

Ratio estimation in SIMS analysis

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ABSTRACT

The determination of an isotope ratio by secondary ion mass spectrometry (SIMS) traditionally involves averaging a number of ratios collected over the course of a measurement. We show that this method leads to an additive positive bias in the expectation value of the estimated ratio that is approximately equal to the true ratio divided by the counts of the denominator isotope of an individual ratio. This bias does not decrease as the number of ratios used in the average increases. By summing all counts in the numerator isotope, then dividing by the sum of counts in the denominator isotope, the estimated ratio is less biased: the bias is approximately equal to the ratio divided by the summed counts of the denominator isotope over the entire measurement. We propose a third ratio estimator (Beale's estimator) that can be used when the bias from the summed counts is unacceptably large for the hypothesis being tested. We derive expressions for the variance of these ratio estimators as well as the conditions under which they are normally distributed. Finally, we investigate a SIMS dataset showing the effects of ratio bias, and discuss proper ratio estimation for SIMS analysis.

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1. Introduction

Ratio estimation is broadly used in all scientific disciplines. Unfortunately, the mathematical issues associated with calculating meaningful ratios from experimental data are frequently ignored or misunderstood. In experiments dealing with blood sera in 1909, Greenwood and White [1] observed that a distribution of ratios (calculated as a mean of a number of individual ratios), randomly generated, was positively skewed. Karl Pearson explained the effect mathematically [2]. The statistical properties of ratio estimation were well-explored in the 1950s and 1960s (e.g., [3–5]), and these methods have been applied sparingly to various disciplines (e.g., [6,7]).

The crux of the problem of ratio estimation can be understood by considering a statistical variate z defined in the range $0 < z \leq \infty$ that has a probability $p(z_i)$ of taking on the value z_i . Assuming the expectation values of z and $1/z$, $E\{z\}$ and $E\{1/z\}$, exist, we can calculate:

$$E\{z\}E\{1/z\} = \sum_{\text{all } i} z_i p(z_i) \sum_{\text{all } k} \frac{1}{z_k} p(z_k) \quad (1)$$

If z takes on just one value z_j (such that the probability $p(z_j) = 1$), this reduces to:

$$E\{z\}E\{1/z\} = (z_j p(z_j)) \left(\frac{1}{z_j} p(z_j) \right) = p(z_j)^2 = 1 \quad (2)$$

If z takes on more than one value (z_1, z_2, \dots), we expand the terms in Eq. (1):

$$E\{z\}E\{1/z\} = (z_1 p(z_1) + z_2 p(z_2) + \dots) \left(\frac{1}{z_1} p(z_1) + \frac{1}{z_2} p(z_2) + \dots \right) \quad (3)$$

Collecting the squared terms and the cross terms for all possible values of z :

$$E\{z\}E\{1/z\} = \left(\sum_{\text{all } k} p(z_k)^2 \right) + \left(\sum_{\substack{\text{all } m, n \\ m \neq n}} \left(\frac{z_m}{z_n} + \frac{z_n}{z_m} \right) p(z_m) p(z_n) \right) \quad (4)$$

We wish to compare this with 1, which we can write as the square of the sum of all possible probabilities $p(z_i)$:

$$1 = \left(\sum_{\text{all } i} p(z_i) \right) \left(\sum_{\text{all } i} p(z_i) \right) = \left(\sum_{\text{all } k} p(z_k)^2 \right) + \left(\sum_{\substack{\text{all } m, n \\ m \neq n}} 2 p(z_m) p(z_n) \right) \quad (5)$$

Using the fact:

$$\left(\frac{z_m}{z_n} + \frac{z_n}{z_m} \right) > 2 \quad \text{for } z_m, z_n > 0 \text{ and } z_m \neq z_n \quad (6)$$

we see that if z that takes on more than one value, Eq. (4) has a larger value than Eq. (5) and:

$$E\{z\}E\{1/z\} > 1 \quad (7)$$

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Therefore we have shown that if z is not single-valued:

$$E\{1/z\} > 1/E\{z\} \tag{8}$$

as also given in [8]. It follows that if z is an unbiased estimator of some quantity θ , $1/z$ cannot be an unbiased estimator of $1/\theta$ [8]. For two positive, statistically independent random variables X and Y :

$$E\{Y/X\} = E\{Y\}E\{1/X\} > E\{Y\}/E\{X\} \tag{9}$$

showing that this ratio estimator is biased: an experiment attempting to measure $E\{Y\}/E\{X\}$ using $E\{Y/X\}$ will have an expectation value larger than the true value.

The current methods for determining isotope ratios in natural samples by secondary-ion mass spectrometry (SIMS) have their roots in other types of mass spectrometry. In mass spectrometry, it is typically harder to determine the absolute abundance of an element than it is to determine the relative abundances of the isotopes of that element. This naturally leads to reporting isotope data as ratios, eliminating the need to know the absolute abundance of any species. A critical analytical problem is variation in the strength of the ion beam in the mass spectrometer. If the mass spectrometer has only one collector, it is important to account for changes in ion-beam strength in order to get accurate isotope ratios. Time interpolation is widely used to account for slow drifts in ion-beam strength. Isotopes are measured in sequence (e.g., ^{24}Mg , ^{25}Mg , ^{26}Mg , ^{24}Mg , ...) such that each isotope samples the time variation in the ion-beam strength. To correct for time drift, the signal for one isotope (e.g., ^{24}Mg) is interpolated between two measurements to give the signal at the time when another isotope was measured. The isotope ratio is then calculated for each cycle and the individual ratios are averaged to give the final result. By collecting data as a series of ratios, one can also identify noise bursts or other problems that can affect the data, and cycles affected by the problem can be eliminated from the measurement. If count rates are so low that an isotope gets zero counts in a cycle, calculating by-cycle ratios does not work (i.e., a ratio with a zero denominator is meaningless). In this case, SIMS analysts typically add up all of the counts for each isotope from all of the cycles and calculate a single ratio for the measurement. It is widely believed that the two methods, using the mean of the individual ratios to calculate the final result and using the ratio of the total counts for each isotope to calculate the final result, give the same answer. But this is not the case.

In this paper, we attempt to understand how ratio-estimator bias affects SIMS analysis and provide a framework for calculating isotope ratios with less bias. We will look at three ratio estimators and investigate the first four statistical moments of each estimator: the mean, variance, skewness, and kurtosis. From these statistical moments we can determine the bias (from the first statistical moment, the expectation value), efficiency (from the second statistical moment, the variance), and approach to normality (from the third and fourth statistical moment, the skewness and kurtosis) of each ratio estimator. A specific example of how bias affects SIMS data will be discussed.

2. First statistical moment of ratio estimators r_1 and r_2 : expectation value

We assume two populations X and Y from which n subsamples x and y are randomly drawn. Our goal is to measure the ratio of the population means:

$$R = \frac{\bar{Y}}{\bar{X}} \tag{10}$$

We can immediately think of two ways to estimate R given x and y . The ratio of the sample means:

$$r_1 = \frac{\frac{1}{n} \sum_{i=1}^n y_i}{\frac{1}{n} \sum_{i=1}^n x_i} = \frac{\bar{y}}{\bar{x}} \tag{11}$$

and the mean of the sample ratios:

$$r_2 = \frac{1}{n} \sum_{i=1}^n \frac{y_i}{x_i} = \frac{1}{n} \sum_{i=1}^n \rho_i = \overline{\left(\frac{y}{x}\right)} \tag{12}$$

Under most circumstances in mass spectrometry, r_2 is employed as the ratio estimator.

In this section we wish to calculate the first statistical moment (the expectation value) of each of these ratios ($E\{r_1\}$ and $E\{r_2\}$) to investigate the accuracy of the ratio estimators.

Following [4], we can rewrite r_1 as:

$$r_1 = R \left[1 + \frac{\bar{y} - \bar{Y}}{\bar{Y}} \right] \left[1 + \frac{\bar{x} - \bar{X}}{\bar{X}} \right]^{-1} \tag{13}$$

where \bar{x} and \bar{y} are the sample means, and \bar{X} and \bar{Y} are the population means. The factor

$$\left[1 + \frac{\bar{x} - \bar{X}}{\bar{X}} \right]^{-1} \tag{14}$$

will converge as a geometric series if

$$\left| \frac{\bar{x} - \bar{X}}{\bar{X}} \right| < 1 \tag{15}$$

which is true under the assumption that x 's are positive (in SIMS data, this is equivalent to nonzero counts), and n (the number of measurement cycles) is sufficiently large so that $\bar{x} < 2\bar{X}$ (which is the case in any SIMS dataset where a reasonable measurement is sought).

With these assumptions we can expand Eq. (13) as a geometric series:

$$\begin{aligned} r_1 &= R \left[1 + \frac{\bar{y} - \bar{Y}}{\bar{Y}} \right] \left[1 - \frac{(\bar{x} - \bar{X})}{\bar{X}} + \frac{(\bar{x} - \bar{X})^2}{\bar{X}^2} - \frac{(\bar{x} - \bar{X})^3}{\bar{X}^3} + \dots \right] \\ &= R \left[1 + \frac{\bar{y} - \bar{Y}}{\bar{Y}} \frac{(\bar{x} - \bar{X})}{\bar{X}} - \frac{(\bar{x} - \bar{X})(\bar{y} - \bar{Y})}{\bar{X}\bar{Y}} + \frac{(\bar{x} - \bar{X})^2}{\bar{X}^2} \right. \\ &\quad \left. + \frac{(\bar{x} - \bar{X})^2(\bar{y} - \bar{Y})}{\bar{X}^2\bar{Y}} - \frac{(\bar{x} - \bar{X})^3}{\bar{X}^3} + \frac{(\bar{x} - \bar{X})^3(\bar{y} - \bar{Y})}{\bar{X}^3\bar{Y}} + \frac{(\bar{x} - \bar{X})^4}{\bar{X}^4} + \dots \right] \tag{16} \end{aligned}$$

At this point we assume that our SIMS measurements, the sampled isotope counts x and y , follow two Poisson distributions with expected values equal to the population means \bar{X} and \bar{Y} . The probability that k counts of x occur in a single sampling is $\bar{X}^k e^{-\bar{X}}/k!$; the probability that k counts of y occur in a single sampling is $\bar{Y}^k e^{-\bar{Y}}/k!$. For now we will assume X and Y are not independent.

Before we calculate $E\{r_1\}$, we note the following equalities (some of which are from [8]):

$$\begin{aligned} E\{(\bar{x} - \bar{X})\} &= E\{(\bar{y} - \bar{Y})\} = 0 \\ E\{(\bar{x} - \bar{X})^2\} &= S^2(\bar{x}) = \bar{X}/n \\ E\{(\bar{x} - \bar{X})(\bar{y} - \bar{Y})\} &= S(\bar{x}, \bar{y}) = S(x, y)/n \\ E\{(\bar{x} - \bar{X})^2(\bar{y} - \bar{Y})\} &= (S(x^2, y) - 2\bar{X}S(x, y))/n^2 \\ E\{(\bar{x} - \bar{X})^3(\bar{y} - \bar{Y})\} &= 3\bar{X}S(x, y)/n^2 + O(n^{-3}) \\ E\{(\bar{x} - \bar{X})^3\} &= \bar{X}/n^2 \\ E\{(\bar{x} - \bar{X})^4\} &= 3\bar{X}^2/n^2 + O(n^{-3}) \end{aligned} \tag{17}$$

The population variance of x is denoted by $S^2(x)$, and the population covariance of x and y is denoted by $S(x, y)$.

We can now calculate $E\{r_1\}$ by substituting these expressions into Eq. (16). Keeping terms to order $1/n^2$ (i.e., $O(n^{-2})$):

$$E\{r_1\} \approx R \left[1 + \frac{1}{n} \left(\frac{1}{\bar{X}} - \frac{S(x,y)}{\bar{X}\bar{Y}} \right) + \frac{1}{n^2} \left(\frac{2}{\bar{X}^2} - \frac{S(x,y)}{\bar{X}\bar{Y}} \left(2 + \frac{3}{\bar{X}} \right) + \frac{S(x^2,y)}{\bar{X}^2\bar{Y}} \right) \right] \tag{18}$$

For independent X and Y this reduces to:

$$E\{r_1\} \approx R \left[1 + \frac{1}{n\bar{X}} + \frac{2}{n^2\bar{X}^2} \right] \tag{19}$$

This shows that r_1 , the ratio of the means, is a biased estimator of the ratio, R , consistent with Eq. (9). For r_2 we note:

$$E\{r_2\} = E\left\{ \frac{1}{n} \sum_{i=1}^n \rho_i \right\} = \frac{1}{n} (E\{\rho_1\} + E\{\rho_2\} + \dots + E\{\rho_n\}) \tag{20}$$

$$= \frac{1}{n} (nE\{\rho_i\}) = E\{\rho_i\} = E\{y_i/x_i\}$$

That is, the expectation value of the mean of the ratios is equal to the expectation value of a single ratio. We choose a ratio ρ_i for which the inequality in Eq. (15) holds (in a reasonable SIMS sample set, there would be at least one ratio that satisfies this constraint), and then we can expand r_2 in a power series and obtain an equation like Eq. (16), with individual x_i and y_i replacing the sample means \bar{x} and \bar{y} . Then, to obtain analogous expressions for the expectation values given in Eq. (17), we substitute $n = 1$ and keep terms to $O(\bar{X}^{-2})$. This yields the expectation value for r_2 analogous to Eq. (18):

$$E\{r_2\} \approx R \left[1 + \frac{1}{\bar{X}} + \frac{2}{\bar{X}^2} - \frac{3S(x,y)}{\bar{X}\bar{Y}} \right] \tag{21}$$

For independent X and Y this reduces to:

$$E\{r_2\} \approx R \left[1 + \frac{1}{\bar{X}} + \frac{2}{\bar{X}^2} \right] \tag{22}$$

which shows that r_2 , the mean of the ratios, is also a biased estimator of R , but with a bias that is independent of the number of cycles n . Because of this, the bias of r_2 is usually much larger than r_1 . This arises from Eq. (20): the first-order bias term for r_1 , which is proportional to the variance of the mean of the denominator counts for n cycles, becomes proportional to the variance of an individual measurement x_i for r_2 . Since the variance of the mean of the counts over n cycles is equal to the variance of an individual measurement divided by n , the first-order bias term of r_1 is a factor of $1/n$ smaller than the first-order bias term of r_2 .

The important conclusion is that the expectation values of r_1 and r_2 are both strictly greater than R :

$$E\{r_{1,2}\} > \frac{E\{y\}}{E\{x\}} = \frac{\bar{Y}}{\bar{X}} = R \tag{23}$$

The difference between the expectation value of the ratio estimators r_1 or r_2 and the actual ratio R is the bias:

$$B\{r_1\} = E\{r_1\} - R \approx R \left[\frac{1}{n\bar{X}} + \frac{2}{n^2\bar{X}^2} \right] \tag{24}$$

$$B\{r_2\} = E\{r_2\} - R \approx R \left[\frac{1}{\bar{X}} + \frac{2}{\bar{X}^2} \right]$$

for independent x and y . The bias of r_1 decreases as the total number of denominator counts in the entire measurement $n\bar{X}$ increases, but the bias of r_2 only decreases as the expected number of counts per cycle \bar{X} increases, independent of the total number of cycles n .

Ratio estimators can be constructed with smaller bias than r_1 . Tin [4] looked at four ratio estimators and found that Beale's ratio estimator [9] has the least bias for finite and infinite source populations with x and y following a bivariate normal distribution. Additionally, Tin [4] found that Beale's estimator performed well

for small samples, e.g., $n = 50$. Srivastva et al. [10] confirmed that Beale's estimator outperformed the standard estimator r_1 in a bivariate normal model. Monte Carlo simulations by Hutchison [11] of various ratio estimators and Poisson-distributed random variables showed that Beale's estimator performed best for small n and small \bar{X} . We choose Beale's estimator for these reasons and also because of its simple structure and accuracy compared to the other estimators on very small and/or unusual data sets [4].

Beale's ratio estimator for $R = \bar{Y}/\bar{X}$, assuming infinite source populations, from which x and y are drawn, is:

$$r_3 = r_1 \frac{\left(1 + \frac{\text{cov}(x,y)}{n\bar{x}\bar{y}} \right)}{\left(1 + \frac{\text{var}(x)}{n\bar{x}^2} \right)} \tag{25}$$

where n is the number of sample pairs x_i and y_i , and:

$$\text{cov}(x,y) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \tag{26}$$

$$\text{var}(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \tag{27}$$

are the sample covariance and sample variance, respectively.

The expectation value of r_3 , analogous to Eqs. (18) and (21), is computed by Tin [4]. For our Poisson-distributed x and y and infinite source population, we find that Beale's estimator is unbiased to order $O(n^{-2})$. Beale's estimator, therefore, appears to perform much better than the standard ratio estimators r_1 and r_2 for our circumstances. The order $O(n^{-3})$ terms of r_3 assuming Poisson distributions are very complicated, but we note that the Poisson distribution is well-approximated by the normal (Gaussian) distribution for large mean values. The theoretical expectation value of r_3 assuming x and y are normally distributed only differs by 0.01% when $\bar{X} = 10$ from the expectation value assuming Poisson-distributed x and y . We report the expectation value of r_3 for \bar{X} and \bar{Y} larger than 10 using a normal approximation to order $O(n^{-3})$ as deduced by Tin [4] and Srivastva et al. [10]:

$$E\{r_3\} \approx R \left[1 - \frac{2}{n^2\bar{X}} \left(\frac{1}{\bar{X}} - \frac{S(x,y)}{\bar{X}\bar{Y}} \right) \left(1 + \frac{13}{2n} + \frac{8}{n\bar{X}} \right) \right] \tag{28}$$

For normally distributed and independent X and Y , the bias of r_3 is:

$$B\{r_3\} = E\{r_3\} - R \approx -R \left[\frac{2}{n^2\bar{X}^2} \left(1 + \frac{13}{2n} + \frac{8}{n\bar{X}} \right) \right] \tag{29}$$

Unlike r_1 and r_2 , the bias of r_3 is negative.

2.1. Computer simulations of the expectation values of r_1 , r_2 , and r_3

The bias in these ratio estimators can be shown to approximately agree with the above equations by simulating a large number of experiments computationally. We calculate r_1 , r_2 , and r_3 using Poisson-distributed counts randomly generated by `poissrnd` in MATLAB version 7.10.0. For simplicity, we assume $R = \bar{Y}/\bar{X} = 1$ for all computer simulations (the results are the same for all values of R). For each \bar{X} and number of cycles $n = 10, 100, 1000$, we generate $10^9, 10^8$, and 10^7 sets of simulated counts of x and y drawn from their associated Poisson distributions. For each of these sets, we calculate the ratio estimator r_1, r_2 , and r_3 . We use the maximum likelihood estimator `mle` in MATLAB to determine the mean of the $10^9, 10^8$, and 10^7 sets (for a given \bar{X} and n) and 95% confidence interval of the mean. We compare these simulated values to the derived equations for the biases $B\{r_{1,2,3}\}$.

For moderately large n and \bar{X} , we approximate the Poisson distribution by a normal distribution, which greatly speeds up the

random-number generation in MATLAB. To estimate the smallest biases, such as those for r_3 , we make a further assumption that the means of the counts follow a normal distribution, as do the sample variances and covariances of these counts [4]. The computer simulations closely follow the theoretical values, as shown in Fig. 1.

2.2. Time-varying count rate in SIMS data

The count rate of individual isotopes in a SIMS measurement can vary by large amounts over the course of a measurement. One contributing factor to this phenomenon is variation in primary beam intensity. In addition, as more primary ions become implanted into the sample, the ionization efficiency increases, and the measured count rate in the electron multiplier or Faraday cup increases. If these types of experimental variations are large compared to the statistical variance of the data, the isotope counts show a strong sample covariance. However, the statistical behavior of the data actually reflects a time-varying population mean: \bar{X} and \bar{Y} are no longer constant, but are time-dependent, and each subsample x_i and y_i samples from $\bar{X}(t)$ and $\bar{Y}(t)$ over the n cycles of the measurement.

We model this phenomenon with computer simulations by assuming the count rate changes (over a range from ~10% to several hundred percent) for the numerator and denominator over the course of the measurement following a sigmoidal function (continuously increasing) or various sinusoidal functions (increasing and decreasing). Our simulations show that the expectation value of r_1 and r_3 for these types of measurements should be calculated by substituting the measurement-time average of $\bar{X}(t)$ for \bar{X} in Eq. (19) and Eq. (28). The analogous expectation value of r_2 is computed differently: the time-varying expectation value is calculated from Eq. (22) by substituting $\bar{X}(t)$ for \bar{X} , and then the expectation value over the entire measurement is deduced by computing the average of this varying expectation value. The variances of

r_1, r_2 , and r_3 can be calculated in a similar way using the equations of Section 3. Using these methods, the statistical properties of a measurement with significant time-dependence can be understood.

3. Second statistical moment of ratio estimators r_1, r_2, r_3 : variance

We wish to measure a ratio (biased, as it is) as precisely as possible given a fixed number of cycles and isotope counts per cycle. The most efficient ratio estimator will have the lowest statistical variance, given by the second statistical moment of the ratio estimator.

The variance of r_1 , the ratio of the means, is:

$$V\{r_1\} = E\{(r_1 - E\{r_1\})^2\} = E\{r_1^2\} - E\{r_1\}^2 \tag{30}$$

Expanding Eqs. (16) and (18), while substituting the expressions in Eq. (17), and keeping terms of $O(n^{-2})$:

$$V\{r_1\} \approx R^2 \left[\frac{1}{n} \left(\frac{1}{\bar{X}} + \frac{1}{\bar{Y}} - \frac{2S(x,y)}{\bar{X}\bar{Y}} \right) + \frac{1}{n^2} \left(\frac{6}{\bar{X}^2} + \frac{3}{\bar{X}\bar{Y}} + S(x,y) \left(\frac{4}{\bar{Y}^2} - \frac{8}{\bar{X}\bar{Y}} - \frac{16}{\bar{X}^2\bar{Y}} + \frac{5S(x,y)}{\bar{X}^2\bar{Y}^2} \right) + \frac{4S(x^2,y)}{\bar{X}^2\bar{Y}} - \frac{2S(x,y^2)}{\bar{X}\bar{Y}^2} \right) \right] \tag{31}$$

which in the case of independent X and Y reduces to:

$$V\{r_1\} \approx R^2 \left[\frac{1}{n} \left(\frac{1}{\bar{X}} + \frac{1}{\bar{Y}} \right) + \frac{1}{n^2} \left(\frac{6}{\bar{X}^2} + \frac{3}{\bar{X}\bar{Y}} \right) \right] \tag{32}$$

Since r_2 is the mean of n ratios, we can write its variance, assuming the individual ratios y_i/x_i are uncorrelated, as:

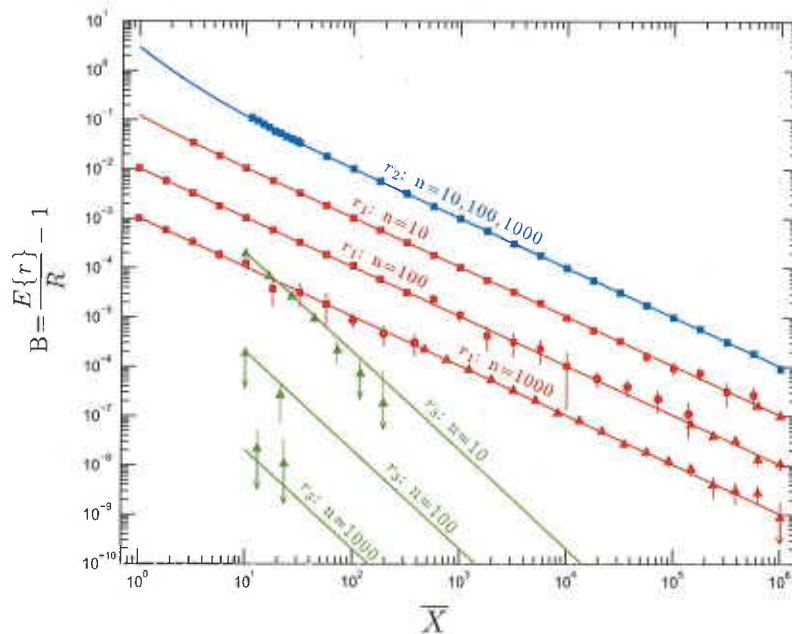


Fig. 1. The calculated bias of r_1, r_2 , and r_3 (Eqs. (24) and (29)) as a function of counts in the denominator for $n=10, 100$, and 1000 cycles are shown by curves. (The bias of estimator r_2 is independent of the number of samples n .) The computer simulations for Poisson-distributed x and y are given for each ratio estimator and n by squares, the 95% confidence interval of the bias are given by vertical lines when the uncertainty is larger than the size of the marker. Circles indicate computations where x and y are normally distributed, triangles indicate computations where \bar{X} , and \bar{Y} , as well as sample variances and covariances, were drawn from a normal distribution. Contributions from terms $O(\bar{X}^{-3})$ and smaller are evident for $B\{r_2\}$ near $\bar{X} = 10$ where simulation points are slightly more biased than the line of calculated bias. For small \bar{X} , calculation of r_2 results in divide-by-zero, so those values are excluded. Since the bias of r_3 is negative, the opposite of $B\{r_3\}$ is plotted.

$$V\{r_2\} = V\left\{\left(\frac{Y}{X}\right)\right\} = \frac{1}{n^2} V\left\{\sum_{i=1}^n \frac{y_i}{x_i}\right\} = \frac{1}{n^2} \sum_{i=1}^n V\left\{\frac{y_i}{x_i}\right\} = \frac{1}{n} V\left\{\frac{y}{x}\right\} \quad (33)$$

If the individual ratios y_i/x_i are not independent:

$$V\{r_2\} = \left(\frac{1}{n} + \frac{(n-1)\rho}{n}\right) V\left\{\frac{y}{x}\right\} \quad (34)$$

where ρ is the average covariance between individual ratios. However, in the application of SIMS measurements, the individual ratios will in general be uncorrelated. We substitute Eq. (31) with $n = 1$ into Eq. (33) and keep terms to $O(\bar{X}^{-2})$ which yields an approximation for the variance of r_2 :

$$V\{r_2\} \approx \frac{R^2}{n} \left[\frac{1}{\bar{X}} + \frac{1}{\bar{Y}} - \frac{10S(x,y)}{\bar{X}\bar{Y}} + \frac{6}{\bar{X}^2} + \frac{3}{\bar{X}\bar{Y}} + \frac{4S(x,y)}{\bar{Y}^2} \right] \quad (35)$$

In the case of independent X and Y , this reduces to:

$$V\{r_2\} \approx \frac{R^2}{n} \left[\frac{1}{\bar{X}} + \frac{1}{\bar{Y}} + \frac{6}{\bar{X}^2} + \frac{3}{\bar{X}\bar{Y}} \right] \quad (36)$$

For Beale's estimator r_3 , we assume the general case where x and y are Poisson-distributed. The variance of r_3 to order $O(n^{-2})$ as derived by Tin [4] is:

$$V\{r_3\} \approx R^2 \left[\frac{1}{n} \left(\frac{1}{\bar{X}} + \frac{1}{\bar{Y}} - \frac{2S(x,y)}{\bar{X}\bar{Y}} \right) + \frac{1}{n^2} \left(\frac{2}{\bar{X}^2} - \frac{4S(x,y)}{\bar{X}^2\bar{Y}} + \frac{S(x,y)^2}{\bar{X}^2\bar{Y}^2} + \frac{1}{\bar{X}\bar{Y}} \right) \right] \quad (37)$$

which in the case of independent X and Y reduces to:

$$V\{r_3\} \approx R^2 \left[\frac{1}{n} \left(\frac{1}{\bar{X}} + \frac{1}{\bar{Y}} \right) + \frac{1}{n^2} \left(\frac{2}{\bar{X}^2} + \frac{1}{\bar{X}\bar{Y}} \right) \right] \quad (38)$$

For a large number of counts, the $O(n^{-1})$ approximation shows that all three ratio estimators r_1, r_2, r_3 have equal variance:

$$V\{r_{1,2,3}\} \approx R^2 \left[\frac{1}{n} \left(\frac{1}{\bar{X}} + \frac{1}{\bar{Y}} - \frac{2S(x,y)}{\bar{X}\bar{Y}} \right) \right] \quad (39)$$

which is the expected and familiar result [12].

3.1. Computer simulations of the variances of r_1, r_2 , and r_3

Similarly, the variances $V\{r_{1,2,3}\}$ for the derived equations and computer simulations (calculated with 10^6 sets) are shown in Fig. 2. For more than a few tens of counts per cycle, the variance follows the familiar expression given in Eq. (39). The variance of the mean of the ratios, r_2 , is substantially more than this for less than a few tens of counts per cycle. If a 1σ uncertainty for r_2 is computed from the square root of the approximation given in Eq. (39), this value will significantly underestimate the actual uncertainty of the r_2 ratio estimator. The variance of the ratio of means, r_1 , is very slightly larger than Eq. (39) for very low counts.

4. Third and fourth statistical moment of ratio estimators of r_1, r_2, r_3 : skewness and kurtosis

The third and fourth statistical moments, the skewness (γ_1) and kurtosis (γ_2), determine if a random variate is approximately normally distributed. Assuming normally distributed x and y , following Tin [4], to first order in $1/n$ the skewness of the ratio estimators r_1 and r_3 are:

$$\gamma_1\{r_1\} = \frac{E\{(r_1 - E\{r_1\})^3\}}{(V\{r_1\})^{3/2}} \approx \left(\frac{\bar{Y}\Omega}{(n\bar{X}\bar{Y}^2\Omega^2 + \bar{X}^2\bar{Y})^{1/2}} \right) \left(6 + \left(44 + \frac{1}{\bar{Y}\Omega^2/\bar{X} + 1} \right) \frac{1}{n\bar{X}} \right) \quad (40)$$

$$\gamma_1\{r_3\} = \frac{E\{(r_3 - E\{r_3\})^3\}}{(V\{r_3\})^{3/2}} \approx \left(\frac{\bar{Y}\Omega}{(n\bar{X}\bar{Y}^2\Omega^2 + \bar{X}^2\bar{Y})^{1/2}} \right) \left(6 + \left(26 - \frac{1}{\bar{Y}\Omega^2/\bar{X} + 1} \right) \frac{1}{n\bar{X}} \right) \quad (41)$$

with:

$$\Omega = (1 - \bar{X}S(x,y))$$

The skewness of r_2 can be found (analogous to our derivation of the variance of r_2) by setting $n = 1$ in the expression for $\gamma_1\{r_1\}$ and then multiplying this expression by $1/\sqrt{n}$

$$\gamma_1\{r_2\} = \frac{E\{(r_2 - E\{r_2\})^3\}}{(V\{r_2\})^{3/2}} \approx \left(\frac{\bar{Y}\Omega}{(n\bar{X}\bar{Y}^2\Omega^2 + n\bar{X}^2\bar{Y})^{1/2}} \right) \left(6 + \left(44 + \frac{1}{\bar{Y}\Omega^2/\bar{X} + 1} \right) \frac{1}{\bar{X}} \right) \quad (42)$$

However, this expression underestimates the skewness of r_2 for \bar{X} less than a few hundred counts. This is because there are more terms in the right-hand factor of Eq. (42) (of order $1/\bar{X}^2, 1/\bar{X}^3, \dots$) that increase the skewness of r_2 quickly for small \bar{X} . Computer simulations show that the $1/\bar{X}^2$ term is important for predicting the skewness of r_2 down to tens of counts in \bar{X} .

We calculate the kurtosis of the ratio estimators, again following Tin [4], and find that similar to the second and third statistical moments (variance and skewness), the kurtosis of r_2 is very close to the kurtosis of r_1 and r_3 down to a few tens of counts in \bar{X} . Below that, the kurtosis of r_2 increases extremely rapidly for $n = 1000, 100, 10$ as shown by computer simulations. We do not report details of the kurtosis calculations or simulations here as they are similar to the results for skewness of the ratio estimators.

4.1. Computer simulations of the skewness of r_1, r_2 , and r_3

A normal distribution of x and y was assumed. Simulated random counts were created in MATLAB as for the simulations of expectation value and variance (Sections 2.1 and 3.1), and the skewness and kurtosis of 10^4 sets of ratios for each ratio estimator r_1, r_2 , and r_3 were computed by MATLAB's `skewness` and `kurtosis` functions. The mean and its 95% confidence interval of 1000 kurtosis and skewness values were computed using MATLAB's `mean` function. The results of the computer simulations for skewness of the three ratio estimators are shown in Fig. 3.

4.2. Approach to normality of r_1, r_2 , and r_3

Ratio estimators in SIMS analysis are typically treated as variates following a normal distribution. Such a treatment greatly simplifies the task of calculating the statistical properties of a measurement, such as its statistical uncertainty. However, if the underlying distribution of the variate is not normal, it is not valid to treat it as such. Here we investigate under what conditions of

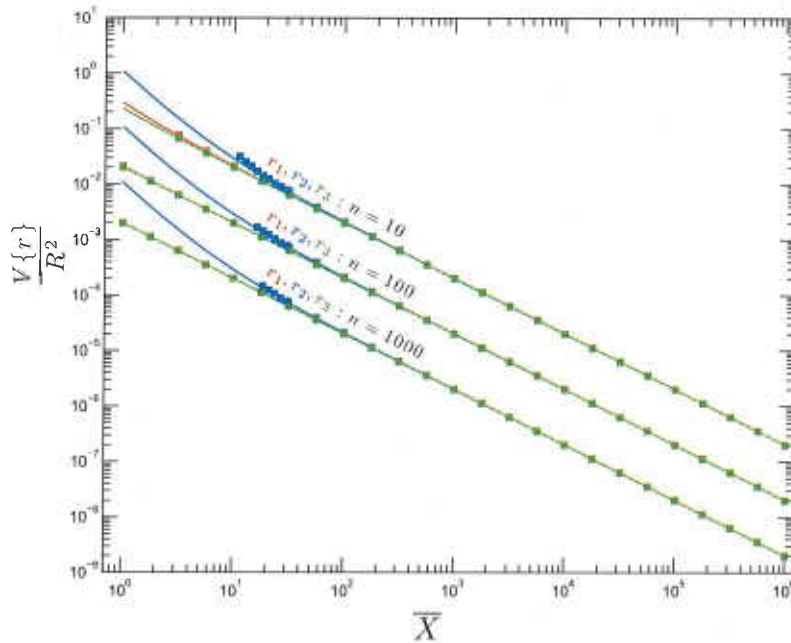


Fig. 2. The calculated variance of r_1 , r_2 , and r_3 (Eqs. (32), (36), (38)) divided by R^2 as a function of counts in the denominator for 10, 100, and 1000 cycles are shown by curves. The computer simulations for Poisson-distributed x and y are given for each ratio estimator and n by squares, the 95% confidence interval of the variance is smaller than the size of the marker in all cases. More simulations were run for tens of counts in \bar{X} to show the deviation of variance in r_2 from r_1 and r_3 . Contributions from terms $O(\bar{X}^{-3})$ and smaller are evident for $V\{r_2\}/R^2$ near $\bar{X} = 10$ and $n = 10$ where the simulation points show slightly higher variance than the line of calculated variance. For \bar{X} smaller than about 10, calculation of r_2 results in divide-by-zero, so those values are excluded from the plot. The calculated variance and computer simulations for r_1 are very close to r_3 and are thus obscured in this plot.

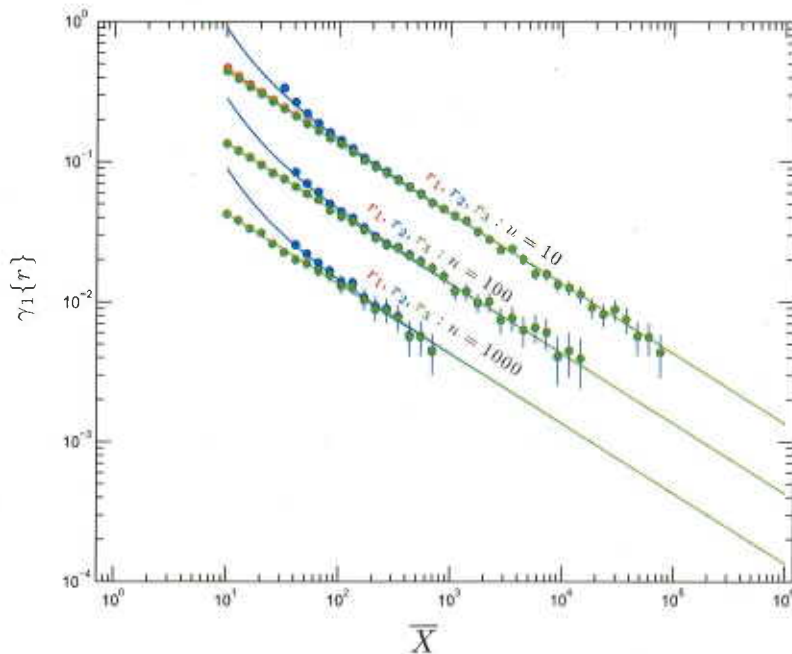


Fig. 3. The calculated skewness of r_1 , r_2 , and r_3 (Eqs. (40), (42), and (41)) as a function of counts in the denominator \bar{X} (equal to counts in the numerator \bar{Y} for these calculations) for 10, 100, and 1000 samples are given by curves. An approximation for the $1/\bar{X}^2$ term is included for the calculated skewness of r_2 . The computer simulations for normally distributed x and y (so $\bar{X} \geq 10$) are given for each ratio estimator and n by circles, the 95% confidence interval of the skewnesses are given by error bars for some values of r_2 , the uncertainties for r_1 and r_3 are always smaller than the size of the marker. For small \bar{X} , calculation of r_2 results in divide-by-zero, so those values are excluded.

the average number of counts per cycle (\bar{X}) and number of cycles (n) this assumption is valid for the three ratio estimators r_1 , r_2 , and r_3 .

The Jarque–Bera hypothesis test uses the sample kurtosis and skewness to determine the departure from normality of a given distribution [13]. The test statistic is:

$$JB = \frac{\eta}{6} \left(\gamma_1^2 + \frac{(\gamma_2 - 3)^2}{4} \right) \quad (43)$$

where η is the sample size, γ_1 is the skewness and γ_2 is the kurtosis. We employ MATLAB's `jbtest` function to determine when a ratio estimator (r_1 , r_2 , or r_3) with a given \bar{X} and n is normally distributed. If there is less than a 5% chance that the data is normally distributed, `jbtest` returns one, otherwise it returns zero. For $\eta > 2000$ the statistic follows a χ^2 distribution, for smaller η , `jbtest` uses results from Monte-Carlo simulations. We performed the Jarque–Bera test 1000 times on each set of $\eta = 10^4$ ratios of r_1 , r_2 , and r_3 for a given \bar{X} and n . From these 1000 tests, we calculated the mean of the Jarque–Bera test; results are shown in Fig. 4.

Where the Jarque–Bera test returns zero, the distribution is normal. We assign \bar{X}_{norm} to the number of counts per cycle where the average of the 1000 Jarque–Bera tests falls below 0.5 in our simulations. At this number of counts per cycle for a given number of cycles n , the ratio estimator can be considered normal, and standard normal statistics can be applied. The \bar{X}_{norm} values for the three ratio estimators and $n = 10, 100, 1000$ are given in Table 1.

5. An example analysis of SIMS data using ratio estimators r_1 , r_2 , and r_3

It is important to note that the biases derived in Section 2 are for the expectation values of the ratio estimators: it is not true that for any given measurement $r_2 > r_1 > r_3$, only that the long-run average (if the measurement is performed an infinite number of times) of these ratio estimators will show this behavior. If the bias is small compared to the variance of a given measurement, the calculated ratio estimators have a reasonably high probability of not being ordered as $r_2 > r_1 > r_3$. Simply subtracting the bias from r_2 to reduce the bias in the calculated ratio is not a good strategy: the bias depends on the population mean \bar{X} which is estimated in an unbiased way by \bar{x} . However \bar{x} has a variance of \bar{X}/n which

Table 1

The minimum values required for normality of the three ratio estimators for number of cycles = 10, 100, 1000 as determined by computer simulations and the Jarque–Bera test.

r	n	\bar{X}_{norm}
r_1	10	599
r_2	10	611
r_3	10	601
r_1	100	64
r_2	100	76
r_3	100	64
r_1	1000	< 10
r_2	1000	30
r_3	1000	< 10

means the estimation of the bias is uncertain, and the experimenter is better off using a ratio estimator with lower bias to begin with.

As an example of the dangers of using the ratio estimator r_2 , we look at secondary ion mass spectrometry (SIMS) isotope measurements of troilite (FeS) in the LL3.1 ordinary chondrite Krymka. The presence of extinct short-lived radionuclides in meteoritic material can be explained by production due to energetic particle spallation or contribution from a nearby stellar source, such as a supernova. However, ^{60}Fe is not efficiently produced by spallation, so the presence of its daughter ^{60}Ni would be evidence of a stellar source. The Krymka meteorite has experienced little metamorphism to disturb the Fe–Ni isotopic system. When $^{60}\text{Ni}/^{61}\text{Ni}$ is plotted against $^{56}\text{Fe}/^{61}\text{Ni}$, the SIMS measurements will fall on a line (an isochron), with slope equal to the initial abundance of $^{60}\text{Fe}/^{56}\text{Fe}$. Measurements of the Fe–Ni system in Krymka troilite were acquired on a Cameca ims 6f at Arizona State University. The ratios $^{60}\text{Ni}/^{61}\text{Ni}$ and $^{56}\text{Fe}/^{61}\text{Ni}$ were initially calculated using r_2 , the mean of ratios, for 100 or 200 cycles with 15–100 counts per cycle of ^{61}Ni (\bar{X}). Using r_2 , the inferred initial abundance of $^{60}\text{Fe}/^{56}\text{Fe}$ in troilite was $(1.0\text{--}1.8) \times 10^{-7}$ [14]. When these ratios are calculated using

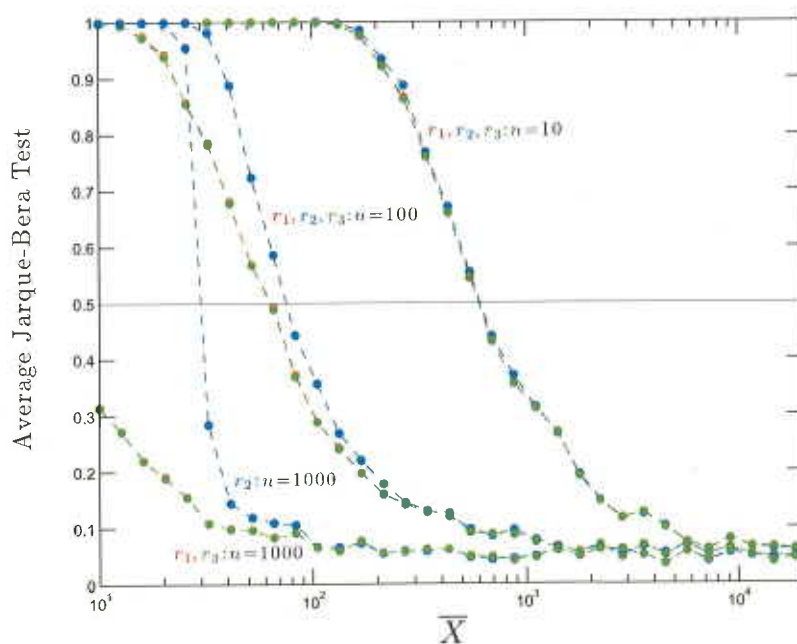


Fig. 4. The average Jarque–Bera test for 1000 simulations of r_1 , r_2 , and r_3 , with Poisson-distributed counts, as a function of average counts in the denominator \bar{X} . A value of zero means the null hypothesis “the data is normally distributed” cannot be rejected at the 5% significance level, a value of one means this hypothesis can be rejected at the 5% significance level. The \bar{X} value at which the curve becomes less than 0.5 is the number of counts we require for a given ratio estimator (with a given n) to be considered normally distributed. Values for r_1 are similar to values for r_3 so are mostly obscured.

r_1 , the ratio of means, this detection goes away, and the initial abundance of $^{60}\text{Fe}/^{56}\text{Fe}$ in the Krymka troilite grains is no longer resolved from zero (see Fig. 5).

6. Proper ratio calculations in mass spectrometry

We have shown that the ratio estimator r_2 , the mean of individually measured ratios, is poorly behaved in its first four statistical moments, especially with low counts. Most importantly, its expectation value is heavily biased and this bias is independent of the number of cycles over which the mean of the ratios is calculated. The bias of r_2 can be as much as several percent for tens of counts in the denominator (reference) isotope. The variance of r_2 can be significantly worse than other ratio estimators for tens of isotope counts, resulting in much larger uncertainties on the measured ratio. Additionally, r_2 cannot be treated as a normally distributed variate for ~ 30 counts per cycle unless ~ 1000 cycles are measured, a factor of ~ 3 more cycles than are needed for the two other ratio estimators.

The bias of r_2 is strictly positive; it cannot be thought of as another source of random error. It simply results in a calculated ratio higher than the actual source ratio. In many experiments, calculating r_2 instead of r_1 or r_3 is experimentally easier because instrumental conditions change over the course of the measurement. Time interpolation of the ratios is a standard method of dealing with this problem. However, one can time-interpolate the ion signals without calculating individual ratios for each cycle. For example, if one wishes to calculate r_1 , the ratio of the mean count rates, one can interpolate the signal for the denominator such that each measurement is modeled to have occurred at the same time as the measurement of the numerator. The mean count rates can then be determined for the measured signal in the numerator and the calculated signal in the denominator, and the final ratio r_1 can be calculated.

In experiments where the ratio estimator r_2 , the mean of ratios, must be used instead of r_1 , the ratio of means, the effect of ratio bias on the final result of the measurement can be understood in terms of the relative bias of the ratio divided by the relative standard deviation of the ratio. For r_2 , the relative bias of the ratio is independent of the number of cycles n , but the relative statistical standard deviation decreases as n increases. If the expected bias is much smaller than the statistical uncertainty of the measurement, the bias can be ignored. We define the relative bias of the ratio divided by the relative standard deviation (the square root of the variance) for r_2 . Using Eqs. (24) and (36), and keeping only first order terms:

$$\frac{B\{r_2\}/R}{\sqrt{V\{r_2\}/R}} \approx \frac{\frac{1}{\bar{X}}}{\sqrt{\frac{1}{n}\left(\frac{1}{\bar{X}} + \frac{1}{\bar{Y}}\right)}} = \sqrt{\frac{nR}{\bar{X} + \bar{Y}}} \quad (44)$$

For the ratio bias associated with using r_2 to be insignificant, we require this quantity to be small. For example, if we require the bias to be less than 10% of the statistical standard deviation of the measurement, we can deduce an upper bound on how many cycles n can be used to calculate r_2 :

$$n < \frac{(\bar{X} + \bar{Y})}{100R} \quad (45)$$

That is, for the ratio bias associated with using r_2 (the mean of ratios) to be insignificant relative to the statistical uncertainty of the measurement, the number of cycles (n) must be smaller than the sum of expected counts per cycle ($\bar{X} + \bar{Y}$) divided by 100 times the expected ratio (R).

In multi-collector inductively coupled plasma mass spectrometry (MC-ICP-MS), count rates are typically higher than SIMS and, correspondingly, the measured ratios are usually more precise. Isotopes of Pb are measured using MC-ICP-MS with a precision of ~ 50 ppm or better (e.g., [15]). The ratio estimator r_2 is typically

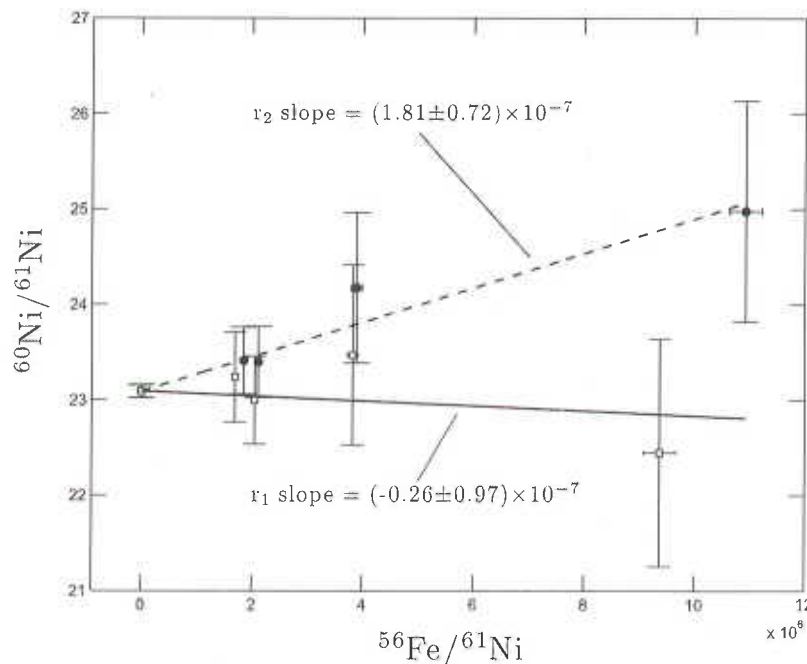


Fig. 5. Fe–Ni isotopic measurements of troilite (FeS) in the Krymka LL3,1 ordinary chondrite made by a Cameca ims 6f secondary ion mass spectrometer (SIMS) at Arizona State University [14]. Shown are two ratio estimators: r_2 (filled circles; data from [14]) and r_1 (unfilled squares). Lines are fits to the data points using weighted total least-squares, the slope of which is equal to the inferred initial $^{60}\text{Fe}/^{56}\text{Fe}$ in the troilite. A significant bias is associated with analyzing the data using the mean of ratios (r_2), resulting in a false detection of extinct ^{60}Fe . This false detection disappears when a less-biased ratio estimator, r_1 , is used on the same data. Isotopes were measured for 100 or 200 cycles (n) with 15–100 counts per cycle of ^{61}Ni (\bar{X}).

used in ICP-MS. In the case of Pb, the counts of all the isotopes in the collectors are so high ($\bar{X}, \bar{Y} \approx 10^9$) that the ratio bias from calculating r_2 over ~ 100 cycles is insignificant (Eqs. (44) and (45)).

In studies of radioactive nuclides and their daughter isotopes using SIMS, the daughter isotope is often in the numerator, and a high calculated ratio is often a positive detection and of scientific interest. A significant positive bias from calculating the ratio as r_2 is therefore potentially misleading and should be avoided whenever possible. The final ratio should only be calculated as r_2 if the bias can be estimated to be sufficiently small compared to the statistical uncertainty of the measurement (Eqs. (44) and (45)) and any additional systematic uncertainty resulting from the use of r_1 or r_3 is significant.

The ratio of means, r_1 is also biased but the bias tends to zero as the number of cycles n goes to infinity. For certain high-accuracy measurements, the bias of r_1 , the ratio of the means, may still be too large to test the hypothesis at hand with sufficiently high confidence. In this case, Beale's estimator r_3 should be used, which has $\sim 1\%$ or less of the bias of r_1 , and similar variance and approach to normality.

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