \in \mathcal{U}} \|u - v\|^2 + \|\nabla u\|^2$$

- **Variables x,y regarded as a scalar function**
- **Defintion of the variational cost:**

$$E(u,v) = \int_{p \in \Omega} \|u(p) - v(p)\|^2 \, dp + \int_{p \in \Omega} \|\nabla u\|^2 \, dp$$

- **Gradient-based minimization**

$$u^{(k+1)} = u^{(k)} - \lambda \cdot \nabla_u E(u^{(k)}, v)$$

Minimization issue: variational formulation

Euler-Lagrange: Calculus of Variations (Gateau derivative)

$$\nabla_u E(u,v) = \frac{\partial E}{\partial u} - div\left(\frac{\partial E}{\partial \nabla u}\right)$$

$$\nabla_u E(u,v) = \frac{\partial E}{\partial u} - \sum_{x_i} \frac{\partial}{\partial x_i}\left(\frac{\partial E}{\partial u_{x_i}}\right)$$

Diffusion equation as continuous-case of the gradient descent

$$\frac{\partial u}{\partial t} = -\lambda \nabla_u E(u,v)$$

Minimization issue: variational formulation

Eueler-Lagrange: Quadratic case

$$\widehat{u} = \arg\min_{u \in \mathcal{U}} \|u - v\|^2 + \alpha \|\nabla u\|^2 \qquad E(u, v) = \|u - v\|^2 + \alpha \|\nabla u\|^2$$

Minimization issue: variational formulation

Eueler-Lagrange: Quadratic case

$$\widehat{u} = \arg\min_{u \in \mathcal{U}} \|u - v\|^2 + \alpha \|\nabla u\|^2 \qquad E(u, v) = \|u - v\|^2 + \alpha \|\nabla u\|^2$$

$$\nabla_u E(u, v) = 2 \cdot (u - v) - \alpha \Delta u$$

$$u^{(k+1)} = u^{(k)} - \lambda \left(2 \cdot \left(u^{(k)} - v \right) - \alpha \Delta u^{(k)} \right)$$

$$\frac{\partial u}{\partial t} = 2 \cdot \lambda \left(\left(v - u^{(k)} \right) + \alpha \Delta u^{(k)} \right)$$

Diffusion equation

Diffusion process using the heat equation

t=0 t=3 t=25

From Nielsen, Lauze and Kornprobst, Computer Vision Course, 2006

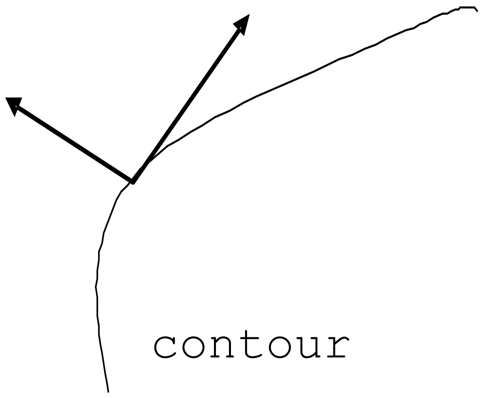
Minimization issue: variational formulation

Euler-Lagrange: Quadratic case. Heat equation

$$\frac{\partial u}{\partial t} = \alpha \Delta u$$

Invariance of the Laplacian operator w.r.t. the local Frame

$$\Delta u = u_{TT} + u_{NN}$$



Green solution of the heat equation

$$u(p,t) = G_t * u(p,0) \quad \textbf{with} \quad G(t,p) = \frac{1}{4\pi\alpha t}\exp\left[-\frac{p^t p}{4\alpha t}\right]$$

Anisotropic Diffusion

Principle of the anisotropic diffusion

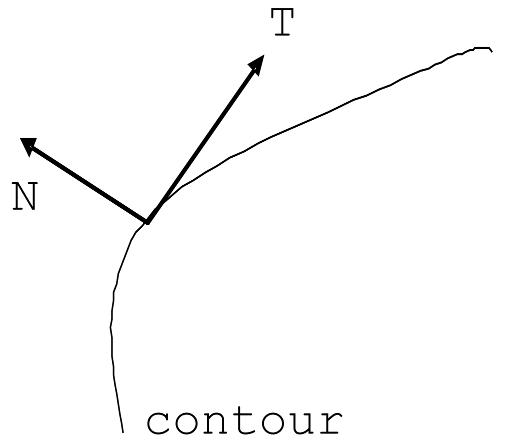
- Heat equation: $$\frac{\partial u}{\partial t} = \alpha \Delta u$$

- Anisotropic diffusion:

$$\frac{\partial u}{\partial t} = c_{TT}\left(\nabla u\right) u_{TT} + c_{NN}\left(\nabla u\right) u_{NN}$$

$$\lim_{\|\nabla u\| \to 0} c_{TT}\left(\nabla u\right) = \lim_{\|\nabla u\| \to 0} c_{NN}\left(\nabla u\right)$$

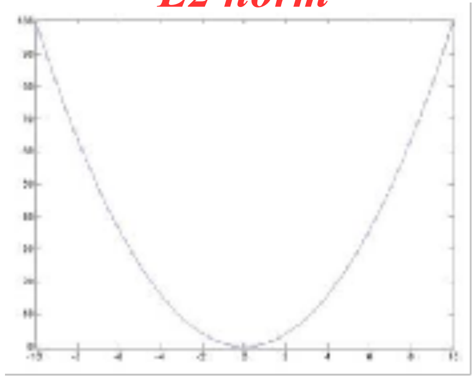
$$\lim_{\|\nabla u\| \to \infty} c_{NN}\left(\nabla u\right) / c_{TT}\left(\nabla u\right) = 0$$



Total variation & curvature motion

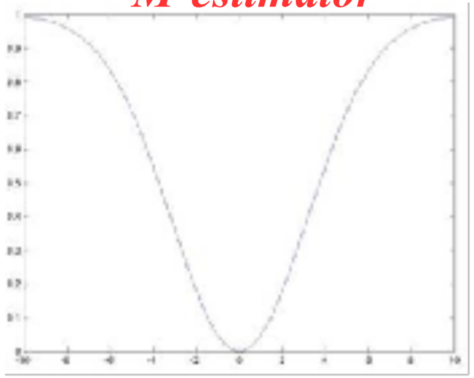
- Preserving contours

L2 norm



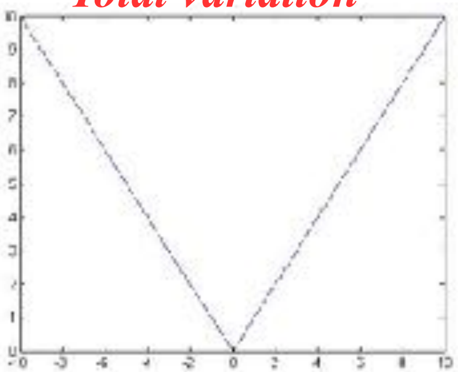
$$\rho(t) = t^2$$

M-estimator



$$\rho(t) = 1 - \exp\left[-\lambda t^2\right]$$

Total variation



$$\rho(t) = |t|$$

- Modified variational cost

$$E(u,v) = \int_{p\in\Omega} \|u(p) - v(p)\|^2 \, dp + \int_{p\in\Omega} \rho\left(\|\nabla u\|\right) dp$$

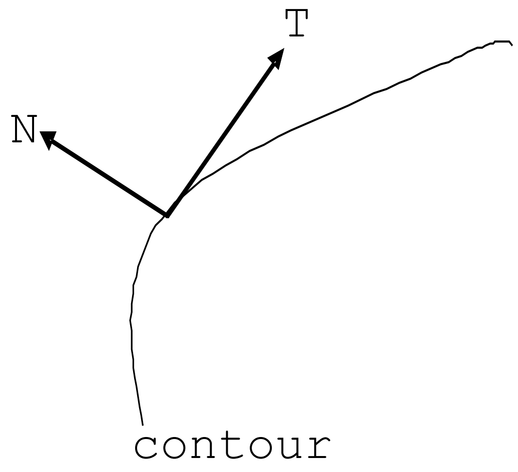
Total variation & curvature motion

- Geometrical interpretation of the regularization

$$div\left[\frac{\nabla u}{\|\nabla u\|}\rho'\left(\|\nabla u\|\right)\right] = \frac{1}{\|\nabla u\|}\rho'\left(\|\nabla u\|\right)u_{TT} + \rho''\left(\|\nabla u\|\right)u_{NN}$$

- Total variation example $\rho(t) = |t|$

$$div\left[\frac{\nabla u}{\|\nabla u\|}\rho'\left(\|\nabla u\|\right)\right] = div\left[\frac{\nabla u}{\|\nabla u\|}\right] = \kappa$$



- TV diffusion $$\frac{\partial u}{\partial t} = \kappa$$

Variation totale, mouvement par courbure

- Curvature of image level-lines

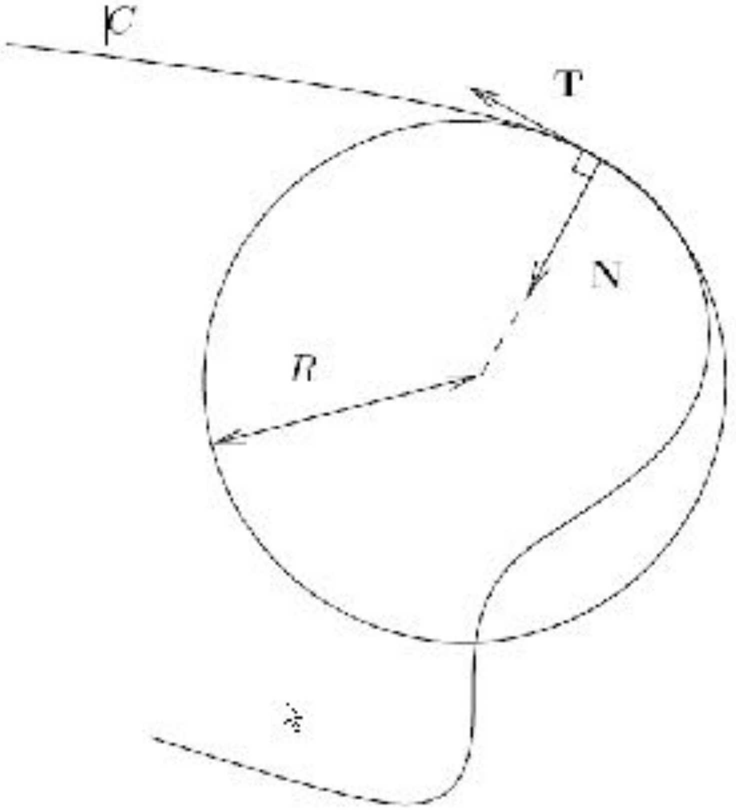


Fig. 2.1. Tangent vector, normal vector and curvature. The curvature κ is equal to $1/R$. In this case, the normal points inward the osculating circle, and the curvature is positive

$$div\left[\frac{\nabla u}{\|\nabla u\|}\right] = \kappa = \frac{u_{TT}}{u_{NN}}$$

From «Geometric Curve Evolution and Image Processing», F. Cao, 2002

Total variation & curvature motion

- TV denoising

$\beta = 20$

$\beta = 6$

$\beta = 1$

From Nielsen, Lauze and Kornprobst, Computer Vision Course, 2006

Anisotropic diffusion: other examples

- Perona-Malik diffusion

$$\frac{\partial u}{\partial t} = div\left[\frac{\nabla u}{\|\nabla u\|}\rho'\left(\|\nabla u\|\right)\right] = div\left[c\left(\|\nabla u\|\right)\nabla u\right]$$

- Examples of functions for c

$$c(t) = \exp\left[-t^2/K^2\right] \qquad c(t) = \left[1 + t^2/K^2\right]^{-1}$$

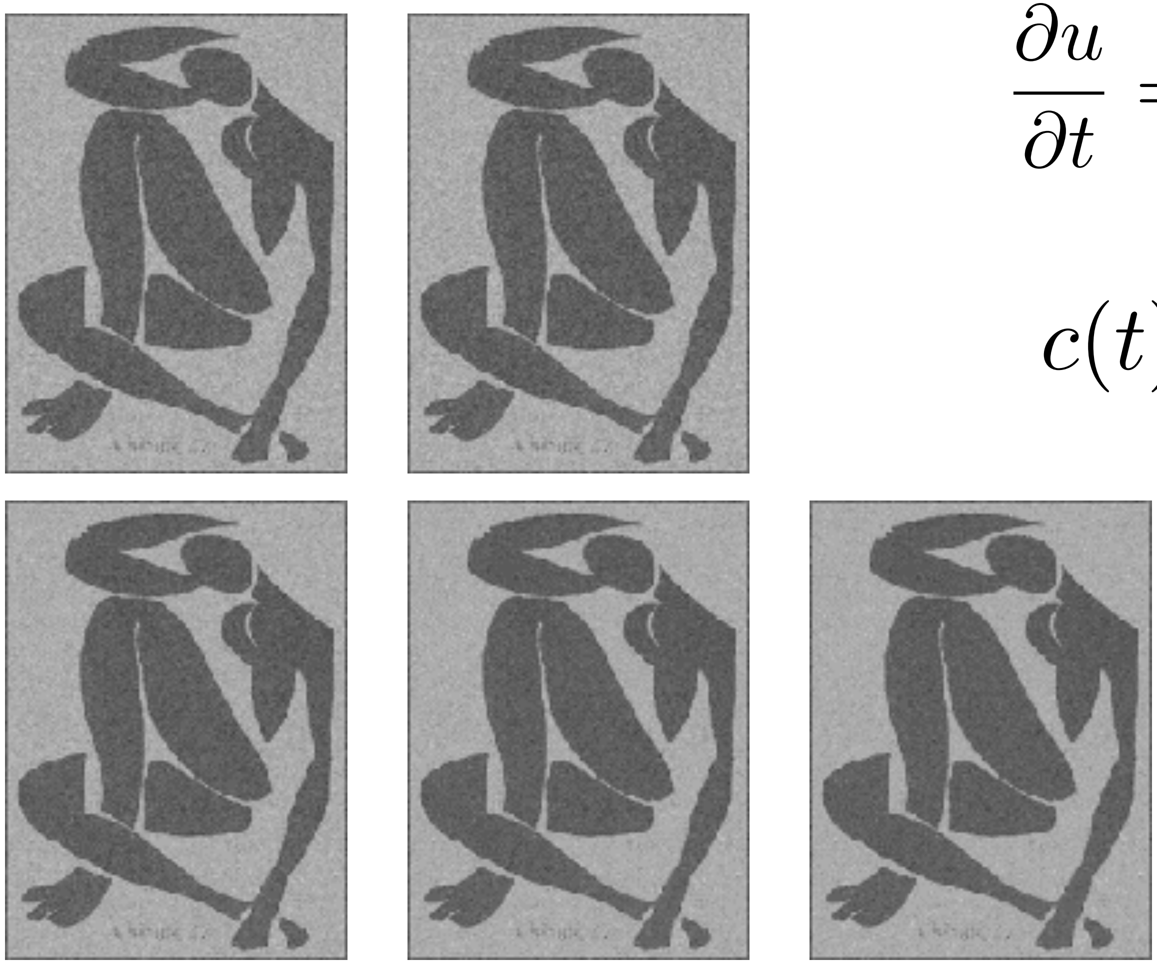
$$c(t) = \left[1 + t^2/K^2\right]^{-1/2}$$

From Nielsen, Lauze and Kornprobst, Computer Vision Course, 2006

Anisotropic diffusion: Perona-Malik diffusion

$$\frac{\partial u}{\partial t} = div \left[c \left(\|\nabla u\| \right) \nabla u \right]$$

$$c(t) = \exp\left[-t^2/K^2\right]$$



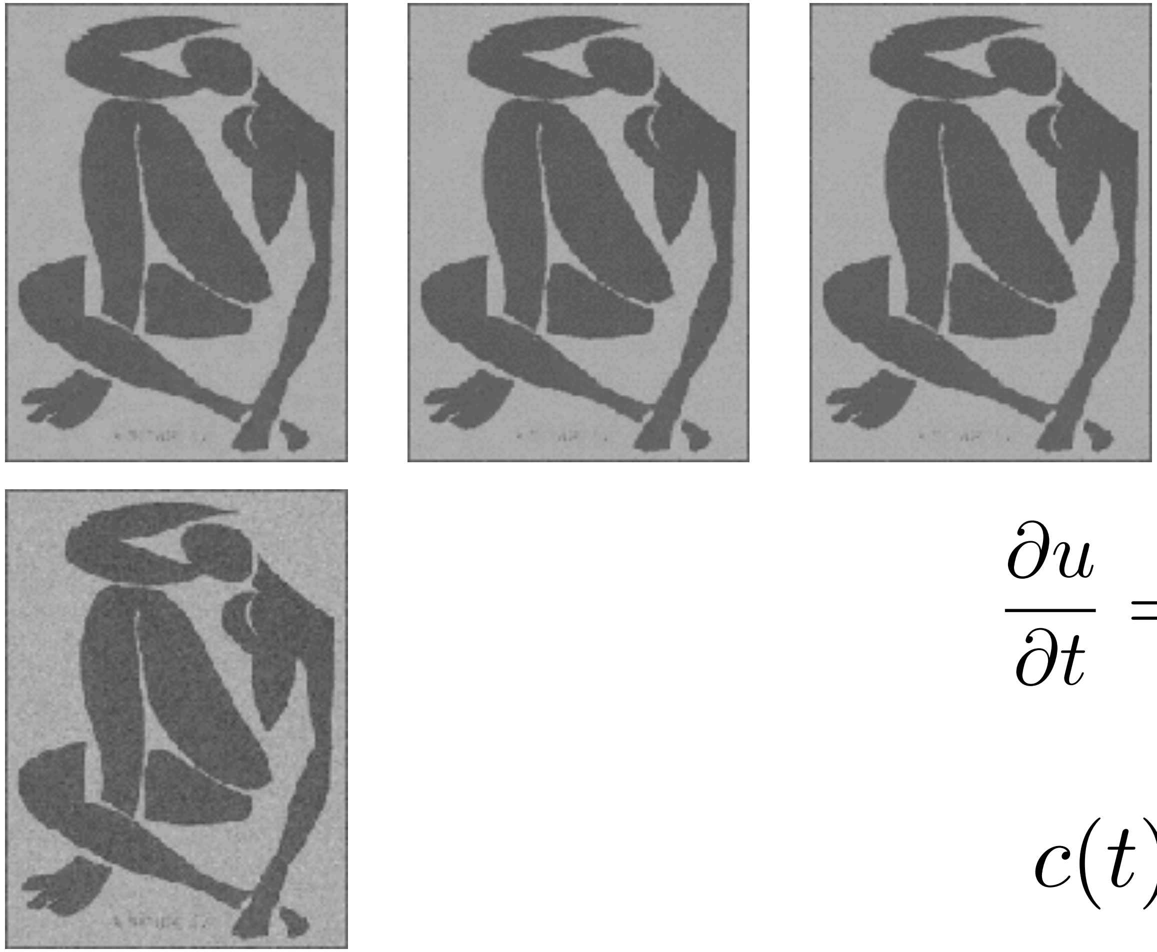
K=1

K=10

t=10 t=30 t=100

From Nielsen, Lauze and Kornprobst, Computer Vision Course, 2006

Anisotropic diffusion: Perona-Malik diffusion



K=20

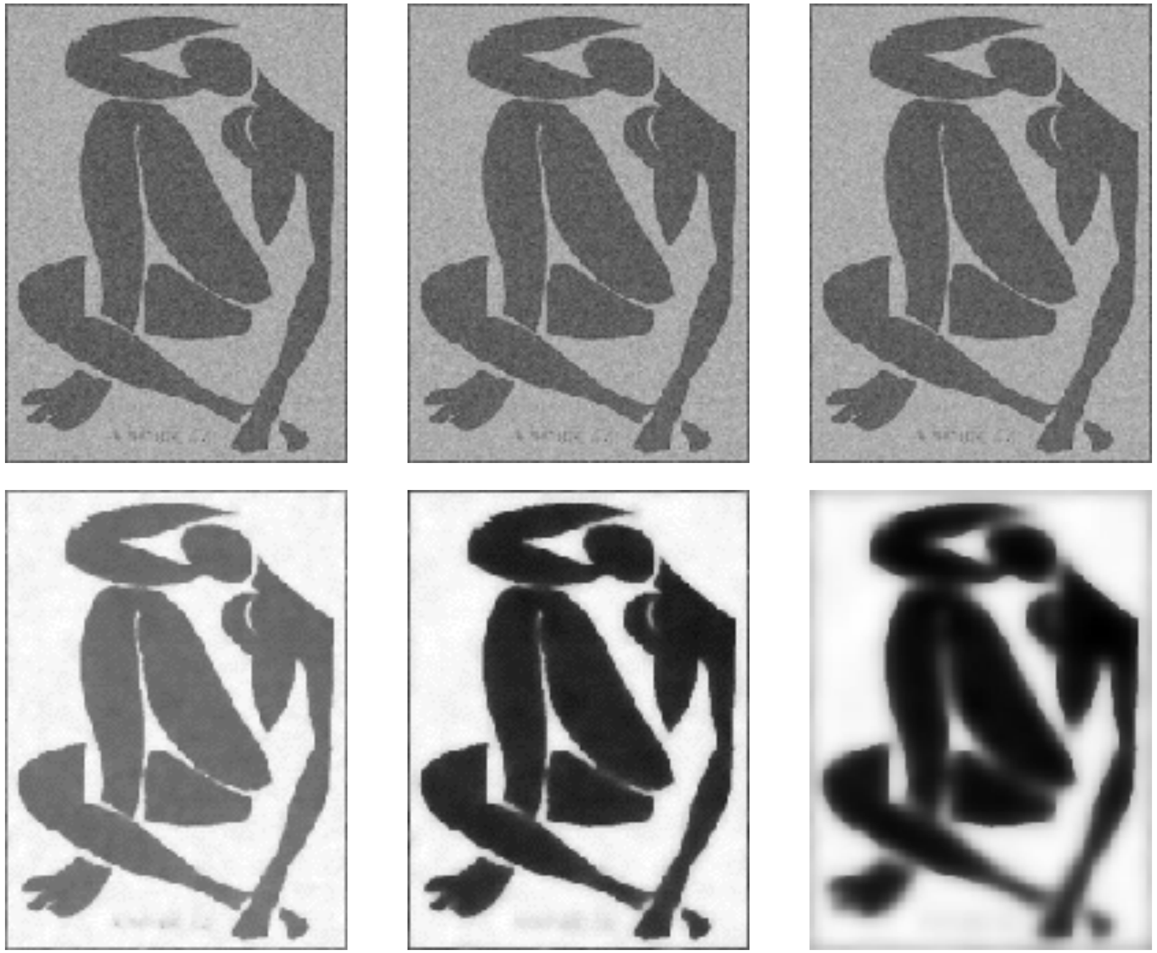
K=50

t=10 t=30 t=50

$$\frac{\partial u}{\partial t} = div\left[c\left(\|\nabla u\|\right)\nabla u\right]$$

$$c(t) = \exp\left[-t^2/K^2\right]$$

Anisotropic diffusion: Perona-Malik diffusion



K=1

K=10

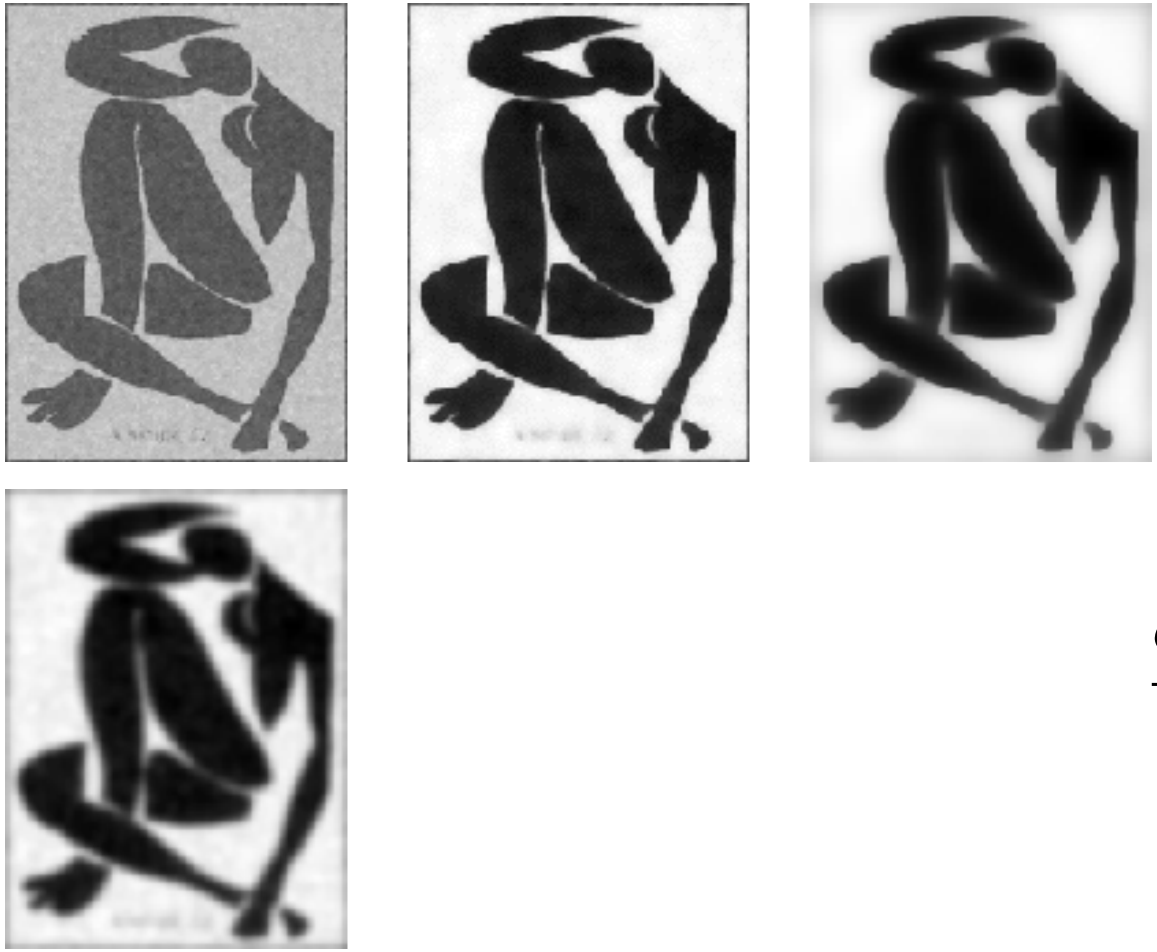
t=10 t=20 t=50

$$\frac{\partial u}{\partial t} = div\left[c\left(\|\nabla u\|\right)\nabla u\right]$$

$$c(t) = \left[1 + t^2/K^2\right]^{-1}$$

From Nielsen, Lauze and Kornprobst, Computer Vision Course, 2006

Anisotropic diffusion: Perona-Malik diffusion



K=1

K=50

t=10 t=30 t=100

$$\frac{\partial u}{\partial t} = div\left[c\left(\|\nabla u\|\right)\nabla u\right]$$

$$c(t) = \left[1 + t^2/K^2\right]^{-1}$$

Mean curvature flow (Guichard et al.)

Geometrical evolution of curve vs. Geometrical evolution of the level-lines of an image

$$\frac{\partial C}{\partial t} = F(\kappa, \kappa', \kappa'', \ldots)\vec{N} \longrightarrow \frac{\partial I}{\partial t} = -g\left(\frac{I_{TT}}{I_N}, \ldots, \ldots\right) I_N$$

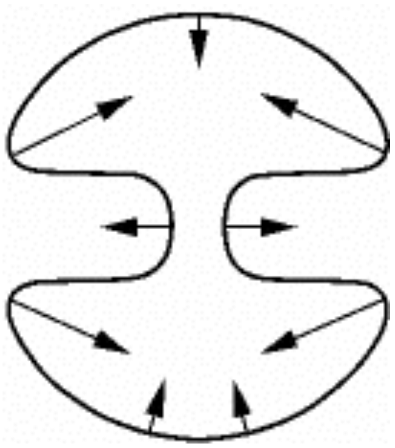
$$\frac{\partial C}{\partial t} = -\frac{1}{I_N} div\left(F\left[I_i, I_{ij}, \ldots\right]\right)\vec{N} \longleftarrow \frac{\partial I}{\partial t} = div\left(F\left[I_i, I_{ij}, \ldots, \ldots\right]\right)$$

Curve evolution **Level-line evolution**

Mean curvature flow (Guichard et al.)

Shape evolution equation:

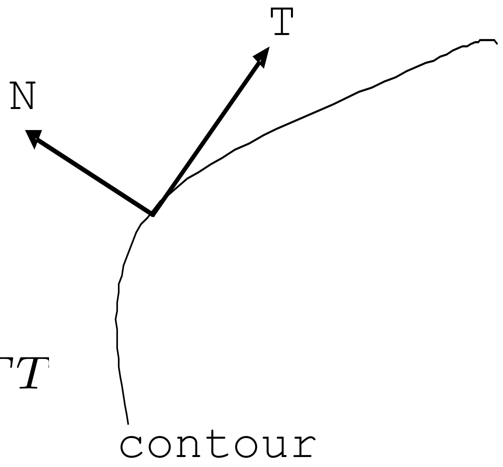
$$\min \int_C ds \quad \longrightarrow \quad \frac{\partial C}{\partial t} = \kappa \vec{N}$$



Application to image level-lines:

$$\frac{\partial C}{\partial t} = g\vec{N} \quad \longrightarrow \quad \frac{\partial \phi}{\partial t} = g\|\nabla \phi\|$$

$$\frac{\partial u}{\partial t} = \kappa \|\nabla u\| \quad \longrightarrow \quad \frac{\partial u}{\partial t} = div\left[\frac{\nabla u}{\|\nabla u\|}\right]\|\nabla u\| = u_{TT}$$



Asymptotic case of an iterated median filter

Mean curvature flow (Guichard et al.)

Examples

t=10

t=50

t=100

$$\frac{\partial u}{\partial t} = div\left[\frac{\nabla u}{\|\nabla u\|}\right]\|\nabla u\| = u_{TT}$$

From Nielsen, Lauze and Kornprobst, Computer Vision Course, 2006

References

References

- **Scale-space and edge detection using anisotropic diffusion. P Perona, J Malik - IEEE Trans. on Pattern Analysis and Machine, 1990**

- **Axioms and fundamental equations of image processing: Multiscale analysis and P.D.E, L. Alvarez, F. Guichard, P. Lions, and J. Morel Archive for Rational Mechanics and Analysis, 16(9), 200-257, 1993**

- **PDE-based methods in image processing, G. Aubert, P. Kornprobst, Springer**

- **A General Framework for Geometry-Driven Evolution Equations, International Journal of Computer Vision 21(3), 187 – 205, W. J. Niessen, B. M. Ter Haar Romeny, L. M. J. Florack, M. A. Viergever, 2007.**