One grid cell wide channel analytical solution for the C-grid and CD-grid

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We consider an east-west one grid cell wide channel with cyclic boundary conditions and wind blowing from the west. We assume that the ocean is at rest and that the sea surface tilt term is zero. We further simplify the problem by setting the Coriolis parameter to zero. With these simplifications, $v = 0$ and the u-momentum equation becomes

$$m\frac{\partial u}{\partial t} = a\tau_a + a\tau_w + \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y}, \tag{1}$$

where m is the total mass of ice and snow and a is the ice concentration.

Because of the cyclic boundary conditions, $\frac{\partial \sigma_{11}}{\partial x} = 0$. At steady-state, the u-momentum equation therefore becomes

$$a\tau_a + a\tau_w + \frac{\partial \sigma_{12}}{\partial y} = 0 \tag{2}$$

Discretizing equation 2 at the e-point we obtain

$$a_e\tau_a + a_e\tau_w + \frac{\sigma_{12u}(i,j) - \sigma_{12u}(i,j-1)}{\Delta y} = 0, \tag{3}$$

where τ_a and τ_w are evaluated at the e-point.

The shear stresses are given by

$$\sigma_{12} = 2\eta\dot{\epsilon}_{12}, \tag{4}$$

where η is the shear viscous coefficient and $\dot{\epsilon}_{12}$ is the shear strain rate. Note that in CICE, $D_S = 2\dot{\epsilon}_{12} = \partial u/\partial y + \partial v/\partial x$.

Given the ellipse aspect ratio e_r, η is expressed as

$$\eta = \frac{e_r^{-2}P}{2\triangle^*}, \tag{5}$$

where $P = P^* h e^{-C^*(1-a)}$ is the ice strength (h is the ice volume) and $\triangle^* = \max(\triangle, \triangle_{min})$.

Because $\dot{\epsilon}_{11}$ and $\dot{\epsilon}_{22}$ are zero, $\triangle = e_r^{-1}|D_S|$.

With $\tau_a = \rho_a C_{da} u_a^2$ and $\tau_w = -\rho_w C_{dw} u_e^2(i,j)$ we have

$$a_e \rho_a C_{da} u_a^2 - a_e \rho_w C_{dw} u_e^2(i,j) + \frac{\eta_u(i,j) D_S(i,j) - \eta_u(i,j-1) D_S(i,j-1)}{\Delta y} = 0. \tag{6}$$

With the no-slip boundary condition we can find that

$$D_S(i,j) = \frac{0 - u_e(i,j)}{\Delta y/2}, \tag{7}$$

and

$$D_S(i,j-1) = \frac{u_e(i,j) - 0}{\Delta y/2}, \tag{8}$$

which means that $D_S(i,j) < 0$ and $D_S(i,j-1) = -D_S(i,j)$.

We want to solve equation 6 for $u_e(i,j)$. For simplicity, we drop (i,j), i.e. $u_e(i,j) = u_e$. With strong winds, the ice is in the plastic regime, that is $\triangle^* = \triangle = e_r^{-1}|D_S|$. We can write equation 6 as

$$a_e \rho_a C_{da} u_a^2 - a_e \rho_w C_{dw} u_e^2 - \frac{P}{e_r \Delta y} = 0. \tag{9}$$

The transition between the plastic and viscous regimes occur for a wind velocity $u_a = u_{a*}$. When the transition between these regimes occur, $\triangle = \triangle_{min}$. This leads to a sea ice velocity equal to $e_r \triangle_{min} \Delta y/2$. Replacing u_e in equation 9 by that expression gives

$$a_e \rho_a C_{da} u_{a*}^2 - a_e \rho_w C_{dw} \left[\frac{e_r \triangle_{min} \Delta y}{2}\right]^2 - \frac{P}{e_r \Delta y} = 0. \tag{10}$$

Solving for u_{a*} we get

$$u_{a*} = \left[\frac{\rho_w C_{dw}}{\rho_a C_{da}} \left(\frac{e_r \triangle_{min} \Delta y}{2}\right)^2 + \frac{P}{a_e \rho_a C_{da} e_r \Delta y}\right]^{1/2}. \tag{11}$$

If $u_a > u_{a*}$ the ice is in the plastic regime and u_e can be found by solving equation 9. We obtain

$$u_e = \left[\frac{\rho_a C_{da} u_a^2}{\rho_w C_{dw}} - \frac{P}{a_e \rho_w C_{dw} e_r \Delta y}\right]^{1/2}, \tag{12}$$

where the first term is the freedrift velocity while the second one, which is due to the rheology, slows down the ice. In the plastic regime, the shear stresses $\sigma_{12u}(i,j)$ and $\sigma_{12u}(i,j-1)$ are respectively $-e_r^{-1}P/2$ and $e_r^{-1}P/2$.

On the other hand, if the wind is too weak (i.e, $u_a < u_{a*}$), the ice is in the viscous regime. In this case $\triangle^* = \triangle_{min}$ and equation 6 is written as

$$a_e\rho_a C_{da}u_a^2 - a_e\rho_w C_{dw}u_e^2 - \frac{2Pu_e}{e_r^2\triangle_{min}\Delta y^2} = 0, \tag{13}$$

which can be rewritten as

$$u_e^2 + \frac{2P}{a_e\rho_w C_{dw}e_r^2\triangle_{min}\Delta y^2}u_e - \frac{\rho_a C_{da}u_a^2}{\rho_w C_{dw}} = 0. \tag{14}$$

The solution of equation 14 is thus

$$u_e = -\frac{P}{a_e\rho_w C_{dw}e_r^2\triangle_{min}\Delta y^2} + \sqrt{\left(\frac{P}{a_e\rho_w C_{dw}e_r^2\triangle_{min}\Delta y^2}\right)^2 + \frac{\rho_a C_{da}u_a^2}{\rho_w C_{dw}}} \tag{15}$$