Review for manuscript on High-Order IIM to solve interface problems

In general, the paper is well written and covers an interesting topic of numerical methods for solving PDE problems with discontinuities in one dimension. Although the quality of the paper is already very good, there are some minor changes suggested (see attached file) to improve the readability of the paper and improve some small English errors.

I strongly suggest the acceptance of the paper after these minor changes have been covered.

ANSWER:
We would like to thank the reviewer for the constructive comments. We have addressed all of them and modified the paper accordingly. Please note that the comments are in black, while our answers are in blue.

Location	Observed error or suggested change
p. 1	Third paragraph is not clearly written. After defining what a IFD and a IIM is, three redaction mentions a combination of both without using the acronyms. Also, no new acronym is declared for the combined method. **ANSWER:** Based on the reviewer's feedback, we have made slight modifications to this paragraph in order to clarify the definition of IFD-IIM. The revised version is presented below: This paper focuses on the basic ideas of combining the IFD and IIM to achieve high-order approximations for second-order derivatives of both continuous and discontinuous real-valued functions. The IFD scheme offers a highly accurate numerical method [3,19], while the IIM handles discontinuities through minimal adjustments made exclusively at grid points where the stencil intersects the interface [32,6], yielding additional terms known as jump contributions. The resulting method will be named as implicit finite-difference immersed interface method (IFD-IIM).
p. 2 and after	There is no consequent use of previously defined acronyms for IFD, IIM and their combined version. **ANSWER:** We appreciate this observation, and we have taken action accordingly. In the revised manuscript, we have replaced all names with their corresponding acronyms as suggested.
p. 2	First paragraph: must say “provide”. **ANSWER:** This typo has been corrected.

p. 2	Fourth paragraph: there is a mention of some intervals, but points x_i are defined. **ANSWER:** We changed the paragraph, as follows: We approximate the numerical solution on the domain $[\mathfrak{a}, \mathfrak{b}]$ that is divided into *N* sub-intervals using the points x_i, as follows $$x_i = \mathfrak{a} + (i-1)h, \qquad i = 1,2,\ldots,N,N+1,$$ where the grid size is given by $h = (\mathfrak{b} - \mathfrak{a})/N$.
p. 3	Before equation (4) must say “and the” **ANSWER:** This typo has been corrected.
p. 3	There is no clear separation of what sentences belong to theorem statement and to its proof. Applies for both theorems in this page. **ANSWER:** We appreciate this valuable remark. It has come to our attention that the Scipedia platform does not recognize the proof environment. As a result, we have taken steps to manually ensure that the word "proof" is included in the new manuscript, making it clear and identifiable for both sections.
p. 4	Third equation has a first appearance of u with a subindex using a calligraphic l without any mention to its meaning. **ANSWER:** The calligraphy ℓ in subindex of *u*, defined at the begging of the proof, is just to identify the new extended function. It does not carry any specific or unique significance beyond this contextual labeling.
p. 4	Section 2.2 starts with a wrong English construction. **ANSWER:** This paragraph has been modified as follows: Before applying previous results to approximate differential equations, it would be useful to express the operator $\mathfrak{D}^2 u$ using finite differences for a real-valued function *u* rather than its second-order derivative u_{xx}.

p. 5	In the reviewed PDF, it appears that the letter x used in x\in\{a,b\} has an unnecessary bold style. **ANSWER:** This typo has been corrected.
p. 8	Third paragraph of Sect. starts with a wrong English construction. **ANSWER:** This paragraph has been modified as follows: By employing the IFD method (2) and approximation (12), equation (33) can be approximated for regular points as follows …
p. 9	I strongly recommend that the complete analytical solution is written in the body of the paper for the used examples, as the reader might need a connection between a known function u (as in equation (40)) and a properly posed partial differential equation for which this equation is a solution. My suggestion is that you include the analytic construction of your example for the Poisson equation and also for one of the time-dependent problems. Otherwhise, it might become cumbersome for the reader to easily understand what the PDE problems being solved are. **ANSWER:** In accordance to the reviewer's suggestion, we have included the complete problem descriptions for Examples 1, 2, 4 and 5. For instance, Example 1 is now described as follows: **5.1 Example 1. Poison equation with smooth solution** Example 1 considers the one-dimensional Poisson problem $$\begin{aligned} u_{xx}(x) &= f(x), \quad x \in (0,1), \\ u(0) &= e^{-\pi/4}, \\ u(1) &= e^{-9\pi/4}, \end{aligned} \qquad (40)$$ where the right-hand side in (40) is given by $$f(x) = 4\pi(16\pi x^2 - 8\pi x + \pi - 2)e^{-4\pi(x-1/4)^2}.$$ The exact solution of problem (40) is an infinitely smooth and differentiable function given by the following expression $$u(x) = e^{-4\pi(x-1/4)^2}. \qquad (41)$$ Note that the function f and Dirichlet boundary conditions are obtained directly from (41). Due to the regularity of the solution, the jump contributions $\mathcal{C}_I$ and $\mathcal{C}_{I+1}$ are equal to zero.

Tables 1-6	For each value of N, the reader sees 8 columns, 4 of them are named “Order” and the other are pairs of L_\infty and L_2 norms, but there is no clear labeling of the columns or lines splitting the column groups to be understandable. For Table 1 it seems that columns 2-5 refer to the case using b=0 while the others refer to b=1/12. For the following tables they refer all to the case with b=1/12 but for two different interfaces. Make this clearer! **ANSWER:** There was an issue with the Scipedia platform. We have corrected the problem by ensuring that the tables are now displayed with their proper labels. Thank you for addressing this matter.
Fig 2(b)	It would help to mention for this value of N=40, the exact location of the x_I point, as from the plot it seems that the dashed vertical line at x_\alpha has the same value as the left grid point. **ANSWER:** Indeed, this observation is included in the manuscript following the presentation of Example 2. However, we have decided to incorporate it into the label of Figure 2 to enhance comprehension. For the case $x_\alpha = 0.4$, we always have $h_R/h = 1$ for $N = 10 \times 2^n$ $(n = 0,1,2,\ldots)$; thus the interface is always located at one grid point of that resolution.
Fig 3	What is the need for the line labeled as “4.00”? **ANSWER:** As the numerical order does not always follow a straight line due to the location of the interface, this solid black line shows the numerical order calculated by the regression-line slope based on a least squares method of the L_∞-norm. This comment has been integrated into the text, as well as into the label of Figure 3, to provide a more comprehensive explanation of its significance.
p. 10	The data in Table 1 consider values of N only until 160, but the plots contain values up to N=320. Make them consistent. **ANSWER:** As highlighted by the reviewer, the values of N in the tables are specified as 10, 20, 40, 80, and 160. However, in the plots of Figures 3, 4 and 5, we have included a range of points from 10 to 320 (a total of 310 N values) to provide a more comprehensive view of the behavior.

p. 11	Change “random” for “scattered”, as at least part of this distribution can be well explained through the location of the interface w.r.t. the grid. **ANSWER:** This word has been changed in the revised manuscript.
p. 12	Figure 6(b) or the text do not mention which values of N were used. **ANSWER:** These values are now integrated in the caption of Figure 6.
p. 13	The last paragraph mentions Figures 8 and 9 very briefly, but these figures present a middle plot that has not been shown before and needs some explanations. What is the reader supposed to see in these plots 8(b) and 9(b). **ANSWER:** As suggested by the reviewer, we extent our analysis of these figures as follows: … On the other hand, the one-dimensional results at *T = 0.5* are presented in Figs. 8 and 9. As expected, we can accurately recover the exact solution using the proposed method for *N* = 40 and considering different interface values. Additionally, we provide the absolute error for both cases, noting that the maximum error is found in close proximity to the point x_α. This behavior is also expected since the local truncation error for irregular points is less accurate than for regular points. Finally, the convergence analysis using *N* = 10, 20, 40, 80 and 160 confirms the fourth-order global convergence of the proposed method.
p. 15	Again, an explanation about the plots in Figure 11(b) and 12(b) would strongly improve the manuscript. **ANSWER:** As suggested by the reviewer, we extent our analysis of these figures as follows: … More in-depth analysis of the results at *T* = 0.5 are illustrated in Figs. 11 and 12, corresponding to values of $x_\alpha = 0.4$ and $x_\alpha = 0.63$, respectively. First, the numerical solution is accurately approximated using the proposed implicit method for *N* = 40, regardless of the interface locations. It is worth noting that the maximum error for $x_\alpha = 0.63$ is concentrated near to this point; however, this is not observed for $x_\alpha = 0.4$. This difference in behavior can be attributed to the interplay between the interface location ($h_R/h = 1$ and $x_\alpha = 0.4$) and the non-zero right-hand side values, which contribute to the local truncation errors. Grid refinement analyses are also provided in these figures using *N* = 10, 20, 40, 80 and 160. As anticipated, the IFD-IIM method demonstrates fourth-order accuracy, a validation that is further detailed in Table 6.

p. 17	Must be corrected as “downloaded” **ANSWER:** The word “downloaded” was originally in the Appendix. Following the suggestion of another reviewer, we have removed the Appendix, as it cannot be properly displayed on the Scipedia platform.